Thesis

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CYCLIC SINGULAR HOMOLOGY

BY

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Preface

This thesis completes my masters degree in mathematics at the University of Oslo, and has an extent of 60 credits.

The project has given me a glimpse into the world of algebraic topology, and has been very inspiring to work on. It is a sad day, now that it all comes to an end.

First, I wish to thank my supervisor Paul Arne Østvær, who not only came up with this exiting project, but who also introduced me to the realms of algebraic topology during the spring of 2006. Thank you for guiding me through the mazes, and for sharing your knowledge. (I advice anyone who has the chance: Attend one of his algebraic topology courses, they are brilliant).

A great thanks to Tron Omland, who had the patience to go through the text, weeding out misspellings and stray symbols. Since I have been studying part time these years, I would also like to thank my employer Telenor, for giving me the opportunity to follow lectures colliding with working hours.

And last, but most of all, I want to thank my dearest Rigmor, whose support is invaluable, and which I could never have completed this project without. All these years you have put up with me spending afternoons, weekends and vacations stooping over math books, when we should have been going to fancy restaurants.

University of Oslo, May 2008

Daniel Bakkelund

Contents

Introduction

Homology theory is probably the most studied and well-understood branch of algebraic topology, rooting back at least to the works of Poincaré, and singular homology was described by Eilenberg in 1940 [3]. The axiomatic approach to the subject, first announced in Eilenberg and Steenrods 1945-paper [5], and exhaustively treated in their celebrated book "Foundations of algebraic topology" [4] from 1952, introduced what has become the Eilenberg-Steenrod axioms for homology and cohomology theories. One of the main achievements of this work was the separation of homological algebra from algebraic topology [9, §2], whereafter homological algebra took on a life of its own.

Cyclic homology is a branch of homological algebra, discovered independently by Connes [1], Feigin and Tsygan [6], and Loday and Quillen [11], all in 1983. It is a modification of Hochschild homology, a homology theory for algebras, and it has found some of its main applications within noncommutative geometry and K-theory. Over the 25 years that has passed, almost no visible attempt has been made to construct a cyclic singular homology theory, an observation that alone justifies this thesis. The only exception is the 2006-paper [16] of Jinhyun Park, which was the inspiration of this project.

In the first three sections of [16], Park reviews cyclic homology in terms of precyclic objects, and as an example application, he shows that the singular chain complex $S(X)$ of a topological space X is precyclic. He defines cyclic singular homology, and calculates the cyclic singular homology groups of a point:

$$HC_n(*) \;\cong\; \begin{cases} \mathbb{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

In this thesis, we prove that if $X^\bullet$ is a pre-cocyclic object in an arbitrary category $\mathbf{C}$, and D is some object in $\mathbf{C}$, the free abelian groups $Ab(Hom_{\mathbf{C}}(X^\bullet, D))$ are precyclic objects, allowing us to define a cyclic double complex $CC(D) = CC(Ab(Hom_{\mathbf{C}}(X^\bullet, D)))$. We may then define the cyclic homology of D to be the total homology of this double complex:

$$HC_n(D) \;=\; H_n(Tot(CC(D))), \quad \text{for } n \geq 0.$$

If in addition $X^\bullet$ is equipped with a certain last degeneracy, this will give rise to Connes periodicity exact sequences

$$\cdots \xrightarrow{B} H_n(D) \xrightarrow{I} HC_n(D) \xrightarrow{S} HC_{n-2}(D) \xrightarrow{B} H_{n-1}(D) \xrightarrow{I} \cdots,$$

where $H_\bullet(D)$ is the homology of $Ab(Hom_{\mathbf{C}}(X^\bullet, D))$ considered as a chain complex as described in section 1.1. To achieve this as a general result, we have to choose a last degeneracy differently from Park's, and also different from the extra degeneracy of Loday [10].

Next, we show that the topological n-simplicies Δ^n for $n \geq 0$ make up a pre-cocyclic object with the last degeneracy. Since the singular chain complex of a topological space X is given by the free abelian groups over the Hom-sets $Hom_{\mathbf{Top}}(\Delta^n, X)$ for $n \geq 0$, the singular chain complex is indeed a precyclic object with the last degeneracy. This leads naturally to the same definition of cyclic singular homology of topological pairs (X, A) as that of Parks definition of cyclic singular homology of topological spaces:

$$HC_n(X, A) \quad = \quad H_n(Tot(CC(S(X, A)))) \quad \text{for } n \geq 0.$$

We proceed by proving that the above construction is functorial, and that the Eilenberg-Steenrod axioms for generalized homology theories hold. We also define reduced cyclic singular homology, and compute the cyclic singular homology groups of some classical spaces, such as the Torus:

$$HC_n(T) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } n > 0. \end{cases}$$

The last part of the thesis presents an attempted construction of a cyclic homology theory on Δ-complexes, and to achieve this, we introduce cyclic Δ-complexes. Somewhat surprisingly, it turns out that the cyclic homology of a cyclic Δ-complex of a topological space X is isomorphic to the singular homology of X.

The layout of the thesis is as follows: In Chapter 1 we recall singular homology and the Eilenberg-Steenrod axioms for generalized homology theories. Chapter 2 recalls double complexes and cyclic homology, and this is where we construct the framework for cyclic homology over Hom-sets. Chapter 3 defines cyclic singular homology in terms of the framework we established in Chapter 2, prove that the Eilenberg-Steenrod axioms hold, and calculate cyclic singular homology for spheres and orientable and nonorientable surfaces. The last chapter, Chapter 4, presents the attempted construction of a cyclic homology theory on Δ-complexes.

Chapter 1

Homology Theory

In this chapter we first recall the constructions and main properties of singular homology and relative singular homology functors. Second, we recall the axioms of generalized homology theories.

1.1 Pre-cosimplicial objects

The construction of singular homology is usually done from the simplicial category $\mathbf{\Delta}$. However, the initial constructions required only make use of a subcategory. The following two definitions are based on [16, §1]:

Definition 1.1.1 *The **presimplicial category** $\mathbf{\Delta_{pre}}$ has objects $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and morphisms are generated by the **coface morphisms** $\delta^i : [n-1] \to [n]$ for $0 \leq i \leq n$, defined by*

$$\delta^i(k) = \begin{cases} k & \text{for } k < i \\ k+1 & \text{for } k \geq i. \end{cases} \tag{1.1.1}$$

The coface morphisms can easily be seen to satisfy the relation

$$\delta^j \delta^i = \delta^i \delta^{j-1} \text{ for } i < j. \tag{1.1.2}$$

Definition 1.1.2 *A **pre-cosimplicial object** in a category $\mathbf{C}$ is a functor $X : \mathbf{\Delta_{pre}} \longrightarrow \mathbf{C}$, alternatively a set of objects $X^\bullet = \{X^n\}_{n\geq 0}$ in $\mathbf{C}$ together with morphisms $\partial^i = X(\delta^i) : X^{n-1} \longrightarrow X^n$ for $0 \leq i \leq n$, called **coface operators**. From (1.1.2), we deduce*

$$\partial^j \partial^i = \partial^i \partial^{j-1} \text{ for } i < j. \tag{1.1.3}$$

Suppose $\mathbf{C}$ is a preadditive category [13, §I.8], so that Hom-sets in $\mathbf{C}$ are additive abelian groups and composition of morphisms is bilinear, i.e.

$$(f + f') \circ (g + g') = f \circ g + f \circ g' + f' \circ g + f' \circ g'. \tag{1.1.4}$$

The following construction will be used to define the singular chain complex of topological spaces in Definition 1.2.3.

Lemma-definition 1.1.3 *Given a pre-cosimplicial object $X^\bullet$ together with an object C in a preadditive category $\mathbf{C}$, we can define maps $\partial_n : Hom_{\mathbf{C}}(X^n, C) \longrightarrow Hom_{\mathbf{C}}(X^{n-1}, C)$ for $n \geq 1$ by*

$$\partial_n(f) \;=\; \sum_{i=0}^{n}(-1)^i f \circ \partial^i, \tag{1.1.5}$$

where f is an element in $Hom_{\mathbf{C}}(X^n, C)$. Then ∂_n is a group homomorphism, and $\partial_{n-1} \circ \partial_n = 0$.

Proof The Hom sets in $\mathbf{C}$ are abelian groups by assumption, and ∂_n is a homomorphism due to (1.1.4):

$$\partial_n(f+g) = \sum_{i=0}^{n}(-1)^i (f+g) \circ \partial^i = \sum_{i=0}^{n}(-1)^i (f \circ \partial^i + g \circ \partial^i) = \partial_n(f) + \partial_n(g).$$

By a standard calculation based on properties (1.1.3) and (1.1.4), we have

$$\begin{aligned}
\partial_{n-1} \circ \partial_n(f) &= \partial_{n-1}\left(\sum_{j=0}^{n}(-1)^j f \circ \partial^j\right) \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j} f \circ \partial^j \partial^i \\
&= \sum_{0 \leq i < j \leq n} (-1)^{i+j} f \circ \partial^j \partial^i \;+\; \sum_{0 \leq j \leq i \leq n-1} (-1)^{i+j} f \circ \partial^j \partial^i \\
&= \sum_{0 \leq i < j \leq n} (-1)^{i+j} f \circ \partial^i \partial^{j-1} \;+\; \sum_{0 \leq j \leq i \leq n-1} (-1)^{i+j} f \circ \partial^j \partial^i \\
&= \sum_{0 \leq k \leq l \leq n-1} (-1)^{k+l+1} f \circ \partial^k \partial^l \;+\; \sum_{0 \leq j \leq i \leq n-1} (-1)^{i+j} f \circ \partial^j \partial^i \\
&= -\sum_{0 \leq k \leq l \leq n-1} (-1)^{k+l} f \circ \partial^k \partial^l \;+\; \sum_{0 \leq j \leq i \leq n-1} (-1)^{i+j} f \circ \partial^j \partial^i \\
&= 0,
\end{aligned}$$

where the change of indices in the fifth equality is according to the identities $i = k$ and $j = l + 1$. □

Corollary 1.1.4 *Given a pre-cosimplicial object $X^\bullet$ and an object C in a preadditive category $\mathbf{C}$ together with the map described in Lemma-definition 1.1.3, the pair $(Hom_{\mathbf{C}}(X^\bullet, C), \partial_\bullet)$ is a chain complex of abelian groups:*

$$\cdots \longrightarrow Hom_{\mathbf{C}}(X^n, C) \xrightarrow{\partial_n} Hom_{\mathbf{C}}(X^{n-1}, C) \xrightarrow{\partial_{n-1}} \cdots$$
$$\cdots \xrightarrow{\partial_2} Hom_{\mathbf{C}}(X^1, C) \xrightarrow{\partial_1} Hom_{\mathbf{C}}(X^0, C).$$

It is possible to generalize this construction to any category $\mathbf{D}$ by replacing the Hom-sets by the free abelian groups generated by these sets,

$Ab(Hom_{\mathbf{D}}(X^\bullet, D))$. The elements in $Ab(Hom_{\mathbf{D}}(X^n, D))$, called **n-chains**, are formal sums of the form

$$\sum_{\sigma \in Hom_{\mathbf{D}}(X^n, D)} \alpha_\sigma \sigma,$$

where $\alpha_\sigma \in \mathbb{Z}$ and all but finitely many α_σ's are zero. Addition of chains is performed term wise, and since (1.1.5) is linear the above results hold.

1.2 Singular homology

We now construct singular homology (or more precisely, singular n-chains) based on pre-cosimplicial objects and Hom-sets. We start by defining the standard topological n-simplex, this definition is from [15, §16.1].

Definition 1.2.1 *For the **standard topological n-simplicies***

$$\Delta^n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \middle| \sum x_i = 1,\ 0 \leq x_i \leq 1,\ 0 \leq i \leq n\right\} \text{ for } n \geq 0,$$

*we define **cosimplicial face maps** $\partial^i : \Delta^{n-1} \longrightarrow \Delta^n$ as*

$$\partial^i(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}) \quad \text{for } 0 \leq i \leq n. \tag{1.2.1}$$

The cosimplicial face maps can easily be seen to satisfy the identity $\partial^j \partial^i = \partial^i \partial^{j-1}$ when $i < j$, and hence $(\Delta^\bullet, \partial^\bullet)$ makes up a pre-cosimplicial object. Since **Top** (the category of topological spaces and continuous maps) is not preadditive, we apply the strategy described after Corollary 1.1.4 to obtain a group structure on the Hom-sets. The following definitions are from [8, p108]:

Definition 1.2.2 *Given a topological space X, a **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \longrightarrow X$. The free abelian group $C_n(X)$, generated by the set of all singular n-simplices $\Delta^n \longrightarrow X$, is **the group of singular n-chains** in X.*

Since $C_n(X)$ is the free abelian group generated by $Hom_{\mathbf{Top}}(\Delta^n, X)$, the construction in Lemma-definition 1.1.3 can be applied directly to obtain a homomorphism, the **boundary map** $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$, defined by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \partial^i \tag{1.2.2}$$

(equivalent to the definition in [15, §16.1] and obtained as a special case of Corollary 1.1.4). In particular $\partial_{n-1} \circ \partial_n = 0$. The two next definitions are also from [8, p108]:

Definition 1.2.3 *The **singular chain complex** $S(X)$ of a topological space X is given by*

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Definition 1.2.4 *the **n-th singular homology group** $H_n(X)$ (or $H_n^{sing}(X)$) of a topological space X is the n-th homology group of the singular chain complex:*

$$H_n(X) = H_n(S(X)) \quad \text{for } n \geq 0.$$

A trivial calculation

We will now compute the singular homology of a point $*$. Since there is only one map $\Delta^n \longrightarrow *$ for any $n \geq 0$, all $C_n(*)$ are free abelian groups of one generator, hence, isomorphic to $\mathbb{Z}$. The maps ∂_n are then trivial for every odd n and isomorphisms for every even n, except $\partial_0 : C_0 \longrightarrow 0$, which is also trivial. The resulting homology is

$$H_n(*) \cong \begin{cases} \mathbb{Z} & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n \neq 0. \end{cases} \tag{1.2.3}$$

1.2.1 Relative singular homology

Definition 1.2.5 *A* ***topological pair*** *(or simply a* ***pair****)* (X, A) *is a topological space* X *together with a subspace* $A \subset X$*. A* ***map of pairs*** *is a map* $f : (X, A) \longrightarrow (Y, B)$ *such that* $f : X \longrightarrow Y$ *is a continuous map and* $f(A) \subset B$*.* (X, A) *is said to be a* ***disjoint union of pairs*** *if there are pairs* (X_α, A_α) *such that* $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$*.*

For simplicity of notation, the pair $(X, \emptyset)$ *will mostly be denoted only by* X*.*

Singular homology, as described above, can be generalized to topological pairs (cf. [8, p115]): Given a pair (X, A), the group $C_n(A)$ is an abelian subgroup of $C_n(X)$, so we can form the quotient groups $C_n(X, A) = C_n(X)/C_n(A)$. In this quotient, the trivial simplicies are the simplicies in A, and since ∂_n takes $C_n(A)$ into $C_{n-1}(A)$ it induces a homomorphism $C_n(X, A) \longrightarrow C_{n-1}(X, A)$ by applying the boundary homomorphism to representatives in the equivalence classes, i.e $\partial_n([\sigma]) = [\partial_n(\sigma)]$. We obviously get $\partial_{n-1} \circ \partial_n([\sigma]) = 0$, so we have a new chain complex $S(X, A) = (C_\bullet(X, A), \partial_\bullet)$, the **relative singular chain complex on** (X, A). The **relative singular homology** of the pair (X, A) is the homology of this chain complex:

$$H_n(X, A) \quad = \quad H_n(C_\bullet(X, A)) \quad \text{for } n \geq 0,$$

where the relation to ordinary singular homology is given by the isomorphism

$$H_n(X) \quad \cong \quad H_n(X, \emptyset).$$

For a topological pair (X, A), there is a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(A) \xrightarrow{\ i\ } C_\bullet(X) \xrightarrow{\ q\ } C_\bullet(X, A) \longrightarrow 0,$$

where i is induced by the inclusion $A \hookrightarrow X$ and q is the canonical quotient homomorphism. Such a short exact sequence of chain complexes gives rise to a long exact sequence in homology [8, p117]:

$$\cdots \longrightarrow H_n(A) \xrightarrow{\ i_*\ } H_n(X) \xrightarrow{\ q_*\ } H_n(X, A) \xrightarrow{\ \partial_*\ } H_{n-1}(A) \longrightarrow \cdots. \tag{1.2.4}$$

If $\overline{z}$ is an element in $H_n(X, A)$ the the **connecting homomorphism** $\partial_* : H_n(X, A) \longrightarrow H_{n-1}(A)$ is defined by applying the boundary map

$\partial_n : C_n(A) \longrightarrow C_{n-1}(A)$ to $\overline{z}$ and taking the homology class of the resulting element: $\partial_*(\overline{z}) = \overline{\partial_n(z)}$. The construction is obtained by applying the Snake lemma [19, Lemma 1.3.2] to the short exact sequence of chain complexes (cf. [8, p117]).

1.2.2 $H_\bullet$ as a functor

Definition 1.2.6 *The category* **PTop** *of topological pairs has as objects all topological pairs and the morphisms are maps of pairs.*

This category has **Top** as a full subcategory, where the inclusion functor is given by $X \longmapsto (X, \emptyset)$ on objects (definition on morphisms follows trivially).

$H_\bullet^{sing}$ is a functor **PTop** $\longrightarrow$ **Ab** (and thus also a functor from **Top**): Each pair (X, A) is assigned the abelian group $H_n(X, A)$, and a map of pairs $f : (X, A) \longrightarrow (Y, B)$ is sent to a homomorphism $H_n(f) : H_n(X, A) \longrightarrow H_n(Y, B)$ in the following manner:

If $\overline{\alpha}$ is an element in $H_n(X, A)$, we define $H_n(f)(\overline{\alpha}) = \overline{f \circ \alpha}$ in $H_n(Y, B)$. By conventional abuse of notation, we write f_* in place of $H_n(f)$.

Also, the connecting homomorphism ∂_* in the long exact sequence (1.2.4) is natural in the sense that given a map $f : (X, A) \longrightarrow (Y, B)$, the diagram below commutes [8, p127]:

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{\partial_*} & H_{n-1}(A) \\ \Big\downarrow{f_*} & & \Big\downarrow{f_*} \\ H_n(Y,B) & \xrightarrow{\partial_*} & H_{n-1}(B). \end{array}$$

1.2.3 Reduced singular homology

Top$_*$ is the subcategory of **Top** where the objects are based topological spaces and the morphisms are basepoint preserving maps. **Top**$_*$ is thus the full subcategory of **PTop** with all pairs $(X, *)$ as objects, where $*$ is the basepoint in X.

The following definition of reduced singular homology [15, §14.1] is somewhat less generic than what is found in e.g. [8, p110], but allows us to reuse the definition directly in section 3.3.

Definition 1.2.7 *The* ***reduced singular homology*** $\widetilde{H}_n(X)$ *of a based topological space* X *is given by*

$$\widetilde{H}_n(X) \;=\; H_n(X, *) \quad \text{for } n \geq 0,$$

where $*$ *is the basepoint in* X*.*

The relation between reduced and nonreduced singular homology is given by the following Lemma [15, §14.1]:

Lemma 1.2.8 *For a based topological space X we have*

$$\widetilde{H}_n(X) \oplus H_n(*) \;\cong\; H_n(X) \quad \text{for } n \geq 0.$$

Proof The long exact homology sequence of the pair $(X, *)$ takes the form

$$\cdots \xrightarrow{\partial_*} H_n(*) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} \widetilde{H}_n(X) \xrightarrow{\partial_*} \cdots .$$

The map i_* is clearly split injective, since it is induced by the inclusion $i : * \hookrightarrow X$ which together with the retract $r : X \longrightarrow *$ makes up the identity on $*$, hence $r_* \circ i_* = id_*$. □

As a direct consequence, the reduced singular homology groups of a point are trivial in all degrees (cf. (1.2.3)).

1.3 Axioms for generalized homology theories

We have seen that the theory of singular homology gives rise to a functor **PTop** $\longrightarrow$ **Ab**. Conversely, any homology theory on topological spaces can be determined by functors, as was first shown by Samuel Eilenberg and Norman Steenrod in their 1945 paper *Axiomatic approach to homology theory* [5], and later in their celebrated book *Foundations of Algebraic Topology* from 1952 [4].

The functor $(H_\bullet, \partial_\bullet)$ fulfills the Eilenberg-Steenrod axioms for homology theories [4, §I.3]. Later we will see that cyclic singular homology fulfills all of these except the dimension axiom, and so is a generalized homology theory. The axioms below are from [15, §13.1].

Definition 1.3.1 *A **generalized homology theory on*** **PTop** *consists of functors H_n :* **PTop** $\longrightarrow$ **Ab** *together with natural maps $\partial_* : H_n(X, A) \longrightarrow H_{n-1}(A)$ for $n \geq 0$, such that the following axioms are fulfilled:*

A1 - *Homotopy invariance:* *Homotopic maps induce the same maps in homology:*

$$f \simeq g \;\Rightarrow\; f_* = g_*$$

A2 - *Excision:* *Given spaces $B \subset A \subset X$ in* **PTop** *such that the closure of B is contained in the interior of A, the inclusion map $(X - B, A - B) \hookrightarrow (X, A)$ induces isomorphisms in homology:*

$$H_n(X - B, A - B) \;\cong\; H_n(X, A)$$

A3 - *Exactness:* *Given a topological pair (X, A), the inclusions $A \overset{i}{\hookrightarrow} X$ and $(X, \emptyset) \overset{j}{\hookrightarrow} (X, A)$ give rise to a long exact sequence of homology groups:*

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \longrightarrow \cdots$$

A4 - *Additivity:* *If* (X, A) *is a disjoint union of pairs* (X_α, A_α)*, the inclusions* $i_\alpha : (X_\alpha, A_\alpha) \hookrightarrow (X, A)$ *induce isomorphisms in homology:*

$$\bigoplus_\alpha H_n(X_\alpha, A_\alpha) \;\cong\; H_n(X, A).$$

1.3.1 Mayer-Vietoris sequences

As proved in e.g. [15, §14.5], the axioms of a generalized homology theory give rise to Mayer-Vietoris sequences.

