PRICING RATE OF RETURN GUARANTEES IN A HEATH-JARROW-MORTON FRAMEWORK

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ABSTRACT. Rate of return guarantees are included in many financial products, for example life insurance contracts or guaranteed investment contracts issued by investment banks. The holder of such a contract is guaranteed a fixed periodically rate of return rather than—or in addition to—a fixed absolute amount at expiration.

We consider rate of return guarantees where the underlying rate of return is either (i) the rate of return on a stock investment or (ii) the short-term interest rate. Various types of these rate of return guarantees are priced in a general no-arbitrage Heath-Jarrow-Morton framework. We show that despite fundamental differences in the underlying rate of return processes ((i) or (ii)), the resulting pricing formulas for the guarantees are remarkably similar.

Finally, we show how the term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985) occur as special cases in our more general framework based on the model of Heath, Jarrow, and Morton (1992).

1. INTRODUCTION

Interest rate guarantees are included in several financial products. For example many life insurance contracts guarantee the policy holder a fixed minimum annual percentage return. Another example is guaranteed investment contracts sold by investment banks, cf., e.g., Walker (1992).

In principle, a guarantee may be connected to any specified rate of return, referred to as the *rate of return process* or simply the *return process*. Real-life examples include rate of returns of stocks and mutual funds, various indexes, or interest rates. In this treatment, we consider (i) guarantees on return processes connected to assets traded in financial markets and (ii) guarantees on the short-term interest rate process. Guarantees on stock returns are obvious examples of the first kind of guarantee and we sometimes refer to the underlying financial asset simply as a stock in that case.

The very existence of guaranteed return contracts reflects the volatile nature of rates of return. It is reasonable to expect that the interest rates in the economy influence any rate of return process. A proper valuation model should accordingly include a consistent model of the stochastic behavior of the interest rate. We work in a Heath-Jarrow-Morton framework. This is a rather general framework and we show how the popular term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985) occur as special cases.

Boyle and Hardy (1997) deal with *long guarantees* or *maturity guarantees*, i.e. guarantees effective only at the point of expiration of the contract and compare different approaches to pricing these guarantees. The cashflows connected to maturity guarantees are related to cashflows of European options. Thus, market prices of long guarantees may readily be expressed in terms of known results for European options. We also include some results for long guarantees, in particular the structure of the resulting pricing formula

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is identical for deterministic and stochastic interest rates for guarantees on stock returns and guarantees on interest rates. This result is perhaps surprising since stock market returns are of unbounded variation whereas accumulated interest rates are of bounded variation.

In the case of annual guarantees we present a rather general expression for its date zero market value. As opposed to the case of maturity guarantees the structure of this formula is much simpler in the case of stock guarantees and deterministic interest rates than in the case of stochastic interest rates. For one special case including deterministic interest rates and guarantees connected to stock returns, our expression specializes to the formula by Hipp (1996). For another special case limited to only two periods and guarantees on interest rates the formula of Persson and Aase (1997) is rediscovered. As a third special case we present a new closed form solution in the two period case for guarantees on stock returns in a model with stochastic interest rates.

Pedersen and Shiu (1994) and Grosen and Jørgensen (1997) deal with other aspects of guaranteed investment contracts and interest rate guarantees.

The paper is organized as follows: In Section 2 the set-up is explained. We show how the term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985) occur as special cases in our more general framework based on the model of Heath, Jarrow, and Morton (1992) in Section 3. In Section 4 pricing results for European call options and long guarantees are obtained. These results are generalized to multi period guarantees in Section 5. Section 6 contains some concluding remarks.

2. The Model

The model of Heath, Jarrow, and Morton (1992) is based on the definitional relationship between forward rates and market prices of unit discount bonds

$$P(t,T) = e^{-\int_t^T f(t,s)ds}.$$

The major primitive is the family of continuously compounded forward rates $f(t,s)$, $0 \le t \le s \le T$, given under an equivalent martingale measure by Itô-processes of the form

$$f(t,s) = f(0,s) + \int_0^t \sigma_f(v,s) \int_v^s \sigma_f(v,u)du dv + \int_0^t \sigma_f(v,s)dW_v.$$

Here W_t, $0 \le t \le T$ is a, possibly multi-dimensional, standard Brownian motion defined on a given filtered probability space. The volatility process $\sigma_f(t,s)$, $0 \le t \le s \le T$, satisfies some technical regularity conditions, cf. Heath, Jarrow, and Morton (1992). The short-term interest rate (spot rate) in the economy is given by

$$r_t = f(t,t) = f(0,t) + \int_0^t \sigma_f(v,t) \int_v^t \sigma_f(v,u)du dv + \int_0^t \sigma_f(v,t)dW_v \tag{1}$$

under an equivalent martingale measure.

When considering the return process of an asset traded in a financial market, we assume that the underlying market price process of the asset under an equivalent martingale measure satisfies the stochastic differential equation

$$S(t) = S(0) + \int_0^t r_v S(v)dv + \int_0^t S(v)\sigma_S(v)dW_v.$$

In most of the paper we will assume that the volatility process $\sigma_S(t)$, $0 \le t \le T$, is deterministic. Possible correlation between the return process and the interest rate process comes via the specification of the diffusion terms (σ_f and σ_S), since it is the same (multi-dimensional) Brownian motion, W, that is used in both SDEs. Later we consider a deterministic interest rate process, r_t, as a special case. In that case $S(t)$ for any fixed t is log-normally distributed.

