On C*-algebras Related to the Roe Algebra

by

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Contents

Introduction

In this thesis we consider a group action of G on a set X. We then construct five closely related C*-algebras, namely $\mathcal{R}_u(G,X)$, $\mathcal{R}_{con}(G,X)$, $\mathcal{R}_r(G,X)$, $\mathcal{R}_c(G,X)$ and $\mathcal{R}_r^\sigma(G,X)$.

The algebra $\mathcal{R}_c(G,X)$ will be the uniform Roe algebra as defined in [16]. In Chapter 2 we highlight when the different algebras coincide. In particular, we see will see that

$$\mathcal{R}_c(G,G) \simeq \mathcal{R}_r(G,G) \simeq \mathcal{R}_{con}(G,G).$$

Hence motivating the the study of $\mathcal{R}_r(G,X)$ and $\mathcal{R}_{con}(G,X)$ as an analogue of the uniform Roe algebra.

We will particularly be interested in how well our constructed algebras carries information of the group. However, this approach is not new. A lot of information regarding group properties can already be deduced from these constructions. The construction $\mathcal{R}_r(G,X)$ is particularly well-studied. We will thus be interested in seeing how properties of $\mathcal{R}_r(G,X)$ carry over to $\mathcal{R}_{con}(G,X)$.

In Chapter 3 we investigate how well the "twisted Roe algebra", $\mathcal{R}_r^\sigma(G,G)$, carries information of the group. We are here inspired by the well-known result that states that $\mathcal{R}_r(G,G)$ is nuclear if and only if G is exact, and produce a natural generalization to $\mathcal{R}_r^\sigma(G,G)$. As we shall see, no information regarding exactness of G is lost when passing to the twisted setting. The results thus motivates $\mathcal{R}_r^\sigma(G,X)$ as a potential candidate for a generalized Roe algebra.

Whenever G acts freely on X, the algebras $\mathcal{R}_{con}(G,X)$, $\mathcal{R}_r(G,X)$ and $\mathcal{R}_r^\sigma(G,X)$ behave nicely when G is exact.

There has been much recent work on the algebra $\mathcal{R}_r(G,X)$, for instance in the article [17]. In Chapter 4 we (partially) extend one of these results to $\mathcal{R}_{con}(G,X)$. Often we shall see that $\mathcal{R}_{con}(G,X)$ is easy to work with, and the last result in Chapter 4 will motivate further study of this algebra. In particular we see that the link between Følner nets, non-paradoxicality and non-proper infinite projections form a tight bond with $\mathcal{R}_{con}(G,X)$.

In the last chapter we will study an interesting C*-subalgebra of $\mathcal{R}_r(G,G)$, namely $\mathcal{AP}(G) \rtimes_{\tau_G,r} G$, the one obtained from the almost periodic functions. In the Abelian case, we shall give a characterization of a class of C*-subalgebras of this subalgebra. In the last section we give some motivation as to why $\mathcal{AP}(G) \rtimes_{\tau_G,r} G$ becomes a highly interesting C*-subalgebra.

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Chapter 1

Fundamentals

1.1 Preliminaries

In this section we shall review some fundamental definitions, as well as establish the basic notation to be used in the rest of the thesis.

1.1.1 Group actions

Most of the discussion here can be found in any book on algebra, see for instance [13]. For a set X, we let $\mathrm{Perm}(X)$ be the group of permutations on X.

Definition 1.1.1. A *left group action* of a group G on a set X is a homomorphism $\pi : G \to \mathrm{Perm}(X)$. In this setting we say that G acts on X from the left.

When there is no risk of ambiguity, we will often just refer to a left group action as a group action.

Whenever we have a group action $\pi : G \to \mathrm{Perm}(X)$ we will use the suggestive notation

$$gx = \pi(g)x \qquad x \in X, g \in G.$$

We shall say that a group action $\pi : G \to \mathrm{Perm}(X)$ is *free* whenever for all $g, h \in G$ and $x \in X$ the equality $\pi(g)x = \pi(h)x$ implies $g = h$. Whenever we have a group G the action of G on itself will be by left multiplication. It is easy to see that this becomes a free group action.

We may also extend this discussion to the case of C*-algebras, a dicusion of which may be found in say [7] or [21].

Definition 1.1.2. Let A be a C*-algebra and G a discrete group. A group action of G on A is a homomorphism $\alpha : G \to \mathrm{Aut}(A)$.

To simplify notation later, we will use the following definition.

Definition 1.1.3. A C*-*dynamical system* is a triple (A, G, α) where A is a C*-algebra, G a discrete group and α an action of G on A.

1.1.2 Vector spaces associated with discrete groups and sets

Let X be any discrete set. We may always form the ℓ^p-space of X, defined as

$$\ell^p(X) = \begin{cases} \{f : X \to \mathbb{C} \mid \sum_{x\in X} |f(x)|^p < \infty\} & 1 \le p < \infty \\ \{f : X \to \mathbb{C} \mid \sup_{x\in X} |f(x)| < \infty\} & p = \infty. \end{cases}$$

We will also equip the ℓ^p-spaces with the familiar norm

$$||f||_{\ell^p(X)} = \begin{cases} \left(\sum_{x\in X} |f(x)|^p\right)^{1/p} & 1 \le p < \infty \\ \sup\{|f(x)| \mid x \in X\} & p = \infty \end{cases} \qquad f \in \ell^p(X).$$

It is relatively easy to show that $\ell^p(X)$ becomes a complete normed vector space with the above norm, though we will omit a proof here.

We also define the δ-functions as follows:

$$\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise.} \end{cases}$$

A case which we will be particularly interested in (partly explained by Proposition 1.1.7) is the case $p = 2$. We may equip $\ell^2(X)$ with an inner-product $\langle \cdot, \cdot \rangle$, defined as

$$\langle \xi, \zeta \rangle = \sum_{x\in X} \xi(x)\overline{\zeta(x)} \qquad \xi, \zeta \in \ell^2(X).$$

Notice also that the set $\{\delta_x \mid x \in X\}$ becomes an orthonormal basis for $\ell^2(X)$.

It is well-known $\ell^\infty(X)$ becomes a C*-algebra with the pointwise product and involution defined as conjugation. Also note that whenever we have a left group action of a discrete group G on X, we get an induced group action $\tau_X : G \to \mathrm{Aut}(\ell^\infty(X))$ defined by

$$\tau_X(g)f(x) = f(g^{-1}x).$$

Whenever we are in the setting of a group action of G on X, we shall equip $\ell^\infty(X)$ with the action τ_X.

For any Hilbert space H we shall as usual let $\mathcal{B}(H)$ denote the space of bounded linear operators on H and $\mathcal{U}(H)$ the group of unitaries on H.

We start of by a surprisingly useful lemma.

Lemma 1.1.4. *Let X be a set and $\phi \in \mathrm{Perm}(X)$ such that then the map $\tilde{\phi} : \ell^2(X) \to \ell^2(X)$ defined by*

$$\tilde{\phi}(\xi) = \xi \circ \phi \qquad \xi \in \ell^2(X),$$

is a unitary operator on $\ell^2(X)$.

As a result the map $\mathrm{ad}(\tilde{\phi}) = \kappa : \mathcal{B}(\ell^2(X)) \to \mathcal{B}(\ell^2(X))$ is a $*$-isomorphism.

Proof. Let $\xi, \zeta \in \ell^2(X)$, then

$$\begin{aligned} \langle \tilde{\phi}(\xi), \zeta \rangle &= \sum_{x\in X} \xi(\phi(x))\overline{\zeta(x)} \\ &= \sum_{x\in X} \xi(x)\overline{\zeta(\phi^{-1}(x))}. \end{aligned}$$

Hence $\tilde{\phi} = \widetilde{\phi^{-1}}$. It is easy to see that

$$\tilde{\phi} \circ \widetilde{\phi^{-1}} = \widetilde{\phi^{-1}} \circ \tilde{\phi} = I,$$

so $\tilde{\phi}$ is indeed unitary.

The last assertion follows immediately. □

Remark 1.1.5. As a result of the above lemma (and a small computation), we see that we get a group action

$$\alpha_X : \mathrm{Perm}(X) \to \mathrm{Aut}(\mathcal{B}(\ell^2(X)))$$

given by

$$\alpha_X(\phi) = \mathrm{ad}(\tilde{\phi}) \qquad \phi \in \mathrm{Perm}(X).$$

1.1.3 Representations of groups and C*-algebra

Definition 1.1.6. Let A be a $*$-algebra. A *$*$-representation* of A is a pair (π, H) where H is a Hilbert space and $\pi : A \to B(H)$ is a $*$-homomorphism. We say that π is a representation of A on H.

In this case we say that a $*$-representation (π, H) is *faithful* if π is injective.

Proposition 1.1.7. *Let X be a discrete set. There is a faithful $*$-representation of $\ell^\infty(X)$ on $\ell^2(X)$, given by the mapping*

$$M_X : \ell^\infty(X) \to B(\ell^2(X)),$$

where

$$[M_X(f)(\xi)](t) = f(t)\xi(t) \quad \textit{for } \xi \in \ell^2(X), t \in X.$$

Proof. The mapping is well-defined as

$$\sum_{x\in X} |f(x)h(x)|^2 \leq ||f||^2_{\ell^\infty(X)} \sum_{x\in X} |h(x)|^2 = ||f||^2_{\ell^\infty(X)} ||h||^2_{\ell^2(X)},$$

for $f \in \ell^\infty(X)$, $h \in \ell^2(X)$. Further, we easily see that the mapping is linear and multiplicative. To see that it respects the $*$-operation, let $f \in \ell^\infty(X)$ and $h_1, h_2 \in \ell^2(X)$, then there is the simple, though dull, task of writing out the inner product in ℓ^2:

$$\langle M_X(f)(h_1), h_2\rangle = \sum_{t\in X} f(t)h_1(t)\overline{h_2(t)} = \sum_{t\in X} h_1(t)\overline{\overline{f(t)}h_2(t)} = \langle h_1, M(\overline{f})(h_2)\rangle,$$

and hence $(M_X(f))^* = M_X(\overline{f})$.

We also see that M_X is injective. Indeed, suppose $\ell^\infty(X) \ni f \neq 0$, and pick $x \in X$ such that $f(x) \neq 0$, then

$$[M_X(f)(\delta_x)](x) = f(x)\delta_x(x) = f(x) \neq 0,$$

and hence we get a faithful $*$-representation of $\ell^\infty(G)$ on $\ell^2(X)$. □

Using the above proposition, we shall sometimes abuse notation and consider $\ell^\infty(X) \subset B(\ell^2(X))$.

Definition 1.1.8. Let G be a discrete group. A *unitary representation* of G is a pair (π, H) where $\pi : G \to \mathcal{U}(H)$ is a group homomorphism. We say that the representation (π, H) is finite dimensional if H is finite dimensional.

Proposition 1.1.9. *Let G be a discrete group acting on a set X from the left. The map $\lambda_X : G \to \mathcal{B}(\ell^2(X))$ given by $\lambda_X(g)\xi(x) = \xi(g^{-1}x)$ is a unitary representation of G in $\mathcal{B}(\ell^2(X))$ with $\lambda_X(g)^* = \lambda_X(g^{-1})$.*

Proof. It is easy to see that λ_X becomes a homomorphism. Obviously, if we have $\lambda_X(g)^* = \lambda_X(g^{-1})$ for each $g \in G$, then $\lambda_X(g)$ is unitary. This is just a tedious calculation of the inner product: For $\xi, \zeta \in \ell^2(X)$ and $g \in G$ we have

$$\begin{aligned}\langle \lambda_X(g)(\zeta), \xi \rangle &= \sum_{x \in X} \zeta(g^{-1}x)\overline{\xi(x)} \\ &= \sum_{x \in X} \zeta(g^{-1}(gx)\overline{\zeta(gx)} \\ &= \sum_{x \in X} \zeta(x)\overline{\lambda_X g^{-1})\xi(x)} \\ &= \langle \zeta, \lambda_X(g^{-1})\xi \rangle. \end{aligned}$$ □

Whenever we have a C*-dynamical system (A, G, α), we want to know when the representations of G and the $*$-representations of A play nicely together. More precisely we make the following definition.

Definition 1.1.10. Let (A, G, α) be a C*-dynamical system. A *covariant representation* of (A, G, α) is a triple (u, π, H) such that (u, H) is a unitary representation of G on H and π is a $*$-representation of A on H satisfying

$$u(g)\pi(a)u(s)^* = \pi(\alpha(g)a) \qquad \text{for all } a \in A \text{ and } g \in G.$$

It is quite easy to come up with examples of covariant representations, in fact we have already seen one.

Lemma 1.1.11. *Let G be a discrete group acting on a set X from the left. Then the triple $(\lambda_X, M_X, \ell^2(X))$ is a covariant representation for the C*-dynamical system $(\ell^\infty(X), G, \tau_X)$.*

Proof. This is again just writing out the definitions:

$$\begin{aligned}\left[\lambda_X(g)M_X(f)\lambda_X(g)^*(\xi)\right](x) &= f(g^{-1}x)\xi(x) \\ &= \left[M_X(\tau_X(g)(f))\xi\right](x),\end{aligned}$$

for $x \in X,\ f \in \ell^\infty(X),\ \xi \in \ell^2(X)$. As such we see that

$$\lambda_X(g)M_X(f)\lambda_X(g)^* = M_X(\tau_X(g)f),$$

and we are done. □

1.2 Crossed products

In this section we consider a discrete group G with an action $\alpha : G \to \mathrm{Aut}(A)$ on some unital C*-algebra A. Now given the action, we wish to form a C*-algebra containing both A and the action of G on A. As a first step we shall construct a intermediate $*$-algebra which we will complete in two different norms to obtain the full and reduced crossed product. Most of our definitions are acquired from [7, Section 4.1].

Consider the vector space $C_c(G, A)$ consisting of finitely supported[1] functions from G to A. We make the convention that for any $g \in G$ we let $g \in C_c(G, A)$ be the function mapping g to $1_A \in A$ and everything else to $0 \in A$. Using this convention we may view $C_c(G, A)$ as the set of finite sums

$$\sum_{g \in G} a_g g$$

where $a_g \in A$ for all g and all but finitely many of the a_g are non-zero.

We define a product $*_\alpha$ between two such finite sums as

$$\left(\sum_{g \in G} a_g g \right) *_\alpha \left(\sum_{h \in G} b_h h \right) = \sum_{g,h \in G} a_g \alpha(g)(b_h) gh,$$

and their involution as

$$\left(\sum_{g \in G} a_g g \right)^* = \sum_{g \in G} \alpha(g^{-1})(a_g^*) g^{-1}.$$

Given these operations, one easily checks that $C_c(G, A)$ becomes a $*$-algebra.

Given a covariant representation (u, π, H) of (A, G, α), it is easy to construct a $*$-representation $\pi \times u : C_c(G, A) \to \mathcal{B}(H)$ by

$$(\pi \times u)(\sum_{g \in G} a_g g) = \sum_{g \in G} \pi(a_g) u(g) \qquad \sum_{g \in G} a_g g \in C_c(G, A).$$

A quick check shows that this indeed becomes a $*$-representation of $C_c(G, A)$.

1.2.1 Full (universal) crossed product

We use the following definition from [7].

Definition 1.2.1. Let G be a discrete group with an action $\alpha : G \to \mathrm{Aut}(A)$ on some C*-algebra A. We define the *universal norm* on $C_c(G, A)$ to be

$$||x||_u = \sup \big\{ ||\pi(x)|| \bigm| \pi : C_c(G, A) \to B(H),\ H \text{ is a Hilbert space}, \\ \pi \text{ is a } * -\text{homomorphism} \big\}.$$

We furthermore define the *full crossed product*, $A \rtimes_\alpha G$ to be the completion of $C_c(G, A)$ with respect to $|| \cdot ||_u$.

Now, $|| \cdot ||_u$ does indeed become a norm on $C_c(G, A)$ (and not just a semi-norm), as we can always construct a faithful $*$-representation of $C_c(G, A)$ whenever we have a faithful $*$-representation of A (this will become clear when we discuss reduced crossed products). We also get the following useful universal property.

[1] actually, compactly supported continuous functions, but as G is discrete, this reduces to finitely supported functions

Theorem 1.2.2. *Let (A, G, α) be a C^*-dynamical system, and assume (u, π, H) is covariant representation for (A, G, α). Then $\pi \times u$ extends uniquely to a $*$-homomorphism $A \rtimes_\alpha G \to \mathcal{B}(H)$.*

Proof. See e.g. [7, Proposition 4.1.3]. □

An immediate consequence of this is the following corollary.

Corollary 1.2.3. *Let G be a discrete group with an action $\alpha : G \to \mathrm{Aut}(A)$ for some C^*-algebra A. Suppose (u, π, H) is a covariant representation of (A, G, α) on some Hilbert space H, and let C be the C^*-algebra generated by $u(G)$ and $\pi(A)$ in $\mathcal{B}(H)$. Then there is a surjective $*$-homomorphism $\sigma : A \rtimes_\alpha G \to C$.*

Whenever G is a discrete group with a left-action on a set X, we shall denote $\mathcal{R}_u(G, X) = \ell^\infty(X) \rtimes_{\tau_X} G$. If G acts on itself by left translation, we let $\mathcal{R}_u(G) = \mathcal{R}_u(G, G)$.

1.2.2 Reduced crossed product

Because of the universal norm, the universal crossed product is often difficult to work with. We thus introduce the reduced crossed product.

Let G be a discrete group with an action $\alpha : G \to \mathrm{Aut}(A)$ on some C^*-algebra A. Let $\pi : A \to \mathcal{B}(H)$ be a $*$-representation of A on a Hilbert space H. We define a new $*$-representation, ρ_π of A on $H \otimes \ell^2(G)$ by

$$\rho_\pi(a)(\xi \otimes \delta_g) = \pi(\alpha(g^{-1})(a))(\xi) \otimes \delta_g.$$

By an easy calculation one sees that the representations $1 \otimes \lambda_G$ and ρ_π form a covariant representation for the action of G on A. We thus get an induced $*$-representation of $C_c(G, A)$ on $H \otimes \ell^2(G)$ with $\rho_\pi \times (1 \otimes \lambda_G)$.

Definition 1.2.4. Let G be a discrete group with an action $\alpha : G \to \mathrm{Aut}(A)$ for some C^*-algebra A. If $\pi : A \to \mathcal{B}(H)$ is a faithful $*$-representation of A, we define the reduced crossed product, $A \rtimes_{\alpha,r} G$ as the closure of $\rho_\pi \times (1 \otimes \lambda_G)(C_c(G, A))$ in $\mathcal{B}(H \otimes \ell^2(G))$.

It can be shown that the reduced crossed product is in fact independent of the choice of $*$-representation π, see for instance [7, Proposition 4.1.5]. Using the map $a \mapsto \rho_\pi(a)$ we may always identify A as a subalgebra of $A \rtimes_{\alpha,r} G$, so we will sometimes abuse notation and assume $A \subset A \rtimes_{\alpha,r} G$.

Owing to Corollary 1.2.3, we see that the reduced crossed product is just a quotient of the full crossed product.

One of the benefits with the reduced crossed product is that we get a conditional expectation, as the next proposition shows.

Proposition 1.2.5. *Let (A, G, α) be a dynamical system. Then there is a faithful conditional expectation $E : A \rtimes_{\alpha,r} G \to A$ such that*

$$E(\sum_{g \in G} a_g g) = a_e \qquad \sum_{g \in G} a_g g \in C_c(G, A).$$

Proof. See [7, Proposition 4.1.9]. □

Whenever G is a discrete group with a left-action on a set X, we shall denote $\mathcal{R}_r(G,X) = \ell^\infty(G) \rtimes_{\tau_X,r} G$. If G acts on itself by left translation, we let $\mathcal{R}_r(G) = \mathcal{R}_r(G,G)$. This is one of the usual ways of defining the Roe algebra in the language of C*-algebras.

