STATISTICAL RESEARCH REPORT No 11
Institute of Mathematics 1971
University of Oslo

# INVARIANT TESTS MAXIMIZING AVERAGE POWER

by

Emil Spjøtvoll

ABSTRACT

It is well known that if there exists a right invariant probability measure over an invariant set of alternatives, then a uniformly most powerful invariant test maximizes the average power over these alternatives. This result is generalized to the case when there exists only a right invariant measure over the set of alternatives. The method of proof is very similar to the proof of the Hunt-Stein theorem; instead of averaging over the group we average over the set of alternatives.

1. Introduction. It is well known that most powerful invariant tests have a number of optimum properties (see e.g., [5]). Among these is the property that if there exists a right invariant probability measure over an invariant set of alternatives, then the uniformly most powerful invariant test maximizes the average power over these alternatives. Blackwell and Girschick ([1], pp. 233-236) have given an example that suggest that in a certain sense this may also hold in cases when there doesn't exist any invariant probability measure over the set of alternatives. In this note we shall prove a general result corresponding to the example of Blackwell and Girschick.

2. The theorem. Let $\mathcal{P} = \{P_\theta : \theta \in \Omega\}$ be a family of distributions over a Euclidean space $(\mathcal{X}, \mathcal{A})$ dominated by a σ-finite measure $\mu$, and let $G$ be a group of transformations of $(\mathcal{X}, \mathcal{A})$ such that the induced group $\overline{G}$ of transformations of $\Omega$ leaves $\Omega$ invariant. Let $p_\theta = dP_\theta/d\mu$.

Theorem. Let $\overline{G}$ be transitive over $\Omega$, and let $\theta_0$ be a given element of $\Omega$. Define $g_\theta$ by $\overline{g}_\theta^{-1}\theta = \theta_0$. Let $\mathcal{C}$ be a σ-field of subsets of $\Omega$ such that for any $A \in \mathcal{A}$ the set of pairs $(x, \theta)$ with $g_\theta x \in A$ is in $\mathcal{A} \times \mathcal{B}$ and for any $C \in \mathcal{C}$ and $\overline{g} \in \overline{G}$ the set $C\overline{g}$ is in $\mathcal{C}$. Let $\nu$ be σ-finite right invariant measure over $\mathcal{C}$.

Then given any test function $\varphi$, there exists an almost invariant test function $\psi$ such that for any sequence $\omega_1 \subset \omega_2 \subset \ldots$ such that $\omega_n \subset \Omega$, $\nu(\omega_n) < \infty$, $\bigcup_{n=1}^{\infty} \omega_n = \Omega$ and

$$(1) \quad \lim_{n\to\infty} \frac{\nu(\omega_n \Delta \omega_n \overline{g})}{\nu(\omega_n)} = 0 \quad \text{for all } \overline{g} \in \overline{G}$$

we have

$$\lim_{n\to\infty} \frac{1}{\nu(\omega_n)} \int_{\omega_n} E_\theta \psi(X) d\nu(\theta) \geq \lim\sup \frac{1}{\nu(\omega_n)} \int_{\omega_n} E_\theta \varphi(X) d\nu(\theta) .$$

Proof. Let

$$\gamma = \lim\sup \frac{1}{\nu(\omega_n)} \int_{\omega_n} \{ \int \varphi(x) p_\theta(x) d\mu(x) \} d\nu(\theta)$$

Introduce the notation $p_o = p_{\theta_o}$ .

We have, see [4, Problem 16, p. 252],

$$\int \varphi(x) p_\theta(x) d\mu(x) = \int \varphi(g_\theta x) p_o(x) d\mu(x) ,$$

and hence

$$\gamma = \lim\sup \int \{ \frac{1}{\nu(\omega_n)} \int_{\omega_n} \varphi(g_\theta x) d\nu(\theta) \} p_o(x) d\mu(x) .$$

Let $\psi_n$ be defined by

$$\psi_n(x) = \frac{1}{\nu(\omega_n)} \int_{\omega_n} \varphi(g_\theta x) d\nu(\theta) ,$$

then $\psi_n$ is measurable and between 0 and 1 .