For our purpose it is convenient to define the associated cumulative return process $\delta(t)$ as

$$\delta(t) = \int_0^t \left(r_v - \frac{1}{2}\sigma_S(v)^2\right) dv + \int_0^t \sigma_S(v) dW_v. \tag{2}$$

Then the familiar relationship from deterministic models between market price and return, $S(t) = S(0)e^{\delta(t)}$, also holds in this stochastic environment.

3. Relation to Earlier Models

In this section we will show how to fit our general no-arbitrage model into two well-known models from the literature, the Vasicek model and the Cox-Ingersoll-Ross model. In this section the Brownian motion is only one-dimensional.

3.1. The Vasicek (1977) Model.

The Vasicek (1977) Model. Under an equivalent martingale measure the SDE of the spot interest rate is given by

$$r_t = r_0 + \int_0^t \kappa_v(\hat{\theta}_v - r_v) dv + \int_0^t \sigma_v dW_v, \tag{3}$$

where $\hat{\theta}_t = \theta_t - \frac{\sigma_t \lambda_t}{\kappa_t}$ is the risk-adjusted mean reversion level and λ_t is the market price of interest rate risk. This SDE can be solved as

$$r_t = r_0 e^{-\int_0^t \kappa_u du} + \int_0^t e^{-\int_v^t \kappa_u du}(\kappa_v \theta_v - \sigma_v \lambda_v) dv + \int_0^t e^{-\int_v^t \kappa_u du} \sigma_v dW_v. \tag{4}$$

On the other hand the solution for the SDE for r from the Heath-Jarrow-Morton model is given by

$$r_t = f(t,t) = f(0,t) + \int_0^t \mu_f(v,t) dv + \int_0^t \sigma_f(v,t) dW_v. \tag{5}$$

Moreover, under an equivalent martingale measure the drift of the forward rate, μ_f, is determined as

$$\mu_f(v,t) = \sigma_f(v,t) \int_v^t \sigma_f(v,u) du \tag{6}$$

by the Heath-Jarrow-Morton drift restriction, cf. equation (1). Comparing r_t and $f(t,t)$ from equations (4) and (5) gives that σ_f must be specified as

$$\sigma_f(v,t) = e^{-\int_v^t \kappa_u du} \sigma_v.$$

Hence the drift, μ_f, can be derived from equation (6)

$$\mu_f(v,t) = \sigma_v^2 e^{-\int_v^t \kappa_u du} \int_v^t e^{-\int_v^s \kappa_u du} ds.$$

Using these specifications of $\mu_f(t,s)$ and $\sigma_f(t,s)$ the SDE for $f(t,t)$ may now be written as

$$f(t,t) = f(0,0) + \int_0^t \kappa_v \left(\frac{1}{\kappa_v} \int_0^v \sigma_s^2 e^{-2\int_s^v \kappa_u du} ds - f(v,v)\right) dv + \int_0^t \sigma_v dW_v, \tag{7}$$

clearly demonstrating the same mean reverting structure as the Vasicek specification introduced earlier. Observe that the Vasicek parameter $\hat{\theta}_v$ is given by $\frac{1}{\kappa_v}\int_0^v \sigma_s^2 e^{-2\int_s^v \kappa_u du} ds$ in the HJM specification, cf. equations (3) and (7).

Matching drift terms in the Heath-Jarrow-Morton model and the Vasicek model under an equivalent martingale measure yields, cf. equations (4) and (5)

$$f(0,t) + \int_0^t \sigma_v^2 e^{-\int_v^t \kappa_u du} \int_v^t e^{-\int_v^s \kappa_u du} ds\, dv = f(0,0)e^{-\int_0^t \kappa_u du} + \int_0^t e^{-\int_v^t \kappa_u du}(\kappa_v \theta_v - \sigma_v \lambda_v)dv.$$

Multiplying this equation with $e^{\int_0^t \kappa_u du}$ and differentiating with respect to t yields

$$\kappa_t e^{\int_0^t \kappa_u du} f(0,t) + e^{\int_0^t \kappa_u du} \frac{\partial}{\partial t} f(0,t) + \int_0^t \sigma_v^2 e^{\int_0^v \kappa_u du} e^{-\int_v^t \kappa_u du} dv = e^{\int_0^t \kappa_u du}(\kappa_t \theta_t - \sigma_t \lambda_t).$$

From this equation we can find an expression for the market price of risk, λ_t, as

$$\lambda_t = \frac{\kappa_t}{\sigma_t}\big(\theta_t - f(0,t)\big) - \frac{1}{\sigma_t}\Big(\frac{\partial}{\partial t} f(0,t) + \int_0^t \sigma_v^2 e^{-2\int_v^t \kappa_u du} dv\Big).$$

If we restrict the parameters κ, θ, and σ of the Vasicek model to be constant, the market price of risk can alternatively be derived as

$$\lambda_t = \frac{\kappa}{\sigma}\Big(\theta - \frac{f(0,t) - f(0,0)e^{-\kappa t}}{1 - e^{-\kappa t}}\Big) - \frac{\sigma}{2\kappa}(1 - e^{-\kappa t}).$$

Many applications in finance, furthermore, restrict the market price of risk, λ, to be constant. By imposing this assumption, the initial forward rates may be determined in terms of the parameters of the Vasicek model as

$$f(0,t) = f(0,0)e^{-\kappa t} + \Big(\theta - \frac{\sigma\lambda}{\kappa}\Big)\big(1 - e^{-\kappa t}\big) - \frac{\sigma^2}{2\kappa^2}\big(1 - e^{-\kappa t}\big)^2.$$

According to Jarrow (1997) this problem has been studied by Robin Brenner, unfortunately, we have not been able to trace specific references.