1.3 Crossed products concretely represented on $\ell^2(X)$

We shall see that there is a more natural approach to the construction of the crossed product in the cases we are interested in.

Indeed, whenever we have a group action of G on X, we get the covariant representation (λ_X, M_X) of $(\ell^\infty(X), G, \tau_X)$ on $\ell^2(X)$ (by Lemma 1.1.11), so it seems unecessary to form the covariant representation $\rho_{M_X} \times (1 \otimes \lambda_G)$ on $\ell^2(X) \otimes \ell^2(G)$.

Definition 1.3.1. Let G be a discrete group with a left action on a set X. We define the concretely represented Roe algebra on $\ell^2(X)$, denoted $\mathcal{R}_{con}(G,X)$, as the closure of $(M_X \times \lambda_X)(C_c(G, \ell^\infty(X)))$ in $\mathcal{B}(\ell^2(X))$. Whenever G acts on itself by left translation, we shall let $\mathcal{R}_{con}(G) = \mathcal{R}_{con}(G,G)$

As in the case of the reduced crossed product, we have a conditional expectation of $\mathcal{R}_{con}(G,X)$ onto $\ell^\infty(X)$.

Proposition 1.3.2. *Let G be a discrete group with a left action on a set X. The map $\tilde{E} = M_X \circ F : \mathcal{R}_{con}(G,X) \to M_X(\ell^\infty(X))$ where $F : \mathcal{R}_{con}(G,X) \to \ell^\infty(X)$ is defined by*

$$[F(\xi)](x) = \langle \xi(\delta_x), \delta_x \rangle \qquad \xi \in \mathcal{R}_{con}(G,X),\ x \in X,$$

is a faithful conditional expectation of $\mathcal{R}_{con}(G,X)$ onto $M_X(\ell^\infty(X)) \subset \mathcal{R}_{con}(G,X)$.

Proof. Now, $\tilde{E}$ is obviously a projection, as

$$[F(M_X(f))](x) = \langle [M_X(f)](\delta_x), \delta_x \rangle = f(x) \qquad f \in \ell^\infty(X),\ x \in X.$$

Using elementary properties of the inner product and the fact that M_X is contractive, it is quite easy to see that $\tilde{E}$ is contractive, hence we may conclude that $\tilde{E}$ is a conditional expectation by [7, Theorem 1.5.10].

To see that it is faithful, observe that whenever $\mathcal{R}_{con}(G,X) \ni T \geq 0$ in $B(\ell^2(X))$ and $\mathcal{E}(T) = 0$, then $\langle T\delta_x, \delta_x \rangle = 0$ for all $x \in X$, so $T = 0$. □

Lemma 1.3.3. *Let G be a discrete group acting freely on a set X. Then the conditional expectation $\tilde{E} : \mathcal{R}_{con}(G,X) \to M_X(\ell^\infty(X))$ satisfies*

$$\tilde{E}(\lambda_X(g)) = \begin{cases} 1 & g = e_G \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This is an easy calculation, as for $x \in X$ we have

$$\begin{aligned} F(\lambda(g))(x) &= \langle \lambda_X(g)\delta_x, \delta_x \rangle \\ &= \langle \delta_{gx}, \delta_x \rangle \\ &= \begin{cases} 1 & gx = x \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & g = e_G \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

since the group action was free. □

As in the case of the reduced crossed product, we may view $\mathcal{R}_{con}(G, X)$ as a quotient of $\mathcal{R}_u(G, X)$, since $(\lambda_X, M_X, \ell^2(X))$ is a covariant representation of $(\ell^\infty(X), G, \tau_X)$, so Corollary 1.2.3 gives us a surjection

$$\mathcal{R}_u(G, X) \to \mathcal{R}_{con}(G, X).$$

1.4 Twisted crossed products and Roe algebras

One may generalize the construction of a crossed product to a case when we do not directly have a group action on a unital C*-algebra. Rather we shall be concerned with the case when we up to a twist have a group action on a C*-algebra.

Definition 1.4.1. Let G be a discrete group. A *cocycle-crossed action* of G on a unital C*-algebra A is a tuple (α, σ) where $\alpha : G \to \mathrm{Aut}(A)$ and $\sigma : G \times G \to \mathcal{U}(A)$ satisfy

1. $\alpha(g)\alpha(k) = \mathrm{ad}(\sigma(g, k))\alpha(gk)$ for all $g, k \in G$
2. $\sigma(g, h)\sigma(gh, k) = [\alpha(g)(\sigma(h, k))]\, \sigma(g, hk)$ for all $g, h, k \in G$
3. $\sigma(g, e_G) = \sigma(e_G, g) = 1$ for all $g \in G$.

Definition 1.4.2. A *twisted* C*-*dynamical system* is a triple (A, G, α, σ) where A is a unital C*-algebra, G a discrete group and (α, σ) a cocycle-crossed action of G on A.

Remark 1.4.3. Notice that whenever we have a cocyle-crossed (α, σ) action of a group G on some commutative C*-algebra A (which is the case we are interested in), or more generally if σ takes values in the center of A, $\mathcal{Z}(A)$, α is a group action on A as

$$\begin{aligned} [\alpha(g)\alpha(k)]\,(a) &= [\mathrm{ad}(\sigma(g, k))\alpha(gk)]\,(a) \\ &= \sigma(g, k)\alpha(gk)(a)\sigma(g, k)^* \\ &= \sigma(g, k)\sigma(g, k)^*\alpha(gk)(a) \\ &= \alpha(gk)(a) \end{aligned}$$

for all $g, k \in G$ and all $a \in A$.

Owing to the above remark, we shall be chiefly interested in appending some "twist" σ to an already given group action. This will luckily make things a lot easier for us, but it will also produce an extra layer of generalization for our definition of the Roe algebra.

We shall consider $C_c(G, A)$ equipped with a new conjugation and multiplication. Namely, we shall define for finite sums $\sum_{g\in G} a_g g, \sum_{h\in G} b_h h \in C_c(G, A)$

$$\left(\sum_g a_g g\right)^* = \sum_g \sigma(g, g^{-1})^* \alpha(g)(a^*_{g^{-1}}) g$$

and

$$\left(\sum_g a_g g\right) * \left(\sum_h b_h h\right) = \sum_{g,h\in G} a_g \alpha(g)(b_h)\sigma(g, h) gh.$$

As we only specifically consider the reduced twisted crossed product, we will not define the universal twisted crossed product, but the reader should rest assured that it can be done quite similar to Definition 1.2.1.

As in the case of the (untwisted) reduced crossed product, we start with a faithful representation $\pi : A \to \mathcal{B}(H)$ for some Hilbert space H. We then define the representation $\rho_\pi : A \to \mathcal{B}(H \otimes \ell^2(G))$ as we did for the untwisted case. Then we define the representation $\lambda_\sigma : G \to \mathcal{B}(H \otimes \ell^2(G))$ by

$$\lambda_\sigma(g)(\xi \otimes \delta_x) = \left[\pi(\sigma(x^{-1}g^{-1}, g))\xi\right] \otimes \delta_{gx} \qquad \xi \in H, g, x \in G.$$

Lastly we define the representation $P_{\pi,\sigma} : C_c(G, A) \to B(H \otimes \ell^2(G))$ by

$$P_{\pi,\sigma}\left(\sum_{g\in G} a_g g\right) = \sum_g \rho_\pi(a_g)\lambda_\sigma(g) \qquad \sum_{g\in G} a_g g \in C_c(G, A).$$

Definition 1.4.4. Let G be a discrete group with cocycle-crossed action (α, σ) on a C*-algebra A faithfully represented on a Hilbert space H via π. We define the *reduced twisted crossed product* $A \rtimes^\sigma_{\alpha,r} G$ as the completion of $P_{\pi,\sigma}(C_c(G, A))$ in $\mathcal{B}(H \otimes \ell^2(G))$.

As in the case of the (untwisted) reduced crossed product, the definition of $A \rtimes^\sigma_{\alpha,r} G$ is actually independent of faithful representation.

Suppose G acts on a set X. We wish to find $\sigma : G \times G \to \mathcal{U}(\ell^\infty(X))$ such that (τ_X, σ) form a cocycle-crossed action of G on $\ell^\infty(X)$. As $\ell^\infty(X)$ is commutative, part one of Definition 1.4.1 is trivially satisfied. An easy choice of σ is to let $\sigma(g, h) \in \mathbb{T}$ be constant for each $g, h \in G$. As $\tau_X(g)(f) = f$ for all constant functions $f \in \ell^\infty(X)$ and all $g \in G$, part two of Definition 1.4.1 reduces to

$$\sigma(g, h)\sigma(gh, k) = \sigma(h, k)\sigma(g, hk) \qquad g, h, k \in G.$$

This equation is just the requirement for σ to be a scalar-valued 2-cocycle. More precisely, we use the definition found in [3, Definition 2.1].

Definition 1.4.5. Let G be a discrete group. A normalized 2-cocycle on G is a map $\sigma : G \times G \to \mathbb{T}$ such that

$$\sigma(g, h)\sigma(gh, k) = \sigma(h, k)\sigma(g, hk) \qquad g, h, k \in G$$

and

$$\sigma(g, e_G) = \sigma(e_G, g) = 1 \qquad g \in G.$$

We will primarily be interested in the case when $\sigma : G \times G \to U(\ell^\infty(G))$ is obtained through some 2-cocycle, that is when

$$\sigma(g,h) = \tilde{\sigma}(g,h)1_{\ell^\infty(X)} \tag{1.1}$$

for some 2-cocycle $\tilde{\sigma}$. We shall abuse notation somewhat and call $\sigma : G \times G \to U(\ell^\infty(X))$ a 2-cocycle whenever it obeys (1.1) for some 2-cocycle $\tilde{\sigma}$.

In the case when (λ, σ) form a cocycle-crossed action of G on $\ell^\infty(X)$ we shall denote $\mathcal{R}_r^\sigma(G,X) = \ell^\infty(X) \rtimes_{\tau_X,r}^\sigma G$. Whenever we are in the case when G acts on itself by left translation, we set $\mathcal{R}_r^\sigma(G) = \mathcal{R}_r^\sigma(G,X)$.

1.5 Coarse geometry

The "traditional way" of defining the Roe algebra is through coarse geometry, which perhaps more closely links the Roe algebra to John Roe.

The exposition here could be made a lot more general, but we shall concentrate on the situation of C*-algebras. Moreover the reader should note that one may make broader definitions in most cases, see for instance [16].

We shall presently move our attention elsewhere, and from now on we shall emphasize less that the action of a group gives rise to an operator on a Hilbert space. Rather, the role of the group action shall be a little more subtle. Though before we bring group actions into this, we shall need to define some extra notation.

Let X be any set, and let E_1, E_2 be subsets of $X \times X$. We define the *inverse* of E_1, denoted E_1^{-1}, by

$$E_1^{-1} = \{(x_1, x_2) \mid (x_2, x_1) \in E_1\},$$

and we also define the *composition* of E_1 and E_2, denoted $E_1 \circ E_2$ by

$$E_1 \circ E_2 = \{(x_1, x_2) \mid \exists x \in X \text{ such that } (x_1, x) \in E_1,\ (x, x_2) \in E_2\}.$$

Similar to the definition of a topological structure, we make the core definition for coarse spaces.

Definition 1.5.1. Let X be a set, and $\mathcal{E}$ a family of subsets of $X \times X$. We say that $\mathcal{E}$ is a coarse structure for X if $\mathcal{E}$ contains the diagonal and is closed under finite unions, inverses, compositions and subsets.

As in the case of topology, we may also consider coarse structures generated by a family of subsets.

Proposition 1.5.2. *Let X be a set, and a $\mathcal{E}$ a family of subset of $X \times X$. Then there is a smallest family $\tilde{\mathcal{E}}$ of subsets of $X \times X$ containing $\mathcal{E}$ such that $\tilde{\mathcal{E}}$ is a coarse structure on X.*

Proof. The proof is the standard one using intersection, see for instance [16, Proposition 2.12]. □

Whenever we are in the situation of Proposition 1.5.2 we shall call $\tilde{\mathcal{E}}$ the *coarse structure generated by $\mathcal{E}$.*

We are now ready to introduce the group action into the picture of coarse geometry.

Definition 1.5.3. Let G be a discrete group with an action on a set X. We define the *G-saturation* of a subset F of $X \times X$ to be $\{(gx, gy) \mid (x, y) \in F, g \in G\}$.

Whenever G is a discrete group acting on a set X, we shall equip X with the coarse structure $\mathcal{E}_G$ generated by the G-saturations of finite subsets of $X \times X$.

For an arbitrary set X, we define $\operatorname{supp} T$ for an operator $T \in \mathcal{B}(\ell^2(X))$ by

$$\operatorname{supp} T = \{(x, y) \in X \times X \mid T(\delta_y)(x) \neq 0\}.$$

Whenever a set X has a coarse structure $\mathcal{E}$, we shall let

$$\operatorname{Ctrl}(\mathcal{E}) = \left\{T \in \mathcal{B}(\ell^2(X)) \mid \operatorname{supp} T \in \mathcal{E}\right\}.$$

We say that $\operatorname{Ctrl}(\mathcal{E})$ are the operators on $\ell^2(X)$ with *controlled propagation* with respect to $\mathcal{E}$.

Lemma 1.5.4. *Let X be a set with a coarse structure $\mathcal{E}$. Then $\operatorname{Ctrl}(\mathcal{E})$ becomes a $*$-algebra.*

Proof. It is easy to see that all linear combinations of operators in $\operatorname{Ctrl}(\mathcal{E})$ are contained in $\operatorname{Ctrl}(\mathcal{E})$. Furthermore, for $T, S \in \operatorname{Ctrl}(\mathcal{E})$ we see that

$$\operatorname{supp} TS \subset \underbrace{\operatorname{supp} T}_{\in \mathcal{E}} \circ \underbrace{\operatorname{supp} S}_{\in \mathcal{E}} \in \mathcal{E}.$$

So $\operatorname{supp} TS \in \mathcal{E}$, hence $TS \in \operatorname{Ctrl}(\mathcal{E})$.

Furthermore, we see that if $(x, y) \in X \times X$, then

$$T(\delta_x)(y) = \langle T\delta_x, \delta_y \rangle = \langle \delta_x, T^*\delta_y \rangle = \overline{T^*(\delta_y)(x)}.$$

Hence $\operatorname{supp} T = (\operatorname{supp} T^*)^{-1}$, so $T^* \in \operatorname{Ctrl}(\mathcal{E})$ whenever $T \in \operatorname{Ctrl}(\mathcal{E})$. □

We define the *uniform Roe algebra*, $C_u^*(\mathcal{E})$ as the closure of $\operatorname{Ctrl}(\mathcal{E})$ in $\mathcal{B}(\ell^2(X))$ with respect to the operator norm. Owing to the above lemma, we see that $C_u^*(\mathcal{E})$ becomes a C*-algebra.

Whenever we are in the setting of a group G acting on a set X from the left, we shall denote $\mathcal{R}_c(G, X) = C_u^*(\mathcal{E}_G)$. Whenever G acts on itself by left translation, we shall let $\mathcal{R}_c(G) = \mathcal{R}_c(G, G)$.

Chapter 2

The link between the constructions

There are special cases when the various definitions of the crossed products and the uniform Roe algebra coincide. The only time we can guarantee they all coincide will be in the case of an amenable group acting on itself by left translation.

2.1 Coarse geometry and $\mathcal{R}_{con}(G)$

Our first proposition will reveal as how the controlled operators on $\ell^2(G)$ are related to $C_c(G, \ell^\infty(G))$.

We first need to make a little lemma. We use # to denote set cardinality.

Lemma 2.1.1. *Let G be a discrete group. Then*

$$\mathcal{E}_G = \mathcal{E}' = \{F \subset G \times G \mid F \text{ satisfy (2.1)}\}$$

where

$$\#\{x^{-1}y \mid (x, y) \in F\} < \infty. \tag{2.1}$$

Proof. To see this, observe that if $E \subset G \times G$ is finite, we have

$$\{(gx)^{-1}gx' \mid g \in G,\ (x, x') \in E\} = \{x^{-1}x' \mid (x, x') \in E\},$$

and the latter set is obviously finite as E is finite. This shows that the G-saturations of E is in $\mathcal{E}'$. We also see that whenever $F \subset G \times G$ satisfies condition (2.1) we may define $E = \{(e_G, x^{-1}y) \mid (x, y) \in F\}$, which is finite. Then

$$\{(gx, gx') \mid (x, x') \in E,\ g \in G\} = \{(g, gx^{-1}y) \mid (x, y) \in F,\ g \in G\} \supset F,$$

and so we see that all the elements in $\mathcal{E}'$ must be in $\mathcal{E}_G$, i.e. $\mathcal{E}' \subset \mathcal{E}_G$.

Now, as the diagonal obviously fulfils condition (2.1), and the action of inverses, composition and finite unions also preserves condition (2.1), we see that $\mathcal{E}'$ becomes a coarse structure on G. Hence $\mathcal{E}_G \subset \mathcal{E}'$ and we are done. □

Theorem 2.1.2. *Whenever G is a discrete group, we have a $*$-isomorphism*

$$\mathcal{R}_{con}(G) \simeq \mathcal{R}_c(G).$$

Proof. We first need to shift $\mathcal{R}_c(G)$ by inversion. So we define the map

$$\phi : G \to G$$

by $\phi(g) = g^{-1}$ and define $\kappa : \mathcal{B}(\ell^2(G)) \to \mathcal{B}(\ell^2(G))$ as in Lemma 1.1.4, that is

$$\kappa(T)(\xi)(x) = T(\xi \circ \phi)(\phi(x)) \qquad \xi \in \ell^2(G), x \in G, T \in \mathcal{B}(\ell^2(G)).$$

With these definitions $\mathrm{Ctrl}(\mathcal{E}_G) \simeq \kappa(\mathrm{Ctrl}(\mathcal{E}_G))$. Notice that $T \in \kappa(\mathrm{Ctrl}(\mathcal{E}_G))$ whenever

$$\#\{xy^{-1} \mid T(\delta_{y^{-1}})(x^{-1}) \neq 0\} < \infty$$

We shall first show that $M_X \times \lambda_G(C_c(G, \ell^\infty(G))) = \kappa(\mathrm{Ctrl}(\mathcal{E}_G))$. This is, however, easy. Let $T = \sum_{g \in F} f_g \lambda_G(g) \in B(\ell^\infty(G))$ be a finite formal sum where $\{f_g\}_g \in \ell^\infty(G)$. For $x, y \in G$ we have

$$(\sum_{g \in G} f_g \lambda_G(g)\delta_y)(x) = (\sum_{g \in G} f_g(gy)\delta_{gy})(x) = f_{xy^{-1}}(x).$$

So

$$\{xy^{-1} \mid T(\delta_y)(x) \neq 0\} = \{xy^{-1} \mid f_{xy^{-1}}(x) \neq 0\}.$$

The latter set is finite, as there are only finitely many $g \in G$ such that $f_g \neq 0$, hence $T \in \kappa(\mathrm{Ctrl}(\mathcal{E}_G))$.

Conversely, assume $T \in \kappa(\mathrm{Ctrl}(\mathcal{E}_G))$. For $g \in G$ define $f_g \in \ell^\infty(G)$ by $f_g(x) = T(\delta_{g^{-1}x})(x)$. We see that $f_g \neq 0$ for only finitely many $g \in G$, as

$$\begin{aligned} \infty &> \#\{xy^{-1} \mid T(\delta_y)(x) \neq 0\} \\ &\geq \#\{xx^{-1}g \mid T(\delta_{g^{-1}x})(x) \neq 0\} \\ &= \#\{g \in G \mid f_g \neq 0\}. \end{aligned}$$

So by a simple calculation we may decompose T as a finite sum

$$T = \sum_{g \in G} f_g \lambda_G(g).$$

Thus $T \in M_X \times \lambda_G(C_c(G, \ell^\infty(G)))$, so we get equality of $M_X \times \lambda_G(C_c(G, \ell^\infty(G)))$ and $\kappa(\mathrm{Ctrl}(\mathcal{E}_G))$. We thus produce

$$\mathcal{R}_c(G) \simeq \kappa(\mathcal{R}_c(G)) = \overline{\kappa(\mathrm{Ctrl}(\mathcal{E}_G))} = \mathcal{R}_{con}(G). \qquad \square$$

Remark 2.1.3. The role of κ in the above proof is rather subtle. Basically what we have done is to move the operators in $\mathrm{Ctrl}(\mathcal{E}_G)$ to operators whose support are contained in sets of the form

$$\#\{xy^{-1} \mid (x, y) \in \operatorname{supp} T\} < \infty.$$

This actually gives us another coarse structure on G, but it coincides with the right group action of G on itself.

