Homogeneous spaces
B. Komrakov seminar

THREE-DIMENSIONAL ISOTROPICALLY–FAITHFUL HOMOGENEOUS SPACES

VOLUME I

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Volume I

Foreword

We consider classification of lower-dimensional homogeneous spaces an immediate continuation and global version of classification results obtained by Sophus Lie. Two-dimensional homogeneous spaces were classified locally by Sophus Lie [L1] and globally by G.D. Mostow [M]. (See also our preprint [KTD], where the complete classification of two-dimensional homogeneous spaces, both locally and globally, is presented.) S. Lie also obtained some results in classification of three-dimensional homogeneous spaces and described all subalgebras in the Lie algebra $\mathfrak{gl}(3,\mathbb{C})$. A detailed account of these classifications can be found in [L2].

The problem of finding the complete description of three- and four-dimensional homogeneous spaces as pairs, (group, subgroup) or even (algebra, subalgebra), is extremely important and rich in applications, but it is a very difficult one: "The description of arbitrary transitive actions on manifolds M, where $\dim M \geqslant 3$, presently seems to be unattainable."([GO], p. 232)

Minimal transitive actions, that is, those that have no proper transitive subgroups, on three-dimensional manifolds were classified in [G]. The problem of local classification of three- and four-dimensional homogeneous spaces was chosen by one of the authors, B. Komrakov, as the topic of Dr. Sci. thesis for A. Tchourioumov, the other author. (Some of the results can be found in [Tch].)

An important subclass in all homogeneous spaces is formed by isotropically-faithful spaces. In particular, it contains all homogeneous spaces that admit an invariant affine connection. The present preprint gives the local classification of three-dimensional isotropically-faithful homogeneous spaces.

In 1990, the International Sophus Lie Centre, jointly with the University of Belarus, organized an experimental group of 25 students majoring in mathematics and working in accordance with a special syllabus oriented to modern differential-geometric methods in the study of nonlinear differential equations. The following idea arose: to split up the classification problem mentioned above into smaller parts and give each part to a student; in the process of learning new material, the student will then try to apply his newly acquired knowledge to this problem as an illustration.

Suppose, for example, that the student is learning about differential equations; he then writes out trajectories of one-parameter subgroups on the specific manifold that he has been given. Studying differential geometry, he computes invariant affine connections, metrics, curvature tensors, geodesics, etc., with special emphasis on his example, and so on.

In their first year, the students all took an advanced course in Lie algebras and the main part of the work on all these "smaller parts" was completed by 12 students. We had no time to give our students an introductory course in cohomologies of Lie algebras, and although their computation constitutes a considerable part of the work, we do not use this language.

This work was started in Tartu University, Estonia (August 1991), continued at the Institute of Astrophysics and Atmosphere Physics in Tõravere, Estonia (December 1991 to March 1992), then at the "Bears' Lakes" Space Center of the Special Research Bureau of Moscow Power Engineering Institute (August 1993), and finished at the University of Oslo and the Center for Advanced Study (SHS) at the

Norwegian Academy of Science and Letters. (Naturally, most of the time from August 1991 to November 1993 was spent in Minsk, Belarus.) The story of this work was rich in experiences and events only indirectly connected with mathematics, something we will not here dwell on at length. We would, however, like to express our gratitude to those who directly or indirectly made it possible for us to complete this work.

In the future, we are going to proceed with the study of geometry of three-dimensional homogeneous spaces in the following directions:

- description of invariant affine connections on three-dimensional homogeneous spaces together with their curvature and torsion tensors, holonomy groups, geodesics, etc.;
- description of invariant tensor geometric structures and their properties;
- global classification of three-dimensional isotropically-faithful homogeneous spaces and description of inclusions among the corresponding transformation groups;
- description of differential invariants for the homogeneous spaces to be found and of the corresponding invariant differential equations;
- description of discrete subgroups in transformation groups together with description of the corresponding topological factor spaces.

Introduction

It is known that the problem of classification of homogeneous spaces $(\overline{G}, M)$ is equivalent to the classification (up to equivalence) of pairs of Lie groups $(\overline{G}, G)$ such that $G \subset \overline{G}$. Two pairs $(\overline{G}_1, G_1)$ and $(\overline{G}_2, G_2)$ are said to be equivalent if there exists an isomorphism of Lie groups $\pi\colon \overline{G}_1 \to \overline{G}_2$ such that $\pi(G_1) = G_2$.

By linearization, the problem can be reduced to the problem of classification of pairs of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$ viewed up to equivalence of pairs. The structure of all pairs of Lie groups $(\overline{G}, G)$ corresponding to a given pair of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$ was described in [M]. In the study of homogeneous spaces it is important to consider not the group $\overline{G}$ itself, but its image in $\mathrm{Diff}(M)$. In other words, it is sufficient to consider only the effective action of the group $\overline{G}$ on the manifold M. In terms of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, this condition is equivalent to the condition for $\mathfrak{g}$ to contain no proper ideals of $\bar{\mathfrak{g}}$. In this case we say that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *effective.*

In the present work we classify all isotropically–faithful pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 3.

Definition. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropically-faithful* if the natural $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ is faithful.

We say that a homogeneous space $(\overline{G}, M)$ is isotropically-faithful if so is the corresponding pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. From geometrical point of view it means that the natural action of the stabilizer $\overline{G}_x$ of an arbitrary point $x \in M$ on T_xM has discrete kernel.

We divide the solution of our problem into the following parts:

(1) We classify (up to isomorphism) all faithful three–dimensional $\mathfrak{g}$-modules U. This is equivalent to classifying all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ viewed up to conjugation.

(2) For each $\mathfrak{g}$-module U obtained in (1) we classify (up to equivalence) all pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the $\mathfrak{g}$-modules $\bar{\mathfrak{g}}/\mathfrak{g}$ and U are isomorphic.

In Chapter I we give basic definitions and introduce the notation to be employed. Here we also solve part (1) of the problem by classifying subalgebras in $\mathfrak{gl}(3, \mathbb{R})$.

In Chapter II we develop methods for constructing pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ given a three-dimensional faithful $\mathfrak{g}$-module U. This involves computation of the first cohomology space of $\mathfrak{g}$ with values in the natural module $\mathcal{L}(U, \mathfrak{g})$. A series of techniques described in Chapter II allows, in some cases, to simplify the computation considerably.

Finally, Chapter III gives the classification of three-dimensional isotropically-faithful pairs itself.

CHAPTER I

ISOTROPICALLY-FAITHFUL PAIRS

1. Basic definitions

It is assumed that the reader is familiar with the concept of a smooth manifold and basic definitions of the theory of Lie groups and algebras.

In the sequel all manifolds (including Lie groups) to be considered are connected real manifolds.

Definition. Suppose G is a Lie group and M is a manifold. An *action of G on M* is a homomorphism of groups $\sigma : G \to \mathrm{Diff}(M)$ such that the mapping $G \times M \to M$ given by

$$(g, m) \mapsto \sigma(g)(m) \qquad (g \in G, m \in M)$$

is smooth.

We write $g.m$ instead of $\sigma(g)(m)$ when no confusion is possible.

Definition. The action $\sigma : G \to \mathrm{Diff}(M)$ is called *transitive* if for any $x, y \in M$ there exists a $g \in G$ such that $g.x = y$; if this element g is unique, the action is called *simply transitive.*

Definition. Suppose G is a Lie group, M is a manifold, and σ is a transitive action of G on M. The triple (G, M, σ) is called a *homogeneous space.*

The *dimension of a homogeneous space* is the dimension of the corresponding manifold M.

Two homogeneous spaces (G_1, M_1, σ_1) and (G_2, M_2, σ_2) are said to be *equivalent,* if there exists a pair of mappings (π, τ), where
$\pi : G_1 \to G_2$ is an isomorphism of Lie groups,
$\tau : M_1 \to M_2$ is a homeomorphism of manifolds,
such that $\tau(g.x) = \pi(g).\tau(x)$ for all $g \in G_1$, $x \in M_1$.

Let us recall some well-known results of Lie group theory.

Proposition 1. *Let σ be an action of a Lie group G on a manifold M. Then for any point $x \in M$ the stabilizer*

$$G_x = \{g \in G \mid g.x = x\}$$

is a Lie subgroup of G.

Proposition 2. *Suppose G is a Lie group and H is a Lie subgroup of G. There exists a unique smooth manifold structure on the set G/H of left cosets such that the canonical action of G on G/H (by means of left shifts) is smooth.*

Let G be a Lie group and H a Lie subgroup of G. In the sequel we assume that the triple $(G, G/H, \tau)$ denotes the homogeneous space where G acts transitively on G/H by means of left shifts.

Proposition 3. *Suppose (G, X, σ) is a homogeneous space. Then for every $x \in X$ the mapping $\alpha_x : G/G_x \to X$, $gG_x \longmapsto \sigma(g)(x)$ is a diffeomorphism of the manifolds G/G_x and X; and the pair of mappings (id_G, α_x) establishes the equivalence of the homogeneous spaces (G, X, σ) and $(G, G/G_x, \tau)$.*

So each pair $(\overline{G}, G)$, where $\overline{G}$ is a Lie group and G is a Lie subgroup of $\overline{G}$, defines a homogeneous space. From Proposition 3 it follows that in this way we can obtain all homogeneous spaces (viewed up to equivalence).

Note that pairs $(\overline{G}_1, G_1)$ and $(\overline{G}_2, G_2)$ define equivalent homogeneous spaces if and only if there exists an isomorphism of Lie groups $\pi : \overline{G}_1 \to \overline{G}_2$ such that $\pi(G_1) = G_2$. In this case we say that the pairs $(\overline{G}_1, G_1)$ and $(\overline{G}_2, G_2)$ are *equivalent.*

Definition. Suppose (G, M, σ) is a homogeneous space. The action $\sigma : G \to \mathrm{Diff}(M)$ is called *effective* if σ is an injection. The kernel of the homomorphism σ is called the *kernel of ineffectiveness of the action σ.*

Proposition 4. *Suppose $H = \ker \sigma$ is the kernel of ineffectiveness of the action $\sigma : G \to \mathrm{Diff}(M)$. Then H is a Lie subgroup of G.*

Proof. Indeed, by definition $H = \cap_{x \in M} G_x$. Since for each $x \in H$ the subgroup G_x is a Lie subgroup of G, we see that H is also a Lie subgroup of G.

It is easily proved that if the action $\sigma : G \to \mathrm{Diff}(M)$ is not effective, then the action $\bar{\sigma} : G/\ker \sigma \to \mathrm{Diff}(M)$ is effective.

Proposition 5. *Let $\overline{G}$ be a Lie group and G a Lie subgroup of $\overline{G}$. The canonical action τ of $\overline{G}$ on $\overline{G}/G$ is effective if and only if the subgroup G contains no nontrivial normal Lie subgroups of $\overline{G}$.*

Proof. Suppose H is a normal Lie subgroup of $\overline{G}$ and $H \subset G$. Then for $\overline{G} \in \overline{G}, h \in H$ we have

$$h(\overline{G}G) = (h\overline{G})G = \overline{G}(\overline{G}^{-1} h\overline{G})G = \overline{G}G.$$

It follows that H belongs to the kernel of ineffectiveness of the action σ.

Conversely, suppose H is the kernel of ineffectiveness of the action σ. Then H is a normal Lie subgroup of $\overline{G}$. On the other hand, $hG = G$ for all $h \in H$. Therefore $H \subset G$.

Let us introduce the following

Definition. Suppose $\overline{G}$ is a Lie group and G is a Lie subgroup of $\overline{G}$. The pair $(\overline{G}, G)$ is said to be *effective* if G contains no nontrivial normal Lie subgroups of $\overline{G}$.

If we are interested in geometry, it is important to consider not a group G that acts on a manifold M but the image of G in $\mathrm{Diff}(M)$. That is why studying homogeneous spaces from this point of view, it is possible to restrict oneself to effective actions.

2. Linearization of the problem

Suppose $\overline{G}$ is a Lie group and G is a Lie subgroup of G. Consider the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to G. We say that the pair $(\overline{G}, G)$ is *associated* with the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$.

By analogy with Lie groups, a pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *effective* if $\mathfrak{g}$ contains no nonzero ideals of the Lie algebra $\bar{\mathfrak{g}}$.

Proposition 6. *If a pair $(\overline{G}, G)$ is effective, then the corresponding pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is effective.*

Proof. Indeed, assume that $\mathfrak{a}$ is a nonzero ideal in the Lie algebra $\bar{\mathfrak{g}}$ such that $\mathfrak{a} \subset \mathfrak{g}$. Then the Maltsev closure of the ideal $\mathfrak{a}$ is the ideal $\mathfrak{a}^M$ lying in $\mathfrak{g}$ ([OV], Ch. I, §4). Suppose H is the Lie subgroup of $\overline{G}$ corresponding to the subalgebra $\mathfrak{a}^M$. Then H is normal and $H \subset G$. But the pair $(\overline{G}, G)$ is effective. We come to a contradiction. Therefore the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is effective.

Generally speaking, the converse is false. But the following statement is true.

Proposition 7. *Suppose $(\bar{\mathfrak{g}}, \mathfrak{g})$ is an effective pair and $(\overline{G}, G)$ is the pair associated with $(\bar{\mathfrak{g}}, \mathfrak{g})$. Then if H is a normal Lie subgroup of $\overline{G}$ and $H \subset G$, the subgroup H is discrete.*

Proof. It is clear.

The description of effective pairs $(\overline{G}, G)$ associated with a given pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ was given by G.D. Mostow in [M].

The basic results of the paper are as follows.

Proposition 8. *Suppose $(\bar{\mathfrak{g}}, \mathfrak{g})$ is an effective pair and $\dim \bar{\mathfrak{g}} - \dim \mathfrak{g} \leqslant 4$. Then there exists an effective pair $(\overline{G}, G)$ associated with the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$.*

Proposition 9. *Suppose $(\bar{\mathfrak{g}}, \mathfrak{g})$ is an effective pair and there exists at least one effective pair $(\overline{G}, G)$ associated with the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. Then there exists a unique effective pair $(\overline{G}^*, G^*)$ associated with the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the group G^* is connected and the manifold $\overline{G}^*/G^*$ is simply connected.*

Proposition 10. *Suppose $(\overline{G}^*, G^*)$ is an effective pair such that G^* is connected and $\overline{G}^*/G^*$ is simply connected. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be the pair corresponding to $(\overline{G}^*, G^*)$. Now suppose Z^* is the center of $\overline{G}^*$ and $N(G^*)$ is the normalizer of G^* in $\overline{G}^*$. A necessary and sufficient condition for an effective pair $(\overline{G}, G)$ to be associated with the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is that $(\overline{G}, G)$ be equivalent to the pair $(\overline{G}^*/(S^* \cap Z^*),\ G^*/(S^* \cap Z^*))$, where S^* is a closed subgroup of $N(G^*)$ such that $G^* \subset S^*$ and the Lie group S^*/G^* is discrete.*

3. Outline of classification of pairs

All vector spaces (including Lie algebras) to be considered are finite-dimensional vector spaces over an arbitrary field k.

Definition. A *generalized module* is a pair $(\mathfrak{g}, U)$, where $\mathfrak{g}$ is a Lie algebra and U is a $\mathfrak{g}$-module.

Generalized modules $(\mathfrak{g}_1, U_1)$ and $(\mathfrak{g}_2, U_2)$ are called *isomorphic* if there exists a pair of mappings (f, F) such that

$f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism of Lie algebras,

$F : U_1 \to U_2$ is an isomorphism of vector spaces,

and for all $x \in \mathfrak{g}_1, u \in U_1$ the following condition holds:

$$F(x.u) = f(x).F(u).$$

Then the pair (f, F) is called an *isomorphism of the generalized modules* $(\mathfrak{g}_1, U_1)$ and $(\mathfrak{g}_2, U_2)$.

A generalized module $(\mathfrak{g}, U)$ is said to be *faithful* if the $\mathfrak{g}$-module U is faithful. The *dimension of a generalized module* $(\mathfrak{g}, U)$ is the dimension of the vector space U.

Definition. Assume that V is a vector space and $\mathfrak{g}$ is a subspace of V. The pair $(V, \mathfrak{g})$ supplied with a bilinear mapping $B : \mathfrak{g} \times V \to V$, $(x, v) \mapsto x.v$ is called a *virtual pair* if the following conditions are satisfied:

(1) $\mathfrak{g}.\mathfrak{g} \subset \mathfrak{g}$;
(2) the restriction of B to $\mathfrak{g} \times \mathfrak{g}$ provides $\mathfrak{g}$ with the structure of a Lie algebra ($[x, y] = x.y$);
(3) V is a $\mathfrak{g}$-module with respect to B.

To any virtual pair $(V, \mathfrak{g})$ we can naturally assign the generalized module $(\mathfrak{g}, V/\mathfrak{g})$, which is said to be *associated with the virtual pair* $(V, \mathfrak{g})$.

Suppose $(V_1, \mathfrak{g}_1)$ and $(V_2, \mathfrak{g}_2)$ are two virtual pairs and $H : V_1 \to V_2$ is an isomorphism of vector spaces. The mapping H is called an *isomorphism of the virtual pairs* $(V_1, \mathfrak{g}_1)$ and $(V_2, \mathfrak{g}_2)$ if

(*a*) $H(\mathfrak{g}_1) = \mathfrak{g}_2$;
(*b*) $H(x.v) = H(x).H(v)$ for all $x \in \mathfrak{g}_1, v \in V_1$.

Suppose H is an isomorphism of virtual pairs $(V_1, \mathfrak{g}_1)$ and $(V_2, \mathfrak{g}_2)$. Let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ be the restriction of H to $\mathfrak{g}_1$ and let $F : V_1/\mathfrak{g}_1 \to V_2/\mathfrak{g}_2$ be the mapping defined by

$$F(v + \mathfrak{g}_1) = H(v) + \mathfrak{g}_2 \quad \text{for all } v \in V_1.$$

Then f is an isomorphism of Lie algebras and F is an isomorphism of vector spaces. Thus the pair (f, F) is an isomorphism of the generalized modules $(\mathfrak{g}_1, V_1/\mathfrak{g}_1)$ and $(\mathfrak{g}_2, V_2/\mathfrak{g}_2)$. In this case we say that (f, F) is *associated with* H.

Definition. The *isotropic representation of a virtual pair* $(V, \mathfrak{g})$ is the mapping

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V/\mathfrak{g})$$

defined by

$$\rho(x)(v + \mathfrak{g}) = x.v + \mathfrak{g} \quad \text{for all } v \in V, x \in \mathfrak{g}.$$

The virtual pair $(V, \mathfrak{g})$ is called *isotropically-faithful* if the homomorphism ρ is injective.

It is obvious that a virtual pair $(V, \mathfrak{g})$ is isotropically-faithful if and only if the associated generalized module $(\mathfrak{g}, V/\mathfrak{g})$ is faithful.

Suppose $\bar{\mathfrak{g}}$ is a finite-dimensional Lie algebra and $\mathfrak{g}$ is a subalgebra of $\bar{\mathfrak{g}}$. For the sake of simplicity we then simply say that the *pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ is given. The *codimension of the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the codimension of $\mathfrak{g}$ in $\bar{\mathfrak{g}}$.

Two pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are said to be *equivalent* if there exists an isomorphism of Lie algebras $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_2$ such that $\pi(\mathfrak{g}_1) = \mathfrak{g}_2$.

Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ can be regarded as a virtual pair with respect to ordinary commutation restricted to $\mathfrak{g} \times \bar{\mathfrak{g}}$. The *isotropic representation of a pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the isotropic representation of the corresponding virtual pair. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is called *isotropically-faithful* if its isotropic representation is an injection.

Further we shall be interested in the following

Problem. *To classify all real isotropically-faithful pairs of codimension three (viewed up to equivalence of pairs).*

According to our previous considerations, we divide the solution of the problem into the following parts:

1°. To classify (up to isomorphism) all real faithful three-dimensional generalized modules $(\mathfrak{g}, U)$.

2°. For each generalized module $(\mathfrak{g}, U)$ obtained in 1° to classify (up to isomorphism) all virtual pairs $(V, \mathfrak{g})$ such that the generalized modules $(\mathfrak{g}, U)$ and $(\mathfrak{g}, V/\mathfrak{g})$ are isomorphic.

3°. For each virtual pair $(V, \mathfrak{g})$ obtained in 2° to classify (up to equivalence) all pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the virtual pairs $(V, \mathfrak{g})$ and $(\bar{\mathfrak{g}}, \mathfrak{g})$ are isomorphic.

4. Subalgebras of the Lie algebra $\mathfrak{gl}(3, \mathbb{R})$

The classification (up to conjugation) of subalgebras of the Lie algebra $\mathfrak{gl}(3, \mathbb{R})$.

Preliminaries:

1. In the sequel we consider only proper subalgebras of $\mathfrak{gl}(3, \mathbb{R})$.

2. For the sake of simplicity instead of the standard notation for a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ such as

$$\mathfrak{b} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & \lambda x & 0 \\ 0 & 0 & \mu x \end{pmatrix} \middle| \; x \in \mathbb{R} \right\},$$

where $\lambda, \mu \in \mathbb{R}$ and $\lambda\mu > 0, \lambda \leqslant \mu \leqslant 1$, we use the following notation:

$$\mathfrak{b} = \boxed{\begin{matrix} x & & \\ & \lambda x & \\ & & \mu x \end{matrix}} \quad \begin{matrix} \lambda\mu > 0 \\ \lambda \leqslant \mu \leqslant 1. \end{matrix}$$

Here we imply that variables denoted by Latin letters run through $\mathbb{R}$ and that parameters are denoted by small Greek letters.

Theorem 1. *Suppose $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{gl}(3,\mathbb{R})$. Then $\mathfrak{g}$ is conjugate to one and only one of the following subalgebras:*

$\underline{\dim \mathfrak{g} = 1}$

1. $\begin{pmatrix} x & & \\ & \lambda x & \\ & & \end{pmatrix}$ $|\lambda| \leqslant 1$;
2. $\begin{pmatrix} x & & \\ & \lambda x & \\ & & \mu x \end{pmatrix}$ $\lambda \leqslant \mu \leqslant 1$, $\lambda\mu > 0$;
3. $\begin{pmatrix} \lambda x & x & \\ -x & \lambda x & \\ & & \end{pmatrix}$ $\lambda \geqslant 0$;
4. $\begin{pmatrix} \lambda x & x & \\ -x & \lambda x & \\ & & \mu x \end{pmatrix}$ $\mu > 0$;
5. $\begin{pmatrix} & & x \\ & & \\ & & \end{pmatrix}$;
6. $\begin{pmatrix} & & x \\ & x & \\ & & \end{pmatrix}$;
7. $\begin{pmatrix} x & & x \\ & \lambda x & \\ & & x \end{pmatrix}$;
8. $\begin{pmatrix} & x & \\ & & x \\ & & \end{pmatrix}$;
9. $\begin{pmatrix} x & x & \\ & x & x \\ & & x \end{pmatrix}$.

$\underline{\dim \mathfrak{g} = 2}$

1. $\begin{pmatrix} x & & \\ & \lambda x & \\ & & y \end{pmatrix}$ $|\lambda| \leqslant 1$;
2. $\begin{pmatrix} x+y & & \\ & \lambda x & \\ & & \mu y \end{pmatrix}$ $-1 \leqslant \mu \leqslant \lambda$, $\lambda\mu > 0$;
3. $\begin{pmatrix} \lambda x & x & \\ -x & \lambda x & \\ & & y \end{pmatrix}$ $\lambda \geqslant 0$;
4. $\begin{pmatrix} y & x & \\ -x & y & \\ & & \lambda x+\mu y \end{pmatrix}$ $\lambda \geqslant 0$;
5. $\begin{pmatrix} y & & x+\lambda y \\ & x & \\ & & y \end{pmatrix}$;
6. $\begin{pmatrix} y & & y \\ & x & \\ & & y \end{pmatrix}$;
7. $\begin{pmatrix} & & x \\ & & \\ & & y \end{pmatrix}$;
8. $\begin{pmatrix} & & x \\ & y & \\ & & \lambda y \end{pmatrix}$;
9. $\begin{pmatrix} y & & x \\ & \lambda y & \\ & & \mu y \end{pmatrix}$;
10. $\begin{pmatrix} x & y & x \\ & x & y \\ & & x \end{pmatrix}$;
11. $\begin{pmatrix} x & y & -x \\ & x & y \\ & & x \end{pmatrix}$;
12. $\begin{pmatrix} x & y & \\ & x & y \\ & & x \end{pmatrix}$;
13. $\begin{pmatrix} & y & x \\ & & y \\ & & \end{pmatrix}$;
14. $\begin{pmatrix} y & y & x \\ & y & y \\ & & y \end{pmatrix}$;
15. $\begin{pmatrix} & & x \\ & y & y \\ & & y \end{pmatrix}$;
16. $\begin{pmatrix} y & & x \\ & \lambda y & \\ & & \lambda y \end{pmatrix}$;
17. $\begin{pmatrix} & & x \\ & & y \\ & & \end{pmatrix}$;
18. $\begin{pmatrix} & y & x \\ & & \\ & & y \end{pmatrix}$;

19. $\boxed{\begin{array}{ccc} y & y & x \\ & y & \\ & & \lambda y \end{array}}$;
20. $\boxed{\begin{array}{ccc} & y & x \\ & & \\ & & \end{array}}$;

21. $\boxed{\begin{array}{ccc} x & y & \\ & \lambda x & y \\ & & (2\lambda-1)x \end{array}}\ \lambda\neq 1$;
22. $\boxed{\begin{array}{ccc} & y & \\ & x & y \\ & & 2x \end{array}}$.

$\underline{\dim \mathfrak{g} = 3}$

1. $\boxed{\begin{array}{ccc} x & & \\ & y & \\ & & z \end{array}}$;
2. $\boxed{\begin{array}{ccc} y & x & \\ -x & y & \\ & & z \end{array}}$;
3. $\boxed{\begin{array}{ccc} x & y & \\ z & -x & \\ & & \end{array}}$;

4. $\boxed{\begin{array}{ccc} x & y & \\ z & & y \\ & z & -x \end{array}}$;
5. $\boxed{\begin{array}{ccc} & y & x \\ -y & & z \\ -x & -z & \end{array}}$;
6. $\boxed{\begin{array}{ccc} & & x \\ & y & \\ & & z \end{array}}$;

7. $\boxed{\begin{array}{ccc} y & & x \\ & \lambda y & \\ & & z \end{array}}$;
8. $\boxed{\begin{array}{ccc} y & & x \\ & z & \\ & & \lambda y + \mu z \end{array}}$;
9. $\boxed{\begin{array}{ccc} x & y & z \\ & x & y \\ & & x \end{array}}$;

10. $\boxed{\begin{array}{ccc} x & & z \\ & y & \lambda x + y \\ & & y \end{array}}$;
11. $\boxed{\begin{array}{ccc} x & & z \\ & y & x \\ & & y \end{array}}$;
12. $\boxed{\begin{array}{ccc} & & z \\ & & y \\ & & x \end{array}}$;

13. $\boxed{\begin{array}{ccc} x & & z \\ & \lambda x & y \\ & & \mu x \end{array}}\ -1<\lambda\leqslant 1$;
14. $\boxed{\begin{array}{ccc} x & & z \\ & -x & y \\ & & \mu x \end{array}}\ \mu\geqslant 0$;
15. $\boxed{\begin{array}{ccc} \lambda x & x & z \\ -x & \lambda x & y \\ & & \end{array}}\ \lambda\geqslant 0$;

16. $\boxed{\begin{array}{ccc} \lambda x & x & z \\ -x & \lambda x & y \\ & & \mu x \end{array}}\ \mu > 0$;
17. $\boxed{\begin{array}{ccc} x & \lambda x + y & z \\ & x & \\ & & y \end{array}}$;
18. $\boxed{\begin{array}{ccc} x & x & z \\ & x & \\ & & y \end{array}}$;

19. $\boxed{\begin{array}{ccc} & y & z \\ & x & \\ & & \lambda x \end{array}}\ |\lambda|\leqslant 1$;
20. $\boxed{\begin{array}{ccc} x & y & z \\ & \lambda x & \\ & & \mu x \end{array}}\ \lambda\leqslant\mu$;
21. $\boxed{\begin{array}{ccc} & y & z \\ & \lambda x & x \\ & -x & \lambda x \end{array}}\ \lambda\leqslant 0$;

22. $\boxed{\begin{array}{ccc} \lambda x & y & z \\ & \mu x & x \\ & -x & \mu x \end{array}}\ \lambda > 0$;
23. $\boxed{\begin{array}{ccc} x & y & z \\ & \lambda x & y \\ & & (2\lambda-1)x \end{array}}\ \lambda\neq 1$;
24. $\boxed{\begin{array}{ccc} & y & z \\ & x & y \\ & & 2x \end{array}}$;

25. $\begin{pmatrix} & x & z \\ & & y \\ & & \end{pmatrix}$; 26. $\begin{pmatrix} & x & z \\ & y & y \\ & & y \end{pmatrix}$; 27. $\begin{pmatrix} y & x & z \\ & \lambda y & y \\ & & \lambda y \end{pmatrix}$;

28. $\begin{pmatrix} & x & z \\ & & y \\ & & x \end{pmatrix}$; 29. $\begin{pmatrix} x & x & z \\ & x & y \\ & & \lambda x \end{pmatrix}$; 30. $\begin{pmatrix} x & x & z \\ & x & x+y \\ & & x \end{pmatrix}$;

31. $\begin{pmatrix} x+y & z & \\ & x & z \\ & & x-y \end{pmatrix}$.

$\underline{\dim \mathfrak{g} = 4}$

1. $\begin{pmatrix} x & z & \\ u & y & \\ & & \end{pmatrix}$; 2. $\begin{pmatrix} \lambda x+y & z & \\ u & \lambda x-y & \\ & & x \end{pmatrix}$; 3. $\begin{pmatrix} x+y & z & \\ u & x & z \\ & u & x-y \end{pmatrix}$;

4. $\begin{pmatrix} x & & u \\ & y & \\ & & z \end{pmatrix}$; 5. $\begin{pmatrix} x & z & y \\ -z & x & u \\ -y & -u & x \end{pmatrix}$; 6. $\begin{pmatrix} x & & u \\ & \lambda x & z \\ & & y \end{pmatrix}$ $-1 \leqslant \lambda \leqslant 1$;

7. $\begin{pmatrix} \lambda x & x & u \\ -x & \lambda x & z \\ & & y \end{pmatrix}$ $\lambda \geqslant 0$; 8. $\begin{pmatrix} x & & u \\ & y & z \\ & & \lambda x+\mu y \end{pmatrix}$ $\lambda \leqslant \mu$; 9. $\begin{pmatrix} y & x & u \\ -x & y & z \\ & & \lambda x+\mu y \end{pmatrix}$ $\lambda \geqslant 0$;

10. $\begin{pmatrix} & z & u \\ & x & \\ & & y \end{pmatrix}$; 11. $\begin{pmatrix} x & z & u \\ & y & \\ & & \lambda x+\mu y \end{pmatrix}$ $-1 \leqslant \mu < 1$; 12. $\begin{pmatrix} x & z & u \\ & y & \\ & & \lambda x+y \end{pmatrix}$ $\lambda \geqslant 0$;

13. $\begin{pmatrix} x & z & u \\ & \lambda y & y \\ & -y & \lambda y \end{pmatrix}$ $\lambda \geqslant 0$; 14. $\begin{pmatrix} \lambda x+\mu y & z & u \\ & y & x \\ & -x & y \end{pmatrix}$ $\lambda \geqslant 0$; 15. $\begin{pmatrix} x & z & u \\ & y & \lambda x+y \\ & & y \end{pmatrix}$ $\lambda \geqslant 0$;

16. $\begin{pmatrix} x & z & u \\ & y & x \\ & & y \end{pmatrix}$; 17. $\begin{pmatrix} x & \lambda x+y & u \\ & x & z \\ & & y \end{pmatrix}$; 18. $\begin{pmatrix} x & x & u \\ & x & z \\ & & y \end{pmatrix}$;

19. $\begin{pmatrix} & y & u \\ & & z \\ & & x \end{pmatrix}$; 20. $\begin{pmatrix} & y & u \\ & x & z \\ & & \lambda x \end{pmatrix}$; 21. $\begin{pmatrix} x & y & u \\ & \lambda x & z \\ & & \mu x \end{pmatrix}$;

22. $\begin{pmatrix} x+y & z & u \\ & x & z \\ & & x-y \end{pmatrix}$.

<u>$\dim \mathfrak{g} = 5$</u>

$$1.\ \begin{array}{|ccc|}\hline x & u & \\ v & y & \\ & & z \\ \hline\end{array}\ ; \quad 2.\ \begin{array}{|ccc|}\hline x & u & v \\ z & -x & u \\ & & \\ \hline\end{array}\ ; \quad 3.\ \begin{array}{|ccc|}\hline & u & v \\ & x & y \\ & z & -x \\ \hline\end{array}\ ;$$

$$4.\ \begin{array}{|ccc|}\hline x & & v \\ & y & u \\ & & z \\ \hline\end{array}\ ; \quad 5.\ \begin{array}{|ccc|}\hline x & y & v \\ -y & x & u \\ & & z \\ \hline\end{array}\ ; \quad 6.\ \begin{array}{|ccc|}\hline x & u & v \\ & y & \\ & & z \\ \hline\end{array}\ ;$$

$$7.\ \begin{array}{|ccc|}\hline x & u & v \\ & y & z \\ & -z & y \\ \hline\end{array}\ ; \quad 8.\ \begin{array}{|ccc|}\hline & z & v \\ & x & u \\ & & y \\ \hline\end{array}\ ; \quad 9.\ \begin{array}{|ccc|}\hline x & z & v \\ & \lambda x & u \\ & & y \\ \hline\end{array}\ ;$$

$$10.\ \begin{array}{|ccc|}\hline x & z & v \\ & y & u \\ & & \lambda x + \mu y \\ \hline\end{array}\ .$$

<u>$\dim \mathfrak{g} = 6$</u>

$$1.\ \begin{array}{|ccc|}\hline x & z & w \\ u & y & v \\ & & \\ \hline\end{array}\ ; \quad 2.\ \begin{array}{|ccc|}\hline \lambda x + y & z & w \\ u & \lambda x - y & v \\ & & x \\ \hline\end{array}\ ; \quad 3.\ \begin{array}{|ccc|}\hline & v & w \\ & x & z \\ & y & u \\ \hline\end{array}\ ;$$

$$4.\ \begin{array}{|ccc|}\hline x & v & w \\ & \lambda x + y & z \\ & u & \lambda x - y \\ \hline\end{array}\ ; \quad 5.\ \begin{array}{|ccc|}\hline x & u & w \\ & y & v \\ & & z \\ \hline\end{array}\ .$$

<u>$\dim \mathfrak{g} = 7$</u>

$$1.\ \begin{array}{|ccc|}\hline x & u & t \\ v & y & w \\ & & z \\ \hline\end{array}\ ; \quad 2.\ \begin{array}{|ccc|}\hline x & w & t \\ & y & u \\ & v & z \\ \hline\end{array}\ .$$

<u>$\dim \mathfrak{g} = 8$</u>

$$1.\ \begin{array}{|ccc|}\hline x & z & v \\ w & y - x & u \\ t & s & -y \\ \hline\end{array}\ .$$

Here subalgebras of the same number but with different values of parameters are not conjugate to each other.

Remark. To refer to subalgebras determined in Theorem 1 we use the following notation:

$$d.n,$$

where
d is the dimension of the subalgebra;
n is the number of the subalgebra in Theorem 1.

Proof. Fix the standard basis of the space $V = \mathbb{R}^3$:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We identify the Lie algebras $\mathfrak{gl}(V)$ and $\mathfrak{gl}(3,\mathbb{R})$.

For a subalgebra $\mathfrak{g} \subset \mathfrak{gl}(3,\mathbb{R})$ by $A(\mathfrak{g})$ denote the following subgroup of $GL(3,\mathbb{R})$:

$$A(\mathfrak{g}) = \{X \in GL(3,\mathbb{R}) \mid X\mathfrak{g}X^{-1} = \mathfrak{g}\}.$$

We divide the classification of all subalgebras of the Lie algebra $\mathfrak{gl}(3,\mathbb{R})$ into three parts:
I. Classification of commutative subalgebras
II. Classification of solvable non-commutative subalgebras
III. Classification of unsolvable subalgebras

Lemma 1. *Any commutative subalgebra of $\mathfrak{gl}(3,\mathbb{R})$ is conjugate to one and only one of the following subalgebras:*

$1.1 - 1.9$	3.1
$2.1 - 2.6$	3.2
$2.8\ (\lambda = 0)$	$3.8\ (\lambda = 1, \mu = 0)$
$2.9\ (\mu = 1)$	3.9
$2.10 - 2.14$	$3.13\ (\lambda = \mu = 1)$
$2.16\ (\lambda = 1)$	$3.20\ (\lambda = \mu = 1)$
2.17	
$2.19\ (\lambda = 1)$	
2.20	

For the proof we need the following Lemma.