3.2. **The Cox, Ingersoll, and Ross (1985) Model.** A similar analysis is performed on the Cox-Ingersoll-Ross model in Heath, Jarrow, and Morton (1992, Section 8). Therefore, we will just present how to specify the volatility function of the forward rate process to get the Cox-Ingersoll-Ross model as a special case of the Heath-Jarrow-Morton model.

Under an equivalent martingale measure the SDE of the spot interest rate is given by

$$r_t = r_0 + \int_0^t \kappa(\hat{\theta}_v - r_v)dv + \int_0^t \sigma\sqrt{r_v}\, dW_v,$$

where $\hat{\theta}_t$ is the risk-adjusted mean reversion level. This SDE has a solution but it cannot be written in an explicit form. Cox, Ingersoll, and Ross (1985) show that the zero-coupon bond prices can be calculated as

$$P(t,T) = A(t,T)e^{-B(t,T)r_t},$$

where $B(t,T)$ is given by

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + \kappa + \lambda)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

and $\gamma = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}$. λ is related to the market price of risk. $A(t,T)$ is not important for our purpose.

By Itô's lemma the SDE of the zero-coupon bond prices are

$$P(t,T) = P(0,T) - \int_0^t \big(B(v,T)P(v,T)\kappa(\hat{\theta}_v - r_v) - \frac{1}{2}B(v,T)^2 P(v,T)\sigma^2 r_v - e^{-B(v,T)r_v}\frac{\partial}{\partial v}A(v,T) + r_v P(v,T)\frac{\partial}{\partial v}B(v,T)\big)\,dv - \int_0^t B(v,T)P(v,T)\sigma\sqrt{r_v}\,dW_v.$$

On the other hand the SDE of the zero-coupon bond prices by the Heath-Jarrow-Morton model is given by

$$P(t,T) = P(0,T) + \int_0^t P(v,T)\Big(f(v,v) - \int_v^T \mu_f(v,s)ds + \frac{1}{2}\Big(\int_v^T \sigma_f(v,s)ds\Big)^2\Big)dv - \int_0^t P(v,T)\Big(\int_v^T \sigma_f(v,s)ds\Big)dW_v.$$

Hence, by matching diffusion terms in these two SDEs yields

$$B(t,T)\sigma\sqrt{f(t,t)} = \int_t^T \sigma_f(t,s)ds.$$

Differentiating with respect to T gives the expression of how to specify the diffusion term of the Heath-Jarrow-Morton model to get the Cox-Ingersoll-Ross model

$$\begin{aligned}\sigma_f(t,s) &= \sigma\sqrt{f(t,t)}\frac{\partial}{\partial s}B(t,s)\\ &= \frac{4\sigma\gamma^2 e^{\gamma(s-t)}}{\big((\gamma+\kappa+\lambda)(e^{\gamma(s-t)}-1)+2\gamma\big)^2}\sqrt{f(t,t)}.\end{aligned}$$

Finally, the drift, μ_f, is given by

$$\mu_f(t,s) = \frac{8\sigma^2\gamma^2 e^{\gamma(s-t)}(e^{\gamma(s-t)}-1)}{\big((\gamma+\kappa+\lambda)(e^{\gamma(s-t)}-1)+2\gamma\big)^3}f(t,t)$$

with a little help from Mathematica.

4. Closed Form Solutions for Long Guarantees

A *long* or a *one period* guarantee guarantees the holder a minimum average return in the contract period. As we demonstrate below, the payoffs of long guarantees are very similar to payoffs of European options. Thus, pricing formulas for long guarantees follow directly from known pricing results for European options. Multi-period guarantees are treated in Section 5.

We work with two different underlying assets for the long interest rate guarantees. First, a *stock market account* is defined as

$$A_\delta(t) = e^{\delta(t)}.$$

Let $\beta(t)$ denote the cumulated return of the short-term interest rate process, i.e.,

$$\beta(t) = \int_0^t r_s ds = \int_0^t f(s,s)ds. \tag{8}$$

The similar account corresponding to the stock market account involving the short-term interest rate is defined as

$$A_\beta(t) = e^{\beta(t)}$$

and is termed *the savings account.*

The payoffs including long guarantees on these accounts are given as

$$A_\alpha(t) \vee e^{gt},$$

where g represents the constant guaranteed minimum rate of return and $\alpha \in \{\delta, \beta\}$. Moreover, $X \vee Y = \max(X, Y)$.

Now consider European call options on the stock market and savings accounts with payoffs at date t

$$\big(A_\alpha(t) - K\big)^+,$$

where K represents the constant exercise price and $\alpha \in \{\delta, \beta\}$. Furthermore, $(Z)^+ = \max(Z, 0)$.

Observe the simple relationship between the European call option with maturity date t and the long guarantee,

$$A_\alpha(t) \vee e^{gt} = K + \big(A_\alpha(t) - K\big)^+, \tag{9}$$

where the exercise price of the option is $K = e^{gt}$ and $\alpha \in \{\delta, \beta\}$. The date zero market price for the European option payable at date t is

$$V^\alpha = E^Q\big[e^{-\int_0^t r_s\,ds}\big(A_\alpha(t) - K\big)^+\big], \tag{10}$$

according to the standard results of Harrison and Kreps (1979) and Harrison and Pliska (1981). Hence, using equation (9), the market price of the long guarantee is

$$\pi^\alpha = P(0, t)e^{gt} + V^\alpha. \tag{11}$$

For the rest of this section we assume that forward rates as well as stock market returns are Gaussian, i.e., $\sigma_f(t, s)$, $0 \le t \le s \le T$ and $\sigma_S(t)$, $0 \le t \le T$ are deterministic processes.