Owing to the above remark we define a new coarse structure on G by

$$\mathcal{E}_G^{-1} = \{E \subset G \times G \mid \#\{xy^{-1} \mid (x, y) \in E\} < \infty\}.$$

We will need this structure in a later chapter.

2.2 $\mathcal{R}_{con}(G,X)$ and $\mathcal{R}_r(G,X)$

We shall show that when things play nicely, $\mathcal{R}_{con}(G,X) \simeq \mathcal{R}_r(G,X)$ for suitable G and X. Nevertheless we start with an example where things go wrong:

Example 2.2.1. Set $G = \mathbb{Z}/n$ for some $1 < n \in \mathbb{N}$ and let G act on $X = \{1, \dots, n\}$ with the trivial action, that is

$$gx = x \qquad \text{for all } x \in X, g \in G.$$

Then, as the operator $\lambda_X(g)$ on $\ell^2(X)$ is the identity for all $g \in G$ we see that

$$\mathcal{R}_{con}(G,X) = M_X(\ell^\infty(X)) \subset B(\ell^2(X)),$$

so

$$\mathcal{R}_{con}(G,X) = M_X(\ell^\infty(X)) \simeq \{A \in M_{n,n}(\mathbb{C}) \mid A \text{ is a diagonal matrix}\}.$$

Meanwhile we see that

$$\dim C_c(G, \ell^\infty(X)) = n \cdot n > \dim \mathcal{R}_{con}(G,X)$$

hence

$$\mathcal{R}_{con}(G,X) \not\simeq \mathcal{R}_r(G,X).$$

What failed in the above example is simply that the action, in some sense, did not describe the group G in any meaningful way. Our intuition leads us to the case of a free action of G on X. It will actually be Lemma 1.3.3 that saves the day.

Theorem 2.2.2. *Let G be a discrete group with a free action on a set X. Then*

$$\mathcal{R}_{con}(G,X) \simeq \mathcal{R}_r(G,X).$$

Proof. Let $\phi : \mathcal{R}_u(G,X) \to \mathcal{R}_{con}(G,X)$ and $\psi : \mathcal{R}_u(G,X) \to \mathcal{R}_r(G,X)$ be the surjective $*$-homomorphisms we get from Corollary 1.2.3. We will show that $\ker\phi = \ker\psi$ and thus get that

$$\mathcal{R}_{con}(G,X) \simeq \mathcal{R}_u(G,X)/\ker\phi = \mathcal{R}_u(G,X)/\ker\psi \simeq \mathcal{R}_r(G,X).$$

Let $E : \mathcal{R}_r(G,X) \to \ell^\infty(G)$ be the conditional expectation in Proposition 1.2.5 and let $F : \mathcal{R}_{con}(G,X) \to \ell^\infty(X)$ be as in Proposition 1.3.2. Consider

$$\sum_{g \in G} f_g g \in C_c(G, \ell^\infty(X)).$$

Then

$$F(\psi(\sum_{g\in G} f_g g)) = F(\sum_{g\in G} M_X(f_g)\lambda_X(g)) = f_{e_G}$$

(by Lemma 1.3.3 and the fact that F is a conditional expectation), likewise we have

$$E(\phi(\sum_{g\in G} f_g g)) = f_{e_G}.$$

We thus see that $E \circ \phi|_{C_c(G,\ell^\infty(X))} = F \circ \psi|_{C_c(G,\ell^\infty(X))}$. So as $C_c(G, \ell^\infty(X))$ is dense in $\mathcal{R}_u(G, X)$ we see that $E \circ \phi = F \circ \psi$.

Assume $x \in \ker \phi$, and assume $x \geq 0$. Then

$$0 = E(0) = E(\phi(x)) = F(\psi(x)),$$

but as $\psi(x) \geq 0$, we must have $\psi(x) = 0$ since F was faithful, so $x \in \ker \psi$.

Conversely, assume $x \in \ker \psi$ and $x \geq 0$, then again we get by the same argument that $x \in \ker \phi$. Hence we have

$$\ker \psi \cap \{x \in \mathcal{R}_u(G, X) \mid x \geq 0\} = \ker \phi \cap \{x \in \mathcal{R}_u(G, X) \mid x \geq 0\},$$

but then $\ker \psi = \ker \phi$, and we are done. □

Corollary 2.2.3. *Let G be a discrete group. Then*

$$\mathcal{R}_r(G) \simeq \mathcal{R}_c(G) \simeq \mathcal{R}_{con}(G).$$

Proof. Immediate from Theorem and Theorem 2.1.2. □

2.3 $\mathcal{R}_r(G)$ and $\mathcal{R}_u(G)$

This material is quite classical and essentially well known. We will therefore mostly just refer to proofs found in other articles where needed. Basically we will see that $\mathcal{R}_r(G, X) \simeq \mathcal{R}_u(G, X)$ are isomorphic whenever G is amenable.

There are several (equivalent) definitions of amenability, but we will use the following.

Definition 2.3.1. Let G be a discrete group. We say that G is amenable if there exists a linear functional $m : \ell^\infty(G) \to \mathbb{C}$ such that

1. $m(\tau_G(g)f) = m(f)$ for all $g \in G$ and $f \in \ell^\infty(G)$ (m is τ_G- or simply G-invariant);
2. m is a state.

We are now ready to state our main theorem in this section(without proof):

Theorem 2.3.2. *Let G be an amenable group with a left action on a set X. Then*

$$\mathcal{R}_r(G, X) \simeq \mathcal{R}_u(G, X).$$

Proof. Easy consequence of [7][Theorem 4.2.6]. □

We will give another characterization of this theorem in Corollary 3.4.4.

Chapter 3

On nuclearity

In this chapter we shall extend a result known in the case of (untwisted) reduced crossed product to the case of twisted crossed products. We will also investigate how well amenability of τ_G extends to amenbility of τ_X. Though before we do this, we need to extend our vocabulary.

3.1 Group C*-algebras

We are going to review the concept of group C*-algebras, a concept usually introduced without the crossed product construction. But to save time, we will just make the constructions through the reduced crossed product.

Let G be a discrete group. We notice that we may let G act trivially on the C*-algebra $\mathbb{C}$. We call this action ι_G.

Definition 3.1.1. Let σ be a scaler 2-cocycle. We define the *reduced σ-twisted group* C*-*algebra*, denoted $C_r^*(G,\sigma)$, to be

$$C_r^*(G,\sigma) = \mathbb{C} \rtimes_{\iota_G, r} G.$$

Whenever $\sigma = 1$ we set

$$C_r^*(G) = C_r^*(G,\sigma).$$

Identifying $\mathbb{C}1_{\ell^\infty(G)} \subset \ell^\infty(G)$ as a G-invariant C*-subalgebra of $\ell^\infty(G)$, we see that we may consider $C_r^*(G,\sigma)$ as a C*-subalgebra of $\mathcal{R}_r^\sigma(G)$.

3.2 Nuclear and exact C*-algebras

There are several equivalent definitions of nuclear C*-algebras, but because it turns out it is the best fitting for our purposes we shall use a variant of the one found in [7].

Definition 3.2.1. Let A be a C*-algebra. We say that A is nuclear if there exists a sequence of contractive completely positive maps $\phi_n : M_{k_n,k_n}(\mathbb{C}) \to A$ and $\psi_n : A \to M_{k_n,k_n}(\mathbb{C})$ such that for all $a \in A$

$$||\phi_n \circ \psi_n(a) - a|| \to 0.$$

A classical problem in C*-theory has been determining when the algebraic tensor product of two C*-algebras has a unique C*-norm. This is known to be true in the case where at least one of the two C*-algebras is nuclear, and the common definition of nuclearity of a C*-algebra is that A is nuclear if and only if $A \odot B$ has a unique C*-norm for all C*-algebras B. The two definitions are equivalent, as seen in [7, Theorem 3.8.7].

A weaker notion than a nuclear C*-algebra is the notion of an exact C*-algebra.

Definition 3.2.2. Let A be a C*-algebra. We say that A is exact whenever there exists a faithful $*$-representation $\pi : A \to \mathcal{B}(H)$ with contractive completely positive maps $\phi_n : A \to M_{k_n,k_n}(\mathbb{C})$ and $\psi_n : M_{k_n,k_n}(\mathbb{C}) \to \mathcal{B}(H)$ such that

$$||\psi_n \circ \phi_n(a) - \pi(a)|| \to 0 \qquad \text{for all } a \in A.$$

Definition 3.2.3. We say that a group G is *exact* whenever $C^*_r(G)$ is exact.

As the next lemma shows, being exact is weaker than being nuclear:

Lemma 3.2.4. *Let A be a nuclear* C*-*algebra, and suppose $B \subset A$ is a subalgebra, then B is exact.*

Proof. Pick a faithful representation $\pi : A \to \mathcal{B}(H)$. As A is nuclear, we obviously see that $\pi(A)$ is nuclear since π becomes a $*$-isomorphism onto its image, and it is enough to show that $\pi(B)$ is an exact C*-algebra.

By nuclearity of A we may pick maps $\phi_n : \pi(A) \to M_{n_k,n_k}(\mathbb{C})$ and $\psi_n : M_{n_k,n_k}(\mathbb{C}) \to \pi(A)$ according to Definition 3.2.1. Now, the map $\phi_n|_{\pi(B)}$ is obviously still contractive completely positive for each n. We may view the inclusion $i : \pi(B) \hookrightarrow \mathcal{B}(H)$ as a faithful representation of $\pi(B)$ on H. Moreover we may consider ψ_n as map into $\mathcal{B}(H)$ for each n, and it will still be contractive completely positive. Lastly, we see

$$||\psi_n \circ \phi_n(\pi(b)) - i(\pi(b))|| \to 0 \qquad \text{for all } b \in B,$$

and hence we are done. □

There is a canonical way to prove Lemma 3.2.4, namely to first show that a subalgebra of an exact C*-algebra is exact, and that a nuclear C*-algebra is exact. We will no't need these results, so our direct proof will suffice.

The following, which is a known result, shed some light on the relation between nuclearity, exactness and groups.

Theorem 3.2.5. *Let G be a discrete group. Then the following are equivalent:*

1. *G is exact.*
2. *$\mathcal{R}_r(G)$ is nuclear.*

Proof. See for instance [7, Theorem 5.1.6]. □

Our goal in this chapter is to extend this result to the case of twisted actions.

3.3 Technicalities

Before we start out on our main proof, we are going to need some technicalities.

3.3.1 Hilbert C*-modules

We shall make a slight digression into the world of Hilbert C*-modules, simply to establish notation and vocabulary. For a (close to) complete discussion one should consult [12].

Definition 3.3.1. Let A be a C*-algebra. A *right A-module* is a vector space X with a multiplication $\cdot : X \times A \to X$ such that

1. $x \cdot (ab) = (x \cdot a) \cdot b$ for all $a, b \in A$ and $x \in X$.
2. $(x + y) \cdot a = x \cdot a + y \cdot a$ for all $a \in A$ and $x, y \in X$.
3. $x \cdot (a + b) = x \cdot a + x \cdot b$ for all $x \in X$ and $a, b \in A$
4. $\lambda(x \cdot a) = (\lambda x) \cdot a = x \cdot (\lambda a)$ for all $a \in A$, $x \in X$ and $\lambda \in \mathbb{C}$.

The last point in the above definition is often omitted, but we include it to simplify later definitions.

Definition 3.3.2. Let A be a C*-algebra. A *pre-Hilbert A-module* is a right A-module equipped with a map $\langle\ ,\ \rangle_X : X \times X \to A$ satisfying the following

1. $\langle y, \lambda x + \nu y\rangle_X = \lambda\langle y, x\rangle_X + \nu\langle y, z\rangle_X$ for all $x, y, z \in X$ and $\lambda, \nu \in \mathbb{C}$.
2. $\langle x, y\rangle_X = \langle y, x\rangle_X^*$ for $x, y \in X$.
3. $\langle x, y \cdot a\rangle_X = \langle x, y\rangle_X \cdot a$ for $x, y \in X$ and $a \in A$.
4. $\langle x, x\rangle_X \geq 0$ for all $x \in X$, with equality if and only if $x = 0$.

We shall omit the subscript X on the map $\langle\ ,\ \rangle$ whenever the space X is clear from the setting.

It can be shown that whenever A is a C*-algebra and X is a pre-Hilbert A module, then the map $\langle\ ,\ \rangle$ satisfy the Cauchy-Schwartz inequality, that is

$$||\langle x, y\rangle||^2 \leq ||\langle x, x\rangle||\,||\langle y, y\rangle|| \qquad x, y \in X,$$

see for instance [12, Proposition 1.1] for a proof. An easy consequence of this is that the function $||\cdot||_X : X \to \mathbb{R}$ defined by

$$||x||_X = \sqrt{||\langle x, x\rangle||} \qquad x \in X$$

is a norm on X.

Definition 3.3.3. Let A be a C*-algebra. A *Hilbert A-module* is a pre-Hilbert A-module X which is complete with respect to the norm $||\cdot||_X$.

Example 3.3.4. Let A be a C*-algebra. The most trivial example of a Hilbert A-module is A itself with multiplication defined as

$$x \cdot a = xa \qquad x, a \in A,$$

and $\langle\ ,\ \rangle_A : A \times A \to A$ defined as

$$\langle x, y\rangle_A = x^* y \qquad x, y \in A.$$

It is easy to check that this is becomes a pre-Hilbert A-module, with

$$||a||_A = \sqrt{||\langle a, a\rangle||} = \sqrt{||a^*a||} = ||a|| \qquad a \in A,$$

Hence the norm induced by $\langle\ ,\ \rangle_A$ coincides with the original norm on A. Thus A becomes a Hilbert A-module.

Definition 3.3.5. Let A be a C*-algebra, and X, Y two Hilbert A-modules. We say that a map $T : X \to Y$ is *adjointable* if there exists a map $T^* : Y \to X$ such that

$$\langle y, T(x)\rangle_Y = \langle T^*(y), x\rangle_X \qquad x \in X, y \in Y.$$

Following the notation from [4], we define a couple of spaces associated to a Hilbert A-module. Whenever X is a Hilbert A-module, we define

$$\mathcal{L}(X) = \{T : X \to X \mid T \text{ adjointable}\},$$

and

$$\mathcal{I}(X) = \{T : X \to X \mid T \text{ is linear, bounded and invertible}\}.$$

In addition whenever G is a group, we define

$$X^G = \{T : G \to X \mid \sum_{g\in G} \langle T(g), T(g)\rangle_X \text{ converges in the } ||\cdot|| \text{ norm of } A\}.$$

We can make X^G into a Hilbert A-module with the inner product defined as

$$\langle T, S\rangle_{X^G} = \sum_{g\in G} \langle T(g), S(g)\rangle_X \qquad T, S \in X^G,$$

the multiplication being defined by

$$(T \cdot a)(g) = T(g) \cdot a \qquad g \in G, a \in A, T \in X^G.$$

Also note that $C_c(G, X)$, the set of X-valued function from G with finite support, is trivially contained in X^G.

We borrow the definition of amenability for group actions on a C*-algebra from [7, Definition 4.3.1].

Definition 3.3.6. Let G be a discrete group. An action $\alpha : G \to \mathrm{Aut}(A)$ on a unital C*-algebra A is *amenable* if there exists a net $\{T_i\}_i \subset C_c(G, A)$ such that

1. $0 \le T_i(g) \in \mathcal{Z}(A)$ for all i and $g \in G$.
2. $\lim_i ||(s *_\alpha T_i) - T_i||_{A^G} = 0$ for all $s \in G$.
3. $\langle T_i, T_i\rangle_{A^G} = 1$ for all i.

The first theorem sheds some light on the relation between exactness and amenability.

Theorem 3.3.7. *Let G be a discrete group. The following are equivalent:*

1. *G is exact.*

2. *The action of G on $\ell^\infty(G)$ is amenable.*

Proof. See [7, Proof of Theorem 5.1.7]. □

An importance consequence of amenability of an action is the following theorem, which should be compared to Theorem 2.3.2 to see the link between amenability of groups and C*-algebras.

Theorem 3.3.8. *Let (A, G, α) be a* C*-*dynamical system. If G acts amenably on A then $A \rtimes_{\alpha,r} G \simeq A \rtimes_\alpha G$.*

Proof. See [7, Theorem 4.3.4]. □

Before we introduce our next result, we need to define some notation that is found in [4].

Definition 3.3.9. Let (α, σ) be a cocycle crossed action of a group G on a C*-algebra A. An *equivariant representation* of the twisted dynamical system (A, G, α, σ) on a Hilbert A-module X is a pair (ρ, v) where $\rho : A \to \mathcal{L}(X)$ and $v : G \to \mathcal{I}(X)$ such that

1. $\rho(\alpha(g)(a)) = v(g)\rho(a)v(g)^{-1}$ for $g \in G$, $a \in A$
2. $v(g)v(h) = \mathrm{ad}_\rho(\sigma(g, h))v(gh)$ for $g, h \in G$
3. $\alpha(g)(\langle x, x'\rangle) = \langle v(g)x, v(g)x'\rangle$, for $g \in G$ and $x, x' \in X$
4. $v(g)(x \cdot a) = (v(g)x) \cdot \alpha(g)(a)$ for $g \in G$, $x \in X$ and $a \in A$,

where
$$\mathrm{ad}_\rho(a)x = \rho(a)x \cdot a^* \qquad x \in X, a \in \mathcal{U}(A).$$

Example 3.3.10. It is easy to come up with an example of an equivariant representation for a twisted dynamical system (A, G, α, σ). Let $X = A$ and equip A with its canonical A-module structure. Then we may define
$$\rho : A \to \mathcal{L}(A)$$
by $\rho(a)b = ab$ for $a \in A$ and $v : G \to \mathcal{I}(A)$ by $v(g) = \alpha(g)$ for $g \in G$. Then trivially all the conditions for an equivariant representation are satisfied.

We then define a weaker notion than amenability, but as we will see, it will suffice for our purposes.

Definition 3.3.11. Let (α, σ) be a cocycle crossed action of a group G on a C*-algebra A. We say that the dynamical system (A, G, α, σ) has the *weak approximation property* if there exist an equivariant representation (ρ, v) of (A, G, α, σ) on some A-module X and nets $\{\xi\}_i$ and $\{\eta_i\}_i$ in X^G satisfying

1. there exists some $M > 0$ such that $||\xi_i|| \cdot ||\eta_i|| \leq M$ for all i.
2. for all $g \in G$ and $a \in A$ we have
$$\lim_i \sum_{h \in G} \left\langle \xi_i(h), \rho(a)v(g)\eta_i(g^{-1}h) \right\rangle = a.$$

The following is a rather useful consequence of the weak approximation property.

Theorem 3.3.12. *Assume (A, G, α, σ) is a twisted* C*-*dynamical system with the weak approximation property. Then $A \rtimes_{\alpha,r} G$ is nuclear if and only if A is nuclear.*

Proof. See [4, Theorem 5.11]. □

We are going to need the following useful observation.

Proposition 3.3.13. *Suppose (A, G, α, σ) is a twisted* C*-*dynamical system such that*

$$\sigma(G, G) \subset \mathcal{Z}(A).$$

If the action α is amenable, then (A, G, α, σ) has the weak approximation property.