Lemma. *Any maximal commutative subalgebra of $\mathfrak{gl}(3,\mathbb{R})$ is conjugate to one and only one of the following subalgebras:*

$$3.1 \begin{array}{|ccc|} \hline x & & \\ & y & \\ & & z \\ \hline \end{array}; \qquad 3.2 \begin{array}{|ccc|} \hline y & x & \\ -x & y & \\ & & z \\ \hline \end{array}; \qquad 3.8 \begin{array}{|ccc|} \hline y & & x \\ & z & \\ & & y \\ \hline \end{array};$$

$$3.9 \begin{array}{|ccc|} \hline x & y & z \\ & x & y \\ & & x \\ \hline \end{array}; \qquad \underset{(\lambda=\mu=1)}{3.13} \begin{array}{|ccc|} \hline x & & z \\ & x & y \\ & & x \\ \hline \end{array}; \qquad \underset{(\lambda=\mu=1)}{3.20} \begin{array}{|ccc|} \hline x & y & z \\ & x & \\ & & x \\ \hline \end{array}.$$

Proof of the Lemma. Suppose $\mathfrak{g}$ is a maximal commutative subalgebra of $\mathfrak{gl}(3,\mathbb{R})$. The maximality of $\mathfrak{g}$ implies that $\mathfrak{g}$ is an associative subalgebra. Since for any endomorphism φ its nilpotent and semisimple parts are polynomials in φ, we see that $\mathfrak{g}$ is a separating subalgebra of $\mathfrak{gl}(3,\mathbb{R})$.

Let $\mathfrak{a}$ be the set of semisimple and $\mathfrak{n}$ the set of nilpotent elements of the Lie algebra $\mathfrak{g}$. Then $\mathfrak{a}$ and $\mathfrak{n}$ are ideals in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}$ ([Bou], Ch. VII, §5, Pr. 5). In addition $\mathfrak{a}$ and $\mathfrak{n}$ are associative algebras. Since $\mathfrak{a}$ consists of semisimple endomorphisms only, we see that the $\mathfrak{a}$–module V is semisimple ([Bou], Ch. I, §6, Th. 4).

Suppose $V = \oplus_{i=1}^{k} V_i$ is a direct sum of isotypical components. Since any element x of the Lie algebra $\mathfrak{g}$ is an endomorphism of the $\mathfrak{a}$–module V, we see that x leaves isotypical components invariant. It follows that V_i is a submodule of the $\mathfrak{g}$–module V for $i = 1, \dots, k$.

Suppose $(p_i)_{1\leqslant i\leqslant k}$ is a system of projections corresponding to the decomposition $V = \oplus_{i=1}^{k} V_i$ and $\mathfrak{g}_i = p_i\mathfrak{g}p_i$ for $i = 1, \dots, k$. It is obvious that $p_i \in \mathfrak{g}$, $\mathfrak{g} = \oplus_{i=1}^{k}\mathfrak{g}_i$, and any subalgebra $\mathfrak{g}_i$ can be identified with some maximal commutative subalgebra of $\mathfrak{gl}(3,\mathbb{R})$, and also $\mathfrak{a} \cap \mathfrak{g}_i \ni p_i$.

Thus the problem of finding maximal commutative subalgebras is reduced to finding all maximal commutative subalgebras of $\mathfrak{gl}(n,\mathbb{R})$, $n = 1, 2, 3$, that contain no projections except 0 and E_n.

For $n = 1$ the problem is trivial.

For $n = 2$ we have two subalgebras satisfying the required conditions:

$$\boxed{\begin{matrix} x & -y \\ y & x \end{matrix}} \quad \text{and} \quad \boxed{\begin{matrix} x & y \\ & x \end{matrix}}\,.$$

Suppose $n = 3$ and $\mathfrak{g}$ is a maximal commutative subalgebra of $\mathfrak{gl}(3,\mathbb{R})$ such that $\mathfrak{g}$ does not contain projections different from 0 and E_3. Then the $\mathfrak{a}$–module V contains exactly one isotypical component, otherwise the subalgebra $\mathfrak{g}$ would contain projections different from 0 and E_3. Note that the $\mathfrak{a}$–module V is a direct sum of isomorphic simple submodules and each of them is either two-dimensional or one-dimensional. Since $\dim V = 3$, all simple submodules of the $\mathfrak{a}$–module V are one-dimensional and $\mathfrak{a} = \mathbb{R}.E_3$.

There exists a basis of V such that $\mathfrak{n}$ is a subalgebra of the Lie algebra

$$\mathfrak{n}_3(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \quad x, y, z \in \mathbb{R} \right\}.$$

Since $\mathfrak{n}_3(\mathbb{R})$ is non-commutative and its center has the form

$$\mathbb{Z}(\mathfrak{n}_3(\mathbb{R})) = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\},$$

we conclude that $\dim \mathfrak{n} = 2$ and the Lie algebra $\mathfrak{n}$ is conjugate to the subalgebra

$$\left\{ \begin{pmatrix} 0 & \alpha y & z \\ 0 & 0 & \beta y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\}, \text{ where } \alpha, \beta \in \mathbb{R} \text{ and } \alpha^2 + \beta^2 \neq 0.$$

Then the Lie algebra $\mathfrak{g}$ (viewed up to conjugation by diagonal matrices) has one of the following forms:

$$(i) \quad \alpha \neq 0, \; \beta \neq 0 \quad \begin{array}{|ccc|} \hline x & y & z \\ & x & y \\ & & x \\ \hline \end{array}\;;$$

$$(ii) \quad \alpha \neq 0, \; \beta = 0 \quad \begin{array}{|ccc|} \hline x & y & z \\ & x & \\ & & x \\ \hline \end{array}\;;$$

$$(iii) \quad \alpha = 0, \; \beta \neq 0 \quad \begin{array}{|ccc|} \hline x & & z \\ & x & y \\ & & x \\ \hline \end{array}\;.$$

Now show that these subalgebras are not conjugate to each other. For the subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ define by induction the following sequence of subalgebras of V:

$$V_0 = V, \; V_{n+1} = \mathfrak{n}(V_n) \text{ for } n \geq 0.$$

Then we have

$$(i) \; \dim V_1 = 2, \; \dim V_2 = 1;$$
$$(ii) \; \dim V_1 = 1, \; \dim V_2 = 0;$$
$$(iii) \; \dim V_1 = 2, \; \dim V_2 = 0.$$

Therefore the obtained subalgebras are not conjugate to each other. The proof of the Lemma is complete.

Proof of Lemma 1. Every commutative subalgebra of $\mathfrak{gl}(3,\mathbb{R})$ is contained in a certain maximal commutative subalgebra and, therefore, is conjugate to some subalgebra of one of the maximal commutative subalgebras determined in the preceding Lemma.

Since any vector subspace of a commutative algebra is a subalgebra, the problem reduces to description (up to conjugation by elements of $A(\mathfrak{g})$) of all vector subspaces for each maximal commutative subalgebra $\mathfrak{g}$ determined in the Lemma. This way we can obtain all commutative subalgebras of $\mathfrak{gl}(3,\mathbb{R})$ viewed up to conjugation. (Note that the same subalgebra can be contained in different maximal subalgebras). Finally we determine which of the obtained subalgebras are conjugate to each other. The one- and three-dimensional cases are trivial, and the problem causes no difficulties when the dimension of a subalgebra is equal to 2.

For instance consider the following case:

$$\mathfrak{g} = \boxed{\begin{matrix} x & & \\ & y & \\ & & z \end{matrix}}\,.$$

It is easy to see that the $\mathfrak{g}$–module V is a direct sum of three non-isomorphic simple submodules. For $\sigma \in \mathbb{S}_3$, by $\tilde{\sigma}$ denote the automorphism of V defined by

$$\tilde{\sigma}(e_i) = e_{\sigma(i)} \text{ for } i = 1, 2, 3.$$

Then

$$A(\mathfrak{g}) = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \circ \tilde{\sigma} \;\middle|\; x, y, z \in \mathbb{R}^*, \sigma \in \mathbb{S}_3 \right\}.$$

Suppose $\mathfrak{b}$ ic a one-dimensional subspace of $\mathfrak{g}$ spanned by an arbitrary nonzero element $e \in \mathfrak{g}$. If e is a degenerate matrix, without loss of generality it can be assumed that

$$e = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } a^2 + b^2 \neq 0 \text{ and } |a| \geqslant |b|.$$

Then $a \neq 0$ and

$$\mathfrak{b} = \boxed{\begin{matrix} x & & \\ & \lambda x & \\ & & \end{matrix}}\,, \text{ where } \lambda = \frac{b}{a}, \text{ and therefore } |\lambda| \leqslant 1.$$

It is possible to show that subalgebras $\mathfrak{b}$ corresponding to different values of the parameter λ are not conjugate to each other.

Now suppose

$$e = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \text{ where } abc \neq 0.$$

Then thre are two alternatives.

1° The numbers a, b, and c are of the same sign. Then (up to conjugation by elements of $A(\mathfrak{g})$) it can be assumed that $|a| \geqslant |c| \geqslant |b|$. And

$$\mathfrak{b} = \boxed{\begin{matrix} x & & \\ & \lambda x & \\ & & \mu x \end{matrix}}\,, \text{ where } \lambda = ba^{-1},\ \mu = ca^{-1}, \text{ and therefore } 0 < \lambda \leqslant \mu \leqslant 1.$$

2° Two of the numbers a, b, c are of the same sign. Then it can be assumed that $bc > 0$ and $|b| \geqslant |c|$. So

$$\mathfrak{b} = \boxed{\begin{matrix} x & & \\ & \lambda x & \\ & & \mu x \end{matrix}}\,, \text{ where } \lambda = \frac{b}{a},\ \mu = \frac{c}{a}, \text{ and therefore } \lambda \leqslant \mu < 0.$$

Thus any one-dimensional subspace of $\mathfrak{g}$ is conjugate (with respect to $A(\mathfrak{g})$) to one of the following subspaces:

$$1.1 \begin{array}{|ccc|} \hline x & & \\ & \lambda x & \\ & & \\ \hline \end{array}, \quad |\lambda| \leqslant 1; \quad 1.2 \begin{array}{|ccc|} \hline x & & \\ & \lambda x & \\ & & \mu x \\ \hline \end{array} \lambda\mu > 0,\ \lambda \leqslant \mu \leqslant 1.$$

It can be shown that subspaces corresponding to different values of parameter λ are not conjugate to each other.

In order to classify (up to conjugation by elements of $A(\mathfrak{g})$) all two-dimensional subspaces of $\mathfrak{g}$ it is sufficient to note that any two-dimensional subspace of $\mathfrak{g}$ is uniquely determined by a one-dimensional subspace of $\mathfrak{g}^*$.

The classification of one-dimensional subspaces in $\mathfrak{g}^*$ consides with that in $\mathfrak{g}$. Thus any two-dimensional subspace of $\mathfrak{g}$ (up to conjugation by elements of $A(\mathfrak{g})$) has the form:

$$2.1 \begin{array}{|ccc|} \hline x & & \\ & \lambda x & \\ & & y \\ \hline \end{array}, |\lambda| \leqslant 1; \quad \text{or} \quad 2.2 \begin{array}{|ccc|} \hline x & & \\ & \lambda x & \\ & & \mu x \\ \hline \end{array} \lambda\mu > 0,\ -1 \leqslant \mu \leqslant \lambda.$$

There exists a unique three-dimensional subspace of $\mathfrak{g}$ and it, of course, coincides with $\mathfrak{g}$:

$$3.1 \begin{array}{|ccc|} \hline x & & \\ & y & \\ & & z \\ \hline \end{array}.$$

After similar classification of all subspaces for other maximal commutative subalgebras determined in Lemma we find all non-conjugate subalgebras among obtained ones.

Lemma 2. *Any subalgebra non-commutative subalgebra of* $\mathfrak{gl}(3,\mathbb{R})$ *is conjugate to one and only one of the following subalgebras:*

2.7	3.8 ($\lambda = 1$ *or* $\mu = 0$)
2.8 ($\lambda \neq 0$)	3.10 – 3.12
2.9 ($\mu \neq 1$)	3.13 ($\lambda \neq 1$ *or* $\mu \neq 1$)
2.15	3.14 – 3.19
2.16 ($\lambda \neq 1$)	3.20 ($\lambda \neq 1$ *or* $\mu \neq 1$)
2.18	3.21 – 3.31
2.19 ($\lambda \neq 1$)	4.4
2.21	4.6 – 4.22
2.22	5.4 – 5.10
3.3 – 3.7	6.5

Proof of Lemma 2. Suppose $\mathfrak{g}$ is a solvable non-commutative subalgebra of the Lie algebra $\mathfrak{gl}(3,\mathbb{R})$ and $\mathfrak{n} = \mathfrak{D}\mathfrak{g}$ is the commutant of $\mathfrak{g}$. Then it can be assumed that $\mathfrak{n}$ consists of nilpotent elements of $\mathfrak{n}$ is a matrix with zeros on and below the diagonal. So it can be assumed that $\mathfrak{n}$ is a subalgebra of

$$\mathfrak{n}_3 = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

It is easy to see that if $\dim \mathfrak{n} \leq 2$, then $\mathfrak{n}$ is commutative. From the classification of commutative Lie algebras it follows that $\mathfrak{n}$ is conjugate to one of the following subalgebras:

$$a)\ \boxed{\begin{matrix} & x & y \\ & & z \\ & & \end{matrix}}\ ;\quad b)\ \boxed{\begin{matrix} & x & y \\ & & x \\ & & \end{matrix}}\ ;\quad c)\ \boxed{\begin{matrix} & x & y \\ & & \\ & & \end{matrix}}\ ;$$

$$d)\ \boxed{\begin{matrix} & & x \\ & & y \\ & & \end{matrix}}\ ;\quad e)\ \boxed{\begin{matrix} & x & \\ & & x \\ & & \end{matrix}}\ ;\quad f)\ \boxed{\begin{matrix} & & x \\ & & \\ & & \end{matrix}}\ .$$

Since $\mathfrak{n}$ is an ideal in $\mathfrak{g}$, we have $\mathfrak{g} \in N(\mathfrak{n})$. Thus description (up to conjugation) of all non-commutative solvable subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ reduces to classification (up to conjugation by elements of $A(\mathfrak{n})$) of all subalgebras $\mathfrak{g}$ in $N(\mathfrak{n})$ such that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}$ for each $\mathfrak{n}$ specified above.

Below $N(\mathfrak{n})$ and $A(\mathfrak{n})$ for each $\mathfrak{n}$ are written:

a) $A(\mathfrak{n}) = T(3, \mathbb{R})$, $\quad N(\mathfrak{n}) = t(3, \mathbb{R})$;

$$b)\quad A(\mathfrak{n}) = \left\{ \begin{pmatrix} x^2y & z & u \\ 0 & xy & t \\ 0 & 0 & y \end{pmatrix} \;\middle|\; \begin{matrix} x, y \in \mathbb{R}^* \\ z, t, u \in \mathbb{R} \end{matrix} \right\}, \qquad N(\mathfrak{n}) = \boxed{\begin{matrix} x & z & u \\ & y & t \\ & & 2x - y \end{matrix}}\ ;$$

$$c)\quad A(\mathfrak{n}) = \left\{ \begin{pmatrix} x & Y \\ 0 & Z \end{pmatrix} \;\middle|\; \begin{matrix} x \in \mathbb{R}^*,\ Y \in \mathrm{Mat}_{1\times 2}(\mathbb{R}) \\ Y \in \mathrm{GL}(2, \mathbb{R}) \end{matrix} \right\}, \qquad N(\mathfrak{n}) = \boxed{\begin{matrix} x & y & z \\ & t & u \\ & v & w \end{matrix}}\ ;$$

$$d)\quad A(\mathfrak{n}) = \left\{ \begin{pmatrix} X & Y \\ 0 & z \end{pmatrix} \;\middle|\; \begin{matrix} X \in \mathrm{GL}(2, \mathbb{R}) \\ Y \in \mathrm{Mat}_{2\times 1}(\mathbb{R}),\ z \in \mathbb{R}^* \end{matrix} \right\}, \qquad N(\mathfrak{n}) = \boxed{\begin{matrix} x & y & u \\ z & t & v \\ & & w \end{matrix}}\ ;$$

$$e)\quad A(\mathfrak{n}) = \left\{ \begin{pmatrix} x^2y & 0 & z \\ 0 & xy & 0 \\ 0 & 0 & y \end{pmatrix} \;\middle|\; \begin{matrix} x, y \in \mathbb{R}^* \\ z \in \mathbb{R} \end{matrix} \right\}, \qquad N(\mathfrak{n}) = \boxed{\begin{matrix} x & z & t \\ & y & z \\ & & 2x - y \end{matrix}}\ ;$$

$$f)\quad A(\mathfrak{n}) = \left\{ \begin{pmatrix} x & u & w \\ 0 & y & v \\ 0 & 0 & z \end{pmatrix} \middle| \begin{array}{c} x,y,z \in \mathbb{R}^* \\ u,v,w \in \mathbb{R} \end{array} \right\}, \qquad N(\mathfrak{n}) = \boxed{\begin{matrix} x & u & w \\ & y & v \\ & & z \end{matrix}}\ ;$$

Consider the case b):

$$\mathfrak{n} = \boxed{\begin{matrix} & x & y \\ & & x \\ & & \end{matrix}}\ .$$

Then $2 = \dim\ \mathfrak{n} \leqslant \dim\ \mathfrak{g} \leqslant \dim\ N(\mathfrak{n}) = 5, \mathfrak{D}\mathfrak{n} \neq \mathfrak{n}$, and $\mathfrak{D}(N(\mathfrak{n})) \neq \mathfrak{n}$. Therefore the dimension of $\mathfrak{g}$ is either 3 or 4.

Suppose $\dim \mathfrak{g} = 3$. Then there exists a unique one-dimensional subspace $\mathbb{R}.e$ of $\mathfrak{g}$ complementary to $\mathfrak{n}$ such that

$$e = \begin{pmatrix} a & c & 0 \\ 0 & b & 0 \\ 0 & 0 & 2b-a \end{pmatrix}.$$

Let P be an element of $A(\mathfrak{n})$ and

$$P = \begin{pmatrix} x & z & 0 \\ 0 & y & 0 \\ 0 & 0 & t \end{pmatrix}, \text{ where } t = y^2x^{-1},\ x,y \in \mathbb{R}^*,\ z \in \mathbb{R}.$$

Then

$$PeP^{-1} = \begin{pmatrix} a & c\frac{x}{y} + (b-a)\frac{z}{y} & 0 \\ 0 & b & 0 \\ 0 & 0 & 2b-a \end{pmatrix}.$$

It follows that the element e is diagonalizable by means of conjugation by elements of $A(\mathfrak{n})$ whenever $a \neq b$. If $a = b$ we have $\mathfrak{D}\mathfrak{g} \neq \mathfrak{n}$.

Therefore, it can be assumed that

$$e = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 2b-a \end{pmatrix}, \text{ where } a^2 + b^2 \neq 0.$$

If $a \neq 0$, the subalgebra $\mathfrak{g}$ has the form:

$$3.23\ \boxed{\begin{matrix} x & y & z \\ 0 & \lambda x & y \\ 0 & 0 & (2\lambda - 1)x \end{matrix}}\ , \text{ where } \lambda = \frac{b}{a}.$$

The subalgebras corresponding to different values of parameter λ are not conjugate to each other.

If $a = 0$, the subalgebra $\mathfrak{g}$ has the form:

$$3.24 \quad \boxed{\begin{array}{ccc} 0 & y & z \\ 0 & x & y \\ 0 & 0 & 2x \end{array}}\ .$$

It is possible to show that the subalgebras 3.23 and 3.24 are not conjugate.

Now suppose $\dim \mathfrak{g} = 4$. Similarly we can show that without loss of generality it can be assumed that the subspace of $\mathfrak{g}$ complementary to $\mathfrak{n}$ is diagonal, and $\mathfrak{g}$ has the form:

$$4.22 \quad \boxed{\begin{array}{ccc} x+y & z & u \\ 0 & x & z \\ 0 & 0 & x-y \end{array}}\ .$$

We consider other cases in a similar way and finally obtain the results of the Lemma 2.

Lemma 3. *Any unsolvable subalgebra of the Lie algebra* $\mathfrak{gl}(\mathbb{R})$ *is conjugate to one and only one of the following subalgebras:*

$3.3 - 3.5$	$6.1 - 6.4$
$4.1 - 4.3$	7.1
4.5	7.2
$5.1 - 5.3$	8.1

Proof of Lemma 3. Let $\mathfrak{g}$ be an unsolvable subalgebra of $\mathfrak{gl}(3, \mathbb{R})$. Then $\mathfrak{g}$ contains the semisimple Levi subalgebra $\mathfrak{a}$, where $\mathfrak{a} = [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{sl}(3, \mathbb{R})$.

Any semisimple subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ is conjugate to one of the following subalgebras:

$$(i) \boxed{\begin{array}{ccc} x & y & \\ z & -x & \\ & & \end{array}}\ ; \quad (ii) \boxed{\begin{array}{ccc} & x & y \\ -x & & z \\ -y & -z & \end{array}}\ ; \quad (iii) \boxed{\begin{array}{ccc} x & y & \\ z & & y \\ & z & -x \end{array}}\ ; \quad (iv)\ \mathfrak{sl}(3, \mathbb{R}).$$

Subalgebras (ii) and (iii) are maximal in $\mathfrak{sl}(3, \mathbb{R})$. Therefore, if $\mathfrak{a}$ is conjugate to subalgebra (ii) or (iii), then $\mathfrak{g}$ has one of the following forms:

$$3.4 \boxed{\begin{array}{ccc} x & y & \\ z & & y \\ & z & -x \end{array}}\ ; \quad 3.5 \boxed{\begin{array}{ccc} & x & y \\ -x & & z \\ -y & -z & \end{array}}\ ;$$

$$4.3 \boxed{\begin{array}{ccc} x+y & z & \\ u & x & z \\ & u & x-y \end{array}}\ ; \quad 4.5 \boxed{\begin{array}{ccc} x & y & z \\ -y & x & u \\ -z & -u & x \end{array}}\ .$$

If the subalgebra $\mathfrak{a}$ is conjugate to $\mathfrak{sl}(3, \mathbb{R})$, then $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ or $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{R})$.

Consider in detail the case when $\mathfrak{a}$ is conjugate to subalgebra (i). Then the $\mathfrak{a}$–module $\mathfrak{gl}(3,\mathbb{R})$ is a direct sum of isotypical components:

$$\mathfrak{gl}(3,\mathbb{R}) = \mathfrak{a} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where modules $\mathfrak{m}_1$ and $\mathfrak{m}_2$ in a suitable basis have the form:

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}; \quad \mathfrak{m}_2 = \left\{ \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & w \\ s & t & 0 \end{pmatrix} \;\middle|\; s, t, v, w \in \mathbb{R} \right\}.$$

Since $\mathfrak{g}$ is a submodule of the $\mathfrak{a}$–module $\mathfrak{gl}(3,\mathbb{R})$, we see that $\mathfrak{g}$ is a direct sum of intersections:

$$\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{a}) \oplus (\mathfrak{g} \cap \mathfrak{m}_1) \oplus (\mathfrak{g} \cap \mathfrak{m}_2).$$

If $\mathfrak{g} \cap \mathfrak{m}_2 = \{0\}$, then $\mathfrak{g} \subset \mathfrak{a} \oplus \mathfrak{m}_1$. Therefore $\mathfrak{g}$ is a reductive subalgebra. Note that the submodule $\mathfrak{m}_1$ is invariant under conjugations preserving the subalgebra $\mathfrak{a}$. This implies that the subalgebra $\mathfrak{g}$ is conjugate to one and only one of the following subalgebras:

$$\mathfrak{g} \cap \mathfrak{m}_1 = \{0\}: \quad 3.3 \begin{array}{|ccc|}\hline x & y & \\ z & -x & \\ & & \\ \hline\end{array}.$$

$$\dim(\mathfrak{g} \cap \mathfrak{m}_1) = 1: \quad 4.1 \begin{array}{|ccc|}\hline x & z & \\ u & y & \\ & & \\ \hline\end{array}; \quad 4.2 \begin{array}{|ccc|}\hline \lambda x + y & z & \\ u & \lambda x - y & \\ & & x \\ \hline\end{array}.$$

$$\dim(\mathfrak{g} \cap \mathfrak{m}_1) = 2: \quad 5.1 \begin{array}{|ccc|}\hline x & u & \\ v & y & \\ & & z \\ \hline\end{array}.$$

Since $\mathfrak{m}_2$, as a subset of $\mathfrak{gl}(3,\mathbb{R})$, generates the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$, we have $\mathfrak{g} \not\supset \mathfrak{m}_2$.

The $\mathfrak{a}$–module $\mathfrak{m}_2$ is a direct sum of two isomorphic simple submodules:

$$\mathfrak{m}_2 = \mathfrak{n}_1 \oplus \mathfrak{n}_2,$$

where

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; v, w \in \mathbb{R} \right\}, \quad \mathfrak{n}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s & t & 0 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\},$$

and the isomorphism $\pi : \mathfrak{n}_1 \longrightarrow \mathfrak{n}_2$ is defined by

$$\pi \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -y & x & 0 \end{pmatrix}.$$

Thus, if $\mathfrak{m}_2 \cap \mathfrak{g} \neq \{0\}$, then

$$\mathfrak{m}_2 \cap \mathfrak{g} = (\alpha + \beta\pi)(\mathfrak{n}_1) = \left\{ \begin{pmatrix} 0 & 0 & \alpha x \\ 0 & 0 & \alpha y \\ -\beta y & \beta x & 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\},$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 \neq 0$. However, if $\alpha\beta \neq 0$, this set also generates the whole Lie algebra $\mathfrak{sl}(3, \mathbb{R})$. Therefore $\alpha = 0$ or $\beta = 0$. Then $\mathfrak{g} \cap \mathfrak{m}_2$ is the nilpotent radical of $\mathfrak{g}$. In a suitable basis it has the form:

$$a) \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} \text{ or } \quad b) \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}.$$

Since $\mathfrak{g} \subset N(\mathfrak{g} \cap \mathfrak{m}_2)$, for cases $a)$ and $b)$ we have:

$$a)\ \mathfrak{g} \subset \boxed{\begin{array}{ccc} x & w & t \\ & y & u \\ & v & z \end{array}}\,; \quad b)\ \mathfrak{g} \subset \boxed{\begin{array}{ccc} x & u & t \\ v & y & w \\ & & z \end{array}}\,.$$

Further it is easily proved that $\mathfrak{g}$ is conjugate to one and only one of the following Lie algebras:

5.2, 5.3, 6.1–6.4, 7.1, 7.2.

The proof of the Lemma 3 is complete.

The results of the theorem are immediate from Lemmas 1,2 and 3.

CHAPTER II

METHODS OF CLASSIFICATION OF PAIRS

1. Structure of virtual pairs

Let $(V, \mathfrak{g})$ be a virtual pair and $U = V/\mathfrak{g}$. Suppose $\pi : V \to U$ is the canonical surjection and $s : U \to V$ is an arbitrary section of the surjection π (in other words, s is a linear mapping such that $\pi \circ s = id_U$). Consider the mapping $H_s : V \to \mathfrak{g} \times U$ defined by

$$H_s(v) = (v - s \circ \pi(v), \pi(v)) \quad \text{for } v \in V.$$

Since

$$\pi(v - s \circ \pi(v)) = \pi(v) - \pi(v) = 0 \quad \text{for } v \in V,$$

we see that $v - s \circ \pi(v) \in \mathfrak{g}$, and therefore H_s is well-defined.

Proposition 1. *The mapping H_s is an isomorphism of the vector spaces V and $\mathfrak{g} \times U$, and also the following condition holds:*

$$H_s(\mathfrak{g}) = \mathfrak{g} \times \{0\}.$$

Proof. Let us prove that the mapping $G : \mathfrak{g} \times U \to V$ given by

$$G(x, u) = x + s(u) \quad \text{for all } (x, u) \in \mathfrak{g} \times U$$

is inverse to the mapping H_s. Indeed, for any $v \in V$ we have

$$G \circ H_s(v) = G(v - s \circ \pi(v), \pi(v)) = v - s \circ \pi(v) + s \circ \pi(v) = v.$$

Therefore $G = H_s^{-1}$ and H_s is an isomorphism. Moreover $\pi(x) = 0$ and $H_s(x) \in \mathfrak{g} \times \{0\}$ for all $x \in \mathfrak{g}$. Conversely, if $(x, 0) \in \mathfrak{g} \times \{0\}$, then $H_s^{-1}(x, 0) = x \in \mathfrak{g}$. It follows that $H_s(\mathfrak{g}) = \mathfrak{g} \times \{0\}$.

The pair of vector spaces $(\mathfrak{g} \times U, \mathfrak{g} \times \{0\})$ is canonically supplied with the structure of a virtual pair isomorphic to the virtual pair $(V, \mathfrak{g})$:

$$(x, 0).(y, u) = H_s(H_s^{-1}(x, 0).H^{-1}(y, u)) \quad \text{for all } x, y \in \mathfrak{g}, u \in U.$$

In the sequel we identify $\mathfrak{g}$ and $\mathfrak{g} \times \{0\}$.

Proposition 2. *There exists a linear mapping* $q_s : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ *such that for all* $x, y \in \mathfrak{g}, u \in U$ *we have*

$$x.(y,u) = ([x,y] + q_s(x)(u), x.u), \tag{1}$$

and for all $x, y \in \mathfrak{g}$ *the following condition holds:*

$$q_s([x,y]) = x.q_s(y) - y.q_s(x). \tag{2}$$

Proof. Indeed, we have

$$x.(y,u) = H_s(x.(y + s(u))) = H_s([x,y] + x.s(u)) = ([x,y] + x.s(u) - s(x.u), x.u).$$

Put $q_s(x)(u) = x.s(u) - s(x.u)$ for $x \in \mathfrak{g}, u \in U$. In other words $q_s(x) = x.s$. Then it is clear that

$$q_s([x,y]) = [x,y].s = x.(y.s) - y.(x.s) = x.q_s(y) - y.q_s(x).$$

Definition. Suppose $(\mathfrak{g}, U)$ is a generalized module and $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is a linear mapping such that

$$q([x,y]) = x.q(y) - y.q(x) \quad \text{for all } x, y \in \mathfrak{g}. \tag{2'}$$

Then the mapping q is called a *virtual structure* on the generalized module $(\mathfrak{g}, U)$.

Proposition 3. *Suppose* q *is a virtual structure on a generalized module* $(\mathfrak{g}, U)$. *Put* $V_q = \mathfrak{g} \times U$. *Then the bilinear mapping* $\mathfrak{g} \times V_q \to V_q$ *given by*

$$x.(y,u) = ([x,y] + q(x)(u), x.u) \quad \textit{for all } x, y \in \mathfrak{g}, u \in U \tag{3}$$

defines the virtual pair $(V_q, \mathfrak{g})$.

Proof. Indeed, for $x_1, x_2, y \in \mathfrak{g}$ and $u \in U$ we have

$$[x_1, x_2].(y,u) = \big([[x_1,x_2],y] + q([x_1,x_2])(u), [x_1,x_2].u\big)$$

$$= \big([x_1,[x_2,y]] - [x_2,[x_1,y]] + [x_1, q(x_2)(u)] - q(x_2)(x_1.u) - [x_2, q(x_1)(u)]$$

$$+ q(x_1)(x_2.u), x_1.(x_2.u) - x_2.(x_1.u)\big) = x_1.\big([x_2,y] + q(x_2)(u), x_2.u\big)$$

$$- x_2.\big([x_1,y] + q(x_1)(u), x_1.u\big) = x_1.(x_2.(y,u)) - x_2.(x_1.(y,u)).$$

So, to any virtual structure on a generalized module $(\mathfrak{g}, U)$ we assign the virtual pair $(\mathfrak{g} \times U, \mathfrak{g})$ defined by formula (3). Moreover, any virtual pair $(V, \mathfrak{g})$ with the associated generalized module $(\mathfrak{g}, U)$ can be constructed in this way.

Definition. Suppose q_1 and q_2 are virtual structures on a generalized module $(\mathfrak{g}, U)$. We say that q_1 and q_2 are *equivalent* if the virtual pairs $(V_{q_1}, \mathfrak{g})$ and $(V_{q_2}, \mathfrak{g})$ are isomorphic.

Proposition 4. *Virtual structures q_1 and q_2 on a generalized module $(\mathfrak{g}, U)$ are equivalent if and only if there exist an automorphism (f, p) of the generalized module $(\mathfrak{g}, U)$ and a linear mapping $h : U \to \mathfrak{g}$ such that the following condition holds:*

$$q_2(x) = f \circ q_1(f^{-1}(x)) \circ p^{-1} - x.h \quad \text{for all } x \in \mathfrak{g}. \tag{4}$$

Proof. Let $H : (V_{q_1}, \mathfrak{g}) \to (V_{q_2}, \mathfrak{g})$ be an isomorphism of virtual pairs. Then we can uniquely define mappings $f \in GL(\mathfrak{g}), p \in GL(U)$, and $h \in \mathcal{L}(U, \mathfrak{g})$ such that

$$H(y, u) = (f(y) + h \circ p(u), p(u)) \quad \text{for all } (y, u) \in V_{q_1}. \tag{5}$$

In this case $H^{-1}(y, u) = (f^{-1}(y) + f^{-1} \circ h(u), p^{-1}(u))$.
Then we have

$$H(x.(y, u)) = H([x, y] + q_1(x)(u), x.u) = (f([x, y]) + f(q_1(x)(u)) + h \circ p(x.u), p(x.u)).$$

On the other hand

$$\begin{aligned} H(x, 0).H(y, u) &= f(x).(f(y) + h \circ p(u), p(u)) \\ &= ([f(x), f(y)] + [f(x), h \circ p(u)] + q_2(f(x))(p(u)), f(x).p(u)) \end{aligned}$$

It follows that $f(x).p(u) = p(x.u)$ for all $x \in g, u \in V$. Thus (f, p) is an automorphism of the generalized module $(\mathfrak{g}, U)$. Putting $u = 0$ we obtain $f([x, y]) = [f(x), f(y)]$ for all $x, y \in \mathfrak{g}$. Therefore f is an automorphism of the Lie algebra $\mathfrak{g}$. Further

$$q_2(x)(u) = f(q_1(f^{-1}(x))(p^{-1}(u))) + h \circ p(f^{-1}(x).p^{-1}(u)) - [x, h(u)]$$

$$= f \circ q_1(f^{-1}(x)) \circ p^{-1}(u) + h(x.u) - [x, h(u)] = f \circ q_1(f^{-1}(x)) \circ p^{-1}(u) - (x.h)(u).$$

Therefore

$$q_2(x) = f \circ q_1(f^{-1}(x)) \circ p^{-1} - x.h$$

Conversely, suppose that there exist an automorphism (f, p) of the generalized module $(\mathfrak{g}, U)$ and a mapping $h \in \mathcal{L}(U, \mathfrak{g})$ satisfying condition (4). Then the mapping $H : V_{q_1} \to V_{q_2}$ defined by (5) is an isomorphism of the virtual pairs $(V_{q_1}, \mathfrak{g})$ and $(V_{q_2}, \mathfrak{g})$.

Corollary 1. *Suppose q_1 and q_2 are virtual structures on a generalized module $(\mathfrak{g}, U)$ and there exists a mapping $h \in \mathcal{L}(U, \mathfrak{g})$ such that $q_1(x) - q_2(x) = x.h$ for all $x \in \mathfrak{g}$. Then the virtual structures q_1 and q_2 are equivalent.*

Proof. It is sufficient to put $p = id_U$ and $f = id_{\mathfrak{g}}$ in (4).

Thus, classification (up to isomorphism) of all virtual pairs $(V, \mathfrak{g})$ for a given generalized module $(\mathfrak{g}, U)$ reduces to classification of all virtual structures on the generalized module $(\mathfrak{g}, U)$ (viewed up to equivalence).

2. Matrix form

Let $(\mathfrak{g}, U)$ be a faithful three-dimensional generalized module over the field $\mathbb{R}$. Suppose $\mathcal{E} = \{e_1, \dots, e_n\}$ is a basis of the Lie algebra $\mathfrak{g}$ ($n = \dim \mathfrak{g}$) and $\mathcal{U} = \{u_1, u_2, u_3\}$ a basis of the vector space U.

For $x \in \mathfrak{g}$, by $A(x)$ and $B(x)$ denote the matrices of the mappings

$$\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g} \quad \text{and} \quad x_U : U \to U$$

in the bases $\mathcal{E}$ and $\mathcal{U}$ respectively. Then $A(x) \in \mathrm{Mat}_{n\times n}(\mathbb{R}), B(x) \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$, and the mapping

$$\rho : \mathfrak{g} \to \mathfrak{gl}(3, \mathbb{R}), \ x \mapsto B(x)$$

is an injection. This allows to identify the Lie algebra $\mathfrak{g}$ with a certain subalgebra of the Lie algebra $\mathfrak{gl}(3, \mathbb{R})$. Without loss of generality it can be assumed that $\mathfrak{g}$ is one of the subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ determined in Theorem 1.