4.1. **European Call Option and Long Guarantee on the Stock Market Account—Deterministic Interest Rates.** The first case we consider is a European call option on the stock market account payable at date t, where the short-term interest rate r_t is deterministic, i.e., $\sigma_f(t, s) = 0$, $0 \le t \le s \le T$. By this assumption market prices of bonds are given by the formula $P(t, T) = e^{-\int_t^T r_s\,ds}$.

From equation (2) and the assumption of deterministic interest rates the variance of the cumulated return process $\delta(t)$ is

$$\sigma_d(t)^2 = \int_0^t \big(\sigma_S(v)\big)^2 dv.$$

The price of this claim is well-known:

Proposition 4.1. *The date zero market price of a European call option on the stock market account payable at date t under deterministic interest rate is*

$$V^\delta = \Phi\Big(\frac{-\ln K - \ln P(0, t)}{\sigma_d(t)} + \frac{1}{2}\sigma_d(t)\Big) - KP(0, t)\Phi\Big(\frac{-\ln K - \ln P(0, t)}{\sigma_d(t)} - \frac{1}{2}\sigma_d(t)\Big).$$

Proof. The payoff of this contract is identical to the payoff of a standard European call option where the initial price of the stock is normalized to 1. The result follows from Black and Scholes (1973) or Merton (1973). □

The following corollary follows immediately from the stated relation (11) between the payoffs of European call options and the long guarantees.

Corollary 4.2. *The market price at date zero of the claim $A_\delta(t) \vee e^{gt}$ under deterministic interest rate is*

$$\pi^\delta = \Phi\Big(\frac{-gt - \ln P(0,t)}{\sigma_d(t)} + \frac{1}{2}\sigma_d(t)\Big) + e^{gt}P(0,t)\Phi\Big(\frac{gt + \ln P(0,t)}{\sigma_d(t)} + \frac{1}{2}\sigma_d(t)\Big).$$

4.2. **European Call Option and Long Guarantee on the Savings Account.** The next case we consider is the case treated by Persson and Aase (1997) involving the payoff $\big(A_\beta(t) - K\big)^+$. Using equation (10), the date zero market value of a European call option on the savings account is

$$V^\beta = E^Q\big[e^{-\int_0^t r_s ds}\big(A_\beta(t) - K\big)^+\big] = E^Q\big[\big(1 - Ke^{-\beta(t)}\big)^+\big].$$

Here we remark that

$$\beta(t) = \int_0^t r_s ds = -\ln P(0,t) + \frac{1}{2}\sigma_\beta(t)^2 + \int_0^t \int_v^t \sigma_f(v,u)du dW_v,$$

where

$$\sigma_\beta(t)^2 = \int_0^t \Big(\int_v^t \sigma_f(v,u)du\Big)^2 dv$$

represents the variance of the cumulative return process $\beta(t)$ for the savings account.

Proposition 4.3. *The date zero market price of a European call option with expiration at date t on the savings account is*

$$V^\beta = \Phi\Big(\frac{-\ln K - \ln P(0,t)}{\sigma_\beta(t)} + \frac{1}{2}\sigma_\beta(t)\Big) - KP(0,t)\Phi\Big(\frac{-\ln K - \ln P(0,t)}{\sigma_\beta(t)} - \frac{1}{2}\sigma_\beta(t)\Big).$$

Proof. The result follows by straightforward calculations. □

Corollary 4.4. *The market price at date zero of the claim $A_\beta(t) \vee e^{gt}$ is*

$$\pi^\beta = \Phi\Big(\frac{-gt - \ln P(0,t)}{\sigma_\beta(t)} + \frac{1}{2}\sigma_\beta(t)\Big) + e^{gt}P(0,t)\Phi\Big(\frac{gt + \ln P(0,t)}{\sigma_\beta(t)} + \frac{1}{2}\sigma_\beta(t)\Big).$$

4.3. **European Call Option and Long Guarantee on the Stock Market Account—Stochastic Interest Rates.** The last case we consider is a European call option on the stock market account payable at date t, where the short-term interest rate r_t is stochastic.

From equation (2) and the assumption of stochastic interest rate the variance of the cumulated return process $\delta(t)$ is

$$\sigma_\delta(t)^2 = \sigma_d(t)^2 + 2\int_0^t \sigma_S(v) \int_v^t \sigma_f(v,u)du dv + \sigma_\beta(t)^2.$$

Proposition 4.5. *The date zero market price of a European call option with expiration at date t on the stock market account under stochastic interest rate is*

$$V = \Phi\Big(\frac{-\ln K - \ln P(0,t)}{\sigma_\delta(t)} + \frac{1}{2}\sigma_\delta(t)\Big) - KP(0,t)\Phi\Big(\frac{-\ln K - \ln P(0,t)}{\sigma_\delta(t)} - \frac{1}{2}\sigma_\delta(t)\Big).$$

Proof. Cf. Merton (1973) and Amin and Jarrow (1992). □

Corollary 4.6. *The market price at date zero of the claim $A_\delta(t) \vee e^{gt}$ under stochastic interest rate is*

$$\pi = \Phi\Big(\frac{-gt - \ln P(0,t)}{\sigma_\delta(t)} + \frac{1}{2}\sigma_\delta(t)\Big) + e^{gt}P(0,t)\Phi\Big(\frac{gt + \ln P(0,t)}{\sigma_\delta(t)} + \frac{1}{2}\sigma_\delta(t)\Big).$$

Note the similar structure in corollaries 4.2, 4.4, and 4.6—only the σ parameter changes. In all cases the σ^2 parameter represents the variance of the accumulated return from date zero to the maturity of the contract.