Before we prove the proposition, we make a small note. By Remark 1.4.3 we know that α indeed becomes an ordinary group action on the C*-algebra A, hence it makes sense to talk about amenability of α.

Proof. Let $\{T_i\}_i \subset C_c(G, A)$ be as in Definition 3.3.6 for the action α. Further define the equivariant representation of (A, G, α, σ) on A as in Example 3.3.10. Put $\{\xi_i\}_i = \{\eta_i\}_i = \{T_i\}_i$. Now, obviously the nets are in A^G as they specifically lie in $C_c(G, A)$. Furthermore, as $T_i(g) \in \mathcal{Z}(A)$ for all $g \in G$, the second part of in Definition 3.3.9 just becomes

$$||1_A - \sum_{h \in G} \xi_i(h)^* \alpha(g) \eta_i(g^{-1}h)|| \to 0 \qquad \text{for all } g \in G.$$

Observe that for $g \in G$ we have by a direct computation

$$\begin{aligned} ||1_A - \sum_{h \in G} \xi_i(gh)^* \alpha(g)(\eta_i(h))|| &= ||\sum_{h \in H} T_i(g)^2 - \sum_{h \in G} \xi_i(gh)^* \alpha(g)(\eta_i(h))|| \\ &= ||\sum_{h \in G} T_i(h)^2 - \sum_{h \in G} T_i(gh) \alpha(g)(T_i(h))|| \\ &= ||\sum_{h \in G} T_i(h)^2 - \sum_{h \in G} T_i(h) \alpha(g)(T_i(g^{-1}h))|| \\ &= ||\sum_{h \in G} T_i(h) \left(T_i(h) - \alpha(g)(T_i(g^{-1}h))\right)|| \\ &= ||\langle T_i, T_i - g *_\alpha T_i \rangle|| \\ &\le ||T_i||_{A^G} ||T_i - g *_\alpha T_i||_{A^G} \to 0 \end{aligned}$$

Now, as

$$\langle T_i, T_i \rangle = 1 \qquad \text{for all } i \in \mathbb{N}$$

we obviously have that $||\xi_i|| \cdot ||\eta_i||$ is uniformly bounded.

We have thus produced an equivariant representation of (A, G, α, σ) with the desired nets $\{\xi_i\}$ and $\{\eta_i\}$ and hence we are done. □

Lemma 3.3.14. *Let G be an exact group, and let $\sigma : G \times G \to \mathcal{U}(\ell^\infty(G))$ be a 2-cocycle. Then $\mathcal{R}_r^\sigma(G)$ is nuclear.*

Proof. By Theorem 3.3.7 we known that the action of G on $\ell^\infty(G)$ is amenable, and hence by Proposition 3.3.13 the system $(\ell^\infty(G), G, \tau_G, \sigma)$ has the weak approximation property. Since $\ell^\infty(G)$ is nuclear (it is Abelian) we get by Theorem 3.3.12 that $\mathcal{R}_r^\sigma(G)$ is nuclear. □

We also need to go the other way around, that is to show that G is exact whenever $\mathcal{R}_r^\sigma(G)$ is nuclear. Actually, we will go a bit further and show that $C_r^*(G,\sigma)$ is exact if and only if $C_r^*(G)$ is exact. We shall follow the method used in [7] closely, but we are going to be rather careful when adding the twist. First, we need a little definition, again borrowed form [7].

Definition 3.3.15. Let G be a discrete group. A *postive definite kernel* is a bounded function $k : G \times G \to \mathbb{C}$ such that the matrix

$$(k(s,t))_{s,t\in F}$$

is positive for any finite subset $F \subset G$.

Brown shows the following theorem.

Theorem 3.3.16. *Let G be a discrete group, the following are equivalent:*

1. *G is exact;*
2. *For any finite subset $E \subset G$ and $\epsilon > 0$ there exists a positive definite kernel $k : G \times G \to \mathbb{C}$ such that*

$$\{(x,y) \in G \times G \mid k(x,y) \neq 0\} \in \mathcal{E}_G^{-1}$$

and

$$\sup\{|k(s,t) - 1| \mid st^{-1} \in E\} < \epsilon.$$

Proof. See [7, Theorem 5.1.6]. □

We produce a simple lemma, the proof of which follows closely to that of the proof of [7, Theorem 5.1.6], we only make slight adjustments where needed.

Lemma 3.3.17. *Let G be a discrete group and $\sigma : G\times G \to \mathbb{T}$ a scalar 2-cocycle. If $C_r^*(G,\sigma)$ is exact, then $C_r^*(G)$ is exact.*

Proof. We will show that item ii) of 3.3.16 holds, and thus get the desired result.
Let E be a finite subset of G. Define $K : G \times G \to C_r^*(G,\sigma)$ by

$$K(s,t) = \sigma(t,t^{-1})\sigma(s^{-1},s)^*\lambda_\sigma(s)\lambda_\sigma(t)^* \qquad s,t \in G.$$

Set

$$F = \mathrm{Span}\{K(s,t), K(s^{-1},t), K(s,t^{-1}), K(s^{-1},t^{-1}) \mid s,t \in G, st^{-1} \in E\}.$$

Then we may, by [7, Exercise 3.9.5] find a finite subset $E' \subset G$ containing E such that we have a unitary completely positive map $\phi : \mathcal{B}(\ell^2(E')) \to \mathcal{B}(\ell^2(G))$ satisfying

$$||x - \phi(p_{E'}xp_{E'})|| \leq \epsilon||x|| \qquad x \in F,$$

where $p_{E'} : \ell^2(G) \to \ell^2(E')$ is the projection. Set $\psi : \mathcal{B}(\ell^2(G)) \to \mathcal{B}(\ell^2(G))$ as $\psi(x) = \phi(p_{E'} x p_{E'})$. Then define $k : G \times G \to \mathbb{C}$ by

$$k(s,t) = \langle \psi(K(s,t))\delta_t, \delta_s \rangle \qquad s,t \in G.$$

Pick a finite subset $S = \{s_1, \ldots, s_n\} \in G$, and set $A = (k(s,t))_{s,t \in S}$, then for $x \in \mathbb{C}^n$

$$\begin{aligned}
\langle Ax, x \rangle &= \sum_{i,j} k(s_i, s_j)\overline{x_i} x_j \\
&= \sum_{i,j} \langle \psi(K(s_i, s_j))\delta_{s_j}, \delta_{s_i} \rangle \overline{x_i} x_j \\
&= \sum_{i,j} \langle \psi(K(s_i, s_j)) x_j \delta_{s_j}, x_i \delta_{s_i} \rangle \\
&= \langle \psi((K(s_i, s_j))_{i,j}) \begin{pmatrix} x_1 \delta_{s_1} \\ \vdots \\ x_n \delta_{s_n} \end{pmatrix}, \begin{pmatrix} x_1 \delta_{s_1} \\ \vdots \\ x_n \delta_{s_n} \end{pmatrix} \rangle.
\end{aligned}$$

To see that this is non-negative, observe first that the matrix $(K(s_i, s_j))_{i,j}$ is positive in $M_{n,n}(C_r^*(G,\sigma))$ since we may decompose

$$K(s_i, s_j) = \underbrace{\sigma(s_i^{-1}, s_i)^* \lambda_\sigma(s_i)}_{a_i} \overbrace{\sigma(s_j, s_j^{-1}) \lambda_\sigma(s_j)^*}^{a_j^*} \qquad i,j = 1, \ldots, n,$$

so [7, Example 1.5.13] tells us that $(K(s_i, s_j))_{i,j}$ is positive. Furthermore, ψ is completely positive, being the composition of two completely positive maps, hence we see that $\langle Ax, x \rangle \geq 0$.

Observe that for $s, t \in G$ we have $k(s,t) = 0$ if $E' \cap (st^{-1}E') = \emptyset$, hence

$$\begin{aligned}
\operatorname{supp} k = \{s,t \in G \mid k(s,t) \neq 0\} &\subset \{s,t \in G \mid st^{-1}x = y \text{ for some } x,y \in E'\} \\
&= \{s,t \in G \mid st^{-1} = x^{-1}y \text{ for some } x,y \in E'\} \\
&= \{s,t \in G \mid st^{-1} \in E'^{-1}E'\},
\end{aligned}$$

but as $E'^{-1}E'$ is finite, we see that $\operatorname{supp} k \in \mathcal{E}_G^{-1}$.

At last, a simple calculation tells us that

$$\begin{aligned}
K(s,t)\delta_t &= \sigma(t,t^{-1})\sigma(s^{-1},s)^* \lambda_\sigma(s) \lambda_\sigma(t)^* \delta_t \\
&= \sigma(t,t^{-1})\sigma(s^{-1},s)^* \lambda_\sigma(s) \sigma(t,t^{-1})^* \lambda_\sigma(t^{-1}) \delta_t \\
&= \sigma(t,t^{-1})\sigma(s^{-1},s)^* \sigma(t,t^{-1})^* \sigma(s,t^{-1}) \lambda_\sigma(st^{-1}) \delta_t \\
&= \sigma(t,t^{-1})\sigma(s^{-1},s)^* \sigma(t,t^{-1}) \sigma(t^{-1}ts^{-1}, st^{-1}) \sigma(s,t^{-1}) \delta_{st^{-1}t} \\
&= \sigma(t,t^{-1})\sigma(s^{-1},s)^* \sigma(t,t^{-1}) \underbrace{\sigma(s^{-1}, st^{-1})\sigma(s,t^{-1})}_{\sigma(s^{-1},s)} \delta_s \\
&= \sigma(t,t^{-1})\sigma(t,t^{-1})^* \sigma(s^{-1},s)^* \sigma(s^{-1},s) \delta_s \\
&= \delta_s.
\end{aligned}$$

So whenever $s, t \in G$ and $st^{-1} \in E$ we have

$$\begin{aligned}|k(s,t) - 1| &= |\langle \psi(K(s,t))\delta_t, \delta_s\rangle - 1| \\ &= |\langle \psi(K(s,t))\delta_t, \delta_s\rangle - \langle \delta_s, \delta_s\rangle| \\ &= |\langle \psi(K(s,t))\delta_t - \delta_s, \delta_s\rangle| \\ &\leq ||\psi(K(s,t))\delta_t - \delta_s||_{\ell^2(G)} \\ &= ||(\psi(K(s,t)) - K(s,t))\delta_t||_{\ell^2(G)} \\ &\leq ||\psi(K(s,t)) - K(s,t)|| < \epsilon.\end{aligned}$$

Where the last inequality follows by choice of ϕ. □

Corollary 3.3.18. *Let G be a discrete group and $\sigma : G \times G \to \mathbb{T}$ a scalar 2-cocycle. Then the following are equivalent:*

1. *$C^*_r(G)$ is exact;*
2. *$C^*_r(G, \sigma)$ is exact.*

Proof. Suppose $C^*_r(G)$ is exact, then $\mathcal{R}^\sigma_r(G)$ is nuclear by Lemma 3.3.14, hence $C^*_r(G, \sigma)$ is exact (as $C^*_r(G, \sigma)$ is a C*-subalgebra of $\mathcal{R}^\sigma_r(G)$). The opposite direction is just Lemma 3.3.17. □

And now to our concluding theorem of this section.

Theorem 3.3.19. *Let G be a discrete group and $\sigma : G \times G \to \mathbb{T}$ a scalar 2-cocycle. Then the following are equivalent.*

1. *G is exact*
2. *$\mathcal{R}^\sigma_r(G)$ is nuclear.*

Proof. Immediate from the previous results. □

3.4 Amenability of free group actions

Our goal is to extend one direction of Theorem 3.3.7 to the case of a group G acting freely on a set X.

Proposition 3.4.1. *Let G be an exact group with an action on a set X, and suppose there exists a map $\phi : X \to G$ where $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X$. The action of G on $\ell^\infty(X)$ is then amenable.*

Proof. By Theorem 3.3.7, we know that the action of G on $\ell^\infty(G)$ is amenable. So pick $\{T_i\}_i \subset C_c(G, \ell^\infty(G))$ according to Definition 3.3.6. Then by using the map ϕ we may define for each $T \in C_c(G, \ell^\infty(G))$ a $\tilde{T} \in C_c(G, \ell^\infty(X))$ by $\tilde{T}(g)(x) = T(g)(\phi(x))$. Consider the collection $\{\tilde{T}_i\}_i \subset C_c(G, \ell^\infty(X))$. Obviously $\tilde{T}_i(g) \geq 0$ for all $g \in G$ as $T_i(g) \geq 0$. And we see that for all $x \in X$ we have

$$\langle \tilde{T}_i, \tilde{T}_i\rangle_{\ell^\infty(X)^G}(x) = \sum_{g\in G}\left(\left[\tilde{T}_i(g)\right](x)\right)^2 = \sum_{g\in G}\left([T_i(g)](\phi(x))\right)^2 = 1$$

in other words

$$\langle \tilde{T}_i, \tilde{T}_i \rangle_{\ell^\infty(X)^G} = 1_{\ell^\infty(X)}.$$

For $f \in \ell^\infty(G)$ define $\tilde{f} \in \ell^\infty(X)$ as $\tilde{f}(x) = f(\phi(x))$. Notice that

$$\tau_X(g)(\tilde{f})(x) = f(\phi(g^{-1} \cdot x)) = f(g^{-1}(\phi(x))) = \widetilde{\tau_G(g)f}(x),$$

for $g \in G, x \in X, f \in \ell^\infty(G)$.

Now for i express T_i as a finite sum

$$T_i = \sum_{g \in G} f_{i,g} g$$

for $\{f_{i,g}\}_{g \in G} \subset \ell^\infty(G)$. We then see that

$$s * \tilde{T}_i = \sum_{g \in G} \tau_X(s)(\tilde{f}_{i,g}) sg = \sum_{g \in G} (\widetilde{\tau_G(s) f_{i,g}}) sg = \widetilde{s * T_i}.$$

Further we see that for $T \in C_c(G, \ell^\infty(G))$ we have

$$\begin{aligned} ||\tilde{T}||^2_{\ell^\infty(X)^G} &= || \sum_{g \in G} \tilde{T}(g)^* \tilde{T}(g) ||_{\ell^\infty(X)} = || \sum_{g \in G} T(g)^* T(g) ||_{\ell^\infty(\phi(X))} \\ &\leq || \sum_{g \in G} T(g)^* T(g) ||_{\ell^\infty(G)} = ||T||^2_{\ell^\infty(G)^G}. \end{aligned}$$

Thus for $s \in G$ we produce

$$\begin{aligned} \lim_i ||s * \tilde{T}_i - \tilde{T}_i||_{C_c(G, \ell^\infty(X))} &= \lim_i ||\widetilde{s * T_i - T_i}||_{C_c(G, \ell^\infty(X)} \\ &\leq \lim_i ||s * T_i - T_i||_{C_c(G, \ell^\infty(G))} = 0. \end{aligned}$$

Hence G acts amenably on $\ell^\infty(X)$. □

We shall need a little technical lemma (which luckily for us is easy to prove and state!):

Lemma 3.4.2. *Let G be a discrete group acting freely on a set X. Then there is is a map $\phi : X \to G$ such that $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X$.*

Proof. We define an equivalence relation $\sim_G$ on X by

$$x \sim_G y \Leftrightarrow x = gy \text{ for some } g \in G.$$

Then we may partition X into equivalence classes under $\sim_G$, and by the axiom of choice, we may form the set U consisting of one element from each equivalence class. Then we may for $x \in U$ define a bijective G-map $\phi_x : Gx \to G$ by

$$\phi_x(y) = g \text{ where } y = gx \text{ for some } g \in G.$$

This is well-defined as $y = gx = g'x$ implies $g' = g$ by the freeness of the action. Furthermore, we see that

$$\phi_x(h(gx)) = \phi_x((hg)x) = hg = h\phi_x(gx) \quad \text{for } h, g \in G.$$

Then we may define the map $\phi : X \to G$ as follows:

$$\phi(y) = \phi_x(y) \qquad \text{if } y \in Gx \text{ for } x \in U.$$

Now, for $g \in G$ and $x \in X$, we have

$$\phi(gx) = g\phi(x)$$

since this equality holds for each ϕ_x. □

Remark 3.4.3. The assumptions in the above lemma can not be loosened. Indeed, suppose G acts on a set X and suppose there is a map $\phi : X \to G$ such that $\phi(gx) = g\phi(x)$ for all $g \in G$ and all $x \in X$. Then if the action of G on X is not free, we may pick $x_0 \in X$ and $g \in G$ where $g \neq e_G$ such that $gx_0 = x_0$. Then $\phi(x_0) = \phi(gx_0) = g\phi(x_0)$, but this implies $g = e_G$, a contradiction.

Thus we are able to produce the following corollary.

Corollary 3.4.4. *Let G be a discrete, exact group acting freely on a set X. Then the following holds.*

1. *The action of G on $\ell^\infty(X)$ is amenable;*
2. *$\mathcal{R}_r(G, X) \simeq \mathcal{R}_{con}(G, X) \simeq \mathcal{R}_u(G, X)$ is nuclear;*
3. *if $\sigma : G \times G \to \mathcal{U}(\ell^\infty(X))$ is a 2-cocycle, then $\mathcal{R}_r^\sigma(G, X)$ is nuclear.*

Proof. Combine Proposition 3.4.1 with Lemma 3.4.2 to get the first assertion. And combine the first assertion with Theorem 3.3.12, Theorem 3.3.8 and Proposition 3.3.13 to get the second assertion.

The last assertion is obtained by the first assertion, Proposition 3.3.13 and Theorem 3.3.12. □

Chapter 4

Some properties of $\mathcal{R}_{con}(G, X)$

In this chapter we will primarily be interested in some fundamental properties of $\mathcal{R}_{con}(G, X)$ deduced from Følner nets.

4.1 Følner nets and Szegö-pairs

4.1.1 Følner nets

For an action of a discrete group G on a set X, we want to see what happens when X has a Følner net. We will use the following definition of a Følner net from [18].

Definition 4.1.1. Let G be a discrete group acting on a set X. We say that a net of finite, non-empty subsets $\{F_i\}_{i \in I} \subset X$ is a *Følner net* for the action of G on X if

$$\lim_i \frac{\#(gF_\alpha \Delta F_\alpha)}{\#(F_\alpha)} = 0 \quad \text{for all } g \in G.$$

Here

$$A \Delta B = (A \cup B) \setminus (A \cap B) \qquad \text{for subsets } A \text{ and } B \text{ of X.}$$

We shall also use the following definition for a Følner net in a C*-algebra, borrowed from [2].

Definition 4.1.2. Let A be a C*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H, and suppose A contains the identity operator of $\mathcal{B}(H)$. Let $|| \cdot ||_1$ be the the trace class norm on $\mathcal{B}(H)$ as defined in [15] , in other words

$$||T||_1 = \sum_{x \in E} \langle |T|x, x\rangle \qquad T \in \mathcal{B}(H),$$

where E is an orthonormal basis for H. We say that a net of non-zero finite dimensional projections $\{p_\alpha\}_\alpha \subset \mathcal{B}(H)$ is a Følner net for $A \subset \mathcal{B}(H)$ if for all $a \in A$

$$\lim_\alpha \frac{||ap_\alpha - p_\alpha a||_1}{||p_\alpha||_1} = 0. \tag{4.1}$$

The following lemma is quite useful for us, but it also shows (some) of the relation between Følner nets for group actions and Følner nets for C*-algebras.

Lemma 4.1.3. *Let G be a discrete group acting on a set X, and suppose $\{F_\alpha\}_\alpha$ is a Følner net for the action of G on X. For each α, let $p_\alpha : \ell^2(X) \to \ell^2(X)$ be the projection of $\ell^2(X)$ onto $\ell^2(F_\alpha)$, i.e.*

$$p_\alpha(f)(x) = \begin{cases} f(x) & x \in F_\alpha \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } x \in X, f \in \ell^2(X).$$

Then $\{p_\alpha\}$ is a Følner net for $\mathcal{R}_{con}(G,X) \subset B(\ell^2(X))$.

