Dept. of Math. University of Oslo
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Estimating the discharge rating curve by nonlinear regression - The frequentist approach

Trond Reitan and Asgeir Petersen-Øverleir

ABSTRACT: This report provides a discussion about the fundamentals of the frequentist approach to the classical nonlinear least squares head – discharge power-law rating curve model, which is a vital procedure in practical hydrology. It is shown that the multivariate minimization problem of the classical nonlinear least squares rating curve model is equivalent to the maximization of a single argument function. We propose four general criteria which the discharge measurements should meet if a trustable frequentist least squares rating curve estimate should exist. The proposed criteria are applied to a large number of real-life discharge measurements, which suggest that the criteria are particularly useful in practice. We also show that the breakdown of one of the criteria implies an exponential law relationship between the head and the discharge. Numerical maximization of the single argument function and inference are discussed.

1 Introduction

The rating curve is very important in surface hydrology, as quality discharge data is highly dependent on a satisfactory head – discharge relationship at the gauging station (the term head is used for the water level relative to the gauging staff or another fixed datum). For establishing this relationship, the rating curve method is the most widespread technique and has been applied on thousands of gauging stations worldwide for decades. Amazingly, a thoroughly theoretical treatment on the most recommended rating curve regression model still seems to be missing.

The head – discharge equation at a gauging station is invariably based on the power relationship

$$Q = A(h + c)^b \tag{1}$$

where Q is the discharge, h the relative head and (A, b, c) are parameters. Thus, assuming negligible velocity head, $h + c$ is the total head. Eq.1 is recommended many places in the hydrometric literature (e.g. Herschy, 1999; ISO, 1998; WMO, 1994; Mosley and

McKerchar, 1993; Lambie, 1978). In compound rating curve problems two or more similar equations (often referred to as segments), each relating to a portion of the head range, must be applied.

It is easy to see that eq.1 is based on the hydraulically motivated Manning equation which frequently is used as the governing equation for steady uniform flow problems; $Q = N^{-1}S^{5/3}P^{-2/3}s_0^{1/2}$, where N is the Manning's roughness coefficient, s_0 the slope, S the area and P the wetted perimeter. However, eq.1 is a simplification of the Manning equation, as eq.1 assumes that the conveyance function $S^{5/3}P^{-2/3}$ can be described by a simple power function of the total head.

Sometimes the parameter c is determined by considering the level on the cross section in relation to zero flow. Some of the above mentioned references encourage this. Obviously, trying to estimate c in advance will have a detrimental effect on the flexibility of eq.1, due to the loss of one degree of freedom. More important, as geometric idealisations originally have been made, it is unlikely that the point of zero flow of the natural channel is the same as for the idealised control. Furthermore, the point of zero flow can be difficult to determine in deep rivers, at controls which consist of rocky profiles and controls which are defined by a reach without a definite edge. Hence, the result can be an inflexible and biased rating curve.

Standard hydrometric literature, of which the most important are mentioned above, recommend the method of nonlinear least squares for constructing the rating curve. The methods recommended are, however, often based on graphical and/or trial and error techniques. Firstly, such techniques can produce very inaccurate results. Secondly, it is impossible to estimate uncertainty in estimated discharge, due to rating curve variability, if the rating curve is constructed by subjective methods. Thirdly, even though the manual methods recommended claim to be based on formal least squares techniques, an analytical solution to the nonlinear least squares problem may not exist. This kind of hydrometric practice is unfortunate.

The aim of this paper is to theoretically investigate the classical nonlinear least squares rating curve model. It is shown that the multivariate minimisation problem is equivalent to maximising a single argument function of the parameter c. Furthermore, it is proposed that a trustable nonlinear least squares rating curve estimate exists if the discharge measurements satisfy four criteria. The four criteria are validated by applying them to over thousand discharge measurement datasets. The study further discusses maximisation and confidence intervals before some practical examples are given.

2 The nonlinear regression model

2.1 Problem formulation

Having $n \geq 3$ head - discharge measurements available with dissimilar head observations, the rating curve regression model proposed is

$$Q_i = A(h_i + c)^b (1 + \varepsilon_i), \quad \varepsilon_i \sim N(0, \sigma^2) \quad i = 1, \cdots, n \qquad (2)$$

The size of σ typically is in the area of 0.03 – 0.07. However, experiences from NVE (Norwegian Water and Energy Directorate) suggests that a σ of 0.01 is attainable, while unsuitable measuring conditions combined with an unwise choice of measurement method can produce standard deviations of up to 0.15 – 0.20.

Eq.2 requires at least 3 measurements with dissimilar heads in order to avoid an ill-defined regression. Ostensibly, an even better criterion would be to require more than 3 measurements with dissimilar heads, and hence avoid the trivial solution where the variance of the residuals becomes zero. However, we shall see later in the study that this is not a necessary criterion in order to establish a proper theoretical assessment of the rating curve fitting.

We assume the variance, which is due to discharge measurement uncertainty, varies as a linear function of the expected discharge. Furthermore, by utilising the first order Taylor approximation $\log(1 + \varepsilon_i) \approx \varepsilon_i$, assuming $|\varepsilon_i| << 1$, and easing the notation, we obtain

$$q_i = a + bx_i + \varepsilon_i \qquad (3)$$

where $q_i = \log(Q_i)$, $a = \log(A)$ and $x_i = \log(h_i + c)$. Note that the dependency on *c* still is assumed in the last expression. The model in eq.3 is recommended in most hydrometric literature that treats rating curve construction. Firstly, the model in eq.3 is a more suitable model than eq.2, since the model of eq.2 allows for a possibility of negative discharge, which is in contradiction with the definition of stream flow. Secondly, Petersen-Øverleir (2004) showed that the logarithmic transformation in many cases fails to stabilise the variance, leading to heteroscedasticity unaccounted for. However, this paper only treats situations where the residuals in eq.3 are homogenously distributed.

Now, the residual sum of squares is given by

$$S(\boldsymbol{\theta} \mid \boldsymbol{M}) = \sum_{i=1}^{n} [q_i - a - bx_i]^2 \qquad (4)$$

where ***M*** is the set of *n* log-transformed head – discharge measurements.

The parameters $\boldsymbol{\theta} = (a, b\ c)$ which minimise eq.4 are the NLS estimated rating curve parameters. It is also worth mentioning the fact that the least squares estimator of $\boldsymbol{\theta}$ is also the maximum likelihood estimator since we have assumed the residuals to be normally distributed.

2.2 Refinements

Minimising eq.4 may seem unpleasant, since $S(\cdot)$ takes values in R^3 which makes it difficult to pre-assess the parameter space. Moreover, estimating three parameters simultaneously requires an advanced numerical scheme. Even if an adequate numerical optimiser is available, one needs to get knowledge about possible local minima and making sure that a solution exists. Fortunately, eq.4 can be simplified so that these problems can be assessed more easily by considering the normal equations of $S(\cdot)$. Denoting the sample mean of any statistical function as $\overline{\ }$ and setting the derivative of $S(\cdot)$ with respect to a and b equal to zero we obtain

$$\frac{\partial S(\theta \mid \boldsymbol{M})}{\partial a} = 2n\left(-\bar{q} + a + b\bar{x}\right) = 0 \Rightarrow a(c) = \bar{q} - b\bar{x} \qquad (5a)$$

$$\frac{\partial S(\theta \mid \boldsymbol{M})}{\partial b} = 2n\left(-\overline{qx} + a\bar{x} + b\overline{x^2}\right) = 2n\left\{-\overline{qx} + \bar{q}\bar{x} + b\left[\overline{x^2} - (\bar{x})^2\right]\right\} = 0 \Rightarrow$$
$$b(c) = \frac{\overline{qx} - \bar{q}\bar{x}}{\overline{x^2} - (\bar{x})^2} = \frac{\hat{\text{cov}}(q,x)}{\hat{\text{var}}(x)} \qquad (5b)$$

where $\hat{\text{var}}(\cdot)$ and $\hat{\text{cov}}(\cdot)$ are defined in eq.8A and eq.10A in the appendix.