Definition 1.3.2 *An* ***excisive triad*** $(X; A, B)$ *is a topological space* X *together with subspaces* A *and* B *where the union of the interiors of* A *and* B *cover* X*.*

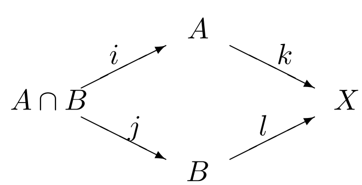
For an excisive triad, the inclusions $(A, A \cap B) \hookrightarrow (X, B)$ and $(B, A \cap B) \hookrightarrow (X, A)$ induce isomorphisms in homology due to excision (cf. [8, Theorem 2.20]).

The following theorem defines the **Mayer-Vietoris sequence** of an excisive triad ([15, §14.5]).

Theorem 1.3.3 *Let* $H_\bullet$ *be a generalized homology theory and* $(X; A, B)$ *an excisive triad. The following long sequence is exact:*

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{\Psi} H_n(A) \oplus H_n(B) \xrightarrow{\Phi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A \cap B) \longrightarrow \cdots \longrightarrow H_0(X) \longrightarrow 0.$$

Considering the diagram of inclusions



the maps Ψ *and* Φ *are defined by*

$$\begin{aligned} \Psi(c) &= (i_*(c), -j_*(c)) \\ \Phi(a, b) &= k_*(a) + l_*(b), \end{aligned}$$

and Δ *is the composition*

$$H_n(X) \longrightarrow H_n(X, B) \cong H_n(A, A \cap B) \xrightarrow{\partial_*} H_{n-1}(A \cap B).$$

Here, the first arrow is induced by the inclusion $(X, \emptyset) \hookrightarrow (X, B)$*, the isomorphism is due to excision, and the map* ∂_* *is the connecting homomorphism in the long exact homology sequence of the pair* $(A, A \cap B)$*.*

There is an important result allowing for an alternative formulation of the excision axiom. Unfortunately, I have not been able to find a published source containing the proof of the proposition, so the only reference I have at present are the lecture notes given in [18, March 22-nd].

Proposition 1.3.4 *Given functors $H_\bullet : \mathbf{PTop} \longrightarrow \mathbf{Ab}$ and natural maps $\partial_* : H_n(X, A) \longrightarrow H_{n-1}(A)$ as in Definition 1.3.1 satisfying axioms A1, A3 and A4, we have the following equivalence:*

$H_\bullet$ fulfills the excision axiom if and only if $H_\bullet$ gives rise to exact Mayer-Vietoris sequences for excisive triads.

Proof We give a concise proof of the if-part. Assume Mayer-Vietoris sequences exist for excisive triads. Given a topological pair (X, A) and an excisive triad $(X; U, V)$, we claim there is a long exact Mayer-Vietoris sequence of relative homology groups

$$\cdots \longrightarrow H_n(U \cap V, U \cap V \cap A) \longrightarrow H_n(U, U \cap A) \oplus H_n(V, V \cap A) \longrightarrow H_n(X, A) \longrightarrow \cdots .$$

The commutative diagram on page 9 has the above sequence in the row labeled $(**)$. All other rows and columns are exact, hence $(**)$ is exact too. This proves the existence of **relative Mayer-Vietoris sequences**.

Now, given spaces $B \subset A \subset X$ such that the closure of B is contained in the interior of A, set $U = A$ and $V = X - B$. This gives

$$\mathrm{int}(U) \cup \mathrm{int}(V) = \mathrm{int}(A) \cup \mathrm{int}(X - B) = \mathrm{int}(A) \cup \mathrm{int}(X - \mathrm{cl}(B)) = X$$

since $\mathrm{cl}(B) \subset \mathrm{int}(A)$. Putting this data into the relative Mayer-Vietoris sequence above yields isomorphisms

$$H_n(X - B, A - B) \;\cong\; H_n(X, A).$$

□

Singular homology of the sphere

The Mayer-Vietoris sequence allows us to compute the singular homology groups for a number of spaces, and in particular spheres. For $n \geq 1$, let $A, B \subset S^n$ be the northern and southern hemispheres, so that $A \cap B \simeq S^{n-1}$. Since A and B are contractible, the Mayer-Vietoris sequence of the excisive triad $(S^n; A, B)$ turns into the exact sequence

$$\cdots \longrightarrow H_q(S^{n-1}) \longrightarrow 0 \longrightarrow H_q(S^n) \longrightarrow H_{q-1}(S^{n-1}) \longrightarrow 0 \longrightarrow \cdots$$

for $q > 1$, hence $H_q(S^n) \cong H_{q-1}(S^{n-1})$ for $q > 1$. In particular, the Mayer-Vietoris sequence for the circle looks like

$$\cdots \longrightarrow 0 \longrightarrow H_1(S^1) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where all groups to the left are trivial. An easy argument reveals that $H_1(S^1) \cong \mathbb{Z}$, so the isomorphism $H_q(S^n) \cong H_{q-1}(S^{n-1})$ for $q \geq 1$ gives

$$H_q(S^n) \;\cong\; \begin{cases} \mathbb{Z} & \text{for} \quad q = 0, n \\ 0 & \text{for} \quad \textit{otherwise.} \end{cases} \tag{1.3.1}$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_n(U \cap V \cap A) & \longrightarrow & H_n(U \cap A) \oplus H_n(V \cap A) & \longrightarrow & H_n(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_n(U \cap V) & \longrightarrow & H_n(U) \oplus H_n(V) & \longrightarrow & H_n(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_n(U \cap V, U \cap V \cap A) & \longrightarrow & H_n(U, U \cap A) \oplus H_n(V, V \cap A) & \longrightarrow & H_n(X, A) & \longrightarrow & \cdots \qquad (**) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_{n-1}(U \cap V \cap A) & \longrightarrow & H_{n-1}(U \cap A) \oplus H_{n-1}(V \cap A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Figure 1.1: A commutative diagram containing the long exact relative Mayer-Vietoris sequence in the $(**)$ row.

Chapter 2

Cyclic Homology

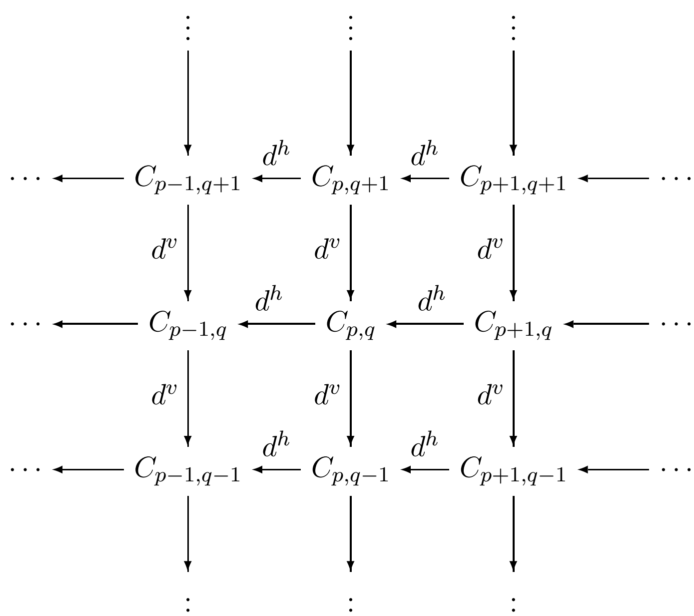
In this chapter, we first recall double complexes and total homology. We then recall cyclic double complexes and cyclic homology. For the latter, we emphasize on precyclic objects equipped with a last degeneracy map. At the end of the chapter, we work out the analogues of the Connes SBI-sequence and the Connes double complex.

2.1 Double complexes and Tot

Definition 2.1.1 *(Cf. [10, §1.0.11]) A* ***double complex*** *(or bicomplex) consists of a set of doubly indexed objects* $C_{\bullet\bullet} = \{C_{p,q}\}_{p,q\in\mathbb{Z}}$ *in an abelian category, together with morphisms* $d^h : C_{p,q} \longrightarrow C_{p-1,q}$ *and* $d^v : C_{p,q} \longrightarrow C_{p,q-1}$ *where the following relations hold:*

$$\begin{aligned} d^v \circ d^v &= d^h \circ d^h = 0 \\ d^v \circ d^h + d^h \circ d^v &= 0 \end{aligned} \tag{2.1.1}$$

Diagrammatically, the double complex is a grid where nodes are the doubly indexed objects and edges are the vertical and horizontal differentials.



From the relations in (2.1.1), every row and column in this grid is a chain complex in its own right, and all the squares anticommute.

A **map of double complexes** $f : C_{\bullet\bullet} \longrightarrow D_{\bullet\bullet}$ is a map from objects of bidegree (p, q) to objects of bidegree (p, q) which commutes with both the horizontal and vertical differentials.

A **first quadrant double complex** is a double complex $C_{\bullet\bullet}$ where $C_{p,q} = 0$ whenever p is negative or q is negative. In this case we have the direct sum

$$Tot(C_{\bullet\bullet})_n \;=\; \bigoplus_{p+q=n} C_{p,q} \quad \text{for } n \geq 0,$$

which is well-defined since it is always finite. Upon these direct sums we can construct morphisms $d : Tot(C_{\bullet\bullet})_n \longrightarrow Tot(C_{\bullet\bullet})_{n-1}$ by $d_n = d^v + d^h$ as indicated in the below diagram:

$$\begin{array}{ccc} C_{p,q+1} & & \\ \Big\downarrow d^v & + & \\ C_{p,q} & \xleftarrow{d^h} & C_{p+1,q}. \end{array} \tag{2.1.2}$$

This map has the property $d \circ d = 0$, following from (2.1.1), and so defines a chain complex, **the total complex** $Tot(C_{\bullet\bullet})$ of $C_{\bullet\bullet}$.

Definition 2.1.2 *The homology groups $H_n(C_{\bullet\bullet})$ of the double complex $C_{\bullet\bullet}$ are the homology groups of the total complex:*

$$H_n(C_{\bullet\bullet}) \;=\; H_n(Tot(C_{\bullet\bullet})) \quad \textit{for } n \geq 0.$$

There is another way of constructing homology groups over $C_{\bullet\bullet}$. Since each column is a chain complex $(C_{p\bullet}, d^v)$, we can define "vertical" homology groups $H_q^v(C_{p\bullet})$ in each column p. The horizontal differentials d^h are still differentials when applied to the cycles in the vertical homology groups, enabling calculation of the horizontal homology of the vertical homology groups: $H_p^h H_q^v(C_{\bullet\bullet})$. The opposite order of homologization yields $H_q^v H_p^h(C_{\bullet\bullet})$.

The relation between these latter definitions of homology and the total homology of a double complex is given by spectral sequences ([10, Appendix D]); namely, there are convergent spectral sequences

$$E^2_{p,q} = H_p^h H_q^v(C_{\bullet\bullet}) \Rightarrow H_{p+q}(Tot(C_{\bullet\bullet}))$$
$$'E^2_{p,q} = H_q^v H_p^h(C_{\bullet\bullet}) \Rightarrow H_{p+q}(Tot(C_{\bullet\bullet})).$$

The abutments are the same, although the top sequence is defined by filtration on columns, while the bottom sequence is defined by filtration on rows.

2.2 Precyclic objects and cyclic homology

We shall now establish a framework for construction of double complexes based on precyclic objects. The construction will give rise to cyclic homology in a natural way. This time, we make use of a subcategory of the cyclic category $\mathbf{\Delta C}$, and the following two definitions are based on definitions 6.1.1 and 6.1.2.1 in [10], alternatively the definitions in [16, §1], except from the last degeneracy which differs from the extra degeneracy of both Loday's [10, §1.1.12] and Park's [16, eqn. (1.20)]. (See the remark on page 14 for details and motivation regarding choice of degeneracy).

Definition 2.2.1 *The* ***precyclic category*** $\mathbf{\Delta C_{pre}}$ *is the presimplicial category together with* ***cyclic morphisms*** $\tau^n : [n] \to [n]$ *satisfying the following relations:*

$$(\tau^n)^{n+1} = id \tag{2.2.1}$$
$$\tau^n \delta^i = \delta^{i-1}\tau^{n-1} \quad \text{for} \quad 1 \le i \le n. \tag{2.2.2}$$

The ***precyclic category with the last degeneracy***, $\mathbf{\Delta C_{last}}$, *is obtained by adding morphisms* $\sigma^n : [n+1] \to [n]$ *to* $\mathbf{\Delta C_{pre}}$ *such that the following additional relations hold:*

$$\sigma^n \delta^i = \begin{cases} \delta^i \sigma^{n-1} & \text{for} \quad 0 \le i < n \\ id & \text{for} \quad i = n, n+1. \end{cases}$$

For $i = 0$ the relation $\tau^n \delta^0 = \delta^n$ follows from (2.2.2) by the calculation

$$\delta^n = (\tau^n)^{n+1}\delta^n = (\tau^n)^n \delta^{n-1}\tau^{n-1} = \cdots = \tau^n \delta^0 (\tau^{n-1})^n = \tau^n \delta^0.$$

Definition 2.2.2 *A* ***precyclic object*** $X_\bullet$ *in a category* $\mathbf{D}$ *is a functor* $(\mathbf{\Delta C_{pre}})^{\mathbf{op}} \longrightarrow \mathbf{D}$, *alternatively a set of objects* $X_\bullet = \{X_n\}_{n \ge 0}$ *in* $\mathbf{D}$ *together with* ***face operators*** $d_i : X_n \longrightarrow X_{n-1}$ *and* ***cyclic operators*** $T_n : X_n \longrightarrow X_n$ *where the following relations hold:*

$$d_i d_j = d_{j-1} d_i \quad \text{for } i < j \tag{2.2.3}$$
$$T_n^{n+1} = id \tag{2.2.4}$$
$$d_i T_n = T_{n-1} d_{i-1} \ \text{for } 1 \le i \le n. \tag{2.2.5}$$

Again, the property $d_0 T_n = d_n$ *follows from* (2.2.5).
If $X_\bullet$ *is a functor* $(\mathbf{\Delta C_{last}})^{\mathbf{op}} \longrightarrow \mathbf{D}$, *we say that* $X_\bullet$ *is* ***equipped with the last degeneracy***, *and we get additional morphisms* $s_n : X_n \longrightarrow X_{n+1}$ *for which the following relations hold:*

$$d_i s_n = \begin{cases} s_{n-1} d_i & \text{for} \quad 0 \le i < n \\ id & \text{for} \quad i = n, n+1. \end{cases} \tag{2.2.6}$$

Dually, a functor $\mathbf{\Delta C_{pre}} \longrightarrow \mathbf{D}$ yields a **pre-cocyclic object**, which we may see as a set of objects $X^\bullet = \{X^n\}_{n\geq 0}$ in $\mathbf{D}$ together with morphisms $\partial^i : X^{n-1} \longrightarrow X^n$ for $0 \leq i \leq n$ and $\Gamma^n : X^n \longrightarrow X^n$ such that

$$\partial^j \partial^i = \partial^i \partial^{j-1} \quad \text{for } i < j \tag{2.2.7}$$

$$(\Gamma^n)^{n+1} = id \tag{2.2.8}$$

$$\Gamma^n \partial^i = \partial^{i-1}\Gamma^{n-1} \quad \text{for } 0 < i \leq n, \tag{2.2.9}$$

and as before we have $\Gamma^n \partial^0 = \partial^n$. If the functor is from $\mathbf{\Delta C_{last}}$, we get additional morphisms $\rho^n : X^{n+1} \longrightarrow X^n$ with relations

$$\rho^n \partial^i = \begin{cases} \partial^i \rho^{n-1} & \text{for } 0 \leq i < n \\ id & \text{for } i = n, n+1. \end{cases} \tag{2.2.10}$$

A remark on the choice of degeneracy In the coming construction of cyclic homology, we follow the approach presented in [10], with only one small difference: when choosing an appropriate degeneracy, Loday introduces the *extra degeneracy* $s_n : A^{\otimes n} \longrightarrow A^{\otimes n+1}$ [10, §1.1.12]. This map has the properties

$$d_i s_n = \begin{cases} s_{n-1} d_{i-1} & \text{for } 1 \leq i \leq n-1 \\ id & \text{for } i = 0, \end{cases}$$

and in [16] Park picks a map with similar properties (equation (1.20)).

The degeneracies chosen by Loday and Park do not seem to have any corresponding counterpart on the standard topological n-simplex, i.e a map $\rho^n : \Delta^{n+1} \longrightarrow \Delta^n$ such that

$$\rho^n \partial^i = \begin{cases} \partial^{i-1} \rho^{n-1} & \text{for } 1 \leq i \leq n \\ id & \text{for } i = 0. \end{cases}$$

Our choice of degeneracy, however, is picked from the generating set of morphisms in the simplicial category, and it will thus be available in any pre-cosimplicial and presimplicial object equipped with the last degeneracy, and, in particular, in the standard topological n-simplex. The degeneracy is needed in the construction of a contracting homotopy of the bar complex (Lemma 2.3.2) which ensures the existence of SBI-sequences and that the Connes double complex is quasi-isomorphic to the cyclic double complex (sections 2.3 and 2.4).

A natural question is whether it makes any sense to restrict ourselves to functors from the precyclic category with the last degeneracy rather than just considering functors from the cyclic category $\mathbf{\Delta C}$. There are two rationales for our choice. First of all we do not need the remaining morphisms to set up the framework. Secondly, as the following proposition shows, dragging along those extra morphisms will make our proofs longer (more identities to prove). Together, this means we alleviate ourselves from doing unnecessary work by restricting our attention to precyclic objects with the last degeneracy.

Proposition 2.2.3 *If $X^\bullet$ is a pre-cocyclic object in a preadditive category* $\mathbf{D}$ *and D is any object in* $\mathbf{D}$*, then the set* $\{Hom_{\mathbf{D}}(X^n, D)\}_{n\geq 0}$ *is a precyclic object. Moreover, if $X^\bullet$ is equipped with the last degeneracy, so is* $\{Hom_{\mathbf{D}}(X^n, D)\}_{n\geq 0}$.

Proof The result is not surprising since $Hom_{\mathbf{D}}(-, D)$ is a contravariant functor, and precomposing with the pre-cocyclic object $\mathbf{\Delta C_{pre}} \longrightarrow \mathbf{D}$ should give a precyclic object. However, we shall invest the effort of giving an explicit construction, since we will need it later.

Assume $X^\bullet$ is pre-cocyclic so that the morphisms in $X^\bullet$ are determined by equations (2.2.7), (2.2.8) and (2.2.9). We must construct morphisms

$$\begin{array}{rlcll} d_i: & Hom_{\mathbf{D}}(X^n, D) & \longrightarrow & Hom_{\mathbf{D}}(X^{n-1}, D) & \text{for} \quad 0 \leq i \leq n \\ T_n: & Hom_{\mathbf{D}}(X^n, D) & \longrightarrow & Hom_{\mathbf{D}}(X^n, D), & \end{array}$$

so that relations (2.2.3), (2.2.4) and (2.2.5) all hold. In the following, f is an element in $Hom_{\mathbf{D}}(X^n, D)$.

Define $d_i(f) = f \circ \partial^i$ for $0 \leq i \leq n$. When $i < j$ we get

$$d_i d_j(f) \;=\; f \circ \partial^j \circ \partial^i \;\stackrel{(2.2.7)}{=}\; f \circ \partial^i \circ \partial^{j-1} \;=\; d_{j-1} d_i(f),$$

hence, relation (2.2.3) holds. Defining $T_n(f) = f \circ \Gamma^n$ yields

$$(T_n)^{n+1}(f) = f \circ (\Gamma^n)^{n+1} \stackrel{(2.2.8)}{=} f \circ id = f,$$

assuring (2.2.4), and for $1 \leq i \leq n$ we get

$$d_i T_n(f) \;= f \circ \Gamma^n \circ \partial^i \stackrel{(2.2.9)}{=} f \circ \partial^{i-1} \circ \Gamma^{n-1} \;= T_{n-1} d_{i-1}(f).$$

Thus (2.2.5) holds, and $\{Hom_{\mathbf{D}}(X^n, D)\}_{n\geq 0}$ is a precyclic object.