Proof. Consider an element of the form $a = f\lambda_X(g) \in \mathcal{R}_{con}(G,X)$ where $f \in \ell^\infty(X)$ and $g \in G$. For arbitrary α and $x \in X$ we have

$$[(f\lambda_X(g))p_\alpha](\delta_x) = \begin{cases} f(gx)\delta_{g\cdot x} & x \in F_\alpha \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[p_\alpha(f\lambda_X(g))](\delta_x) = \begin{cases} f(g\cdot x)\delta_{gx} & gx \in F_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$(ap_\alpha - p_\alpha a)(\delta_x) = \begin{cases} 0 & x \notin F_\alpha \cup (g^{-1}F_\alpha) \\ 0 & x \in (g^{-1}F_\alpha \cap F_\alpha) \\ f(gx)\delta_{gx} & x \in F_\alpha \setminus (g^{-1}F_\alpha) \\ -f(gx)\delta_{gx} & x \in (g^{-1}F_\alpha) \setminus F_\alpha. \end{cases}$$

So then

$$\begin{aligned} ||ap_\alpha - p_\alpha a||_1 = \sum_{x\in X}(|ap_\alpha - p_\alpha a|\delta_x, \delta_x) &\le \sum_{x\in X} ||\, |ap_\alpha - p_\alpha a|\delta_x|| \\ &= \sum_{x\in X} ||(ap_\alpha - p_\alpha a)\delta_x|| \\ &= \sum_{x\in F_\alpha\Delta(g^{-1}F_\alpha)} |f(g\cdot x)| \\ &\le \#(F_\alpha\Delta(g^{-1}F_\alpha))||f||_{\ell^\infty(X)}. \end{aligned}$$

It is quite easy to verify that

$$||p_\alpha||_1 = \#(F_\alpha).$$

Hence we see that

$$\lim_\alpha \frac{||ap_\alpha - p_\alpha a||_1}{||p_\alpha||_1} \le \lim_\alpha \frac{\#((F_\alpha)\Delta(g^{-1}\cdot F_\alpha))||f||_{\ell^\infty(X)}}{\#(F_\alpha)} = 0.$$

Now consider any finite sum of the form

$$a = \sum_{g\in G} f_g\lambda_X(g) \in M_X \times \lambda_X(C_c(G, \ell^\infty(X))).$$

By computation we see that

$$\begin{aligned}
\frac{||ap_\alpha - p_\alpha a||_1}{||p_\alpha||_1} &= \lim_\alpha \frac{||\sum_{g\in G}(f_g\lambda_X(g)p_\alpha - p_\alpha f_g\lambda_X(g))||_1}{||p_\alpha||_1} \\
&\leq \sum_{g\in G} \lim_\alpha \frac{||f_g\lambda_X(g)p_\alpha - p_\alpha f_g\lambda_X(g)||_1}{||p_\alpha||_1} \\
&= 0.
\end{aligned}$$

Where we were able to exchange limits as the sum was finite.

To extend the result for the whole of $\mathcal{R}_{con}(G, X)$ take an arbitrary element $a \in \mathcal{R}_{con}(G, X)$ and pick a sequence $\{a_n\}_{n\in\mathbb{N}} \subset M_X \times \lambda_X(C_c(G, \ell^\infty(X)))$ such that $a_n \to a$. We readily see that for any α and $n \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{||ap_\alpha - p_\alpha a||_1}{||p_\alpha||_1} &\leq \frac{||(ap_\alpha - p_\alpha a) - (a_np_\alpha - p_\alpha a_n)||_1}{||p_\alpha||_1} + \frac{||a_np_\alpha - p_\alpha a_n||_1}{||p_\alpha||_1} \\
&= \frac{||(a - a_n)p_\alpha - p_\alpha(a - a_n)||_1}{||p_\alpha||_1} + \frac{||a_np_\alpha - p_\alpha a_n||_1}{||p_\alpha||_1} \\
&\leq \frac{2||a - a_n||\,||p_\alpha||_1}{||p_\alpha||_1} + \frac{||a_np_\alpha - p_\alpha a_n||_1}{||p_\alpha||_1} \\
&= ||a - a_n|| + \frac{||a_np_\alpha - p_\alpha a_n||_1}{||p_\alpha||_1},
\end{aligned}$$

in other words we have

$$\lim_\alpha \frac{||ap_\alpha - p_\alpha a||_1}{||p_\alpha||_1} \leq ||a - a_n||.$$

So

$$\lim_\alpha \frac{||ap_\alpha - p_\alpha a||_1}{||p_\alpha||_1} = 0,$$

since $\lim_{n\to\infty} ||a - a_n|| = 0$. □

4.1.2 Szegö-pairs

We use the definition of a Szegö-pair found in [2].

Definition 4.1.4. Let A be a C*-subalgebra of $\mathcal{B}(H)$ containing the identity operator. A Szegö-pair for $A \subset \mathcal{B}(H)$ is a pair $(\{p_\alpha\}_\alpha, \phi)$ where $\{p_\alpha\}_\alpha$ is a net of finite dimensional orthogonal projections in $\mathcal{B}(H)$ and ϕ is a state on A such that

$$\lim_\alpha \sum_{i=1}^{n_\alpha} \frac{g(\lambda_i^\alpha)}{n_\alpha} = \phi(g(a)) \qquad g \in C_0(\mathbb{R})$$

for all self-adjoint $a \in A$ where $\{\lambda_1^\alpha, \dots, \lambda_{n_\alpha}^\alpha\}$ is the eigenvaluelist of $p_\alpha a|_{p_\alpha H}$ (with repetitions).

We shall make good use of the following theorem

Theorem 4.1.5. *Suppose A is a C*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. Then a pair $(\{p_\alpha\}, \phi)$ is a Szegö pair for $A \subset \mathcal{B}(H)$ if and only if the following two conditions hold*

1. $\{p_\alpha\}$ *is a Følner net for* $A \subset \mathcal{B}(H)$.
2. $\phi(a) = \lim_\alpha \frac{1}{n_\alpha} \operatorname{Tr}(p_\alpha a)$ *for all* $a \in A$.

Proof. See [2, Theorem 6 i)] □

We wish to make a stronger connection between Følner nets for group actions and Szegö-pairs. For any discrete group G acting on a set X with a Følner net $\{F_\alpha\}$, we define the net of states $\{\phi_{F_\alpha}\}$ on $\ell^\infty(X)$ by

$$\phi_{F_\alpha}(f) = \frac{1}{\#(F_\alpha)} \sum_{x \in F_\alpha} f(x) \qquad f \in \ell^\infty(X).$$

We can not always guarantee that the net converges, but as the following lemma shows, we can always pick out a convergent subnet.

Lemma 4.1.6. *Let* G *be a discrete group acting on a set* X*, and suppose* $\{F_\alpha\}_\alpha$ *is a Følner net for the action of* G *on* X*. Then we may pick a subnet* $\{F_{\alpha_i}\}$ *of* $\{F_\alpha\}$ *such that* $\phi_{F_{\alpha_i}}$ *converges in the weak*-topology to a* G*-invariant state* ϕ *on* $\ell^\infty(X)$.

Proof. We easily deduce that

$$||\phi_{F_\alpha}|| = 1$$

for each α. So $\{\phi_{F_\alpha}\}_\alpha$ is contained in a weak*-compact set by Alaoglu's theorem, and hence we may pick a weak*-convergent subnet $\{\phi_{F_{\alpha_i}}\}_i$ of $\{\phi_{F_\alpha}\}_\alpha$. Then we obviously have that $\{\phi_{F_{\alpha_i}}(f)\}_i$ converges for each $f \in \ell^\infty(X)$. And for $g \in G$ and i we have

$$\begin{aligned} |\phi_{F_{\alpha_i}}(\tau_X(g)f) - \phi_{F_{\alpha_i}}(f)| &\leq \frac{1}{\#F_{\alpha_i}} \sum_{x \in F_{\alpha_i} \Delta g^{-1} F_{\alpha_i}} |f(x)| \\ &\leq \frac{\#(F_{\alpha_i} \Delta g^{-1} F_{\alpha_i})}{\#F_{\alpha_i}} ||f||_{\ell^\infty(X)} \to 0. \end{aligned}$$

Hence

$$\phi(\tau_X(g)f) = \lim_i \phi_{F_{\alpha_i}}(\tau_X(g)f) = \lim_i \frac{1}{\#(F_{\alpha_i})} \sum_{x \in g^{-1}F_{\alpha_i}} f(x) = \lim_i \phi_{F_{\alpha_i}}(f) = \phi(f),$$

so ϕ is indeed G-invariant.

To see that it is a state, observe that for each i we have

$$\phi_{F_{\alpha_i}}(1) = 1 = ||\phi_{F_{\alpha_i}}||.$$

Thus $\phi(1) = 1$ and hence $\phi(1) = ||\phi||$, so ϕ is a state. □

Lemma 4.1.7. *Let* G *be a discrete group acting on a set* X*, and suppose* $\{F_\alpha\}_\alpha$ *is a Følner net for the action of* G *on* X *such that the net* $\{\phi_{F_\alpha}(f)\}_\alpha$ *converges for each* $f \in \ell^\infty(X)$*. Define the state* ϕ *as*

$$\phi(f) = \lim_\alpha \phi_{F_\alpha}(f) \qquad f \in \ell^\infty(X).$$

Then the pair $(\{F_\alpha\}_\alpha, \phi \circ F)$ *is a Szegö-pair for* $\mathcal{R}_{con}(G, X)$.

Proof. By Theorem 4.1.5 it suffices to show that

$$\phi(F(a)) = \lim_\alpha \frac{1}{n_\alpha} \operatorname{Tr}(p_\alpha a) \qquad \text{for all } a \in \mathcal{R}_{con}(G,X),$$

where p_α is the orthogonal projection of $\ell^2(X)$ onto $\ell^2(F_\alpha)$. But this is almost immediately true, since for each α we have for $a \in \mathcal{R}_{con}(G,X)$

$$\begin{aligned}\operatorname{Tr}(p_\alpha a) &= \sum_{x\in X} \langle p_\alpha a \delta_x, \delta_x\rangle \\ &= \sum_{x\in F_\alpha} \langle a\delta_x, \delta_x\rangle.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\phi(F(a)) &= \lim_\alpha \left[\frac{1}{\#(F_\alpha)} \sum_{x\in F_\alpha} [F(a)]\,(x)\right] \\ &= \lim_\alpha \left[\frac{1}{\#(F_\alpha)} \sum_{x\in F_\alpha} \langle a\delta_x, \delta_x\rangle\right].\end{aligned}$$

Combining these two gives us

$$\phi(F(a)) = \lim_\alpha \frac{1}{n_\alpha} \operatorname{Tr}(p_\alpha a) \qquad \text{for all } a \in \mathcal{R}_{con}(G,X),$$

so we are done. □

One can combine Lemma 4.1.7 with Lemma 4.1.6 to see that any Følner net for a group action may be subnetted to produce a Szegö pair for $\mathcal{R}_{con}(G,X)$.

4.2 Traces

Proposition 4.2.1. *Let G be a group acting on a set X, and let*

$$F : \mathcal{R}_{con}(G,X) \to \ell^\infty(X)$$

be defined as in Proposition 1.3.2. Then for any τ_X-invariant state ϕ on $\ell^\infty(X)$, $\phi \circ F$ becomes a tracial state on $\mathcal{R}_{con}(G,X)$.

Moreover, whenever ψ is a tracial state on $\mathcal{R}_{con}(G,X)$ the function $\psi \circ M_X : \ell^\infty(X) \to \mathbb{C}$ becomes a τ_G-invariant state on $\ell^\infty(X)$

Proof. We divide the proof of the first assertion into two steps.

Tracial: Let

$$S = \sum_{g\in G} f_g \lambda_X(g), T = \sum_{k\in G} h_k \lambda_X(k) \in M_X \times \lambda_X(C_c(G, \ell^\infty(X))).$$

Then

$$\left(\sum_{g\in G} f_g \lambda_X(g)\right)\left(\sum_{k\in G} h_k \lambda_X(k)\right) = \sum_{g\in G}\sum_{k\in G} f_g \tau_X(g)(h_k)\lambda_X(gk),$$

while

$$\left(\sum_{k\in G} h_k\lambda_X(k)\right)\left(\sum_{g\in G} f_g\lambda_X(g)\right) = \sum_{g\in G}\sum_{h\in G}\tau_X(k)(f_g)h_k\lambda_X(kg).$$

So all we need to check is that $\phi(F(f\tau_X(g)(h)\lambda_X(gk))) = \phi(F(\tau_X(k)(f)h\lambda_X(kg)))$ for all $g,k\in G$ and $f,h\in\ell^\infty(X)$. But for $x\in X$ we see that

$$\begin{aligned} F(f\tau_X(g)(h)\lambda_X(gk))(x) &= \langle\delta_x, f\tau_X(g)(h)\lambda_X(gk)\delta_x\rangle \\ &= \sum_{y\in X}\delta_x(y)\overline{f(y)h(g^{-1}y)(\lambda_X(gk)(\delta_x))(y)} \\ &= \sum_{y\in X}\delta_x(y)\overline{f(y)h(g^{-1}y)\delta_{(gk)x}(y)} \\ &= \overline{f(x)h(g^{-1}x)}\delta_{(gk)x}(x). \end{aligned}$$

Likewise, we see that:

$$F(\tau_X(k)(f)h\lambda_X(kg))(x) = \overline{f(k^{-1}x)h(x)}\delta_{(kg)x}(x).$$

Notice that

$$\overline{f(k^{-1}x)h(x)}\delta_{(gk)x}(x) = \begin{cases}\overline{f(k^{-1}x)h(x)} & x=(kg)x\\ 0 & \text{otherwise,}\end{cases}$$

while

$$\overline{f(x)h(g^{-1}x)}\delta_{(kg)x}(x) = \begin{cases}\overline{f(x)h(g^{-1}x)} & x=(gk)x\\ 0 & \text{otherwise.}\end{cases}$$

From this we see that

$$\begin{aligned} \tau_X(k)\left(F(\tau_X(k)(f)h\lambda_X(kg))\right)(x) &= \begin{cases}\overline{f(k^{-1}x)h(g^{-1}(k^{-1}x))} & k^{-1}x=(gk)(k^{-1}x)\\ 0 & \text{otherwise.}\end{cases} \\ &= \begin{cases}\overline{f(k^{-1}x)h(x)} & x=(kg)x\\ 0 & \text{otherwise.}\end{cases} \\ &= F(f\tau_X(g)(h)\lambda_X(gk))(x) \end{aligned}$$

Combining this we get

$$\begin{aligned} \phi\left[F\left(TS\right)\right] &= \sum_{g,k\in G}\phi(F(f_g\tau_X(g)(h_k)\lambda_X(gk))) \\ &= \sum_{g,k\in G}\phi(\tau_X(k)(F(f_g\tau_X(g)(h_k)\lambda_X(gk)))) \\ &= \sum_{g,k\in G}\phi(F(\tau_X(k)(f)h\lambda_X(kg))) \\ &= \phi\left[F\left((\sum_{k\in G}h_k\lambda_X(k))(\sum_{g\in G}f_g\lambda_X(g))\right)\right] \\ &= \phi[F(ST)] \end{aligned}$$

as ϕ is invariant under the action of G.

Now take any two element $\xi, \zeta \in \mathcal{R}_{con}(G, X)$. We may find sequences

$$\{\xi_n\}_{n\in\mathbb{N}} \subset M_X \times \lambda_X(C_c(G, \ell^\infty(X))) \text{ and } \{\zeta_n\}_{n\in\mathbb{N}} \subset M_X \times \lambda_X(C_c(G, \ell^\infty(X)))$$

such that $\xi_n \to \xi$, $\zeta_n \to \zeta$. Then we see

$$\begin{aligned} \phi(F(\xi\zeta)) &= \phi(F((\lim_{n\to\infty} \xi_n)(\lim_{n\to\infty} \zeta_n))) \\ &= \phi(F(\lim_{n\to\infty} \xi_n\zeta_n)) \\ &= \lim_{n\to\infty} \phi(F(\xi_n\zeta_n)) \\ &= \lim_{n\to\infty} \phi(F(\zeta_n\xi_n)) \\ &= \phi(F(\lim_{n\to\infty} \zeta_n\xi_n)) \\ &= \phi(F(\zeta\xi)). \end{aligned}$$

Hence $\phi \circ F$ is tracial.

State: We shall show that $1 = \phi(F(1)) = ||\phi \circ F||$. But $\phi(F(1)) = \phi(1) = ||\phi|| = 1$. As F is a norm one projection, we get

$$\begin{aligned} ||\phi \circ F|| &= \sup \big\{ |(\phi \circ F)(\xi)| \mid \xi \in \mathcal{R}_{con}(G, X), ||\xi|| = 1 \big\} \\ &= \sup \big\{ |\phi(f)| \mid f \in \ell^\infty(X), ||f|| = 1 \big\} \\ &= ||\phi|| = 1 \end{aligned}$$

Hence $\phi \circ F$ is a state.

To show the last assertion, assume ψ is a tracial state on $\mathcal{R}_{con}(G, X)$. Then $\psi \circ M_X$ becomes a state on $\ell^\infty(X)$ since

$$||\psi \circ M_X|| \geq \psi \circ M_X(1) = \psi(1) = ||\psi|| = 1 \geq ||\psi \circ M_X||.$$

So $1 = \psi \circ M_X(1) = ||\psi \circ M_X||$.

Finally, we use the fact that $(M_X, \lambda_X, \ell^2(X))$ form a covariant representation of $(\ell^\infty(X), G, \tau_X)$ to show that $\psi \circ M_X$ is G-invariant. Let $g \in G$ and $f \in \ell^\infty(X)$, then

$$\begin{aligned} \psi \circ M_X(\tau_X(g)f) &= \psi(\lambda_X(g) M_X(f) \lambda_X(g)^*) \\ &= \psi(\lambda_X(g)\lambda_X(g)^* M_X(f)) \\ &= \lambda(M_X(f)). \end{aligned}$$

So $\psi \circ M_X$ is indeed G-invariant. □

4.2.1 Paradoxicality, traces and properly infinite projections

We conclude this chapter with a nice theorem reviewing the role of $\mathcal{R}_{con}(G, X)$ as a measure of finiteness. This will be made precise when we are ready to state our theorem.

First we need to review the basic concept of a paradoxical decomposition. For a full treatment of the subject, see e.g. [20] or [19].

Definition 4.2.2. Let G act on a set X. We say that $E \subset X$ is *G-paradoxical* if there exists pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m$ of E and elements $g_1, \ldots, g_n, h_1, \ldots, h_m$ in G such that

$$E = \bigcup_{i=1}^{n} g_i A_i \qquad \text{and} \qquad E = \bigcup_{i=1}^{m} h_i B_i.$$

By Tarski's Theorem (see [20]) we know that E is G-paradoxical if and only if there is a G-invariant finite additive set function $\mu : \mathfrak{P}(X) \to [0, \infty]$ such that $\mu(E) \in (0, \infty)$.

Before we can state our theorem, we need a little definition found in [17].

Definition 4.2.3. Let A be a C*-algebra. We say that a projection $p \in A$ is *properly infinite* if there exist partial isometries x and y in A such that

$$x^*x = y^*y = p \qquad \text{and} \qquad xx^* + yy^* \leq x.$$

Theorem 4.2.4. *Let G be a discrete group acting on a set X. Then the following are equivalent.*

1. *There is a Følner net for the action of G on X;*
2. *$\ell^\infty(X)$ has a G-invariant state;*
3. *$\mathcal{R}_{con}(G, X)$ has a tracial state;*
4. *X is not G-paradoxical;*
5. *1_X is not properly infinite in $\mathcal{R}_{con}(G, X)$.*

Remark 4.2.5. The implications 3)$\Leftrightarrow$ 4) $\Leftrightarrow$ 5) is known for $\mathcal{R}_r(G, X)$ in a more general setting, see [17, Proposition 5.5]. The interesting case is thus when G does not act freely on X.