These results are in fact equivalent to the well known solutions for an ordinary least squares problem. Substituting eq.5a into eq.4 yields

$$S(\theta \mid \boldsymbol{M}) = \sum_{i=1}^{n}\left[q_i - \bar{q} - b(x_i - \bar{x})\right]^2 = n\left[\overline{(q - bx)^2} - \left(\overline{q - bx}\right)^2\right] =$$
$$= n\left[\overline{q^2} - 2b\overline{qx} + b^2\overline{x^2} - (\bar{q})^2 + 2b\bar{q}\bar{x} - b^2(\bar{x})^2\right] \qquad (6)$$

Furthermore, by substituting eqs.5b into eq.6 the sum of squares becomes a single argument function:

$$S(c \mid \boldsymbol{M}) = n\left[\overline{q^2} - (\bar{q})^2\right] - n\frac{\left[\overline{qx} - \bar{q}\bar{x}\right]^2}{\overline{x^2} - (\bar{x})^2} \qquad (7)$$

which can be written as

$$S(c \mid \boldsymbol{M}) = n\hat{\text{var}}(q) - n\frac{\hat{\text{cov}}^2(q,x)}{\hat{\text{var}}(x)} \qquad (8)$$

2.3 Exploring the solution space of the sum of squares

The behaviour of $S(\cdot)$ needs to be studied. The main theoretical problems are existence and uniqueness of estimates. First, we define the function

$$r(c) = \frac{\hat{\text{cov}}^2(q,x)}{\hat{\text{var}}(x)} \qquad (9)$$

By its definition $S(\cdot)$ is bounded from below since $S(\cdot) \geq 0$. We also see that $S(\cdot)$ is bounded from above, since eq.8 consists of a constant minus a positive function of c. Due to the existence of this finite upper limit, global criteria theorems such as found in Demidenko (2000) do not apply to our problem. The upper limit of $S(\cdot)$ can be found, since it is easily seen that

$$\begin{aligned} &S(c \mid \boldsymbol{M}) = n\,\hat{\text{var}}(q)\left[1 - \hat{\text{cor}}^2(q,x)\right] \\ &0 \leq S(c \mid \boldsymbol{M}) \leq n\,\hat{\text{var}}(q) \end{aligned} \qquad (10)$$

Now, the only dependency on c in $S(\cdot)$ is contained in the purely negative function $-r(c)$. Consequently, minimising $S(\cdot)$ is equivalent to maximising $r(c)$. It is also worth noting that this is not surprisingly equivalent to maximizing $\hat{\text{cor}}(q,x)$ when eq.17 later on is assumed.

The c which maximises $r(c)$ cannot be evaluated in closed form. However, a lot of information on the optimisation problem can be provided by investigating the normal equation of $r(c)$ more closely. Firstly, we need the following results:

$$\frac{\partial \bar{x}}{\partial c} = \frac{\partial}{\partial c}\left(\frac{1}{n}\sum_{i=1}^{n} \log(h_i + c)\right) = \overline{\exp(-x)} \qquad (11a)$$

$$\frac{\partial \overline{x^2}}{\partial c} = \frac{\partial}{\partial c}\left(\frac{1}{n}\sum_{i=1}^{n} \log^2(h_i + c)\right) = 2\overline{x\exp(-x)} \qquad (11b)$$

$$\frac{\partial \overline{qx}}{\partial c} = \frac{\partial}{\partial c}\left(\frac{1}{n}\sum_{i=1}^{n}\log(h_i + c)q_i\right) = \overline{q\exp(-x)} \qquad (11c)$$

Inserting eqs.(11a) – (11c) into the expression for the derivative of $r(c)$ yields

$$\frac{\partial r(c)}{\partial c} = 2\frac{\overline{qx} - \bar{q}\bar{x}}{\left[\overline{x^2} - (\bar{x})^2\right]^2}\left\{\left[\overline{q\exp(-x)} - \bar{q}\,\overline{\exp(-x)}\right]\left[\overline{x^2} - (\bar{x})^2\right] - \left[\overline{qx} - \bar{q}\bar{x}\right]\left[\overline{x\exp(-x)} - \bar{x}\,\overline{\exp(-x)}\right]\right\} \qquad (12)$$

which furthermore can be written as

$$\frac{\partial r(c)}{\partial c} = 2\frac{\hat{\mathrm{c}}\mathrm{ov}(q,x)}{\hat{\mathrm{v}}\mathrm{ar}^2(x)}\left[\hat{\mathrm{c}}\mathrm{ov}(q,\exp(-x))\hat{\mathrm{v}}\mathrm{ar}(x) - \hat{\mathrm{c}}\mathrm{ov}(q,x)\hat{\mathrm{c}}\mathrm{ov}(x,\exp(-x))\right] \qquad (13)$$

Possible finite roots of eq.13 origins either from

$$f(c) = \hat{\mathrm{c}}\mathrm{ov}(q,x) = b(c)\hat{\mathrm{v}}\mathrm{ar}(x) = 0 \qquad (14)$$

or

$$A(c) = \hat{\mathrm{c}}\mathrm{ov}(q,\exp(-x))\hat{\mathrm{v}}\mathrm{ar}(x) - \hat{\mathrm{c}}\mathrm{ov}(q,x)\hat{\mathrm{c}}\mathrm{ov}(x,\exp(-x)) = 0 \qquad (15)$$

The left hand side of eq.14 being negative or zero for some reasonable estimate of c is very unlikely, since that would imply serious violations of the hydraulic and hydrometric model assumptions on which eq.2 are based. In such a situation no trustable rating curve can be constructed at the gauging station. Hence, we would like to require that the right hand side of eq.14 is positive for all finite values of c. Therefore, the solution we seek implies eq.15 being zero.

Maximising $r(c)$ implies that we are looking for arguments of finite c of which $\partial r(c)/\partial c = 0$ and $\partial^2 r(c)/\partial c^2 < 0$, meaning that $\partial r(c)/\partial c$ goes from being positive to being negative. This may happened several times. Primarily, we want this to happen only once for a finite c. This is a requirement which is difficult to ensure. However, establishing requirements ensuring that $\partial r(c)/\partial c$ changes sign an odd number of times, where the first crossing produces a maximum, is easier, and hopefully sufficient.

If $\partial r(c)/\partial c$ crosses (not touching) the line of zero an even number of times $r(c)$ has no max/min (no crossings), one maximum and one minimum (two crossings), etc. Obviously, these are situations we are hoping to avoid.

Denoting $h_m = \min(h_1, h_2, \dots, h_n,)$, we see from eq.2 that c spans $\langle -h_m, \infty \rangle$. Now, if $\partial r(c)/\partial c = 0$ for an odd number of finite arguments c, of which the first crossing produces a maximum, we must require that

$$\lim_{c \to -h_m} \frac{\partial r(c)}{\partial c} \in \langle 0, \infty] \text{ and } \lim_{c \to \infty} \frac{\partial r(c)}{\partial c} \in [-\infty, 0\rangle \qquad (16)$$

by the *Intermediate-Value Theorem of Continuous Calculus*, since $\partial r(c)/\partial c$ by lemma 5 in the appendix is continuous for $c \in \langle -h_m, \infty \rangle$.

If we require that

$$f(c) > 0 \;\forall c \in \langle -h_m, \infty \rangle \qquad (17)$$

the criteria of eq.16 can be satisfied in only one way:

$$\lim_{c \to -h_m} A(c) \in \langle 0, \infty] \text{ and } \lim_{c \to \infty} A(c) \in [-\infty, 0\rangle \qquad (18)$$

Eq.17 permits the limits of $f(c)$ to be zero. However, as we have argued, $f(c) > 0$ for finite values of c is a very reasonable criterion. Moreover, we see from eq.9 and eq.14 that if there should exists a solution where $f(c) = 0$ for a finite c, this maxima must found where also $r(c) = 0$. However, we have concluded that $r(c)$ cannot be negative, which means that the situation would be that $r(c)$ is overall zero. Thus, we want to ensure that $f(c) > 0$ for finite values of c is the case.

A sufficient set of requirements for ensuring that eq.16 is true can be achieved by introducing the following theorem:

Theorem 1

If

i. $\bar{q}_{i\square\square} < \bar{q}_{i\square\square}$ (19a)

ii. $\hat{\text{cov}}(q,h) > 0$ (19b)

iii. $\hat{\text{cov}}\left(q, \log_{i \neq m}(h - h_m)\right) > 0$ (19c)

iv. $\hat{\text{cov}}(q,h^2) < \left(\frac{\hat{\text{skew}}(h)}{\hat{\text{var}}(h)} + 2\bar{h}\right)\hat{\text{cov}}(q,h)$ (19d)

then r(c) has an odd number of stationary points where $\partial^2 r(c)/\partial c^2 \neq 0$ *, of which at least one is a maximum.*

Here $i \neq m$ symbolises that the actual function is evaluated not using data from the measurements containing h_m (formal definition, see eq.A3) and $\hat{\text{skew}}(h)$ is the estimated third order central moment (formal definition, see eq.A9).