5. Multi period guarantees

In this section we consider guarantees over more than one period. Imagine the time horizon T divided into N sub-periods of length Δ with possibly a different guaranteed return in each sub-period. A sub-period typically corresponds to a year in potential applications. Only deterministic guarantees are considered. However, the derived results will generalize to guarantees which are not known until the beginning of the period where they become effective.

An investment of one unit of account at date zero into a *general* account with periodical returns α_j and periodical minimum guarantees g_j, $j = 1, \ldots, n$, will, at the end of period n, $1 \leq n \leq N$, be

$$C_n^\alpha = e^{\sum_{j=1}^n (\alpha_j \vee g_j \Delta)}. \tag{12}$$

For $n = 1$ the multi period guaranteed payoff C_1^α is identical to the one period guaranteed payoff $A_\alpha(\Delta) \vee e^{g\Delta}$.

Proposition 5.1. *The market price at date zero of the claim C_n^α from equation* (12) *is*

$$\pi_n^\alpha = \sum_{\omega \in \Omega} A^\omega Q_\omega(\gamma_j(i_j) > \gamma_j(1 - i_j),\ j = 1, \ldots, n),$$

where $\sum_{\omega \in \Omega}$ represents the sum over all elements of

$$\Omega = \{(i_1, \ldots, i_n) \mid i_j \in \{0, 1\},\ j = 1, \ldots, n\}.$$

Moreover,

$$\gamma_j(i_j) = \begin{cases} \alpha_j, & \text{if } i_j = 0, \\ g_j \Delta, & \text{if } i_j = 1, \end{cases}$$

$$A^\omega = E^Q[e^{\sum_{j=1}^n (\gamma_j(i_j) - \beta_j)}],$$

and Q_ω is the pricing measure corresponding to using the security with price process A^ω as numeraire.

Proof. The price of the guaranteed investment C_n^α can be derived as

$$\begin{aligned}
\pi_n^\alpha &= E^Q[e^{-\sum_{j=1}^n \beta_j} e^{\sum_{j=1}^n (\alpha_j \vee g_j \Delta)}] \\
&= \sum_{\omega \in \Omega} E^Q[e^{-\sum_{j=1}^n \beta_j} e^{\sum_{j=1}^n (\alpha_j \vee g_j \Delta)} 1_{\{\gamma_j(i_j) > \gamma_j(1-i_j),\ j=1,\ldots,n\}}] \\
&= \sum_{\omega \in \Omega} E^Q[e^{-\sum_{j=1}^n \beta_j} e^{\sum_{j=1}^n \gamma_j(i_j)} 1_{\{\gamma_j(i_j) > \gamma_j(1-i_j),\ j=1,\ldots,n\}}] \\
&= \sum_{\omega \in \Omega} A^\omega E^Q[\frac{e^{-\sum_{j=1}^n \beta_j} e^{\sum_{j=1}^n \gamma_j(i_j)}}{A^\omega} 1_{\{\gamma_j(i_j) > \gamma_j(1-i_j),\ j=1,\ldots,n\}}] \\
&= \sum_{\omega \in \Omega} A^\omega E^Q[\frac{dQ_\omega}{dQ} 1_{\{\gamma_j(i_j) > \gamma_j(1-i_j),\ j=1,\ldots,n\}}] \\
&= \sum_{\omega \in \Omega} A^\omega Q_\omega(\gamma_j(i_j) > \gamma_j(1 - i_j),\ j = 1, \ldots, n).
\end{aligned}$$

□

Each element in the set Ω represents a particular sequence of γ_j's over the term of the contract, i.e., a specification of periods in which the guarantees are effective. The pricing formula is a sum over all possible sequences of γ_j's. In each term of this sum A^ω is a date zero market price of the financial asset with the return process corresponding to the particular sequence of γ_j's. Moreover, Q_ω is the pricing measure corresponding to using A^ω as numeraire, cf. Geman, El Karoui, and Rochet (1995).

For the rest of this section we assume that forward rates and stock market returns are Gausssian in order to get some more specific results.

5.1. **Multi Period Guarantees on the Stock Market Account—Deterministic Interest Rates.** First consider the return of the risky asset in period n, $1 \leq n \leq N$, under an equivalent martingale measure,

$$\delta_n = \delta(n\Delta) - \delta((n-1)\Delta) = \int_{(n-1)\Delta}^{n\Delta} \Big(r_v - \frac{1}{2}\sigma_S(v)^2 \Big) dv + \int_{n-1}^{n} \sigma_S(v) dW_v . \tag{13}$$

From this equation and the assumption of deterministic interest rates it follows that the variance of δ_n, $1 \leq n \leq N$, is

$$(\sigma_d^n)^2 = \int_{(n-1)\Delta}^{n\Delta} \sigma_S(v)^2 dv.$$

Define $F_n = \frac{P(0,n\Delta)}{P(0,(n-1)\Delta)}$. Observe that $F_1 = P(0,\Delta)$. Thus, F_n may be interpreted as the *forward price* at date zero of a unit discount bond expiring at date $n\Delta$ for delivery at date $(n-1)\Delta$.

Proposition 5.2. *The market price at date zero of the claim C_n^δ described above under deterministic interest rate is*

$$\pi_n^\delta = \prod_{j=1}^{n} \Big(\Phi\big(\frac{-g_j\Delta - \ln(F_j) + \frac{1}{2}(\sigma_d^j)^2}{\sigma_d^j}\big) + e^{g_j\Delta} F_j \Phi\big(\frac{g_j\Delta + \ln(F_j) + \frac{1}{2}(\sigma_d^j)^2}{\sigma_d^j}\big) \Big).$$

Proof. The result follows by straightforward calculations using Proposition 5.1 with $\alpha_j = \delta_j$. □

A similar result for equity linked life insurance is independently derived by Hipp (1996) for the case of constant interest rate, guarantee and volatility. Also observe that in this situation the futures price equals the forward price since the interest rate is deterministic.