Now, assume $X^\bullet$ is also equipped with the last degeneracy so that (2.2.10) holds, and define $s_n : Hom_{\mathbf{D}}(X^n, D) \longrightarrow Hom_{\mathbf{D}}(X^{n+1}, D)$ by $s_n(f) = f \circ \rho^n$. When $0 \leq i < n$ this gives us

$$d_i s_n(f) \;=\; f \circ \rho^n \circ \partial^i \stackrel{(2.2.10)}{=} f \circ \partial^i \circ \rho^{n-1} \;=\; s_{n-1} d_i(f),$$

and when $i = n$ or $n + 1$ we get

$$d_i s_n(f) \;=\; f \circ \rho^n \circ \partial^i \stackrel{(2.2.10)}{=} f \circ id \;=\; f.$$

This proves that the properties in (2.2.6) hold, and completes the proof. □

Corollary 2.2.4 *The results in Proposition 2.2.3 hold when applied to the free abelian groups over the Hom-sets* $\{Hom_{\mathbf{D}}(X^n, D))\}_{n\geq 0}$ *in any category* $\mathbf{D}$*. That is,* $X_\bullet = \{Ab(Hom_{\mathbf{D}}(X^n, D))\}_{n\geq 0}$ *is a precyclic object when* $X^\bullet = \{X^n\}_{n\geq 0}$ *is pre-cocyclic, and if $X^\bullet$ has the last degeneracy, so has* $X_\bullet$.

Proof The result follows immediately by defining the required homomorphisms as the linear extensions of the maps defined in the proof of Proposition 2.2.3. □

The aim now is to fit the precyclic objects into a double complex. In the following, any precyclic object is assumed to have the morphisms described in Definition 2.2.2. The next lemma-definition is based on [10, §2.1.0 and §2.1.1].

Lemma-definition 2.2.5 *Let $X_\bullet$ be a precyclic object in an abelian category $\mathbf{A}$. Define the **signed cyclic operator** $t_n : X_n \longrightarrow X_n$ by*

$$t_n \;=\; (-1)^n T_n$$

*and the **norm map** $N_n : X_n \longrightarrow X_n$ by*

$$N_n \;=\; 1 + t_n + \cdots + (t_n)^n.$$

(Here, 1 is the element id_{X_n} in the abelian group $Hom_{\mathbf{A}}(X_n, X_n)$, and -1 is its additive inverse). We also define maps $\partial_n, \partial'_n : X_n \longrightarrow X_{n-1}$ by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i \quad \text{and} \quad \partial'_n = \sum_{i=0}^{n-1} (-1)^i d_i. \tag{2.2.11}$$

For these maps, the following relations hold:

$$\begin{aligned}
\partial_{n-1} \circ \partial_n &= \partial'_{n-1} \circ \partial'_n = 0 && (2.2.12)\\
N_n(1 - t_n) &= (1 - t_n)N_n = 0 && (2.2.13)\\
\partial_n(1 - t_n) &= (1 - t_{n-1})\partial'_n && (2.2.14)\\
\partial'_n N_n &= N_{n-1}\partial_n. && (2.2.15)
\end{aligned}$$

Before proving the lemma, a quick check of the relations between the new morphisms is in place:

$$\left.\begin{aligned}
d_i t_n &= d_i(-1)^n T_n\\
&= (-1)^n T_{n-1} d_{i-1}\\
&= -t_{n-1} d_{i-1}
\end{aligned}\right\} \text{ for } 1 \le i \le n \tag{2.2.16}$$

$$(t_n)^{n+1} = (-1)^{n^2+n} T_n^{n+1} = id, \tag{2.2.17}$$

and as before, we can deduce the relation $d_0 t_n = (-1)^n d_n$ from (2.2.16).

Proof (Lemma-definition 2.2.5) The properties in (2.2.12) follow from property (2.2.3) and an argument more or less identical to the proof of Lemma-definition 1.1.3.

For property (2.2.13) we note that

$$N_n(1 - t_n) \;=\; N_n - t_n - (t_n)^2 - \cdots - (t_n)^{n+1} \;=\; N_n - N_n \;=\; 0,$$

where the opposite composition is equally easy.

For property (2.2.14), we have the following calculation:

$$
\begin{aligned}
\partial_n(1-t_n) &= \sum_{i=0}^{n}(-1)^i(d_i - d_i t_n)\\
&= (d_0 - d_0 t_n) + \sum_{i=1}^{n-1}(-1)^i(d_i - d_i t_n) \;+\; (-1)^n(d_n - d_n t_n)\\
&= d_0 - (-1)^n d_n + \sum_{i=1}^{n-1}(-1)^i(d_i + t_{n-1}d_{i-1}) \;+ (-1)^n(d_n + t_{n-1}d_{n-1})\\
&= d_0 \;+\; \sum_{i=1}^{n-1}(-1)^i(d_i + t_{n-1}d_{i-1}) \;+\; (-1)^n t_{n-1}d_{n-1} \qquad (*)\\
&= \sum_{i=0}^{n-1}(-1)^i(d_i - t_{n-1}d_i) \;=\; (1-t_{n-1})\partial'_n.
\end{aligned}
$$

The next to last equality is justified by considering $(*)$ in its expanded form, pairing the elements in which the d_i match:

$$
\begin{aligned}
&d_0 - d_1 - t_{n-1}d_0 + d_2 + t_{n-1}d_1 - \;\cdots\\
&\qquad\cdots\; + (-1)^{n-1}d_{n-1} + (-1)^{n-1}t_{n-1}d_{n-2} + (-1)^n t_{n-1}d_{n-1}.
\end{aligned}
$$

The cumbersome part is to prove property (2.2.15). We first note that property (2.2.16) gives us the identities

$$
\begin{aligned}
d_i t_n^j &= (d_i t_n)t_n^{j-1}\\
&= \begin{cases} -t_{n-1}d_{i-1}t_n^{j-1} & \text{for } 0 < i \le n\\ (-1)^n d_n t_n^{j-1} & \text{for } i = 0, \end{cases}
\end{aligned}
$$

where the top one of the two latter transforms further as (cf. [17, §6.1.11])

$$
\begin{aligned}
d_i t_n^j &= \begin{cases} (-1)^j t_{n-1}^j d_{i-j} & \text{for } j \le i\\ (-1)^i t_{n-1}^i d_0 t_n^{j-i} & \text{for } j > i \end{cases}\\
&= \begin{cases} (-1)^j t_{n-1}^j d_{i-j} & \text{for } j \le i\\ (-1)^{i+n} t_{n-1}^i d_n t_n^{j-i-1} & \text{for } j > i \end{cases}\\
&= \begin{cases} (-1)^j t_{n-1}^j d_{i-j} & \text{for } j \le i\\ (-1)^{n+j+1} t_{n-1}^{j-1} d_{n+i-j+1} & \text{for } j > i. \end{cases}
\end{aligned}
$$

These identities allow for the computation

$$
\begin{aligned}
\partial'_n N_n \;&=\; \left(\sum_{i=0}^{n-1}(-1)^i d_i\right)\left(\sum_{j=0}^{n} t_n^j\right)\\
&=\; \sum_{0\le j\le i\le n-1}(-1)^{i-j}t_{n-1}^j d_{i-j} + \sum_{0\le i<j\le n}(-1)^{n+1+i-j}t_{n-1}^{j-1}d_{n+i-j+1},
\end{aligned}
$$

and in this sum the coefficient of $(-1)^q d_q$, for $0 \le q \le n$, is given by

$$
\sum_{0\le j\le n-1-q} t_{n-1}^j + \sum_{n-q\le j-1\le n-1} t_{n-1}^{j-1} \;=\; N_{n-1}.
$$

Hence, $\partial'_n N_n = N_{n-1}\partial_n$ (cf. [10, §2.1.1]). □

This actually completes the framework required to build double complexes from precyclic objects [10, §2.1.2]:

Definition 2.2.6 *The* ***cyclic double complex*** $CC(X_\bullet)$ *of a precyclic object* $X_\bullet$ *in an abelian category is the first quadrant double complex in the below diagram.*

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle \partial_3} & & \downarrow{\scriptstyle -\partial'_3} & & \downarrow{\scriptstyle \partial_3} & & \downarrow{\scriptstyle -\partial'_3} & & \\
X_2 & \xleftarrow{1-t_2} & X_2 & \xleftarrow{N_2} & X_2 & \xleftarrow{1-t_2} & X_2 & \xleftarrow{N_2} & \cdots \\
\downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle -\partial'_2} & & \downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle -\partial'_2} & & \\
X_1 & \xleftarrow{1-t_1} & X_1 & \xleftarrow{N_1} & X_1 & \xleftarrow{1-t_1} & X_1 & \xleftarrow{N_1} & \cdots \\
\downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle -\partial'_1} & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle -\partial'_1} & & \\
X_0 & \xleftarrow{1-t_0} & X_0 & \xleftarrow{N_0} & X_0 & \xleftarrow{1-t_0} & X_0 & \xleftarrow{N_0} & \cdots
\end{array}
$$

This is well-defined according to Lemma-definition 2.2.5.

The cyclic double complex is sometimes referred to as Tsygan's double complex after its first discoverer B. Tsygan, whereafter it was independently discovered by Loday and Quillen (see the introduction for references to the original publications).

Definition 2.2.7 *(Cf. [10, §2.1.3]) The* ***cyclic homology groups*** $HC_n(X_\bullet)$ *of a precyclic object* $X_\bullet$ *in an abelian category are the homology groups of the total complex of the associated cyclic double complex:*

$$HC_n(X_\bullet) \;=\; H_n(Tot(CC(X_\bullet))) \quad \text{for } n \geq 0.$$

The notation HC comes from the French *homologie cyclique*.

By placing an additional requirement on maps between precyclic objects we can prove some general results that will help us later.

Definition 2.2.8 *A* ***proper map of precyclic objects*** $f : X_\bullet \longrightarrow Y_\bullet$ *is a set of maps* $f_n : X_n \longrightarrow Y_n$ *for* $n \geq 0$ *where each* f_n *commutes with the operators* T_n *and* d_i *for* $0 \leq i \leq n$*. If* $X_\bullet$ *is equipped with the last degeneracy, we also require that* f_n *commutes with* s_n*.*

Proposition 2.2.9 *If* $f : A \longrightarrow B$ *is a morphism in a preadditive category* $\mathbf{C}$ *and* $X^\bullet$ *is a pre-cocyclic object in* $\mathbf{C}$*, possibly equipped with the last degeneracy, then there are induced proper maps* $f_n : Hom_{\mathbf{C}}(X^n, A) \longrightarrow Hom_{\mathbf{C}}(X^n, B)$ *defined by*

$$f_n(\sigma) \;=\; f \circ \sigma, \tag{2.2.18}$$

where σ *is an element of* $Hom_{\mathbf{C}}(X^n, A)$*.*

Proof Here, the precyclic objects $\{Hom_{\mathbf{C}}(X^n, A)\}_{n\geq 0}$ and $\{Hom_{\mathbf{C}}(X^n, B)\}_{n\geq 0}$ have the structure given in the proof of Proposition 2.2.3. We have the following identities

$$\begin{array}{rcl} T_n \circ f_n(\sigma) &=& (f \circ \sigma) \circ \Gamma^n \;=\; f \circ (\sigma \circ \Gamma^n) \;=\; f_n \circ T_n(\sigma) \\ d_i \circ f_n(\sigma) &=& (f \circ \sigma) \circ \partial^i \;=\; f \circ (\sigma \circ \partial^i) \;=\; f_{n-1} \circ d_i(\sigma), \end{array}$$

and if $X^\bullet$ is equipped with the last degeneracy, we also have

$$s_n \circ f_n(\sigma) = (f \circ \sigma) \circ \rho^n = f \circ (\sigma \circ \rho^n) = f_{n+1} \circ s_n(\sigma).$$

Hence, commutativity is ensured. □

In particular, (2.2.18) can be extended linearly to a homomorphism $f_n : Ab(Hom_{\mathbf{C}}(X^n, A)) \longrightarrow Ab(Hom_{\mathbf{C}}(X^n, B))$ for any category $\mathbf{C}$, where f_n is defined on n-chains by

$$f\left(\sum \alpha_\sigma \sigma\right) = \sum \alpha_\sigma f \circ \sigma.$$

Lemma 2.2.10 *Let* $0 \longrightarrow X_\bullet \xrightarrow{f} Y_\bullet \xrightarrow{g} Z_\bullet \longrightarrow 0$ *be a short exact sequence of proper maps of precyclic objects in an abelian category. Then there is a short exact sequence of cyclic double complexes*

$$0 \longrightarrow CC(X_\bullet) \xrightarrow{\hat{f}} CC(Y_\bullet) \xrightarrow{\hat{g}} CC(Z_\bullet) \longrightarrow 0.$$

Proof This follows directly from Definition 2.2.6 since $CC_{p,q}(X_\bullet) = X_q$. □

A nice property of the cyclic double complex we will use several times is the following theorem. The result is mentioned in a comment after Definition 9.6.7 in [19].

Proposition 2.2.11 *Let* A*,* B *and* C *be precyclic objects in an abelian category. A short exact sequence of cyclic double complexes*

$$0 \longrightarrow CC(A) \xrightarrow{\alpha} CC(B) \xrightarrow{\beta} CC(C) \longrightarrow 0$$

induces a long exact sequence in cyclic homology

$$\cdots \longrightarrow HC_n(A) \xrightarrow{\alpha_*} HC_n(B) \xrightarrow{\beta_*} HC_n(C) \xrightarrow{\partial} HC_{n-1}(A) \longrightarrow \cdots.$$

Proof The key point is to notice that the short exact sequence of cyclic double complexes gives a short exact sequence of total complexes, wherefrom both exactness and the indicated maps follow from homological algebra [19, §1.3].

Since α and β are maps of double complexes, we can define maps

$$\alpha_n = \bigoplus_{p+q=n} \alpha_{p,q} : TotA_n \longrightarrow TotB_n \qquad \text{for } n \geq 0$$

by termwise application of α, that is $\alpha_{p,q} : A_{p,q} \longrightarrow B_{p,q}$ is defined by $\alpha_{p,q}(a) = \alpha(a)$. This map is a chain map as is illustrated by the following computation for $a \in CC_{p,q}(A)$:

$$\begin{aligned} d_n \circ \alpha_n(a) &= (d^h_{p,q}(\alpha(a)), d^v_{p,q}(\alpha(a))) \\ &= (\alpha(d^h_{p,q}(a)), \alpha(d^v_{p,q}(a))) \\ &= \alpha_{n-1} \circ d_n(a). \end{aligned}$$

The differentials in the cyclic double complex differ depending on whether p is odd or even, but since α sends elements from bidegree (p, q) to bidegree (p, q), the maps will be the same at "both sides" of α.

The same goes for β, and the composition $\beta_n \circ \alpha_n$ is exact since $\beta \circ \alpha$ is exact. □

2.3 Connes' periodicity exact sequence

This short section introduces Connes' periodicity exact sequence (commonly called the SBI-sequence), and gives a proof of its existence in the context of precyclic objects with the last degeneracy.

Recall the complex $(X_\bullet, \partial_\bullet)$ for a precyclic object $X_\bullet$ in an abelian category, where the differential is given by (2.2.11). The following theorem is from [10, §2.2.1]:

Theorem 2.3.1 *Let $X_\bullet$ be a precyclic object in an abelian category, and let $X_\bullet$ be equipped with the last degeneracy. If $H_n(X_\bullet)$ denotes the n-th homology group of the complex $(X_\bullet, \partial_\bullet)$, then there is a long exact sequence*

$$\cdots \longrightarrow H_n(X_\bullet) \xrightarrow{I} HC_n(X_\bullet) \xrightarrow{S} HC_{n-2}(X_\bullet) \xrightarrow{B} H_{n-1}(X_\bullet) \longrightarrow \cdots$$

*called **the SBI-sequence** of Connes.*

Proof Let $CC(X_\bullet)[2, 0]$ denote the cyclic double complex obtained by shifting every object two steps right and zero steps up, that is $C_{p,q} \mapsto C_{p+2,q}$, and let $CC(X_\bullet)^{\{2\}}$ denote the cyclic double complex where only the two first columns are preserved and the remaining columns are zeroed out. This gives us a short exact sequence of cyclic double complexes

$$0 \longrightarrow CC(X_\bullet)^{\{2\}} \xrightarrow{i} CC(X_\bullet) \xrightarrow{s} CC(X_\bullet)[2, 0] \longrightarrow 0.$$

Now, if the cyclic homology of $CC(X_\bullet)^{\{2\}}$ is isomorphic to $H_n(X_\bullet)$, the long exact homology sequence induced by the short exact sequence above is exactly the SBI-sequence. This isomorphism is assured by the following lemma [10, §1.1.12]:

Lemma 2.3.2 *If $X_\bullet$ is a precyclic object with the last degeneracy in an abelian category, the complex $(X_\bullet, \partial'_\bullet)$ is acyclic.*

($\partial'_\bullet$ is the differential in the odd numbered columns in the cyclic double complex given in (2.2.11)). The consequence of the lemma is that in total homology, the second column in $CC(X_\bullet)^{\{2\}}$ vanishes, and hence,

$$HC_n(CC(X_\bullet)^{\{2\}}) \;\cong\; H_n(X_\bullet) \quad \text{for } n \geq 0,$$

which concludes the proof of Theorem 2.3.1. □

Next we prove Lemma 2.3.2. The following proof can be found as the second proof of Lemma 2.2 in [16].

Proof (of Lemma 2.3.2) We will construct a contracting homotopy of the chain complex $(X_\bullet, \partial'_\bullet)$, and the construction makes use of the last degeneracy. Define $s'_n : X_n \longrightarrow X_{n+1}$ by $s'_n = (-1)^n s_n$. Then

$$\begin{aligned} \partial'_{n+1} s'_n &= (-1)^n \sum_{i=0}^{n} (-1)^i d_i s_n = (-1)^n \sum_{i=0}^{n-1} (-1)^i s_{n-1} d_i + id \\ &= -(-1)^{n-1} s_{n-1} \sum_{i=0}^{n-1} (-1)^i d_i + id = -s'_{n-1}\partial'_n + id. \end{aligned}$$

□

2.4 The Connes double complex

For a precyclic object $X_\bullet$ in an abelian category and equipped with the last degeneracy, we define the Connes double complex $\mathcal{B}(X_\bullet)$ and show that its total homology $H_n(Tot(\mathcal{B}(X_\bullet)))$ is isomorphic to $H_n(Tot(CC(X_\bullet)))$. The approach we present here is from [12, §1], although we provide some more details.

For $p \geq 0$ even, define maps $B_{p,q} : CC_{p,q}(X_\bullet) \longrightarrow CC_{p-2,q+1}(X_\bullet)$ by

$$B_{p,q} = (1 - t_{q+1}) s'_q N_q,$$

where s'_q is the contracting homotopy from the proof of Lemma 2.3.2. This immediately yields

$$B_{p-2,q+1} B_{p,q} = (1 - t_{q+2}) s'_{q+1} N_{q+1} (1 - t_{q+1}) s'_q N_q = 0$$

since $N_{q+1}(1 - t_{q+1}) = 0$ by (2.2.13). We also have

$$\partial_{q+1} B_{p,q} + B_{p,q-1} \partial_q = 0$$

from the calculation

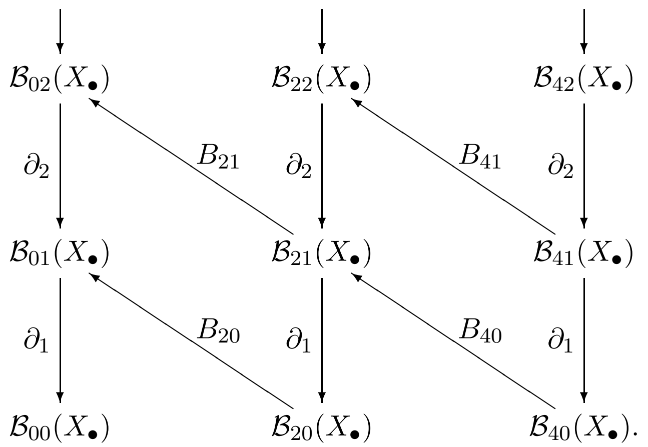
$$\begin{aligned} \partial_{q+1} B_{p,q} + B_{p,q-1} \partial_q &= \partial_{q+1} (1 - t_{q+1}) s'_q N_q + (1 - t_q) s'_{q-1} N_{q-1} \partial_q \\ &= (1 - t_q) \partial'_{q+1} s'_q N_q + (1 - t_q) s'_{q-1} \partial'_q N_q \\ &= (1 - t_q)(\partial'_{q+1} s'_q + s'_{q-1} \partial'_q) N_q \\ &= (1 - t_q) N_q \\ &= 0, \end{aligned}$$

where $\partial'_{q+1} s'_q + s'_{q-1} \partial'_q = id$ due to Lemma 2.3.2.