Recall that for $\mathfrak{g} \subset \mathfrak{gl}(3, \mathbb{R})$, by $A(\mathfrak{g})$ we denote the following subgroup of $\mathfrak{gl}(3, \mathbb{R})$:

$$A(\mathfrak{g}) = \left\{X \in \mathfrak{gl}(3, \mathbb{R}) \mid X\mathfrak{g}X^{-1} \subset \mathfrak{g}\right\}.$$

Consider the homomorphism of groups

$$\varphi : A(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g})$$

defined by

$$\varphi(P) : x \mapsto PxP^{-1} \quad \text{for } x \in \mathfrak{g}, P \in A(\mathfrak{g}).$$

Proposition 5. *Suppose (f, p) is an automorphism of the generalized module $(\mathfrak{g}, U)$ and P is the matrix of the mapping p. Then $P \in A(\mathfrak{g})$ and $f = \varphi(p)$.*

Proof. For all $x \in \mathfrak{g}, u \in U$ we have

$$p(x.u) = f(x).p(u)$$

or alternatively

$$p \circ x_U(u) = f(x)_U \circ p(u).$$

In matrix form it is equivalent to

$$PB(x) = B(f(x))P.$$

We identify x with $B(x)$ and $f(x)$ with $B(f(x))$. Then for any $x \in \mathfrak{g}$ we have $PxP^{-1} \in \mathfrak{g}$. Hence $P \in A(\mathfrak{g})$. Moreover $f(x) = PxP^{-1}$ and therefore $f = \varphi(P)$.

There is a one-to-one correspondence between the set of mappings $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ and the set of mappings $C : \mathfrak{g} \to \mathrm{Mat}_{n\times 3}(\mathbb{R})$, where $C(x)$ is the matrix of the mapping $q(x)$ in the bases fixed before.

Allowing a certain freedom of expression, we say that a mapping $C : \mathfrak{g} \to \mathrm{Mat}_{n\times 3}(\mathbb{R})$ is a virtual structure on the generalized module $(\mathfrak{g}, U)$ if the corresponding mapping q is a virtual structure.

Proposition 6. *A necessary and sufficient condition for a mapping*

$$C : \mathfrak{g} \to \mathrm{Mat}_{n\times 3}(\mathbb{R})$$

to be a virtual structure on the generalized module $(\mathfrak{g}, U)$ *is that the following condition hold:*

$$C([x,y]) = A(x)C(y) - C(y)B(x) - A(y)C(x) + C(x)B(y) \quad \text{for } x,y \in \mathfrak{g} \tag{6}$$

Proof. Indeed, we have

$$q([x,y]) = x.q(y) - y.q(x).$$

In other words

$$q([x,y])(u) = [x, q(y)(u)] - q(y)(x.u) - [y, q(x)(u)] + q(x)(y.u) = \mathrm{ad}\, x \circ q(y)(u)$$
$$-q(y) \circ x_u(u) - \mathrm{ad}\, y \circ q(x)(u) + q(x) \circ y_u(u) \quad \text{for all } u \in U.$$

This implies equivalence of conditions (2') and (6)

Proposition 7. *Suppose C_1 and C_2 are virtual structures on the generalized module $(\mathfrak{g}, U)$. C_1 and C_2 are equivalent if and only if there exist matrices $P \in A(\mathfrak{g})$ and $H \in \mathrm{Mat}_{n\times 3}(\mathbb{R})$ such that the following condition holds:*

$$C_2(x) = FC_1(\varphi^{-1}(x))P^{-1} - A(x)H + HB(x) \quad \text{for } x \in \mathfrak{g}, \tag{7}$$

where $\varphi = \varphi(P)$ and F is the matrix of the mapping φ.

Proof. Indeed, suppose q_1 and q_2 are the virtual structures on the generalized module $(\mathfrak{g}, U)$ corresponding to C_1 and C_2, respectively. Then from Proposition 4 it follows that there exist an automorphism (f, p) of the generalized module $(\mathfrak{g}, U)$ and a mapping $h \in \mathcal{L}(U, \mathfrak{g})$ such that for all $x \in \mathfrak{g}$ we have

$$g_2(x) = f \circ q_1(f^{-1}(x)) \circ p^{-1} - x.h.$$

Let H and P be matrices of the mappings h and p respectively. Then $P \in A(\mathfrak{g})$ and $f = \varphi(P)$. Further, the matrix of the mapping $x.h$ is equal to $A(x)H - HB(x)$. Therefore condition (7) holds.

Conversely, assume that condition (7) is satisfied. Then we can uniquely define mappings $p : U \to U$ and $h : U \to \mathfrak{g}$ such that their matrices are equal to P and H respectively. Then putting $f = \varphi(P)$, we see that the pair (f, p) is an automorphism of the generalized module $(\mathfrak{g}, U)$ and condition (4) is satisfied. Hence the virtual structures are equivalent.

Corollary 2. *Suppose C_1 and C_2 are virtual structures on the generalized module $(\mathfrak{g}, U)$ and there exists a matrix $H \in \mathrm{Mat}_{n\times 3}(\mathbb{R})$ such that for all $x \in \mathfrak{g}$ the following condition holds:*

$$C_1(x) - C_2(x) = A(x)H - HB(x). \tag{8}$$

Then C_1 and C_2 are equivalent.

Proof. It is sufficient to put $P = E_3$. Then $\varphi(P) = id_{\mathfrak{g}}$ and $F = E_3$. Therefore condition (7) is satisfied.

Remark. Note that all expressions in (6), (7), and (8) are linear in $x, y \in \mathfrak{g}$. Therefore, in order to ensure that these conditions are satisfied for all $x, y \in \mathfrak{g}$, we must only check that they hold for $x, y \in \mathcal{E} = \{e_1, \ldots, e_n\}$.

3. Primary virtual structures

Suppose $(V, \mathfrak{g})$ is a virtual pair and $(\mathfrak{g}, U)$, where $U = V/\mathfrak{g}$, is the generalized module associated with $(V, \mathfrak{g})$.

Proposition 8. *Let $\mathfrak{h}$ be a nilpotent subalgebra of $\mathfrak{g}$.*

(1) *A necessary and sufficient condition for the $\mathfrak{h}$-module V to be a direct sum of primary components is that the $\mathfrak{h}$-modules $\mathfrak{g}$ and U be direct sums of primary components.*

(2) *There exists a section $s : U \to V$ of the canonical surjection $\pi : V \to U$ such that for every $\alpha \in \mathfrak{h}^*$ the following condition holds:*

$$s(U^\alpha(\mathfrak{h})) \subset V^\alpha(\mathfrak{h}) \tag{9}$$

Proof.

(1) Let us remark that for any $x \in \mathfrak{h}$ the endomorphism x_V can be reduced to triangular form if and only if this can be done for the endomorphisms $x_\mathfrak{g} = \operatorname{ad} x$ and x_U. Indeed, suppose x_V can be reduced to triangular form; then there exists a Jordan-Hölder sequence for x_V:

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n+k}$$

($n = \dim \mathfrak{g}$, $k = \dim U$) such that $\dim V_i = i, 0 \leqslant i \leqslant n + k$, and $V_n = \mathfrak{g}$. Then the Jordan-Hölder sequences for $x_\mathfrak{g}$ and x_U have the form:

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathfrak{g} \quad \text{and}$$

$$\{0\} = V_n/W \subset V_{n+1}/W \subset \cdots \subset V_{n+k}/W = U$$

respectively. Therefore the endomorphisms $x_\mathfrak{g}$ and x_U can also be reduced to triangular form. Conversely, let the Jordan-Hölder sequence for $x_\mathfrak{g}$ and x_U have the form:

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g} \quad \text{and}$$

$$\{0\} = U_0 \subset U_1 \subset \cdots \subset U_k = U, \quad \text{where}$$

$$\dim \mathfrak{g}_i = i, 0 \leqslant i \leqslant n, \quad \text{and} \quad \dim U_i = i, 0 \leqslant i \leqslant k.$$

Then the sequence

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n \subset \pi^{-1}(U_1) \subset \cdots \subset \pi^{-1}(U_k) = V$$

is a Jordan-Hölder sequence for x_V such that all quotients of the sequence are one-dimensional. Therefore the endomorphism x_V can be reduced to triangular form. This proves the first part of the Proposition ([Bou], Ch. VII, §1, Pr. 9(i)).

(2) Let

$$U_1 = \sum_{\alpha \in \mathfrak{h}^*} U^\alpha(\mathfrak{h}), \qquad V_1 = \sum_{\alpha \in \mathfrak{h}^*} V^\alpha(\mathfrak{h}).$$

Let U_0 be a subspace of V complementary to U_1. Since the mapping $\pi : V \to U$ is a surjection, we have $\pi(V^\alpha(\mathfrak{h})) = U^\alpha(\mathfrak{h})$ for all $\alpha \in \mathfrak{h}^*$. ([Bou], Ch. VII, §1, Corollary 3 of Th. 1). Then it is obvious that $\pi(V_1) = U_1$ and $\pi^{-1}(U_0) + V_1 = V$. Therefore there exists a subspace V_0 of the vector space V such that V_0 is complimentary to V_1 and $V_0 \subset \pi^{-1}(U_0)$. In this case $\pi(V_0) = U_0$. For each $\alpha \in \mathfrak{h}^*$ consider the mapping $\pi^\alpha : V^\alpha(\mathfrak{h}) \to U^\alpha(\mathfrak{h})$ such that π^α is the restriction of the mapping π to $V^\alpha(\mathfrak{h})$. Consider also the mapping $\pi_0 : V_0 \to U_0$ such that π_0 is the restriction of the mapping π to V_0. For $\alpha \in \mathfrak{h}^*$, let s^α be an arbitrary section of the surjection π^α and let s_0 be an arbitrary section of the surjection π_0. Now consider the mapping $s : U \to V$ defined by

$$s(u_0 + \sum_{\alpha\in\mathfrak{h}^*} u^\alpha) = s_0(u_0) + \sum_{\alpha\in\mathfrak{h}^*} s^\alpha(u^\alpha),$$

where $u_0 \in U_0, u^\alpha \in U^\alpha(\mathfrak{h})$ for all $\alpha \in \mathfrak{h}^*$. The mapping s is a section of the surjection π and

$$s(U^\alpha(\mathfrak{h})) \subset V^\alpha(\mathfrak{h}) \quad \text{for all } \alpha \in \mathfrak{h}^*.$$

Definition. Suppose s is a section of the canonical surjection $\pi : V \to U$. We say that s is *consistent with the subalgebra* $\mathfrak{h}$ if

$$s(U^\alpha(\mathfrak{h})) \subset V^\alpha(\mathfrak{h}) \quad \text{for all } \alpha \in \mathfrak{h}^*.$$

From Proposition 8(2) it follows that there always exists such a section.

Proposition 9. *Suppose s is a section of the canonical surjection $\pi : V \to U$ consistent with the subalgebra $\mathfrak{h}$. Then the corresponding virtual structure $q_s : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ on the generalized module $(\mathfrak{g}, U)$ satisfies the following condition:*

$$q_s(\mathfrak{g}^\alpha(\mathfrak{h}))(U^\beta(\mathfrak{h})) \subset \mathfrak{g}^{\alpha+\beta}(\mathfrak{h}) \quad \textit{for } \alpha, \beta \in \mathfrak{h}^*. \tag{10}$$

Proof. Indeed, by definition we have $q_s(x)(u) = x.s(u) = x.s(u) - s(x.u)$ for all $x \in \mathfrak{g}, u \in U$. Suppose $x \in \mathfrak{g}^\alpha(\mathfrak{h})$ and $u \in U^\beta(\mathfrak{h})$ for $\alpha, \beta \in \mathfrak{h}^*$. Then

$$q(x)(u) = x.s(u) - s(x.u) \subset x.s(U^\beta(\mathfrak{h})) - s(x.U^\beta(\mathfrak{h})) \subset$$

$$\subset x.V^\beta(\mathfrak{h}) - s(U^{\alpha+\beta}(\mathfrak{h})) \subset V^{\alpha+\beta}(\mathfrak{h}) - V^{\alpha+\beta}(\mathfrak{h}) \subset V^{\alpha+\beta}(\mathfrak{h}).$$

On the other hand $q_s(x)(u) \in \mathfrak{g}$. Therefore $q_s(x)(u) \in \mathfrak{g} \cap V^{\alpha+\beta}(\mathfrak{h}) = \mathfrak{g}^{\alpha+\beta}(\mathfrak{h})$. This completes the proof of the Proposition.

Definition. We say that a virtual structure q on $(\mathfrak{g}, U)$ is *primary* (with respect to $\mathfrak{h}$) if q satisfies condition (10).

From Propositions 8(2) and 9 it follows that every virtual structure is equivalent to a certain primary virtual structure.

Proposition 10. *Suppose q is a primary (with respect to $\mathfrak{h}$) virtual structure on the generalized module $(\mathfrak{g}, U)$ and $(V_q, \mathfrak{g})$ is the corresponding virtual pair. Then*

$$V_q^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \quad \textit{for all } \alpha \in \mathfrak{h}^*.$$

Proof. It is obvious that $\mathfrak{g}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times \{0\}$ is contained in $V_q^{\alpha}(\mathfrak{h})$ for all $\alpha \in \mathfrak{h}^*$. Now let us prove the embedding $\{0\} \times U^{\alpha}(\mathfrak{h}) \subset V_q^{\alpha}(\mathfrak{h})$. Indeed, suppose $(0, u) \in U^{\alpha}(\mathfrak{h})$ and x is an arbitrary element of $\mathfrak{h}$. Then there exists an $n \in \mathbb{N}$ such that $(x_U - \alpha(x))^n(u) = 0$. Let us prove by induction on m that for any $m \in \mathbb{N}$ the following condition holds:

$$(x_V - \alpha(x))^m(0, u) - (0, (x_U - \alpha(x))^m(u)) \in \mathfrak{g}^{\alpha}(\mathfrak{h}). \tag{11}$$

The condition is evidently true for $m = 0$. Assuming that (11) is true for $m = p$, we have

$$(x_V - \alpha(x))^p(0, u) = (y, (x_U - \alpha(x))^p(u)),$$

where $y \in \mathfrak{g}^{\alpha}(\mathfrak{h})$. Then

$$(x_V - \alpha(x))^{p+1}(0, u) = (x_V - \alpha(x))(y, (x_U - \alpha(x))^p(u))$$

$$= (x, 0).\big(y, (x_U - \alpha(x))^p(u)\big) - \alpha(x)\big(y, (x_U - \alpha(x))^p(u)\big)$$

$$= \big((x_{\mathfrak{g}} - \alpha(x))(y) + q(x)((x_U - \alpha(x))^p(u), (x_U - \alpha(x))^{p+1}(u)\big).$$

Since $x \in \mathfrak{g}^0(\mathfrak{h})$, we see that the spaces $U^{\alpha}(\mathfrak{h})$ and $\mathfrak{g}^{\alpha}(\mathfrak{h})$ are invariant under x_U and $x_{\mathfrak{h}}$ respectively. Hence $(x_U - \alpha(x))^p(u) \in U^{\alpha}(\mathfrak{h})$ and $(x_{\mathfrak{g}} - \alpha(x))(y) \in \mathfrak{g}^{\alpha}(\mathfrak{h})$. Since the virtual structure q is primary, we have

$$q(x)((x_U - \alpha(x))^p(u) \in \mathfrak{g}^{\alpha}(\mathfrak{h}).$$

Therefore

$$((x_U - \alpha(x))^{p+1}(0, u) - (0, (x_U - \alpha(x))^{p+1}(u)) \in \mathfrak{g}^{\alpha}(\mathfrak{h}).$$

Thus embedding (11) is true.

Put $m = n$ in (11). Since $(x_U - \alpha(x))^n(u) = 0$, we obtain

$$(x_V - \alpha(x))^n(0, u) \in \mathfrak{g}^{\alpha}(\mathfrak{h}).$$

This immediately implies that $(0, u) \in V_q^{\alpha}(\mathfrak{h})$ and finally

$$\mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \subset V_q^{\alpha}(\mathfrak{h}).$$

Conversely, suppose $(x, u) \in V_q^{\alpha}(\mathfrak{h})$ and $\pi : V_q \to U$ is the canonical surjection. Then $\pi(x, u) = u \in U^{\alpha}(\mathfrak{h})$ and, as we saw above, $(0, u) \in V_q^{\alpha}(\mathfrak{h})$. Therefore $(x, 0) \in V_q^{\alpha}(\mathfrak{h})$. It follows that $x \in \mathfrak{g}^{\alpha}(\mathfrak{h}) = \mathfrak{g} \cap V_q^{\alpha}(\mathfrak{h})$ and $V_q^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h})$.

4. Trivial cases

Definition. A virtual pair $(V, \mathfrak{g})$ is said to be *trivial* if there exists a submodule U of the $\mathfrak{g}$-module V such that $V = U \oplus \mathfrak{g}$.

Note that a trivial virtual pair $(V, \mathfrak{g})$ is uniquely defined (up to isomorphism) by the corresponding generalized module $(\mathfrak{g}, V/\mathfrak{g})$.

Proposition 11. *Let q be a virtual structure on the generalized module $(\mathfrak{g}, U)$. Then a necessary and sufficient condition for the virtual pair $(V_q, \mathfrak{g})$ to be trivial is that q be equivalent to the zero mapping of $\mathfrak{g}$ into $\mathcal{L}(U, \mathfrak{g})$.*

Proof. It is obvious that the zero mapping of $\mathfrak{g}$ into $\mathcal{L}(U, \mathfrak{g})$ is a virtual structure and defines the trivial virtual pair $(V_0, \mathfrak{g})$ (the submodule $\{0\} \times U$ is complementary to $\mathfrak{g}$).

Suppose $(V, \mathfrak{g})$ is a trivial virtual pair and M is a submodule of V complimentary to $\mathfrak{g}$. If $\pi : V \to U$ is the canonical surjection (here $U = V/\mathfrak{g}$), then we have $\pi(M) = U$. Therefore, there exists a section $s : U \to V$ of the surjection π such that $s(U) \subset M$. Then for any $x \in \mathfrak{g}, u \in U$ we have

$$q_s(x)(u) = (x.s)(u) = x.s(u) - s(x.u) \subset x.M - s(U) \subset M.$$

On the other hand $q_s(x)(u) \in \mathfrak{g}$. Therefore $q_s(x)(u) = 0$ for all $x \in \mathfrak{g}, u \in U$ and q_s is the zero mapping. Thus, any virtual structure on $(\mathfrak{g}, U)$ defining the trivial virtual pair is equivalent to the zero mapping of $\mathfrak{g}$ into $\mathcal{L}(U, \mathfrak{g})$.

Corollary 3. *Suppose $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is a virtual structure on the generalized module $(\mathfrak{g}, U)$ and there exists a mapping $s \in \mathcal{L}(U, \mathfrak{g})$ such that $q(x) = x.s$ for all $x \in \mathfrak{g}$. Then the virtual pair $(V_q, \mathfrak{g})$ is trivial.*

Proposition 12. *Suppose $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is a virtual structure on the generalized module $(\mathfrak{g}, U)$ and $\mathfrak{a}$ is a semisimple subalgebra of the Lie algebra $\mathfrak{g}$. Then there exists a virtual structure $\tilde{q}$ equivalent to q such that $\tilde{q}(\mathfrak{a}) = \{0\}$.*

Proof. Indeed, suppose $(V, \mathfrak{g})$ is the virtual pair defined by q. Since $\mathfrak{a}$ is a semisimple subalgebra of $\mathfrak{g}$, we see that the $\mathfrak{a}$-module V is also semisimple. But $\mathfrak{g}$ is a submodule of the $\mathfrak{a}$-module V. Therefore there exists a submodule M of the $\mathfrak{a}$-module V such that $V = M \oplus \mathfrak{g}$. Suppose s is a section of the canonical surjection $\pi : V \to U$ such that $s(U) \subset M$. Calculation similar to that from the proof of the previous Proposition shows that $q_s(x)(u) = 0$ for all $x \in \mathfrak{a}, u \in U$, i.e., $q_s(\mathfrak{a}) = \{0\}$. The virtual structures q and q_s are equivalent.

Corollary 4. *If $\mathfrak{g}$ is a semisimple Lie algebra, then every virtual pair $(V, \mathfrak{g})$ is trivial.*

Definition. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is called *trivial* if there exists a commutative ideal $\mathfrak{a}$ in the Lie algebra $\bar{\mathfrak{g}}$ such that $\mathfrak{g} \oplus \mathfrak{a} = \bar{\mathfrak{g}}$.

Suppose $(\bar{\mathfrak{g}}, \mathfrak{g})$ is a trivial pair. This obviously implies that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is also trivial, but not conversely. A trivial pair is uniquely defined (up to equivalence) by the corresponding generalized module $(\mathfrak{g}, \bar{\mathfrak{g}}/\mathfrak{g})$

Proposition 13. *Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an isotropically-faithful pair and $(\mathfrak{g}, U)$ the corresponding generalized module ($U = \bar{\mathfrak{g}}/\mathfrak{g}$). Suppose that there exists $x \in \mathfrak{g}$ such that $x_U = id_U + \varphi$, where φ is a nilpotent endomorphism; then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.*

Proof. By $\mathfrak{h}$ denote the nilpotent subalgebra of $\mathfrak{g}$ spanned by x. We identify functions from $\mathfrak{h}^*$ with their values on the element x. It is clear that $U = U^1(\mathfrak{h})$. Since φ is nilpotent, we see that the endomorphism $\mathrm{ad}_{\mathfrak{gl}(U)} \varphi$ is also nilpotent. Further, since the $\mathfrak{g}$-module U is faithful and $\mathrm{ad}_{\mathfrak{gl}(U)} \varphi = \mathrm{ad}_{\mathfrak{gl}(U)} x_U$, we see that the endomorphism $\mathrm{ad}_{\mathfrak{g}}$ is also nilpotent and $\mathfrak{g} = \mathfrak{g}^0(\mathfrak{h})$. From Proposition 8(2) it follows that $\dim \bar{\mathfrak{g}}^1(\mathfrak{h}) \geqslant \dim U^1(\mathfrak{h})$, where $\bar{\mathfrak{g}}^0(\mathfrak{h}) = \mathfrak{g}$. From the embedding

$$[\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}), \bar{\mathfrak{g}}^{\beta}(\mathfrak{h})] \subset \bar{\mathfrak{g}}^{\beta+\alpha}(\mathfrak{h}) \quad \alpha, \beta \in \mathfrak{h}^*$$

it follows that $\bar{\mathfrak{g}}^1(\mathfrak{h})$ is a commutative ideal in $\bar{\mathfrak{g}}$. This proves the Proposition.

5. Real and complex virtual pairs

In this section we put $k = \mathbb{R}$. For any real vector space V, by $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of V. In a similar manner we define complexifications of other algebraic structures such as Lie algebras, modules, and so on.

We identify the vector space V with the subset $V \otimes_{\mathbb{R}} \mathbb{R}$ of $V^{\mathbb{C}}$. Then $V^{\mathbb{C}} = V \dot{+} (iV)$ and $V \cap (iV) = \{0\}$.

Suppose $\mathfrak{h}$ is a nilpotent real Lie algebra and V is a finite-dimensional real $\mathfrak{h}$-module. Then the vector space $V^{\mathbb{C}}$ is a $\mathfrak{h}^{\mathbb{C}}$-module. Since the field $\mathbb{C}$ is algebraically closed, we see that the $\mathfrak{h}^{\mathbb{C}}$-module $V^{\mathbb{C}}$ is a direct sum of primary components. A linear function $\alpha \in (\mathfrak{h}^{\mathbb{C}})^*$ such that $(V^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) \neq \{0\}$ is called a *weight* of the $\mathfrak{h}^{\mathbb{C}}$-module $V^{\mathbb{C}}$. Denote by $\Delta^{\mathbb{C}}$ the set of all weights of $V^{\mathbb{C}}$. Then

$$V^{\mathbb{C}} = \bigoplus_{\alpha \in \Delta^{\mathbb{C}}} (V^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}).$$

For $\lambda \in \Delta^{\mathbb{C}}$, by $V^{\lambda}(\mathfrak{h})$ denote the subspace of V defined by

$$V^{\lambda}(\mathfrak{h}) = V \cap \left((V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}}) + (V^{\mathbb{C}})^{\bar{\lambda}}(\mathfrak{h}^{\mathbb{C}})\right).$$

Remark. If $\lambda \in \Delta^{\mathbb{C}}$ and $\lambda = \bar{\lambda}$, then $\lambda \in \mathfrak{h}^*$ and

$$(V^{\lambda}(\mathfrak{h}))^{\mathbb{C}} = (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}}).$$

(Bourbaki, N. "Groupes et algèbres de Lie", Chapter VII, §1). So the new definition for $V^{\lambda}(\mathfrak{h})$ is in full agreement with the old one whenever $\lambda = \bar{\lambda}$.

Proposition 14.

(1) $\Delta^{\mathbb{C}} = \bar{\Delta}^{\mathbb{C}}$;

(2) $V^{\lambda}(\mathfrak{h}) = V^{\bar{\lambda}}(\mathfrak{h})$ *and*
$(V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}}) + (V^{\mathbb{C}})^{\bar{\lambda}}(\mathfrak{h}^{\mathbb{C}}) = (V^{\lambda}(\mathfrak{h}))^{\mathbb{C}}$ *for all* $\lambda \in \Delta^{\mathbb{C}}$;

(3) *for any* $\lambda \in \Delta^{\mathbb{C}}$ *the subspace* $V^{\lambda}(\mathfrak{h})$ *is invariant under the action of* $\mathfrak{h}$ *and*

$$V = \sum_{\lambda \in \Delta^{\mathbb{C}}} V^{\lambda}(\mathfrak{h});$$

(4) *if* $\lambda \in \Delta^{\mathbb{C}}$ *and* $\lambda \neq \bar{\lambda}$, *then the submodule* $V^{\lambda}(\mathfrak{h})$ *of the* $\mathfrak{h}$*-module* V *possesses a unique complex structure* J^{λ} *such that*

$$v + iJ^{\lambda}(v) \in (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}}) \quad \text{for all } v \in V^{\lambda}(\mathfrak{h}).$$

Proof. If $x \in \mathfrak{h}^{\mathbb{C}}, \lambda \in (\mathfrak{h}^{\mathbb{C}})^*, n \in \mathbb{N}$, and $v \in V^{\mathbb{C}}$, then

$$\overline{(x_{V^{\mathbb{C}}} - \lambda(x))^n(v)} = (\bar{x}_{V^{\mathbb{C}}} - \bar{\lambda}(\bar{x}))^n(\bar{v}).$$

This proves statements (1) and (2). Then statement (3) is obvious.

Suppose $\lambda \in \Delta^{\mathbb{C}}$ and $\lambda \neq \bar{\lambda}$. Then

$$V \cap (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}}) = (iV) \cap (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}}) = \{0\}.$$

Indeed, if $v \in V \cap (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}})$, then $v = \bar{v} \in (V^{\mathbb{C}})^{\bar{\lambda}}(\mathfrak{h}^{\mathbb{C}})$ and $v \in (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}}) \cap (V^{\mathbb{C}})^{\bar{\lambda}}(\mathfrak{h}^{\mathbb{C}}) = \{0\}$. In a similar way we see that the second equality is true.

By V^{λ} denote a subset of V such that for any $v \in V^{\lambda}$ there exists $u \in V$ such that $v + iu \in (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}})$. From condition (12) it follows that for any $v \in V^{\lambda}$ this element u is unique. Put $J^{\lambda}(v) = u$

Let us show that V^{λ} is a subspace of the vector space V and the mapping $J^{\lambda}(v)$ is linear. Indeed, if $v_1, v_2 \in V^{\lambda}$, then $(v_1 + v_2) + i(J^{\lambda}(v_1) + J^{\lambda}(v_2)) = (v_1 + iJ^{\lambda}(v_1)) + (v_2 + iJ^{\lambda}(v_2)) \in (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}})$. Therefore $v_1 + v_2 \in V^{\lambda}$ and $J^{\lambda}(v_1 + v_2) = J^{\lambda}(v_1) + J^{\lambda}(v_2)$. Similarly, if $v \in V^{\lambda}, a \in \mathbb{R}$, then $av \in V^{\lambda}$ and $J^{\lambda}(av) = aJ^{\lambda}(v)$.

Let us show that $J^{\lambda}(V^{\lambda}) \subset V^{\lambda}$. Indeed, if $v \in V^{\lambda}$, then $-i(v + iJ^{\lambda}(v)) = J^{\lambda}(v) - iv \in (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}})$ and $J^{\lambda}(v) \in V^{\lambda}$. In addition $J^{\lambda}(J^{\lambda}(v)) = -v$.

Let us show that $J^{\lambda}(x.v) = x.J^{\lambda}(v)$ for all $x \in \mathfrak{h}, v \in V^{\lambda}$. Indeed $x.(v + iJ^{\lambda}(v)) = x.v + ix.J^{\lambda}(v) \in (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}})$. Therefore $J^{\lambda}(x.v) = x.J^{\lambda}(v)$.

Let us now show that $V^{\bar{\lambda}} = V^{\lambda}$. Indeed, if $v \in V^{\lambda}$, then $v - iJ^{\lambda}(v) = \overline{v + iJ^{\lambda}(v)} \in (V^{\mathbb{C}})^{\bar{\lambda}}(\mathfrak{h}^{\mathbb{C}})$. Therefore $V^{\lambda} \subset V^{\bar{\lambda}}$. Similarly $V^{\bar{\lambda}} \subset V^{\lambda}$. This implies $V^{\lambda} = V^{\bar{\lambda}}$ and $J^{\lambda} = -J^{\bar{\lambda}}$.

It remains to note that

$$(V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}}) + (V^{\mathbb{C}})^{\bar{\lambda}}(\mathfrak{h}^{\mathbb{C}}) = (V^{\lambda})^{\mathbb{C}}$$

and, therefore, $V^{\lambda}(\mathfrak{h}) = V^{\lambda}$.

Definition. Suppose Δ is a subset of $\Delta^{\mathbb{C}}$ such that for every $\lambda \in \Delta^{\mathbb{C}}$ the set $\Delta \cap \{\lambda, \bar{\lambda}\}$ contains exactly one element. Then

$$V = \bigoplus_{\lambda \in \Delta} V^{\lambda}(\mathfrak{h})$$

and the decomposition itself is called a *generalized primary decomposition.*

Remark. Generally speaking, the set Δ is not defined uniquely. Nevertheless the terms of the generalized primary decomposition are independent of Δ.

Proposition 15. *Suppose $(\mathfrak{g}, U)$ is a real generalized module, $\mathfrak{h}$ is a nilpotent subalgebra of the Lie algebra $\mathfrak{g}$, and q is a virtual structure on $(\mathfrak{g}, U)$. Then there exists a virtual structure $\bar{q}$ equivalent to q such that $\bar{q}^{\mathbb{C}}$ is a primary (with respect to $\mathfrak{h}^{\mathbb{C}}$) virtual structure on the complex generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$.*

Proof. Let $(V, \mathfrak{g})$ be the real virtual pair defined by q. Suppose $\Delta_V^{\mathbb{C}}, \Delta_{\mathfrak{g}}^{\mathbb{C}}$, and $\Delta_U^{\mathbb{C}}$ are the sets of all weights of the $\mathfrak{h}$-modules $V^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}$, and $U^{\mathbb{C}}$ respectively. Then $\Delta_V^{\mathbb{C}} = \Delta_{\mathfrak{g}}^{\mathbb{C}} \cup \Delta_U^{\mathbb{C}}$. We have

$$\mathfrak{g}^{\lambda}(\mathfrak{h}) = \mathfrak{g} \cap V^{\lambda}(\mathfrak{h})$$

and

$$U^{\lambda}(\mathfrak{h}) = \pi(V^{\lambda}(\mathfrak{h}))$$

for all $\lambda \in \Delta_V^{\mathbb{C}}$, where $\pi : V \to U$ is the canonical surjection. Let Δ_V be a subset of $\Delta_V^{\mathbb{C}}$ such that

$$V = \bigoplus_{\lambda \in \Delta_V} V^{\lambda}(\mathfrak{h})$$

is the generalized primary decomposition of the $\mathfrak{h}$-module V. Put $\Delta_{\mathfrak{g}} = \Delta_V \cap \Delta_{\mathfrak{g}}^{\mathbb{C}}$ and $\Delta_U = \Delta_V \cap \Delta_U^{\mathbb{C}}$. Then

$$\mathfrak{g} = \bigoplus_{\lambda \in \Delta_{\mathfrak{g}}} \mathfrak{g}^{\lambda}(\mathfrak{h})$$

and

$$U = \bigoplus_{\lambda \in \Delta_U} U^{\lambda}(\mathfrak{h}).$$

For $\lambda \in \Delta_U$, let $\pi^{\lambda} : V^{\lambda}(\mathfrak{h}) \to U^{\lambda}(\mathfrak{h})$ be the restriction of the mapping π to $V^{\lambda}(\mathfrak{h})$. If $\bar{\lambda} = \lambda$, then by s^{λ} denote an arbitrary section of the surjection π^{λ}.

Suppose $\bar{\lambda} \neq \lambda$ and $J_V^{\lambda}, J_U^{\lambda}$ are the complex structures on $V^{\lambda}(\mathfrak{h})$ and $U^{\lambda}(\mathfrak{h})$, respectively, determined in Proposition 15(4). By $V^{\lambda}(\mathfrak{h})(J_V^{\lambda})$ and $U^{\lambda}(\mathfrak{h})(J_U^{\lambda})$ denote the corresponding complex vector spaces. Then

$$\pi^{\lambda} : V^{\lambda}(\mathfrak{h})(J_V^{\lambda}) \to U^{\lambda}(\mathfrak{h})(J_U^{\lambda})$$

is a surjection of complex spaces. By s^{λ} denote an arbitrary $\mathbb{C}$-linear section of the surjection π^{λ}.

Consider the mapping $s : U \to V$ defined by

$$s\Big(\sum_{\alpha \in \Delta_U} u^{\lambda}\Big) = \sum_{\alpha \in \Delta_U} s^{\lambda}(u^{\lambda})$$

for $u^{\lambda} \in U^{\lambda}(\mathfrak{h}), \lambda \in \Delta_U$. The mapping s is a section of π. Moreover, if $\lambda \in \Delta_U^{\mathbb{C}}$ and $\lambda \neq \bar{\lambda}$, then

$$J_V^{\lambda}(s(u)) = s(J_V^{\lambda}(u)) \quad \text{for all } u \in U^{\lambda}(\mathfrak{h}).$$

Suppose q_s is the corresponding virtual structure on $(\mathfrak{g}, U)$. It is obvious that $q_s^{\mathbb{C}}$ is a virtual structure on the complex generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$ and $q_s^{\mathbb{C}} = q_{s^{\mathbb{C}}}$. Let us show that the section $s^{\mathbb{C}}$ is consistent with $\mathfrak{h}^{\mathbb{C}}$. Indeed, if $\bar{u} \in (U^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}})$ for $\lambda \in \Delta_U^{\mathbb{C}}$, then there exists $u \in U^{\lambda}(\mathfrak{h})$ such that $\bar{u} = u + iJ_U^{\lambda}(u)$ and $s^{\mathbb{C}}(\bar{u}) = s(u) + iJ_V^{\lambda}(s(u))$. Since $s(u) \in V^{\lambda}(\mathfrak{h})$, we obtain $s^{\mathbb{C}}(\bar{u}) \in (V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}})$.

From Proposition 9 it follows that $q_s^{\mathbb{C}}$ is a primary virtual structure. Since the virtual pairs $(V_q, \mathfrak{g})$ and $(V_{q_s}, \mathfrak{g})$ are isomorphic, we see that q and q_s are equivalent. The proof of the Proposition is complete.

CHAPTER III

THE CLASSIFICATION OF PAIRS

Preliminaries

1. Let $\mathfrak{g}$ be one of the subalgebras of the Lie algebra $\mathfrak{gl}(3,\mathbb{R})$ determined in Theorem 1. We assume that the Lie algebra $\mathfrak{g}$ acts naturally on $\mathbb{R}^3$; then $(\mathfrak{g},\mathbb{R}^3)$ is a faithful generalized module. The enumeration of the generalized modules obtained in this way coincides with that of the corresponding subalgebras of $\mathfrak{gl}(3,\mathbb{R})$ in Theorem 1.

We say that a pair $(\bar{\mathfrak{g}},\mathfrak{g})$ (a virtual pair $(V,\mathfrak{g})$) has type $(n.m)$, if the corresponding generalized module $(\mathfrak{g},\bar{\mathfrak{g}}/\mathfrak{g})$ (respectively, $(\mathfrak{g},V/\mathfrak{g})$) is isomorphic to the generalized module $n.m$, i.e., to the generalized module $(\mathfrak{g},\mathbb{R}^3)$, where $\mathfrak{g}$ is the subalgebra of $\mathfrak{gl}(3,\mathbb{R})$ supplied with the number $n.m$ in Theorem 1.

2. Let $(V,\mathfrak{g})$ be a virtual pair of type $n.m$. Then without loss of generality we can identify the Lie algebra $\mathfrak{g}$ with the subalgebra $n.m$ of the Lie algebra $\mathfrak{gl}(3,\mathbb{R})$.