5.2. **Multi Period Guarantees on the Savings Account.** Now a similar claim on the savings account is studied. Denote the return on the savings account in period n by β_n, i.e.,

$$\beta_n = \beta(n\Delta) - \beta((n-1)\Delta) = \int_{(n-1)\Delta}^{n\Delta} r_u du.$$

Some calculations based on equation (8) and linearity of integrals yield

$$\begin{aligned} \beta_n &= -\ln F_n + \frac{1}{2}(\sigma_\beta^n)^2 + c_n \\ &\quad + \int_0^{(n-1)\Delta} \int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u) du dW_v + \int_{(n-1)\Delta}^{n\Delta} \int_v^{n\Delta} \sigma_f(v,u) du dW_v , \end{aligned} \tag{14}$$

where

$$(\sigma_\beta^n)^2 = \int_0^{(n-1)\Delta} \Big(\int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u) du \Big)^2 dv + \int_{(n-1)\Delta}^{n\Delta} \Big(\int_v^{n\Delta} \sigma_f(v,u) du \Big)^2 dv$$

represents the variance of β_n, and

$$c_n = \int_0^{(n-1)\Delta} \Big(\int_v^{(n-1)\Delta} \sigma_f(v,u) du \Big) \Big(\int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u) du \Big) dv$$

represents the covariance between β_n and $\beta((n-1)\Delta)$. By definition $c_1 = 0$.

Observe that, in general, $c_n \neq 0$, for $n > 1$, hence, different β_n's are not, in general, independent and a simple closed-form solution as in the previous subsection is not immediately attainable.

In order to obtain some insights for the multi period case, we study the case $n = 2$. At the end of period two an investment of one unit of account at date zero in the savings account is given by

$$C_2^\beta = e^{(\beta_1 \vee g_1 \Delta) + (\beta_2 \vee g_2 \Delta)}.$$

Here β_1 and β_2 are bivariate normally distributed with expectations, variances, and covariance

$$(-\ln P(0,\Delta) + \frac{1}{2}(\sigma_\beta^1)^2, -\ln F_2 + \frac{1}{2}(\sigma_\beta^2)^2 + c, (\sigma_\beta^1)^2, (\sigma_\beta^2)^2, c),$$

respectively, where $c = c_2$ and $(\sigma_\beta^1)^2, (\sigma_\beta^2)^2$, and c_2 are given above.

Let $\Phi(a, b, p)$ denote the bivariate standard normal cumulative distribution function evaluated at the point (a, b), of the bivariate standard normal probability density function with correlation coefficient p.

The solution to the pricing problem is given in the following proposition.

Proposition 5.3. *The market price at date zero of the claim C_2^β described above is*

$$\pi_2^\beta = \Phi(-a_1, -b_1, \rho) + F_2 e^{g_2\Delta - \rho\sigma_\beta^1\sigma_\beta^2}\Phi(-a_2, b_2, -\rho)$$
$$+ P(0,\Delta)e^{g_1\Delta}\Phi(a_3, -b_3, -\rho) + P(0, 2\Delta)e^{(g_1+g_2)\Delta}\Phi(a_4, b_4, \rho),$$

where

$$\rho = \frac{c}{\sigma_\beta^1\sigma_\beta^2},$$

$$a_1 = \frac{g_1\Delta + \ln P(0,\Delta) - \frac{1}{2}(\sigma_\beta^1)^2}{\sigma_\beta^1}, \qquad a_2 = a_1 + \rho\sigma_\beta^2, \qquad a_3 = a_1 + \sigma_\beta^1, \qquad a_4 = a_1 + \rho\sigma_\beta^2 + \sigma_\beta^1,$$

$$b_1 = \frac{g_2\Delta + \ln F_2 - \frac{1}{2}(\sigma_\beta^2)^2}{\sigma_\beta^2} - \rho\sigma_\beta^1, \qquad b_2 = b_1 + \sigma_\beta^2, \qquad b_3 = b_1 + \rho\sigma_\beta^1, \qquad b_4 = b_1 + \rho\sigma_\beta^1 + \sigma_\beta^2.$$

Proof. Cf. Persson and Aase (1997). □

Note that with the Gaussian assumptions the date zero *futures price* of a bond with expiration at date 2Δ for delivery at date Δ is given by $G_2 = F_2 e^{-\rho\sigma_\beta^1\sigma_\beta^2}$. Hence, the futures price naturally enters this formula.

5.3. Multi Period Guarantees on the Stock Market Account—Stochastic Interest Rates.

In this case we consider the payoff given by equation (12) with $\alpha_j = \delta_j$ in a stochastic interest rate framework for the special case $n = 2$.