The proof of this theorem is found in appendix. We mention that criteria i.-ii. are related to eq.17, while criteria iii.-iv. are related to eq.18.

Note that theorem 1 excludes saddle points from the class of stationary points considered. On the other hand, theorem 1 does not exclude saddle points from being present in $r(c)$. However, the presence of one or more saddle points for finite arguments c would indicate an ill-defined and oversensitive regression, where adding/removing one measurement could change the whole outcome of the estimation. Hence, no robust rating curve could then be constructed. Nevertheless, one or more saddle points in addition to a maximum can perhaps be the case, even though the authors have never come across such a situation.

2.4 Discussion of Theorem 1

The three first criteria of theorem 1 are reasonable and logical requirements, as they all reflect the basic hydraulic assumption behind the rating curve. If one or more of criteria (i. – iii.) are to be violated, and the measurement data is correctly collected and processed, then we have a situation where we seem to measure less discharge as the water level increases. This is contradictory and indicates either; (1) few and very uncertain measurements done at almost the same water level or; (2) extreme hydraulic conditions (e.g. the control is near a hydraulic jump).

Criterion iv. is the less intuitive of the four criteria. Experience has shown the authors that this criterion can be broken even if things seem to be arranged for an easy rating curve construction. Firstly, note that criterion iv. can, assuming that criterion ii. is met, be written as

$$\frac{\hat{\text{cov}}(q,h^2)}{\hat{\text{cov}}(q,h)} < \frac{\hat{\text{skew}}(h)}{\hat{\text{var}}(h)} + 2\bar{h} \quad (20)$$

The right side of eq.20 is given by the way measurements are carried out, while the left hand side describes the natural statistical relationship between the head and the discharge. We can control the right hand side of eq.20 by designing the measurements. However, the left hand side of eq.20 cannot fully be designed due to the randomness involved with the discharge measurements and inaccurate model assumptions. Thus is it impossible to securely fulfil criterion iv. by experimental design alone. Secondly, there is strong indication that the ostensible mysterious criterion iv. is connected with the breakdown of a power-law relationship between the head and the discharge. If we assume that all criteria of theorem 1 are met except iv., and furthermore that $r(c)$ is a monotonically increasing function with no roots (as in fig.2c), it is easy to see that the least squares solution of c is infinite. At first, this situation appears unfavourable. However, by lemma 6 in the appendix, we see that accepting an infinite c as an estimate leads to an estimated predicted discharge given by

$$\hat{Q}^{+} = \exp(\hat{\alpha})\exp(\hat{\beta}h^{+}) \quad \hat{\alpha} = \bar{q} - \bar{h}\frac{\hat{\mathrm{cov}}(q,h)}{\hat{\mathrm{var}}(h)}, \quad \hat{\beta} = \frac{\hat{\mathrm{cov}}(q,h)}{\hat{\mathrm{var}}(h)} \tag{21}$$

Hence, in such a situation, the data seem to support an exponential-law relationship between the head and the discharge. The model of eq.21 has no point of zero flow, which makes it difficult to accept as a lower segment part of a rating curve. However, there should be no obstacle to accepting the exponential-law relationship in the upper segments. Recall that the power-law relationship is only motivated by idealising the channel geometry of the Manning equation. If the channel area and wetted perimeter is changing drastically as functions of the water level, which can happen when the river has topped the banks and spreads out over flat fields, the exponential-law is not unlikely.

2.5 Verification of the usefulness of Theorem 1

If the criteria in theorem 1 are satisfied we know that one or more finite solutions to the problem exist. On the other hand, if theorem 1 is not satisfied, one or more global maxima together with at least one minimum could still exist. So how useful is really theorem 1 in practice? This is a question that needs answering. The theorem was applied to 1176 head – discharge measurement datasets of the database of NVE. The data origins from almost all gauging stations ever established in Norway. No assessment of segmentation, time dependent trend/breaks or data quality was performed in advance, and all 1176 datasets were directly applied to theorem 1 and subsequently to the one-segmented rating curve model presented in this study. Hence, the intention was to perform a very comprehensive test of theorem 1, including everything from the most unsuitable to the most ideal datasets available.

Table 1. Statistics from the application of theorem 1 and the classical rating curve model to 1176 discharge measurement datasets.

	Number of datasets	No solution	Solution
All criteria met	989 (84 % of total)	0	989

One or more of criteria i. – iii. not met	58 (5 % of total)	46	12
Only criterion iv. not met	129 (11 % of total)	129	0

Table 1 shows the results from the practical study. Firstly, note that for all the 989 cases where all four criteria of theorem 1 were met a least squares solution existed. This is a powerful empirical justification for that meeting the criteria of theorem 1 implies the existence of a rating curve estimate. Secondly, note that of the 58 cases where one or more of the criteria i. – iii. (at times also with criterion iv.) were not met, only 12 still had a solution. These 12 cases were investigated. We found that in 7 of the 12 cases the estimated exponent b was negative (criterion iii. not met), which is unwanted from a hydraulic point of view and should lead to a rejection of the data or the gauging station. The other 5 cases also produced unrealistic exponents far outside the interval (1, 4). Erroneous data processing was the most dominant reason for being able to produce an unrealistic rating curve estimate despite breaking the criteria of theorem 1. Two examples are shown in fig.1, where; (a) a wrongly observed or punched h_m resulted in criteria i. and iv. being broken; and (b) a small dataset of which two heads having been assigned the wrong sign caused criteria ii. and iii. to be broken. 3 of the 12 above mentioned cases were bimodal, apparently due to two different datasets having been wrongly stored on the database as one dataset. Also note that in the 129 cases were only criterion iv. was broken, no rating curve estimate existed. Once again the mysterious criterion iv. seems to be a key criterion, because when an ostensible adequate dataset has no solution, it is usually this criterion that is broken.

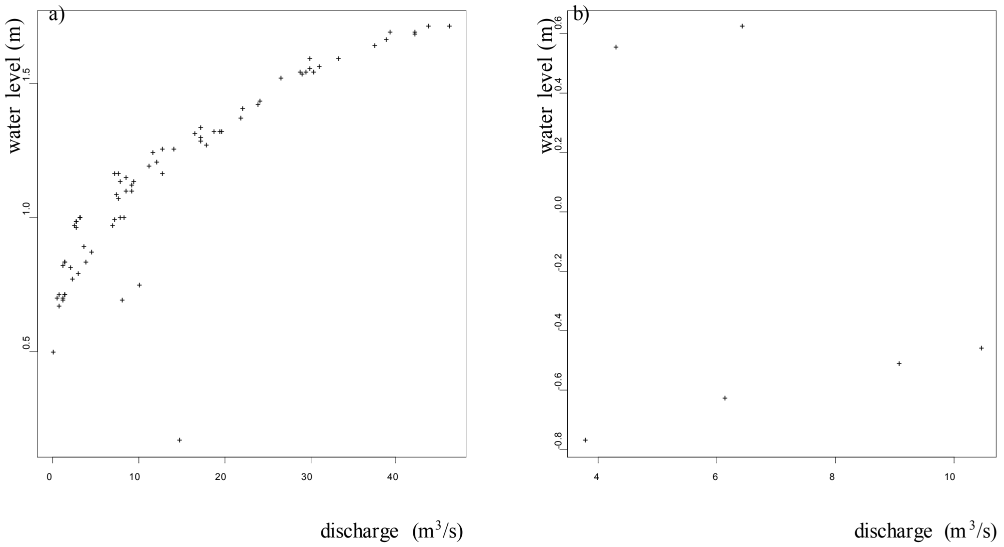

Figure 1. Examples from the 12 out of 1176 head – discharge measurement datasets which had a rating curve estimate despite not meeting the criteria of theorem 1.

Based on the results of this section, we state the following proposition which is the main result of the present study:

Proposition 1

If one or more of the criteria of theorem 1 are broken, a trustable rating curve estimate based on frequentist estimation of a power-law relationship between the head and discharge, does not exist.

2.6 Maximising *r*(*c*)

If having concluded that $r(c)$ is unimodal, numerical optimisation is required in order to obtain an estimate $\hat{c}$ of c. Maximising the single argument function in eq.9 may seem straightforward by solving eq.15 numerically using Newton-Raphson iteration. However, practice has shown the authors that the Newton-Raphson iteration method can occasionally be unreliable when initial values must be supplied automatically, due to the fact that $\partial^2 r(c)/\partial c^2$ changes sign for a $\tilde{c} > \hat{c}$, which is a consequence of the fact that $r(c)$ has a limit as $c \to \infty$ (given explicitly in corollary 5 in the appendix). Clearly, if a Newton-Raphson scheme is supplied with an initial value above $\tilde{c}$, the iteration will take $\hat{c}$ towards infinity and thus not converge. In order to avoid this runaway situation, a scheme based on the golden section method (see Cheney and Kincaid, 1999 for details) seems to work well for maximising $r(c)$. This technique is known to be secure when $r(c)$ is unimodal, although it converges relatively slowly compared to the Newton-Raphson iteration which converges remarkably fast. In any case, with modern computer power available, using the golden section method should be no problem.