We suppose that $U=\mathbb{R}^3$. Let $\{u_1,u_2,u_3\}$ be the standard basis of U:

$$u_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad u_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad u_3=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

3. Allowing a certain freedom of notation, we define a pair $(\bar{\mathfrak{g}},\mathfrak{g})$ by the commutation table of the Lie algebra $\bar{\mathfrak{g}}$ only. Here by $\{e_1,\dots,e_n,u_1,u_2,u_3\}$ we denote a basis of $\bar{\mathfrak{g}}$ ($n=\dim\mathfrak{g}$). We assume that the Lie algebra $\mathfrak{g}$ is generated by $e_1,\dots,e_n$ unless the contrary is stated.

In addition, λ and μ are the parameters of the corresponding generalized module. If there are some complementary conditions on λ and μ (comparing with Theorem 1), they are indicated before the table. By α,β,γ, etc. we denote the parameters appearing in the process of the classification. If there are some complementary conditions on them, it is indicated just after the table. Otherwise we assume that these parameters run through $\mathbb{R}$.

4. We make use of the following notation:

$\mathcal{D}^n\mathfrak{g}$ are the elements of the derived series of a Lie algebra $\mathfrak{g}$;

$\mathcal{C}^n\mathfrak{g}$ are the elements of the lower central series of $\mathfrak{g}$;

$\mathfrak{r}(\mathfrak{g})$ is the radical of $\mathfrak{g}$;

$\mathcal{Z}\mathfrak{g}$ is the center of $\mathfrak{g}$;

$\operatorname{ad}_{\mathfrak{a}} x$, where x is an element of $\mathfrak{g}$, and $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, denotes the restriction of the endomorphism $\operatorname{ad} x$ to $\mathfrak{a}$.

5. In the trivial case $\mathfrak{g}=\{0\}$ the classification of isotropically-faithful pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the classification (up to isomorphism) of all three-dimensional Lie algebras $\bar{\mathfrak{g}}$. It can be found, for example, in [J].

Proposition 0.1. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 0.1 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	u_1	u_2	u_3
u_1	0	0	0
u_2	0	0	0
u_3	0	0	0

2.

$[\,,]$	u_1	u_2	u_3
u_1	0	0	0
u_2	0	0	u_1
u_3	0	$-u_1$	0

3.

$[\,,]$	u_1	u_2	u_3
u_1	0	0	u_1
u_2	0	0	αu_2
u_3	$-u_1$	$-\alpha u_2$	0

, $|\alpha| \leqslant 1,$

4.

$[\,,]$	u_1	u_2	u_3
u_1	0	0	$\alpha u_1 + u_2$
u_2	0	0	$-u_1 + \alpha u_2$
u_3	$-\alpha u_1 - u_2$	$u_1 - \alpha u_2$	0

, $\alpha \geqslant 0,$

5.

$[\,,]$	u_1	u_2	u_3
u_1	0	0	u_1
u_2	0	0	$u_1 + u_2$
u_3	$-u_1$	$-u_1 - u_2$	0

6.

$[\,,]$	u_1	u_2	u_3
u_1	0	$2u_2$	$-2u_3$
u_2	$-2u_2$	0	u_1
u_3	$2u_3$	$-u_1$	0

7.

$[\,,]$	u_1	u_2	u_3
u_1	0	u_3	$-u_2$
u_2	$-u_3$	0	u_1
u_3	u_2	$-u_1$	0

1. One-dimensional case

Proposition 1.1. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.1 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	0
u_1	$-u_1$	0	0	0
u_2	$-\lambda u_2$	0	0	0
u_3	0	0	0	0

2.

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	0
u_1	$-u_1$	0	0	0
u_2	$-\lambda u_2$	0	0	u_2
u_3	0	0	$-u_2$	0

3. $\lambda = -1$

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	$-u_2$	0
u_1	$-u_1$	0	e_1	0
u_2	u_2	$-e_1$	0	0
u_3	0	0	0	0

4. $\lambda = -1$

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	$-u_2$	0
u_1	$-u_1$	0	u_3	0
u_2	u_2	$-u_3$	0	0
u_3	0	0	0	0

5. $\lambda = -1$

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	$-u_2$	0
u_1	$-u_1$	0	$e_1 + u_3$	0
u_2	u_2	$-e_1 - u_3$	0	0
u_3	0	0	0	0

6. $\lambda = 1$

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	u_2	0
u_1	$-u_1$	0	0	0
u_2	$-u_2$	0	0	u_1
u_3	0	0	$-u_1$	0

7. $\lambda = 1$

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	u_2	0
u_1	$-u_1$	0	0	$-u_2$
u_2	$-u_2$	0	0	u_1
u_3	0	u_2	$-u_1$	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 1.1 is isomorphic to one of the following:*

a) $\lambda \neq 0$

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & p_3 \end{pmatrix}, \quad p_3 \in \mathbb{R};$$

b) $\lambda = 0$

$$C_2(e_1) = \begin{pmatrix} 0 & p_2 & p_3 \end{pmatrix}, \quad p_2, p_3 \in \mathbb{R}.$$

Proof. Any virtual structure q has the form

$$C(e_1) = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}.$$

Suppose $\lambda \neq 0$. Put

$$H = \begin{pmatrix} c_1 & c_2/\lambda & 0 \end{pmatrix},$$

and $C_1(x) = C(x) - A(x)H + HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & c_3 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtuial structures C and C_1 are equivalent.

Now suppose $\lambda = 0$. Similarly, putting

$$H = \begin{pmatrix} c_1 & 0 & 0 \end{pmatrix},$$

and $C_2(x) = C(x) - A(x)H + HB(x)$ for $x \in \mathfrak{g}$, we see that

$$C_2(e_2) = \begin{pmatrix} 0 & c_2 & c_3 \end{pmatrix}.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 1.1. Thus it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma.

Consider the following cases:

1°. $\lambda \neq 0$. Then the vectors $[e_1, u_j]$ $1 \leqslant j \leqslant 3$ have the form:

$$\begin{aligned} [e_1, u_1] &= u_1, \\ [e_1, u_2] &= \lambda u_2, \qquad p_3 \in \mathbb{R} \\ [e_1, u_1] &= p_3 e_1. \end{aligned}$$

Put

$$\begin{aligned} [u_1, u_2] &= a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\ [u_1, u_3] &= b_1 e_1 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\ [u_2, u_3] &= c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3. \end{aligned}$$

Let us check the Jacobi identity for the triples (e_1, u_j, u_k), $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_2, [e_1, u_1]] + [u_1, [u_2, e_1]] = 0$

$\alpha_1 u_1 + \lambda \alpha_2 u_2 + p_3 \alpha_3 e_1 - (\lambda + 1)(a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) = 0$

1. $p_3\alpha_3 - (\lambda+1)a_1 = 0$
2. $\alpha_1 = 0$
3. $\alpha_2 = 0$
4. $(\lambda+1)\alpha_3 = 0$

2. $[e_1,[u_1,u_3]] + [u_3,[e_1,u_1]] + [u_1,[u_3,e_1]] = 0$
$\beta_1 u_1 + \lambda\beta_2 u_2 + p_3\beta_3 e_1 + p_3 u_1 - (b_1 e_1 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3) = 0$

5. $b_1 = 0$
6. $p_3 = 0$
7. $(\lambda - 1)\beta_2 = 0$
8. $\beta_3 = 0$

3. $[e_1,[u_2,u_3]] + [u_3,[e_1,u_2]] + [u_2,[u_3,e_1]] = 0$
$\gamma_1 u_1 + \lambda\gamma_2 u_2 - (c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3) = 0$

9. $c_1 = 0$
10. $(\lambda - 1)\gamma_1 = 0$
11. $\gamma_3 = 0$

4. $[u_1,[u_2,u_3]] + [u_3,[u_1,u_2]] + [u_2,[u_3,u_1]] = 0$
$(\gamma_2 + \beta_1)(a_1 e_1 + \alpha_3 u_3) = 0$

12. $(\beta_1 + \gamma_2)a_1 = 0$
13. $(\beta_1 + \gamma_2)\alpha_3 = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has one of the following forms

1.1°. $\lambda \neq \pm 1$

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	0
u_1	$-u_1$	0	0	$\beta_1 u_1$
u_2	$-\lambda u_2$	0	0	$\gamma_2 u_2$
u_3	0	$-\beta_1 u_1$	$-\gamma_2 u_2$	0 .

1.2°. $\lambda = -1$

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	$-u_2$	0
u_1	$-u_1$	0	$a_1 e_1 + \alpha_3 u_3$	$\beta_1 u_1$
u_2	u_2	$-a_1 e_1 - \alpha_3 u_3$	0	$\gamma_2 u_2$
u_3	0	$-\beta_1 u_1$	$-\gamma_2 u_2$	0 ,

where $a_1(\beta_1 + \gamma_2) = \alpha_3(\beta_1 + \gamma_2) = 0$.

1.3°. $\lambda = 1$

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	u_2	0
u_1	$-u_1$	0	0	$\beta_1 u_1 + \beta_2 u_2$
u_2	$-u_2$	0	0	$\gamma_1 u_1 + \gamma_2 u_2$
u_3	0	$-\beta_1 u_1 - \beta_2 u_2$	$-\gamma_1 u_1 - \gamma_2 u_2$	0 .

Consider the corresponding cases.

1.1°. $\lambda \neq \pm 1$. If $\gamma_2 - \lambda\beta_1 = 0$, then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi: \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= u_3 - \beta_1 e_1. \end{aligned}$$

If $\gamma_2 - \lambda\beta_1 \neq 0$, then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi: \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= (u_3 - \beta_1 e_1)/(\lambda\beta_1 - \gamma_2). \end{aligned}$$

1.2°. $\lambda = -1$. If $\beta_1 + \gamma_2 \neq 0$, then $a_1 = \alpha_3 = 0$ and as in the case 1.1° we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ or $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ for $\lambda = -1$.

Now suppose $\beta_1 + \gamma_2 = 0$. The mapping $\pi: \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= u_3 - \beta_1 e_1. \end{aligned}$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has the form:

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	$-u_2$	0
u_1	$-u_1$	0	$a_1 e_1 + \alpha_3 u_3$	0
u_2	u_2	$-a_1 e_1 - \alpha_3 u_3$	0	0
u_3	0	0	0	0.

If $a_1 \neq 0$ and $\alpha_3 \neq 0$, then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi: \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= a_1^{-1} u_2, \\ \pi(u_3) &= \alpha_3 a_1^{-1} u_3. \end{aligned}$$

Smilarly, the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to one of the pairs $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$, $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ or $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)_{\lambda=-1}$ when $(a_1 \neq 0, \alpha_3 = 0)$, $(a_1 = 0, \alpha_3 \neq 0)$, or $(a_1 = \alpha_3 = 0)$, respectively.

1.3°. $\lambda = 1$. In this case each pair is determined by a matrix of the form $A = \begin{pmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}$. It easy to show that if two matrises A and A' of this form are conjugate then, the corresponding pairs are equivalent. So, we can assume that A has one of the followig forms:

$$i)\ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\quad a, b \in \mathbb{R}; \qquad ii)\ \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix},\quad a \in \mathbb{R}; \qquad iii)\ \begin{pmatrix} a & -b \\ b & a \end{pmatrix},\quad a, b \in \mathbb{R}.$$

In case *i*) as in 1.1° we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to either $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ or $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ for $\lambda = 1$.

In case *ii*) the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi: \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= u_3 - ae_1. \end{aligned}$$

In case *iii*) it can be assumed that $b \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi: \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= b^{-1}(u_3 - ae_1). \end{aligned}$$

2°. $\lambda = 0$. In this case after checking the Jacoby identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the following form:

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	0	0
u_1	$-u_1$	0	$\alpha_1 u_1$	$\beta_1 u_1$
u_2	0	$-\alpha_1 u_1$	0	$c_1 e_1 + \gamma_2 u_2 + \gamma_3 u_3$
u_3	0	$-\beta_1 u_1$	$-c_1 e_1 - \gamma_2 u_2 - \gamma_3 u_3$	0

,

where $c_1 = \alpha_1 \gamma_2 + \beta_1 \gamma_3$. The mapping $\pi: \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2 + \alpha_1 e_1, \\ \pi(u_3) &= u_3 + \beta_1 e_1, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$ where the latter has the form:

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	0	0
u_1	$-u_1$	0	0	0
u_2	0	0	0	$\gamma_2 u_2 + \gamma_3 u_3$
u_3	0	0	$-\gamma_2 u_2 - \gamma_3 u_3$	0

.

It is easy to show that the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to either $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ or $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ for $\lambda = 0$.

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Since $\mathcal{Z}(\bar{\mathfrak{g}}_1) \neq \{0\}$ and $\mathcal{Z}(\bar{\mathfrak{g}}_2) = \{0\}$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

Note that $\mathcal{D}^2\bar{\mathfrak{g}}_1 = \mathcal{D}^2\bar{\mathfrak{g}}_2 = \{0\}$ but $\mathcal{D}^2\bar{\mathfrak{g}}_i \neq \{0\}$ for $i = 3, 4, 5$. It follows that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ $(\lambda = -1)$ are not equivalent to any of the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 3, 4, 5$. Since

$$\mathcal{D}\bar{\mathfrak{g}}_3 \cap \mathfrak{g}_3 \neq \{0\}, \quad \mathcal{D}\bar{\mathfrak{g}}_3 \cap \mathcal{Z}(\bar{\mathfrak{g}}_3) = \{0\};$$

$$\mathcal{D}\bar{\mathfrak{g}}_4 \cap \mathfrak{g}_4 = \{0\}, \quad \mathcal{D}\bar{\mathfrak{g}}_4 \cap \mathcal{Z}(\bar{\mathfrak{g}}_4) \neq \{0\};$$

$$\mathcal{D}\bar{\mathfrak{g}}_5 \cap \mathfrak{g}_5 = \{0\}, \quad \mathcal{D}\bar{\mathfrak{g}}_5 \cap \mathcal{Z}(\bar{\mathfrak{g}}_5) = \{0\},$$

we see that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 3, 4, 5$ are not equivalent to each other.

Let $\lambda = 1$. For the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$, $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$, and $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ consider the homomorphisms $f_i \colon \bar{\mathfrak{g}}_i \to \mathfrak{gl}(2, \mathbb{R})$ $(i = 1, 2, 6, 7)$, where $f_i(x)$ is the matrix of the mapping $\operatorname{ad} x|_{\mathcal{D}\bar{\mathfrak{g}}_i}$ in the basis $\{u_1, u_2\}$, $x \in \bar{\mathfrak{g}}_i$. Then

$$f_1(\bar{\mathfrak{g}}_1) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \middle| \ x \in \mathbb{R} \right\};$$

$$f_2(\bar{\mathfrak{g}}_2) = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| \ x, y \in \mathbb{R} \right\};$$

$$f_6(\bar{\mathfrak{g}}_6) = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \middle| \ x, y \in \mathbb{R} \right\};$$

$$f_7(\bar{\mathfrak{g}}_7) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \middle| \ x \in \mathbb{R} \right\}.$$

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$ $(i = 1, 2, 6, 7)$, are not conjugate, we conclude that the corresponding pairs are not equivalent.

This completes the proof of the Proposition.

Proposition 1.2. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.2 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	μu_3
u_1	$-u_1$	0	0	0
u_2	$-\lambda u_2$	0	0	0
u_3	$-\mu u_3$	0	0	0

2. $\mu = \lambda + 1$, $\lambda < -1$

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	$(\lambda + 1)u_3$
u_1	$-u_1$	0	u_3	0
u_2	$-\lambda u_2$	$-u_3$	0	0
u_3	$-(\lambda + 1)u_3$	0	0	0

3. $\mu = 1 - \lambda,\ 0 < \lambda \leqslant 1/2$

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	$(1-\lambda)u_3$
u_1	$-u_1$	0	0	0
u_2	$-\lambda u_2$	0	0	u_1
u_3	$(\lambda-1)u_3$	0	$-u_1$	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 1.2 is trivial.*

Proof. Any virtual structure q has the form

$$C(e_1) = (\, c_1 \quad c_2 \quad c_3 \,).$$

Put

$$H = (\, c_1 \quad c_2/\lambda \quad c_3/\mu \,).$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_1) = (\, 0 \quad 0 \quad 0 \,).$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Then

$$[e_1, u_1] = u_1,$$
$$[e_1, u_2] = \lambda u_2,$$
$$[e_1, u_3] = \mu u_3.$$

Put

$$[u_1, u_2] = a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3,$$
$$[u_1, u_3] = b_1 e_1 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3,$$
$$[u_2, u_3] = c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.$$

Let us check the Jacobi identity for the triples (e_1, u_i, u_j), $1 \leqslant i < j \leqslant 3$ and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_1, [u_2, e_1]] + [u_2, [e_1, u_1]] = 0$

$\alpha_1 u_1 + \lambda \alpha_2 u_2 + \mu \alpha_3 u_3 - (\lambda + 1)(a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) = 0$

1. $(\lambda + 1)a_1 = 0$
2. $\alpha_1 = 0$
3. $\alpha_2 = 0$
4. $(\mu - \lambda - 1)\alpha_3 = 0$

2. $[e_1,[u_1,u_3]]+[u_1,[u_3,e_1]]+[u_3,[e_1,u_1]]=0$
$\beta_1u_1+\lambda\beta_2u_2+\mu\beta_3u_3-(\mu+1)(b_1e_1+\beta_1u_1+\beta_2u_2+\beta_3u_3)=0$
5. $(\mu+1)b_1=0$
6. $\beta_1=0$
7. $\beta_2=0$
8. $\beta_3=0$

3. $[e_1,[u_2,u_3]]+[u_2,[u_3,e_1]]+[u_3,[e_1,u_2]]=0$
$\gamma_1u_1+\lambda\gamma_2u_2+\mu\gamma_3u_3-(\mu+\lambda)(c_1e_1+\gamma_1u_1+\gamma_2u_2+\gamma_3u_3)=0$
9. $c_1=0$
10. $(\mu+\lambda-1)\gamma_1=0$
11. $\gamma_2=0$
12. $\gamma_3=0$

4. $[u_1,[u_2,u_3]]+[u_2,[u_3,u_1]]+[u_3,[u_1,u_2]]=0$
$\lambda b_1u_2-\mu a_1u_3=0$
13. $b_1=0$
14. $a_1=0$

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has one of the following forms:
1°. $\mu+\lambda\neq1$ and $\mu-\lambda\neq1$

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	μu_3
u_1	$-u_1$	0	0	0
u_2	$-\lambda u_2$	0	0	0
u_3	$-\mu u_3$	0	0	0

2°. $\mu-\lambda=1$

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	$(\lambda+1)u_3$
u_1	$-u_1$	0	α_3u_3	0
u_2	$-\lambda u_2$	$-\alpha_3u_3$	0	0
u_3	$-(\lambda+1)u_3$	0	0	0

3°. $\mu+\lambda=1$

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	$(1-\lambda)u_3$
u_1	$-u_1$	0	0	0
u_2	$-\lambda u_2$	0	0	γ_1u_1
u_3	$(\lambda-1)u_3$	0	$-\gamma_1u_1$	0

Consider the following cases:

1°. $\mu+\lambda\neq1$ and $\mu-\lambda\neq1$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$.

2°. $\mu-\lambda=1$. 2.1°. $\alpha_3=0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$.

2.2°. $\alpha_3 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= \alpha_3 u_2, \\ \pi(u_3) &= u_3. \end{aligned}$$

3°. $\mu + \lambda = 1$.

3.1°. $\gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

3.2°. $\gamma_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= \gamma_1 u_2, \\ \pi(u_3) &= u_3. \end{aligned}$$

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 = 0$, $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_2 = 1$, and $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_3 = 1$, we see that the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ is not equivalent to either of the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$.

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{R})$ $(i = 2, 3)$, where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}|_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in the basis $\{u_1, u_2, u_3\}$, $x \in \bar{\mathfrak{g}}_i$. Since the subalgebras $f_2(\bar{\mathfrak{g}}_2)$ and $f_3(\bar{\mathfrak{g}}_3)$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent. This completes the proof of the Proposition.

Proposition 1.3. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.3 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	u_1	u_2	u_3
e_1	0	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
u_1	$u_2 - \lambda u_1$	0	0	0
u_2	$-u_1 - \lambda u_2$	0	0	0
u_3	0	0	0	0

2.

[,]	e_1	u_1	u_2	u_3
e_1	0	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
u_1	$u_2 - \lambda u_1$	0	0	u_1
u_2	$-u_1 - \lambda u_2$	0	0	u_2
u_3	0	$-u_1$	$-u_2$	0

3. $\lambda = 0$

[,]	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	$e_1 + u_3$	0
u_2	$-u_1$	$-e_1 - u_3$	0	0
u_3	0	0	0	0

4. $\lambda = 0$

[,]	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	$-e_1 + u_3$	0
u_2	$-u_1$	$e_1 - u_3$	0	0
u_3	0	0	0	0

5. $\lambda = 0$

[,]	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	e_1	0
u_2	$-u_1$	$-e_1$	0	0
u_3	0	0	0	0

6. $\lambda = 0$

[,]	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	$-e_1$	0
u_2	$-u_1$	e_1	0	0
u_3	0	0	0	0

7. $\lambda = 0$

[,]	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	u_3	0
u_2	$-u_1$	$-u_3$	0	0
u_3	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A(e_1) = (0)$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 1.3 is isomorphic to one of the following:*

$$C(e_1) = \begin{pmatrix} 0 & 0 & p \end{pmatrix}, \; p \in \mathbb{R}.$$

Proof. Any virtual structure q has the form

$$C(e_1) = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} h_1 & h_2 & 0 \end{pmatrix},$$

where the set of coefficients h_1 and h_2 is a solution of the following system:

$$\begin{cases} -\lambda h_1 + h_2 = c_1 \\ -h_1 - \lambda h_2 = c_2 \end{cases}$$

Note that the solution exists, since the matrix of the system is nondegenerate. Now put $C_1(x) = C(x) - A(x)H + HB(x)$ for $x \in \mathfrak{g}$. Then

$$C(e_1) = \begin{pmatrix} 0 & 0 & c_3 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtuial structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 1.3. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma.

Then the vectors $[e_1, u_j]$ $1 \leqslant j \leqslant 3$ have the form:

$$\begin{aligned} [e_1, u_1] &= \lambda u_1 - u_2, \\ [e_1, u_2] &= u_1 + \lambda u_2, \qquad p \in \mathbb{R} \\ [e_1, u_1] &= p e_1. \end{aligned}$$

Put

$$\begin{aligned} [u_1, u_2] &= a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\ [u_1, u_3] &= b_1 e_1 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\ [u_2, u_3] &= c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3. \end{aligned}$$

Let us check the Jacobi identity for the triples (e_1, u_j, u_k), $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_2, [e_1, u_1]] + [u_1, [u_2, e_1]] = 0$
1. $\alpha_3 p - 2\lambda a_1 = 0$
2. $\alpha_2 = \lambda \alpha_1$
3. $\alpha_1 = -\lambda \alpha_2$
4. $\lambda \alpha_3 = 0$

2. $[e_1, [u_1, u_3]] + [u_3, [e_1, u_1]] + [u_1, [u_3, e_1]] = 0$
5. $\beta_3 p - \lambda b_1 + c_1 = 0$
6. $\beta_2 + \gamma_1 + p\lambda = 0$
7. $\gamma_2 - \beta_1 - p = 0$
8. $\gamma_3 = \lambda \beta_3$

3. $[e_1, [u_2, u_3]] + [u_3, [e_1, u_2]] + [u_2, [u_3, e_1]] = 0$
9. $\gamma_3 p - b_1 - \lambda c_1 = 0$
10. $\gamma_2 - \beta_1 + p = 0$
11. $\lambda a_1 = 0$
12. $\gamma_1 + \beta_2 - p\lambda = 0$

4. $[u_1, [u_2, u_3]] + [u_3, [u_1, u_2]] + [u_2, [u_3, u_1]] = 0$
13. $\beta_1 a_1 = 0$
14. $\beta_1 \alpha_3 = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

$[,]$	e_1	u_1	u_2	u_3
e_1	0	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
u_1	$u_2 - \lambda u_1$	0	$a_1 e_1 + \alpha_3 u_3$	$\beta_1 u_1 + \beta_2 u_2$
u_2	$-\lambda u_2 - u_1$	$-a_1 e_1 - \alpha_3 u_3$	0	$-\beta_2 u_1 + \beta_1 u_2$
u_3	0	$-\beta_1 u_1 - \beta_2 u_2$	$-\beta_1 u_2 + \beta_2 u_1$	0

where the set of coefficients satisfies the following system of linear equations:

$$\begin{cases} \lambda a_1 = 0, \\ \lambda \alpha_3 = 0, \\ \beta_1 a_1 = 0, \\ \beta_1 \alpha_3 = 0. \end{cases}$$

The isomorophism $\pi : \mathfrak{g}' \to \mathfrak{g}$ such that

$$\pi(e_1) = e_1, \pi(u_1) = u_1, \pi(u_2) = u_2, \pi(u_3) = u_3 + \beta_2 e_1$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has the form:

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
u_1	$u_2 - \lambda u_1$	0	$a_1 e_1 + \alpha_3 u_3$	$\beta_1' u_1$
u_2	$-\lambda u_2 - u_1$	$-a_1 e_1 - \alpha_3 u_3$	0	$\beta_1' u_2$
u_3	0	$-\beta_1' u_1$	$-\beta_1' u_2$	0 ,

where $\beta_1' = \beta_1 + \lambda \beta_2$.
Consider the following cases:

1°. $\lambda = 0$.

$$[u_1, u_2] = a_1 e_1 + \alpha_3 u_3,$$
$$[u_1, u_3] = \beta_1 u_1,$$
$$[u_2, u_3] = \beta_1 u_2.$$

1.1°. $a_1 \alpha_3 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ or the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{3(4)} \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = \sqrt{|a_1|}u_1,$$
$$\pi(u_2) = \sqrt{|a_1|}u_2,$$
$$\pi(u_3) = \frac{|a_1|}{\alpha_3}u_3.$$

The pairs $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ are not equivalent, since a Levi subalgebra of $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ is isomorphic to $\mathfrak{su}(2)$, and a Levi subalgebra of $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

1.2°. $\alpha_3 = 0, \ a_1 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ or $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{5(6)} \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = \sqrt{|a_1|}u_1,$$
$$\pi(u_2) = \sqrt{|a_1|}u_2,$$
$$\pi(u_3) = u_3.$$

The pairs $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ and $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ are not equivalent, since a Levi subalgebra of $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ is isomorphic to $\mathfrak{su}(2)$, and a Levi subalgebra of $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. These pairs are not equivalent to the pairs $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$, since $\dim \mathcal{D}\bar{\mathfrak{g}}_{5,6} \cap \mathfrak{g} \neq 0$ and $\dim \mathcal{D}\bar{\mathfrak{g}}_{3,4} \cap \mathfrak{g} = 0$.

1.3°. $\alpha_3 \neq 0$, $a_1 = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \alpha_3^{-1} u_3.$$

The pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ is not equivalent to any of the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, ..., 6$, since its nilpotent radical has a nontrivial submodule.

1.4°. $\alpha_3 = a_1 = 0$, $\beta_1 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ $\lambda = 0$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \beta_1 u_3.$$

1.5°. $\alpha_3 = a_1 = \beta_1 = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

2°. $\lambda \neq 0$.

2.1°. $\beta_1 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \beta_1 u_3.$$

2.2°. $\beta_1 = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

The pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent, since the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ has the nontrivial center $\mathcal{Z}(\bar{\mathfrak{g}}_1) = \mathbb{R}(u_3)$.

This completes the proof of the Proposition.

Proposition 1.4. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	u_1	u_2	u_3
e_1	0	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	μu_3
u_1	$-\lambda u_1 + u_2$	0	0	0
u_2	$-\lambda u_2 - u_1$	0	0	0
u_3	$-\mu u_3$	0	0	0

2. $\mu = 2\lambda$

$[,]$	e_1	u_1	u_2	u_3
e_1	0	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	$2\lambda u_3$
u_1	$-\lambda u_1 + u_2$	0	u_3	0
u_2	$-\lambda u_2 - u_1$	$-u_3$	0	0
u_3	$-2\lambda u_3$	0	0	0 , $\lambda > 0.$

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 1.4 is trivial.*

Proof. Any virtual structure q has the form:

$$C(e_1) = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} \frac{c_2+\lambda c_1}{1+\lambda^2} & \frac{-c_1+\lambda c_2}{1+\lambda^2} & \frac{c_3}{\mu} \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Then the vectors $[e_1, u_i], 1 \leqslant i \leqslant 3$, have the form:

$$[e_1, u_1] = \lambda u_1 - u_2,$$
$$[e_1, u_2] = u_1 + \lambda u_2,$$
$$[e_1, u_3] = \mu u_3.$$

Put

$$[u_1, u_2] = a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3,$$
$$[u_1, u_2] = b_1 e_1 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3,$$
$$[u_2, u_3] = c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.$$

Let us check the Jacobi identity for the triples (e_1, u_j, u_k), $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_2, [e_1, u_1]] + [u_1, [u_2, e_1]] = 0$

$\alpha_1(\lambda u_1 - u_2) + \alpha_2(u_1 + \lambda u_2) + \alpha_3 \mu u_3 - 2\lambda(a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) = 0$

1. $\lambda a_1 = 0$
2. $\alpha_1 = \alpha_2 = 0$
3. $(\mu - 2\lambda)\alpha_3 = 0$

2. $[e_1,[u_1,u_3]]+[u_3,[e_1,u_1]]+[u_1,[u_3,e_1]]=0$

$$\beta_1(\lambda u_1-u_2)+\beta_2(u_1+\lambda u_2)+\beta_3\mu u_3-(\lambda+\mu)(b_1e_1+\beta_1u_1+\beta_2u_2+\beta_3u_3)+$$
$$+c_1e_1+\gamma_1u_1+\gamma_2u_2+\gamma_3u_3=0$$

4. $c_1=(\lambda+\mu)b_1$
5. $\beta_2+\gamma_1=\mu\beta_1$
6. $-\beta_1+\gamma_2=\mu\beta_2$
7. $\gamma_3=\lambda\beta_3$

3. $[e_1,[u_2,u_3]]+[u_3,[e_1,u_2]]+[u_2,[u_3,e_1]]=0$

$$\gamma_1(\lambda u_1-u_2)+\gamma_2(u_1+\lambda u_2)+\gamma_3\mu u_3-(\lambda+\mu)(c_1e_1+\gamma_1u_1+\gamma_2u_2+\gamma_3u_3)-$$
$$-b_1e_1-\beta_1u_1-\beta_2u_2+\beta_3u_3=0$$

8. $b_1=-(\lambda+\mu)c_1$
9. $-\beta_1+\gamma_2=\mu\gamma_1$
10. $\beta_2+\gamma_1=-\mu\gamma_2$
11. $\beta_3=-\lambda\beta_3$

4. $[u_1,[u_2,u_3]]+[u_3,[u_1,u_2]]+[u_2,[u_3,u_1]]=0$

$-\mu a_1=0$

12. $a_1=0$

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	λu_1-u_2	$u_1+\lambda u_2$	μu_3
u_1	$-\lambda u_1+u_2$	0	α_3u_3	0
u_2	$-\lambda u_2-u_1$	$-\alpha_3u_3$	0	0
u_3	$-\mu u_3$	0	0	0 ,

where $\alpha_3(\lambda-2\mu)=0$.

Consider the following cases:

1°. $\mu\neq 2\lambda$.

Then $\alpha_3=0$ and the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$.

2°. $\mu=2\lambda$.

2.1°. $\alpha_3\neq 0$. The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi:\bar{\mathfrak{g}}_2\to\mathfrak{g}$, where

$$\pi(e_1)=e_1,$$
$$\pi(u_1)=u_1,$$
$$\pi(u_2)=u_2,$$
$$\pi(u_3)=\alpha_3u_3.$$

2.2°. $\alpha_3=0$. The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$.

Since $\dim\mathcal{D}^2\bar{\mathfrak{g}}_1\neq\dim\mathcal{D}^2\bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ are not equivalent to each other.

This completes the proof of the Proposition.

Proposition 1.5. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.5 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	0
u_2	0	0	0	0
u_3	$-u_1$	0	0	0

2.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	u_1
u_2	0	0	0	0
u_3	$-u_1$	$-u_1$	0	0

3.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	$e_1+\alpha u_1$
u_2	0	0	0	0
u_3	$-u_1$	$-e_1-\alpha u_1$	0	0

4.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	$-e_1+\alpha u_1$
u_2	0	0	0	0
u_3	$-u_1$	$e_1-\alpha u_1$	0	0

5.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	u_1
u_2	0	0	0	e_1
u_3	$-u_1$	$-u_1$	$-e_1$	0

6.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	0
u_2	0	0	0	e_1
u_3	$-u_1$	0	$-e_1$	0

7.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	u_1
u_2	0	0	0	u_1+u_2
u_3	$-u_1$	$-u_1$	$-u_1-u_2$	0

8.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	$\alpha e_1+\beta u_1$
u_2	0	0	0	u_2
u_3	$-u_1$	$-\alpha e_1-\beta u_1$	$-u_2$	0

9.

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	$\alpha e_1+(1-\alpha)u_1$
u_2	0	0	0	e_1+u_2
u_3	$-u_1$	$-\alpha e_1+(\alpha-1)u_1$	$-e_1-u_2$	0

, $\alpha \neq 0$

10.

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	$-u_1$	0
u_2	0	u_1	0	u_3
u_3	$-u_1$	0	$-u_3$	0

11.

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	u_2
u_2	0	0	0	u_2
u_3	$-u_1$	$-u_2$	$-u_2$	0

12.

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	u_2
u_2	0	0	0	0
u_3	$-u_1$	$-u_2$	0	0

13. $\bar{\mathfrak{g}} = \langle f_1, f_2, f_3, f_4\rangle$, $\mathfrak{g} = \langle f_2 + f_3\rangle$

$[\,,]$	f_1	f_2	f_3	f_4
f_1	0	0	0	f_1
f_2	0	0	0	f_1+f_2
f_3	0	0	0	αf_3
f_4	$-f_1$	$-f_1-f_2$	$-\alpha f_3$	0

, $\alpha \neq 1$

14. $\bar{\mathfrak{g}} = \langle f_1, f_2, f_3, f_4\rangle$, $\mathfrak{g} = \langle f_3\rangle$

$[\,,]$	f_1	f_2	f_3	f_4
f_1	0	0	0	f_1
f_2	0	0	0	f_1+f_2
f_3	0	0	0	f_2+f_3
f_4	$-f_1$	$-f_1-f_2$	$-f_2-f_3$	0

15. $\bar{\mathfrak{g}} = \langle f_1, f_2, f_3, f_4\rangle$, $\mathfrak{g} = \langle f_1 + f_3\rangle$

$[\,,]$	f_1	f_2	f_3	f_4
f_1	0	0	0	αf_1-f_2
f_2	0	0	0	$f_1+\alpha f_2$
f_3	0	0	0	βf_3
f_4	$f_2-\alpha f_1$	$-f_1-\alpha f_2$	$-\beta f_3$	0

, $\alpha > 0$

16. $\bar{\mathfrak{g}} = \langle f_1, f_2, f_3, f_4\rangle$, $\mathfrak{g} = \langle f_1 + f_3\rangle$

$[\,,]$	f_1	f_2	f_3	f_4
f_1	0	0	0	$-f_2$
f_2	0	0	0	f_1
f_3	0	0	0	αf_3
f_4	f_2	$-f_1$	$-\alpha f_3$	0

, $\alpha \geqslant 0$

17. $\bar{\mathfrak{g}} = \langle f_1, f_2, f_3, f_4 \rangle$, $\mathfrak{g} = \langle f_1 + f_2 + f_3 \rangle$

[,]	f_1	f_2	f_3	f_4
f_1	0	0	0	f_1
f_2	0	0	0	αf_2
f_3	0	0	0	0
f_4	$-f_1$	$-\alpha f_2$	0	0

$0 <\mid \alpha \mid \leqslant 1, \alpha \neq 0$

18. $\bar{\mathfrak{g}} = \langle f_1, f_2, f_3, f_4 \rangle$, $\mathfrak{g} = \langle f_1 + f_2 + f_3 \rangle$

[,]	f_1	f_2	f_3	f_4
f_1	0	0	0	f_1
f_2	0	0	0	αf_2
f_3	0	0	0	βf_3
f_4	$-f_1$	$-\alpha f_2$	$-\beta f_3$	0

$\alpha\beta > 0, \alpha < \beta < 1$

19.