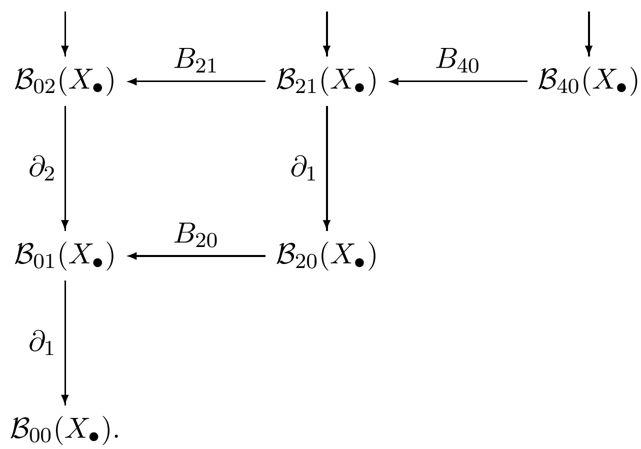
For $p \geq 0$ even we define

$$\mathcal{B}_{p,q}(X_\bullet) = CC_{p,q}(X_\bullet),$$

and together with the fact that $\partial_q \partial_{q+1} = 0$, we have a new double complex



This diagram is usually rearranged by shifting the columns up to obtain the more comprehendible diagram



The double complex is referred to as **the double complex of Connes** [12, §1], and from [12, Proposition 1.5] we have:

Proposition 2.4.1 *The total complexes of the double complexes $CC(X_\bullet)$ and $\mathcal{B}(X_\bullet)$ are quasi-isomorphic.*

Proof First define a map $f : \mathcal{B}(X_\bullet) \longrightarrow CC(X_\bullet)$ by sending α in $\mathcal{B}_{p,q}(X_\bullet)$ to $f_{p,q}(\alpha) = (s'_q N_q(\alpha), \alpha)$ in $CC_{p-1,q+1}(X_\bullet) \oplus CC_{p,q}(X_\bullet)$ (note that p is even). We now have two total complexes at our hands, namely $(Tot(\mathcal{B}(X_\bullet)), d')$ and $(Tot(CC(X_\bullet)), d)$, and we claim that f is a map of total complexes. For if p is even and α is an element in $\mathcal{B}_{p,q}(X_\bullet)$ we get

$$\begin{aligned} d \circ f(\alpha) &= d(s'_q N_q(\alpha), \alpha) \\ &= ((1 - t_{q+1}) s'_q N_q(\alpha), -\partial'_{q+1} s'_q N_q(\alpha) + N_q(\alpha), \partial_q(\alpha)) \\ &= (B_{p,q}(\alpha), -\partial'_{q+1} s'_q N_q(\alpha) + N_q(\alpha), \partial_q(\alpha)) \\ &\quad \in CC_{p-2,q+1}(X_\bullet) \oplus CC_{p-1,q}(X_\bullet) \oplus CC_{p,q-1}(X_\bullet). \end{aligned}$$

On the other hand, we have

$$d'(\alpha) = (B_{p,q}(\alpha), \partial_q(\alpha)) \;\in\; \mathcal{B}_{p-2,q+1}(X_\bullet) \oplus \mathcal{B}_{p,q-1}(X_\bullet)$$

giving us

$$\begin{aligned} f \circ d'(\alpha) &= f(B_{p,q}(\alpha), \partial_q(\alpha)) \\ &= (s'_{q+1}N_{q+1}B_{p,q}(\alpha), B_{p,q}(\alpha), s'_{q-1}N_{q-1}\partial_q(\alpha), \partial_q(\alpha)) \\ &= (0, B_{p,q}(\alpha), s'_{q-1}N_{q-1}\partial_q(\alpha), \partial_q(\alpha)) \\ &\quad \in CC_{p-3,q+2}(X_\bullet) \oplus CC_{p-2,q+1}(X_\bullet) \oplus CC_{p-1,q}(X_\bullet) \oplus CC_{p,q-1}(X_\bullet), \end{aligned}$$

where the zero on the left comes from the calculation

$$s'_{q+1}N_{q+1}B_{p,q} \;=\; s'_{q+1}N_{q+1}(1-t_{q+1})s'_qN_q \;\overset{(2.2.13)}{=}\; 0.$$

The only position in which the results now differ is in the $CC_{p-1,q}(X_\bullet)$ term, and for this we have the following remedy:

$$\begin{aligned} -\partial'_{q+1}s'_qN_q(\alpha) + N_q(\alpha) \;&=\; (1-\partial'_{q+1}s'_q)N_q(\alpha) \\ \overset{*}{=}\; s'_{q-1}\partial'_qN_q(\alpha) \;&=\; s'_{q-1}N_{q-1}\partial_q(\alpha), \end{aligned}$$

where $\overset{*}{=}$ is due to s'_q being a homotopy of the identity. We conclude that $f \circ d' = d \circ f$, and thus f is a map of the given total complexes.

But f is also a quasi-isomorphism. To see this, consider the filtrations of $\mathcal{B}(X_\bullet)$ and $CC(X_\bullet)$ by columns:

$$F_i\mathcal{B}(X_\bullet) = \bigoplus_{\substack{p\leq i \\ p \text{ even}}} \mathcal{B}_{i,\bullet}(X_\bullet) \qquad\qquad F_iCC(X_\bullet) = \bigoplus_{p\leq i} CC_{i,\bullet}(X_\bullet).$$

Both filtrations are canonically bounded giving spectral sequences [19, Theorem 5.5.1]

$$\begin{aligned} E^1_{p,q} \;&=\; H_{p+q}(F_p(\mathcal{B}(X_\bullet))/F_{p-1}(\mathcal{B}(X_\bullet))) \quad\;\; \Rightarrow H_{p+q}(\mathcal{B}(X_\bullet)) \\ {}'E^1_{p,q} \;&=\; H_{p+q}(F_p(CC(X_\bullet))/F_{p-1}(CC(X_\bullet))) \;\Rightarrow H_{p+q}(CC(X_\bullet)). \end{aligned}$$

We can view f as a map sending α from $\mathcal{B}_{p,q}(X_\bullet)$ to $(f_1(\alpha), f_2(\alpha))$ in $CC_{p-1,q+1}(X_\bullet) \oplus CC_{p,q}(X_\bullet)$ where $f_1 : \mathcal{B}_{p,q}(X_\bullet) \longrightarrow CC_{p-1,q+1}(X_\bullet)$ and $f_2 : \mathcal{B}_{p,q}(X_\bullet) \longrightarrow CC_{p,q}(X_\bullet)$, and where f_2 is the identity morphism. At the E^1-page, ${}'E^1_{p-1,q+1}$ is trivial if p is even, thus the map induced by f_1 is trivial. Hence, f induces a map of spectral sequences on the E^1-pages that is an isomorphism

$$f^1_{p,q} : E^1_{p,q} \xrightarrow{\cong} {}'E^1_{p,q},$$

but then it is an isomorphism on the abutments as well [19, Lemma 5.2.4]. □

Chapter 3

Cyclic Singular Homology

In this chapter, we first prove that the relative singular chain complex $S(X, A)$ of a topological pair is precyclic with the last degeneracy, and thus fits into a cyclic double complex giving rise to cyclic singular homology. We give some basic properties before proving that the axioms for generalized homology theories hold for this construction. Next, we define reduced cyclic singular homology, and the last section presents computations of reduced cyclic singular homology groups for the spheres and the orientable and non-orientable surfaces, together with some more general results.

3.1 Definition and basic properties

We have $C_n(X) = Ab(Hom_{\mathbf{Top}}(\Delta^n, X))$, and for a topological pair (X, A), $C_n(X, A)$ is the quotient $C_n(X)/C_n(A)$. We claim that if $S(X)$ is precyclic with the last degeneracy for any topological space X, then so is $S(X, A)$ for any topological pair (X, A), by applying the presimplicial and cyclic operators to representatives of equivalence classes. I.e, if φ is any one of the operators in question we define

$$\varphi([\alpha]) \;=\; [\varphi(\alpha)],$$

so, for instance when $i < j$, we get

$$d_i d_j([\alpha]) \;=\; [d_i d_j(\alpha)] \;=\; [d_{j-1} d_i(\alpha)] \;=\; d_{j-1} d_i([\alpha]).$$

If $[\alpha] = [\beta]$ in $C_n(X, A)$, choose $\gamma \in C_n(A)$ such that $\alpha - \beta = \gamma$. Since $C_\bullet(A)$ is precyclic we have $d_i(\gamma) \in C_{n-1}(A)$, and since d_i is linear we get $d_i(\alpha) - d_i(\beta) \in C_{n-1}(A)$. Hence, the suggested definition is independent of choice of representative, and since the other operators are linear too, the claim holds.

It remains to prove that $S(X)$ is precyclic with the last degeneracy for any topological space X [16, Proposition 3.2]. Since $C_n(X) = Ab(Hom_{\mathbf{Top}}(\Delta^n, X))$, according to Proposition 2.2.3 and Corollary 2.2.4, this amounts to prove the following lemma:

Lemma-definition 3.1.1 $\Delta^\bullet = \{\Delta^n\}_{n\geq 0}$ *is pre-cocyclic with the last degeneracy.*

The operators are as follows:
The ***cosimplicial face maps*** $\partial^i : \Delta^{n-1} \longrightarrow \Delta^n$ *for* $0 \leq i \leq n$*, where*

$$\partial^i(x_0, \dots, x_{n-1}) = (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}),$$

inserting a zero in the i-th position as before. The ***cosimplicial cyclic operator*** $\Gamma^n : \Delta^n \longrightarrow \Delta^n$*, shifting the coordinates one place to the left:*

$$\Gamma^n(x_0, \dots, x_n) = (x_1, \dots, x_n, x_0). \tag{3.1.1}$$

And finally, ***the last cosimplicial degeneracy map*** $\rho^n : \Delta^{n+1} \longrightarrow \Delta^n$ *is defined by adding the last two elements in the (n+1)-tuple:*

$$\rho^n(x_0, \dots, x_{n+1}) = (x_0, \dots, x_{n-1}, x_n + x_{n+1}).$$

Proof We must prove that the given operators satisfy identities (2.2.7), (2.2.8), (2.2.9) and (2.2.10). In the following, upper indices indicate the elements position in the tuple. Assume $i < j$:

$$\begin{aligned} \partial^j\partial^i(x_0^0, \dots, x_{n-1}^{n-1}) &= \partial^j(x_0^0, \dots, 0^i, \dots, x_{n-1}^n) \\ &= (x_0^0, \dots, 0^i, \dots, 0^j, \dots, x_{n-1}^{n+1}) \\ &= \partial^i(x_0^0, \dots, \dots, 0^{j-1}, \dots, x_{n-1}^n) \\ &= \partial^i\partial^{j-1}(x_0^0, \dots, x_{n-1}^{n-1}). \end{aligned}$$

This proves property (2.2.7).

Property (2.2.8) requires that $(\Gamma^n)^{n+1} = id_{\Delta^n}$, which is obvious. If $1 \leq i \leq n$ we have

$$\begin{aligned} \Gamma^n\partial^i(x_0^0, \dots, x_{n-1}^{n-1}) &= \Gamma^n(x_0^0, \dots, 0^i, \dots, x_{n-1}^n) \\ &= (x_1^0, \dots, 0^{i-1}, \dots, x_{n-1}^{n-1}, x_0^n) \\ &= \partial^{i-1}(x_1^0, \dots, x_{n-1}^{n-2}, x_0^{n-1}) \\ &= \partial^{i-1}\Gamma^{n-1}(x_0^0, \dots, x_{n-1}^{n-1}), \end{aligned}$$

proving property (2.2.9).

For the interaction with the last degeneracy, assume first $0 \leq i < n$:

$$\begin{aligned} \rho^n\partial^i(x_0^0, \dots, x_n^n) &= \rho^n(x_0^0, \dots, 0^i, \dots, x_n^{n+1}) \\ &= (x_0^0, \dots, 0^i, \dots, (x_{n-1} + x_n)^n) \\ &= \partial^i(x_0^0, \dots, (x_{n-1} + x_n)^{n-1}) \\ &= \partial^i\rho^{n-1}(x_0^0, \dots, x_{n-1}^{n-1}, x_n^n), \end{aligned}$$

and for $i = n$ we have

$$\begin{aligned} \rho^n\partial^n(x_0^0, \dots, x_n^n) &= \rho^n(x_0^0, \dots, 0^n, x_n^{n+1}) \\ &= (x_0^0, \dots, (0 + x_n)^n) \\ &= (x_0^0, \dots, x_n^n). \end{aligned}$$

Since $i = n+1$ obviously yields the same result, this proves property (2.2.10), and thus completes the proof. □

So far, we have proved the following:

Corollary 3.1.2 *For a topological pair (X,A), $C_\bullet(X,A)$ is precyclic with the last degeneracy and each $C_n(X,A)$ is a free abelian group.*

Definition 3.1.3 *The **cyclic singular double complex***

$$CC(X,A) = CC^{sing}(X,A)$$

of a topological pair (X,A) is the first quadrant double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\Big\downarrow{\scriptstyle \partial_3} & & \Big\downarrow{\scriptstyle -\partial_3'} & & \Big\downarrow{\scriptstyle \partial_3} & & \\
C_2(X,A) & \xleftarrow{1-t_2} & C_2(X,A) & \xleftarrow{N_2} & C_2(X,A) & \xleftarrow{1-t_2} & \cdots \\
\Big\downarrow{\scriptstyle \partial_2} & & \Big\downarrow{\scriptstyle -\partial_2'} & & \Big\downarrow{\scriptstyle \partial_2} & & \\
C_1(X,A) & \xleftarrow{1-t_1} & C_1(X,A) & \xleftarrow{N_1} & C_1(X,A) & \xleftarrow{1-t_1} & \cdots \\
\Big\downarrow{\scriptstyle \partial_1} & & \Big\downarrow{\scriptstyle -\partial_1'} & & \Big\downarrow{\scriptstyle \partial_1} & & \\
C_0(X,A) & \xleftarrow{1-t_0} & C_0(X,A) & \xleftarrow{N_0} & C_0(X,A) & \xleftarrow{1-t_0} & \cdots
\end{array}$$

This is well-defined from Corollary 3.1.2, and the signed cyclic operator $t_n : C_n(X,A) \longrightarrow C_n(X,A)$ is given by

$$t_n([\sigma]) \;=\; (-1)^n T_n([\sigma]) \;=\; (-1)^n[\sigma \circ \Gamma^n].$$

Definition 3.1.4 *(Cf. [16, §3.2]) The **n-th relative cyclic singular homology group** $HC_n(X,A)$ of a topological pair (X,A) is the n-th total homology group of the singular cyclic double complex $CC(X,A)$:*

$$HC_n(X,A) \;=\; HC_n^{sing}(X,A) \;=\; H_n(Tot(CC(X,A))_\bullet) \quad \textit{for } n \geq 0.$$

For a topological space X we define non-relative cyclic singular homology by

$$HC_n(X) \;=\; HC_n(X,\emptyset) \quad \textit{for } n \geq 0.$$

Comparing (2.2.11) and (1.2.2), we recognize the complex $(C_\bullet(X,A), \partial_\bullet)$ in the even-numbered columns of $CC(X,A)$ as the relative singular chain complex of (X,A) from section 1.2.1. According to Theorem 2.3.1, since $S(X,A)$ is precyclic with the last degeneracy, we have long exact SBI-sequences for topological pairs (X,A)

$$\longrightarrow H_n(X,A) \xrightarrow{I} HC_n(X,A) \xrightarrow{S} HC_{n-2}(X,A) \xrightarrow{B} H_{n-1}(X,A) \longrightarrow,$$

where the $H_q(X,A)$ are the singular homology groups of the pair.

The odd-numbered columns are acyclic due to Lemma 2.3.2, and for the sake of completeness we give an explicit description of the last degeneracy:

Definition 3.1.5 *The* ***last degeneracy of the cyclic singular chain complex*** $s_q : CC_{\bullet,q}(X,A) \longrightarrow CC_{\bullet,q+1}(X,A)$ *for* $q \geq 0$ *is given by*

$$s_q(\sigma) \;=\; \sigma \circ \rho^q \quad \textit{for } q \geq 0.$$

The SBI-sequence allows us to deduce our first result:

Lemma 3.1.6 *For a topological pair* (X, A)*, we have*

$$HC_0(X,A) \;\cong\; H_0(X,A).$$

Proof For degree zero, the tail of the SBI-sequence contains the fragment $HC_{-1} \longrightarrow H_0 \longrightarrow HC_0 \longrightarrow HC_{-2}$ and since all negative degree groups are trivial we have an exact sequence $0 \longrightarrow H_0 \longrightarrow HC_0 \longrightarrow 0$. □

From section 2.4, we know that we have a Connes double complex $\mathcal{B}(X,A)$ for any topological pair (X,A), and that its total homology is isomorphic to that of $CC(X,A)$. The previous result can also be deduced from considering the Connes double complex

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
C_2(X,A)_{0,2} & \xleftarrow{B_{2,1}} & C_1(X,A)_{2,1} & \xleftarrow{B_{4,0}} & C_0(X,A)_{4,0} \\
\downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle \partial_1} & & \\
C_1(X,A)_{0,1} & \xleftarrow{B_{2,0}} & C_0(X,A)_{2,0} & & \\
\downarrow{\scriptstyle \partial_1} & & & & \\
C_0(X,A)_{0,0}. & & & &
\end{array}$$

The homology in total degree zero is $C_0(X,A)/\operatorname{im}(\partial_1)$, which is $H_0(X,A)$. So if X is a nonempty, path-connected topological space, we have $HC_0(X) \cong \mathbb{Z}$ ([8, Proposition 2.7]).

As indicated by Park [16, Exercise 3.3], we have the following result, although we also provide the computation:

Lemma 3.1.7 *The cyclic singular homology of a point is given by*

$$HC_n(*) \cong \begin{cases} \mathbb{Z} & \textit{for } n \textit{ even} \\ 0 & \textit{for } n \textit{ odd.} \end{cases}$$

Proof Since the singular homology of a point is trivial for $n > 0$, $n \geq 2$ gives us fragments of the SBI-sequence looking like

$$0 \longrightarrow HC_n(*) \longrightarrow HC_{n-2}(*) \longrightarrow 0.$$

Hence all odd-degree, respectively all even-degree, cyclic singular homology groups of a point are isomorphic. Since a point is nonempty and path-connected, we have $HC_n(*) \cong \mathbb{Z}$ for n even ((1.2.3) and Lemma 3.1.6).

For the odd-degree groups, observe that the SBI-sequence contains the fragment $H_1(*) \longrightarrow HC_1(*) \longrightarrow HC_{-1}(*)$, where both $H_1(*)$ and $HC_{-1}(*)$ are trivial, hence $HC_1(*)$ is trivial too. We thus conclude that $HC_n(*)$ is trivial for all odd n. $\square$

This can also be seen from the Connes double complex $\mathcal{B}(*)$: The E^1-page of the associated spectral sequence with filtration on columns turns out as

$$E^1_{**}: \quad \begin{array}{|lllll} 0_{0,2} & \longleftarrow & 0_{2,1} & \longleftarrow & \mathbb{Z}_{4,0} \\ & & & & \\ 0_{0,1} & \longleftarrow & \mathbb{Z}_{2,0} & & \\ & & & & \\ \mathbb{Z}_{0,0} & & & & \\ \hline \end{array}$$

where the indices are the bidegrees. The only nontrivial terms are base terms, thus the spectral sequence collapses, and the result follows immediately.

The above spectral sequence will be used frequently in the remainder of the text, so we name it the **standard spectral sequence** of the topological pair (X, A). I.e. it is the convergent, first quadrant spectral sequence associated with the Connes double complex $\mathcal{B}(X, A)$ with filtrations on columns starting at E^0:

$$E^0_{p,q} = \mathcal{B}_{p,q}(X, A) \;\Rightarrow\; HC_{p+q}(X, A) \quad \text{for } p \text{ even.}$$

In this spectral sequence, the terms on the E^1-page are the singular homology groups of the pair, and the relation is given by

$$E^1_{p,q}(X, A) \cong H_q(X, A).$$

In particular, if the standard spectral sequence of the topological pair (X, A) collapses on the E^1-page, we have

$$HC_n(X, A) \;\cong\; \bigoplus_{\substack{p+q=n \\ p \text{ even}}} E^1_{p,q} \;\cong\; \bigoplus_{i=0,2,4,\cdots} H_{n-i}(X, A). \tag{3.1.2}$$

For if F_i is the canonically bounded filtration on columns, we have isomorphisms

$$E^1_{p,q} \;\cong\; F_p HC_{p+q}/F_{p-1}HC_{p+q},$$

which is nontrivial only if p is even, so we get

$$HC_n \;\cong\; \bigoplus_{\substack{0 \le i \le n \\ i \text{ even}}} F_i HC_n / F_{i-1} HC_n \;\cong\; \bigoplus_{\substack{0 \le i \le n \\ i \text{ even}}} E^1_{i,n-i}.$$

Lemma 3.1.8 *If $f : (X, A) \longrightarrow (Y, B)$ is a map of pairs, then there are induced proper maps of precyclic objects with the last degeneracy $\hat{f}_n : C_n(X, A) \longrightarrow C_n(Y, B)$ for $n \geq 0$. If $[\alpha]$ is an element in $C_n(X, A)$, then $\hat{f}_n$ is defined by*

$$\hat{f}_n([\alpha]) \quad = \quad [f \circ \alpha].$$

Proof Assume $[\alpha] = [\beta]$ in $C_n(X, A)$. Then there is a γ in $C_n(A)$ such that $\alpha - \beta = \gamma$. Since $f \circ \gamma \in C_n(B)$ and $f \circ \gamma = f \circ \alpha - f \circ \beta$ due to linearity, we have $[f \circ \alpha] = [f \circ \beta]$ in $C_n(Y, B)$, so the map is well-defined.

Commutativity with the d_i follows from the calculation

$$\hat{f}_n(d_i([\alpha])) \;=\; \hat{f}_n([\alpha \circ \partial^i]) \;=\; [(f \circ \alpha) \circ \partial^i] \;=\; d_i([f \circ \alpha]) \;=\; d_i(\hat{f}_n([\alpha])),$$

where the key in the commutativity is found in the definition of d_i by precomposing with ∂^i. Since both T_n and s_n are defined in the same manner, commutativity with T_n and s_n follows, and $\hat{f}_\bullet = \{\hat{f}_n\}_{n \geq 0}$ is a proper map of precyclic objects with the last degeneracy. □

Corollary 3.1.9 *Maps of pairs induce homomorphisms in relative cyclic singular homology.*

Proof Maps of pairs induce homomorphisms on the cyclic double complexes. Constructing the map in the total complex according to the proof of Proposition 2.2.11, the result follows immediately. □

Corollary 3.1.10 *A map of pairs induces isomorphisms in relative cyclic singular homology if and only of it induces isomorphisms in relative singular homology.*

Proof Let $f : X \longrightarrow Y$ be the isomorphism-inducing map. In this proof we consider the non-relative case, but the relative case is identical, except the diagrams take more room.