2.7 Inference

No rating curve and discharge estimates should be presented without information about its uncertainty. This is something that seems to be forgotten by many hydrometric offices.

There are several methods available for obtaining confidence intervals about the estimated rating curve parameters and any predicted discharge. Resampling methods like the bootstrap are often useful in nonlinear regression problems (Efron and Tibshirani, 1993; Huet et al., 1990). The method of bootstrap can handle aspects like unaccounted for heteroscedasticity, bias and curvature effects. However, applying bootstrap techniques to the maximisation of eq.9 is not straightforward. The authors have experienced that occasionally some resamples fail to satisfy one or more criteria of theorem 1 (especially criteria iv.), leading to no finite solution situations while resampling. This produces an unclear situation. However, if only criterion iv. is broken while resampling, the predicted discharge estimate will exist although the estimated c is infinite, and bootstrap methods are applicable for inference around predicted discharge values.

Classical asymptotic nonlinear least squares results (see Seber and Wild, 1989) are certainly available for inference about $\hat{c}$, and subsequently the delta method can be applied in order to obtain approximate confidence intervals for $\hat{b}$, $\hat{a}$ and any predicted log-discharge $\hat{q}^+$. However, typical normal assumptions can be violated by curvature effects. The authors experience is that some moderate curvature effects are present in $r(c)$ for most real-life rating curve problems. An example is seen in fig.1a where $r(c)$ for the lower segment Corrientes data is slightly unsymmetrical around its maxima. However, the curvature effects are mostly small and not serious, and normal approximations normally provide trustworthy information about the accuracy of the rating curve. Note the constraint that both A and Q must be greater than zero indicates that it is unwise to construct confidence intervals on $\hat{A}$ and $\hat{Q}^+$ itself. In addition to this, the assumptions of $\hat{a} - a$ and $\hat{q}^+ - q^+$ being normally distributed will not be valid in other scales than the logarithmic. The appropriate thing to do is to do all the calculations on the logarithmic scale and transform the approximated confidence limits obtained at the end.

We also mention that maximum likelihood based methods are applicable for obtaining confidence intervals in rating curve estimates. Clarke (1999) applied the asymptotic normal results for the likelihood function for providing confidence intervals to predicted discharges. However, the profile likelihood approach is perhaps the most attractive likelihood method (Huet et al. 2004), as it handles curvature effects rather well.

3 A practical example

The rating curve data from Rio Paraná at Corrientes, Argentina, consists of 368 discharge measurements performed between 1981 and 2002. The control at Corrientes is compound, and consists of one main channel segment ($h < 6$ m) and one segment from where the river overtops its banks ($h \geq 6$ m.) (Clarke et al. 2000). The discharge was measured solely by current meter from 1981 to 1996. ADCP (Acoustic Doppler Current Profiler) was introduced in 1996. There are 233 measurements available at the lower segment. We see from table 2 that these 233 discharge measurements satisfy all four criteria in theorem 1, indicating that a least squares rating curve solution exists for the lower segment. This indication is confirmed in fig.2a and fig.2b, where we see that $r(c)$ for the lower segment has one unique maxima and $\partial r(c)/\partial c$ has a single root at $c = 5.43$ m. Fig.3a shows the discharge measurements for the lower segment along with the estimated rating curve. After regression diagnostics shown in fig.3b suggests that the variance was stabilised rather adequately by the logarithmic transformation and that only one segment is in operation for $h < 6$ m. However, fig.3c indicates that the steady-state assumption is somewhat violated. The normalised residuals seem to be time dependent, suggesting that the hydraulic properties of the control of the lower segment have changed over time. The measurements of the upper segment along with an exponential-based rating curve estimate are shown in fig.3d. Visually, the data work satisfactory, and constructing a power-law based rating curve seems feasible. However, table 2 shows that criteria iv. of theorem 1 is not met by the data for the upper segment. Fig.2c gives us the truth of the matter: $r(c)$ for the upper segment has no stationary point and consequently $\partial r(c)/\partial c$ does not cross the line of zero as shown in fig.2d. For that reason it is impossible to obtain a power-law based least squares estimated rating curve for the upper segment. However, the exponential model in eq.21 was satisfactory applied to the upper segment data. The estimated lower and upper rating curve segments merge at $h = 6.1423\cdots$, which is slightly off the presumed phase-shift. Note the surprisingly small approximated standard deviation of the upper-segment exponential model parameters, which suggests that the exponential-law model really suits the data. Fig.2e indicates that no unaccounted for heteroscedasticity is present in the exponential model, while fig.2f shows evidence of change of hydraulic properties of the upper segment over time.

Table 2. Results from fitting the Corrientes data to power-law and/or exponential-law rating curve models. Approximated standard deviations of estimates are given in parenthesis

	Criteria of theorem 1 satisfied?				Estimated rating curve parameters			
	i.	ii.	iii.	iv.	$\hat{c}$	$\hat{b}$	$\hat{A}$	$\square$
Lower segment	Yes	Yes	Yes	Yes	5.43~(0.17)	1.88~(0.03)	269~(364)	0.052~(0.002)
Upper segment	Yes	Yes	Yes	No	Na	Na	Na	Na
					Estimated rating curve parameters, exponential-law model			
					$\exp(\square)$	$\hat{\square}$		$\square$
Upper segment					8.689~(0.047)	0.247~(0.007)		0.062~(0.004)

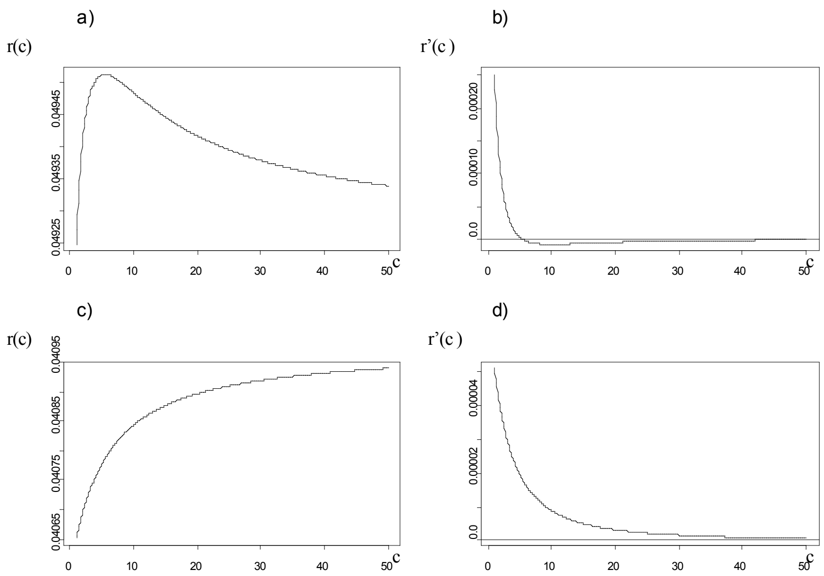

Figure 2. $r(c)$ and $\partial r(c)/\partial c$ for the Corrientes data, of which (a) and (b) are related to the lower segment and (c) and (d) are related to the upper segment.

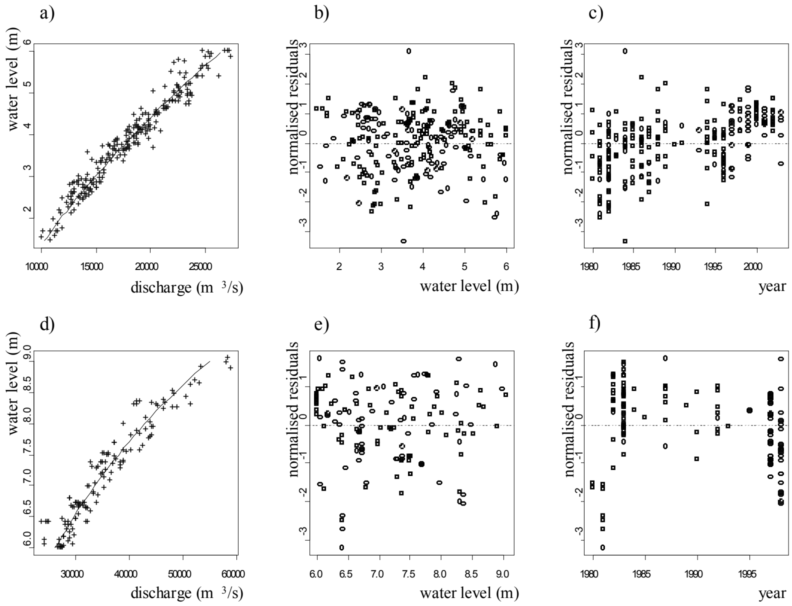

Figure 3. Head – discharge measurements and rating curve estimated by the classical power-law model (a) along with predictor-residual plot (b) and time-residual plot (c) for the lower segment at Corrientes. Head – discharge measurements and rating curve estimated by an exponential-law model (e) along with predictor-residual plot (f) and time-residual plot (g) for the upper segment at Corrientes.