Proof. The implications 1) $\Rightarrow$ 2) $\Leftrightarrow$ 3) is clear from Lemma 4.1.6 and Proposition 4.2.1. The direction 2)$\Rightarrow$ 1) is given by [18, Theorem 2.3]. 4)$\Leftrightarrow$ 2) is a consequence of Tarski's theorem.

To see that 3) $\Rightarrow$ 5), assume for contradiction that 1_X is properly infinite in $\mathcal{R}_{con}(G, X)$ and $\mathcal{R}_{con}(G, X)$ has a tracial state ϕ. Pick x and y according to Definition 4.2.3, then

$$0 \leq \phi(1_X - xx^* - yy^*) = \phi(1_X) - \phi(xx^*) - \phi(yy^*) = 1 - \phi(x^*x) - \phi(y^*y) = -1.$$

So we get a contradiction.

The direction 5)$\Rightarrow$ 4) is also proved by contradiction. Assume 1_X is not properly infinite, but X is G paradoxical. Then [17, Proposition 4.3] asserts that we may find $x, y \in C_c(G, \ell^\infty(X)^+)$ satisfying

$$x^*x = y^*y = 1_X \text{ and } xx^* + yy^* \leq 1_X.$$

So 1_X is properly infinite in $\mathcal{R}_{con}(G, X)$, a contradiction. □

Chapter 5

Almost periodic functions

The concept of almost periodic functions was first introduced by Harald Bohr, and the early formulations were something like that found in [6]:

> "Almost periodicity of a function $f(x)$ in general is defined by this property:
>
> The equation
> $$f(x+\tau) = f(x)$$
> is satisfied with an arbitrary degree of accuracy by infinitely many values of τ, these values being spread over the whole range from $-\infty$ to $+\infty$ in such a way as not to leave empty intervals of arbitrarily length"

Unfortunately, the definition does not make sense for an arbitrary topological group G, but there is a useful generalization which is found in for instance [11].

Definition 5.0.6. Let G be a group. We say that $f \in \ell^\infty(G)$ is almost periodic if the set

$$\mathrm{Hull}(f) = \overline{\{\tau_G(g)f \mid g \in G\}}$$

is compact in $\ell^\infty(G)$ with respect to $||\cdot||_{\ell^\infty(G)}$.

We shall denote the set of almost periodic functions in $\ell^\infty(G)$ by $\mathcal{AP}(G)$. Whenever G is a topological group with a topology $\mathcal{T}$, we denote by $\mathcal{AP}(G,\mathcal{T})$ the almost periodic continuous functions on G.

For a group G and an element $g \in G$ we define $\Delta(g) : \ell^\infty(G) \to \ell^\infty(G \times G)$ by

$$[\Delta(g)(f)]\,(x,y) = f(xgy) \qquad x,y,g \in G,\ f \in \ell^\infty(G).$$

Furthermore we define $\rho_G : G \to \mathrm{Aut}(\ell^\infty(G))$ by

$$\rho_G(g)f(x) = f(xg) \qquad g,x \in G.$$

The following properties are essentially well known.

Proposition 5.0.7. *Let G be a group, and $f \in \ell^\infty(G)$ The following are equivalent:*

1. *f is almost periodic,*

2. *$\overline{\{\rho_G(g)f \mid g \in G\}}$ is compact in $\ell^\infty(G)$;*

3. *$\overline{\{\Delta(g)f \mid g, h \in G\}}$ is compact in $\ell^\infty(G \times G)$;*

Proof. See for instance [11, Theorem 18.1]. □

For our purposes the following is rather important.

Theorem 5.0.8. *Let G be a group. Then $\mathcal{AP}(G)$ becomes a τ_G-invariant unital* C**-subalgebra of $\ell^\infty(G)$.*

Proof. This is just [11, Theorem 18.3] translated into C*-language. □

An important consequence of the above theorem is that we get an action of G on $\mathcal{AP}(G)$ by restricting the action $\tau_G : G \to \mathrm{Aut}(\ell^\infty(G))$ to $\mathcal{AP}(G)$. Hence we are able to form the reduced crossed product $\mathcal{AP}(G) \rtimes_{\tau_G, r} G$.

A surprisingly useful result about almost periodic functions is the following theorem found in [14].

Theorem 5.0.9. *Let G be a discrete group and $f \in \mathcal{AP}(G)$. Then the convex hull generated by $\{\tau_G(g)f \mid g \in G\}$ contains exactly one constant function.*

Proof. See [14, page 169]. □

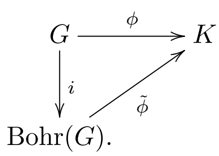
Remark 5.0.10. An important consequence of the above theorem is that whenever $f \in \mathcal{AP}(G)$ is a non-zero function, the τ_G-invariant C*-algebra generated by f must contain the constant functions.

5.1 The Bohr compactification

The main results here are essentially well-known. We will give the abstract definition of the Bohr compactification, then show in a later section that it does in fact exist.

Before we begin with the main theory of this section, we need to make a small remark. Whenever G is a topological group (not necessarily discrete), we get a continuous left action $\tau_G : C(G) \to C(G)$ given by $\tau_G(f)(x) = f(g^{-1}x)$ for $g, x \in G$.

Definition 5.1.1. Let G be a discrete group. The *Bohr compactification* of G, denoted $\mathrm{Bohr}(G)$, is a compact Hausdorff group $\mathrm{Bohr}(G)$ with a continuous homomorphism $i : G \to \mathrm{Bohr}(G)$ such that $i(G)$ is dense in $\mathrm{Bohr}(G)$ and such that for all continuous group homomorphisms $\phi : G \to K$ where K is a compact Hausdorff group, there is a continuous homormophism $\tilde{\phi} : \mathrm{Bohr}(G) \to K$ making the following diagram commute



Bohr(G) can be constructed in a rather easy manner using Tychonoff's theorem and the Peter-Weyl theorem, though we are actually going to construct the Bohr compactification in a less obvious way, so we will hold of any proof of existence as of now. Rather, we will use the remainder of this section to show some nice properties of the Bohr compactification.

Using the definition, it is easy to see that the Bohr compactification of a group must be unique up to unique isomorphism.

Before we state our next proposition, we need a little technical lemma.

Lemma 5.1.2. *Let G be a topological group. Suppose there is a continuous homomorphism $i : G \to K$ for some compact Hausdorff group K. If $f \in C(K)$, then $f \circ i \in \mathcal{AP}(G)$.*

Proof. Consider the map $\psi : C(K) \to \ell^\infty(G)$ given by $\psi(f) = f \circ i$. Note that ψ is continuous with respect to the uniform topology on $C(K)$ and $\ell^\infty(G)$.

Suppose $f \in C(K)$, and consider the set of translates of $\psi(f)$,

$$\begin{aligned}\{\tau_G(g)\psi(f) \mid g \in G\} &= \{\psi(\tau_K(i(g))f) \mid g \in G\} \\ &\subset \psi(\{\tau_K(x)f \mid x \in K\})\end{aligned}$$

Define $\Gamma : K \to \{\tau_K(x)f \mid x \in K\}$ by

$$\Gamma(k) = \tau_K(k)f \qquad k \in K.$$

Then Γ is continuous (when we equip $\{\tau_K(x)f \mid x \in K\}$ with the topology it inherits from the uniform norm). We deduce that

$$\{\tau_G(g)\psi(f) \mid g \in G\} \subset \Gamma(K).$$

But as K is compact and Γ and ψ are continuous, we know that $\psi(\Gamma(K))$ is compact, and being a subset of a Hausdorff space, we know that $\phi(\Gamma(K))$ is closed. As $\psi(\Gamma(K))$ is closed, we see that

$$\mathrm{Hull}(f) \subset \psi(\Gamma(K)).$$

Hence Hull(f) is compact. □

The next proposition is well-known, see for instance [10, Thérème 16.2.1]

Proposition 5.1.3. *Let G be a discrete group. Then there is a $*$-isomorphism $\psi : C(\mathrm{Bohr}(G)) \to \mathcal{AP}(G)$ given by*

$$\psi(f)(x) = f(i(x)) \qquad f \in C(\mathrm{Bohr}(G)), x \in G.$$

Proof. We divide the proof into several steps.

Well-defined: This is just 5.1.2.

Injectivity: Let $f_1, f_2 \in C(\mathrm{Bohr}(G))$ where $f_1 \neq f_2$. Then we may pick an open set $U \subset \mathrm{Bohr}(G))$ such that $0 \notin (f_1 - f_2)(U)$, but since $i(G)$ is dense in Bohr(G), $U \cap i(G) \neq \emptyset$ so $f_1 \circ i \neq f_2 \circ i$.

Surjectivity: Let $f \in \mathcal{AP}(G)$. Then we may consider the group

$$\left\{\tau_G(g)|_{\mathrm{Hull}(f)} \mid g \in G\right\} \subset \mathrm{Iso}(\mathrm{Hull}(f)),$$

where Iso(Hull(f)) is the set of isometries from Hull(f) to Hull(f). Since Hull(f) is a compact metric space, Iso(Hull(f)) becomes compact Hausdorff space, hence we may form the completion $F = \overline{\{\tau_G(g)|_{\text{Hull}(f)} \mid g \in G\}}$, which is then a compact Hausdorff group. Now, consider the continuous group homomorphism $r : G \to F$ given by

$$r(g) = \tau_G(g)|_{\text{Hull}(f)} \qquad g \in G.$$

By the universal property of the Bohr compactification, we may pick $\tilde{r} : \text{Bohr}(G) \to F$ such that $\tilde{r} \circ i = r$.

We may consider the continuous map $q : F \to \mathbb{C}$ given by $q(S) = S(f)(e_G)$ for $S \in F$, hence we may define $\tilde{f} \in C(\text{Bohr}(G))$ by $\tilde{f}(x) = q(\tilde{r}(x^{-1}))$ for $x \in \text{Bohr}(G)$. Then we see that for $g \in G$

$$\tilde{f}(i(g)) = q(\tilde{r}(i(g)^{-1})) = q(r(g^{-1})) = \tau_G(g^{-1})(f)(e_G) = f(g)$$

whence $\psi(\tilde{f}) = f$. □

Remark 5.1.4. The above proposition actually completely characterizes the Bohr compactification of a discrete group, as we will see later in Theorem 5.2.20.

We will be able construct a "compactification" of G for smaller subalgebras of $\mathcal{AP}(G)$. These compactifications will replace Bohr(G) in the sense of the above proposition. This will be made clear in the next section.

5.2 The hull

For a discrete group G we wish to solve the following problem: Given a set $F \subset \mathcal{AP}(G)$, we wish to give a description a C*-algebra which is the smallest C*-algebra containing F and all of it translates under G. This leads us to the construction of the hull.

5.2.1 A motivating example

The following example is rather simple, but carries with it a lot of the motivation for what we are going to do later.

Example 5.2.1. Consider the group $\mathbb{Z}$ acting on itself by left translation. Define the function $f : \mathbb{Z} \to \mathbb{C}$ given by

$$f(n) = e^{ni\alpha} \qquad \alpha \in \mathbb{R},\ n \in \mathbb{Z}.$$

First we will show that

$$\text{Hull}(f) \simeq \overline{f(\mathbb{Z}))} \subset S_1,$$

and conclude that f is almost periodic (as $\overline{f(\mathbb{Z})}$ is compact). Define the map $\psi : \text{Hull}(f) \to \overline{f(\mathbb{Z})}$ by $\psi(\xi) = \xi(0)$ for $\xi \in \text{Hull}(f)$. We will show that ψ is an isometry, hence a homeomorphism (as we get both injectivity and surjectivity from the isometry property).

Whenever $n, x \in \mathbb{Z}$ we have the equality

$$\tau_{\mathbb{Z}}(n)f(x) = e^{(x-n)i\alpha} = e^{xi\alpha}e^{-ni\alpha} = f(x)\tau(n)f(0).$$

So as a result we have

$$||\tau_{\mathbb{Z}}(n)f|| = ||f||\,|\tau_{\mathbb{Z}}(n)f(0)| = |\tau_{\mathbb{Z}}(n)f(0)|.$$

In other words

$$||\tau_{\mathbb{Z}}(n)f|| = |\psi(\tau_{\mathbb{Z}}(n)f(0))|.$$

Now take any $\psi \in \mathrm{Hull}(f)$ and pick a sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{Z}$ such that

$$\tau_{\mathbb{Z}}(n_k)f \to \xi.$$

Owing to the above calculations, we have

$$||\xi|| = \lim_{k\to\infty} ||\tau_{\mathbb{Z}}(n_k)f|| = \lim_{k\to\infty} |\tau_{\mathbb{Z}}(n_k)f(0)| = |\xi(0)|.$$

As a result we have $||\xi|| = |\xi(0)| = |\psi(\xi)|$. So we get an isometry. Hence $\mathrm{Hull}(f) \simeq \overline{f(\mathbb{Z})}$.

Trivially,

$$\mathrm{Sp}(f) = \overline{f(\mathbb{Z})}.$$

Hence by the Gelfand theorem we have a $*$-isomorphism

$$\mathrm{C}^*(f) \simeq C(\overline{f(\mathbb{Z})}).$$

But for $n \in \mathbb{Z}$, we have

$$(\tau_{\mathbb{Z}}(n)f(0)f)(k) = e^{(-n+k)i\alpha} = \tau_{\mathbb{Z}}(n)f(k) \qquad k \in \mathbb{Z}.$$

And since $\tau_{\mathbb{Z}}(n)f(0)f \in \mathrm{C}^*(f)$, we must have $\tau_{\mathbb{Z}}(n)f \in \mathrm{C}^*(f)$. In other words, we must have that the smallest C*-algebra containing $\tau_{\mathbb{Z}}(n)f$ for all $n \in \mathbb{Z}$ is (*-isomorphic to) $C(\overline{f(\mathbb{Z})})$ (since $\mathrm{Hull}(f)$ obviously is dense in $\overline{f(\mathbb{Z})}$). Hence

$$\mathrm{C}^*(\{\tau_{\mathbb{Z}}(n)f \mid n \in \mathbb{Z}\}) \simeq C(\overline{f(\mathbb{Z})}) \simeq C(\mathrm{Hull}(f)).$$

5.2.2 The construction

Inspired by [5, Section 2.3] we make the following definition.

Definition 5.2.2. Let G be a discrete group. Let $F \subset \ell^\infty(G)$ and define

$$S(F) = \{\{g_i\}_i \subset G \mid \{g_i\}_i \text{ is a net such that } \Delta(g_i)(f) \text{ converges in } \ell^\infty(G\times G) \text{ for all } f \in F\}. \tag{5.1}$$

Define a relation $\sim_F$ on $S(F)$ by

$$\{g_i\}_{i\in I} \sim_F \{h_j\}_{j\in J} \Leftrightarrow \lim_{(i,j)} ||\Delta(g_i)(f) - \Delta(h_j)(f)|| = 0 \text{ for all } f \in F,$$

where $I \times J$ is ordered by the product order, that is $(i,j) \le (i',j')$ if and only if $i \le i'$ and $j \le j'$.

Remark 5.2.3. The observant reader might notice that the condition (5.1) is a tad stronger than the one found in [5], but the two are equivalent in the Abelian case.

A slight generalization of [5, Proposition 1] yields the following result.

Proposition 5.2.4. *Let G be a discrete group. The set $S(F)$ becomes a semigroup under the multiplication*

$$\{g_i\} \cdot \{h_j\} = \{g_i h_j\}_{(i,j)} \qquad \{g_i\}, \{h_j\} \in S(F).$$

And the relation $\sim_F$ is an equivalence relation that respects the operation on $S(F)$. Moreover $S(F)/\sim_F$ becomes a group with inverses given by componentwise inversion, identity element being the equivalence class of the constant sequence $g_n = e_G$ for all $n \in \mathbb{N}$.

Proof. First we wish to show that whenever $\{g_i\}_i$ and $\{h_j\}_j$ are elements of $S(F)$, then $\{g_i h_j\}_{i,j}$ is an element of $S(F)$. So pick an arbitrary $f \in F$ and let $\epsilon > 0$. As the nets $\{\Delta(g_i)f\}_i$ and $\{\Delta(h_j)f\}_j$ are Cauchy in $\ell^\infty(G)$, we may pick (i_0, j_0) such that for all $(i,j) \geq (i_0, j_0)$ and $(i', j') \geq (i_0, j_0)$ we have

$$||\Delta(g_i)f - \Delta(g_{i'})f|| < \epsilon \text{ and } ||\Delta(h_j)f - \Delta(h_{j'})f|| < \epsilon.$$

Then we have

$$||\Delta(g_i h_j)f - \Delta(g_{i'} h_{j'})f|| \leq ||\Delta(g_i h_j)f - \Delta(g_{i'} h_j)f|| + ||\Delta(g_{i'} h_j)f - \Delta(g_{i'} h_{j'})f||.$$

Looking at each of the terms on the right hand side, we have for $x, y \in G$

$$|f(xg_i h_j y) - f(xg_{i'} h_j y)| = |f(xg_i(h_j y)) - f(xg_{i'}(h_j y))| < \epsilon,$$

as $||\Delta(g_i)f - \Delta(g_{i'})f|| < \epsilon$. And likewise, as $||\Delta(h_j)f - \Delta(h_{j'})f|| < \epsilon$ we obtain

$$|f(xg_{i'} h_j y) - f(xg_{i'} h_{j'} y)| = |f((xg_{i'})h_j y) - f((xg_{i'})h_{j'} y| < \epsilon.$$

Thus we get for $x, y \in G$ and (i,j), (i',j') as above that

$$|\left[\Delta(g_i h_j)(f)\right](x,y) - \left[\Delta(g_{i'} h_{j'})(f)\right](x,y)| \leq |\left[\Delta(g_i h_i)(f)\right](x,y) - \left[\Delta(g_{i'} h_j)(f)\right](x,y)| + |\left[\Delta(g_{i'} h_j)(f)\right](x,y) - \left[\Delta(g_{i'} h_{j'})(f)\right](x,y)| < 2\epsilon,$$

and hence

$$||\Delta(g_i h_j)(f) - \Delta(g_{i'} h_{j'})(f)|| < 2\epsilon.$$

As the net $\{\Delta(g_i h_j)f\}_{i,j}$ is Cauchy, and $\ell^\infty(G \times G)$ is complete, it converges in $\ell^\infty(G \times G)$. As f was arbitrary, we see that $\{g_i h_j\}_{i,j} \in S(F)$.

Now, it is easy to see that the multiplication is associative, so we see that we get a semigroup.

Before we show the last statement of the proposition, we show an a little remark. Now for $\{g_i\}_i \in S(F)$ we wish to show that $\{g_i^{-1}\}_i \in S(F)$. Pick i_0 such that $||\Delta(g_i)f - \Delta(g_j)f|| < \epsilon$ for all $i, j \geq i_0$. Let the map $\beta : G \times G \to \mathrm{Aut}(\ell^\infty(G \times G))$ be defined as

$$\left[\beta(x,y)H\right](g,h) = H(gx, yh),$$

for $x, y, g, h \in G$, $H \in \ell^\infty(G \times G)$. We readily see that for each $x, y \in G$ the map $\beta(x,y)$ is a $*$-automorphism with $\beta(x,y)^{-1} = \beta(x^{-1}, y^{-1}))$. Hence we get

$$||\Delta(g_i^{-1})f - \Delta(g_j^{-1})f|| = ||\beta(g_i, g_j)(\Delta(g_i^{-1})f - \Delta(g_j^{-1})f)|| = ||\Delta(g_j)f - \Delta(g_i)f|| < \epsilon.$$

The last equality follows as

$$\begin{aligned}\left[\beta(g_i, g_j)\left(\Delta(g_i^{-1})f - \Delta(g_j^{-1})f\right)\right](x,y) &= f(xg_ig_i^{-1}g_jy) - f(xg_ig_j^{-1}g_jy)\\ &= f(xg_jy) - f(xg_iy)\\ &= [\Delta(g_j)(f)](x,y) - [\Delta(g_i)(f)](x,y)\end{aligned}$$

for $x, y \in G$.