The "only-if" part follows from the commutative diagram below and the Five-lemma, where $f_\bullet : HC_\bullet(X) \longrightarrow HC_\bullet(Y)$ is an isomorphism. The top and bottom rows are from the SBI-sequences of X and Y.

$$\begin{array}{ccccccccc}
HC_{n+2}(X) & \longrightarrow & HC_{n+1}(X) & \longrightarrow & H_n(X) & \longrightarrow & HC_n(X) & \longrightarrow & HC_{n-2}(X) \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
HC_{n+2}(Y) & \longrightarrow & HC_{n+1}(Y) & \longrightarrow & H_n(Y) & \longrightarrow & HC_n(Y) & \longrightarrow & HC_{n-2}(Y)
\end{array}$$

Now assume f induces an isomorphism $f_\bullet : H_\bullet(X) \longrightarrow H_\bullet(Y)$. From the SBI-sequence we have a commutative diagram

$$\begin{array}{ccc}
H_0(X) & \xrightarrow{\cong} & HC_0(X) \\
f_* \downarrow \cong & & \downarrow f_* \\
H_0(Y) & \xrightarrow{\cong} & HC_0(Y).
\end{array}$$

The horizontal isomorphisms are due to Lemma 3.1.6 and the left hand side map is an isomorphism by assumption, hence $f_* : HC_0(X) \longrightarrow HC_0(Y)$ is an isomorphism as well. Assume by induction that $f_* : HC_{n-1}(X) \longrightarrow HC_{n-1}(Y)$ is an isomorphism for $n \geq 0$. By applying the Five-lemma to the diagram

$$\begin{array}{ccccccccc} HC_{n-1}(X) & \longrightarrow & H_n(X) & \longrightarrow & HC_n(X) & \longrightarrow & HC_{n-2}(X) & \longrightarrow & H_{n-1}(X) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ HC_{n-1}(X) & \longrightarrow & H_n(X) & \longrightarrow & HC_n(X) & \longrightarrow & HC_{n-2}(X) & \longrightarrow & H_{n-1}(X), \end{array}$$

we conclude that $f_* : HC_n(X) \longrightarrow HC_n(Y)$ is an isomorphism too. □

The map $f_* : HC_n(X, A) \longrightarrow HC_n(Y, B)$

To see what the induced map $f_* : HC_n(X, A) \longrightarrow HC_n(Y, B)$ does, let $\overline{z}$ in $HC_n(X, A)$ be represented by the element $[z_i]$ in $Tot(CC(X, A))_n$. Overline indicates elements in homology groups, while square brackets indicate equivalence classes in the quotient group $Tot(CC(X, A))_n$. We get:

$$f_*(\overline{z}) \;=\; \overline{f_n([z_i])} \;=\; \overline{[\hat{f}_i(z_i)]} \;=\; \overline{[f \circ z_i]} \;=\; \overline{f \circ z}.$$

3.2 $HC_\bullet$ is a homology functor

We now prove that the axioms for a generalized homology theory in Definition 1.3.1 hold for $HC_\bullet$. For the excision axiom, we have chosen to prove the existence of Mayer-Vietoris sequences, according to Proposition 1.3.4.

3.2.1 Functoriality

Lemma 3.2.1 *Cyclic singular homology is a functor $HC_\bullet :$ **PTop** $\longrightarrow$ **Ab**.*

Proof We have already seen that $HC_\bullet$ sends topological pairs to abelian groups, and that maps of pairs are sent to group homomorphisms. Now assume we have maps $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ and that $\overline{z}$ is an element in $HC_n(X, A)$ represented by the cycle z. Then $(g \circ f)_* = g_* \circ f_*$ since

$$g_*(f_*(\overline{z})) \;=\; g_*(\overline{f \circ z}) \;=\; \overline{g \circ f \circ z} \;=\; (g \circ f)_*(\overline{z}),$$

and the map induced by the identity on (X, A) is sent to the identity on $HC_\bullet(X, A)$:

$$id_*(\overline{z}) \;=\; \overline{id \circ z} \;=\; \overline{z}.$$

□

Corollary 3.2.2 *For a nonempty topological space X, all even degree cyclic singular homology groups of X contain a subgroup isomorphic to $\mathbb{Z}$.*

Proof Since X is nonempty, there is an inclusion $i : * \longrightarrow X$ and a retract $r : X \longrightarrow *$ so that the composition $r \circ i$ is the identity on $*$. Since $HC_\bullet$ is a functor, we have a composition

$$HC_n(*) \xrightarrow{i_*} HC_n(X) \xrightarrow{r_*} HC_n(*),$$

where $r_* \circ i_* = id_{HC_n(*)}$. Since the identity on $HC_n(*)$ factors through $HC_n(X)$, the latter must contain a subgroup isomorphic to $HC_n(*)$ for all $n \geq 0$. The result then follows from Lemma 3.1.7. □

This shows that there are a lot of non-trivial cyclic singular homology groups for any nonempty topological space, and since a great deal of this structure will be equal (up to isomorphism) for all path-connected spaces it does not reveal any new information. It may thus be more enlightening to study the reduced cyclic singular homologies of these spaces. Reduced cyclic singular homology is described in section 3.3.

3.2.2 Disjoint unions split into direct sums

Lemma 3.2.3 *If $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$ is a disjoint union of topological pairs, the inclusions $i_\alpha : X_\alpha \hookrightarrow X$ induce isomorphisms*

$$\bigoplus_\alpha HC_n(X_\alpha, A_\alpha) \;\cong\; HC_n(X, A) \quad \text{for } n \geq 0.$$

Proof Consider the convergent spectral sequences

$$\begin{aligned} E^1_{p,q} &= H_q(\mathcal{B}_{p,\bullet}(X, A)) &&\Rightarrow HC_{p+q}(X, A) \\ {}'E^1_{p,q} &= \textstyle\bigoplus_\alpha H_q(\mathcal{B}_{p,\bullet}(X_\alpha, A_\alpha)) &&\Rightarrow \textstyle\bigoplus_\alpha HC_{p+q}(X_\alpha, A_\alpha). \end{aligned}$$

Since the terms on the E^1-pages are the singular homology groups of the given pairs, the inclusions induce an isomorphism $i^1_* : {}'E^1_{**} \longrightarrow E^1_{**}$ [2, §4.12]. But then the map induced on the abutments is an isomorphism as well. □

3.2.3 Long exact sequences in homology

Lemma 3.2.4 *For a topological pair (X, A), we have a long exact sequence of cyclic singular homology groups*

$$\cdots \longrightarrow HC_n(A) \xrightarrow{i_*} HC_n(X) \xrightarrow{j_*} HC_n(X, A) \xrightarrow{\partial_*} HC_{n-1}(A) \longrightarrow \cdots,$$

where the maps i_* and j_* are induced by the inclusions $A \xhookrightarrow{i} X$ and $(X, \emptyset) \xhookrightarrow{j} (X, A)$, respectively.

Proof We have a short exact sequence of precyclic objects with the last degeneracy

$$0 \longrightarrow C_\bullet(A) \xrightarrow{\hat{i}} C_\bullet(X) \xrightarrow{\hat{j}} C_\bullet(X, A) \longrightarrow 0. \qquad (3.2.1)$$

The maps in the sequence are proper maps due to Lemma 3.1.8, and by Lemma 2.2.10 this yields a long exact sequence in homology. □

3.2.4 Naturality of the connecting homomorphism

The connecting homomorphism $\partial_* : HC_n(X,A) \longrightarrow HC_{n-1}(A)$ is obtained by applying the Snake-lemma [19, Lemma 1.3.2] to the short exact sequence of total complexes arising from (3.2.1), and it is induced by the map $d_n : Tot(CC(A))_n \longrightarrow Tot(CC(A))_{n-1}$. If $\overline{\alpha} \in HC_n(X,A)$ is represented by the cycle α, then $\partial_*(\overline{\alpha}) = \overline{d_n(\alpha)}$ (cf. [8, p117]).

Lemma 3.2.5 *Given a map of pairs $f : (X,A) \longrightarrow (Y,B)$, the connecting homomorphism $\partial_* : HC_n(X,A) \longrightarrow HC_n(A)$ in the long exact homology sequence is natural in the sense that the diagram below commutes for all $n \geq 0$:*

$$\begin{array}{ccc} HC_n(X,A) & \xrightarrow{\partial_*} & HC_{n-1}(A) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ HC_n(Y,B) & \xrightarrow{\partial_*} & HC_{n-1}(B). \end{array}$$

Proof Assume $\overline{z}$ is an element in $HC_n(X,A)$. Since f induces proper maps $f_n : Tot(CC(A))_n \longrightarrow Tot(CC(B))_n$, and since the connecting homomorphism is induced by $d_n : Tot(CC(A))_n \longrightarrow Tot(CC(A))_{n-1}$, we have the following calculation based on the description of ∂_* given prior to the lemma:

$$\begin{aligned} f_*(\partial_*(\overline{z})) &= f_*(\overline{d_n(z)}) = \overline{f_{n-1}(d_n(z))} \overset{*}{=} \overline{d_n(f_n(z))} \\ &= \partial_*(\overline{f \circ z}) = \partial_*(f_*(\overline{z})), \end{aligned}$$

where $\overset{*}{=}$ is due to f_n being proper. □

3.2.5 Homotopy invariance

Lemma 3.2.6 *Homotopic maps of pairs induce equal maps in cyclic singular homology:*

$$f \simeq g : (X,A) \longrightarrow (Y,B) \;\Rightarrow\; HC_n(f) = HC_n(g) \quad \textit{for } n \geq 0.$$

Proof Consider the standard spectral sequences

$$\begin{aligned} E^0_{p,q} &= \mathcal{B}(X,A) \Rightarrow HC_{p+q}(X,A) \\ {}'E^0_{p,q} &= \mathcal{B}(Y,B) \Rightarrow HC_{p+q}(Y,B). \end{aligned}$$

By [8, Proposition 2.19], f and g induce coinciding maps in relative singular homology, hence the induced maps on the E^1-pages coincide:

$$f^1_* = g^1_* : E^1_{p,q} \longrightarrow {}'E^1_{p,q}.$$

But then they coincide at the abutments too, for the terms on the successive pages are just successive subquotients of the terms on the E^1-page, and the successive maps are induced by the two identical maps on the E^1-pages. □

3.2.6 Mayer-Vietoris sequences

Lemma 3.2.7 *Given an excisive triad* $(X; A, B)$*, there is a long exact Mayer-Vietoris sequence*

$$\cdots \longrightarrow HC_n(A \cap B) \xrightarrow{\Psi} HC_n(A) \oplus HC_n(B) \xrightarrow{\Phi} HC_n(X) \xrightarrow{\Delta} HC_{n-1}(A \cap B) \longrightarrow \cdots \longrightarrow HC_0(X) \longrightarrow 0.$$

Proof We will show that there is a short exact sequence of cyclic double complexes that is quasi-isomorphic to the short exact sequence

$$0 \longrightarrow CC(A \cap B) \xrightarrow{\psi} CC(A) \oplus CC(B) \xrightarrow{\phi} CC(X) \longrightarrow 0, \qquad (3.2.2)$$

where the maps ψ and ϕ come from the commutative diagram of inclusions

$$\begin{array}{ccccc} & & A & & \\ & \overset{i}{\nearrow} & & \overset{k}{\searrow} & \\ A \cap B & & & & X \\ & \underset{j}{\searrow} & & \underset{l}{\nearrow} & \\ & & B & & \end{array}$$

and are defined by

$$\begin{aligned} \psi(c) &= (i_*(c), -j_*(c)) \\ \phi(a, b) &= k_*(a) + l_*(b). \end{aligned}$$

From [8, p149] we know that such maps exist for a short exact sequence quasi-isomorphic to the short exact sequence

$$0 \longrightarrow C_\bullet(A \cap B) \xrightarrow{\hat{\psi}} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{\hat{\phi}} C_\bullet(X) \longrightarrow 0,$$

and all the maps are proper due to Lemma 3.1.8. By Lemma 2.2.10 this gives us a short exact sequence of cyclic double complexes quasi-isomorphic to (3.2.2). □

The two corollaries to follow are direct consequences of Proposition 1.3.4.

Corollary 3.2.8 *Given a topological pair* (X, A) *and an excisive triad* $(X; U, V)$*, then there is a long exact* ***relative Mayer-Vietoris sequence***

$$\cdots \longrightarrow HC_n(U \cap V, U \cap V \cap A) \xrightarrow{\Psi} HC_n(U, U \cap A) \oplus HC_n(V, V \cap A) \xrightarrow{\Phi} HC_n(X, A) \xrightarrow{\Delta} HC_{n-1}(U \cap V, U \cap V \cap A) \longrightarrow \cdots .$$

Corollary 3.2.9 *Given spaces* $B \subset A \subset X$ *such that the closure of* B *is contained in the interior of* A*, the inclusion map* $(X - B, A - B) \hookrightarrow (X, A)$ *induces isomorphisms in cyclic singular homology:*

$$HC_n(X, A) \;\cong\; HC_n(X - B, A - B) \quad \text{for } n \geq 0.$$

This completes the proofs of the generalized homology axioms.

3.3 Reduced cyclic singular homology

For a generalized reduced homology theory one generally wants the spaces under consideration to be well-based, i.e. the inclusion $* \hookrightarrow X$ of the basepoint must be a cofibration [15, §14.4]. But since every CW-complex is well-based, and since every topological space can be approximated by a CW-complex up to a homotopy equivalence [15, §10.5], this is no real limitation.

We will, however, not make this requirement, and wherever we need results that are possible to derive directly from the axioms for generalized reduced homology theories, and where these axioms require well-based spaces we shall prove the results directly.

Recall from section 1.2.3 that $\mathbf{Top}_*$ is the full subcategory of **PTop** with the topological pairs of the form $(X, *)$ as objects, where $*$ is the basepoint in X. Restricting $HC_\bullet$ to this subcategory, we obtain the following homology functor:

Definition 3.3.1 *The* ***reduced cyclic singular homology*** $\widetilde{HC}_n(X)$ *of a topological space* X *in* $\mathbf{Top}_*$ *is given by*

$$\widetilde{HC}_n(X) \;=\; HC_n(X, *) \quad \textit{for } n \geq 0,$$

where $*$ *is the basepoint in* X.

For the topological pair $(X, *)$, the SBI-sequence looks like

$$\cdots \longrightarrow H_k(X,*) \xrightarrow{I} HC_k(X,*) \xrightarrow{S} HC_{k-2}(X,*) \xrightarrow{B} H_{k-1}(X,*) \longrightarrow \cdots,$$

which immediately gives rise to **the relative SBI-sequence of** X:

$$\cdots \longrightarrow \widetilde{H}_k(X) \xrightarrow{I} \widetilde{HC}_k(X) \xrightarrow{S} \widetilde{HC}_{k-2}(X) \xrightarrow{B} \widetilde{H}_{k-1}(X) \longrightarrow \cdots.$$

We also have a **reduced Connes double complex** $\widetilde{\mathcal{B}}(X) = \mathcal{B}(X, *)$.

Lemma 3.3.2 *The relation between reduced and nonreduced cyclic singular homology is given by:*

$$\widetilde{HC}_n(X) \oplus HC_n(*) \;\cong\; HC_n(X) \quad \textit{for } n \geq 0. \tag{3.3.1}$$

Proof The proof is identical to the proof of Lemma 1.2.8, using the functor properties from Lemma 3.2.1. ☐

Lemma-definition 3.3.3 *Given an excisive triad* $(X; A, B)$ *where* $A \cap B$ *is nonempty we have a* ***reduced Mayer-Vietoris sequence***

$$\cdots \longrightarrow \widetilde{HC}_n(A \cap B) \xrightarrow{\Psi} \widetilde{HC}_n(A) \oplus \widetilde{HC}_n(B) \xrightarrow{\Phi} \widetilde{HC}_n(X) \xrightarrow{\Delta} \cdots.$$

Proof We can pick a basepoint $*$ inside $A \cap B$ and form the topological pair $(X, *)$. The relative Mayer-Vietoris sequence from Corollary 3.2.8 then takes the form

$$\begin{aligned} \cdots \longrightarrow HC_n(A \cap B, A \cap B \cap *) &\xrightarrow{\Psi} \\ HC_n(A, A \cap *) \oplus HC_n(B, B \cap *) &\xrightarrow{\Phi} \\ HC_n(X, *) \xrightarrow{\Delta} HC_{n-1}(A \cap B, A \cap B \cap *) &\longrightarrow \cdots, \end{aligned}$$

but this is the same as

$$\begin{aligned} \cdots \longrightarrow HC_n(A \cap B, *) \xrightarrow{\Psi} HC_n(A, *) \oplus HC_n(B, *) &\xrightarrow{\Phi} \\ HC_n(X, *) \xrightarrow{\Delta} HC_{n-1}(A \cap B, *) &\longrightarrow \cdots, \end{aligned}$$

giving us the desired sequence. □

3.4 Calculations in cyclic singular homology

This section contains calculations of the reduced cyclic singular homology groups of the spheres and the orientable and non-orientable surfaces, as well as proving a general result for reduced cyclic singular homology groups of wedge sums.

Lemma 3.4.1 *Suppose X is an $(n-1)$-connected, based space, i.e $\widetilde{H}_i(X) = 0$ for $0 \leq i < n$ and $\widetilde{H}_n(X)$ is nontrivial. Then $\widetilde{HC}_q(X)$ is trivial for $0 \leq q < n$ and $\widetilde{HC}_n(X) \cong \widetilde{H}_n(X)$.*

Proof Consider the **reduced standard spectral sequence** of X, given by the standard spectral sequence of $\mathcal{B}(X, *)$. The E^1-page turns out as

$$E^1_{**}: \quad \begin{array}{|cccccc} \widetilde{H}_{n+1}(X)_{0,n+1} & \longleftarrow & \widetilde{H}_n(X)_{2,n} & \longleftarrow & 0_{4,n-1} & \longleftarrow \\ \widetilde{H}_n(X)_{0,n} & \longleftarrow & 0_{2,n-1} & \longleftarrow & 0_{4,n-2} & \longleftarrow \\ 0_{0,n-1} & \longleftarrow & 0_{2,n-2} & \longleftarrow & 0_{4,n-3} & \longleftarrow \\ \vdots & & \vdots & & \vdots & \\ \hline \end{array}$$

The $E^1_{0,n}$-term is clearly stable, and it is the only term contributing to the homology of total degree n. All reduced cyclic singular homology groups in lower degrees are trivial. □

Thus, if the premises of the lemma hold for a topological space, the (nonreduced) cyclic singular homology of the space is trivial in all odd degrees and isomorphic to $\mathbb{Z}$ in all even degrees in the range $0 \leq q < n$.

We have already stated that for X nonempty and path-connected, we have $HC_0(X) \cong H_0(X)$. The following result, which is due to reduced cyclic singular homology, is in the same lane:

Corollary 3.4.2 *For a nonempty, path-connected based space X, we have*

$$\widetilde{HC}_1(X) \cong \widetilde{H}_1(X).$$

Proof Since $\widetilde{H}_0(X) = 0$, the result follows directly from Lemma 3.4.1. □

Lemma 3.4.3 *Suppose $\widetilde{H}_i(X) = 0$ for $i > n$ and $\widetilde{H}_n(X)$ is nontrivial. Then $\widetilde{HC}_q(X)$ for $q \geq n$ is given by*

$$\widetilde{HC}_q(X) \cong \begin{cases} \widetilde{HC}_n(X) & \text{for } q = n+2, n+4\dots \\ \widetilde{HC}_{n+1}(X) & \text{for } q = n+3, n+5\dots \end{cases}$$

Proof For $q > n$ the SBI-sequence yields fragments

$$\widetilde{H}_{q+1}(X) \longrightarrow \widetilde{HC}_{q+1}(X) \longrightarrow \widetilde{HC}_{q-1}(X) \longrightarrow \widetilde{H}_q(X).$$

The reduced singular homology groups at the ends are trivial, giving us the desired isomorphisms. □

3.4.1 The cyclic singular homology of the spheres

Lemma 3.4.4 *The reduced cyclic singular homology of the sphere S^k for $k \geq 0$ is given by*

$$\widetilde{HC}_q(S^k) \cong \begin{cases} \mathbb{Z} & \text{for } q = k, k+2, k+4, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Before proving the lemma, recall the reduced singular homology of S^k for $k \geq 0$ (cf. [8, Corollary 2.14]):

$$\widetilde{H}_q(S^k) \cong \begin{cases} \mathbb{Z} & \text{for } q = k \\ 0 & \text{otherwise.} \end{cases}$$

Proof (of Lemma 3.4.4) We consider the zero-sphere to be the disjoint union of two points $S^0 = * \coprod *$, giving $HC_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$. Passing to reduced homology one copy of the integers disappears, giving $\widetilde{HC}_0(S^0) \cong \mathbb{Z}$, and for the higher degree groups we have

$$\widetilde{HC}_1(S^0) = \widetilde{HC}_1(* \coprod *) = 0,$$

wherefrom the result for S^0 follows according to Lemma 3.4.3.