Conclusions

The work presented shows in detail how the original multivariate minimisation problem reduces to the maximisation of a single argument function. Utilizing that result and by reasonable mathematical approaches it is shown that the frequentist approach to the classical nonlinear least squares rating curve model rests upon four criteria which the head – discharge measurements should meet. The usefulness of the four criteria is supported by an application to over thousand real-life datasets. It is shown that in some instances an exponential-law relationship between the head and the discharge is motivated in conjunction with the traditional power-law relationship having no rating curve estimate. Maximisation of the single argument resolvent function is discussed, and the study arrives at a conclusion that the golden section method is trustworthy if a maximum exists. It is argued that the classical asymptotic nonlinear least squares methods provide clear and adequate confidence intervals for the estimated rating curve parameters and predicted discharges.

The present study presents fundamental knowledge about fitting the discharge rating curve to a set of head – discharge measurements using the classical frequentist method of nonlinear least squares. However, there still remains scope for further analysis. There are grounds in hydraulics for providing a priori knowledge about some of the rating curve parameters, and a Bayesian approach may turn out to be a viable method for obtaining a rating curve estimate. Traditionally the power-law relationship between the head and the discharge is assumed for all segments of a rating curve. In contrast, the theoretically statistical results of the present study bring up the possibility of an exponential-law relationship. This matter requires further investigation.

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Appendix

A1 Definitions

Let M_i be the ith log-transformed rating curve regression pair such that

$$M_i = (\log(Q_i), \log(h_i + c)) \equiv (q_i, x_i) \qquad \text{(A1)}$$

The set of all $n \geq 3$ log-transformed rating curve regression pairs is thus defined as

$$\boldsymbol{M} = \{M_1, M_2, \cdots, M_n\} \qquad \text{(A2)}$$

Let   be any statistical function of q_i and/or x_i. Then

$$\square_{i \square \square} \tag{A3}$$

is evaluated using all h and/or q in $\boldsymbol{M}$ except the k pairs containing h_m, which is defined as

$$h_m = \min(h_1, h_2, \cdots, h_n) \tag{A4}$$

so that

$$h_j = h_m \;\square\; j \square \square \tag{A5}$$

On the other hand

$$\square_{i \square \square} \tag{A6}$$

means an evaluation of   using only the h and/or q of the k pairs in $\boldsymbol{M}$ containing h_m.
Let the estimated expectation value be defined as

$$\bar{\square} = \frac{1}{n} \sum_{i=1}^{n} \square_i \tag{A7}$$

We then define the estimated variance $\overline{(\square \square \bar{\square})^2}$ as

$$\hat{\mathrm{v}}\mathrm{ar}(\square) \equiv \overline{\square^2} \,\square\, (\bar{\square})^2 \tag{A8}$$

and the estimated third order central moment $\overline{(\square \square \bar{\square})^3}$ as

$$\hat{\mathrm{s}}\mathrm{kew}(\square) \equiv \overline{\square^3} \,\square\, 3\bar{\square}\overline{\square^2} + 2(\bar{\square})^3 \tag{A9}$$

Furthermore, let   be any statistical function of q_i and/or x_i. We then define the estimated covariance $\overline{(\square\square\bar{\square})(\square\square\bar{\square})}$ as

$$\hat{\text{c}}\text{ov}(\square,\square) \equiv \overline{\square\square}\square\bar{\square}\bar{\square} \tag{A10}$$

We also define

$$y_i \equiv \begin{cases} x_i, & i \square \square \\ 0, & i \square \square \end{cases} \tag{A11}$$

and

$$w_i \equiv \begin{cases} q_i, & i \square \square \\ 0, & i \square \square \end{cases} \tag{A12}$$

and

$$z_i \equiv \begin{cases} \exp(\square x_i), & i \square \square \\ 0, & i \square \square \end{cases} \tag{A13}$$

Note that $\exp(\square x_i) = 1/(h_i + c)$.

We also define two requirements on the set of predictors h:

$$n \square k \geq 2 \text{ and } \square j \square \square \mid \bar{h} \neq h_j \tag{A14}$$

Eq.A14 ensures that there are at least two dissimilar values of h in $\boldsymbol{M}$ greater than h_m.

We also state here the presently important function $A(c)$

$$A(c) = \hat{\text{c}}\text{ov}(q, \exp(\square x))\hat{\text{v}}\text{ar}(x) \square \hat{\text{c}}\text{ov}(q, x)\hat{\text{c}}\text{ov}(x, \exp(\square x)) \tag{A15}$$

A2 Lemmas

Lemma 1

$$\lim_{c \to -h_m} \hat{\mathrm{var}}(x) \to \infty \qquad \text{(A16)}$$

Proof

Let $d = h_m + c$. Then we can write

$$\bar{x} = \frac{k}{n}\log(d) + \sum_{i \neq m} \log(h_i - h_m + d) \qquad \text{(A17)}$$

Using Taylor expansion on the last expression in eq.A17 yields

$$\bar{x} = \frac{k}{n}\log(d) + \sum_{i \neq m} \log(h_i - h_m) + d\sum_{i \neq m} \frac{1}{h_i - h_m} + o(d) \qquad \text{(A18)}$$

By the definition of *d*, we have that $\lim_{c \to -h_m} = \lim_{d \to 0^+}$, thus, for a very small and positive *d*, we obtain the first order Taylor approximation

$$\bar{x} \approx \frac{k}{n}\log(d) + \bar{y} \qquad \text{(A19)}$$

Furthermore, for a very small positive *d*, we obtain

$$\left(\bar{x}\right)^2 \approx \frac{k^2}{n^2}\log^2(d) + \frac{k}{n}2\bar{y}\log(d) + \left(\bar{y}\right)^2 \qquad \text{(A20)}$$

and

$$\overline{x^2} \approx \frac{k}{n}\log^2(d) + \overline{y^2} \qquad \text{(A21)}$$

which gives us

$$\hat{\mathrm{v}}\mathrm{ar}(x) \Box \Box 1 \Box \frac{k}{n} \Box \frac{k}{n} \log^2(d) \Box \frac{k}{n} 2\log(d)\overline{y} + \hat{\mathrm{v}}\mathrm{ar}(y) \qquad \text{(A22)}$$

From the definition/requirement in eq.A14 we know that $\hat{\mathrm{v}}\mathrm{ar}(y)$ is finite. Thus, by eq.A22 we arrive at

$$\lim_{c \Box \Box h_m} \hat{\mathrm{v}}\mathrm{ar}(x) \Box$$

Lemma 2

$$\lim_{c \Box \Box h_m} \hat{\mathrm{c}}\mathrm{ov}(q,x) \Box \quad \Box \quad \overline{q}_{i \Box \Box} < \overline{q_{i \Box \Box}} \qquad \text{(A23)}$$

Proof

For a very small positive d we have that

$$\overline{qx} \Box \frac{k}{n} \overline{q}_{i \Box \Box} \log(d) + \overline{qy} \qquad \text{(A24)}$$

and

$$\overline{q}\,\overline{x} \quad \Box \frac{k}{n} \overline{q} \log(d) + \overline{q}\,\overline{y} \qquad \text{(A25)}$$

This gives us

$$\hat{\mathrm{c}}\mathrm{ov}(q,x) \Box \frac{k}{n} \Box \overline{q}_{i \Box \Box} \Box \overline{q} \Box \log(d) + \hat{\mathrm{c}}\mathrm{ov}(q,y) \qquad \text{(A26)}$$