[,]	e_1	u_1	u_2	u_3
e_1	0	e_1	0	u_1
u_1	$-e_1$	0	0	u_3
u_2	0	0	0	0
u_3	$-u_1$	$-u_3$	0	0

20.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	e_1	u_1
u_1	0	0	u_1	e_1
u_2	$-e_1$	$-u_1$	0	0
u_3	$-u_1$	$-e_1$	0	0

21.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	e_1	u_1
u_1	0	0	u_1	$-e_1$
u_2	$-e_1$	$-u_1$	0	0
u_3	$-u_1$	e_1	0	0

22.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	e_1	u_1
u_1	0	0	αu_1	0
u_2	$-e_1$	$-\alpha u_1$	0	$(1-\alpha)u_3$
u_3	$-u_1$	0	$(\alpha-1)u_3$	0

23.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	e_1	u_1
u_1	0	0	$2u_1$	0
u_2	$-e_1$	$-2u_1$	0	$e_1 - u_3$
u_3	$-u_1$	0	$u_3 - e_1$	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma 1. *Any virtual structure q on generalized module 1.5 is equivalent to one and only one of the following:*

a)
$$C_1(x) = (\,0 \quad 0 \quad 0\,);$$

b)
$$C_2(x) = (\,1 \quad 0 \quad 0\,);$$

c)
$$C_3(x) = (\,0 \quad 1 \quad 0\,).$$

Proof. Any virtual structure q has the form:

$$C(x) = (\,c_1 \quad c_2 \quad c_3\,).$$

By corollary 2, Chapter II, virtual structures C and C' are equivalent if and only if there exist $P \in \mathcal{A}(\mathfrak{g})$ and $H \in \mathrm{Mat}_{1\times 3}(\mathbb{R})$ such that for any $x \in \mathfrak{g}$ the following condition holds:

$$C'(x) = FC(\varphi^{-1}(x))P^{-1} - A(x)H + HB(x),$$

where $\varphi(x) = PxP^{-1}$ and F is the matrix of the mapping φ.

We have $\varphi(e_1) = ae_1$, where $a \in \mathbb{R}^*$. Then $F = a$ and P has the form:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & \frac{1}{a}x_{11} \end{pmatrix},$$

where $x_{11}x_{22} \neq 0$. Put $H = (\,h_1 \quad h_2 \quad h_3\,)$. Then $C'(e_1) = (\,c_1' \quad c_2' \quad c_3'\,)$, where

$$\begin{aligned} c_1' &= x_{11}^{-1}c_1, \\ c_2' &= x_{22}^{-1}c_2 - \frac{x_{12}}{x_{11}x_{22}}c_1, \\ c_3' &= ax_{11}^{-1}c_3 - a\frac{x_{23}}{x_{11}x_{22}}c_2 + \frac{a}{x_{11}^2x_{22}}(x_{12}x_{23} - x_{22}x_{13})c_1 + h_1. \end{aligned} \qquad (*)$$

Consider the following cases:

1°. $c_1 = c_2 = 0$. Putting, $h_1 = -ax_{11}^{-1}c_3$ in (*) we obtain $C'(e_1) = (\,0 \quad 0 \quad 0\,)$. Put $C_1 = C'$.

2°. $c_1 \neq 0$. Putting $x_{11} = c_1$, $x_{22} = 1$, $x_{12} = c_2$, $x_{23} = x_{13} = 0$ and $h_1 = -ac_1^{-1}c_3$, we obtain $C' = (\,1 \quad 0 \quad 0\,)$. Put $C_2 = C'$.

3°. $c_1 = 0$ and $c_2 \neq 0$. Putting $x_{22} = c_2$, $x_{23} = 0$ and $h_1 = -ax_{11}^{-1}c_3$, we obtain $C'(e_1) = (\,0 \quad 1 \quad 0\,)$. Put $C_3 = C'$.

Using $(*)$ it is easy to see that the virtual structures C_1, C_2, and C_3 are not equivalent to each other. The proof of the Lemma is complete.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 1.5. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in Lemma 1. Put

$$\begin{aligned} [u_1, u_2] &= a_1e_1 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3, \\ [u_1, u_2] &= b_1e_1 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3, \\ [u_2, u_3] &= c_1e_1 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3. \end{aligned}$$

Consider the following cases:

1°. Suppose the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by C_2. Then

$$[e_1, u_1] = e_1,$$
$$[e_1, u_2] = 0,$$
$$[e_1, u_3] = u_1.$$

Let us check the Jacobi identity for the triples (e_1, u_j, u_k), $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_2, [e_1, u_1]] + [u_1, [u_2, e_1]] = 0$
$\alpha_1 e_1 + \alpha_3 u_1 = 0$
1. $\alpha_1 = 0$
2. $\alpha_3 = 0$

2. $[e_1, [u_1, u_3]] + [u_3, [e_1, u_1]] + [u_1, [u_3, e_1]] = 0$
$\beta_1 e_1 + \beta_3 u_1 - u_1 = 0$
3. $\beta_1 = 0$
4. $\beta_3 - 1 = 0$

3. $[e_1, [u_2, u_3]] + [u_3, [e_1, u_2]] + [u_2, [u_3, e_1]] = 0$
$\gamma_1 e_1 + \gamma_3 u_1 + a_1 e_1 + \alpha_2 u_2 = 0$
5. $\alpha_2 = 0$
6. $\gamma_3 = 0$
7. $\gamma_1 + a_1 = 0$

4. $[u_1, [u_2, u_3]] + [u_3, [u_1, u_2]] + [u_2, [u_3, u_1]] = 0$
$-c_1 e_1 + \gamma_2 a_1 e_1 - (c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2) - a_1 u_1 = 0$
8. $c_1 = 0$
9. $\gamma_2 = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	e_1	0	u_1
u_1	$-e_1$	0	$a_1 e_1$	$b_1 e_1 + \beta_2 u_2 + u_3$
u_2	0	$-a_1 e_1$	0	$-a_1 u_1$
u_3	$-u_1$	$-b_1 e_1 - \beta_2 u_2 - u_3$	$a_1 e_1$	0

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{19}, \mathfrak{g}_{19})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{19} \to \bar{\mathfrak{g}}$ such that

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2 + a_1 e_1,$$
$$\pi(u_3) = u_3 + \beta_2 u_2 + \frac{1}{2}(b_1 + \beta_2 a_1) e_1.$$

2°. Suppose the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by C_3. Then

$$[e_1, u_1] = 0,$$
$$[e_1, u_2] = e_1,$$
$$[e_1, u_3] = u_1.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	u_1	u_2	u_3
e_1	0	0	e_1	u_1
u_1	0	0	$a_1e_1+\alpha_1u_1$	$b_1e_1+\beta_1u_1$
u_2	$-e_1$	$-a_1e_1-\alpha_1u_1$	0	$c_1e_1+\gamma_1u_1-a_1u_2+(1-\alpha_1)u_3$
u_3	$-u_1$	$-b_1e_1-\beta_1u_1$	$-c_1e_1-\gamma_1u_1+a_1u_2+(\alpha_1-1)u_3$	0

,

where

$$\begin{cases} 2b_1(1-\alpha_1)+\beta_1a_1-a_1^2=0, \\ \beta_1(1-\alpha_1)-a_1(1+\alpha_1)=0. \end{cases}$$

2.1°. $\alpha_1 = 1$. Then $a_1 = 0$. Put $t = b_1 + \frac{\beta_1^2}{4}$.

2.1.1°. $t = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{22}, \mathfrak{g}_{22})$ $\alpha = 1$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{22} \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1 - \frac{\beta_1}{2}e_1,$$
$$\pi(u_2) = u_2 - \gamma_1 e_1,$$
$$\pi(u_3) = u_3 + c_1e_1 - \frac{\beta_1}{2}(u_2 - \gamma_1 e_1).$$

2.1.2°. $t > 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{20}, \mathfrak{g}_{20})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{20} \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = \frac{1}{\sqrt{t}}(u_1 - \frac{\beta_1}{2}e_1),$$
$$\pi(u_2) = u_2 - \gamma_1 e_1,$$
$$\pi(u_3) = \frac{1}{\sqrt{t}}(u_3 + c_1e_1 - \frac{\beta_1}{2}(u_2 - \gamma_1 e_1)).$$

2.1.3°. $t < 0$. Then we similarly obtain the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_{21}, \mathfrak{g}_{21})$.

Let us show that the pairs $(\bar{\mathfrak{g}}_{20}, \mathfrak{g}_{20})$, $(\bar{\mathfrak{g}}_{21}, \mathfrak{g}_{21})$, and $(\bar{\mathfrak{g}}_{22}, \mathfrak{g}_{22})$ $(\alpha = 1)$ are not equivalent to each other. Indeed, put

$$W_i = [\mathfrak{g}_i, \bar{\mathfrak{g}}_i], \quad i \in \{20, 21, 22\},$$

and consider the homomorphisms

$$f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(2, \mathbb{R}),$$

where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{W_i} x$ in the basis $\{u_1, e_1\}$. Then

$$f_{20}(\bar{\mathfrak{g}}_{20}) = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\},$$

$$f_{21}(\bar{\mathfrak{g}}_{21}) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\},$$

$$f_{22}(\bar{\mathfrak{g}}_{22}) = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}.$$

Since the subalgebras $f_{20}(\bar{\mathfrak{g}}_{20})$, $f_{21}(\bar{\mathfrak{g}}_{21})$, and $f_{22}(\bar{\mathfrak{g}}_{22})$ of the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ are not conjugate to each other, we conclude that the pairs $(\bar{\mathfrak{g}}_{20}, \mathfrak{g}_{20})$, $(\bar{\mathfrak{g}}_{21}, \mathfrak{g}_{21})$, and $(\bar{\mathfrak{g}}_{22}, \mathfrak{g}_{22})$ $\alpha = 1$ are not equivalent.

2.2°. $\alpha_1 \neq 1$. Then

$$b_1 = -\frac{\alpha_1}{(1-\alpha_1)^2} a_1^2 \quad \text{and} \quad \beta_1 = \frac{1+\alpha_1}{1-\alpha_1} a_1.$$

2.2.1°. $\alpha_1 \neq 2$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{22}, \mathfrak{g}_{22})$ $\alpha \neq 1, 2$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{22} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= u_1 - \frac{a_1}{1-\alpha_1} e_1, \\
\pi(u_2) &= u_2 - \gamma_1 e_1, \\
\pi(u_3) &= u_3 + \frac{c_1 - a_1\gamma_1}{2-\alpha_1} e_1 - \frac{a_1}{1-\alpha_1}(u_2 - \gamma_1 e_1).
\end{aligned}$$

Here $\alpha = \alpha_1$.

2.2.2°. $\alpha_1 = 2$, $c_1 - a_1\gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{22}, \mathfrak{g}_{22})$ $\alpha = 2$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{22} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= u_1 + a_1 e_1, \\
\pi(u_2) &= u_2 - \gamma_1 e_1, \\
\pi(u_3) &= u_3 + a_1(u_2 - \gamma_1 e_1).
\end{aligned}$$

2.2.3°. $\alpha_1 = 2$, $c_1 - a_1\gamma_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{23}, \mathfrak{g}_{23})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{23} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= \frac{1}{c_1 - a_1\gamma_1} u_1 + \frac{a_1}{c_1 - a_1\gamma_1} e_1, \\
\pi(u_2) &= u_2 - \gamma_1 e_1, \\
\pi(u_3) &= \frac{1}{c_1 - a_1\gamma_1}(u_3 + a_1 u_2 - a_1\gamma_1 e_1).
\end{aligned}$$

Consider the homomorphisms

$$f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{R}),$$

where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{D\bar{\mathfrak{g}}_i}$, $i = 22, 23$, in the basis $\{u_1, e_1, u_2\}$. Then

$$f_{22}(\bar{\mathfrak{g}}_{22}) = \left\{ \begin{pmatrix} \alpha x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & (\alpha - 1)x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\},$$

$$f_{23}(\bar{\mathfrak{g}}_{23}) = \left\{ \begin{pmatrix} 2x & y & 0 \\ 0 & x & -x \\ 0 & 0 & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}.$$

It is clear that the subalgebras $f_{22}(\bar{\mathfrak{g}}_{22})$ and $f_{23}(\bar{\mathfrak{g}}_{23})$ are not conjugate and that the subalgebras $f_{22}(\bar{\mathfrak{g}}_{22})$ with different values of the parameter α are also not conjugate. Therefore, all pairs obtained in section 2.2.2° are not equivalent to each other.

Note that no one of the subalgebras 1.5.20, 1.5.21, 1.5.22 ($\alpha = 1$) is equivalent to any of the subalgebras 1.5.22 $\alpha \neq 1$, 1.5.23. To prove this, it is sufficient to compare the dimensions of the commutants.

3°. Suppose the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by C_1. Then

$$[e_1, u_1] = 0,$$
$$[e_1, u_2] = 0,$$
$$[e_1, u_3] = u_1.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	$\alpha_1 u_1$	$b_1 e_1 + \beta_1 u_1 + \beta_2 u_2$
u_2	0	$-\alpha_1 u_1$	0	$c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2 - \alpha_1 u_3$
u_3	$-u_1$	$-b_1 e_1 - \beta_1 u_1 - \beta_2 u_2$	$-c_1 e_1 - \gamma_1 u_1 - \gamma_2 u_2 + \alpha_1 u_3$	0

,

where

$$\alpha_1 b = \alpha_1(\gamma_2 - \beta_1) = \alpha_1 \beta_2 = 0.$$

3.1°. If $\alpha_1 = \beta_2 = 0$, then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, \ldots, 9$; if $\alpha_1 \neq 0$, then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$. The proof is similar to that for case 2.°.

3.2°. Consider in detail the case where: $\alpha_1 = 0$ and $\beta_2 \neq 0$.

Then the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\pi(e_1) = \beta_2 e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2 + \frac{b_1}{\beta_2} e_1 + \frac{\beta_1}{\beta_2} u_1,$$
$$\pi(u_3) = \frac{1}{\beta_2} u_3$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has the form:

[,]	e_1	u_1	u_2	u_3
e_1	0	0	0	u_1
u_1	0	0	0	u_2
u_2	0	0	0	$t_1e_1+t_2u_1+t_3u_2$
u_3	$-u_1$	$-u_2$	$-t_1e_1-t_2u_1-t_3u_2$	0

, (**)

where

$$t_1 = \frac{1}{\beta_2}(c_1 - \frac{\gamma_2\beta_1}{\beta_2}),$$
$$t_2 = \frac{1}{\beta_2}(\beta_2\gamma_1 - \gamma_2\beta_1 + b_1),$$
$$t_3 = \frac{1}{\beta_2}(\gamma_2 + \beta_1).$$

We see that t_1, t_2, and t_3 are arbitrary real numbers.

Suppose

$$V = \mathcal{Z}(D\bar{\mathfrak{g}}) \quad \text{and} \quad \mathfrak{a} = \{\operatorname{ad}_V x \mid x \in \bar{\mathfrak{g}}\}.$$

Then $V = \mathbb{R}e_1 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2$ and $\mathfrak{a} = \mathbb{R}(\operatorname{ad}_V u_3)$ is a one-dimensional subalgebra of the Lie algebra $\mathfrak{gl}(V)$.

Let $W = V \bigcap \mathfrak{g} = \mathfrak{g} = \mathbb{R}e_1$. The Lie algebra $\bar{\mathfrak{g}}$ can be identified with the Lie algebra $\mathfrak{a} \leftthreetimes V$. Note that the following condition holds:

$$V = W \oplus \mathfrak{a}(W) \oplus \mathfrak{a}\big(\mathfrak{a}(W)\big).$$

Conversely, suppose $V = \mathbb{R}^3$ and $\mathfrak{a}$ is a one-dimensional subalgebra of $\mathfrak{gl}(V)$. Let W be a one-dimensional subspace of V such that $V = W \oplus \mathfrak{a}(W) \oplus \mathfrak{a}\big(\mathfrak{a}(W)\big)$. Put $\bar{\mathfrak{g}} = \mathfrak{a} \times V$ and $\mathfrak{g} = W$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to some pair of form (**).

Therefore there exists a one-to-one correspondence between the set of desired pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and the set of pairs $(\mathfrak{a}, W)$, where $\mathfrak{a}$ is a one-dimensional subalgebra of $\mathfrak{gl}(V)$ and W is a one-dimensional subspace of V such that

$$V = W \oplus \mathfrak{a}(W) \oplus \mathfrak{a}\big(\mathfrak{a}(W)\big).$$

Here $V = \mathbb{R}^3$.

Lemma 2. *Suppose $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are subalgebras of $\mathfrak{gl}(V)$. Then the Lie algebras $\bar{\mathfrak{g}}_1 = \mathfrak{a}_1 \leftthreetimes V$ and $\bar{\mathfrak{g}}_2 = \mathfrak{a}_2 \leftthreetimes V$ are isomorphic if and only if there exists an endomorphism $\varphi \in \mathrm{GL}(V)$ such that $\mathfrak{a}_2 = \varphi\mathfrak{a}_1\varphi^{-1}$.*

Proof. Indeed, suppose there exists a $\varphi \in \mathrm{GL}(V)$ such that $\mathfrak{a}_2 = \varphi\mathfrak{a}_1\varphi^{-1}$. Consider the mapping $f : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_2$ defined by

$$f(x, v) = (\varphi x \varphi^{-1}, \varphi(v)) \text{ for } x \in \mathfrak{a}_1,\ v \in V.$$

It is easy to see that f is an isomorphism of Lie algebras.

The converse statement is obvious.

Lemma 3. *Let* $\bar{\mathfrak{g}}_2 = \mathfrak{a}_1 \leftthreetimes V$, $\mathfrak{g}_1 = W_1$, $\bar{\mathfrak{g}}_2 = \mathfrak{a}_2 \leftthreetimes V$, *and* $\mathfrak{g}_2 = W_2$, *where* $\mathfrak{a}_1$ *and* $\mathfrak{a}_2$ *are subalgebras of* $\mathfrak{gl}(V)$, W_1 *and* W_2 *are one-dimensional subspaces of* V. *Then a necessary and sufficient condition for the pairs* $(\bar{\mathfrak{g}}_1, \mathfrak{g}_2)$ *and* $(\bar{\mathfrak{g}}_1, \mathfrak{g}_2)$ *to be equivalent is that there exist a* $\varphi \in \mathrm{GL}(V)$ *such that* $\mathfrak{a}_2 = \varphi\mathfrak{a}_1\varphi^{-1}$ *and* $\varphi(W_1) = W_2$. *In other words, the group* $\mathrm{GL}(V)$ *acts on the set of pairs* $(\mathfrak{a}, W)$ *and the action is defined by*

$$\varphi : (\mathfrak{a}, W) \to (\varphi\mathfrak{a}\varphi^{-1}, \varphi(W)).$$

Proof. It immediately follows from the previous Lemma.

Let us classify (up to transformations determined before) all pairs $(\mathfrak{a}, W)$.

It is known that any one-dimensional subalgebra $\mathfrak{a}$ of the Lie algebra $\mathfrak{gl}(3, \mathbb{R})$ is equivalent (up to conjugation) to one and only one of the following subalgebras:

$$\mathfrak{a}_1 = \left\{ \left. \begin{pmatrix} x & x & 0 \\ 0 & x & 0 \\ 0 & 0 & \alpha x \end{pmatrix} \right| \in \mathbb{R} \right\},$$

$$\mathfrak{a}_2 = \left\{ \left. \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix} \right| x \in \mathbb{R} \right\},$$

$$\mathfrak{a}_3 = \left\{ \left. \begin{pmatrix} x & x & 0 \\ 0 & x & x \\ 0 & 0 & x \end{pmatrix} \right| x \in \mathbb{R} \right\},$$

$$\mathfrak{a}_4 = \left\{ \left. \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\},$$

$$\mathfrak{a}_5 = \left\{ \left. \begin{pmatrix} \alpha x & x & 0 \\ -x & \alpha x & 0 \\ 0 & 0 & \beta x \end{pmatrix} \right| x \in \mathbb{R}, \alpha > 0 \right\},$$

$$\mathfrak{a}_6 = \left\{ \left. \begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & \alpha x \end{pmatrix} \right| x \in \mathbb{R}, \alpha \geqslant 0 \right\},$$

$$\mathfrak{a}_7 = \left\{ \left. \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\},$$

$$\mathfrak{a}_8 = \left\{ \left. \begin{pmatrix} x & 0 & 0 \\ 0 & \alpha x & 0 \\ 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R}, |\alpha| \leqslant 1 \right\},$$

$$\mathfrak{a}_9 = \left\{ \left. \begin{pmatrix} x & 0 & 0 \\ 0 & \alpha x & 0 \\ 0 & 0 & \beta x \end{pmatrix} \right| x \in \mathbb{R}, \alpha\beta > 0, \alpha \leqslant \beta < 0 \right\}.$$

Consider in detail the case when $\mathfrak{a} = \mathfrak{a}_1$. Recall that

$$\mathcal{A}(\mathfrak{a}) = \left\{ X \in \mathrm{GL}(3, \mathbb{R}) | X\mathfrak{a}X^{-1} = \mathfrak{a} \right\}.$$

Let $\alpha \neq 1$, then

$$\mathcal{A}(\mathfrak{a}) = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, a, c \in \mathbb{R}^*, b \in \mathbb{R} \right\}$$

Any one-dimensional subspace W of V is equivalent (up to the action of elements of $\mathcal{A}(\mathfrak{a})$) to one and only one of the following subspaces:

$$W_1 = \mathbb{R}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad W_2 = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad W_3 = \mathbb{R}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad W_4 = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Note that the condition

$$V = W \oplus \mathfrak{a}(W) \oplus \mathfrak{a}(\mathfrak{a}(W))$$

holds only for W_3. Let $\{f_1, f_2, f_3\}$ be the standard basis of V, that is

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let

$$f_4 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \in \mathfrak{a}.$$

Then

$$W = \mathbb{R}(f_2 + f_3) = \mathfrak{g} \quad \text{and} \quad \bar{\mathfrak{g}} = \mathbb{R}f_1 \oplus \mathbb{R}f_2 \oplus \mathbb{R}f_3 \oplus \mathbb{R}f_4.$$

We obtain

$$\begin{aligned} [f_1, f_4] &= f_1, \\ [f_2, f_4] &= f_1 + f_2, \\ [f_3, f_4] &= \alpha f_3. \end{aligned}$$

The pair is equivalent to the pair $(\bar{\mathfrak{g}}_{13}, \mathfrak{g}_{13})$.

If $\alpha = 1$, then there is no any one-dimensional subspace W of V such that

$$V = W \oplus \mathfrak{a}(W) \oplus \mathfrak{a}(\mathfrak{a}(W)).$$

In a similar way we obtain the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 11, 12, 14, \ldots, 18$.

Proposition 1.6. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.6 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	u_2	u_1
u_1	0	0	0	0
u_2	$-u_2$	0	0	0
u_3	$-u_1$	0	0	0

2.

$[,]$	e_1	u_1	u_2	u_3
e_1	0	0	u_2	u_1
u_1	0	0	0	u_1
u_2	$-u_2$	0	0	0
u_3	$-u_1$	$-u_1$	0	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 1.6 is equivalent to one of the following:*

$$C_1(e_1) = (\,p \quad 0 \quad 0\,), \quad p \in \mathbb{R}.$$

Proof. Any virtual structure q has the form

$$C(e_1) = (\,c_1 \quad c_2 \quad c_3\,).$$

Put

$$H = (\,c_3 \quad c_2 \quad 0\,),$$

and $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_1) = (\,c_1 \quad 0 \quad 0\,).$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

Thus it can be assumed that the corresponding virtual pair is defined by one of the virtual structures determined in the Lemma. Then,

$$[e_1, u_1] = pe_1,$$
$$[e_1, u_2] = u_2,$$
$$[e_1, u_3] = u_1.$$

Put

$$[u_1, u_2] = a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3,$$
$$[u_1, u_3] = b_1 e_1 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3,$$
$$[u_2, u_3] = c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.$$

Let us check the Jacobi identity for the triples (e_1, u_i, u_j), $1 \leqslant i < j \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_1, [u_2, e_1]] + [u_2, [e_1, u_1]] = 0$

$p\alpha_1 e_1 + \alpha_2 u_2 + \alpha_3 u_1 - a_1 e_1 - \alpha_1 u_1 - \alpha_2 u_2 - \alpha_3 u_3 = 0$

1. $p\alpha_1 - a_1 = 0$
2. $\alpha_3 - \alpha_1 = 0$

3. $p = 0$
4. $\alpha_3 = 0$

2. $[e_1,[u_1,u_3]] + [u_1,[u_3,e_1]] + [u_3,[e_1,u_1]] = 0$
$\beta_2 u_2 + \beta_3 u_1 = 0$
5. $\beta_2 = 0$
6. $\beta_3 = 0$

3. $[e_1,[u_2,u_3]] + [u_2,[u_3,e_1]] + [u_3,[e_1,u_2]] = 0$
$\gamma_2 u_2 + \gamma_3 u_1 - c_1 e_1 - \gamma_1 u_1 - \gamma_2 u_2 - \gamma_3 u_3 = 0$
7. $c_1 = 0$
8. $\gamma_1 = 0$
9. $\gamma_3 = 0$

4. $[u_1,[u_2,u_3]] + [u_2,[u_3,u_1]] + [u_3,[u_1,u_2]] = 0$
$b_1 u_2 = 0.$
10. $b_1 = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	u_1	u_2	u_3
e_1	0	0	u_2	u_1
u_1	0	0	0	$\beta_1 u_1$
u_2	$-u_2$	0	0	$\gamma_2 u_2$
u_3	$-u_1$	$-\beta_1 u_1$	$-\gamma_2 u_2$	0

Consider the following cases:

1°. $\beta_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = u_3 - \gamma_2 e_1.$$

2°. $\beta_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = \beta_1 u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \beta_1 u_3 - \gamma_2 \beta_1 e_1.$$

Now it remains to show that the pairs determined by the Proposition are not equivalent to each other.

Since $\dim \mathcal{Z}(\bar{\mathfrak{g}}_1) = 1$ and $\dim \mathcal{Z}(\bar{\mathfrak{g}}_2) = 0$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

Proposition 1.7. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.7 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	λu_2	$u_1 + u_3$
u_1	$-u_1$	0	0	0
u_2	$-\lambda u_2$	0	0	0
u_3	$-u_1 - u_3$	0	0	0

2. $\lambda = 0$

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	0	$u_1 + u_3$
u_1	$-u_1$	0	0	0
u_2	0	0	0	u_1
u_3	$-u_1 - u_3$	0	$-u_1$	0

3. $\lambda = 2$

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	$2u_2$	$u_1 + u_3$
u_1	$-u_1$	0	0	u_2
u_2	$-2u_2$	0	0	0
u_3	$-u_1 - u_3$	$-u_2$	0	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1, that is $\mathfrak{h} = \mathfrak{g}$.

Lemma. *Any virtual structure q on generalized module 1.7 is equivalent to one of the following:*

a) $\lambda = 0$

$$C_1(e_1) = (\,0 \quad p \quad 0\,), \qquad p \in \mathbb{R}.$$

b) $\lambda \neq 0$

$$C_2(e_1) = (\,0 \quad 0 \quad 0\,).$$

Proof. Any virtual structure q has the form:

$$C(e_1) = (\,c_1 \quad c_2 \quad c_3\,).$$

Suppose $\lambda = 0$. Put

$$H = (\,c_1 \quad 0 \quad c_3 - c_1\,)$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then $C_1(x) = (\,0 \quad c_1 \quad 0\,)$. By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

Now suppose $\lambda \neq 0$. Similarly, putting

$$H = (\,c_1 \quad c_2/\lambda \quad c_3 - c_1\,)$$

and $C_2(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$, we see that

$$C_2(e_1) = (0 \quad 0 \quad 0),$$

and the virtual structure C is trivial.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 1.7. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\lambda = 0$. The vectors $[e_1, u_i]$, $1 \leqslant i \leqslant 3$, have the form:

$$[e_1, u_1] = u_1,$$
$$[e_1, u_2] = pe_1,$$
$$[e_1, u_3] = u_1 + u_3.$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h})$$

for all $\alpha \in \mathfrak{h}^*$ (statement 10, Chapter II).

Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_1 \oplus \mathbb{R}u_3$$

and

$$[u_1, u_2] = \alpha_1 u_1 + \alpha_3 u_3,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = \gamma_1 u_1 + \gamma_3 u_3.$$

Let us check the Jacobi identity for the triples $(e_1, u_j, u_k), 1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_2, [e_1, u_1]] + [u_1, [u_2, e_1]] = 0$
$\alpha_1 u_1 + \alpha_3 u_1 + \alpha_3 u_3 + pu_1 - \alpha_1 u_1 - \alpha_3 u_3 = 0$
1. $\alpha_3 + p = 0$

2. $[e_1, [u_1, u_3]] + [u_3, [e_1, u_1]] + [u_1, [u_3, e_1]] = 0$
$0 = 0$

3. $[e_1, [u_2, u_3]] + [u_3, [e_1, u_2]] + [u_2, [u_3, e_1]] = 0$
$\gamma_1 u_1 + \gamma_3 u_1 + \gamma_3 u_3 + \alpha_1 u_1 + \alpha_3 u_3 - \gamma_1 u_1 - \gamma_3 u_3 - pu_1 - pu_3 = 0$
2. $\alpha_3 - p = 0$
3. $\alpha_1 + \gamma_3 - p = 0$

4. $[u_1, [u_2, u_3]] + [u_3, [u_1, u_2]] + [u_2, [u_3, u_1]] = 0$
$0 = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	u_1	u_2	u_3
e_1	0	u_1	0	$u_1 + u_3$
u_1	$-u_1$	0	$\alpha_1 u_1$	0
u_2	0	$-\alpha_1 u_1$	0	$\gamma_1 u_1 - \alpha_1 u_3$
u_3	$-u_1 - u_3$	0	$-\gamma_1 u_1 + \alpha_1 u_3$	0

1.1°. $\alpha_1 + \gamma_1 = 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \mathfrak{g}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= u_2 + \alpha_1 e_1, \\
\pi(u_3) &= u_3.
\end{aligned}$$

1.2°. $\alpha_1 + \gamma_1 \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \mathfrak{g}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= \frac{1}{\gamma_1 + \alpha_1} u_2 + \frac{\alpha_1}{\gamma_1 + \alpha_1} e_1, \\
\pi(u_3) &= u_3.
\end{aligned}$$

Since $\dim Z(\bar{\mathfrak{g}}_1) \neq \dim Z(\bar{\mathfrak{g}}_2)$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

2°. $\lambda \neq 0$.
The vectors $[e_1, u_i], 1 \leqslant i \leqslant 3$, have the form:

$$\begin{aligned}
[e_1, u_1] &= u_1, \\
[e_1, u_2] &= \lambda u_2, \\
[e_1, u_3] &= u_1 + u_3.
\end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1, \ \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \subset \mathbb{R}u_1 \oplus \mathbb{R}u_3, \ \bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}) = \mathbb{R}u_2.$$

Therefore
2.1°. If $\lambda = -1$ then

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1, \\
[u_1, u_3] &= 0, \\
[u_2, u_3] &= c_1 e_1.
\end{aligned}$$

Using the Jacobi identity for the triples (e_1, u_2, u_3) and (u_1, u_2, u_3), we obtain $a_1 = c_1 = 0$, and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2.2°. If $\lambda = 2$ then

$$\begin{aligned}
[u_1, u_2] &= 0, \\
[u_1, u_3] &= \beta_2 u_2, \\
[u_2, u_3] &= 0.
\end{aligned}$$

So, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	$2u_2$	$u_1 + u_3$
u_1	$-u_1$	0	0	$\beta_2 u_2$
u_2	$-2u_2$	0	0	0
u_3	$-u_1 - u_3$	$-\beta_2 u_2$	0	0

2.2.1°. $\beta_2 = 0$.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2.2.2°. $\beta_2 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \mathfrak{g}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = \beta_2 u_2,$$
$$\pi(u_3) = u_3.$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_3$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

2.3°. If $\lambda \neq -1$ and $\lambda \neq 2$ then

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0.$$

It is easy to see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

Thus the proof of the Proposition is complete.

Proposition 1.8. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.8 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	0	0
u_2	$-u_1$	0	0	0
u_3	$-u_2$	0	0	0

2.

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	u_1	u_2
u_2	$-u_1$	$-u_1$	0	u_3
u_3	$-u_2$	$-u_2$	$-u_3$	0

3.

$[\,,]$	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	0	u_1
u_2	$-u_1$	0	0	$\alpha e_1 + u_2$
u_3	$-u_2$	$-u_1$	$-\alpha e_1 - u_2$	0

4.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	0	0
u_2	$-u_1$	0	0	e_1
u_3	$-u_2$	0	$-e_1$	0

5.

[,]	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	0	0
u_2	$-u_1$	0	0	$-e_1$
u_3	$-u_2$	0	e_1	0

Proof. Let $\mathcal{E} = e_1$ be basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 1.8 is equivalent to one of the following:*

$$C(e_1) = (\,p \quad 0 \quad 0\,), \qquad p \in \mathbb{R}.$$

Proof. Any virtual structure q has the form:

$$C(e_1) = (\,c_1 \quad c_2 \quad c_3\,).$$

Put

$$H = (\,c_2 \quad c_3 \quad 0\,).$$

Now put $C_1 = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = (\,c_1 \quad 0 \quad 0\,).$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.
This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 1.8. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma.

Then the vectors $[e_1, u_j]$, $1 \leqslant j \leqslant 3$ have the form

$$[e_1, u_1] = pe_1,$$
$$[e_1, u_2] = u_1,$$
$$[e_1, u_3] = u_2.$$

Put

$$[u_1, u_2] = a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3,$$
$$[u_1, u_3] = b_1 e_1 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3,$$
$$[u_2, u_3] = c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.$$

Let us check the Jacobi identity for the triples (e_1, u_i, u_j), $1 \leqslant i < j \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_1, [u_2, e_1]] + [u_2, [e_1, u_1]] = 0$

$p\alpha_1 e_1 + \alpha_2 u_1 + \alpha_3 u_2 - pu_1 = 0$

1. $p\alpha_1 = 0$
2. $\alpha_3 = 0$
3. $p = \alpha_2$

2. $[e_1, [u_1, u_3]] + [u_1, [u_3, e_1]] + [u_3, [e_1, u_1]] = 0$

$p\beta_1 e_1 + \beta_2 u_1 + \beta_3 u_2 - a_1 e_1 - \alpha_1 u_1 - pu_2 - pu_2 = 0$

4. $a_1 - p\beta_1 = 0$
5. $\alpha_1 - \beta_2 = 0$
6. $2p - \beta_3 = 0$

3. $[e_1, [u_2, u_3]] + [u_2, [u_3, e_1]] + [u_3, [e_1, u_2]] = 0$

$p\gamma_1 e_1 + \gamma_2 u_1 + \gamma_3 u_2 - b_1 e_1 - \beta_1 u_1 - \alpha_1 u_2 - 2pu_3 = 0$

7. $p = 0$
8. $b_1 = 0$
9. $\beta_1 - \gamma_2 = 0$
10. $\alpha_1 - \gamma_3 = 0$

4. $[u_1, [u_2, u_3]] + [u_2, [u_3, u_1]] + [u_3, [u_1, u_2]] = 0$

$\gamma_2 \alpha_1 u_2 + \alpha_1(\gamma_2 u_1 + \alpha_1 u_2) - \alpha_1 \gamma_3 u_1 - \alpha_1(\gamma_2 u_1 + \alpha_1 u_2) = 0$

11. $\alpha_1 \gamma_2 = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	$\alpha_1 u_1$	$\gamma_2 u_1 + \alpha_1 u_2$
u_2	$-u_1$	$-\alpha_1 u_1$	0	A
u_3	$-u_2$	$-\gamma_2 u_1 - \alpha_1 u_2$	$-A$	0

,

where $A = c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2 + \alpha_1 u_3$, $\alpha_1 \gamma_2 = 0$.

The mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = u_3 + \gamma_1 e_1,$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has the form:

	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	$\alpha_1 u_1$	$\gamma_2 u_1 + \alpha_1 u_2$
u_2	$-u_1$	$-\alpha_1 u_1$	0	$c_1 e_1 + \gamma_2 u_2 + \alpha_1 u_3$
u_3	$-u_2$	$-\gamma_2 u_1 - \alpha_1 u_2$	$-c_1 e_1 - \gamma_2 u_2 - \alpha_1 u_3$	0

,

where $\alpha_1\gamma_2 = 0$.

Consider the following cases:

1°. $\alpha_1 = 0,\ \gamma_2 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_i) = \frac{1}{\gamma_2}u_i,\ 1 \leqslant i \leqslant 3,$$
$$\alpha = \frac{c_1}{\gamma_2^2}.$$

Consider the pairs $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ and $(\bar{\mathfrak{g}}_3', \mathfrak{g}_3')$ with parameters α and α' respectively. It is possible to show that these pairs are not equivalent, whenever $\alpha \neq \alpha'$.

2°. $\alpha_1 \neq 0,\ \gamma_2 = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_1) = \frac{1}{\alpha_1}e_1,$$
$$\pi(u_1) = \frac{1}{\alpha_1^2}u_1,$$
$$\pi(u_2) = \frac{1}{\alpha_1}u_2,$$
$$\pi(u_3) = u_3 + \frac{c_1}{\alpha_1}e_1 - \frac{c_1}{2\alpha_1^2}u_1.$$

3°. $\alpha_1 = \gamma_2 = 0$.