Now, let A and B be the northern and southern hemispheres of S^k for $k \geq 1$, so that $S^k = A \cup B$, $A \cap B \simeq S^{k-1}$ and A and B are contractible. Since $k \geq 1$ we have $A \cap B \neq \emptyset$, and can therefore pick a basepoint in the intersection. For $q \geq 1$, the reduced Mayer-Vietoris sequence of $(S^k; A, B)$ takes the form

$$\cdots \longrightarrow \widetilde{HC}_q(*) \oplus \widetilde{HC}_q(*) \longrightarrow \widetilde{HC}_q(S^k) \longrightarrow \widetilde{HC}_{q-1}(S^{k-1}) \longrightarrow \widetilde{HC}_{q-1}(*) \oplus \widetilde{HC}_{q-1}(*) \longrightarrow \cdots,$$

giving isomorphisms

$$\widetilde{HC}_k(S^k) \;\cong\; \widetilde{HC}_{k-1}(S^{k-1}) \;\cong\; \cdots \;\cong\; \widetilde{HC}_0(S^0) \;\cong\; \mathbb{Z},$$

so we have $\widetilde{HC}_k(S^k) \cong \mathbb{Z}$ for $k \geq 0$, and Lemma 3.4.1 yields $\widetilde{HC}_q(S^k) = 0$ when $q < k$.

It remains to show that $\widetilde{HC}_{k+1}(S^k)$ is trivial, wherefrom the result follows due to Lemma 3.4.3. This triviality is guaranteed by the sequence of isomorphisms

$$\widetilde{HC}_{k+1}(S^k) \;\cong\; \widetilde{HC}_k(S^{k-1}) \;\cong\; \cdots \;\cong\; \widetilde{HC}_1(S^0) \;=\; 0.$$

□

The proof only makes use of the reduced Mayer-Vietoris sequence, but the result can also be deduced from the reduced SBI-sequence: We have a fragment

$$\widetilde{HC}_{k-1}(S^k) \xrightarrow{B} \widetilde{H}_k(S^k) \xrightarrow{I} \widetilde{HC}_k(S^k) \xrightarrow{S} \widetilde{HC}_{k-2}(S^k), \tag{3.4.1}$$

where both $\widetilde{HC}_{k-1}(S^k)$ and $\widetilde{HC}_{k-2}(S^k)$ are trivial due to Lemma 3.4.1, hence, $\widetilde{HC}_k(S^k) \cong \widetilde{H}_k(S^k)$. The lower reduced cyclic singular homology groups are trivial by the same lemma, and triviality of the higher homology groups follows from the SBI-sequence fragment

$$\widetilde{H}_{k+1}(S^k) \longrightarrow \widetilde{HC}_{k+1}(S^k) \longrightarrow \widetilde{HC}_{k-1}(S^k),$$

where the left and right hand sides are trivial.

The degree of maps $f : S^n \longrightarrow S^n$

Since $\widetilde{HC}_q(S^n)$ is isomorphic to $\mathbb{Z}$ for $q = n, n+2, n+4, \cdots$, and is trivial otherwise, a homomorphism $f_* : \widetilde{HC}_q(S^n) \longrightarrow \widetilde{HC}_q(S^n)$, induced by a map $f : S^n \longrightarrow S^n$, is determined by multiplication by an integer, just as for reduced singular homology [8, p134]: From (3.4.1) we get a commutative diagram

$$\begin{array}{ccc} \widetilde{H}_n(S^n) & \xrightarrow[\cong]{I} & \widetilde{HC}_n(S^n) \\ {\scriptstyle \widetilde{H}_n(f)}\downarrow & & \downarrow{\scriptstyle \widetilde{HC}_n(f)} \\ \widetilde{H}_n(S^n) & \xrightarrow[\cong]{I} & \widetilde{HC}_n(S^n), \end{array}$$

where the horizontal arrows are isomorphisms. Commutativity yields $\widetilde{HC}_n(f) = \widetilde{H}_n(f)$, hence, the degree in reduced cyclic singular homology is the same as the degree in reduced singular homology. For the other nontrivial reduced cyclic singular homology groups, we have similar commutative diagrams from Lemma 3.4.3:

$$\begin{array}{ccc} \widetilde{HC}_{n+2k}(S^n) & \xrightarrow{\cong} & \widetilde{HC}_n(S^n) \\ {\scriptstyle \widetilde{HC}_{n+2k}(f)}\downarrow & & \downarrow{\scriptstyle \widetilde{HC}_n(f)} \\ \widetilde{HC}_{n+2k}(S^n) & \xrightarrow{\cong} & \widetilde{HC}_n(S^n) \end{array}$$

for $k > 0$. Again, the horizontal arrows are isomorphisms. This shows that all the maps $\widetilde{HC}_q(f)$ for $q = n, n+2, n+4, \ldots$ have the same degree, and that this degree is the same as for $\widetilde{H}_n(f)$. We put this observation up as a definition:

Definition 3.4.5 *The **degree** of a continuous map $f : S^n \longrightarrow S^n$ is the unique integer $\deg(f) \in \mathbb{Z}$ such that all maps $f_* : \widetilde{HC}_q(S^n) \longrightarrow \widetilde{HC}_q(S^n)$ for $q = n, n+2, n+4, \cdots$ are given by $f_*(\alpha) = \deg(f)\, \alpha$.*

3.4.2 The cyclic singular homology of wedge sums

Lemma 3.4.6 *(Cf. [8, Corollary 2.25]) Given a wedge sum of based spaces*

$$(X, *) \;=\; \bigvee\nolimits_{i \in \Lambda} (X_i, *_i),$$

where each basepoint $*_i$ in the factors X_i is a deformation retract of some neighborhood in X_i containing $*_i$, then the reduced cyclic singular homology groups of X split as a direct sum over the reduced cyclic singular homology groups of the factors in the wedge sum:

$$\bigoplus_{i \in \Lambda} \widetilde{HC}_q(X_i) \;\cong\; \widetilde{HC}_q(X),$$

and the isomorphism is induced by the inclusions $(X_i, *_i) \hookrightarrow (X, *)$ for $i \in \Lambda$.

Proof This can be proven by a standard spectral sequence argument, and the result will then hold for arbitrary index sets Λ. We will, however, prove this by induction and by applying the reduced Mayer-Vietoris sequence. This proof will therefore only safeguard the result for countable index sets.

If the cardinality of Λ is one, then the statement clearly holds, so assume that X is a wedge sum of n spaces, that $n > 1$, and that the result holds for wedge sums of less than n spaces.

Define U by forming the wedge sum of all the neighborhoods containing the different basepoints in the respective X_i's. In this way, since $* \sim *_i$ for all $i \in \Lambda$, $*$ is a deformation retract of U, and U intersects each X_i in an open neighborhood containing $*_i$.

Define $A = (\bigvee_{i=0}^{n-1} X_i) \cup U$ and $B = X_n \cup U$. Since the subset of U not intersecting $\bigvee_{n-1} X_i$ is the open subset of X_n that deformation retracts to $*_n \sim *$, we have $A \simeq \bigvee_{n-1} X_i$, and similarly $B \simeq X_n$. We also have $A \cup B = X$ and $A \cap B = U \simeq *$, so the reduced Mayer-Vietoris sequence takes the form

$$\cdots \longrightarrow \widetilde{HC}_q(\bigvee_{n-1} X_i) \oplus \widetilde{HC}_q(X_n) \xrightarrow{\Phi} \widetilde{HC}_q(X) \longrightarrow \widetilde{HC}_{q-1}(*) \longrightarrow \widetilde{HC}_{q-1}(\bigvee_{n-1} X_i) \oplus \widetilde{HC}_{q-1}(X_n) \longrightarrow \cdots .$$

Since $\widetilde{HC}_\bullet(*) = 0$, the sequence splits into isomorphisms

$$\widetilde{HC}_q(\bigvee_{n-1} X_i) \oplus \widetilde{HC}_q(X_n) \xrightarrow{\Phi} \widetilde{HC}_q(X) \quad \text{for } q \geq 0.$$

Thus, due to induction, the reduced cyclic singular homology groups of the wedge sum is isomorphic to the direct sum of the reduced cyclic singular homology groups of the factors in the wedge sum.

According to Lemma 3.2.7, the isomorphism Φ in the Mayer-Vietoris sequence is induced by the two inclusions $\bigvee_{i=0}^{n-1}(X_i, *_i) \hookrightarrow (X, *)$ and $(X_n, *_n) \hookrightarrow (X, *)$. By induction, Φ is induced by the inclusions $(X_i, *_i) \hookrightarrow (X, *)$ as stated. □

For the next calculation, we record the following corollary:

Corollary 3.4.7 *The reduced cyclic singular homology of a wedge sum of k n-spheres for $n \geq 0$ is given by:*

$$\widetilde{HC}_q(\bigvee_k S^n) \cong \begin{cases} \bigoplus_k \mathbb{Z} & \text{for } q = n, n+2, n+4, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

3.4.3 The cyclic singular homology of surfaces

Definition 3.4.8 *An **n-manifold** is a Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$, and a **surface** is a connected, compact 2-manifold without boundary.*

Definition 3.4.9 *The **orientable surface of genus g** for a positive integer g, is the surface given by*

$$M_g = \bigvee_{i=1}^{g} S^1_{a_i} \vee S^1_{b_i} \cup_{\Pi_{i=1}^g [a_i, b_i]} \mathbb{D}^2.$$

That is, we have one 0-cell (the wedge point), $2g$ 1-cells and one 2-cell. The attaching maps of the 1-cells are uniquely determined *by the one 0-cell, and the attaching map of the 2-cell is given by* $\Pi_{i=1}^{g}[a_i, b_i] = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.

The sphere is the orientable surface of genus zero, M_0.

Definition 3.4.10 *The* ***non-orientable surface of genus g*** *for a positive integer g, is the surface given by*

$$N_g = \bigvee_{i=1}^{g} S^1_{a_i} \cup_{\Pi_{i=1}^{g} a_i^2} \mathbb{D}^2.$$

That is, we have one 0-cell, g 1-cells and one 2-cell. The attaching maps of the 1-cells are uniquely determined by the one 0-cell, and the attaching map of the 2-cell is given by $\Pi_{i=1}^{g} a_i^2 = a_1^2 a_2^2 \cdots a_g^2$.

If we remove a point from M_g, we may consider it to be a point in the interior of the 2-cell. The remains of the interior of the cell deformation retracts to the boundary, so all that remains is the 1-skeleton:

$$M_g - \{*\} \simeq \bigvee_{2g} S^1.$$

The following, astounding, result was proven during the twenties through the contribution of many, with one of the turning points being the proof by Radó from 1925, showing that any surface can be triangulated (cf. [14, Theorem 7.2]):

Theorem 3.4.11 (The Classification Theorem for Surfaces) *Any surface X is either homeomorphic to the 2-sphere, or there exists a non-negative integer g, such that X is homeomorphic to either* M_g *or* N_g.

Thus, the calculation of cyclic singular homology groups of surfaces is reduced to computing the cyclic singular homology groups of the 2-sphere, M_g and N_g. Since we have already computed the (reduced) cyclic singular homology of the 2-sphere, the reduced cyclic singular homology of the remaining orientable surfaces is given by the next lemma:

Lemma 3.4.12 *The reduced cyclic singular homology of an oriented surface of genus* $g \geq 1$ *is given by*

$$\widetilde{HC}_q(M_g) \cong \begin{cases} 0 & \text{for } q = 0 \\ \bigoplus_{2g} \mathbb{Z} & \text{for } q \geq 1 \text{ odd} \\ \mathbb{Z} & \text{for } q \geq 2 \text{ even.} \end{cases}$$

Proof Recall the reduced singular homology of M_g [8, Example 2.36]:

$$\widetilde{H}_q(M_g) \cong \begin{cases} \bigoplus_{2g} \mathbb{Z} & \text{for } q = 1 \\ \mathbb{Z} & \text{for } q = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Since M_g is connected, we have $\widetilde{HC}_0(M_g) = 0$, and $\widetilde{HC}_1(M_g) \cong \bigoplus_{2g} \mathbb{Z}$ from Lemma 3.4.1. According to Lemma 3.4.3, what remains is to calculate $\widetilde{HC}_2(M_g)$ and $\widetilde{HC}_3(M_g)$ to determine the rest of the groups. For this, we apply the reduced Mayer-Vietoris sequence:

Pick an open, contractible neighborhood B in M_g, and choose a point x in B. Define $A = M_g - \{x\}$. This gives us $A \simeq \bigvee_{2g} S^1$, $B \simeq *$, $A \cup B = M_g$ and $A \cap B = B - \{x\} \simeq S^1$. The tail of the reduced Mayer-Vietoris sequence turns out at

$$\cdots \longrightarrow \widetilde{HC}_2(\bigvee_{2g} S^1) \oplus \widetilde{HC}_2(*) \longrightarrow \widetilde{HC}_2(M_g) \longrightarrow \widetilde{HC}_1(S^1) \longrightarrow$$
$$\widetilde{HC}_1(\bigvee_{2g} S^1) \oplus \widetilde{HC}_1(*) \longrightarrow \widetilde{HC}_1(M_g) \longrightarrow 0.$$

Substituting with the known groups (Lemma 3.1.7, Corollary 3.4.7 and $\widetilde{HC}_1(M_g) \cong \bigoplus_{2g} \mathbb{Z}$), this yields an exact fragment

$$\cdots \longrightarrow 0 \longrightarrow \widetilde{HC}_2(M_g) \longrightarrow \mathbb{Z} \longrightarrow \bigoplus_{2g} \mathbb{Z} \longrightarrow \bigoplus_{2g} \mathbb{Z} \longrightarrow 0,$$

hence, $\widetilde{HC}_2(M_g) \cong \mathbb{Z}$, and according to Lemma 3.4.3, $\widetilde{HC}_q(M_g) \cong \mathbb{Z}$ for $q = 4, 6, 8, \cdots$.

Finally, for $q = 3$ we have the following fragment of the reduced Mayer-Vietoris sequence:

$$\longrightarrow \widetilde{HC}_4(\bigvee_{2g} S^1) \oplus \widetilde{HC}_4(*) \longrightarrow \widetilde{HC}_4(M_g) \longrightarrow \widetilde{HC}_3(S^1) \longrightarrow$$
$$\widetilde{HC}_3(\bigvee_{2g} S^1) \oplus \widetilde{HC}_3(*) \longrightarrow \widetilde{HC}_3(M_g) \longrightarrow \widetilde{HC}_2(S^1) \longrightarrow$$

Substituting the known groups yields

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \bigoplus_{2g} \mathbb{Z} \longrightarrow \widetilde{HC}_3(M_g) \longrightarrow 0,$$

wherefrom we get $\widetilde{HC}_3(M_g) \cong \bigoplus_{2g} \mathbb{Z}$, and so also for $\widetilde{HC}_q(M_g)$ when $q = 5, 7, 9, \cdots$. □

Corollary 3.4.13 *The reduced cyclic singular homology of the torus $T = S^1 \times S^1$ is given by*

$$\widetilde{HC}_q(T) \cong \begin{cases} 0 & \text{for } q = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } q \geq 1 \text{ odd} \\ \mathbb{Z} & \text{for } q \geq 2 \text{ even.} \end{cases}$$

For the non-orientable surfaces we have the following result:

Lemma 3.4.14 *The reduced cyclic singular homology of the non-oriented surface of genus $g \geq 1$ is given by*

$$\widetilde{HC}_q(N_g) \cong \begin{cases} \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{g-1} \oplus \mathbb{Z}_2 & \text{for } q \geq 1 \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Proof Recalling the reduced singular homology of N_g [8, Example 2.37],

$$\widetilde{H}_q(N_g) \cong \begin{cases} \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{g-1} \oplus \mathbb{Z}_2 & \text{for } q = 1 \\ 0 & \text{otherwise,} \end{cases}$$

since N_g is connected, we have $\widetilde{HC}_0(N_g) = 0$, and again, from Lemma 3.4.1, $\widetilde{HC}_1(N_g) \cong \widetilde{H}_1(N_g)$. To determine the rest of the groups, according to Lemma 3.4.3, we only have to find $\widetilde{HC}_2(N_g)$ (Lemma 3.4.3), For this, we apply the reduced SBI-sequence, containing the fragment

$$\cdots \longrightarrow \widetilde{H}_2(N_g) \longrightarrow \widetilde{HC}_2(N_g) \longrightarrow \widetilde{HC}_0(N_g) \longrightarrow \cdots.$$

Both the left and right groups are trivial, hence, $\widetilde{HC}_2(N_g) = 0$, and we are done. □

Corollary 3.4.15 *The reduced cyclic singular homology of the real projective plane is given by:*

$$\widetilde{HC}_q(\mathbb{R}\mathrm{P}^2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } q \geq 1 \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 4

Cyclic Simplicial Homology

This chapter describes an attempt to create a simplicial cyclic homology theory on Δ-complexes. We first recall Δ-complexes and simplicial homology on Δ-complexes. Second, we extend the Δ-complexes with cyclic operators to obtain cyclic Δ-complexes, which allows us to compute cyclic homology.

Disappointingly, it turns out that the resulting homology theory is isomorphic to singular homology. This correspondence is proved in the last section of the chapter, following a section of example calculations.

4.1 Δ-Complexes and simplicial homology

Rather than using simplicial complexes, we shall apply the somewhat more lenient notion of a Δ-complex structure. This section recalls Δ-complexes and simplicial homology on Δ-complexes. The first definition is an elaborated version of what can be found in [8, p102], details are from [18]:

Definition 4.1.1 *A* ***Δ-complex structure*** $\Delta(X)$ *on a topological space* X *is a union of sets*

$$\Delta_n(X) = \{\sigma_\alpha : \Delta^n \longrightarrow X\}_{\alpha \in \Lambda_n} \quad \text{for } n \geq 0,$$

that is, $\Delta(X) = \bigcup_{n \geq 0} \Delta_n(X)$*, such that the following axioms are fulfilled:*

Δ1 - *For each* $\sigma_\alpha \in \Delta_n(X)$ *for* $n \geq 0$*,* $\sigma_\alpha|_{\Delta^n - \partial\Delta^n}$ *is injective, and* X *is the disjoint union of the images of the* $\sigma_\alpha|_{\Delta^n - \partial\Delta^n}$*:*

$$X = \coprod_{\substack{\sigma_\alpha \in \Delta_n(X) \\ n \geq 0}} \operatorname{im}(\sigma_\alpha|_{\Delta^n - \partial\Delta^n}).$$

Δ2 - *If* $\partial^i : \Delta^{n-1} \longrightarrow \Delta^n$ *for* $0 \leq i \leq n$ *are the cosimplicial face maps from* (1.2.1)*, then*

$$\sigma_\alpha \in \Delta_n(X) \;\Rightarrow\; \sigma_\alpha \circ \partial^i \in \Delta_{n-1}(X) \quad \text{for} \quad 0 \leq i \leq n, \;\; n \geq 1.$$

Δ3 - $U \subset X$ *is open* $\Leftrightarrow$ $\sigma_\alpha^{-1}(U) \subset \Delta^n$ *is open for each* $\sigma_\alpha \in \Delta_n(X)$*,* $n \geq 0$*.*

Given a Δ-complex structure $\Delta(X)$, the evaluation map $\mathrm{ev} : \coprod_{n\geq 0} \Delta_n(X) \times \Delta^n \longrightarrow X$, defined by $(\sigma_\alpha, t) \longmapsto \sigma_\alpha(t)$, fits into a commutative diagram

$$\begin{array}{ccc} \coprod\limits_{n\geq 0} \Delta_n(X) \times \Delta^n & \xrightarrow{\quad \mathrm{ev} \quad} & X \\ & {\scriptstyle q} \searrow \quad \nearrow {\scriptstyle \widetilde{\mathrm{ev}}} \, \approx & \\ & \Big(\coprod\limits_{n\geq 0} \Delta_n(X) \times \Delta^n \Big) \Big/ \sim, & \end{array}$$

where q is the quotient map generated by the equivalence relation $(\sigma_\alpha \circ \partial^i, t) \sim (\sigma_\alpha, \partial^i(t))$, and $\widetilde{\mathrm{ev}}$ is a homeomorphism induced by ev. The sets $\Delta_n(X)$ have the discrete topology, while the topology on Δ^n is the standard one. We will refer to the quotient space $\big(\coprod_{n\geq 0} \Delta_n(X) \times \Delta^n\big)/\sim$ as the **geometric realization** of the Δ-complex, denoted by $|\Delta(X)|$.

We define the **dimension of $\boldsymbol{\Delta(X)}$** [7, §4.4] by:

$$\dim(\Delta(X)) \;=\; \max\{\, n \mid \Delta_n(X) \neq \emptyset \,\},$$

and by convention, $\dim(\Delta(\emptyset)) = -1$.

The following definition could have been introduced before Definition 2.2.2, but we have not had any use for it until now:

Definition 4.1.2 *(Cf. [16, Definition 1.1]) A* ***presimplicial object*** *in a category* $\mathbf{C}$ *is a functor* $(\boldsymbol{\Delta}_{\mathbf{pre}})^{\mathbf{op}} \longrightarrow \mathbf{C}$*, equivalently, a set of objects* $X_\bullet = \{X_n\}_{n\geq 0}$ *in* $\mathbf{C}$ *together with* ***simplicial face operators*** $d_i : X_n \longrightarrow X_{n-1}$ *subject to the* ***presimplicial identity*** (2.2.3).

For completeness, we restate (2.2.3) here:

$$d_i d_j \;=\; d_{j-1} d_i \quad \text{for } i < j.$$

A precyclic object is as such a presimplicial object with cyclic operators. With a little caution, this simple observation allows us to reuse some of the results we have proven for precyclic objects in section 2.2 for presimplicial objects.