There exists a subsequence $\{\psi_{n_i'}\}$ of $\{\psi_n\}$ such that

(2) $\gamma = \lim \int \psi_{n_i'}(x) p_o(x) d\mu(x)$ .

By the weak compactness theorem for test functions [4, p. 354] there exists a test function $\psi$ and a subsequence $\{\psi_{n_i}\}$ of $\{\psi_{n_i'}\}$ such that

$$\lim \int \psi_{n_i}(x) p_o(x) d\mu(x) = \int \psi(x) p_o(x) d\mu(x) .$$

By (2)

$$\gamma = \int \psi(x) p_o(x) d\mu(x) .$$

If we can prove that $\psi$ is almost invariant, we would have

$$\lim\sup \frac{1}{\nu(\omega_n)} \int_{\omega_n} \{ \int \psi(x) p_\theta(x) d\mu(x) \} d\nu(\theta) =$$

$$= \limsup \int \{\frac{1}{\nu(\omega_n)} \int_{\omega_n} \psi(g_\theta x) d\nu(\theta)\} p_o(x) d\mu(x)$$

$$= \limsup \int \{\frac{1}{\nu(\omega_n)} \int_{\omega_n} \psi(x) d\nu(\theta)\} p_o(x) d\mu(x)$$

$$= \int \psi(x) p_o(x) d\mu(x) = \gamma ,$$

and the theorem follows.

Using a technique similar to that of [4, pp. 336-7] we prove that $\psi(x)$ is almost invariant by proving that for all $x$ and $g$

(3) $\psi_{n_i}(gx) - \psi_{n_i}(x) \to 0$ .

For fixed $x$ and any integer $m$, let $\Omega$ be partitioned into mutually exclusive sets

$$B_k = \{\theta:\ a_k < \varphi(g_\theta x) \le a_k + \frac{1}{m}\} \quad k = 0,\dots,m$$

where $a_k = (k-1)/m$ . It is seen from the definition of the sets $B_k$ that

$$\sum_{k=0}^{m} a_k \frac{\nu(B_k \cap \omega_n)}{\nu(\omega_n)} \le \sum_{k=0}^{m} \frac{1}{\nu(\omega_n)} \int_{B_k \cap \omega_n} \varphi(g_\theta x) d\nu(\theta) \le \sum_{k=0}^{m} a_k \frac{\nu(B_k \cap \omega_n)}{\nu(\omega_n)} + \frac{1}{m} ,$$

and analogously that

$$\left| \sum_{k=0}^{m} \int_{B_k \bar{g}^{-1}} \varphi(g_\theta g x) d\nu(\theta) - \sum_{k=0}^{m} a_k \frac{\nu((B_k \bar{g}^{-1}) \cap \omega_n)}{\nu(\omega_n)} \right| \le \frac{1}{m}$$

from which it follows that

(4) $$|\psi_{n_i}(gx) - \psi_{n_i}(x)| \le \sum_{k=0}^{m} a_k \frac{|\nu(B_k \cap \omega_n) - \nu((B_k \bar{g}^{-1}) \cap \omega_n)|}{\nu(\omega_n)} + \frac{2}{m} .$$

We have

$$\frac{|\nu(B_k \cap \omega_n) - \nu(B_k \bar{g}^{-1} \cap \omega_n)|}{\nu(\omega_n)} = \frac{|\nu(B_k \cap \omega_n) - \nu(B_k \cap \omega_n \bar{g})|}{\nu(\omega_n)} \le \frac{\nu(\omega_n \Delta \omega_n \bar{g} \Delta)}{\nu(\omega_n)} .$$

By (1) this tends to zero when $n \to \infty$. From (4) it is now seen that (3) holds. This completes the proof.