3.1°. $c_1 = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

3.2°. $c_1 > 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_i) = \frac{1}{\sqrt{c_1}}u_i,\ i = 1, 2, 3.$$

3.3°. $c_1 < 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_i) = \frac{1}{\sqrt{-c_1}}u_i,\ 1 \leqslant i \leqslant 3.$$

Now it remains to show that the pairs determined by the Proposition are not equivalent to each other.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_4 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_1$, we see that the pairs $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent. Similarly, since $\dim \mathcal{D}\bar{\mathfrak{g}}_5 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_1$, the pairs $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent.

The algebra $\bar{\mathfrak{g}}_4/\mathcal{Z}(\bar{\mathfrak{g}}_4)$ contains a one-dimensional ideal, while algebra $\bar{\mathfrak{g}}_5/\mathcal{Z}(\bar{\mathfrak{g}}_5)$ does not contain any one-dimensional ideal. Hence, the pairs $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ are not equivalent.

Since $\dim \mathcal{Z}(\bar{\mathfrak{g}}_1) = \dim \mathcal{Z}(\bar{\mathfrak{g}}_4) = \dim \mathcal{Z}(\bar{\mathfrak{g}}_5) = 1 \neq \dim \mathcal{Z}(\bar{\mathfrak{g}}_3) = \dim \mathcal{Z}(\bar{\mathfrak{g}}_2) = 0$, we see that no one of the pairs 1.8.1, 1.8.4, and 1.8.5 is equivalent to any of the pairs 1.8.2 and 1.8.3.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_2 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_3$, we see that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

This complete the proof of the Proposition.

Proposition 1.9. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 1.9 is trivial.*

$[,]$	e_1	u_1	u_2	u_3
e_1	0	u_1	$u_1 + u_2$	$u_2 + u_3$
u_1	$-u_1$	0	0	0
u_2	$-u_1 - u_2$	0	0	0
u_3	$-u_2 - u_3$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

And this proves the Proposition.

2. Two-dimensional case

Proposition 2.1. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.1 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	λu_2	0
e_2	0	0	0	0	u_3
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0
u_3	0	$-u_3$	0	0	0

2. $\lambda = -1$.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_3
u_1	$-u_1$	0	0	e_1	0
u_2	u_2	0	$-e_1$	0	0
u_3	0	$-u_3$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $A(e_1) = A(e_2) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Note that $\mathfrak{g}$ is a nilpotent Lie algebra.

Lemma. *Any virtual structure q on generalized module 2.1 is equivalent to one of the following:*

a) $\lambda = 0$

$$C(e_1) = \begin{pmatrix} 0 & p & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & r & 0 \\ 0 & s & 0 \end{pmatrix}.$$

b) $\lambda \neq 0$

$$C(e_1) = C(e_2) = 0.$$

Proof. Let q be a virtual structure on generalized module 2.1. Without loss of generality it can be assumed that q is primary. Consider the following cases:

1°. $\lambda = 0$.

Since

$$\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, \quad U^{(1,0)}(\mathfrak{h}) = \mathbb{R}u_1,$$
$$U^{(0,0)}(\mathfrak{h}) = \mathbb{R}u_2, \quad U^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & c^1_{12} & 0 \\ 0 & c^1_{22} & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c^2_{12} & 0 \\ 0 & c^2_{22} & 0 \end{pmatrix}.$$

Let us check condition (6), Chapter II, for $x, y \in \mathcal{E}$.

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have

$$C([e_1,e_2]) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that:

$$C(e_1) = \begin{pmatrix} 0 & p & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & r & 0 \\ 0 & s & 0 \end{pmatrix}.$$

2°. $\lambda \neq 0$. Since

$$\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, \quad U^{(1,0)}(\mathfrak{h}) \supset \mathbb{R}u_1,$$

$$U^{(\lambda,0)}(\mathfrak{h}) \supset \mathbb{R}u_2, \quad U^{(0,1)}(\mathfrak{h}) \supset \mathbb{R}u_3,$$

we have

$$C(e_1) = C(e_2) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Put

$$\begin{aligned} [u_1,u_2] &= a_1e_1 + a_2e_2 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3, \\ [u_1,u_3] &= b_1e_1 + b_2e_2 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3, \\ [u_2,u_3] &= c_1e_1 + c_2e_2 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3. \end{aligned}$$

Consider the following cases:

1°. $\lambda = 0$. Then

$$\begin{aligned} &[e_1,e_2] = 0, && \\ &[e_1,u_1] = u_1, && [e_2,u_1] = 0, \\ &[e_1,u_2] = pe_1 + qe_2, && [e_2,u_2] = re_1 + se_2, \\ &[e_1,u_3] = 0, && [e_2,u_3] = u_3. \end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	0	0
e_2	0	0	0	0	u_3
u_1	$-u_1$	0	0	α_1u_1	0
u_2	0	0	$-\alpha_1u_1$	0	γ_3u_3
u_3	0	$-u_3$	0	$-\gamma_3u_3$	0

.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(e_2) &= e_2, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= \alpha_1e_1 + u_2 - \gamma_3e_2, \\ \pi(u_3) &= u_3. \end{aligned}$$

2°. $\lambda \neq 0$. Then

$$\begin{aligned} &[e_1, e_2] = 0, \\ &[e_1, u_1] = u_1, \quad &&[e_2, u_1] = 0, \\ &[e_1, u_2] = \lambda u_2, \quad &&[e_2, u_2] = 0, \\ &[e_1, u_3] = 0, \quad &&[e_2, u_3] = u_3. \end{aligned}$$

2.1°. $\lambda = -1$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_3
u_1	$-u_1$	0	0	$a_1 e_1$	0
u_2	u_2	0	$-a_1 e_1$	0	0
u_3	0	$-u_3$	0	0	0.

Consider the following cases.

2.1.1°. $a_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2.1.2°. $a_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(e_2) &= e_2, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= \frac{1}{a_1} u_2, \\ \pi(u_3) &= u_3. \end{aligned}$$

2.2°. $\lambda \neq -1$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_2 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_1$, we see that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent.

This completes the proof of the Proposition.

Proposition 2.2. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.2 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	λu_2	0
e_2	0	0	u_1	0	μu_3
u_1	$-u_1$	$-u_1$	0	0	0
u_2	$-\lambda u_2$	0	0	0	0
u_3	0	$-\mu u_3$	0	0	0

2. $\lambda = \mu = 1$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	u_2	0
e_2	0	0	u_1	0	u_3
u_1	$-u_1$	$-u_1$	0	0	0
u_2	$-u_2$	0	0	0	u_1
u_3	0	$-u_3$	0	$-u_1$	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Then $A(e_1) = A(e_2) = (0)$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Note that $\mathfrak{g}$ is a nilpotent Lie algebra.

Lemma. *Any virtual structure q on generalized module 2.2 is trivial.*

Proof. Suppose q is a virtual structure on generalized module 2.2. Without loss of generality it can be assumed that q is primary. Since

$$\mathfrak{g}^{(0,0)}(\mathfrak{g}) = \mathfrak{g}, \quad U^{(1,1)}(\mathfrak{g}) = \mathbb{R}u_1, \quad U^{(0,\mu)}(\mathfrak{g}) = \mathbb{R}u_3, \quad U^{(\lambda,0)} = \mathbb{R}u_2,$$

we have

$$C(e_1) = C(e_2) = 0,$$

and this completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.2. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure. Then

$$\begin{aligned} &[e_1, e_2] = 0, \\ &[e_1, u_1] = u_1, \quad &&[e_2, u_1] = u_1, \\ &[e_1, u_2] = \lambda u_2, \quad &&[e_2, u_2] = 0, \\ &[e_1, u_3] = 0, \quad &&[e_2, u_3] = \mu u_3. \end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{g}) = \mathfrak{g}^{\alpha}(\mathfrak{g}) \times U^{\alpha}(\mathfrak{g}) \quad \text{for all } \alpha \in \mathfrak{g}^*$$

(Proposition 10, Chapter II). Thus

$$\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{g}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, \; \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{g}) = \mathbb{R}u_1, \; \bar{\mathfrak{g}}^{(\lambda,0)}(\mathfrak{g}) = \mathbb{R}u_2, \; \bar{\mathfrak{g}}^{(0,\mu)}(\mathfrak{g}) = \mathbb{R}u_3$$

and

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(\lambda+1,1)}(\mathfrak{g}), \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(1,\mu+1)}(\mathfrak{g}), \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(\lambda,\mu)}(\mathfrak{g}). \end{aligned}$$

Consider the following cases:

1°. $\lambda \neq 1$ or $\mu \neq 1$. We have

$$\begin{aligned} &[u_1, u_2] = 0, \\ &[u_1, u_3] = 0, \\ &[u_2, u_3] = 0. \end{aligned}$$

It is clear that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $\lambda = \mu = 1$. We have

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = \gamma_1 u_1.$$

So, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	u_2	0
e_2	0	0	u_1	0	u_3
u_1	$-u_1$	$-u_1$	0	0	0
u_2	$-u_2$	0	0	0	$\gamma_1 u_1$
u_3	0	$-u_3$	0	$-\gamma_1 u_1$	0 .

Consider the following cases.

2.1°. $\gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2.2°. $\gamma_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = e_2,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = \frac{1}{\gamma_1} u_2,$$
$$\pi(u_3) = u_3.$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent. And this completes the proof of the Proposition.

Proposition 2.3. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.3 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
e_2	0	0	0	0	u_3
u_1	$u_2 - \lambda u_1$	0	0	0	0
u_2	$-u_1 - \lambda u_2$	0	0	0	0
u_3	0	$-u_3$	0	0	0

2. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	$-u_2$	u_1	0
e_2	0	0	0	0	u_3
u_1	u_2	0	0	e_1	0
u_2	$-u_1$	0	$-e_1$	0	0
u_3	0	$-u_3$	0	0	0

3. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	$-u_2$	u_1	0
e_2	0	0	0	0	u_3
u_1	u_2	0	0	$-e_1$	0
u_2	$-u_1$	0	e_1	0	0
u_3	0	$-u_3$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $A(e_1) = A(e_2) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 2.3 is trivial.*

Proof. Put

$$C(e_i) = \begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} \\ c^i_{21} & c^i_{22} & c^i_{23} \end{pmatrix}, \; i = 1, 2.$$

Let us check condition (6), Chapter II, for $x, y \in \mathcal{E}$.

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} \lambda c^2_{11} - c^2_{12} & c^2_{11} + \lambda c^2_{12} & 0 \\ \lambda c^2_{21} - c^2_{22} & c^2_{21} + \lambda c^2_{22} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \end{pmatrix}.$$

We obtain the system of linear equations:

$$\begin{cases} c^2_{12} = \lambda c^2_{11} \\ c^2_{11} = -\lambda c^2_{12} \\ c^2_{22} = \lambda c^2_{21} \\ c^2_{21} = -\lambda c^2_{22} \\ c^1_{13} = c^1_{23} = 0 \end{cases}$$

It follows that $c^2_{11} = c^2_{12} = c^2_{21} = c^2_{22} = c^1_{13} = c^1_{23} = 0$.

So, any virtual structures q on generalized module 2.3 has the form:

$$c(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & 0 \\ c^1_{21} & c^1_{22} & 0 \end{pmatrix}, \quad c(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} \frac{\lambda c^1_{11} + c^1_{12}}{1+\lambda^2} & \frac{-c^1_{11} + \lambda c^1_{12}}{1+\lambda^2} & c^2_{13} \\ \frac{\lambda c^1_{21} + c^1_{22}}{1+\lambda^2} & \frac{-c^1_{21} + \lambda c^1_{22}}{1+\lambda^2} & c^2_{23} \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = C_1(e_2) = 0.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.3. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is determined by the trivial virtual structure. Then

$$\begin{aligned} &[e_1, e_2] = 0, \\ &[e_1, u_1] = \lambda u_1 - u_2, \quad [e_2, u_1] = 0, \\ &[e_1, u_2] = \lambda u_2 + u_1, \quad [e_2, u_2] = 0, \\ &[e_1, u_3] = 0, \qquad\qquad [e_2, u_3] = u_3. \end{aligned}$$

Put

$$\begin{aligned} [u_1, u_2] &= a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\ [u_1, u_3] &= b_1 e_1 + b_2 e_2 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\ [u_2, u_3] &= c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3. \end{aligned}$$

Let us check the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1, 2$, $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_2, [u_1, u_2]] + [u_2, [e_2, u_1]] + [u_1, [u_2, e_2]] = 0$

$\alpha_3 u_3 = 0$

1. $\alpha_3 = 0$

2. $[e_1, [u_1, u_2]] + [u_2, [e_1, u_1]] + [u_1, [u_2, e_1]] = 0$

$\alpha_1 \lambda u_1 - \alpha_1 u_2 + \alpha_2 \lambda u_2 + \alpha_2 u_1 - 2\lambda(a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2) = 0$

2. $\lambda a_1 = 0$
3. $\lambda a_2 = 0$
4. $\alpha_2 - \lambda \alpha_1 = 0$
5. $\alpha_1 + \lambda \alpha_2 = 0$

3. $[e_2, [u_1, u_3]] + [u_3, [e_2, u_1]] + [u_1, [u_3, e_2]] = 0$

$\beta_3 u_3 - b_1 e_1 - b_2 e_2 - \beta_1 u_1 - \beta_2 u_2 - \beta_3 u_3 = 0$

6. $b_1 = 0$
7. $b_2 = 0$
8. $\beta_1 = 0$
9. $\beta_2 = 0$

4. $[e_1, [u_1, u_3]] + [u_3, [e_1, u_1]] + [u_1, [u_3, e_1]] = 0$

$-\lambda \beta_3 u_3 + c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3 = 0$

10. $\gamma_3 - \lambda \beta_3 = 0$
11. $c_1 = 0$
12. $c_2 = 0$
13. $\gamma_1 = 0$
14. $\gamma_2 = 0$

5. $[e_1, [u_2, u_3]] + [u_3, [e_1, u_2]] + [u_2, [u_3, e_1]] = 0$

$-\beta_3 u_3 - \lambda \gamma_3 u_3 = 0$

15. $\beta_3 + \lambda \gamma_3 = 0$

6. $[e_2, [u_2, u_3]] + [u_3, [e_2, u_2]] + [u_2, [u_3, e_2]] = 0$

$\gamma_3 u_3 - \gamma_3 u_3 = 0$

7. $[u_1,[u_2,u_3]] + [u_3,[u_1,u_2]] + [u_2,[u_3,u_1]] = 0$
$\gamma_3\beta_3 u_3 - \gamma_3\beta_3 u_3 - a_2 u_3 = 0$
16. $a_2 = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
e_2	0	0	0	0	u_3
u_1	$u_2 - \lambda u_1$	0	0	$a_1 e_1$	0
u_2	$-u_1 - \lambda u_2$	0	$-a_1 e_1$	0	0
u_3	0	$-u_3$	0	0	0

,

where $\lambda a_1 = 0$. Consider the following cases:

1°. $\lambda \neq 0$.
Then $a_1 = 0$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $\lambda = 0$.

2.1°. $a_1 = 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2.2°. $a_1 > 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = e_2,$$
$$\pi(u_1) = \frac{1}{\sqrt{a_1}} u_1,$$
$$\pi(u_2) = \frac{1}{\sqrt{a_1}} u_2,$$
$$\pi(u_3) = u_3.$$

2.3°. $a_1 < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \mathfrak{g}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = e_2,$$
$$\pi(u_1) = \frac{1}{\sqrt{-a_1}} u_1,$$
$$\pi(u_2) = \frac{1}{\sqrt{-a_1}} u_2,$$
$$\pi(u_3) = u_3.$$

Now it remains to show that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent to each other whenever $\lambda = 0$.

Indeed, note that the Lie algebra $\bar{\mathfrak{g}}_1$ is solvable and the Lie algebras $\bar{\mathfrak{g}}_2$, $\bar{\mathfrak{g}}_3$ are unsolvable. It follows that the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ is not equivalent to the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$.

Since the Levi subalgebras of $\bar{\mathfrak{g}}_2$ and $\bar{\mathfrak{g}}_3$ are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$, respectively, we see that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

The proof of the Proposition is complete.

Proposition 2.4. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.4 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	u_2	μu_3
e_2	0	0	$-u_2$	u_1	λu_3
u_1	$-u_1$	u_2	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0
u_3	$-\mu u_3$	$-\lambda u_3$	0	0	0 ,

2. $\lambda = 0$, $\mu = 2$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	u_2	$2u_3$
e_2	0	0	$-u_2$	u_1	0
u_1	$-u_1$	u_2	0	u_3	0
u_2	$-u_2$	$-u_1$	$-u_3$	0	0
u_3	$-2u_3$	0	0	0	0 .

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$, the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by e_1.

Lemma. *Any virtual structure q on generalized module 2.4 is equivalent to one of the following:*

a) $\lambda \neq 0$ *or* $\mu \neq 0$

$$C_1(e_1) = C_1(e_2) = 0;$$

b) $\lambda = \mu = 0$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & s \end{pmatrix}.$$

Proof. Let q be a virtual structure on generalized module 2.4. Without loss of generality it can be assumed that q is primary. Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2,$$

$$U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad U^{(\mu)}(\mathfrak{h}) \supset \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \end{pmatrix}.$$

Let us check condition (6), Chapter II, for $x, y \in \mathcal{E}$:

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have

$$0 = 0 - \begin{pmatrix} 0 & 0 & \mu c_{13}^2 \\ 0 & 0 & \mu c_{23}^2 \end{pmatrix} - 0 + \begin{pmatrix} 0 & 0 & \lambda c_{13}^1 \\ 0 & 0 & \lambda c_{23}^1 \end{pmatrix}.$$

We obtain the system of linear equations:

$$\begin{cases} \lambda c_{13}^1 &= \mu c_{13}^2, \\ \lambda c_{23}^1 &= \mu c_{23}^2. \end{cases}$$

So, any virtual structure q on generalized module 2.4 has the form:

$$C(e_1) = \begin{pmatrix} 0 & 0 & c_{13}^1 \\ 0 & 0 & c_{23}^1 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ 0 & 0 & c_{23}^2 \end{pmatrix},$$

where $\lambda c_{13}^1 = \mu c_{13}^2, \lambda c_{23}^1 = \mu c_{23}^2$.

Suppose $\lambda \neq 0$ or $\mu \neq 0$. Put

$$H = \begin{pmatrix} 0 & 0 & h_{13} \\ 0 & 0 & h_{23} \end{pmatrix},$$

where the set of coefficients h_{13}, h_{23} is a solution of the following system:

$$\begin{cases} c_{13}^1 &= \mu h_{13}, \\ c_{23}^1 &= \mu h_{23}, \\ c_{13}^2 &= \lambda h_{13}, \\ c_{23}^2 &= \lambda h_{23}. \end{cases}$$

It is easily proved that the solution exists.

Now put $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_1) = C_1(e_2) = 0.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.4. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\lambda = \mu = 0$. Then

$$\begin{aligned} &[e_1, e_2] = 0, \\ &[e_1, u_1] = u_1, && [e_2, u_1] = -u_2, \\ &[e_1, u_2] = u_2, && [e_2, u_2] = u_1, \\ &[e_1, u_3] = pe_1 + qe_2, && [e_2, u_3] = re_1 + se_2, \end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h})$$

for all $\alpha \in \mathfrak{h}^*$ (Proposition 10, Chapter II). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}u_3, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1 \oplus \mathbb{R}u_2,$$

and

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), & & [u_1, u_2] = 0, \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), & \Rightarrow \quad & [u_1, u_3] = \beta_1 u_1 + \beta_2 u_2, \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), & & [u_2, u_3] = \gamma_1 u_1 + \gamma_2 u_2. \end{aligned}$$

Let us check the Jacobi identity for the triples $(e_i, u_j, u_k), i = 1, 2,\ 1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3):

1. $[e_2, [u_1, u_3]] + [u_1, [u_3, e_2]] + [u_3, [e_2, u_1]] = 0$
$-\beta_1 u_2 + \beta_2 u_1 + r u_1 - s u_2 + \gamma_1 u_1 + \gamma_2 u_2 = 0$
1. $r + \beta_2 + \gamma_1 = 0$
2. $-\beta_1 - s + \gamma_2 = 0$

2. $[e_1, [u_1, u_3]] + [u_1, [u_3, e_1]] + [u_3, [e_1, u_1]] = 0$
$\beta_1 u_1 + \beta_2 u_2 + p u_1 - q u_2 - \beta_1 u_1 - \beta_2 u_2 = 0$
3. $p = 0$
4. $q = 0$

3. $[e_2, [u_2, u_3]] + [u_2, [u_3, e_2]] + [u_3, [e_2, u_2]] = 0$
$-\gamma_1 u_1 + \gamma_2 u_1 + r u_2 + s u_1 - \beta_1 u_1 - \beta_2 u_2 = 0$
5. $\gamma_2 + s - \beta_1 = 0$
6. $-\gamma_1 + r - \beta_2 = 0$

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_i) = e_i, \ i = 1, 2, \\ &\pi(u_1) = u_1, \\ &\pi(u_2) = u_2, \\ &\pi(u_3) = u_3 + \beta_1 e_1 - \beta_2 e_2. \end{aligned}$$

2°. $\lambda \neq 0$ or $\mu \neq 0$. Then

$$\begin{aligned} &[e_1, e_2] = 0, & & \\ &[e_1, u_1] = u_1, & & [e_2, u_1 = -u_2, \\ &[e_1, u_2] = u_2, & & [e_2, u_2] = u_1, \\ &[e_1, u_3] = \mu u_3, & & [e_2, u_3] = \lambda u_3, \end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1 \oplus \mathbb{R}u_2,$$

$$\bar{\mathfrak{g}}^{(\mu)}(\mathfrak{h}) \supset \mathbb{R}u_3,$$

and

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), & & [u_1, u_2] = \alpha_3 u_3 \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(1+\mu)}(\mathfrak{h}), & \Rightarrow \quad & [u_1, u_3] = b_1 e_1 + b_2 e_2 + \beta_1 u_1 + \beta_2 u_2 \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(1+\mu)}(\mathfrak{h}), & & [u_2, u_3] = c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 \end{aligned}$$

Using the Jacobi identity we see that the pair ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	u_2	μu_3
e_2	0	0	$-u_2$	u_1	λu_3
u_1	$-u_1$	u_2	0	$\alpha_3 u_3$	0
u_2	$-u_2$	$-u_1$	$-\alpha_3 u_3$	0	0
u_3	$-\mu u_3$	$-\lambda u_3$	0	0	0 ,

where $\alpha_3(\lambda^2 + (\mu - 2)^2) = 0$.

2.1°. $\alpha_3 = 0$. Then the pair ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) is equivalent to the trivial pair ($\bar{\mathfrak{g}}_1$,$\mathfrak{g}_1$).

2.2°. $\alpha_3 \neq 0$. Then the pair ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) is equivalent to the pair ($\bar{\mathfrak{g}}_2$,$\mathfrak{g}_2$) by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1, \pi(e_2) = e_2, \pi(u_1) = u_1, \pi(u_2) = u_2, \pi(u_3) = \alpha_3 u_3.$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

Thus the proof of the Proposition is complete.

Proposition 2.5. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.5 is trivial.*

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_2	u_1
e_2	0	0	u_1	0	$u_3 + \lambda u_1$
u_1	0	$-u_1$	0	0	0
u_2	$-u_2$	0	0	0	0
u_3	$-u_1$	$-u_3 - \lambda u_1$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & \lambda + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

This proves the Proposition.

Proposition 2.6. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.6 is trivial.*

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	0	u_1+u_3
e_2	0	0	0	u_2	0
u_1	$-u_1$	0	0	0	0
u_2	0	$-u_2$	0	0	0
u_3	$-u_1-u_3$	0	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

This proves the Proposition.

Proposition 2.7. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.7 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1
e_2	$-e_1$	0	0	0	u_3
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	$-u_3$	0	0	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1+e_2
e_2	$-e_1$	0	0	0	u_3
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1-e_2$	$-u_3$	0	0	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1
e_2	$-e_1$	0	0	0	u_3
u_1	0	0	0	u_1	0
u_2	0	0	$-u_1$	0	$-u_3$
u_3	$-u_1$	$-u_3$	0	u_3	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_2.

Lemma. *Any virtual structure q on generalized module 2.7 is equivalent to one of the following:*

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = 0.$$

Proof. Let q be a virtual structure on generalized module 2.7. Without loss of generality it can be assumed that q is primary. Then,

$$C(e_1) = \begin{pmatrix} c_{11}^1 & c_{12}^1 & 0 \\ 0 & 0 & c_{23}^1 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^2 & c_{22}^2 & 0 \end{pmatrix}.$$

Let us check condition (6), Chapter II. Direct calculation shows that

$$C(e_1) = \begin{pmatrix} c_{11}^1 & c_{12}^1 & 0 \\ 0 & 0 & c_{23}^1 \end{pmatrix}, \quad C(e_2) = 0.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ c_{11}^1 & c_{12}^1 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) - A(x)H + HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_1(e_2) = 0.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.7. Then it can be assumed that the corresponding virtual pair is defined by one of the virtual structures determined in the Lemma. Then,

$$\begin{aligned}
&[e_1, e_2] = e_1, & & \\
&[e_1, u_1] = 0, & &[e_2, e_1] = 0, \\
&[e_1, u_2] = 0, & &[e_2, u_2] = 0, \\
&[e_1, u_3] = pe_2 + u_1, & &[e_2, u_3] = u_3.
\end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$

where

$$\begin{aligned}
\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) &= \mathbb{R}e_1, \\
\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) &= \mathbb{R}e_2 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2, \\
\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) &= \mathbb{R}u_3.
\end{aligned}$$

Therefore

$$[u_1, u_2] = a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2,$$
$$[u_1, u_3] = \beta_3 u_3,$$
$$[u_2, u_3] = \gamma_3 u_3.$$

Using the Jacobi identity we see that

$$[u_1, u_2] = \alpha_1 u_2,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = -\alpha_1 u_3,$$

where the coefficients α_1 and p satisfies the equation $\alpha_1 p = 0$.

Consider the following cases:

1°. $\alpha_1 = p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $\alpha_1 = 0,\ p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \mathfrak{g}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = e_2,$$
$$\pi(u_1) = p u_1,$$
$$\pi(u_2) = p u_2,$$
$$\pi(u_3) = p u_3.$$

3°. $\alpha_1 \neq 0,\ p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \mathfrak{g}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = e_2,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = \alpha_1 u_2,$$
$$\pi(u_3) = u_3.$$

Since $\dim \mathcal{Z}(\bar{\mathfrak{g}}_3) = 0$ and $\dim \mathcal{Z}(\bar{\mathfrak{g}}_1) = \dim \mathcal{Z}(\bar{\mathfrak{g}}_2) = 2$, we see that the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ is not equivalent to any of the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ or $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$.

Since $\dim([\mathfrak{g}_1, \bar{\mathfrak{g}}_1] \cap \mathcal{Z}(\bar{\mathfrak{g}}_1)) = 1$ and $\dim([\mathfrak{g}_2, \bar{\mathfrak{g}}_2] \cap \mathcal{Z}(\bar{\mathfrak{g}}_2)) = 0$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

The proof of the Proposition is complete.

Proposition 2.8. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.8 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	λe_1	0	0	u_1
e_2	$-\lambda e_1$	0	0	u_2	λu_3
u_1	0	0	0	0	0
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-\lambda u_3$	0	0	0

2. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	u_2	0
u_1	0	0	0	0	$u_1 + \alpha e_1$
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	0	$-u_1 - \alpha e_1$	0	0

3. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	u_2	0
u_1	0	0	0	0	e_1
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	0	$-e_1$	0	0

4. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	u_2	0
u_1	0	0	0	0	$-e_1$
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	0	e_1	0	0

5. $\lambda = 1$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1
e_2	$-e_1$	0	0	u_2	u_3
u_1	0	0	0	0	u_2
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-u_3$	$-u_2$	0	0

6. $\lambda = -\frac{1}{2}$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-\frac{1}{2}e_1$	0	0	u_1
e_2	$\frac{1}{2}e_1$	0	0	u_2	$-\frac{1}{2}u_3$
u_1	0	0	0	0	0
u_2	0	$-u_2$	0	0	e_1
u_3	$-u_1$	$\frac{1}{2}u_3$	0	$-e_1$	0

7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	λe_1	e_1	0	u_1
e_2	$-\lambda e_1$	0	0	u_2	λu_3
u_1	$-e_1$	0	0	0	u_3
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-\lambda u_3$	$-u_3$	0	0

8. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	u_2	e_1
u_1	0	0	0	0	$u_1 + \alpha e_1$
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-e_1$	$-u_1 - \alpha e_1$	0	0

9. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	u_2	e_1
u_1	0	0	0	0	$-e_1$
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-e_1$	e_1	0	0

10. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	u_2	e_1
u_1	0	0	0	0	0
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-e_1$	0	0	0

11. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	u_2	e_1
u_1	0	0	0	0	e_1
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-e_1$	$-e_1$	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} -\lambda & 0 \\ 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of Lie algebra $\mathfrak{g}$ spanned by e_2.

Lemma. *Any virtual structure q on generalized module 2.8 is equivalent to one of the following:*

a)

$$C_1(e_1) = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = 0;$$

b) $\lambda = 0$

$$C_2(e_1) = \begin{pmatrix} p & 0 & 0 \\ q & 0 & 0 \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & s \end{pmatrix};$$

c) $\lambda = 1$

$$C_3(e_1) = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C_3(e_2) = 0.$$

Proof. Let q be a virtual structure on generalized module 2.8. Without loss of generality it can be assumed that q is primary. Since

$$\mathfrak{g}^{(-\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_1, \quad U^{(0)}(\mathfrak{h}) \supset \mathbb{R}u_1,$$
$$\mathfrak{g}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_2, \quad U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_2,$$
$$U^{(\lambda)}(\mathfrak{h}) \supset \mathbb{R}u_3,$$

we have:

$$C(e_1) = \begin{pmatrix} c^1_{11} & 0 & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{11} & c^2_{12} & c^2_{13} \\ c^2_{21} & 0 & c^2_{23} \end{pmatrix}.$$

Let us check condition (6), Chapter II for e_1, e_2:

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have:

$$\begin{pmatrix} \lambda c^1_{11} & 0 & \lambda c^1_{13} \\ \lambda c^1_{21} & \lambda c^1_{22} & \lambda c^1_{23} \end{pmatrix} = \begin{pmatrix} \lambda c^2_{21} & 0 & \lambda c^2_{23} \\ 0 & 0 & 0 \end{pmatrix} -$$

$$- \begin{pmatrix} \lambda 0 & 0 & c^2_{11} \\ 0 & 0 & c^2_{21} \end{pmatrix} + \begin{pmatrix} \lambda c^1_{11} & 0 & \lambda c^1_{13} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \lambda 0 & 0 & \lambda c^1_{13} \\ 0 & c^1_{22} & \lambda c^1_{23} \end{pmatrix}.$$

We obtain the system of linear equations:

$$\begin{cases} \lambda c^2_{21} = 0 \\ \lambda c^2_{23} - c^2_{11} + \lambda c^1_{13} = 0 \\ \lambda c^1_{21} = 0 \\ \lambda c^1_{22} = c^1_{22} \\ c^2_{21} = 0 \end{cases}$$

So, any virtual structure q has the form:

a) $\lambda \notin \{0, 1\}$

$$C(e_1) = \begin{pmatrix} c^1_{11} & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} \lambda(c^1_{13} + c^2_{23}) & c^2_{12} & c^2_{13} \\ 0 & 0 & c^2_{23} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} -c^1_{13} - c^2_{23} & -\frac{c^2_{12}}{\lambda+1} & -\frac{c^2_{13}}{2\lambda} \\ -c^1_{23} & 0 & -\frac{c^2_{23}}{\lambda} \end{pmatrix}.$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_1) = \begin{pmatrix} c^1_{11} + \lambda c^1_{23} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = 0.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

b) $\lambda = 0$

$$C(e_1) = \begin{pmatrix} c^1_{11} & 0 & c^1_{13} \\ c^1_{21} & 0 & c^1_{23} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} -c^1_{13} & 0 & 0 \\ -c^1_{23} & 0 & 0 \end{pmatrix}.$$

Now put $C_2(x) = C(x) + A(x)H - HB(x)$. Then

$$C_2(e_1) = \begin{pmatrix} c^1_{11} & 0 & 0 \\ c^1_{21} & 0 & 0 \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_2 are equivalent.

c) $\lambda = 1$

$$C(e_1) = \begin{pmatrix} c^1_{11} & 0 & 0 \\ 0 & c^1_{22} & c^1_{23} \end{pmatrix}, \quad C(e_2) = 0.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ -c^1_{23} & 0 & 0 \end{pmatrix}.$$

Now put $C_3(x) = C(x) + A(x)H - HB(x)$. Then

$$C_3(e_1) = \begin{pmatrix} c^1_{11} + c^1_{23} & 0 & 0 \\ 0 & c^1_{22} & 0 \end{pmatrix}, \quad C_3(e_2) = 0.$$

By corollary 2, Chapter II the virtual structures C and C_3 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.8. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\lambda \notin \{0, 1\}$. Then

$$\begin{aligned} &[e_1, e_2] = \lambda e_1, \\ &[e_1, u_1] = pe_1, \quad [e_2, u_1] = 0, \\ &[e_1, u_2] = 0, \quad\ \ [e_2, u_2] = u_2, \\ &[e_1, u_3] = u_1, \quad [e_2, u_3] = \lambda u_3. \end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Thus

$$\begin{aligned} &\bar{\mathfrak{g}}^{(-\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_1, && \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_2, \\ &\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_2 \oplus \mathbb{R}u_1, && \bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}) \supset \mathbb{R}u_3, \end{aligned}$$

and

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), && & [u_1, u_2] &= a_1 e_1 + \alpha_2 u_2 \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}), && \Rightarrow & [u_1, u_3] &= \beta_3 u_3 \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(\lambda+1)}(\mathfrak{h}), && & [u_2, u_3] &= c_1 e_1 + c_2 e_2 + \gamma_1 u_1. \end{aligned}$$

Let us check the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1,2$, $1 \leqslant j < k \leqslant 3$ and (u_1, u_2, u_3):

1. $[e_1,[u_1,u_2]] + [u_1,[u_2,e_1]] + [u_2,[e_1,u_1]] = 0$
$0 = 0$

2. $[e_2,[u_1,u_3]] + [u_1,[u_3,e_2]] + [u_3,[e_2,u_1]] = 0$
$\lambda\beta_3 u_3 - \lambda\beta_3 u_3 = 0$

3. $[e_2,[u_1,u_2]] + [u_1,[u_2,e_2]] + [u_2,[e_2,u_1]] = 0$
$-\lambda a_1 e_1 + \alpha_2 u_2 - a_1 e_1 - \alpha_2 u_2 = 0$
1. $(\lambda+1)a_1 = 0$

4. $[e_1,[u_1,u_3]] + [u_1,[u_3,e_1]] + [u_3,[e_1,u_1]] = 0$
$\beta_3 u_1 - p u_1 = 0$
2. $\beta_3 = p$

5. $[e_1,[u_2,u_3]] + [u_2,[u_3,e_1]] + [u_3,[e_1,u_2]] = 0$
$c_2\lambda e_1 + \gamma_1 p e_1 + \alpha_2 u_2 + a_1 e_1 = 0$
3. $\alpha_2 = 0$
4. $a_1 + c_2\lambda + \gamma_1 p = 0$

6. $[e_2,[u_2,u_3]] + [u_2,[u_3,e_2]] + [u_3,[e_2,u_2]] = 0$
$-c_1\lambda e_1 - \lambda(c_1 e_1 + c_2 e_2 + \gamma_1 u_1) - c_1 e_1 - c_2 e_2 - \gamma_1 u_1 = 0$
5. $(2\lambda+1)c_1 = 0$
6. $(\lambda+1)c_2 = 0$
7. $(\lambda+1)\gamma_1 = 0$

7. $[u_1,[u_2,u_3]] + [u_2,[u_3,u_1]] + [u_3,[u_1,u_2]] = 0$
$-c_1 p e_1 - p(c_1 e_1 + c_2 e_2 + \gamma_1 u_1) - a_1 u_1 = 0$
8. $pc_1 = 0$
9. $pc_2 = 0$
10. $\gamma_1 p + a_1 = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:
$\lambda \neq -1$, $\lambda \neq -\frac{1}{2}$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	λe_1	pe_1	0	u_1
e_2	$-\lambda e_1$	0	0	u_2	λu_3
u_1	$-pe_1$	0	0	0	pu_3
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-\lambda u_3$	$-pu_3$	0	0

$\lambda = -\frac{1}{2}$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$-\frac{1}{2}e_1$	pe_1	0	u_1
e_2	$\frac{1}{2}e_1$	0	0	u_2	$-\frac{1}{2}u_3$
u_1	$-pe_1$	0	0	0	pu_3
u_2	0	$-u_2$	0	0	$c_1 e_1$
u_3	$-u_1$	$\frac{1}{2}u_3$	$-pu_3$	$-c_1 e_1$	0

, $c_1 p = 0$

$\lambda = -1$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	pe_1	0	u_1
e_2	e_1	0	0	u_2	$-u_3$
u_1	$-pe_1$	0	0	a_1e_1	pu_3
u_2	0	$-u_2$	$-a_1e_1$	0	γ_1u_1
u_3	$-u_1$	u_3	$-pu_3$	$-\gamma_1u_1$	0

, $a_1 + \gamma_1 p = 0$

Consider the following cases:

1.1°. $\lambda \notin \{-1, -\frac{1}{2}, 0, 1\}$.