Lemma 4.1.3 *A Δ-complex structure is a presimplicial object.*

Proof Given a Δ-complex structure $\Delta(X)$, define the simplicial face operators $d_i : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ by

$$d_i(\sigma_\alpha) \;=\; \sigma_\alpha \circ \partial^i \quad \text{for } 0 \leq i \leq n.$$

As shown in the proof of Proposition 2.2.3, this definition of the face operators, by precomposing with the cosimplicial face maps, guarantees that they satisfy the presimplicial identity. $\square$

Based on Lemma 4.1.3, axiom Δ2 can be restated as

$$\sigma_\alpha \in \Delta_n(X) \Rightarrow d_i(\sigma_\alpha) \in \Delta_{n-1}(X) \quad \text{for } 0 \le i \le n, \ n \ge 1.$$

In the following, we will stick to this notation, using d_i to denote the presimplicial operators in $\Delta(X)$.

Definition 4.1.4 *(Cf. [8, p106]) The n-th group of the* ***simplicial chain complex*** $(C_\bullet^\Delta(X), \partial_\bullet)$ *is the free abelian group generated by* $\Delta_n(X)$*:*

$$C_n^\Delta(X) \ = \ Ab(\Delta_n(X)) \quad for \ n \ ge0.$$

The face homomorphisms $d_i : C_n^\Delta(X) \longrightarrow C_{n-1}^\Delta(X)$ *for* $0 \le i \le n$ *are the linear extensions of the face operators* $d_i : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$*, and the boundary homomorphism* $\partial_n : C_n^\Delta(X) \longrightarrow C_{n-1}^\Delta(X)$ *is given by*

$$\partial_n(\sigma_\alpha) \ = \ \sum_{i=0}^{n} (-1)^i d_i(\sigma_\alpha).$$

The boundary homomorphism is well-defined by Δ2, and the property $\partial_n \circ \partial_{n+1} = 0$ follows from (2.2.11) and (2.2.12) in Lemma-definition 2.2.5, since these definitions and relations do not depend on the cyclic operator, and thus only on the presimplicial structure of the (precyclic) object.

Definition 4.1.5 *The n-th* ***simplicial homology group*** *of a topological space X is the n-th homology group of the simplicial chain complex:*

$$H_n^\Delta(X) \ = \ H_n((C_\bullet^\Delta(X), \partial_\bullet)) \quad for \ n \ge 0.$$

For a subspace $A \subset X$ with Δ-complex structure $\Delta(A)$ which is a subcomplex of $\Delta(X)$, we define the Δ-complex pair $\Delta(X, A)$. By defining $C_\bullet^\Delta(X, A) = C_\bullet^\Delta(X)/C_\bullet^\Delta(A)$, the above definitions generalize to Δ-complex pairs in the normal way (see the initial paragraphs of section 3.1).

The relation between the singular homology functor and $H_\bullet^\Delta(X, A)$ is described by the following result (cf. [8, Theorem 2.27]):

Theorem 4.1.6 *For a Δ-complex pair* $\Delta(X, A)$*, we have*

$$H_n^\Delta(X, A) \ \cong \ H_n^{sing}(X, A) \quad for \ n \ge 0.$$

In particular, $H_\bullet^\Delta(X, A)$ is independent of the choice of Δ-complex structure on $\Delta(X, A)$.

4.2 Turning $\Delta(X)$ into a precyclic object

Clearly, for $\Delta(X)$ to become precyclic, we need cyclic operators which, together with the face operators in $\Delta(X)$, satisfy relations (2.2.4) and (2.2.5). It seems natural to choose the same cyclic operator as we used with cyclic singular homology (cf. Lemma-definition 3.1.1):

Definition 4.2.1 *A **cyclic Δ-complex structure** on a topological space X,*

$$\Delta^C(X) \;=\; \{\Delta_n^C(X)\}_{n\geq 0},$$

is obtained from a Δ-complex $\Delta(X)$ by adding the n-simplicies $T_n^k(\sigma_\alpha)$ for $1 \leq k \leq n$ for each $\sigma_\alpha \in \Delta_n(X)$ and $n \geq 1$:

$$\Delta_n^C(X) \;=\; \Delta_n(X) \cup \{\, T_n^k(\sigma_\alpha) \mid \sigma_\alpha \in \Delta_n(X) \text{ and } 1 \leq k \leq n \,\}.$$

*Here, the T_n are the **cyclic simplicial operators** $T_n : \Delta_n^C(X) \longrightarrow \Delta_n^C(X)$ for $n \geq 0$, given by*

$$T_n(\sigma_\alpha) \;=\; \sigma_\alpha \circ \Gamma^n,$$

where $\Gamma^n : \Delta^n \longrightarrow \Delta^n$ for $n \geq 0$ are the cosimplicial cyclic operators (3.1.1)*.*

Since the cyclic simplicial operators and the simplicial face operators are both defined exactly as in Proposition 2.2.3, by precomposition with the cosimplicial cyclic and face operators, the following result follows directly:

Lemma 4.2.2 *The simplicial face operators and cyclic simplicial operators satisfy relations* (2.2.3)*,* (2.2.4) *and* (2.2.5)*.* □

It turns out that these extra simplicies is exactly what is required to obtain a precyclic structure:

Lemma 4.2.3 *A cyclic Δ-complex $\Delta^C(X)$ is a precyclic object.*

Proof Assume we have added simplicies as prescribed, so that T_n is defined, and pick $\sigma_\alpha \in \Delta_n(X)$. We have to show that $d_i T_n^j(\sigma_\alpha) \in \Delta_{n-1}(X)$ for $0 \leq i, j \leq n$. According to Lemma 4.2.2, we have the following identities:

$$d_i T_n^j(\sigma_\alpha) \overset{(2.2.5)}{=} \begin{cases} T_{n-1}^j d_{i-j}(\sigma_\alpha) & \text{for } i \geq j \\ T_{n-1}^i d_0 T_n^{j-i}(\sigma_\alpha) & \text{for } i < j. \end{cases}$$

The top map is in $\Delta_{n-1}(X)$, for $d_{i-j}(\sigma_\alpha) = \sigma_\alpha \circ \partial^{i-j} \in \Delta_{n-1}(X)$ by axiom Δ2, and $T_{n-1}^j(\sigma_\alpha \circ \partial^{i-j}) \in \Delta_{n-1}(X)$, according to assumptions. For the bottom map, we recall the identity $d_0 T_n = d_n$ from Definition 2.2.2, and continue the calculation:

$$\begin{aligned} T_{n-1}^i d_0 T_n^{j-i}(\sigma_\alpha) &= T_{n-1}^i d_n T_n^{j-i-1}(\sigma_\alpha) \\ &\overset{(2.2.5)}{=} T_{n-1}^{i+j-i-1} d_{n-j+i+1}(\sigma_\alpha) \;=\; T_{n-1}^{j-1} d_{n-j+i+1}(\sigma_\alpha). \end{aligned}$$

By an identical argument as above, this element is in $\Delta_{n-1}(X)$ too, so the presimplicial face operators are defined also after adding simplicies. We do not have to add any 0-simplicies, since $T_0(\sigma_\alpha) = \sigma_\alpha$. □

On the other hand, if we fail to insert one of the prescribed simplicies the operators T_n will not be defined: None of the $T_n^k(\sigma_\alpha)$ for $1 \leq k \leq n$ and $n \geq 1$ are initially present in $\Delta_n(X)$, for Γ^n is a permutation of Δ^n, so if both σ_α and $\sigma_\alpha \circ \Gamma^n$ are elements in $\Delta_n(X)$, they cover $\mathrm{im}(\sigma_\alpha|_{\Delta^n - \partial\Delta^n})$ twice, violating axiom $\Delta 1$. As a consequence, a cyclic Δ-complex is no longer a covering of X by disjoint sets. However, all of the other requirements of the axioms are still met:

Lemma 4.2.4 *A cyclic Δ-complex $\Delta^C(X)$ satisfies axioms $\Delta 2$ and $\Delta 3$, but not $\Delta 1$: X is no longer the disjoint union of the sets $\mathrm{im}(\sigma_\alpha|_{\Delta^n - \partial\Delta^n})$.*

Proof Axiom $\Delta 2$ holds according to Lemma 4.2.3.

Since $\Gamma^n : \Delta^n \longrightarrow \Delta^n$ is a homeomorphism, and since precomposing an injective map by a homeomorphism produces an injective map, $T_n(\sigma_\alpha)|_{\Delta^n - \partial\Delta^n} = \sigma_\alpha \circ \Gamma^n|_{\Delta^n - \partial\Delta^n}$ is injective, so the remains of $\Delta 1$ is satisfied.

For $\Delta 3$, again the Γ^n are homeomorphisms, so the axiom holds for elements on the form $\sigma_\alpha \circ \Gamma^n$ if it holds for σ_α.

Both of the last results extend to all of the elements of $\Delta_n^C(X)$ for $n \geq 1$ by induction. □

4.3 Cyclic simplicial homology

The precyclic object $\Delta^C(X)$ gives rise to another precyclic object, $C_\bullet^C(X) = \{C_n^C(X)\}_{n\geq 0}$, where $C_n^C(X)$ is the free abelian group generated by $\Delta_n^C(X)$:

$$C_n^C(X) \;=\; Ab(\Delta_n^C(X)) \quad \text{for } n \geq 0.$$

The face homomorphisms $d_i : C_n^C(X) \longrightarrow C_{n-1}^C(X)$ for $0 \leq i \leq n$ and the cyclic homomorphisms $T_n : C_n^C(X) \longrightarrow C_n^C(X)$ are the linear extensions of their counterparts $d_i : \Delta_n^C(X) \longrightarrow \Delta_{n-1}^C(X)$ and $T_n : \Delta_n^C(X) \longrightarrow \Delta_n^C(X)$ in the cyclic Δ-complex.

According to Lemma-definition 2.2.5, the object $C_\bullet^C(X)$ fits into a cyclic double complex $CC(C_\bullet^C(X))$, which allows us to compute cyclic homology:

Definition 4.3.1 *The **cyclic simplicial homology** of a topological space X is the total homology of the cyclic double complex $CC(C_\bullet^C(X))$:*

$$HC_n^\Delta(X) \;=\; H_n(Tot(CC(C_\bullet^C(X)))) \quad \textit{for } n \geq 0.$$

There are several questions following in the wake of this definition, but we put them aside for now, and will look at a couple of calculations motivating the last section.

4.4 Calculations in cyclic simplicial homology

To simplify notation, we introduce the following conventions: 0-simplicies will be denoted $v_0, \cdots, v_n$, 1-simplicies by $a, b, c, \cdots$ and their cyclified counterparts will be written $a', b', c', \cdots$, where $T_1(a) = a'$ and $T_1(a') = a$, and so on. Extending to dimension 2, denoting 2-simplicies by uppercase letters, e.g. L, we have $T_2(L) = L'$ and $T_2^2(L) = L''$. We will not see simplicies in dimension 3 or higher, so we stop here.

In the following, we will denote the free abelian group generated by a set G by $\mathbb{Z}\{G\}$, and relations R will be listed in the usual way, by $\mathbb{Z}\{G|R\}$. In particular, $C_n^C(X) = \mathbb{Z}\{\Delta_n^C(X)\}$ for $n \geq 0$.

4.4.1 The cyclic simplicial homology of the circle with a minimal Δ-complex structure

If we consider S^1 to be composed by one 0-cell and one 1-cell, $\Delta(S^1)$ takes the following form:

- $\Delta_0(S^1) = \{v_0 : \Delta^0 \longrightarrow S^1\}$, where $v_0(0) = *$.
- $\Delta_1(S^1) = \{a : \Delta^1 \longrightarrow S^1\}$, where a sends ∂I to $*$ and $I - \partial I$ invectively onto $S^1 - *$.
- $\Delta_q(S^1) = \emptyset$ for $q > 1$.

The corresponding cyclic simplicial chain complex has the following groups, where we have to add one 1-simplex to generate $\Delta_1^C(S^1)$:

$$\begin{aligned} C_0^C(S^1) &\cong \mathbb{Z}\{v_0\}, \\ C_1^C(S^1) &\cong \mathbb{Z}\{a, a'\}, \end{aligned} \qquad C_q^C(S^1) = 0 \quad \text{for } q > 1.$$

The resulting cyclic double complex $CC(C_\bullet^C(S^1))$ is given by

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z}\{a,a'\} & \xleftarrow{1-t_1} & \mathbb{Z}\{a,a'\} & \xleftarrow{N_1} & \mathbb{Z}\{a,a'\} & \longleftarrow & \\ \partial_1 \downarrow & & -\partial_1' \downarrow & & \partial_1 \downarrow & & \\ \mathbb{Z}\{v_0\} & \xleftarrow{1-t_0} & \mathbb{Z}\{v_0\} & \xleftarrow{N_0} & \mathbb{Z}\{v_0\} & \longleftarrow & , \end{array}$$

where the maps are as prescribed by Lemma-definition 2.2.5. Written out, they take the form

$$\begin{aligned} \partial_1(x) &= x \circ \partial^0 - x \circ \partial^1 & \qquad \partial_1'(x) &= x \circ \partial^0 \\ t_0(x) &= x & t_1(x) &= -x' \\ N_0(x) &= x & N_1(x) &= x - x', \end{aligned}$$

where $t_n = (-1)^n T_n$, as one would expect.

Observation 4.4.1 *The homology of the even-numbered columns is*

$$H_n((C^C_\bullet(S^1), \partial_\bullet)) \cong \begin{cases} \mathbb{Z} & \textit{for } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \textit{for } n = 1 \\ 0 & \textit{for } n > 1. \end{cases}$$

Proof $\partial_1(a) = a \circ \partial^0 - a \circ \partial^1 = v_0 - v_0 = 0$, and the same goes for a', hence, $H_1 \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_0 \cong \mathbb{Z}$. All higher groups are trivial. □

Observation 4.4.2 *The homology of the odd-numbered columns is*

$$H_n((C^C_\bullet(S^1), \partial'_\bullet)) \cong \begin{cases} \mathbb{Z} & \textit{for } n = 1 \\ 0 & \textit{for } n \neq 1. \end{cases}$$

Proof Since $\partial'_1(a) = a \circ \partial^0 = v_0$, all zero-cycles are boundaries, so $H_0 = 0$. Now, $\partial'_1 : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$ is surjective with $\mathrm{im}(\partial'_1) \cong \mathbb{Z}$, so $\ker(\partial'_1) \cong \mathbb{Z}$, and it is generated by $a - a'$. □

Considering the spectral sequence with filtration on columns associated to the above cyclic double complex, the E^1-page looks like (cf. observations 4.4.2 and 4.4.1):

$$E^1_{**} : \quad \begin{array}{|ccccccc} 0_{0,2} & \longleftarrow & 0_{1,2} & \longleftarrow & 0_{2,2} & \longleftarrow & \\ \mathbb{Z} \oplus \mathbb{Z}_{0,1} & \xleftarrow{(1-t_1)_*} & \mathbb{Z}_{1,1} & \xleftarrow{N_{1*}} & \mathbb{Z} \oplus \mathbb{Z}_{2,1} & \longleftarrow & \\ \mathbb{Z}_{0,0} & \longleftarrow & 0_{1,0} & \longleftarrow & \mathbb{Z}_{2,0} & \longleftarrow & \\ \hline \end{array}$$

The diagram stretches infinitely far to the right, but the bi-periodic symmetry of the columns allows us only to look at the first three.

The differential $d^1_{p,1} = N_{1*} : \mathbb{Z} \oplus \mathbb{Z}_{p,1} \longrightarrow \mathbb{Z}_{p-1,1}$, for $p \geq 2$ even, is given by $N_{1*}([x]) = [N_1(x)] = [x - x']$. Since $\mathbb{Z}_{p-1,1}$ is generated by $a - a'$, and since a is a generator of $\mathbb{Z} \oplus \mathbb{Z}_{p,1}$ and $N_1(a) = a - a'$, $d^1_{p,1}$ is surjective, and $E^2_{p-1,1} = 0$. Further, we see that $\ker(d^1_{p,1}) \cong \mathbb{Z}$, and it is generated by $a + a'$.

Applying the other differential of interest, $d^1_{q,1} = (1 - t_1)_*$ for q odd, to the generator of $\mathbb{Z}_{q,1}$, we get $(1 - t_1)_*(a - a') = (a - a') + (a' - a) = 0$. Hence, $E^2_{q-1,1} \cong \ker(d^1_{q-1,1})$. Summing up the homology groups of the E^1-page, we have

$$E^2_{p,0} \cong \begin{cases} \mathbb{Z} & \text{for } p \text{ even} \\ 0 & \text{for } p \text{ odd,} \end{cases} \qquad E^2_{p,1} \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } p = 0 \\ \mathbb{Z} & \text{for } p \geq 2 \text{ even} \\ 0 & \text{for } p \geq 1 \text{ odd.} \end{cases}$$

All other groups are trivial, and the E^2-page looks like:

$$
E^2_{**}: \quad
\begin{array}{|ccccc}
0_{0,2} & 0_{1,2} & 0_{2,2} & 0 & 0 \\
\mathbb{Z}\oplus\mathbb{Z}_{0,1} & 0_{1,1} & \mathbb{Z}_{2,1} & 0_{3,1} & \mathbb{Z}_{4,1} \\
 & d^2_{2,0} & & d^2_{4,0} & \\
\mathbb{Z}_{0,0} & 0_{1,0} & \mathbb{Z}_{2,0} & 0_{3,0} & \mathbb{Z}_{4,0} \\
\hline
\end{array}
$$

The differential of interest on the E^2-page is the homomorphism $d^2_{p,0} : E^2_{p,0} \longrightarrow E^2_{p-2,1}$ for p even. Since $E^2_{p,0} \cong \mathbb{Z}\{v_0\}$ whenever p is even, this map is defined as follows [10, D.4]:

The generator of $\mathbb{Z}_{p,0}$ is v_0, and $N_0(v_0) = v_0$. There is an element α in $C^C_1(S^1)$ such that $\partial'_1(\alpha) = N_0(v_0)$. It turns out that any of a or a' will do, so we pick a. We define $d^2_{p,0}([v_0]) = [(1-t_1)(a)]$, and extend linearly to all of $\mathbb{Z}_{p,0}$.

Since $(1-t_1)(a) = a + a'$, and since $d^2_{p-2,1}$ is trivial, we have two outcomes:

$$
E^3_{p,1} \cong \begin{cases} E^3_{0,1} \cong \mathbb{Z}\{a,a'|a+a'\} \cong \mathbb{Z}\{a\} \cong \mathbb{Z} & \text{for } p=0 \\ E^3_{p,1} \cong \mathbb{Z}\{a+a'|a+a'\} = \mathbb{Z}\{0\} = 0 & \text{for } p \geq 2 \text{ even.} \end{cases}
$$

Also, $d^2_{p,0}$ is injective, with trivial kernel, killing all the nontrivial base terms except for $p = 0$:

$$
E^3_{**}: \quad
\begin{array}{|ccccc}
0_{0,2} & 0_{1,2} & 0_{2,2} & 0 & 0 \\
\mathbb{Z}_{0,1} & 0_{1,1} & 0_{2,1} & 0_{3,1} & 0_{4,1} \\
\mathbb{Z}_{0,0} & 0_{1,0} & 0_{2,0} & 0_{3,0} & 0_{4,0} \\
\hline
\end{array}
$$

The spectral sequence clearly degenerates here. We summarize our findings as an observation:

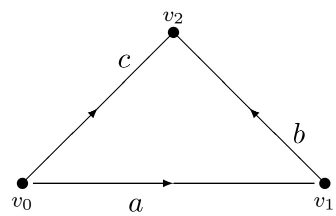
Observation 4.4.3 *The cyclic simplicial homology of the circle composed by one 1-cell and one 2-cell is given by:*

$$
HC^{\Delta}_n(S^1) \cong \begin{cases} \mathbb{Z} & \textit{for } n = 0,1 \\ 0 & \textit{for } n \neq 0,1. \end{cases}
$$

We immediately recognize this as the singular homology of the circle.