To see what condition (1) means, consider the following example. Let $\Omega$ be the real line, $\bar{G}$ the translation group, $\nu$ the Lebesgue measure and $\omega_n = [-n, n]$. Then

$$\frac{\nu(\omega_n \Delta \omega_n \bar{g})}{\nu(\omega_n)} \leq \frac{|\bar{g}|}{n},$$

and (1) is satisfied. If $\bar{G}$ was a group of positive scale changes and $\Omega$ the positive part of the real line, we could let $\nu(C)$ be equal to the Lebesgue measure of $\log C$. Then $\nu$ is right invariant, and with $\omega_n = [n^{-1}, n]$ (1) would be satisfied.

3. Relation to other results. Lehmann [5] has given four conditions under which the uniformly most powerful invariant test" has a number of optimum properties. His condition (ii) is not satisfied for our problem, and hence his results cannot be used. To see this consider the following situation. Let $\varphi_i$, $i = 1,2$ be tests such that

$$\frac{1}{\nu(\omega_n)} \int_{\omega_n} E_\theta \varphi_i(X) d\nu(\theta) = \begin{cases} 1-a & \text{if } n-i \text{ is even} \\ 1-5a & \text{if } n-i \text{ is odd} \end{cases} \qquad i = 1,2$$

where $a$ is a number between 0 and .20. Then

$$(5) \quad \limsup \frac{1}{\nu(\omega_n)} \int_{\omega_n} E_\theta \varphi_i(X) d\nu(\theta) = 1-a, \quad i = 1,2$$

and

$$(6) \quad \limsup \frac{1}{\nu(\omega_n)} \int E_\theta \{\tfrac{1}{2}(\varphi_1 + \varphi_2)\} d\nu(\theta) = 1-3a.$$

Let $\psi$ be a test such that

$$(7) \quad \limsup \frac{1}{\nu(\omega_n)} \int_{\omega_n} E_\theta \varphi_i(X) d\nu(\theta) = 1-2a.$$

Then $\varphi_1$ and $\varphi_2$ are both better than $\psi_1$ in the sense that (5) is greater than (7). But $\frac{1}{2}(\varphi_1+\varphi_2)$ is not as good as $\psi$ since (6) is less than (7). This means that Lehmann's condition (ii) is not satisfied.

One of the optimality properties of the uniformly most powerful invariant test is that it maximizes the minimum power over certain alternatives. That fact does not guarantee that there does not exist a test which has less minimum power than the best invariant test but has power which exceeds the power of the best invariant test by a fixed amount over most of the alternative. It follows from the theorem that this cannot happen.

Lehmann and Stein [3] proved that the uniformly most powerful invariant test is admissible if G is a group of translations or scale changes. Fox and Perng [2] and Perng [6] have shown that if one of the conditions in [3] is not satisfied then the uniformly most powerful invariant test is not necessarily admissible. But it follows from our theorem that although the uniformly most powerful invariant test may not be admissible, it will maximize the average power.

## REFERENCES


[1] BLACKWELL, D. and GIRSCHICK, M.A. (1954). *Theory of Games and Statistical Decisions*. Wiley, New York.

[2] FOX, M. and PERNG, S.G. (1969). Inadmissibility of the best invariant test when the moment is infinite under one of the hypotheses. *Ann. Math. Statist.* **40** 1483-1485.

[3] LEHMANN, E.L. and STEIN, C.M. (1953). The admissibility tests involving a translation parameter. *Ann. Math. Statist.* **24** 473-479.

[4] LEHMANN, E.L. (1959). *Testing Statistical Hypotheses.* Wiley, New York.

[5] LEHMANN, E.L. (1959). Optimum invariant tests. *Ann. Math. Statist.* **30** 881-884.

[6] PERNG, S.G. (1970). Inadmissibility of various "good" statistical procedures which are translation invariant. **41** 1311-1321.