Since $\log(d) < 0$ and $|\log(d)| >> \hat{\mathrm{c}}\mathrm{ov}(q,y)$, it follows that

$$\bar{q}_{i\square\square} \square \bar{q} = \bar{q}_{i\square\square} \square \frac{k}{n} \bar{q}_{i\square\square} \square \frac{n \square k}{n} \bar{q}_{i\square\square} < 0 \square \quad \bar{q}_{i\square\square} < \bar{q}_{i\square\square} \tag{A27}$$

and we reach the equation

$$\lim_{c\square -h_m} \hat{\text{c}}\text{ov}(q, x) \square$$

Lemma 3

$$\lim_{c \square \square h_m} A(c) \square \qquad \square \quad \hat{\text{c}}\text{ov}(q, \log(h \square h_m)_{i\square\square}) > 0 \tag{A28}$$

Proof

By using the definition of z in eq.A12 and d very small and positive we obtain

$$\overline{\exp(\square x)} \square \frac{k}{n} \exp(\square x_m) + \bar{z} \tag{A29}$$

and

$$\overline{q \exp(\square x)} \square \frac{k}{nd} \bar{q}_{i\square\square} + \overline{qz} \tag{A30}$$

Hence, we get

$$\hat{\text{c}}\text{ov}(q, \exp(\square x)) \square \frac{k}{nd} \square \bar{q}_{i\square\square} \square \bar{q} \square + \hat{\text{c}}\text{ov}(q, z) \tag{A31}$$

Furthermore, since

$$\overline{x\exp(-x)} \approx \frac{k}{n}\frac{\log(d)}{d} + \overline{yz} \qquad \text{(A32)}$$

for d very small and positive, it follows that

$$\hat{\mathrm{c}}\mathrm{ov}(x,\exp(-x)) \approx \left(1-\frac{k}{n}\right)\frac{k}{n}\frac{\log(d)}{d} - \frac{k}{n}\frac{\overline{y}}{d} \qquad \text{(A33)}$$

Thus, we obtain

$$\hat{\mathrm{c}}\mathrm{ov}(q,\exp(-x))\hat{\mathrm{v}}\mathrm{ar}(x) \approx \left(1-\frac{k}{n}\right)\left[\frac{k^2}{n^2}\frac{\left(\overline{q}_{i\in\cdot}-\overline{q}\right)\log^2(d)}{d} - \frac{k^2}{n^2}\frac{2\left(\overline{q}_{i\in\cdot}-\overline{q}\right)\log(d)}{d}\overline{y}\right] \qquad \text{(A34)}$$

and

$$\hat{\mathrm{c}}\mathrm{ov}(q,x)\hat{\mathrm{c}}\mathrm{ov}(x,\exp(-x)) \approx \left(1-\frac{k}{n}\right)\frac{k^2}{n^2}\frac{\left(\overline{q}_{i\in\cdot}-\overline{q}\right)\log^2(d)}{d} + \frac{\log(d)}{d}\frac{k}{n}\left[\left(1-\frac{k}{n}\right)\hat{\mathrm{c}}\mathrm{ov}(q,y) - \frac{k}{n}\left(\overline{q}_{i\in\cdot}-\overline{q}\right)\overline{y}\right] \qquad \text{(A35)}$$

Recalling the definition of $A(c)$ in eq.A15, the results of eq.A34 and A.35 enables us to obtain

$$\lim_{c\to -h_m} A(c) = -\frac{k}{n}\frac{\log(d)}{d}\left[\frac{k}{n}\left(\overline{q}_{i\in\cdot}-\overline{q}\right)\overline{y} + \left(1-\frac{k}{n}\right)\hat{\mathrm{c}}\mathrm{ov}(q,y)\right] \qquad \text{(A36)}$$

Since $\lim_{c\to -h_m}\left[-\log(d)/d\right] = \infty$, $\lim_{c\to -h_m} A(c) > 0$ will require that the parentheses in eq.A36 is greater than zero. This implies

$$\frac{k}{n}\left(\overline{q}_{i\in\cdot}-\overline{q}\right)\overline{y} + \left(1-\frac{k}{n}\right)\left(\overline{qy}-\overline{q}\,\overline{y}\right) > 0 \qquad \text{(A37)}$$

Using *w* defined in eq.A12, eq.A37 can be rewritten as

$$\frac{k}{n}\left(1-\frac{k}{n}\right)\left(\bar{q}\,\bar{y}\right)_{i\square\square}\square\left(1-\frac{k}{n}\right)\overline{wy}+\left(1-\frac{k}{n}\right)^{2}\left(\overline{wy}-\overline{wy}\right)\square\frac{k}{n}\left(1-\frac{k}{n}\right)\left(\bar{q}\,\bar{y}\right)_{i\square\square}>0\;\square$$

$$\frac{1}{n-k}\sum_{i=1}^{n}w_i y_i-\frac{1}{n-k}\sum_{i=1}^{n}w_i\,\frac{1}{n-k}\sum_{i=1}^{n}y_i>0\;\square \qquad \text{(A38)}$$

$$\frac{1}{n-k}\sum_{i\square\square}w_i y_i-\frac{1}{n-k}\sum_{i\square\square}w_i\,\frac{1}{n-k}\sum_{i\square\square}y_i>0$$

Thus is it easy to see, since $q_i = w_i \square i \square\square$ and $\lim_{c\square\square h_m} y_i = \log\left(h_i \square h_m\right)$, that the last expression of eq.A38 is identical to

$$\hat{\text{cov}}\left(q,\log\left(h\square h_m\right)_{i\square\square}\right)>0 \qquad \text{(A39)}$$

Corollary 1

Since $b(c)=\dfrac{\hat{\text{cov}}(q,x)}{\hat{\text{var}}(x)}$, we have that $\lim_{c\square\square h_m} A(c)>0 \;\square\; b\left(\square h_m\right)_{i\square\square}>0$ A40)

Lemma 4

$$\lim_{c\square} A(c)\,\square\,0^{\square}\;\square\;\hat{\text{cov}}\left(q,h^2\right)<\left(\frac{\hat{\text{skew}}(h)}{\hat{\text{var}}(h)}+2\bar{h}\right)\hat{\text{cov}}(q,h) \qquad \text{(A41)}$$

Proof

Using Taylor expansion we obtain

$$\lim_{c\to\infty}\bar{x} = \log(c) + \overline{\frac{h}{c} - \frac{h^2}{2c^2} + \frac{h^3}{3c^3} + o\left(\frac{1}{c^3}\right)} = \log(c) + \frac{\bar{h}}{c} - \frac{\overline{h^2}}{2c^2} + \frac{\overline{h^3}}{3c^3} + o\left(\frac{1}{c^3}\right) \quad \text{(A42)}$$

Hence, for c very large, we have the third order Taylor approximations

$$(\bar{x})^2 \approx \log^2(c) + \frac{2\bar{h}\log(c)}{c} - \frac{\overline{h^2}\log(c)}{c^2} + \frac{2\overline{h^3}\log(c)}{3c^3} + \frac{(\bar{h})^2}{c^2} - \frac{\bar{h}\overline{h^2}}{c^3} \quad \text{(A43)}$$

and

$$\overline{x^2} \approx \left(\log(c) + \frac{h}{c} - \frac{h^2}{2c^2} + \frac{h^3}{3c^3}\right)^2 =$$
$$\log^2(c) + \frac{2\bar{h}\log(c)}{c} - \frac{\overline{h^2}\log(c)}{c^2} + \frac{2\overline{h^3}\log(c)}{3c^3} + \frac{\overline{h^2}}{c^2} - \frac{\overline{h^3}}{c^3} \quad \text{(A44)}$$

Thus, we obtain

$$\hat{\mathrm{var}}(x) \approx \frac{\hat{\mathrm{var}}(h)}{c^2} - \frac{\hat{\mathrm{skew}}(h) + 2\bar{h}\,\hat{\mathrm{var}}(h)}{c^3} \quad \text{(A45)}$$

Furthermore, for very large c, we have

$$\overline{\exp(-x)} \approx \frac{1}{c}\overline{\left(1 - \frac{h}{c} + \frac{h^2}{c^2} - \frac{h^3}{c^3}\right)} = \frac{1}{c} - \frac{\bar{h}}{c^2} + \frac{\overline{h^2}}{c^3} - \frac{\overline{h^3}}{c^4} \quad \text{(A46)}$$

and

$$\overline{q\exp(-x)} \approx \frac{\bar{q}}{c} - \frac{\overline{hq}}{c^2} + \frac{\overline{h^2 q}}{c^3} - \frac{\overline{h^3 q}}{c^4} \quad \text{(A47)}$$

which result in

$$\hat{\mathrm{cov}}\left(q,\exp(-x)\right) \approx \frac{\bar{q}}{c} - \frac{\overline{hq}}{c^2} + \frac{\overline{h^2 q}}{c^3} - \frac{\overline{h^3 q}}{c^4} - \bar{q}\left(\frac{1}{c} - \frac{\bar{h}}{c^2} + \frac{\overline{h^2}}{c^3} - \frac{\overline{h^3}}{c^4}\right) = -\frac{\hat{\mathrm{cov}}(q,h)}{c^2} + \frac{\hat{\mathrm{cov}}(q,h^2)}{c^3} - \frac{\hat{\mathrm{cov}}(q,h^3)}{c^4} \tag{A48}$$