4.4.2 The cyclic simplicial homology of the circle with a different Δ-complex structure

A bit astounded by the previous result, we make another attempt with a different Δ-complex structure:



The nontrivial objects in the cyclic Δ-complex structure are

$$\Delta_0^C(S^1) = \{v_0, v_1, v_2\} \qquad \text{and} \qquad \Delta_1^C(S^1) = \{a, a', b, b', c, c'\}.$$

This time, we consider the spectral sequence associated to $CC(C_\bullet^C(S^1))$ with filtration on rows. It turns out that this spectral sequence converges faster than the corresponding spectral sequence with filtration on columns. We start at the E^0-page:

$$E^0_{**}: \quad \begin{array}{|ccccccc} 0_{0,2} & & 0_{1,2} & & 0_{2,2} & & \\ E^0_{0,1} & \xleftarrow{(1-t_1)} & E^0_{1,1} & \xleftarrow{N_1} & E^0_{2,1} & \longleftarrow & \\ E^0_{0,0} & \xleftarrow{(1-t_0)} & E^0_{1,0} & \xleftarrow{N_0} & E^0_{2,0} & \longleftarrow & \\ \hline \end{array}$$

In this diagram,

$$E^0_{\bullet,0} \cong \mathbb{Z}\{v_0, v_1, v_2\} \qquad \text{and} \qquad E^0_{\bullet,1} \cong \mathbb{Z}\{a, a', b, b', c, c'\}.$$

If we start with the bottom row, $(1-t_0) = 0$ and $N_0 = 1$, so the bottom row is exact in all terms but the left-most, which is unchanged:

$$E^1_{p,0} \cong \begin{cases} \mathbb{Z}\{v_0, v_1, v_2\} & \text{for } p = 0 \\ 0 & \text{for } p \neq 0. \end{cases}$$

For the second row, we note that $(1-t_1)(x) = x + x'$, so the image is generated by elements on the form $x + x'$ and the kernel by elements on the form $x - x'$, where $x \in \Delta_1^C(S^1)$. On the other hand, $N_1(x) = x - x' = 0$ if and only if $x = x'$, which is the case only if x is generated by terms on the form $\alpha(a + a')$. This means that $\ker(N_1)$ is exactly $\operatorname{im}(1-t_1)$. Since $N_1(x) = x - x'$, we have $\operatorname{im}(N_1) \cong \ker(1-t_1)$, so the sequence in row 1 is also exact in all terms but the left-most. Now, $\operatorname{im}(1-t_1) \cong \mathbb{Z}\{a+a', b+b', c+c'\}$, giving $E^1_{0,1} \cong \mathbb{Z}\{a, a', b, b', c, c' | a + a', b + b', c + c'\} \cong \mathbb{Z}\{a, b, c\}$:

$$E^1_{p,1} \cong \begin{cases} \mathbb{Z}\{a, b, c\} & \text{for } p = 0 \\ 0 & \text{for } p \neq 0. \end{cases}$$

This leaves us with two nontrivial groups and one differential on the E^1-page:

$$\partial_{1*} : E^1_{0,1} \longrightarrow E^1_{0,0}.$$

Since ∂_1 is induced by the simplicial chain map, we have

$$\operatorname{im}(\partial_{1*}) \cong \mathbb{Z}\{v_1 - v_0, v_2 - v_1, v_2 - v_0\} \cong \mathbb{Z}\{v_1 - v_0, v_2 - v_0\},$$

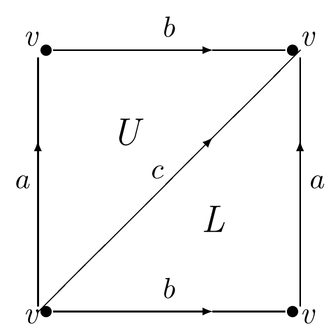
and $\ker(\partial_{1*}) \cong \mathbb{Z}\{a + b + c\}$. This yields

$$\begin{aligned} E^2_{0,1} &\cong \mathbb{Z}\{a + b + c\} \cong \mathbb{Z} \\ E^2_{0,0} &\cong \mathbb{Z}\{v_0, v_1, v_2 | v_1 = v_0, v_2 = v_0\} \cong \mathbb{Z}\{v_0\} \cong \mathbb{Z} \\ E^2_{p,q} &= 0 \quad \text{otherwise.} \end{aligned}$$

The spectral sequence degenerates here, and again we recognize the resulting total homology as the singular homology of S^1.

4.4.3 The cyclic simplicial homology of the torus

We use the "standard" Δ-complex structure on the torus [8, p102]:



Here, U and L represent the 2-simplicies embedding $\Delta^2 - \partial\Delta^2$ onto the interior of the respective triangles. The cyclic Δ-complex structure consequently looks like:

$$\Delta^C_0(T) = \{v_0\}, \qquad \Delta^C_2(T) = \{U, U', U'', L, L', L''\},$$

$$\Delta^C_1(T) = \{a, a', b, b', c, c'\}, \qquad \Delta^C_q(T) = \emptyset \quad \text{for } q > 2.$$

Again, to compute the total homology of $CC(C_\bullet^C(T))$, we consider the associated spectral sequence with filtration on the rows:

$$
E^0_{**}: \quad
\begin{array}{c|ccccccc}
 & 0_{0,2} & & 0_{1,2} & & 0_{2,2} & & \\
 & E^0_{0,2} & \xleftarrow{1-t_2} & E^0_{1,2} & \xleftarrow{N_2} & E^0_{2,2} & \longleftarrow & \\
 & E^0_{0,1} & \xleftarrow{1-t_1} & E^0_{1,1} & \xleftarrow{N_1} & E^0_{2,1} & \longleftarrow & \\
 & E^0_{0,0} & \xleftarrow{1-t_0} & E^0_{1,0} & \xleftarrow{N_0} & E^0_{2,0} & \longleftarrow & \\
\hline
\end{array}
$$

Here, the groups in the diagram are given by

$$E^0_{\bullet,0} \cong \mathbb{Z}\{v_0\}, \quad E^0_{\bullet,1} \cong \mathbb{Z}\{a, a', b, b', c, c'\} \quad \text{and} \quad E^0_{\bullet,2} \cong \mathbb{Z}\{L, L', L'', U, U', U''\},$$

and all other groups are trivial.

Row zero is still exact in all terms but the last, and since $(1 - t_0) = 0$, we have $E^1_{0,0} \cong \mathbb{Z}\{v_0\}$, and $E^1_{p,0} = 0$ for $p \neq 0$. Row one is equal to the same row in the calculation in section 4.4.2, so we can conclude that $E^1_{0,1} \cong \mathbb{Z}\{a, b, c\}$ and $E^1_{p,1} = 0$ for $p \neq 0$.

For row two, we first observe that the unbounded sequence

$$\cdots \xrightarrow{1-t_2} C_2^C(T) \xrightarrow{N_2} C_2^C(T) \xrightarrow{1-t_2} C_2^C(T) \xrightarrow{N_2} \cdots$$

is exact: Since $(1 - t_2)(x) = x - x'$, we must have $(1 - t_2)(x) = 0$ if and only if $x = x'$, hence, $\ker(1 - t_2) \cong \mathbb{Z}\{L + L' + L'', U + U' + U''\}$ since these are the only elements staying fixed under t_n, and $\operatorname{im}(1 - t_2) \cong \mathbb{Z}\{L - L', L - L'', U - U', U - U''\}$. Since $N_2(x) = x + x' + x''$, we immediately see that $\operatorname{im}(N_2) \cong \ker(1 - t_2)$. Due to an argument identical to the discussion of the kernel of N_1 in the previous calculation, we also see that $\ker(N_2) \cong \mathbb{Z}\{L - L', L - L'', U - U', U - U''\}$, wherefrom exactness follows. The only group that is not killed on the E^0-page in row two, is the leftmost group, and

$$
\begin{aligned}
E^1_{0,2} &\cong \mathbb{Z}\{L, L', L'', U, U', U'' | L - L', L - L'', U - U', U - U''\} \\
&\cong \mathbb{Z}\{L, U\}.
\end{aligned}
$$

The E^2-page contains three nontrivial groups in the bottom three rows in the zeroth column looking like

$$0 \longrightarrow \mathbb{Z}\{L, U\} \xrightarrow{\partial_{2*}} \mathbb{Z}\{a, b, c\} \xrightarrow{\partial_{1*}} \mathbb{Z}\{v_0\} \longrightarrow 0,$$

and this we recognize as the simplicial chain complex of $\Delta(T)$. Hence, the total homology of $CC(C_\bullet^C(T))$ is isomorphic to the simplicial (and singular) homology of the torus.

Just as a curiosity, it is well worth mentioning that the above calculation with the spectral sequence with filtration on columns fills no less that five full pages!

4.5 The correspondence between cyclic simplicial homology and singular homology

In this section, we prove the isomorphism between cyclic simplicial homology and singular homology indicated by the foregoing computations.

Proposition 4.5.1 *For any topological space X with a Δ-complex structure, we have*

$$HC_n^{\Delta}(X) \;\cong\; H_n^{sing}(X) \quad \text{for } n \geq 0.$$

We prove the proposition at page 59 at the end of this section, but first we present some intermediate results to aid us. Our first goal is to prove that the unbounded sequence

$$\cdots \xrightarrow{1-t_n} C_n^C(X) \xrightarrow{N_n} C_n^C(X) \xrightarrow{1-t_n} C_n^C(X) \xrightarrow{N_n} \cdots \tag{4.5.1}$$

is exact for all $n \geq 0$.

The result is a consequence of the somewhat special structure of the groups $C_\bullet^C(X)$, so I have not been able to dig up a contracting homotopy of this complex. Rather, we will have to do with a brute force argument, showing that kernels and images coincide, one case at a time.

Since $t_n^\bullet = \{t_n^i\}_{i=0}^n$ is a group under composition, isomorphic to $\mathbb{Z}_{n+1}$, we have a cyclic group acting on $C_n^C(X)$, hence, $C_n^C(X)$ is a $t_n^\bullet$-set. Let $C_n^C(X)^{t_n}$ denote the fixed elements in $C_n^C(X)$ under the action t_n, and denote the orbit of an element σ under t_n by σ^{t_n}.

Also, recall that elements in $C_n^C(X)$ are n-chains generated by $\Delta_n^C(X)$ over $\mathbb{Z}$, so every $x \in C_n^C(X)$ is on the form

$$x \;=\; \sum_{\sigma \in \Delta_n^C(X)} \eta_\sigma \sigma, \tag{4.5.2}$$

where $\eta_\sigma \in \mathbb{Z}$, and at most finitely many η_σ are non-zero.

Lemma 4.5.2 *The image of N_n is the subgroup of $C_n^C(X)$ staying fixed under t_n:*

$$\text{im}(N_n) \;=\; C_n^C(X)^{t_n}.$$

Proof We start by proving the relation $C_n^C(X)^{t_n} \subset \text{im}(N_n)$. If $x \in C_n^C(X)^{t_n}$, writing x on the form of (4.5.2) gives us

$$\sum_{\sigma \in \Delta_n^C(X)} \eta_\sigma \sigma \;=\; x \;=\; t_n(x) \;=\; \sum_{\sigma \in \Delta_n^C(X)} \eta_\sigma t_n(\sigma).$$

Compared to the left hand side sum, the coefficients in the right hand side sum has "changed owner" within an orbit: η_σ becomes assigned to $t_n(\sigma)$, $\eta_{t_n(\sigma)}$ to $t_n^2(\sigma)$ and so on. Since the left and right hand side sums are equal, we have equalities

$$\eta_\sigma = \eta_{t_n(\sigma)} = \eta_{t_n^2(\sigma)} = \cdots = \eta_{t_n^n(\sigma)}.$$

This means that all elements in an orbit σ^{t_n} has the same coefficients when $x \in C_n^C(X)^{t_n}$, so we can rewrite x as follows:

$$x \;=\; \sum_{\sigma\in\Delta_n(X)} \eta_\sigma\Big(\sum_{i=0}^{n} t_n^i(\sigma)\Big) \;=\sum_{\sigma\in\Delta_n(X)} \eta_\sigma N_n(\sigma) \;=\; N_n\Big(\sum_{\sigma\in\Delta_n(X)} \eta_\sigma\sigma\Big),$$

hence, $x \in \mathrm{im}(N_n)$.

For the opposite inclusion, we only have to observe that $x \in C_n^C(X)^{t_n}$ if and only if $(1-t_n)(x) = 0$. Hence, $\ker(1-t_n) = C_n^C(X)^{t_n}$, and of course $\mathrm{im}(N_n) \subset \ker(1-t_n) = C_n^C(X)^{t_n}$, since we have a chain complex. □

The following corollary is a direct consequence, and need no proof:

Corollary 4.5.3 *In the sequence* (4.5.1)*, we have*

$$\ker(1-t_n)/\,\mathrm{im}(N_n) \;=\; 0.$$

This result does not hold for the singular chain complex, as the proof relies on all elements in $\{t_n^i(\sigma)\}_{i=0}^n$ to be distinct for all σ in the generating set of $C_n^C(X)$: Take for instance the singular n-chain $2c$ consisting of the single constant n-simplex c with coefficient $\eta_c = 2$. If n is even, $t_n(2c) = 2c$, so $(1-t_n)(2c) = 0$ and $2c \in \ker(1-t_n)$. But $2c$ is not in the image of N_n for $n > 1$, since $N_n(c) = (n+1)c$.

Before the next lemma, notice that the set $\{T_n^i\}_{i=0}^n$ is also a group under function composition isomorphic to $\mathbb{Z}_{n+1}$, and this group acts on $\Delta_n^C(X)$ in the obvious way. In particular, if the orbit of $\sigma \in \Delta_n^C(X)$ under T_n is denoted by σ^{T_n}, we have

$$\Delta_n^C(X) \;=\; \bigcup_{\sigma\in\Delta_n(X)} \sigma^{T_n}. \tag{4.5.3}$$

Moreover, $T_n^i(\sigma) \neq T_n^j(\sigma)$ whenever $i \neq j$ and $0 \le i,j \le n$, so all orbits are of length $n+1$. Since t_n at most differ from T_n by a sign, we can write the free abelian group $C_n^C(X)$, as

$$C_n^C(X) \;\cong\; \bigoplus_{\sigma\in\Delta_n(X)} \mathbb{Z}\{\sigma^{t_n}\},$$

where each orbit σ^{t_n} generates a subgroup isomorphic to $\oplus_{n+1}\mathbb{Z}$, and the isomorphism on the subgroups, $\theta_\sigma : \mathbb{Z}\{\sigma^{t_n}\} \longrightarrow \oplus_{n+1}\mathbb{Z}$, is given on basis elements by

$$\theta_\sigma(t_n^i(\sigma)) = (0,\cdots,\underset{i}{1},\cdots,0) \quad \text{for } 0 \le i \le n.$$

The inclusion $\iota_\sigma : \oplus_{n+1}\mathbb{Z} \longrightarrow C_n^C(X)$ is given by

$$\iota_\sigma(0,\cdots,\underset{i}{1},\cdots,0) \;=\; t_n^i(\sigma).$$

Lemma 4.5.4 *The kernel of N_n is generated by the set*

$$\bigcup_{\sigma\in\Delta_n(X)} \{\sigma - t_n(\sigma), t_n(\sigma) - t_n^2(\sigma), \cdots, t_n^{n-1}(\sigma) - t_n^n(\sigma)\}. \tag{4.5.4}$$

Proof Let x be any n-chain in $C_n^C(X)$. Writing x on the form of (4.5.2) and applying N_n we get

$$N_n(x) \;=\; \sum_{\sigma\in\Delta_n^C(X)} \eta_\sigma N_n(\sigma). \tag{4.5.5}$$

Now, if $\phi \in \sigma^{t_n}$, we claim that $N_n(\phi) = N_n(\sigma)$. Notice first that if $\phi \in \sigma^{t_n}$, then there is an integer j such that $t_n^j(\sigma) = \phi$. The claim follows from the calculation

$$N_n(\sigma) \;= \sum_{i=0}^{n} t_n^i(\sigma) \stackrel{*}{=} \sum_{i=0}^{n} t_n^{i+j}(\sigma) = \sum_{i=0}^{n} t_n^i(t_n^j(\sigma)) = \sum_{i=0}^{n} t_n^i(\phi) = \; N_n(\phi),$$

where $\stackrel{*}{=}$ is valid since the sum contains all elements in the orbit. We can thus rewrite (4.5.5) as

$$N_n(x) \;=\; \sum_{\substack{\sigma\in\Delta_n(X)\\ \phi\in\sigma^{t_n}}} \eta_\phi N_n(\sigma).$$

Assuming $x \in \ker(N_n)$, the above sum is zero if and only if

$$\sum_{\phi\in\sigma^{T_n}} \eta_\phi \;=\; 0 \quad \text{for all } \sigma \in \Delta_n(X).$$

Since this problem is now restricted to each subgroup $\mathbb{Z}\{\sigma^{t_n}\}$, we may move to $\oplus_{n+1}\mathbb{Z}$ by the isomorphisms θ_σ:

$$\theta_\sigma\Big(\sum_{\phi\in\sigma^{t_n}} \eta_\sigma\Big) \;=\; \sum_{i=0}^{n}(0,\cdots,\eta_{t_n^i(\sigma)},\cdots,0) = \;(\eta_{t_n(\sigma)},\eta_{t_n^2(\sigma)},\cdots,\eta_{t_n^n(\sigma)}).$$

Hence, we are looking for a subgroup $G_\sigma \subset \oplus_{n+1}\mathbb{Z}$ with the properties

$$G_\sigma \;=\; \left\{(a_0,\cdots,a_n)\in \bigoplus_{n+1}\mathbb{Z} \mid \textstyle\sum_{i=0}^n a_i = 0\right\}.$$

A possible basis for G_σ is the set

$$B_\sigma \;=\; \{(1,-1,0,\cdots,0),(0,1,-1,0,\cdots,0),\cdots,(0,\cdots,0,1,-1)\},$$

and applying ι_σ to this basis yields a basis for the desired subgroup of $\mathbb{Z}\{\sigma^{t_n}\}$ given by

$$\iota_\sigma(B_\sigma) \;=\; \{\sigma - t_n(\sigma), t_n(\sigma)-t_n^2(\sigma),\cdots,t_n^{n-1}(\sigma)-t_n^n(\sigma)\}.$$

Taking the union over all σ in $\Delta_n(X)$ gives us the stated generating set of the kernel of N_n. □

Corollary 4.5.5 *In the sequence* (4.5.1)*, we have*

$$\ker(N_n)/\operatorname{im}(1-t_n) \;=\; 0.$$

Proof It is easy to see that the generating set of the kernel of N_n is in the image of $(1-t_n)$:

$$(1-t_n)(t_n^i(x)) \;=\; t_n^i(x) - t_n^{i+1}(x) \quad \text{for all } i \in \mathbb{Z},$$

wherefrom the result follows. □

Corollary 4.5.6 *For a topological space X with a Δ-complex structure, the sequence* (4.5.1) *is exact for all $n \geq 0$.*

Proof This follows directly from Corollaries 4.5.3 and 4.5.5. $\square$

We are now ready to prove our proposition:

Proof (of Proposition 4.5.1) Considering the spectral sequence associated with the cyclic double complex $CC(C^C_\bullet(X)$ with filtration on rows, all rows are on the form

$$0 \longleftarrow C^C_n(X) \xleftarrow{1-t_n} C^C_n(X) \xleftarrow{N_n} C^C_n(X) \xleftarrow{1-t_n} \cdots,$$

and are exact in all terms but the leftmost nontrivial term (Corollary 4.5.6). The homology of this term is $C^C_n(X)/\operatorname{im}(1-t_n)$. By Corollary 4.5.5, we have $\operatorname{im}(1-t_n) \cong \ker(N_n)$, and by Lemma 4.5.4 the generating set of the kernel of N_n is

$$\bigcup_{\sigma \in \Delta_n(X)} \{\sigma - t_n(\sigma), t_n(\sigma) - t_n^2(\sigma), \cdots, t_n^{n-1}(\sigma) - t_n^n(\sigma)\}.$$

Hence, this set contain the relations in the group $C^C_n(X)/\operatorname{im}(1-t_n)$. They clearly yield $\sigma = t_n(\sigma) = \cdots = t_n^n(\sigma)$, hence, all "cyclified" simplicies are killed, and we are left only with the simplicies from $\Delta_n(X)$:

$$E^1_{p,q} \cong \begin{cases} C^\Delta_q(X) & \text{for } p = 0 \\ 0 & \text{for } p \neq 0. \end{cases}$$

The (vertical) differentials on the E^1-page are induced by the simplicial chain maps ∂_n. After evaluating this page, the only terms left are fiber terms, and the spectral sequence degenerates. The resulting total homology is the simplicial homology of X, which, according to Theorem 4.1.6, is isomorphic to the singular homology of X. $\square$

The result of Proposition 4.5.1 was quite a surprise. There are open questions with regard to the above construction concerning e.g. functoriality, but since the result is far from what we were looking for, we will not pursue this any further here.

Bibliography

[1] Alain Connes. Cohomologie cyclique et foncteurs Ext^n. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(23):953–958, 1983.

[2] A. Dold. *Lectures on algebraic topology.* Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 200.

[3] Samuel Eilenberg. Cohomology and continuous mappings. *Ann. of Math. (2)*, 41:231–251, 1940.

[4] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology.* Princeton University Press, Princeton, New Jersey, 1952.

[5] Samuel Eilenberg and Norman E. Steenrod. Axiomatic approach to homology theory. *Proc. Nat. Acad. Sci. U. S. A.*, 31:117–120, 1945.

[6] B. L. Feĭgin and B. L. Tsygan. Cohomology of Lie algebras of generalized Jacobi matrices. *Funktsional. Anal. i Prilozhen.*, 17(2):86–87, 1983.

[7] Rudolf Fritsch and Renzo A. Piccinini. *Cellular structures in topology*, volume 19 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1990.

[8] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[9] J.P. May. Stable algebraic topology 1945-1966. Preprint, December 9, 1998, K-theory Preprint Archives,
http://www.math.uiuc.edu/K-theory/0321/.

[10] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, second edition, 1998.

[11] Jean-Louis Loday and Daniel Quillen. Homologie cyclique et homologie de l'algèbre de Lie des matrices. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(6):295–297, 1983.

[12] Jean-Louis Loday and Daniel Quillen. Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.*, 59(4):569–591, 1984.

[13] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[14] William S. Massey. *A basic course in algebraic topology*, volume 127 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[15] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[16] Jinhyun Park. Algebraic cycles and Connes periodicity. *arXiv:math/0607272v1* [*math.AG*], Jul 2006.

[17] Jonathan Rosenberg. *Algebraic K-theory and its applications*, volume 147 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[18] Paul Arne Østvær. Algebraic Topology I (MAT4530). Lectures given at The University of Oslo, Spring 2006.

[19] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.