The relation $\sim_F$ is obviously reflexive and symmetric, and an easy application of the triangle inequality shows that it is also transitive, hence an equivalence relation.

Now suppose $\{g_i\}_i$, $\{g'_j\}_j$, $\{h_\lambda\}_\lambda$ and $\{h'_\nu\}_\nu$ are elements of $S(F)$ such that

$$\{g_i\}_i \sim_F \{g'_j\}_j \qquad \text{and} \qquad \{h_\lambda\}_\lambda \sim_F \{h'_\nu\}_\nu.$$

We want to show that $\{g_ih_\lambda\}_{i,\lambda} \sim_F \{g'_jh'_\nu\}_{j,\nu}$. So let $\epsilon > 0$, and choose $(i_0, j_0, \lambda_0, \nu_0)$ such that for $(i, j, \lambda, \nu) \geq (i_0, j_0, \lambda_0, \nu_0)$ we have

$$||\Delta(g_i)f - \Delta(g'_j)f|| < \epsilon/2 \qquad \text{and} \qquad ||\Delta(h_\lambda)f - \Delta(h'_\nu)f|| < \epsilon/2.$$

Then we see that for (i, j, λ, ν) as above we have

$$||\Delta(g_ih_\lambda)f - \Delta(g'_jh'_\nu)f|| \leq ||\Delta(g_ih_\lambda)f - \Delta(g'_jh_\lambda)f|| + ||\Delta(g'_jh_\lambda)f - \Delta(g'_jh'_\nu)f||.$$

Moreover for $x, y \in G$ we have

$$|\left[\Delta(g_ih_\lambda)f\right](x,y) - \left[\Delta(g'_jh_\lambda)f\right](x,y)| = |f(xg_ih_\lambda y) - f(xg'_ih_\lambda y)| < \epsilon/2$$

hence $||\Delta(g_ih_\lambda)f - \Delta(g'_jh_\lambda)f|| \leq \epsilon/2$.

Similarly we obtain $||\Delta(g'_jh_\lambda)f - \Delta(g'_jh'_\nu)f|| \leq \epsilon/2$. Thus we have

$$\begin{aligned}||\Delta(g_ih_\lambda)f - \Delta(g'_jh'_\nu)f|| &\leq ||\Delta(g_ih_\lambda)f - \Delta(g'_jh_\lambda)f|| + ||\Delta(g'_jh_\lambda)f - \Delta(g'_jh'_\nu)f||\\ &\leq \epsilon/2 + \epsilon/2\\ &= \epsilon.\end{aligned}$$

So $\lim_{(i,j,\lambda,\nu)} ||\Delta(g_ih_\lambda)f - \Delta(g'_jh'_\nu)f|| = 0$, which implies $\{g_ih_\lambda\}_{i,\lambda} \sim_F \{g'_jh'_\nu\}_{j,\lambda}$. Hence the relation $\sim_F$ respects the semigroup operation. Furthermore we see that whenever $\{g_i\}_i \in S(F)$, we have $\{g_i^{-1}\}_i \in S(F)$, and $\{g_ig_i^{-1}\} = \{g_i^{-1}g_i\} = \{e_G\}_i \sim_F \{e_G\}_{n\in\mathbb{N}}$. So we see that the constant sequence $g_n = e_G$ becomes the identity element in $S(F)/\sim_F$. □

Remark 5.2.5. There is a subtle point in the above proposition. As we are working with arbitrary nets in G, we will not get an identity element of $S(F)$, as different index sets of elements would give different results for the multiplication of an element with its inverse. So we do in fact need to divide out by the equivalence relation to get a group.

We shall denote the group $S(F)/\sim_F$ constructed in Proposition 5.2.4 by Ω_F. For an element $\{g_i\}_i \in S(F)$, we shall denote by (g_i) its image in Ω_F.

For each $f \in F$ we get a pseudometric, d_f on Ω_F defined as

$$d_f((g_i), (h_j)) = \lim_{i,j} ||\Delta(g_i)f - \Delta(h_j)f|| \qquad (g_i), (h_j) \in \Omega_F.$$

It is easy to see that this is well-defined with respect to $\sim_F$ using the continuity of the $||\cdot||_{\ell^\infty(G)}$-norm.

We shall equip Ω_F with the topology with neighborhood basis at $x \in \Omega_F$ given by the sets

$$U_{f_1,\ldots,f_n}(\epsilon, x) = \{y \in \Omega_F \mid d_{f_i}(x,y) < \epsilon \text{ for } i = 1,\ldots,n\}$$

for $f_1,\ldots,f_n \in F$ and $\epsilon > 0$.

This also respects the group structure, as the following lemma shows.

Lemma 5.2.6. *Let G be a discrete group, $F \subset \ell^\infty(G)$. Then Ω_F is a topological Hausdorff group.*

Proof. We want to show that the multiplication and inversion operations are continuous.

Multiplication: Let $\{x_\iota\}, \{y_\lambda\} \in S(F)$, and let $U_{f_1,\ldots,f_m}(\epsilon, (x_\iota y_\lambda))$ be a neighborhood around $(x_\iota y_\lambda)$ for $f_1,\ldots,f_m \in F$ and $\epsilon > 0$. We see that whenever $(x'_j) \in U_{f_1,\ldots,f_m}(\epsilon/4, (x_\iota))$ and $(y'_\nu) \in U_{f_1,\ldots,f_m}(\epsilon/4, (y_\lambda))$ we have for $i = 1,\ldots,m$

$$d_{f_i}((x'_j)(y'_\nu), (x_\iota y_\lambda)) = \lim_{\iota,j,\nu,\lambda} ||\Delta(x'_j y'_\nu) f_i - \Delta(x_\iota y_\lambda) f_i||$$

and for (ι, j, λ, ν)

$$\begin{aligned} ||\Delta(x'_j y'_\nu) f_i - \Delta(x_\iota y_\lambda) f_i|| &= \sup_{a,b \in G} |f_i(a x'_j y'_\nu b) - f_i(a x_\iota y_\lambda b)| \\ &\le \sup_{a,b\in G} |f_i(a x'_j y'_\nu b) - f_i(a x'_j y_\lambda b)| + \sup_{a,b\in G} |f_i(a x'_j y_\lambda b) - f_i(a x_\iota y_\lambda b)| < \epsilon/2, \end{aligned}$$

so $d_{f_i}((x'_j)(y'_\nu), (x_\iota)(y_\lambda)) < \epsilon$ and hence $(x'_j)(y'_\nu) \in U_{f_1,\ldots,f_m}(\epsilon, (x_\iota y_\lambda))$.

Inversion: Let $\{x_\iota\}_\iota \in S(F)$, and pick a neighborhood $U_{f_1,\ldots,f_m}(\epsilon, (x_\iota)^{-1})$ for $f_1,\ldots,f_m \in F$ and $\epsilon > 0$. Then whenever $(x'_j) \in U_{f_1,\ldots,f_m}(\epsilon, (x_\iota)^{-1})$ we have for $i = 1,\ldots,m$

$$\begin{aligned} d_{f_i}((x'_j)^{-1}, (x_\iota)^{-1}) &= \lim_{\iota,j} ||\Delta(x'_j) f_i - \Delta(x_\iota) f_i|| \\ &= \lim_{\iota,j} \sup_{a,b\in G} |f_i(a {x'_j}^{-1} b) - f_i(a x_\iota^{-1} b)| \\ &= \lim_{\iota,j} \sup_{a,b\in G} |f(a x'_j {x'_j}^{-1} x_\iota b) - f(a {x'_j}^{-1} x_\iota^{-1} x_\iota b)| \\ &= \lim_{\iota,j} \sup_{a,b\in G} |f(a x_\iota b) - f(a x'_j b)| < \epsilon \end{aligned}$$

hence $(x'_j)^{-1} \in U_{f_1,\ldots,f_m}(\epsilon, (x_\iota)^{-1})$.

Hausdorff: If $x \neq y$ in Ω_F, then there exists $f \in F$ such that $\epsilon = d_f(x,y) > 0$. Hence $U_f(\epsilon/2, x) \cap U_f(\epsilon/2, y) = \emptyset$ and we have found two open sets separating x and y. □

We also note that we have an inclusion of G into $S(F)$ given by $g \mapsto \{g_n\}_{n\in\mathbb{N}}$ where $g_n = g$ for all $n \in \mathbb{N}$. Thus we also get a homomorphism $i_F : G \to \Omega_F$ induced by this inclusion.

The following is a collection of useful lemmas.

Lemma 5.2.7. *Let G be a discrete group and F a subset of $\ell^\infty(G)$. Then the set $i_F(G)$ is dense in Ω_F.*

Proof. Let $\{g_\iota\}_\iota \in S(F)$, $f_1, \dots, f_m \in F$, and set $h_i = \lim_\iota \Delta(g_\iota)f_i$. Choose ι_0 such that $||\Delta(g_\iota f - h_i|| < \epsilon$ for all $\iota \geq \iota_0$ and $i \in \{1, \dots m\}$. Then

$$\lim_\iota ||\Delta(g_\iota)f_i - \Delta(g_{\iota_0})f_i|| \leq \lim_\iota ||\Delta(g_\iota)f_i - h_i|| + \lim_\iota ||h_i - \Delta(g_{\iota_0})f_i|| < \epsilon.$$

Hence $i_F(g_{\iota_0}) \in U_{f_1,\dots,f_m}(\epsilon, (g_\iota))$. □

Lemma 5.2.8. *Let G be a discrete group, and let $F \subset F' \subset \mathcal{AP}(G)$. Then there is a continuous surjective homomorphism*

$$\rho_{F,F'} : \Omega_{F'} \to \Omega_F$$

given by

$$\rho((g_i)) = (g_i) \qquad \textit{for } (g_i) \in \Omega_{F'}.$$

Proof. First we must show that the map is well-defined. Let $\{g_i\}_i \in S(F')$. Since $\{\Delta(g_i)f\}_i$ converges for all $f \in F'$, it specifically converges for all $f \in F$, hence we get a group homomorphism $S(F') \to S(F)$. And it is easy to see that whenever $\{g_i\}_i \sim_{F'} \{g'_j\}_j$, then $\{g_i\}_i \sim_F \{g'_j\}_j$, hence we get a homomorphism

$$\rho : \Omega_{F'} \to \Omega_F,$$

given by $\rho((g_i)) = (g_i)$. It is clear that the map is Lipschitz for each pseudonorm d_f, it is also continuous.

Now, whenever $x \in \Omega_F$ we may pick a net $\{x_i\}_{i\in I} \subset G$ such that $i_F(x_i)$ converges to x in Ω_F. Define

$$\mathcal{D} = \{E \subset F \mid E \text{ finite}\}$$

and order $\mathcal{D}$ by inclusion. For each $E \in \mathcal{D}$ we may pick a subnet $\{x_{i_E(j)}\}_{j\in J_E}$ such that $\{\Delta(x_{i_E(j)})f\}_j$ is convergent for each $f \in E$ (as $\overline{\{\Delta(g)f \mid g \in G\}}$ is compact). Furthermore, for each $E \in \mathcal{D}$ and $i \in I$ we may choose $\mu_E(i)$ such that $i_E(\mu_E(i)) > i$. Order $\mathcal{D} \times I$ with the product order, and define the net $\{\tilde{x}_{E,i}\}_{(E,i)\in\mathcal{D}\times I}$ by

$$\tilde{x}_{E,i} = x_{i_E(\mu_E(i))}.$$

Then $\{\tilde{x}_{E,i}\} \in S(F')$ as for each $f \in F'$ we see that the net $\{\Delta(x_{E,i})f\}$ converges (just pick $(E, i) > (\{f\}, i_0)$ for some i_0), and we have that

$$\rho_{F,F'}((x_{E,i})) = x$$

by construction. □

Lemma 5.2.9. *Let G be a discrete group and $f \in \mathcal{AP}(G)$. Then $\Omega_{\{f\}} \simeq \overline{\{\Delta(g)f \mid g \in G\}}$.*

Proof. Define $r : \Omega_{\{f\}} \to \overline{\{\Delta(g)f \mid g \in G\}}$ by $r((g_i)) = \lim_i \Delta(g_i)f$. Then it is easy to see that r is both injective and surjective by construction. Furthermore

$$\begin{aligned} d_f((g_i),(h_j)) &= \lim_{(i,j)} ||\Delta(g_i)f - \Delta(h_j)f|| \\ &= ||\lim_{(i,j)}(\Delta(g_i)f - \Delta(h_j)f)|| \\ &= ||r((g_i)) - r((h_j))|| \end{aligned}$$

for $(g_i), (h_i) \in \Omega_{\{f\}}$, hence the map is an isometry. Especially it is a homeomorphism. $\square$

Lemma 5.2.10. *Let G be a discrete group and $F \subset \mathcal{AP}(G)$. Then Ω_F is compact.*

Proof. For each f in F we have a continuous homomorphism

$$\rho_{F,\{f\}} : \Omega_F \to \Omega_{\{f\}}.$$

Furthermore, by Lemma 5.2.9 and Lemma 5.0.7 we see that $\Omega_{\{f\}}$ is compact Hausdorff. The product $\prod_{f\in F} \Omega_{\{f\}}$ thus becomes a compact Hausdorff space. The set

$$C = \overline{\left\{ x_g \in \prod_{f\in F} \Omega_{\{f\}} \mid g \in G, x_g(f) = i_{\{f\}}(g) \right\}}$$

is closed in $\prod_{f\in F} \Omega_{\{f\}}$ and hence compact. Define the map $P : \Omega_F \to C$ by $P(x)(f) = \rho_{F,\{f\}}(x)$. Then P is continuous as each $\rho_{F,\{f\}}$ is continuous. Furthermore, we see that P is injective as

$$P(x) = P(y) \text{ if and only if } d_f(x, y) = 0 \text{ for all } f \in F.$$

Furthermore we see that it is surjective, and so $\Omega_F \simeq C$. But C is compact, hence Ω_F is compact. $\square$

5.2.3 A description of the subalgebras of $\mathcal{AP}(G)$

In [8], H. W. Davis defines a neighborhood system which we shall use in this section. For a discrete group G and $F \subset \mathcal{AP}(G)$, we shall set

$$\mathcal{A}_F = \mathrm{C}^*(\{\tau_G(g)f, \rho_G(g)f \mid g \in G\}) \subset \mathcal{AP}(G).$$

Definition 5.2.11. Let G be a discrete group, and $F \subset \mathcal{AP}(G)$. We define the *topology generated by* F, denoted $\mathcal{T}(F)$ to be the topology generated by the basis of open sets of the form

$$V_{f_1,\dots,f_n}(\epsilon, z) = \{x \in G \mid ||\Delta(x)f_i - \Delta(z)f_i|| < \epsilon, i = 1, \dots, n\},$$

for $\epsilon > 0, f_1, \dots, f_n \in F, z \in G$.

Lemma 5.2.12. *Let G be a discrete group and $F \subset \mathcal{AP}(G)$. Then G equipped with the topology $\mathcal{T}(F)$ is a topological group.*

Proof. We will show that multiplication and inversion is continuous.

Multiplication: Let $x, y \in G$. Consider a fundamental neighborhood of xy of the form $V_{f_1,\dots,f_n}(\epsilon, xy)$ for $f_1, \dots, f_n \in F$ and $\epsilon > 0$. Whenever $x' \in V_{f_1,\dots,f_n}(\epsilon/2, x)$ and $y' \in V_{f_1,\dots,f_n}(\epsilon/2, y)$ we have for $j = 1, \dots, n$:

$$||\Delta(x'y')f_j - \Delta(xy)f_j|| \leq ||\Delta(x'y')f_j - \Delta(xy')f_j|| + ||\Delta(xy')f_j - \Delta(xy)f_j|| < \epsilon$$

so $x'y' \in V_{f_1,\dots,f_n}(\epsilon, xy)$.

Inversion: Let $x \in G$ and pick a fundamental neighborhood of x^{-1} of the form $V_{f_1,\dots,f_n}(\epsilon, x^{-1})$ for $f_1, \dots, f_n \in F$ and $\epsilon > 0$. Then whenever $x' \in V_{f_1,\dots,f_n}(\epsilon, x)$ we have for $j = 1, \dots, n$

$$\begin{aligned} ||\Delta(x'^{-1})f_j - \Delta(x^{-1})f_j|| &= \sup_{a,b} |f_j(ax'x'^{-1}xb) - f_j(ax'x^{-1}x)| \\ &= ||\Delta(x)f_j - \Delta(x')f_j|| \\ &< \epsilon \end{aligned}$$

so $x'^{-1} \in V_{f_1,\dots,f_n}(\epsilon, x^{-1})$. □

We then get a useful description of some of the C*-subalgebras of $\mathcal{AP}(G)$.

Proposition 5.2.13. *Let G be a discrete group, and $F \subset \mathcal{AP}(G)$. Then*

$$\mathcal{A}_F = \mathcal{AP}(G, \mathcal{T}(F)).$$

Proof. From [8, Theorem 3.3] we know that $\mathcal{A}_F = \mathcal{AP}(G, \mathcal{T}(\mathcal{A}_F))$. Moreover we readily see that the topology $\mathcal{T}(F)$ makes all $f \in F$ continuous. Since product, sums, limits (in the uniform topology) and left and right translations of continuous functions are continuous, we see that all functions in $\mathcal{A}_F$ are continuous with respect to $\mathcal{T}(F)$, hence

$$\mathcal{AP}(G, \mathcal{T}(\mathcal{A}_F)) = \mathcal{A}_F \subset \mathcal{AP}(G, \mathcal{T}(F))$$

Meanwhile, as $\mathcal{T}(F) \subset \mathcal{T}(\mathcal{A}_F)$ we have

$$\mathcal{AP}(G, \mathcal{T}(\mathcal{A}_F)) \subset \mathcal{AP}(G, \mathcal{T}(F))$$

and we are done. □

We need two small technical lemmas.

Lemma 5.2.14. *Let G be a discrete group and $F \subset \mathcal{AP}(G)$, and equip G with the topology $\mathcal{T}(F)$. Then the map $i_F : G \to \Omega_F$ becomes continuous.*

Proof. We will show that i_F is continuous around e_G. Let $\epsilon > 0$ and pick $f_1, \dots, f_n \in F$. Then we may consider the neighborhood generated by $f_1, \dots, f_n$, namely

$$U_{f_1,\dots,f_n}(\epsilon, e_G) = \{(x_n) \in \Omega_F \mid d_{f_i}((x_n), (e_G)) < \epsilon \text{ for } i = 1, \dots n\}.$$

Whenever $x \in V_{f_1,\dots,f_n}(\epsilon, e_G)$ we see that

$$i_F(x) \in U_{f_1,\dots,f_n}(\epsilon, e_G)$$

hence we are done. □

And now to our main result of this section.

Theorem 5.2.15. *Let G be a discrete group, and $F \subset \mathcal{AP}(G)$. Then the map*

$$\psi : \mathcal{A}_F \to C(\Omega_F)$$

defined by

$$\psi(f) = f \circ i_F \qquad f \in \mathcal{A}_F.$$

is a $*$-isomorphism.

Proof. The map ψ is well-defined as i_F is continuous with respect to $\mathcal{T}(F)$ and $f \circ i_F$ is an almost periodic function by Proposition 5.1.2. Furthermore we see that as the image of i_F is dense in Ω_F, ψ is injective (just use the same argument as in the proof of Proposition 5.1.3).

For surjectivity, pick $\xi \in \mathcal{A}_F$. We wish to define $\tilde{\xi} : \Omega_F \to \mathbb{C}$ such that $\xi(x) = \tilde{\xi}(i_F(x))$. First we will show that the function $\zeta : i_F(G) \to \mathbb{C}$ defined by

$$\zeta(x) = \xi(y) \qquad \text{if } i_F(y) = x \text{ for } y \in G \text{ and } x \in \Omega_F$$

is both well-defined and continuous. It is well-defined as for $y, y' \in G$

$$i_F(y) \sim_F i_F(y')$$

whenever y and y' can not be separated by a fundamental neighborhood in $\mathcal{T}(F)$. More precisely, whenever $f_1, \ldots, f_n \in F$ and $\epsilon > 0$ we have that $y \in V_{f_1,\ldots,f_n}(\epsilon, y')$, hence any continuous function with respect to the topology $\mathcal{T}(F)$ must be have the same value on y and y'.