Using the same methodology the date zero market value is calculated as

$$\pi_2 = E^Q[e^{-\beta_1 - \beta_2} e^{(\delta_1 \vee g_1\Delta) + (\delta_2 \vee g_2\Delta)})].$$

This calculation involves the four multinormally distributed random variables $(\beta_1, \beta_2, \delta_1, \delta_2)$ with variance-covariance matrix

$$\begin{pmatrix} (\sigma_\beta^1)^2 & c & (\sigma_\beta^1)^2 + k_1 & c \\ c & (\sigma_\beta^2)^2 & c + k_2 & (\sigma_\beta^2)^2 + k_3 \\ (\sigma_\beta^1)^2 + k_1 & c + k_2 & (\sigma_\delta^1)^2 & c + k_2 \\ c & (\sigma_\beta^2)^2 + k_3 & c + k_2 & (\sigma_\delta^2)^2 \end{pmatrix},$$

where

$$(\sigma_\delta^1)^2 = (\sigma_d^1)^2 + (\sigma_\beta^1)^2 + 2k_1,$$
$$(\sigma_\delta^2)^2 = (\sigma_d^2)^2 + (\sigma_\beta^2)^2 + 2k_3,$$
$$k_1 = \int_0^\Delta \sigma_S(v) \int_v^\Delta \sigma_f(v,u)\,du\,dv,$$
$$k_2 = \int_0^\Delta \sigma_S(v) \int_\Delta^{2\Delta} \sigma_f(v,u)\,du\,dv,$$

and

$$k_3 = \int_\Delta^{2\Delta} \sigma_S(v) \int_v^{2\Delta} \sigma_f(v,u)\,du\,dv.$$

We present the date zero market price in the following proposition.

Proposition 5.4. *The market price at date zero under stochastic interest rates of the claim C_2^δ described above is*

$$\pi_2 = \Phi(-\bar{a}_1, -\bar{b}_1, \bar{\rho}) + F_2 e^{g_2\Delta - \bar{\rho}\sigma_\delta^1\sigma_\delta^2}\Phi(-\bar{a}_2, \bar{b}_2, -\bar{\rho})$$
$$+ P(0,\Delta)e^{g_1\Delta}\Phi(\bar{a}_3, -\bar{b}_3, -\bar{\rho}) + P(0,2\Delta)e^{(g_1+g_2)\Delta}\Phi(\bar{a}_4, \bar{b}_4, \bar{\rho}),$$

where

$$\bar{\rho} = \frac{c + k_2}{\sigma_\delta^1\sigma_\delta^2},$$
$$\bar{a}_1 = \frac{g_1\Delta + \ln P(0,\Delta) - \frac{1}{2}(\sigma_\delta^1)^2}{\sigma_\delta^1}, \qquad \bar{a}_2 = \bar{a}_1 + \bar{\rho}\sigma_\delta^2, \qquad \bar{a}_3 = \bar{a}_1 + \sigma_\delta^1, \qquad \bar{a}_4 = \bar{a}_1 + \bar{\rho}\sigma_\delta^2 + \sigma_\delta^1,$$
$$\bar{b}_1 = \frac{g_2\Delta + \ln F_2 - \frac{1}{2}(\sigma_\delta^2)^2}{\sigma_\delta^2} - \bar{\rho}\sigma_\delta^1, \qquad \bar{b}_2 = \bar{b}_1 + \sigma_\delta^2, \qquad \bar{b}_3 = \bar{b}_1 + \bar{\rho}\sigma_\delta^1, \qquad \bar{b}_4 = \bar{b}_1 + \bar{\rho}\sigma_\delta^1 + \sigma_\delta^2.$$

Proof. See Appendix A. □

Note again the similar structure in propositions 5.3 and 5.4—only the σ parameter changes. In both cases the σ^2 parameter represents the variance of the periodical returns. Note carefully that in this situation $F_2 e^{-\bar{\rho}\sigma_\delta^1\sigma_\delta^2}$ *cannot* be interpreted as the similar futures price as explained below Proposition 5.3.

6. Concluding remarks

We have introduced stochastic interest rates into a model dealing with minimum rate of return guarantees using the very general Heath-Jarrow-Morton approach. This approach takes the initial term structure of interest rates as given. Based on this information the future term structures of interest rates is modeled as stochastic processes by no-arbitrage arguments. Well-known models such as the Vasicek model and the Cox-Ingersoll-Ross model are special cases of this model. This approach gave us the opportunity to study single as well as multi period rate of return guarantees based on both stock market return processes and short-term interest rate return processes in a consistent stochastic term structure of interest rate model.

In the paper, we have derived a number of pricing formulas for single and multi period guarantees on both stock market return processes and short-term interest rate return processes. Despite the differences in the underlying return processes, (i.e., stock market return processes are of unbounded variation, whereas cumulated short-term interest rate processes are of bounded variation) the derived pricing formulas for the guarantees are remarkably similar. Moreover, for multi period guarantees, we have shown that the stochastic term structure of interest rate model introduce intertemporal dependencies in the periodical returns which complicates the pricing formulas considerably compared to the case of deterministic interest rates.

We believe that our analysis is relevant for life insurance, since many real-life contracts include similar guarantees to the ones treated here. Our results constitute a natural starting point for pricing such guarantees. The various market prices (denoted by π's with appropriate sub- and superscripts) minus 1 may be interpreted as option premiums for the guarantees and thus provide an economic explanation (quantification) for *loadings* often seen in actuarial literature. To fully incorporate these loadings in premium calculations for life insurance contracts also mortality factors etc. must be included.

Apparently, current practice among life insurance companies does not involve the calculation of explicit market values of such guarantees. In a companion paper Miltersen and Persson (1998) we investigate how this observed practice may be consistent with economic pricing theory if we extend the model to also include a surplus distribution (or bonus) mechanism between the customer and the insurance company.