We then have

$$\hat{\mathrm{cov}}\left(q,\exp(-x)\right)\hat{\mathrm{var}}(x) \approx -\frac{\hat{\mathrm{var}}(h)\hat{\mathrm{cov}}(q,h)}{c^4} + \frac{\hat{\mathrm{var}}(h)\hat{\mathrm{cov}}(q,h^2) + \hat{\mathrm{skew}}(h)\hat{\mathrm{cov}}(q,h) + 2\bar{h}\,\hat{\mathrm{var}}(h)\hat{\mathrm{cov}}(q,h)}{c^5} \tag{A49}$$

Furthermore, for c very large, we obtain

$$\overline{qx} \approx \bar{q}\log(c) + \frac{\overline{qh}}{c} - \frac{\overline{qh^2}}{c^2} + \frac{\overline{qh^3}}{c^3} \qquad \text{A50)}$$

which implies

$$\hat{\mathrm{cov}}(q,x) \approx \frac{\hat{\mathrm{cov}}(q,h)}{c} - \frac{\hat{\mathrm{cov}}(q,h^2)}{2c^2} + \frac{\hat{\mathrm{cov}}(q,h^3)}{3c^3} \tag{A51}$$

Furthermore, for very large c, we have

$$\overline{x\exp(-x)} \approx \left(\log(c) + \frac{h}{c} - \frac{h^2}{2c^2} + \frac{h^3}{3c^3}\right)\left(\frac{1}{c} - \frac{h}{c^2} + \frac{h^2}{c^3} - \frac{h^3}{c^4}\right) = \frac{\log(c)}{c} - \frac{\bar{h}\log(c)}{c^2} + \frac{\bar{h}}{c^2} + \frac{\overline{h^2}\log(c)}{c^3} - \frac{3\overline{h^2}}{2c^3} - \frac{\overline{h^3}\log(c)}{c^4} + \frac{11\overline{h^3}}{6c^4} \tag{A52}$$

Using eq.A42 and eq.A46 we get

$$\bar{x}\,\overline{\exp(-x)} \approx \left(\log(c) + \frac{\bar{h}}{c} - \frac{\overline{h^2}}{2c^2} + \frac{\overline{h^3}}{3c^3}\right)\left(\frac{1}{c} - \frac{\bar{h}}{c^2} + \frac{\overline{h^2}}{c^3} - \frac{\overline{h^3}}{c^4}\right) = \frac{\log(c)}{c} - \frac{\bar{h}\log(c)}{c^2} + \frac{\overline{h^2}\log(c)}{c^3} - \frac{\overline{h^3}\log(c)}{c^4} + \frac{\bar{h}}{c} - \frac{\bar{h}^2}{c^3} + \frac{3\bar{h}\overline{h^2}}{c^4} - \frac{\overline{h^2}}{2c^3} + \frac{\overline{h^3}}{2c^4} \tag{A53}$$

Combining eq.A52 and eq.A53 yields

$$\hat{\text{cov}}(x, \exp(\square x)) \square \square \frac{\hat{\text{var}}(h)}{c^3} + \frac{3}{2c^4}\left(\hat{\text{skew}}(h) + 2\bar{h}\,\hat{\text{var}}(h)\right) \quad \text{(A54)}$$

Thus, it follows that

$$\hat{\text{cov}}(q, x)\hat{\text{cov}}(x, \exp(\square x)) \square$$
$$\square \frac{\hat{\text{cov}}(q,h)\hat{\text{var}}(h)}{c^4} + \frac{1}{c^5}\left[\frac{3}{2}\hat{\text{skew}}(h)\hat{\text{cov}}(q,h) + 3\bar{h}\,\hat{\text{var}}(h)\hat{\text{cov}}(q,h) + \frac{1}{2}\hat{\text{cov}}(q,h^2)\hat{\text{var}}(h)\right]$$

(A55)

Using eq.49 and eq.55 results in

$$A(c) \square \frac{1}{2c^5}\left(\square 2\bar{h}\,\hat{\text{var}}(h)\text{cov}(q,h) + \hat{\text{var}}(h)\text{cov}(q,h^2) \square \hat{\text{skew}}(h)\hat{\text{cov}}(q,h)\right) \quad \text{(A56)}$$

which enables us to see that a negative parenthesis in eq.A56 is equivalent to

$$\hat{\text{cov}}(q,h^2) < \left(\frac{\hat{\text{skew}}(h)}{\hat{\text{var}}(h)} + 2\bar{h}\right)\hat{\text{cov}}(q,h) \quad \text{(A57)}$$

which, if valid, is equivalent to

$$\lim_{c \square} A(c) \square \ 0^{\square} \quad \text{(A58)}$$

Corollary 2

From eq.A45 we see that

$$\lim_{c \square} \hat{\text{var}}(x) = \frac{\hat{\text{var}}(h)}{c^2} + o\left(\frac{1}{c^2}\right) \quad \text{(A59)}$$

which implies that

$$\lim_{c\to\infty} \hat{\text{v}}\text{ar}(x) \to 0^{+} \tag{A60}$$

since we from the definition/requirement in eq.A14 have that $\hat{\text{v}}\text{ar}(h)$ must be greater than zero.

Corollary 3

From eq.A51 we, for a large *c*, see that

$$\lim_{c\to\infty} \hat{\text{c}}\text{ov}(q,x) \to \frac{\hat{\text{c}}\text{ov}(q,h)}{c} + o\left(\frac{1}{c}\right) \tag{A61}$$

which implies that

$$\lim_{c\to\infty} \hat{\text{c}}\text{ov}(q,x) \to 0^{+} \tag{A62}$$

if and only if

$$\hat{\text{c}}\text{ov}(q,h) > 0 \tag{A63}$$

Lemma 5

$r(c)$ and $\partial r(c)/\partial c$ are continuous for $c \in (-h_m, \infty)$

Proof

We have that

$$r(c) = \frac{\hat{\text{cov}}^2(q,x)}{\hat{\text{var}}(x)} \tag{A64}$$

and

$$\frac{\partial r(c)}{\partial c} = 2\frac{\hat{\text{cov}}(q,x)}{\hat{\text{var}}^2(x)}\left[\hat{\text{cov}}(q,\exp(-x))\hat{\text{var}}(x) - \hat{\text{cov}}(q,x)\hat{\text{cov}}(x,\exp(-x))\right] \tag{A65}$$

Firstly, we know that $\hat{\text{var}}(x)$, $\hat{\text{cov}}(q,x)$, $\hat{\text{cov}}(q,\exp(-x))$ and $\hat{\text{cov}}(x,\exp(-x))$ are nothing more than functions that consist of sums or sums of squares of $x = \log(h_j + c)$ and/or $(h_j + c)^{-1}$, of which $\hat{\text{cov}}(q,x)$ and $\hat{\text{cov}}(q,\exp(-x))$ in addition are scaled by scalars which are functions of q_j. Since both $\log(h_j + c)$ and $(h_j + c)^{-1}$ are continuous for all $c > -h_m$, the numerators of eq.A64 and eq.A65 are continuous for $c \in \langle -h_m, \infty\rangle$, since any function consisting of products or sums of continuous functions is continuous. We also know that the denominators of eq.A64 and eq.A65 are continuous and different from zero, since $\hat{\text{var}}(x)$ by the definition/requirement in eq.A14 is strictly positive. Hence, are $r(c)$ and $\partial r(c)/\partial c$ continuous for $c \in \langle -h_m, \infty\rangle$.

Proof of theorem 1

We have that

$$\frac{\partial r(c)}{\partial c} = 2\frac{\hat{\text{cov}}(q,x)}{\hat{\text{var}}^2(x)}A(c) \tag{A66}$$

Since the argument c spans the interval $I \in \langle -h_m, \infty\rangle$ and $\partial r(c)/\partial c$ by lemma 5 is continuous on I, there must exist an odd number of finite arguments c inside I so that $\partial r(c)/\partial c = 0$ for each of these arguments, of which $\partial^2 r(c)/\partial c^2 < 0$ for at least one of these arguments, if

$$\lim_{c \to -h_m} \frac{\partial r(c)}{\partial c} \in \langle 0, \infty \rangle \quad \text{and} \quad \lim_{c \to \infty} \frac{\partial r(c)}{\partial c} \in \langle -\infty, 0 \rangle \tag{A67}$$

Eq.A68 is a consequence of the well known *Intermediate-Value Theorem of Continuous Calculus*.