Moreover we see that it is continuous as

$$x \in V_{f_1,\ldots,f_n}(z,\epsilon) \;\Leftrightarrow\; i_F(x) \in U_{f_1,\ldots,f_n}, i_F(z), \epsilon) \quad z, x \in G, \epsilon > 0.$$

Now, let $x \in \Omega_F$ and pick a net $\{x_i\} \subset G$ such that $\lim_i i_F(x_i) = x$. We want to show that $\{\xi(x_i)\}_i$ converges. For $\epsilon > 0$ we may, as $\mathrm{Hull}(\xi)$ is compact, pick a finite number of elements $g_1, \ldots, g_n \in G$ such that

$$\mathrm{Hull}(\xi) \subset \bigcup_{i=1}^{n} B(\tau_G(g_i^{-1})\xi, \epsilon).$$

Furthermore, we may pick a neighborhood U of e_G in $\mathcal{T}(F)$ such that

$$|\tau_G(g_i^{-1})\xi(z) - \tau_G(g_i^{-1})\xi(e_G)| < \epsilon \qquad i = 1, \ldots, n$$

whenever $z \in U$. Then we may pick i_0 such that $x_i^{-1}x_j \in U$ whenever $i_0 \leq i, j$. We thus see that for $i, j \geq i_0$ we may select $k \in \{1, \ldots, n\}$ such that

$$\tau_G(x_i^{-1})\xi \in B(\tau_G(g_k^{-1})\xi, \epsilon).$$

We then have

$$\begin{aligned} |\xi(x_i) - \xi(x_j)| &= |\tau_G(x_i^{-1})\xi(e_G) - \tau_G(x_i^{-1})\xi(x_i^{-1}x_j)| \\ &\leq |\tau_G(x_i^{-1})\xi(e_G) - \tau_G(g_k^{-1})\xi(e_G)| + |\tau_G(g_k^{-1})\xi(e_G) - \tau_G(g_k^{-1})(x_i^{-1}x_j)| \\ &\qquad + |\tau_G(g_k^{-1})(x_i^{-1}x_j) - \tau_G(x_i^{-1})(x_i^{-1}x_j)| < 3\epsilon. \end{aligned}$$

Hence $\{\xi(x_i)\}_i$ converges in $\mathbb{C}$, so we may define $\tilde{\xi} : \Omega_F \to \mathbb{C}$ by $\tilde{\xi}(x) = \xi(i_F(x_i))$, and hence

$$\psi(\tilde{\xi}) = \xi.$$

□

Remark 5.2.16. Notice that there is an action of G on Ω_F through

$$gx = i_F(g)x \qquad g \in G, x \in \Omega_F.$$

Hence we get an induced action $\tilde{\tau}_F : G \to \mathrm{Aut}(C(\Omega_F))$ by

$$\tilde{\tau}_F(g)f(x) = f(i_F(g)^{-1}x).$$

There is an interesting consequence of the above theorem. We see that whenever we have a chain

$$F_1 \subset F_2 \subset \cdots F_i \subset \cdots \subset \mathcal{AP}(G)$$

of inclusions we get a reverse chain of surjections of compact groups

$$\Omega_{\mathcal{AP}(G)} \twoheadrightarrow \cdots \twoheadrightarrow \Omega_{F_i} \twoheadrightarrow \cdots \twoheadrightarrow \Omega_{F_2} \twoheadrightarrow \Omega_{F_1}.$$

5.2.4 A non-Abelian example

We shall make an example akin to that of Example 5.2.1.

Example 5.2.17. Let G be a discrete group(the interesting case here is when G is non-Abelian), and (u, H) a finite dimensional unitary representation of G. Let $\{e_1, \ldots, e_n\} \subset H$ form an orthonormal basis for H, and define the coefficient functions $c_{i,j} : \mathcal{B}(H) \to \mathbb{C}$ for $i, j = 1, \ldots, n$ by

$$c_{i,j}(T) = \langle Te_i, e_j\rangle \qquad T \in \mathcal{B}(H).$$

Then define $f_{i,j} : G \to \mathbb{C}$ by $f_{i,j} = c_{i,j} \circ u$.

We first show that each $f_{i,j}$ is almost periodic by exhibiting a continuous surjection $U \to \mathrm{Hull}(f)$ for a compact set U. Define U as

$$U = \{u(g)^* \mid g \in G\} \subset \mathcal{U}(H).$$

The closure of U is obviously compact in $\mathcal{U}(H)$ (as it is a closed, bounded subset of a finite dimensional vector space). Furthermore, we may define

$$P : \overline{U} \to \ell^\infty(G)$$

by

$$P(v)(g) = \langle vu(g)e_i, e_j\rangle \qquad v \in U, g \in G.$$

Notice that P is indeed continuous by elementary properties of the inner product. To see that $P(\overline{U}) \subset \mathrm{Hull}(f)$, we calculate for $h, g \in G$ that

$$P(u(g)^*)(h) = \langle u(g)^*u(h)e_i, e_j\rangle = \langle u(g^{-1}h)e_i, e_j\rangle = \tau_G(g)f_{i,j}(h).$$

so $P(U) \subset \mathrm{Hull}(f)$, hence $P(\overline{U}) \subset \mathrm{Hull}(f)$ since P is continuous and $\mathrm{Hull}(f)$ is closed.

To see that P is surjective, take any $\xi \in \mathrm{Hull}(f)$ and pick a sequence $\{g_n\}_{n\in\mathbb{N}} \subset G$ such that $\tau_G(g_n)f \to \xi$. Then, as $\overline{U}$ is compact, we may pick a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ of $\{g_n\}_{n\in\mathbb{N}}$ such that $\{u(g_{n_k})\}_{k\in\mathbb{N}}$ converges in $\overline{U}$. Then for $x \in G$ we have

$$\xi(x) = \lim_{k\to\infty} \langle u(g_{n_k})^*u(x)e_i, e_j\rangle = \lim_{k\to\infty} P(u(g_{n_k}))(x) = P(\lim_{k\to\infty} u(g_{n_k}))(x),$$

so we see that P is surjective. Hence $\mathrm{Hull}(f_{i,j})$ is the image of $\overline{U}$, a compact set, under P, so $\mathrm{Hull}(f_{i,j})$ is compact.

Now, let $F = \{f_{i,j} \mid i,j = 1\dots,n\} \subset \mathcal{AP}(G)$. For each $f_{i,j} \in F$ and $g,h \in G$ we have

$$\begin{aligned}\tau_G(g)f_{i,j}(h) &= \langle u(g)^*u(x)e_i, e_j\rangle \\ &= \langle u(x)e_i, u(g)e_j\rangle \\ &= \langle u(x)e_i, \sum_{k=1}^n \langle u(g)e_j, e_k\rangle e_k\rangle \\ &= \sum_{k=1}^n \langle e_k, u(g)e_j\rangle\langle u(x)e_i, e_k\rangle \\ &= \sum_{k=1}^n \langle e_k, u(g)e_j\rangle f_{i,k}(x)\end{aligned}$$

and likewise we have

$$\rho_G(g)f_{i,j}(h) = \sum_{k=1}^n \langle u(g)e_i, e_k\rangle\overline{\langle u(x)e_k, e_j\rangle} = \sum_{k=1}^n \langle u(x)e_k, e_j\rangle \overline{f}_{k,j}(x).$$

So we see that $\mathrm{C}^*(F)$ contains all the left and right translates of G, so

$$\mathrm{C}^*(F) = \mathcal{A}_F \simeq C(\Omega_F).$$

We may easily compute $\ker i_F$: If $g \in \ker i_F$ then for $i,j = 1,\dots,n$ we produce

$$\langle u(y)u(g)u(x)e_i, e_j\rangle = \langle u(y)u(e_G)u(x)e_i, e_j\rangle = \langle u(x)u(y)e_i, e_j\rangle \qquad x,y \in G.$$

since $\Delta(g)f_{i,j} = \Delta(e_G)f_{i,j}$.

In particular for $y = x = e_G$ this reduces to

$$\langle u(g)e_i, e_j\rangle = \langle e_i, e_j\rangle.$$

Hence $u(g) = I$, so $g \in \ker u$. Conversely, assume $g \in \ker u$, then

$$\langle u(y)u(g)u(x)e_i, e_j\rangle = \langle u(y)u(e_G)u(x)e_i, e_j\rangle$$

for all $x,y \in G$. Hence $g \in \ker i_F$. We thus see that $\ker i_F = \ker u$.

We shall actually come back to this example in the next section.

Remark 5.2.18. The above example actually characterizes the almost periodic functions on G. As shown in [10, Thérème 16.2.1], the almost periodic functions are just the uniform closure of the span of coefficient functions of finite dimensional unitary representations.

5.2.5 The relation to the Bohr compactification

We are actually going to show that the space $\Omega_{\mathcal{AP}(G)}$ becomes the Bohr compactification of G for any discrete group G. But first a little lemma.

Lemma 5.2.19. *Let G be a discrete group, and suppose there is a homomorphism $\phi : G \to K$ such that $\phi(G)$ is dense in K and such that the map $\tilde{\phi} : C(K) \to \mathcal{AP}(G)$ defined by*

$$\tilde{\phi}(f) = f \circ \phi \qquad f \in C(K),$$

is a $*$-isometry. Then for every finite dimensional unitary representation (u, H) of G there is a continuous finite dimensional representation $(\tilde{u}, H)$ of K such that $\tilde{u} \circ \phi = u$.

Proof. Let (u, H) be a finite dimensional unitary representation of G. Let $e_1, \ldots, e_n$ be a orthonormal basis for H. By Example 5.2.17 we know that the coefficient functions $f_{i,j} : G \to \mathbb{C}$ (as defined in Example 5.2.17) are almost periodic. Hence for each $f_{i,j}$ we may pick an extension $\tilde{f}_{i,j} \in C(K)$ such that $\tilde{\phi}(\tilde{f}_{i,j}) = f_{i,j}$. But then we may form the function $\tilde{u} : K \to \mathcal{U}(H)$ by the equation

$$\langle \tilde{u}(k)e_i, e_j \rangle = \tilde{f}_{i,j}(u) \qquad i, j = 1, \ldots, n \text{ and } u \in K.$$

Now, $\tilde{u}$ is well-defined as $\tilde{u}(\psi(g)) \in \mathcal{U}(H)$ for all $g \in G$, and hence by density we see that $\tilde{u}(k) \in \mathcal{U}(H)$ for all $k \in K$. It is a homomorphism as it is a homomorphism on $\psi(G)$, hence we have our desired finite dimensional unitary representation. □

To prove the next theorem, we shall use the idea usually used to construct the Bohr compactification of a group. See for instance [10, Thérème 16.1.1].

Theorem 5.2.20. *Let G, K and ψ be as in Lemma 5.2.19. Then K is the Bohr compactification of G.*

Proof. Let Γ be a compact Hausdorff group, and suppose there is a continuous homomorphism $\phi : G \to \Gamma$. As an immediate consequence of the Peter Weyl theorem, we may pick a collection of finite dimensional representations $\{(u_i, H_i)\}_i$ of Γ such that $\bigcap_i \ker u_i = \{e_\Gamma\}$.

Furthermore, we may consider the continuous homomorphism

$$u : \Gamma \to \prod_i \mathcal{U}(H_i)$$

by

$$u(h)(i) = u_i(h) \qquad h \in \Gamma.$$

As $\ker u = \bigcap_i \ker u_i$, we see that $\Gamma \simeq u(\Gamma)$. Notice also that u becomes a homeomorphism between Γ and C (as it is a continuous bijection from a compact space). Furthermore, we see that $u \circ \phi$ becomes a finite dimensional unitary representation of G, hence we may use Lemma 5.2.19 to produce finite dimensional unitary representation $(\tilde{u}_i, H_i)$ of K. We may again define a continuous homomorphism $\tilde{u} : K \to \prod_i \mathcal{U}(H_i)$ by

$$\tilde{u}(k)(i) = \tilde{u}_i(k) \qquad k \in K.$$

As $\tilde{u}(\psi(g)) = u(\phi(g)) \in C$, we know that $\tilde{u}(K) \subset C$ by density of $\psi(G)$ in K.

At last we define $\tilde{\phi} : K \to \Gamma$ by $\tilde{\phi} = u^{-1} \circ \tilde{u}$, then

$$\tilde{\phi}(\psi(g)) = u^{-1}(\tilde{u}(\psi(g))) = u^{-1}(u(\phi(g))) = \phi(g) \qquad g \in G.$$

So $K \simeq \mathrm{Bohr}(G)$. □

Corollary 5.2.21. *Let G be a discrete group. Then $\Omega_{\mathcal{AP}(G)} \simeq \mathrm{Bohr}(G)$.*

Proof. Easy consequence of Theorem 5.2.20 and Theorem 5.2.15. □

5.2.6 The Abelian case

We started the section with an example involving the Abelian group $\mathbb{Z}$, and we saw the relation between the $\mathbb{Z}$-invariant C*-algebra generated by a character on $\mathbb{Z}$ and the continuous functions on the hull of the character. Later we went on to consider general (possibly non-Abelian) groups, and things got a tad worse. But we see that whenever G is abelian, for $f \in \ell^\infty(G)$ the set $\overline{\{\Delta(g)f \mid g \in G\}}$ isomorphic to Hull(f) through the the map $d : \ell^\infty(G \times G) \to \ell^\infty(G)$ given by $d(H)(x) = H(x, e_G)$ for $H \in \ell^\infty(G \times G)$, so we see that the example can indeed be carried out to full generality. That is, we get the following corollary.

Corollary 5.2.22. *Let G be a discrete Abelian group. If $f \in \mathcal{AP}(G)$ then* $\mathrm{C}^*(\tau_G(g)f \mid g \in G) \simeq C(Hull(f))$.

Proof. This is just a special case of Theorem 5.2.15 and Lemma 5.2.9 (and the fact the Hull$(f) \simeq \overline{\{\Delta(g)f \mid g \in G\}}$. □

5.2.7 On subalgebras of $\mathcal{AP}(G) \rtimes_{\tau_G,r} G$

In the Abelian case we get a nice characterization of some of the C*-subalgebras of $\mathcal{AP}(G) \rtimes_{\tau_G,r} G$.

Corollary 5.2.23. *Let G be an Abelian discrete group. Suppose*

$$B \subset \mathcal{AP}(G) \rtimes_{\tau_G,r} G$$

is a C*-*subalgebra containing the set*

$$\{1 \otimes \lambda(g) \mid g \in G\}.$$

Then

$$B \cap A \simeq C(\Omega_{B\cap A}).$$

In particular

$$B = C(\Omega_{B\cap A}) \rtimes_{\tilde{\tau}_F,r} G$$

Proof. This is the simple matter of observering that whenever B is as in the Corollary, $B \cap A$ is τ_G-invariant. As G was Abelian, we see that $B \cap A$ is also ρ_G-invariant, hence Theorem 5.2.15 states that

$$B \cap A \simeq C(\Omega_{B\cap A}).$$

□

5.3 Ideals in $\mathcal{AP}(G)$ and $\mathcal{AP}(G) \rtimes_{\tau_G,r} G$

We end this chapter with a slight motivation as to why $\mathcal{AP}(G)$ is an interesting subalgebra of $\ell^\infty(G)$.

There is an immediate consequence of theorem 5.0.9 which relates to the ideals of $\mathcal{AP}(G)$.

Lemma 5.3.1. *Let G be a discrete group, then the τ_G-invariant ideals in $\mathcal{AP}(G)$ are precisely $\mathcal{AP}(G)$ and $\{0\}$.*

Proof. Assume $I \in \mathcal{AP}(G)$ is a G-invariant ideal of $\mathcal{AP}(G)$ and $I \neq \{0\}$. Pick $f \in I$ where $f \neq 0$. We may, by possibly replacing f with $\overline{f}f$, assume that $f > 0$. Let C be the convex hull generated by $\{\tau_G(g)f \mid g \in G\}$. Since I is a G-invariant ideal, $C \subset I$. By Theorem 5.0.9 we have a constant function $K \in C \subset I$, but then as $f > 0$, we know that $K > 0$, hence $1 = K^{-1}K \in I$, so $I = \mathcal{AP}(G)$. □

Though the above theorem does suggest that $\mathcal{AP}(G) \rtimes_{\tau_G,r} G$ is simple, we can not make that conclusion without being sure that $\mathcal{AP}(G)$ separates the ideals in $\mathcal{AP}(G) \rtimes_{\tau_G,r} G$. Fortunately, there is a class of groups where we can guarantee that $\mathcal{AP}(G) \rtimes_{\tau_G,r} G$ also becomes simple.

We use the definition of maximally almost periodic found in [9].

Definition 5.3.2. Let G be a Hausdorff topological group. We say that G is *maximally almost periodic* if there exists an injective continuous homomorphism $i : G \to K$ for some compact Hausdorff group K.

The next result is a fun application of our construction of Ω_F, though the result is well known.

Theorem 5.3.3. *Let G be a discrete group, then G is maximally almost periodic if and only if for every $x \in G$ there exists a finite dimensional unitary representation (u, H) of G such that $u(x) \neq 0$.*

Proof. Assume that for each $x \in G$ we may pick a finite dimensional unitary representation (u_x, H_x) such that $u_x(x) \neq 0$. In Example 5.2.17 we saw that each u_x gives rise to a set F_x of almost periodic functions. Furthermore, $\ker i_{F_x} = \ker u_x$. We may form the product

$$\prod_{x \in G} \Omega_{F_x}$$

which is compact by Tychonoff's theorem. And we get a continuous map

$$i : G \to \prod_{x \in G} \Omega_{F_x}$$

by

$$i(g)(x) = i_{F_x}(g) \qquad x \in X.$$

But i is injective, as

$$\ker i = \bigcap_{x \in G} \ker i_{F_x} = \bigcap_{x \in G} \ker u_x = \{0\}.$$

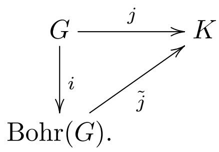
Hence G is maximally almost periodic.

Conversely, assume we have an injective homomorphism $i : G \to K$ for some compact Hausdorff group K. Then as K is Hausdorff, we may for each $x \in X$ pick a unitary finite dimensional representation (u_x, H_x) such that $u_x(x) \neq 0$ (this is again an easy consequence of the Peter-Weyl theorem). Then we get induced representations $(u_x \circ i, H_x)$ of G, and as i was injective, $u_{i(x)}(i(x)) \neq 0$. □

It also turns out it is enough to consider when the map $i : G \to \mathrm{Bohr}(G)$ is injective:

Lemma 5.3.4. *Let G be a discrete group. Then G is maximally almost periodic if and only if the map $i : G \to \mathrm{Bohr}(G)$ is injective.*

Proof. If $i : G \to \mathrm{Bohr}(G)$ is injective, G is maximally almost periodic. Conversely, assume G is maximally almost periodic and pick an injective continuous homomorphism $j : G \to K$ for some compact Hausdorff group K. We may pick $\tilde{j}$ such that the following commutes



But if i is not injective, j can not be injective, so i must be injective. □

Now, we notice that G acts on $\mathrm{Bohr}(G)$ by $gx = i(g)x$ for $g \in G$ and $x \in \mathrm{Bohr}(G)$. So whenever G is maximally almost periodic, we see that the action of G on $\mathrm{Bohr}(G)$ is free. Thus, we get the following theorem.

Theorem 5.3.5. *Let G be a maximally almost periodic group. Then*

$$\mathcal{AP}(G) \rtimes_{\tau_G, r} G$$

is simple.

Proof. Immediate from [1, Theorem 3]. □

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