Appendix A. Proof of Proposition 5.4

Following the recipe from Proposition 5.1 the four possible scenarios for the guarantees are enumerated. Let ω be a sample point of the underlying space of possible outcomes Ω. We define

$$\begin{aligned}
A_1 &= \{\omega : \delta_1 > g_1\Delta, \delta_2 > g_2\Delta\},\\
A_2 &= \{\omega : \delta_1 > g_1\Delta, \delta_2 < g_2\Delta\},\\
A_3 &= \{\omega : \delta_1 < g_1\Delta, \delta_2 > g_2\Delta\},\\
A_4 &= \{\omega : \delta_1 < g_1\Delta, \delta_2 < g_2\Delta\}.
\end{aligned}$$

The event A_1 corresponds to the situation where guarantees are not effective in any period, A_2 represents the situation where a guarantee is effective only in the second period, A_3 represents the situation where a guarantee is effective only in the first period, and A_4 represents the situation where the guarantees are effective in both periods. Let 1_{A_i} be the indicator function of the event A_i. We then write

$$\begin{aligned}
\pi_2 &= E^Q[e^{-\beta_1-\beta_2}e^{(\delta_1\vee g_1\Delta)+(\delta_2\vee g_2\Delta)}]\\
&= E^Q[e^{-\beta_1+\delta_1-\beta_2+\delta_2}1_{A_1}] + e^{g_2\Delta}E^Q[e^{-\beta_1+\delta_1-\beta_2}1_{A_2}]\\
&\quad + e^{g_1\Delta}E^Q[e^{-\beta_1-\beta_2+\delta_2}1_{A_2}] + e^{(g_1+g_2)\Delta}E^Q[e^{-\beta_1-\beta_2}1_{A_4}].
\end{aligned}$$

We now proceed by a distinct change of probability measure for each of these four terms. For the first term we define the probability measure $Q^{\delta\delta}$ by the Radon-Nikodym derivative

$$\frac{dQ^{\delta\delta}}{dQ} = e^{-\beta_1-\beta_2+\delta_1+\delta_2}.$$

For the second term we define the probability measure $Q^{\delta g}$ by

$$\frac{dQ^{\delta g}}{dQ} = \frac{e^{-\beta_1-\beta_2+\delta_1}}{F_2e^{-c-k_2}}.$$

Similarly, for the third and fourth terms the probability measures $Q^{g\delta}$ and Q^{gg} are given by

$$\frac{dQ^{g\delta}}{dQ} = \frac{e^{-\beta_1-\beta_2+\delta_2}}{P(0,\Delta)}.$$

and

$$\frac{dQ^{gg}}{dQ} = \frac{e^{-\beta_1-\beta_2}}{P(0,2\Delta)},$$

respectively. We are now able to write

$$\pi_2 = Q^{\delta\delta}(A_1) + F_2 e^{g_2\Delta - \bar{\rho}\sigma_\delta^1\sigma_\delta^2} Q^{\delta g}(A_2) + P(0,\Delta)e^{g_1\Delta}Q^{g\delta}(A_3) + P(0,2\Delta)e^{(g_1+g_2)\Delta}Q^{gg}(A_4).$$

The expectations of $(\beta_1, \beta_2, \delta_1, \delta_1)$ are calculated by Girsanov's theorem and are presented in Table 1. Finally, the formula in Proposition 5.4 is obtained by recalling that δ_1 and δ_2 are bivariate normally

TABLE 1. Expectations of $(\beta_1, \beta_2, \delta_1, \delta_1)$ under Q^{gg}, $Q^{g\delta}$, $Q^{\delta g}$ and $Q^{\delta\delta}$.

	Q^{gg}	$Q^{g\delta}$	$Q^{\delta g}$	$Q^{\delta\delta}$
β_1	$-\ln P(0,\Delta) - \frac{1}{2}(\sigma_\beta^1)^2 - c$	$-\ln P(0,\Delta) - \frac{1}{2}(\sigma_\beta^1)^2$	$-\ln P(0,\Delta) + \frac{1}{2}(\sigma_\beta^1)^2 - c + k_1$	$-\ln P(0,\Delta) + \frac{1}{2}(\sigma_\beta^1)^2 + k_1$
β_2	$-\ln F_2 - \frac{1}{2}(\sigma_\beta^2)^2$	$-\ln F_2 + \frac{1}{2}(\sigma_\beta^2)^2 + k_3$	$-\ln F_2 - \frac{1}{2}(\sigma_\beta^2)^2 + c + k_2$	$-\ln F_2 + \frac{1}{2}(\sigma_\beta^2)^2 + c + k_2 + k_3$
δ_1	$-\ln P(0,\Delta) - \frac{1}{2}(\sigma_\delta^1)^2 - c - k_2$	$-\ln P(0,\Delta) - \frac{1}{2}(\sigma_\delta^1)^2$	$-\ln P(0,\Delta) + \frac{1}{2}(\sigma_\delta^1)^2 - c - k_2$	$-\ln P(0,\Delta) + \frac{1}{2}(\sigma_\delta^1)^2$
δ_2	$-\ln F_2 - \frac{1}{2}(\sigma_\delta^2)^2$	$-\ln F_2 + \frac{1}{2}(\sigma_\delta^2)^2$	$-\ln F_2 - \frac{1}{2}(\sigma_\delta^2)^2 + c + k_2$	$-\ln F_2 + \frac{1}{2}(\sigma_\delta^2)^2 + c + k_2$

distributed with variances $(\sigma_\delta^1)^2$ and $(\sigma_\delta^2)^2$, respectively and covariance $c + k_2$ under all the above probability measures. The probabilities $Q^{\delta\delta}(A_1), Q^{\delta g}(A_2), Q^{g\delta}(A_3), Q^{gg}(A_4)$ can be expressed by the standard bivariate cumulative distribution by substituting to standard (zero mean, unit variance) random variables and using symmetry properties of standard multivariate normal random variables.

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