1.1.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

1.1.2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1,2,$$
$$\pi(u_1) = \frac{1}{p}u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \frac{1}{p}u_3.$$

1.2°. $\lambda = -\frac{1}{2}$.

1.2.1°. $c_1 = 0,\ p = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

1.2.2°. $c_1 = 0,\ p \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1,2,$$
$$\pi(u_1) = \frac{1}{p}u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \frac{1}{p}u_3.$$

1.2.3°. $c_1 \neq 0$, $p = 0$. The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6,\mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1,2,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = \frac{1}{c_1}u_2,$$
$$\pi(u_3) = u_3.$$

1.3°. $\lambda = -1$.

1.3.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1,2,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2 - \gamma_1 e_1,$$
$$\pi(u_3) = u_3.$$

1.3.2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1,2,$$
$$\pi(u_1) = \frac{1}{p}u_1,$$
$$\pi(u_2) = u_2 - \gamma_1 e_1,$$
$$\pi(u_3) = \frac{1}{p}u_3.$$

2°. $\lambda = 0$. Then

$$\begin{array}{ll} [e_1,e_2] = 0 & \\ [e_1,u_1] = pe_1 + qe_2 & [e_2,u_1] = 0 \\ [e_1,u_2] = 0 & [e_2,u_2] = u_2 \\ [e_1,u_3] = u_1 & [e_2,u_3] = re_1 + se_2 \end{array}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_3, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_2,$$

and

$$\begin{array}{lcl} [u_1,u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), & & [u_1,u_2] = \alpha_2 u_2, \\ [u_1,u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), & \Rightarrow & [u_1,u_3] = b_1e_1 + b_2e_2 + \beta_1u_1 + \beta_3u_3, \\ [u_2,u_3] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), & & [u_2,u_3] = \gamma_2u_2. \end{array}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	pe_1	0	u_1
e_2	0	0	0	u_2	re_1
u_1	$-pe_1$	0	0	0	$b_1e_1+b_2e_2+\beta_1u_1+pu_3$
u_2	0	$-u_2$	0	0	γ_2u_2
u_3	$-u_1$	$-re_1$	$-b_1e_1-b_2e_2-\beta_1u_1-pu_3$	$-\gamma_2u_2$	0

where $\beta_1p=0$, $rp=0$, $b_2=p\gamma_2$.

Consider the following cases.

2.1°. $p = 0$.

Then the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, such that

$$\pi(e_i) = e_i,\ i = 1,2,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = u_3 + \gamma_2 e_2,$$

establishes the equivalence of the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}',\mathfrak{g}')$, where latter has the form:

$[,]$	e_1	e_2	u_1	u_2	u_3
e_1	0		0	0	u_1
e_2	0	0	0	u_2	re_1
u_1	0	0	0	0	$b_1e_1+\beta_1u_1$
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-re_1$	$-b_1e_1-\beta_1u_1$	0	0

2.1.1°. $r=\beta_1=b_1=0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

2.1.2°. $r=\beta_1=0$, $b_1\neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)_{i=3,4}$ by means of the mapping $\pi:\bar{\mathfrak{g}}_i\to\bar{\mathfrak{g}}$, where

$$\pi(e_i)=e_i,\ i=1,2,$$
$$\pi(u_j)=\frac{1}{\sqrt{|b_1|}}u_j,\ j=1,2,3,$$

(if $b_1>0$ then $i=3$, if $b_1<0$ then $i=4$).

2.1.3°. $r=0$, $\beta_1\neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi:\bar{\mathfrak{g}}_2\to\bar{\mathfrak{g}}$, where

$$\pi(e_i)=e_i,\ i=1,2,$$
$$\pi(u_j)=\frac{1}{\beta_1}u_j,\ j=1,2,3.$$

2.1.4°. $r\neq 0$, $\beta_1=b_1=0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)_{i=9,10}$ by means of the mapping $\pi:\bar{\mathfrak{g}}_i\to\bar{\mathfrak{g}}$, where

$$\pi(e_1)=\frac{r}{\sqrt{|b_1|}}e_1,$$
$$\pi(e_2)=e_2,$$
$$\pi(u_1)=\frac{r}{|b_1|}u_1,$$
$$\pi(u_2)=u_2,$$
$$\pi(u_3)=\frac{1}{\sqrt{|b_1|}}u_3,$$

(if $b_1>0$ then $i=9$, if $b_1<0$ then $i=10$).

2.1.5°. $r\neq 0$, $\beta_1=0$, $b_1\neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{11},\mathfrak{g}_{11})$ by means of the mapping $\pi:\bar{\mathfrak{g}}_{11}\to\bar{\mathfrak{g}}$, where

$$\pi(e_i)=e_i,\ i=1,2,$$
$$\pi(u_j)=\frac{1}{r}u_j,\ j=1,2,3.$$

2.1.6°. $r \neq 0$, $\beta_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_8,\mathfrak{g}_8)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_8 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= \frac{r}{\beta_1}e_1,\\
\pi(e_2) &= e_2,\\
\pi(u_1) &= \frac{r}{\beta_1^2}u_1,\\
\pi(u_2) &= u_2,\\
\pi(u_3) &= \frac{1}{\beta_1}u_3.
\end{aligned}$$

2.2°. $p \neq 0$. Then $\beta_1 = r = 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_i) &= e_i, \ i = 1,2,\\
\pi(u_1) &= \frac{1}{p}u_1,\\
\pi(u_2) &= u_2,\\
\pi(u_3) &= \frac{1}{p}(u_3 + \gamma_2 e_2).
\end{aligned}$$

3°. $\lambda = 1$. Then

$$\begin{array}{ll}
[e_1, e_2] = e_1 & \\
[e_1, u_1] = pe_1 & [e_2, u_1] = 0\\
[e_1, u_2] = qe_2 & [e_2, u_2] = u_2\\
[e_1, u_3] = u_1 & [e_2, u_3] = u_3
\end{array}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_1, \quad \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}u_1,$$

$$\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_2 \oplus \mathbb{R}u_3,$$

and

$$\begin{array}{lcl}
[u_1, u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), & & [u_1, u_2] = \alpha_2 u_2 + \alpha_3 u_3,\\
[u_1, u_3] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), & \Rightarrow & [u_1, u_3] = \beta_2 u_2 + \beta_3 u_3,\\
[u_2, u_3] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), & & [u_2, u_3] = 0.
\end{array}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	pe_1	0	u_1
e_2	$-e_1$	0	0	u_2	u_3
u_1	$-pe_1$	0	0	0	$\gamma_2 u_2 + pu_3$
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-u_3$	$-\gamma_2 u_2 - pu_3$	0	0

3.1°. $p = 0$.

3.1.1°. $\gamma_2 = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

3.1.2°. $\gamma_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5,\mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2,\\
&\pi(u_1) = u_1,\\
&\pi(u_2) = \gamma_2 u_2,\\
&\pi(u_3) = u_3.
\end{aligned}$$

3.2°. $p \neq 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2,\\
&\pi(u_1) = \frac{1}{p} u_1,\\
&\pi(u_2) = u_2,\\
&\pi(u_3) = \frac{1}{p^2}(p u_3 + \gamma_2 u_2).
\end{aligned}$$

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_7$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ are not equivalent.

Consider homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(2,\mathbb{R}), i = 1, 2, 3, 4, 8, 9, 10, 11$, where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}|_{\mathcal{D}\bar{\mathfrak{g}}_i} x$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, 2, 3, 4$, are not conjugated, we see that the pairs $(\bar{\mathfrak{g}}_i,\ \mathfrak{g}_i)$, $i = 1, 2, 3, 4$ are not equivalent. Similarly, the pairs $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)$, $i = 8, 9, 10, 11$, are not equivalent to each other.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_6$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ are not equivalent.

Since $\dim \mathcal{Z}(\bar{\mathfrak{g}}_1) \neq \dim \mathcal{Z}(\bar{\mathfrak{g}}_5)$, we see that the pairs $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_5,\mathfrak{g}_5)$ are not equivalent.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_i \neq \dim \mathcal{D}\bar{\mathfrak{g}}_j$, $i = 1, 2, 3, 4$, $j = 8, 9, 10, 11$, we see that the pairs $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)$ and $(\bar{\mathfrak{g}}_j,\mathfrak{g}_j)$ are not equivalent.

Thus the proof of the Proposition is complete.

Proposition 2.9. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.9 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1-\mu)e_2$	u_1	λu_2	μu_3
e_2	$(\mu-1)e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0
u_3	$-\mu u_3$	$-u_1$	0	0	0

2. $\lambda = \mu + 1$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1-\mu)e_2$	u_1	$(\mu+1)u_2$	μu_3
e_2	$(\mu-1)e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	u_2
u_2	$-(\mu+1)u_2$	0	0	0	0
u_3	$-\mu u_3$	$-u_1$	$-u_2$	0	0

3. $\lambda = 1 - 2\mu$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1-\mu)e_2$	u_1	$(1-2\mu)u_2$	μu_3
e_2	$(\mu-1)e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	$(2\mu-1)u_2$	0	0	0	e_2
u_3	$-\mu u_3$	$-u_1$	0	$-e_2$	0

4. $\lambda = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1-\mu)e_2$	u_1	0	μu_3
e_2	$(\mu-1)e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	u_1	0
u_2	0	0	$-u_1$	0	$-u_3$
u_3	$-\mu u_3$	$-u_1$	0	u_3	0 ,

$\mu \neq 1$

5. $\mu = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	λu_2	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	e_2
u_2	$-\lambda u_2$	0	0	0	αu_2
u_3	0	$-u_1$	$-e_2$	$-\alpha u_2$	0

$\alpha \geqslant 0$

6. $\mu = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	λu_2	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	$-e_2$
u_2	$-\lambda u_2$	0	0	0	αu_2
u_3	0	$-u_1$	e_2	$-\alpha u_2$	0

$\alpha \geqslant 0$

7. $\mu = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	λu_2	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	u_2
u_3	0	$-u_1$	0	$-u_2$	0

8. $\mu = 1/2$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	u_1	λu_2	$\frac{1}{2}u_3 + e_2$
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0
u_3	$-\frac{1}{2}u_3 - e_2$	$-u_1$	0	0	0

9. $\lambda = 0,\ \mu = 1/2$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	u_1	0	$\frac{1}{2}u_3 + e_2$
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	e_2
u_3	$-\frac{1}{2}u_3 - e_2$	$-u_1$	0	$-e_2$	0

10. $\lambda = 3/2,\ \mu = 1/2$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	u_1	$\frac{3}{2}u_2$	$\frac{1}{2}u_3 + e_2$
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	u_2
u_2	$-\frac{3}{2}u_2$	0	0	0	0
u_3	$-\frac{1}{2}u_3 - e_2$	$-u_1$	$-u_2$	0	0

11. $\lambda = 0,\ \mu = 1$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	0	u_3
e_2	0	0	0	e_2	u_1
u_1	$-u_1$	0	0	0	0
u_2	0	$-e_2$	0	0	u_3
u_3	$-u_3$	$-u_1$	0	$-u_3$	0

12. $\lambda = -2,\ \mu = 2$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_2$	u_1	$-2u_2$	$2u_3$
e_2	e_2	0	0	0	u_1
u_1	$-u_1$	0	0	e_2	0
u_2	$2u_2$	0	$-e_2$	0	$-e_1$
u_3	$-2u_3$	$-u_1$	0	e_1	0

13. $\lambda = 1,\ \mu = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	u_2	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	u_1
u_2	$-u_2$	0	0	0	e_2
u_3	0	$-u_1$	$-u_1$	$-e_2$	0

14. $\lambda = 1,\ \mu = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	u_2	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	u_2
u_2	$-u_2$	0	0	0	u_2
u_3	0	$-u_1$	$-u_2$	$-u_2$	0

15. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	u_2	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	$e_2 + u_2$
u_2	$-u_2$	0	0	0	αu_2
u_3	0	$-u_1$	$-e_2 - u_2$	$-\alpha u_2$	0

$0 \leqslant \alpha < 1$

16. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	u_2	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	$-e_2 + u_2$
u_2	$-u_2$	0	0	0	αu_2
u_3	0	$-u_1$	$e_2 - u_2$	$-\alpha u_2$	0

$\alpha \geqslant 0$

17. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$2e_2$	u_1	$-2u_2$	$-u_3$
e_2	$-2e_2$	0	0	e_1	u_1
u_1	$-u_1$	0	0	$-u_3$	0
u_2	$2u_2$	$-e_1$	u_3	0	0
u_3	u_3	$-u_1$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1-\mu \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 \\ \mu - 1 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on the generalized module 2.9 is equivalent to one of the following:*

a) $\mu \notin \{0, \frac{1}{2}, 2\}$, $\lambda \neq \pm(1-\mu)$

$$C_1(e_1) = 0, \quad C_1(e_2) = 0;$$

b) $\mu \notin \{0, \frac{1}{2}, 1, 2\}$, $\lambda = 1 - \mu$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}, \quad C_2(e_2) = 0;$$

c) $\mu \notin \{0, \frac{1}{2}, 1, 2\}$, $\lambda = \mu - 1$

$$C_3(e_1) = 0, \quad C_3(e_2) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

d) $\mu = 0$, $\lambda \neq \pm 1$

$$C_4(e_1) = \begin{pmatrix} 0 & 0 & p \\ -p & 0 & 0 \end{pmatrix}, \quad C_4(e_2) = 0;$$

e) $\mu = 0$, $\lambda = 1$

$$C_5(e_1) = \begin{pmatrix} 0 & 0 & p \\ -p & q & 0 \end{pmatrix}, \quad C_5(e_2) = 0;$$

f) $\mu = 0$, $\lambda = -1$

$$C_6(e_1) = \begin{pmatrix} 0 & 0 & p \\ -p & 0 & 0 \end{pmatrix}, \quad C_6(e_2) = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

g) $\mu = \frac{1}{2}$, $\lambda \neq \pm\frac{1}{2}$

$$C_7(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_7(e_2) = 0;$$

h) $\mu = \frac{1}{2}$, $\lambda = \frac{1}{2}$

$$C_8(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & p \end{pmatrix}, \quad C_8(e_2) = 0;$$

i) $\mu = \frac{1}{2}$, $\lambda = -\frac{1}{2}$

$$C_9(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_9(e_2) = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

j) $\mu = 1$, $\lambda = 0$

$$C_{10}(e_1) = \begin{pmatrix} 0 & p & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C_{10}(e_2) = \begin{pmatrix} 0 & r & 0 \\ 0 & s & 0 \end{pmatrix};$$

k) $\mu = 2$, $\lambda \neq \pm 1$

$$C_{11}(e_1) = 0, \quad C_{11}(e_2) = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

l) $\mu = 2$, $\lambda = 1$

$$C_{12}(e_1) = 0, \quad C_{12}(e_2) = \begin{pmatrix} p & q & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

m) $\mu = 2$, $\lambda = -1$

$$C_{13}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C_{13}(e_2) = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Put

$$C(e_i) = \begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} \\ c^i_{21} & c^i_{22} & c^i_{23} \end{pmatrix}, \quad i = 1, 2.$$

By Proposition 9, Chapter II, without loss of generality it can be assumed that q is primary. Then we have:

$$\begin{aligned} &\mathfrak{g}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_1, && U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1, \\ &\mathfrak{g}^{(1-\mu)}(\mathfrak{h}) \supset \mathbb{R}e_2, && U^{(\lambda)}(\mathfrak{h}) \supset \mathbb{R}u_2, \\ & && U^{(\mu)}(\mathfrak{h}) \supset \mathbb{R}u_3, \end{aligned} \tag{*}$$

and $c^1_{11} = c^2_{13} = c^2_{21} = 0$. Checking condition (6), Chapter II, for e_1, e_2, we obtain:

$$\begin{cases} c^1_{21} = (\mu - 1)c^1_{13}, \\ (1-\mu)c^1_{12} = \lambda c^2_{22}. \end{cases} \tag{**}$$

Consider the following cases:

1°. $\mu \notin \{0, \frac{1}{2}, 2\}$, $\lambda \neq \pm(1-\mu)$. Then, from (*) and (**) it follows that

$$C(e_1) = 0, \qquad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad \lambda q = 0.$$

2°. $\mu \neq \{0, \frac{1}{2}, 1, 2\}$, $\lambda = 1 - \mu$. Then from (*) it follows that

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c^1_{22} & 0 \end{pmatrix}, \qquad C(e_2) = 0.$$

3°. $\mu \neq \{0, \frac{1}{2}, 1, 2\}$, $\lambda = \mu - 1$. Then from (*) it follows that

$$C(e_1) = 0, \qquad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}.$$

Similarly we obtain the other results of the Lemma.

Thus it can be assumed that the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures q determined in the Lemma. Put

$$\begin{aligned} [u_1, u_2] &= a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\ [u_1, u_3] &= b_1 e_1 + b_2 e_2 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\ [u_2, u_3] &= c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3. \end{aligned}$$

For example, consider the following cases:

1°. $\mu \notin \{0, \frac{1}{2}, 2\}$, $\lambda \neq \pm(1-\mu)$. Then

$$\begin{aligned} &[e_1, e_2] = (1-\mu)e_2, && \\ &[e_1, u_1] = u_1, && [e_2, u_1] = 0, \\ &[e_1, u_2] = \lambda u_2, && [e_2, u_2] = 0, \\ &[e_1, u_3] = \mu u_3, && [e_2, u_3] = u_1. \end{aligned}$$

Let us check the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1, 2$, $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_1, [u_2, e_1]] + [u_2, [e_1, u_1]] = 0$
$(1-\mu)a_2 e_2 + \alpha_1 u_1 + \lambda\alpha_2 u_2 + \mu\alpha_3 u_3 - (\lambda+1)[u_1, u_2] = 0$

1. $(\lambda+1)a_1 = 0$
2. $(\mu+\lambda)a_2 = 0$
3. $\lambda\alpha_1 = 0$
4. $\alpha_2 = 0$
5. $\alpha_3 = 0$

2. $[e_2, [u_1, u_2]] + [u_1, [u_2, e_2]] + [u_2, [e_2, u_1]] = 0$
$(\mu-1)a_1 e_2 = 0$

6. $(\mu-1)a_1 = 0$

3. $[e_1, [u_1, u_3]] + [u_1, [u_3, e_1]] + [u_3, [e_1, u_1]] = 0$
$(1-\mu)b_2 e_2 + \beta_1 u_1 + \lambda\beta_2 u_2 + \mu\beta_3 u_3 - (\mu+1)[u_1, u_3] = 0$

7. $(\mu+1)b_1 = 0$
8. $b_2 = 0$
9. $\beta_1 = 0$
10. $(\lambda-\mu-1)\beta_2 = 0$
11. $\beta_3 = 0$

4. $[e_2, [u_1, u_3]] + [u_1, [u_3, e_2]] + [u_3, [e_2, u_1]] = 0$
$(\mu-1)b_1 e_2 = 0$

12. $b_1 = 0$

5. $[e_1, [u_2, u_3]] + [u_2, [u_3, e_1]] + [u_3, [e_1, u_2]] = 0$
$(1-\mu)c_2 e_2 + \gamma_1 u_1 + \lambda\gamma_2 u_2 + \mu\gamma_3 u_3 - (\lambda+\mu)[u_2, u_3] = 0$

13. $(\lambda+\mu)c_1 = 0$
14. $(1-\lambda-2\mu)c_2 = 0$
15. $\gamma_1 = 0$
16. $\gamma_2 = 0$
17. $\lambda\gamma_3 = 0$

6. $[e_2, [u_2, u_3]] + [u_2, [u_3, e_2]] + [u_3, [e_2, u_2]] = 0$
$(\mu-1)c_1 e_2 + \gamma_3 u_1 + a_2 e_2 + a_1 e_1 + \alpha_1 u_1 = 0$

18. $a_1 = 0$
19. $a_2 + (\mu-1)c_1 = 0$

20. $\gamma_3 + \alpha_1 = 0$

7. $[u_1, [u_2, u_3]] + [u_2, [u_3, u_1]] + [u_3, [u_1, u_2]] = 0$
$-c_1 e_1 + \gamma_3 \beta_2 u_2 - a_2 u_1 - \alpha_1 \beta_2 u_2 = 0$
21. $c_1 + a_2 = 0$
22. $\beta_2(\gamma_3 - \alpha_1) = 0$
Finally, we have

$\lambda = 0$	$\lambda = \mu + 1$	$\lambda = 1 - 2\mu$	$\lambda \neq \mu + 1$, $\lambda \neq 1 - 2\mu$
$[u_1, u_2] = \alpha_1 u_1,$	$[u_1, u_2] = 0,$	$[u_1, u_2] = 0,$	$[u_1, u_2] = 0,$
$[u_1, u_3] = 0,$	$[u_1, u_3] = \beta_2 u_2,$	$[u_1, u_3] = 0,$	$[u_1, u_3] = 0,$
$[u_2, u_3] = -\alpha_1 u_3,$	$[u_2, u_3] = 0,$	$[u_2, u_3] = c_2 e_2,$	$[u_2, u_3] = 0.$

Consider the following cases:

1.1°. $\lambda = 0$.

1.1.1°. $\alpha_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

1.1.2°. $\alpha_1 \neq 0$. If $\mu \neq 1$, then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mappings $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_i) = e_i, \ i = 1, 2, \\ &\pi(u_1) = u_1, \\ &\pi(u_2) = \alpha_1 u_2, \\ &\pi(u_3) = u_3, \end{aligned}$$

and, in the case of $\mu = 1$, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair by means of the mapping $\pi_{\mu=1} : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi_{\mu=1}(e_i) = e_i, \ i = 1, 2, \\ &\pi_{\mu=1}(u_1) = u_1, \\ &\pi_{\mu=1}(u_2) = u_2 - \alpha_1 e_1, \\ &\pi_{\mu=1}(u_3) = u_3. \end{aligned}$$

1.2°. $\lambda = 1 + \mu$.

1.2.1°. $\beta_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.2.2°. $\beta_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = \beta_2 e_2, \\ &\pi(u_1) = \beta_2 u_1, \\ &\pi(u_2) = u_2, \\ &\pi(u_3) = u_3. \end{aligned}$$

1.3°. $\lambda = 1 - 2\mu$.

1.3.1°. $c_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.3.2°. $c_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2,\\
&\pi(u_1) = u_1,\\
&\pi(u_2) = c_2 u_2,\\
&\pi(u_3) = u_3.
\end{aligned}$$

1.4°. $\lambda \neq 1 + \mu$, $\lambda \neq 1 - 2\mu$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

2°. $\mu = 0$, $\lambda = 1$. Then

$$\begin{aligned}
&[e_1, e_2] = e_2, \\
&[e_1, u_1] = u_1 - pe_2, \quad && [e_2, u_1] = 0,\\
&[e_1, u_2] = qe_2 + u_2, \quad && [e_2, u_2] = 0,\\
&[e_1, u_3] = pe_1, \quad && [e_2, u_3] = u_1
\end{aligned}$$

and

$$\begin{aligned}
&[u_1, u_2] = 0,\\
&[u_1, u_3] = b_2 e_2 + \beta_1 u_1 + \beta_2 u_2,\\
&[u_2, u_3] = c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2.
\end{aligned}$$

Let us check the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1, 2$, $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_3]] + [u_1, [u_3, e_1]] + [u_3, [e_1, u_1]] = 0$
$b_2 e_2 + \beta_1(u_1 - pe_2) + \beta_2(qe_2 + u_2) + p(u_1 - pe_2) + pu_1 - [u_1, u_3] = 0$
1. $p = 0$
2. $q\beta_2 = 0$

5. $[e_1, [u_2, u_3]] + [u_2, [u_3, e_1]] + [u_3, [e_1, u_2]] = 0$
$c_2 e_2 + \gamma_1 u_1 + \gamma_2(qe_2 + u_2) - qu_1 - [u_2, u_3] = 0$
3. $q = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	u_2	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	$b_2e_2+\beta_1u_1+\beta_2u_2$
u_2	$-u_2$	0	0	0	$c_2e_2+\gamma_1u_1+\gamma_2u_2$
u_3	0	$-u_1$	$-b_2e_2-\beta_1u_1-\beta_2u_2$	$-c_2e_2-\gamma_1u_1-\gamma_2u_2$	0

Suppose that

$$V = \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}) \text{ and } \mathfrak{a} = \{\mathrm{ad}_V x \mid x \in \bar{\mathfrak{g}}\}.$$

Then $V = \mathbb{R}e_2 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2$. Since

$$\mathfrak{a} = \mathbb{R}(\operatorname{ad}_V e_1) \oplus \mathbb{R}(\operatorname{ad}_V u_3) = \mathbb{R}(\operatorname{id}_V) \oplus \mathbb{R}(\operatorname{ad}_V u_3),$$

we see that $\mathfrak{a}$ is a two-dimensional subalgebra of $\mathfrak{gl}(V)$ that contains the identity mapping.

Let $W = V \cap \mathfrak{g} = \mathbb{R}e_2$. The Lie algebra $\bar{\mathfrak{g}}$ can be identified with the Lie algebra $\mathfrak{a} \leftthreetimes V$, and $\mathfrak{g}$ can be identified with the vector space $\mathbb{R}(\operatorname{id}_V) \times W$. Also, $\mathfrak{a}(W) \neq W$, that is $\operatorname{ad} u_3(e_2) \not\subset \mathbb{R}e_2$.

Conversely, suppose $V = \mathbb{R}^3$, $\mathfrak{a}$ is a subalgebra of $\mathfrak{gl}(V)$ such that the identity mapping belongs to $\mathfrak{a}$. Let W be a one-dimensional subspace of V and $\mathfrak{a}(W) \neq W$. Putting

$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes V, \quad \mathfrak{g} = \mathbb{R}(\operatorname{id}_V) \times W,$$

we obtain the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.9 ($\lambda = 1, \mu = 0$).

Therefore, there is a one-to-one correspondence between the set of desired pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and the set of pairs $(\mathfrak{a}, W)$, where $\mathfrak{a}$ is a two-dimensional subalgebra of $\mathfrak{gl}(V)$ such that $\operatorname{id}_V \in \mathfrak{a}$, W is a one-dimensional subspace of V such that $\mathfrak{a}(W) \neq W$.

Statement 1. *Suppose $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are subalgebras of $\mathfrak{gl}(V)$. Then the Lie algebras $\bar{\mathfrak{g}}_1 = \mathfrak{a}_1 \leftthreetimes V$ and $\bar{\mathfrak{g}}_2 = \mathfrak{a}_2 \leftthreetimes V$ are isomorphic if and only if there exists an endomorphism $\varphi \in \mathrm{GL}(V)$ such that*

$$\mathfrak{a}_2 = \varphi \mathfrak{a}_1 \varphi^{-1}.$$

Proof. Indeed, suppose there exists $\varphi \in \mathrm{GL}(V)$ such that

$$\mathfrak{a}_2 = \varphi \mathfrak{a}_1 \varphi^{-1}.$$

Consider the mapping $f : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_2$ defined by

$$f(x, v) = (\varphi x \varphi^{-1}, \varphi(v))$$

for $x \in \mathfrak{a}_1$, $v \in V$. It is easy to see that f is an isomorphism.

Statement 2. *Let*

$$\bar{\mathfrak{g}}_1 = \mathfrak{a}_1 \leftthreetimes V, \quad \bar{\mathfrak{g}}_2 = \mathfrak{a}_2 \leftthreetimes V, \quad \mathfrak{g}_1 = \mathbb{R}(\operatorname{id}_V) \times W_1, \quad \textit{and} \quad \mathfrak{g}_2 = \mathbb{R}(\operatorname{id}_V) \times W_2,$$

where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are subalgebras of $\mathfrak{gl}(V)$, W_1 and W_2 are one-dimensional subspaces of V. Then a necessary and sufficient condition for the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ to be equivalent is that there exists $\varphi \in \mathrm{GL}(V)$ such that

$$\mathfrak{a}_2 = \varphi \mathfrak{a}_1 \varphi^{-1} \text{ and } \varphi(W_1) = W_2;$$

in other words, the group $\mathrm{GL}(V)$ acts on the pairs $(\mathfrak{a}, W)$:

$$\varphi : (\mathfrak{a}, W) \mapsto (\varphi \mathfrak{a} \varphi^{-1}, \varphi(W)).$$

Proof. It immediately follows from the previous statement.

Let us classify (up to just determined transformations) all pairs $(\mathfrak{a}, W)$. We have

$$\operatorname{ad} e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \operatorname{ad} u_3 = \begin{pmatrix} 0 & b_2 & c_2 \\ 1 & \beta_1 & \gamma_1 \\ 0 & \beta_2 & \gamma_2 \end{pmatrix}.$$

Up to conjugation and choice of λ ($\lambda \in \mathbb{R}$) the matrix of the endomorphism $\operatorname{ad}_V(u_3 + \lambda e_1)$ has one of the following forms:

$$i)\ X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \alpha \end{pmatrix},\ 0 \leqslant \alpha < 1; \quad ii)\ X_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix},\ \alpha \in \mathbb{R}_+;$$

$$iii)\ X_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad iv)\ X_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad v)\ X_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By definition

$$F_i = \{A \in \mathrm{GL}(3, \mathbb{R}) \mid AX_iA^{-1} = X_i\}, \quad i = 1, \ldots, 5.$$

Up to action of endomorphisms φ such that the matrix of φ belongs to F_i, one-dimensional subspaces $W \subset V$ have the form:

$$i)\ x_1^1 = \mathbb{R}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } x_1^2 = \mathbb{R}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for } X_1;$$

$$ii)\ x_2^1 = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } x_2^2 = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ for } X_2;$$

$$iii)\ x_3^1 = \mathbb{R}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ for } X_3;$$

$$iv)\ x_4^1 = \mathbb{R}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ x_4^2 = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } x_4^3 = \mathbb{R}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ for } X_4;$$

$$v)\ x_5^1 = \mathbb{R}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } x_5^2 = \mathbb{R}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ for } X_5.$$

It remains to write out the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ corresponding to the pair $(\mathfrak{a}, W)$.
Suppose

$$\mathfrak{a} = \mathbb{R}X_1 + \lambda E_3 = \mathbb{R}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} + \lambda E_3, \text{ where } 0 \leqslant \alpha < 1,\ \lambda \in \mathbb{R},$$

$$W = x_1^2 = \mathbb{R}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\operatorname{ad} u_3(e_2) = \begin{pmatrix} 1 \\ -1 \\ \alpha \end{pmatrix}.$$

Put $u_1 = \operatorname{ad} u_3(e_2)$, then

$$\operatorname{ad} u_3(u_1) = \begin{pmatrix} 1 \\ 1 \\ \alpha^2 \end{pmatrix} = e_2 + \begin{pmatrix} 0 \\ 0 \\ \alpha^2 - 1 \end{pmatrix}.$$

Since the vectors e_2, u_1, u_2 are linearly independent, put

$$u_2 = \begin{pmatrix} 0 \\ 0 \\ \alpha^2 - 1 \end{pmatrix}.$$

Then

$$\operatorname{ad} u_3(u_2) = \alpha \begin{pmatrix} 0 \\ 0 \\ \alpha^2 - 1 \end{pmatrix} = \alpha u_2.$$

Finally,

$$\begin{aligned} [e_2, u_3] &= u_1, \\ [u_1, u_3] &= e_2 + u_2, \\ [u_2, u_3] &= \alpha u_2, \end{aligned}$$

where $0 \leqslant \alpha < 1$. So, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{15}, \mathfrak{g}_{15})$.

Similarly,
if $\mathfrak{a} = \mathbb{R}X_1 + \lambda E_3$ and $W = x_1^1$, we get the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ with $\lambda = 1$;
if $\mathfrak{a} = \mathbb{R}X_2 + \lambda E_3$ and $W = x_2^1$, we get the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ with $\lambda = 1$;
if $\mathfrak{a} = \mathbb{R}X_2 + \lambda E_3$ and $W = x_2^2$, we get the pair $(\bar{\mathfrak{g}}_{16}, \mathfrak{g}_{16})$;
if $\mathfrak{a} = \mathbb{R}X_3 + \lambda E_3$ and $W = x_3^1$, we get the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ with $\lambda = 1$, $\mu = 0$;
if $\mathfrak{a} = \mathbb{R}X_4 + \lambda E_3$ and $W = x_4^1$, we get the pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ with $\lambda = 1$;
if $\mathfrak{a} = \mathbb{R}X_4 + \lambda E_3$ and $W = x_4^2$, we get the pair $(\bar{\mathfrak{g}}_{13}, \mathfrak{g}_{13})$;
if $\mathfrak{a} = \mathbb{R}X_4 + \lambda E_3$ and $W = x_4^3$, we get the pair $(\bar{\mathfrak{g}}_{14}, \mathfrak{g}_{14})$;
if $\mathfrak{a} = \mathbb{R}X_5 + \lambda E_3$ and $W = x_5^1$, we get the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ with $\mu = 0$;
if $\mathfrak{a} = \mathbb{R}X_5 + \lambda E_3$ and $W = x_5^2$, we get the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ with $\mu = 0$.
Similarly we obtain the other results of the Proposition.

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

There are only two pairs such that $\dim(\mathcal{D}\bar{\mathfrak{g}}_i) = 5$. These are $(\bar{\mathfrak{g}}_{12}, \mathfrak{g}_{12})$ and $(\bar{\mathfrak{g}}_{17}, \mathfrak{g}_{17})$. Note that if $\mathfrak{a}_1$ is a Levi subalgebra of $\bar{\mathfrak{g}}_{12}$ and $\mathfrak{a}_2$ is a Levi subalgebra of $\bar{\mathfrak{g}}_{17}$, then

$$\dim(\mathfrak{a}_1 \cap \mathfrak{g}_{12}) \neq \dim(\mathfrak{a}_2 \cap \mathfrak{g}_{17}),$$

and the pairs $(\bar{\mathfrak{g}}_{12}, \mathfrak{g}_{12})$ and $(\bar{\mathfrak{g}}_{17}, \mathfrak{g}_{17})$ are not equivalent.

Let $\mathfrak{n}_i$ be a maximal nilpotent ideal of the Lie algebra $\bar{\mathfrak{g}}_i$, $1 \leqslant i \leqslant 17$. Note that $\dim \mathfrak{n}_1 = 4$ and $\mathcal{C}^3\mathfrak{n}_1 = \{0\}$; $\dim \mathfrak{n}_i = 4$ and $\mathcal{C}^3\mathfrak{n}_i \neq \{0\}$, for $i = 2, 3$; $\dim n_i = 3$ for $i = 4, \ldots, 7, 11$. It follows that all pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$ for $i = 2, \ldots, 7, 11$ are not equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$. For the same reason the pairs $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$, $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$ are not equivalent to the pair $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$.

Since $\dim \mathfrak{n}_8 = 4$ and $\mathcal{C}^3\mathfrak{n}_8 = \{0\}$, we see that the pair $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$ is not equivalent to each of the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$ for $i = 2, 3, 4$. Since $\bar{\mathfrak{g}}_i$-module $\mathfrak{n}_i/\mathcal{D}\mathfrak{n}_i$ is semisimple for $i = 1$ and not semisimple for $i = 8$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$ are also not equivalent.

Similarly we can prove that the other pairs determined in the Proposition are not equivalent to each other.

Proposition 2.10. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.10 is trivial.*

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	u_2	$u_1 + u_3$
e_2	0	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0
u_3	$-u_1 - u_3$	$-u_2$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

This proves the Proposition.

Proposition 2.11. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.11 is trivial.*

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	u_2	$u_3 - u_1$
e_2	0	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0
u_3	$u_1 - u_3$	0	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

This proves the Proposition.

Proposition 2.12. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 2.12 is trivial.*

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	u_2	u_3
e_2	0	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0
u_3	$-u_3$	$-u_2$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.
This proves the Proposition.