We see that if criterion i. and criterion iii. of theorem 1 are met, then, by Lemma 1, Lemma 2 and Lemma 3, we have that

$$\lim_{c \to -h_m} \left(\frac{1}{\hat{\mathrm{var}}^2(c)} \cdot \hat{\mathrm{cov}}(q,x) \cdot A(c) \right) = \left(\frac{1}{[illegible]} \right)^2 \cdot [illegible] \cdot [illegible] \tag{A68}$$

which proves that the first requirement of eq.A67 is met. Not necessary to prove but worth noting, is that the limit of eq.A68 in fact is ∞, which is easily seen by considering eq.A22, eq.A26 and eq.A36.

Furthermore, we see that if criterion ii. and criterion iv. of theorem 1 are met, then, by lemma 4, corollary 3 and corollary 4, it follows that

$$\lim_{c \to \infty} \left(\frac{1}{\hat{\mathrm{var}}^2(c)} \cdot \hat{\mathrm{cov}}(q,x) \cdot A(c) \right) = \left(\frac{1}{0^+} \right)^2 \cdot 0^+ \cdot 0^- \tag{A69}$$

which proves that the last requirement of eq.A67 is met (note that the limit of eq.A69 in fact is 0^-, which is a consequence of $r(c)$ having an upper limit).

Thus, if criteria (i. – iv.) of theorem 1 are all met, then eq.A67 is true and $r(c)$ has an odd number of stationary maximum/minimum points of which at least one is a maxima.

Remark 1

The requirement $\partial^2 r(c) / \partial c^2 \neq 0$ will exclude any saddle points from being counted as one of the odd stationary points in theorem 1.

Corollary 4

From eq.A22 and eq.A26, *d* very small and positive, we obtain

$$r(c) \approx \frac{\left[\frac{k}{n}\left(\bar{q}_{i\in\square} - \bar{q}\right)\log(d) + \hat{\operatorname{cov}}(q, y)\right]^2}{\left(1 - \frac{k}{n}\right)\frac{k}{n}\log^2(d) - 2\frac{k}{n}\log(d)\bar{y} + \hat{\operatorname{var}}(y)} \qquad \text{(A71)}$$

Applying L'Hospital's rule twice on eq.A71 we obtain that

$$\lim_{c \to -h_m} \frac{\hat{\operatorname{cov}}^2(q, x)}{\hat{\operatorname{var}}(x)} = \frac{k}{n}\left(1 - \frac{k}{n}\right)\left(\bar{q}_{i\in\square} - \bar{q}_{i\notin\square}\right)^2 \qquad \text{(A72)}$$

which is the left hand limit for *r*(*c*).

Corollary 5

Using the expressions of eq.A45 and eq.A51, *c* very large, we get

$$\frac{\hat{\operatorname{cov}}^2(q, x)}{\hat{\operatorname{var}}(x)} \approx \frac{\left(\frac{\hat{\operatorname{cov}}(q, h)}{c}\right)^2}{\left(\frac{\hat{\operatorname{var}}(h)}{c^2}\right)} \qquad \text{(A73)}$$

implying

$$\lim_{c \to \infty} r(c) = \frac{\hat{\text{c}}\text{ov}^2(q,h)}{\hat{\text{v}}\text{ar}(h)} \tag{A74}$$

which is the right hand limit for $r(c)$.

Lemma 6

Let $\hat{q}^+ = \hat{a} + \hat{b}\log(h^+ + \hat{c})$ be a predicted log-discharge. We then have

$$\lim_{c \to \infty} \hat{q}^+ = \hat{\alpha} + \hat{\beta}h^+ \tag{A75}$$

where

$$\hat{\alpha} = \bar{q} - \bar{h}\frac{\hat{\text{c}}\text{ov}(q,h)}{\hat{\text{v}}\text{ar}(h)}, \quad \hat{\beta} = \frac{\hat{\text{c}}\text{ov}(q,h)}{\hat{\text{v}}\text{ar}(h)} \tag{A76}$$

Proof

Recalling that $b(c) = \dfrac{\hat{\text{c}}\text{ov}(q,x)}{\hat{\text{v}}\text{ar}(x)}$, we get, for a very large c, using Taylor approximation

$$\begin{aligned}
b(c) &\approx \frac{\dfrac{\hat{\text{c}}\text{ov}(q,h)}{c} - \dfrac{\hat{\text{c}}\text{ov}(q,h^2)}{c^2}}{\dfrac{\hat{\text{v}}\text{ar}(h)}{c^2} - \dfrac{\hat{\text{s}}\text{kew}(h) + 2\bar{h}\,\hat{\text{v}}\text{ar}(h)}{c^3}} = \frac{\dfrac{\hat{\text{c}}\text{ov}(q,h)}{c}\left(1 - \dfrac{\hat{\text{c}}\text{ov}(q,h^2)}{2c\,\hat{\text{c}}\text{ov}(q,h)}\right)}{\dfrac{\hat{\text{v}}\text{ar}(h)}{c^2}\left(1 - \dfrac{1}{c}\left(\dfrac{\hat{\text{s}}\text{kew}(h)}{\hat{\text{v}}\text{ar}(h)} + 2\bar{h}\right)\right)} \approx \\
&c\frac{\hat{\text{c}}\text{ov}(q,h)}{\hat{\text{v}}\text{ar}(h)}\left(1 - \frac{\hat{\text{c}}\text{ov}(q,h^2)}{2c\,\hat{\text{c}}\text{ov}(q,h)}\right)\left(1 + \frac{1}{c}\left(\frac{\hat{\text{s}}\text{kew}(h)}{\hat{\text{v}}\text{ar}(h)} + 2\bar{h}\right)\right) \approx \\
&c\frac{\hat{\text{c}}\text{ov}(q,h)}{\hat{\text{v}}\text{ar}(h)}\left(1 + \frac{1}{c}\left(\frac{\hat{\text{s}}\text{kew}(h)}{\hat{\text{v}}\text{ar}(h)} + 2\bar{h} - \frac{\hat{\text{c}}\text{ov}(q,h^2)}{2c\,\hat{\text{c}}\text{ov}(q,h)}\right)\right)
\end{aligned} \tag{A77}$$

We define the constants

$$H_A = \frac{\hat{\text{skew}}(h)}{\hat{\text{var}}(h)} + 2\bar{h} - \frac{\hat{\text{cov}}(q, h^2)}{2c\,\hat{\text{cov}}(q, h)} \tag{A78}$$

and

$$H_B = \frac{\hat{\text{cov}}(q, h)}{\hat{\text{var}}(h)} \tag{A79}$$

Thus, for a very large *c*, we obtain

$$b(c) \approx \frac{c}{H_B}\left(1 + \frac{H_A}{c}\right) = \frac{1}{H_B}(c + H_A) \tag{A80}$$

and by the definition of *a* from eq.5a

$$a(c) = \bar{q} - b(c)\bar{x} \approx \bar{q} - \frac{1}{H_B}(c + H_A)\left[\log(c) + \frac{\bar{h}}{c} - \frac{\overline{h^2}}{2c^2}\right] \approx$$

$$\bar{q} - \frac{1}{H_B}\left[c\log(c) + H_A\log(c) + \bar{h} + \frac{H_A\bar{h} - \frac{1}{2}\overline{h^2}}{c}\right]$$

A81)

which, for *c* very large, gives the estimations

$$\hat{q}^+ = a(c) + b(c)\log(h^+ + c) \approx$$

$$\bar{q} - \frac{1}{H_B}\left[c\log(c) + H_A\log(c) + \bar{h} + \frac{H_A\bar{h} - \frac{1}{2}\overline{h^2}}{c}\right] + \frac{1}{H_B}(c + H_A)\left[\log(c) + \frac{h^+}{c} - \frac{(h^+)^2}{2c^2}\right] =$$

$$\bar{q} + \frac{h^+ - \bar{h}}{H_B} + \frac{1}{H_B c}\left\{H_A(h^+ - \bar{h}) - \frac{1}{2}\left[(h^+)^2 - \overline{h^2}\right]\right\}$$

(A82)

which implies

$$\lim_{c \to \infty} \hat{q}^{+} = \bar{q} + \frac{h^{+} - \bar{h}}{H_B}$$