Proposition 2.13. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	$-u_2$	$-u_1$	0	0	0

2.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1
u_3	$-u_2$	$-u_1$	0	$-e_1$	0

3.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	$-e_1$
u_3	$-u_2$	$-u_1$	0	e_1	0

4.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_2
u_3	$-u_2$	$-u_1$	0	$-e_2$	0

5.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	$e_2 + u_2$
u_3	$-u_2$	$-u_1$	$-u_1$	$-e_2 - u_2$	0

6.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	$\alpha e_1 + u_2$
u_3	$-u_2$	$-u_1$	$-u_1$	$-\alpha e_1 - u_2$	0

7.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	$e_2 + u_1$
u_1	0	0	0	0	αu_1
u_2	$-u_1$	0	0	0	$(1-\alpha)e_1 + e_2 + \alpha u_2$
u_3	$-u_2$	$-e_2 - u_1$	$-\alpha u_1$	$-(1-\alpha)e_1 - e_2 - \alpha u_2$	0

8.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	$e_2 + u_1$
u_1	0	0	0	0	αu_1
u_2	$-u_1$	0	0	0	$\beta e_1 + \alpha u_2$
u_3	$-u_2$	$-e_2 - u_1$	$-\alpha u_1$	$-\beta e_1 - \alpha u_2$	0

9.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	e_2	$u_1 + 2e_1$	u_2
e_2	0	0	0	e_2	u_1
u_1	$-e_2$	0	0	$-u_1$	0
u_2	$-u_1 - 2e_1$	$-e_2$	u_1	0	$2u_3$
u_3	$-u_2$	$-u_1$	0	$-2u_3$	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A(e_1) = 0$, $A(e_2) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma 1. *Any virtual structure q on generalized module 2.13 is equivalent to one of the following:*

$$C_1(e_1) = \begin{pmatrix} p & 0 & 0 \\ r & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & p & s \\ 0 & r & t \end{pmatrix}.$$

Proof. Put

$$C(e_i) = \begin{pmatrix} c_{11}^i & c_{12}^i & c_{13}^i \\ c_{21}^i & c_{22}^i & c_{23}^i \end{pmatrix}, \quad i = 1, 2.$$

Since for any virtual structure q condition (6), Chapter II, is satisfied, after some calculation we obtain:

$$C(e_1) = \begin{pmatrix} c_{11}^1 & c_{12}^1 & c_{13}^1 \\ c_{21}^1 & c_{22}^1 & c_{23}^1 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c_{11}^1 & c_{13}^2 \\ 0 & c_{21}^1 & c_{23}^2 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c^1_{12} & c^1_{13} & 0 \\ c^1_{22} & c^1_{23} & 0 \end{pmatrix}.$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} c^1_{11} & 0 & 0 \\ c^1_{21} & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & c^1_{11} & -c^1_{12} + c^2_{13} \\ 0 & c^1_{21} & -c^1_{22} + c^2_{23} \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.13. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in Lemma 1. Then

$$\begin{aligned}
&[e_1, e_2] = 0, \\
&[e_1, u_1] = pe_1 + re_2, \quad && [e_2, u_1] = 0, \\
&[e_1, u_2] = u_1, && [e_2, u_2] = pe_1 + re_2, \\
&[e_1, u_3] = u_2, && [e_2, u_3] = se_1 + te_2 + u_1.
\end{aligned}$$

Put

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\
[u_1, u_3] &= b_1 e_1 + b_2 e_2 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\
[u_2, u_3] &= c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	re_2	u_1	u_2
e_2	0	0	0	re_2	E
u_1	$-re_2$	0	0	A	B
u_2	$-u_1$	$-re_2$	$-A$	0	C
u_3	$-u_2$	$-E$	$-B$	$-C$	0

where

$$\begin{aligned}
E &= se_1 + te_2 + u_1, \\
A &= a_1 e_1 + a_2 e_2 + \alpha_1 u_1, \\
B &= b_2 e_2 + \beta_1 u_1 + \beta_2 u_2, \\
C &= c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3,
\end{aligned}$$

and

$$\begin{cases} \alpha_1 r - r^2 = 0, \\ a_1 + rs = 0, \\ \beta_1 r - a_2 - rt = 0, \\ \beta_2 - \alpha_1 - r = 0, \\ \beta_2 r + rs = 0, \\ \gamma_1 r - b_2 = 0, \\ \beta_1 r + \beta_2 t + a_2 = 0, \\ \beta_2 s + a_1 - rs = 0, \\ \beta_2 + s + \alpha_1 - r = 0, \\ a_2 r - 2\beta_1 a_1 = 0, \\ a_2 - \alpha_1\beta_1 - \beta_1\beta_2 = 0, \\ a_1 + \alpha_1\beta_2 - \beta_2^2 = 0, \\ a_2 t + c_1 r - b_2 r - 2\beta_1 a_2 + \alpha_1 b_2 - \beta_2 b_2 = 0. \end{cases}$$

Consider the following cases:

1°. $r \neq 0$. Then the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	re_2	u_1	u_2
e_2	0	0	0	re_2	$-2re_1 + u_1$
u_1	$-re_2$	0	0	$2r^2 e_1 + ru_1$	$pre_2 + 2ru_2$
u_2	$-u_1$	$-re_2$	$-2r^2 e_1 - ru_1$	0	A
u_3	$-u_2$	$-2re_1 - u_1$	$-pre_2 - 2ru_2$	$-A$	0

where $A = 2pe_1 + ce_2 + pu_1 + 2ru_3$.

The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_9,\mathfrak{g}_9)$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_9$, where

$$\pi(e_1) = pe_1,$$
$$\pi(e_2) = p^2 e_2,$$
$$\pi(u_1) = \frac{p}{r}u_1 - 2pe_1,$$
$$\pi(u_2) = \frac{1}{r}u_2,$$
$$\pi(u_3) = \frac{1}{r}(e_1 + \frac{c}{3p}e_2 + \frac{1}{p}u_3).$$

2°. $r = 0$. Then we have $a_1 = a_2 = b_2 = \alpha_1 = \beta_2 = s = 0$.

2.1°. $t \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i, \ i = 1, 2,$$
$$\pi(u_i) = \frac{1}{t}u_i, \ 1 \leqslant i \leqslant 3.$$

The pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ has the form (*):

	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	e_2+u_1
u_1	0	0	0	0	αu_1
u_2	$-u_1$	0	0	0	A
u_3	$-u_2$	$-e_2-u_1$	$-\alpha u_1$	$-A$	0

(*)

where $A = \beta e_1 + \gamma e_2 + \delta u_1 + \alpha u_2$.

Note that any pair $(\bar{\mathfrak{g}},\mathfrak{g})$ of the form (*) is uniquely defined by the set of parameters $(\alpha, \beta, \gamma, \delta)$.

Lemma 2. *Two pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}',\mathfrak{g}')$ of the form (*) defined by sets $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$, respectively, are equivalent if and only if there exist $a \in \mathbb{R}^*$, $b, c \in \mathbb{R}$ such that*

$$\begin{cases} \alpha' = \alpha, \\ \beta' = \beta, \\ \gamma' = \frac{b}{a}(\alpha + \beta - 1)r + a\gamma, \\ \delta' = a\delta - \frac{b}{a} + c. \end{cases}$$

Proof. Suppose the pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}',\mathfrak{g}')$ are equivalent by means of a mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}'$. Let $H = (h_{ij})_{1\leqslant i,j\leqslant 5}$ be the matrix of π.

Since $\pi(\mathfrak{g}) = \mathfrak{g}'$, we have $h_{ij} = 0$ whenever $3 \leqslant i \leqslant 5$ and $j = 1, 2$. Since π is an isomorphism of Lie algebras, we have

$$\pi([x, y]) = [\pi(x), \pi(y)] \text{ for } x, y \in \bar{\mathfrak{g}}. \tag{1}$$

Check this condition for vectors of the basis.

After some calculation we obtain that H has the form:

$$H = \begin{pmatrix} a & 0 & 0 & 0c & \\ b & a^2 & 0 & b & d \\ 0 & 0 & a^2 & af+b & e \\ 0 & 0 & 0 & a & f \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Check condition (1) for vectors u_1, u_2, u_3.

1. $\pi([u_1, u_2]) = [\pi(u_1), \pi(u_2)] = 0.$
2. $\pi([u_1, u_3]) = [\pi(u_1), \pi(u_3)] \Rightarrow a^2\alpha u_1 = a^2\alpha' u_1 \Rightarrow$

$$\alpha' = \alpha.$$

3. $\pi([u_2, u_3]) = [\pi(u_2), \pi(u_3)] \Rightarrow a\beta e_1 + (b\beta + a^2\gamma + b\alpha)e_2 + ((af+b)\alpha + a^2\delta)u_1 +$
$+ a\alpha u_2 = a\beta' e_1 + (b + a\gamma')e_2 + ((af+b)\alpha' + a\delta' - ac + b)u_1 + a\alpha' u_2 \Rightarrow$

$$\begin{cases} \beta' = \beta, \\ \gamma' = \frac{b}{a}(\alpha + \beta - 1)r + a\gamma, \\ \delta' = a\delta - \frac{b}{a} + c. \end{cases}$$

So, the classification (up to equivalence) of pairs (*) is reduced to the classification of quadruples $(\alpha, \beta, \gamma, \delta)$ up to transformations determined in Lemma 2.

After elementary calculation, we see that every quadruple is equivalent to one and only one of following:

$$(\alpha,\ \beta\ \ ,0,0),$$
$$(\alpha, 1-\alpha, 1, 0).$$

The corresponding pairs are $(\bar{\mathfrak{g}}_8,\mathfrak{g}_8)$ and $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$ respectively.

2.2°. $t = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form (**):

	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	αu_1
u_2	$-u_1$	0	0	0	A
u_3	$-u_2$	$-u_1$	$-\alpha u_1$	$-A$	0

(**)

where $A = \beta e_1 + \gamma e_2 + \delta u_1 + \alpha u_2$.

Lemma 3. *Two pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}',\mathfrak{g}')$ of the form (**) defined by sets $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ respectively are equivalent if and only if there exist $a, c \in \mathbb{R}^*$, $b, g \in \mathbb{R}$ such that $\alpha' = \frac{1}{c}\alpha$ and*

1) if $\alpha' = \alpha = 0$,

$$\begin{cases} \beta' = \frac{1}{c^2}\beta \\ \gamma' = \frac{1}{a}(b\beta + a^2\gamma)c^2 \\ \delta' = \frac{1}{c}(a\delta - g) \end{cases}$$

2) if $\alpha'\alpha \neq 0$,

$$\begin{cases} \beta' = \beta \\ \gamma' = \frac{1}{a}(b\beta + a^2\gamma) \\ \delta' = a\delta - g \end{cases}$$

Proof.

The proof is similar to that of the previous Lemma.

Since virtual structures of the form (**) are trivial, and virtual structures of the form (*) are non trivial, we see that pairs of forms (*) and (**) are not equivalent.

Classification of quadruples $(\alpha, \beta, \gamma, \delta)$ up to transformations determined in lemma 3 shows that $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1 -- 6$.

Let $(\bar{\mathfrak{g}},\mathfrak{g})$ be a pair of the form (*). Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_9 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}$, we see that the pairs $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ and $(\bar{\mathfrak{g}},\mathfrak{g})$ are not equivalent.

This completes the proof of the Proposition.

Proposition 2.14. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.14 is trivial.*

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	$u_1 + u_2$	$u_2 + u_3$
e_2	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	$-u_1 - u_2$	0	0	0	0
u_3	$-u_2 - u_3$	$-u_1$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

This proves the Proposition.

Proposition 2.15. *Any pair* $(\bar{\mathfrak{g}},\mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1
e_2	$-e_1$	0	0	u_2	u_2+u_3
u_1	0	0	0	0	0
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-u_2-u_3$	0	0	0

2.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1
e_2	$-e_1$	0	0	u_2	u_2+u_3
u_1	0	0	0	0	u_2
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-u_2-u_3$	$-u_2$	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 2.15 is equivalent to one of the following:*

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_1(e_2) = 0.$$

Proof. Put

$$C(e_i) = \begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} \\ c^i_{21} & c^i_{22} & c^i_{23} \end{pmatrix}, \quad i = 1, 2.$$

Let us check condition (6), Chapter II for e_1, e_2.

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2)$$

We have:

$$\begin{pmatrix} c^2_{21} & c^2_{22} & c^2_{23} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & c^2_{11} \\ 0 & 0 & c^2_{21} \end{pmatrix} + \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ 0 & 0 & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & c^1_{12} & c^1_{12} + c^1_{13} \\ 0 & c^1_{22} & c^1_{22} + c^1_{23} \end{pmatrix} = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \end{pmatrix}.$$

We obtain the system of linear equation:

$$\begin{cases} c^1_{11} &= c^2_{21} + c^1_{11}, \\ c^1_{12} &= c^2_{22} + 2c^1_{12}, \\ c^1_{13} &= c^2_{23} - c^2_{11} + c^1_{12} + 2c^1_{13}, \\ c^1_{21} &= 0, \\ c^1_{22} &= c^1_{22}, \\ c^1_{23} &= c^1_{22} - c^2_{21} + c^1_{23}. \end{cases}$$

It follows that:

$$\begin{cases} c^2_{21} &= 0, \\ c^1_{21} &= 0, \\ c^1_{22} &= 0, \\ c^2_{22} &= -c^1_{12}, \\ c^1_{13} &= -c^2_{23} - c^1_{12} + c^2_{11}. \end{cases}$$

So, any virtual structure q on generalized module 2.15 has the form:

$$C(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ 0 & 0 & c^1_{23} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{11} & c^2_{12} & c^2_{13} \\ 0 & -c^1_{12} & c^2_{11} - c^1_{13} - c^1_{12} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c^2_{11} & 1/2c^2_{12} & 1/2c^2_{13} - 1/4c^2_{12} \\ -c^1_{11} & -c^1_{12} & c^2_{11} - c^1_{13} \end{pmatrix}.$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c^1_{23} + c^1_{11} \end{pmatrix}, \quad C_1(e_2) = 0.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}},\mathfrak{g})$ be a pair of type 2.15. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is defined by the virtual structure determined in the Lemma. Then

$$\begin{aligned} &[e_1, e_2] = e_1, \\ &[e_1, u_1] = 0, && [e_2, u_1] = 0, \\ &[e_1, u_2] = 0, && [e_2, u_2] = u_2, \\ &[e_1, u_3] = pe_2 + u_1, && [e_2, u_3] = u_2 + u_3. \end{aligned}$$

Put

$$[u_1, u_2] = a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3,$$
$$[u_1, u_3] = b_1 e_1 + b_2 e_2 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3,$$
$$[u2, u3] = c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.$$

Let us check the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1, 2$, $1 \leqslant j < k \leqslant 3$:

1. $[e_1, [u_1, u_2]] + [u_1, [u_2, e_1]] + [u_2, [e_1, u_1]] = 0$
$a_2 e_1 + \alpha_3(pe_2 + u_1) = 0$
1. $a_2 = 0$
2. $\alpha_3 = 0$

2. $[e_2, [u_1, u_2]] + [u_1, [u_2, e_2]] + [u_2, [e_2, u_1]] = 0$
$-a_1 e_1 + \alpha_2 u_2 - a_1 e_1 - \alpha_1 u_1 - \alpha_2 u_2 = 0$
3. $a_1 = 0$
4. $\alpha_1 = 0$

3. $[e_1, [u_1, u_3]] + [u_1, [u_3, e_1]] + [u_3, [e_1, u_1]] = 0$
$b_2 e_1 + \beta_3(pe_2 + u_1) = 0$
5. $b_2 = 0$
6. $\beta_3 = 0$

4. $[e_2, [u_1, u_3]] + [u_1, [u_3, e_2]] + [u_3, [e_2, u_1]] = 0$
$-b_1 e_1 + \beta_2 u_2 - \alpha_2 u_2 - b_1 e_1 - \beta_1 u_1 - \beta_2 u_2 = 0$
7. $b_1 = 0$
8. $\beta_1 = 0$
9. $\alpha_2 = 0$

5. $[e_1, [u_2, u_3]] + [u_2, [u_3, e_1]] + [u_3, [e_1, u_2]] = 0$
$c_2 e_1 + \gamma_3(pe_2 + u_1) + pu_2 = 0$
10. $c_2 = 0$
11. $\gamma_3 = 0$
12. $p = 0$

6. $[e_2, [u_2, u_3]] + [u_2, [u_3, e_2]] + [u_3, [e_2, u_2]] = 0$
$-c_1 e_1 + \gamma_2 u_2 - 2c_1 e_1 - 2\gamma_1 u_1 - 2\gamma_2 u_2 = 0$
13. $c_1 = 0$
14. $\gamma_1 = 0$
15. $\gamma_2 = 0$

7. $[[u_1, u_2], u_3] + [[u_2, u_3], u_1] + [[u_3, u_1], u_2] = 0$
$0 = 0$

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1
e_2	$-e_1$	0	0	u_2	u_2+u_3
u_1	0	0	0	0	$\beta_2 u_2$
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	$-u_2-u_3$	$-\beta_2 u_2$	0	0

Consider the following cases:

1°. $\beta_2 = 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$.

2°. $\beta_2 \neq 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \mathfrak{g} \to \mathfrak{g}_2$, where

$$\pi(e_1) = \frac{1}{\beta_2}e_1,$$
$$\pi(e_2) = e_2,$$
$$\pi(u_1) = \frac{1}{\beta_2}u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = u_3.$$

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

This completes the proof of the Proposition.

Proposition 2.16. *Any pair $(\bar{\mathfrak{g}},\mathfrak{g})$ of type 2.16 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$(\lambda-1)e_1$	0	0	u_1
e_2	$(1-\lambda)e_1$	0	u_1	λu_2	$u_2+\lambda u_3$
u_1	0	$-u_1$	0	0	0
u_2	0	$-\lambda u_2$	0	0	0
u_3	$-u_1$	$-u_2-\lambda u_3$	0	0	0

2. $\lambda = \frac{1}{3}$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$-\frac{2}{3}e_1$	0	0	u_1
e_2	$\frac{2}{3}e_1$	0	u_1	$\frac{1}{3}u_2$	$u_2+\frac{1}{3}u_3$
u_1	0	$-u_1$	0	0	0
u_2	0	$-\frac{1}{3}u_2$	0	0	e_1
u_3	$-u_1$	$-u_2-\frac{1}{3}u_3$	0	$-e_1$	0

3. $\lambda = \frac{1}{2}$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$-\frac{1}{2}e_1$	0	0	u_1
e_2	$\frac{1}{2}e_1$	0	u_1	$\frac{1}{2}u_2$	$u_2+\frac{1}{2}u_3$
u_1	0	$-u_1$	0	0	0
u_2	0	$-\frac{1}{2}u_2$	0	0	u_1
u_3	$-u_1$	$-u_2-\frac{1}{2}u_3$	0	$-u_1$	0

4. $\lambda = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	0	0	u_1
e_2	e_1	0	u_1	0	u_2
u_1	0	$-u_1$	0	0	αe_1
u_2	0	0	0	0	u_2
u_3	$-u_1$	$-u_2$	$-\alpha e_1$	$-u_2$	0

5. $\lambda = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	0	0	u_1
e_2	e_1	0	u_1	0	u_2
u_1	0	$-u_1$	0	0	e_1
u_2	0	0	0	0	0
u_3	$-u_1$	$-u_2$	$-e_1$	0	0

6. $\lambda = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	0	0	u_1
e_2	e_1	0	u_1	0	u_2
u_1	0	$-u_1$	0	0	$-e_1$
u_2	0	0	0	0	0
u_3	$-u_1$	$-u_2$	e_1	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & \lambda - 1 \\ 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 2.16 is equivalent to one of the following:*

a) $\lambda \notin \{0, \frac{1}{2}, 2\}$

$$C_1(e_1) = C_1(e_2) = 0;$$

b) $\lambda = 2$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix}, \quad C_2(e_2) = 0;$$

c) $\lambda = \frac{1}{2}$

$$C_3(e_1) = 0, \quad C_3(e_2) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

d) $\lambda = 0$

$$C_4(e_1) = 0, \quad C_4(e_2) = \begin{pmatrix} -p & 0 & 0 \\ 0 & 0 & p \end{pmatrix}.$$

Proof. Put

$$C(e_i) = \begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} \\ c^i_{21} & c^i_{22} & c^i_{23} \end{pmatrix}, \quad i = 1, 2.$$

Let us check condition (6), Chapter II, for e_1, e_2:

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2)$$

We have:

$$\begin{pmatrix} (\lambda-1)c^2_{21} & (\lambda-1)c^2_{22} & (\lambda-1)c^2_{23} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & c^2_{11} \\ 0 & 0 & c^2_{21} \end{pmatrix} -$$

$$- \begin{pmatrix} (1-\lambda)c^1_{11} & (1-\lambda)c^1_{12} & (1-\lambda)c^1_{13} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} c^1_{11} & \lambda c^1_{12} & c^1_{12} + \lambda c^1_{13} \\ c^1_{21} & \lambda c^1_{22} & c^1_{22} + \lambda c^1_{23} \end{pmatrix} =$$

$$= (\lambda - 1) \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \end{pmatrix}.$$

We obtain the system of linear equations:

$$\begin{cases} (\lambda-1)c^1_{11} &= (\lambda-1)c^2_{21} + \lambda c^1_{11} \\ (\lambda-1)c^1_{12} &= (\lambda-1)c^2_{22} - c^1_{12} \\ (\lambda-1)c^1_{13} &= (\lambda-1)c^2_{23} - c^2_{11} + (\lambda-1)c^1_{13} + c^1_{12} + \lambda c^1_{13} \\ (\lambda-1)c^1_{21} &= c^1_{21} \\ (\lambda-1)c^1_{22} &= \lambda c^1_{22} \\ (\lambda-1)c^1_{23} &= \lambda c^1_{23} - c^2_{21} + c^1_{22} \end{cases}$$

1°. $\lambda \notin \{0, \frac{1}{2}, 2\}$.
Any virtual structure q on generalized module 2.16 has the form:

$$C(e_1) = \begin{pmatrix} (1-\lambda)c^2_{21} & \frac{1-\lambda}{\lambda}c^2_{22} & c^1_{13} \\ 0 & 0 & c^2_{21} \end{pmatrix},$$

$$C(e_2) = \begin{pmatrix} \frac{1-\lambda}{\lambda}c^2_{22} + \lambda c^1_{13} + (\lambda-1)c^2_{23} & c^2_{12} & c^2_{13} \\ c^2_{21} & c^2_{22} & c^2_{23} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} \frac{(1-\lambda)/\lambda c^2_{22} + (\lambda-1)c^2_{23}}{\lambda} + c^1_{13} & \frac{c^2_{12}}{2\lambda-1} & \frac{c^2_{13}}{2\lambda-1} - \frac{c^2_{12}}{(2\lambda-1)^2} \\ c^2_{21} & \frac{c^2_{22}}{\lambda} & \frac{c^2_{23}}{\lambda} - \frac{c^2_{22}}{(\lambda)^2} \end{pmatrix}.$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$. Then $C_1(e_1) = C_1(e_2) = 0$.
By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

2°. $\lambda = 2$.
Then any virtual structure q on generalized module 2.16 has the form:

$$C(e_1) = \begin{pmatrix} -c^1_{23} & c^1_{12} & c^1_{13} \\ c^1_{21} & 0 & c^1_{23} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{11} & c^2_{12} & c^2_{13} \\ c^1_{23} & -2c^1_{12} & c^2_{11} - c^1_{12} - 2c^1_{13} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} \frac{1}{2}c^2_{11} & \frac{1}{3}c^2_{12} & \frac{1}{3}c^2_{13} - \frac{1}{9}c^2_{12} \\ c^1_{23} & -c^1_{12} & \frac{1}{2}c^2_{11} - c^1_{13} \end{pmatrix}.$$

Now put $C_2(x) = C(x) + A(x)H - HB(x)$. Then

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ c^1_{21} & 0 & 0 \end{pmatrix}, \quad C_2(e_2) = 0.$$

By corollary 2, Chapter II, the virtual structures C and C_2 are equivalent.

3°. $\lambda = \frac{1}{2}$.

Any virtual structure q on generalized module 2.16 has the form:

$$C(e_1) = \begin{pmatrix} \frac{1}{2}c^2_{21} & c^2_{22} & c^1_{13} \\ 0 & 0 & c^2_{21} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{22} + \frac{1}{2}c^1_{13} - \frac{1}{2}c^2_{23} & c^2_{12} & c^2_{13} \\ c^2_{21} & c^2_{22} & c^2_{23} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 2c^2_{22} - c^2_{23} + c^1_{13} & c^2_{13} & 0 \\ c^2_{21} & 2c^2_{22} & 2c^2_{23} - 4c^2_{22} \end{pmatrix}.$$

Now put $C_3(x) = C(x) + A(x)H - HB(x)$. Then

$$C_3(e_1) = 0, \quad C_3(e_2) = \begin{pmatrix} 0 & c^2_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_3 are equivalent.

4°. $\lambda = 0$.

Any virtual structure q on generalized module 2.16 has the form:

$$C(e_1) = \begin{pmatrix} c^2_{21} & c^1_{12} & c^1_{13} \\ 0 & 0 & c^2_{21} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^1_{12} - c^2_{23} & c^2_{12} & c^2_{13} \\ c^2_{21} & 0 & c^2_{23} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & -c^2_{12} & -c^2_{12} - c^2_{13} \\ c^2_{21} & c^1_{12} & c^1_{13} \end{pmatrix}.$$

Now put $C_4(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_4(e_1) = 0, \quad C_4(e_2) = \begin{pmatrix} c^1_{12} - c^2_{23} & 0 & 0 \\ 0 & 0 & c^2_{23} - c^1_{12} \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_4 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.16. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma.

Consider the following cases:

1°. $\lambda = 0$. Then

$$\begin{aligned}
&[e_1, e_2] = -e_1, \\
&[e_1, u_1] = 0, \qquad [e_2, u_1] = -pe_1 + u_1, \\
&[e_1, u_2] = 0, \qquad [e_2, u_2] = 0, \\
&[e_1, u_3] = u_1, \qquad [e_2, u_3] = pe_2 + u_2,
\end{aligned}$$

Put

$$[u_1,u_2] = a_1e_1 + a_2e_2 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3,$$
$$[u_1,u_3] = b_1e_1 + b_2e_2 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3,$$
$$[u_2,u_3] = c_1e_1 + c_2e_2 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3.$$

Let us check the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1,2$, $1 \leqslant j < k \leqslant 3$ and (u_1, u_2, u_3).

1. $[e_1,[u_1,u_2]] + [u_1,[u_2,e_1]] + [u_2,[e_1,u_1]] = 0$
$-a_2e_1 + \alpha_3u_1 = 0$
1. $\alpha_3 = 0$
2. $a_2 = 0$

2. $[e_1,[u_1,u_3]] + [u_1,[u_3,e_1]] + [u_3,[e_1,u_1]] = 0$
$-b_2e_1 + \beta_3u_1 = 0$
3. $\beta_3 = 0$
4. $b_2 = 0$

3. $[e_1,[u_2,u_3]] + [u_2,[u_3,e_1]] + [u_3,[e_1,u_2]] = 0$
$-c_2e_1 + \gamma_3u_1 + a_1e_1 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3 = 0$
5. $a_1 - c_2 = 0$
6. $\alpha_1 + \gamma_3 = 0$
7. $\alpha_2 = 0$

4. $[e_2,[u_1,u_2]] + [u_1,[u_2,e_2]] + [u_2,[e_2,u_1]] = 0$
$a_1e_1 + \alpha_1(-pe_1 + u_1) - a_1e_1 - \alpha_1u_1 = 0$
8. $\alpha_1p = 0$

5. $[e_2,[u_1,u_3]] + [u_1,[u_3,e_2]] + [u_3,[e_2,u_1]] = 0$
$b_1e_1\beta_1(-pe_1 + u_1) - a_1e_1 - \alpha_1u_1 - p^2e_1 + pu_1 - b_1e_1 - \beta_1u_1 - \beta_2u_2 + pu_1 = 0$
9. $\beta_2 = 0$
10. $p^2 + a_1 + p\beta_1 = 0$
11. $2p = \alpha_1$

6. $[e_2,[u_2,u_3]] + [u_2,[u_3,e_2]] + [u_3,[e_2,u_2]] = 0$
$c_1e_1 + \gamma_1(u_1 - pe_1) + \gamma_3(pe_2 + u_2) = 0$
12. $c_1 - p\gamma_1 = 0$
13. $\gamma_3 = 0$
14. $\gamma_1 = 0$

7. $[u_1,[u_2,u_3]] + [u_2,[u_3,u_1]] + [u_3,[u_1,u_2]] = 0$
$0 = 0$

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	0	0	u_1
e_2	e_1	0	u_1	0	u_2
u_1	0	$-u_1$	0	0	$b_1e_1+\beta_1u_1$
u_2	0	0	0	0	γ_2u_2
u_3	$-u_1$	$-u_2$	$-b_1e_1-\beta_1u_1$	$-\gamma_2u_2$	0

1.1°. $\gamma_2 = 0,\ b_1 = 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_1 + e_2,\\
\pi(u_1) &= -\frac{\beta_1}{2}e_1 + u_1,\\
\pi(u_2) &= u_2,\\
\pi(u_3) &= \frac{\beta_1}{2}(e_1 + e_2) - u_1 + u_3.
\end{aligned}$$

1.2°. $\gamma_2 = 0,\ b_1 > 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5,\mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_1 + e_2,\\
\pi(u_1) &= \frac{1}{\sqrt{b_1}}(-\frac{\beta_1}{2}e_1 + u_1),\\
\pi(u_2) &= \frac{1}{\sqrt{b_1}}u_2,\\
\pi(u_3) &= \frac{1}{\sqrt{b_1}}(u_3 - u_1 + \frac{\beta_1}{2}(e_1 + e_2)).
\end{aligned}$$

1.3°. $\gamma_2 = 0,\ b_1 < 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6,\mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_1 + e_2,\\
\pi(u_1) &= \frac{1}{\sqrt{-b_1}}(-\frac{\beta_1}{2}e_1 + u_1),\\
\pi(u_2) &= \frac{1}{\sqrt{-b_1}}u_2,\\
\pi(u_3) &= \frac{1}{\sqrt{-b_1}}(u_3 - u_1 + \frac{\beta_1}{2}(e_1 + e_2)).
\end{aligned}$$

1.4°. $\gamma_2 \neq 0$.

Then the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, such that

$$\begin{aligned}
\pi(e_1) &= \gamma_2 e_1, \\
\pi(e_2) &= e_1 + e_2, \\
\pi(u_1) &= -\frac{\beta_1}{2} e_1 + u_1, \\
\pi(u_2) &= \frac{1}{\gamma_2} u_2, \\
\pi(u_3) &= \frac{1}{\gamma_2}(\frac{\beta_1}{2}(e_1 + e_2) - u_1 + u_3)
\end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}_4,\mathfrak{g}_4)$.

2°. $\lambda \notin \{0, \frac{1}{2}, 2\}$. Then

$$\begin{aligned}
&[e_1, e_2] = (\lambda - 1)e_1, && \\
&[e_1, u_1] = 0, && [e_2, u_1] = u_1, \\
&[e_1, u_2] = 0, && [e_2, u_2] = \lambda u_2, \\
&[e_1, u_3] = u_1, && [e_2, u_3] = u_2 + \lambda u_3,
\end{aligned}$$

Put

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\
[u_1, u_3] &= b_1 e_1 + b_2 e_2 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\
[u_2, u_3] &= c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$\lambda \neq \frac{1}{3}$	e_1	e_2	u_1	u_2	u_3
e_1	0	$(\lambda - 1)e_1$	0	0	u_1
e_2	$(1 - \lambda)e_1$	0	u_1	λu_2	$u_2 + \lambda u_3$
u_1	0	$-u_1$	0	0	0
u_2	0	$-\lambda u_2$	0	0	0
u_3	$-u_1$	$-u_2 - \lambda u_3$	0	0	0

and

$\lambda = \frac{1}{3}$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-\frac{2}{3}e_1$	0	0	u_1
e_2	$\frac{2}{3}e_1$	0	u_1	$\frac{1}{3}u_2$	$u_2 + \frac{1}{3}u_3$
u_1	0	$-u_1$	0	0	0
u_2	0	$-\frac{1}{3}u_2$	0	0	$c_1 e_1$
u_3	$-u_1$	$-u_2 - \frac{1}{3}u_3$	0	$-c_1 e_1$	0

2.1°. $\lambda \neq \frac{1}{3}$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

2.2°. $\lambda = \frac{1}{3}$.

2.2.1°. $c_1 = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$.

2.2.2°. $c_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}\pi(e_1) &= c_1 e_1,\\ \pi(e_2) &= e_2,\\ \pi(u_1) &= c_1 u_1,\\ \pi(u_2) &= u_2,\\ \pi(u_3) &= u_3.\end{aligned}$$

3°. $\lambda = 2$. Then

$$\begin{aligned}&[e_1, e_2] = e_1, && \\ &[e_1, u_1] = pe_2, && [e_2, u_1] = u_1,\\ &[e_1, u_2] = 0, && [e_2, u_2] = 2u_2,\\ &[e_1, u_3] = u_1, && [e_2, u_3] = u_2 + 2u_3,\end{aligned}$$

Put

$$\begin{aligned}[u_1, u_2] &= a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3,\\ [u_1, u_3] &= b_1 e_1 + b_2 e_2 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3,\\ [u_2, u_3] &= c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1
e_2	$-e_1$	0	u_1	$2u_2$	$u_2 + 2u_3$
u_1	0	$-u_1$	0	0	0
u_2	0	$-2u_2$	0	0	0
u_3	$-u_1$	$-u_2 - 2u_3$	0	0	0

We see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

4°. $\lambda = \frac{1}{2}$. Then

$$\begin{aligned}&[e_1, e_2] = -\frac{1}{2}e_1, && \\ &[e_1, u_1] = 0, && [e_2, u_1] = u_1,\\ &[e_1, u_2] = 0, && [e_2, u_2] = pe_1 + \frac{1}{2}u_2,\\ &[e_1, u_3] = u_1, && [e_2, u_3] = u_2 + \frac{1}{2}u_3.\end{aligned}$$

Put

$$\begin{aligned}[u_1, u_2] &= a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3,\\ [u_1, u_3] &= b_1 e_1 + b_2 e_2 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3,\\ [u_2, u_3] &= c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-\frac{1}{2}e_1$	0	0	u_1
e_2	$\frac{1}{2}e_1$	0	u_1	$\frac{1}{2}u_2$	$u_2+\frac{1}{2}u_3$
u_1	0	$-u_1$	0	0	0
u_2	0	$-\frac{1}{2}u_2$	0	0	$\gamma_1 u_1$
u_3	$-u_1$	$-u_2-\frac{1}{2}u_3$	0	$-\gamma_1 u_1$	0

4.1°. $\gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$.

4.2°. $\gamma_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3,\mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= \gamma_1 e_1, \\ \pi(e_2) &= e_2, \\ \pi(u_1) &= \gamma_1 u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= u_3. \end{aligned}$$

Now it remains to show that the pairs determined by Proposition 2.16 are not equivalent to each other.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ are not equivalent.

The mapping $\pi : \bar{\mathfrak{g}}_3' \to \bar{\mathfrak{g}}_3$ such that

$$\begin{aligned} \pi(e_i) &= e_i, \; i = 1,2, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2 - e_1, \\ \pi(u_3) &= u_3, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}_3,\mathfrak{g}_3)$ and $(\bar{\mathfrak{g}}_3',\mathfrak{g}_3')$, where the latter has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-\frac{1}{2}e_1$	0	0	u_1
e_2	$\frac{1}{2}e_1$	0	u_1	$\frac{1}{2}u_2$	$u_2+\frac{1}{2}u_3+e_1$
u_1	0	$-u_1$	0	0	0
u_2	0	$-\frac{1}{2}u_2$	0	0	0
u_3	$-u_1$	$-u_2-\frac{1}{2}u_3-e_1$	0	0	0

Consider homomorphisms

$$f : \bar{\mathfrak{g}}_1/(\mathcal{D}^2\bar{\mathfrak{g}}_1) \to \mathfrak{gl}(3,\mathbb{R}),$$

$$f' : \bar{\mathfrak{g}}_3'/(\mathcal{D}^2\bar{\mathfrak{g}}_3') \to \mathfrak{gl}(3,\mathbb{R}),$$

where $f(x)$ and $f'(x)$ are the matrices of the mappings $\operatorname{ad}_{\mathcal{D}\bar{\mathfrak{g}}_1} x$ and $\operatorname{ad}_{\mathcal{D}\bar{\mathfrak{g}}_3'} x$ in basis the $\{e_1+\mathfrak{a}, e_2+\mathfrak{a}, u_2+\mathfrak{a}, u_3+\mathfrak{a}\}$, $\mathfrak{a} = \bar{\mathfrak{g}}_1/\mathcal{D}^2\bar{\mathfrak{g}}_1 = \bar{\mathfrak{g}}_3'/\mathcal{D}^2\bar{\mathfrak{g}}_3'$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i/(\mathcal{D}^2\bar{\mathfrak{g}}_i)), i = 1,3$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

Since $\dim \mathcal{Z}(\bar{\mathfrak{g}}_i) \neq \dim \mathcal{Z}(\bar{\mathfrak{g}}_4), i = 1,5,6$, we see that the pairs $(\bar{\mathfrak{g}}_4,\mathfrak{g}_4)$ and $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)$ $i = 1,5,6$, are not equivalent.

Consider homomorphisms

$$f_\alpha : \bar{\mathfrak{g}}_4^\alpha \to \mathfrak{gl}(3,\mathbb{R}),$$

$$f_\beta : \bar{\mathfrak{g}}_4^\beta \to \mathfrak{gl}(3,\mathbb{R}),$$

where $f_\alpha(x), f_\beta(x)$ are the matrices of the mappings $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_4^\alpha}$ and $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_4^\beta}$ in the basis fixed before.

Since the subalgebras $f_\alpha(\bar{\mathfrak{g}}_4^\alpha)$ and $f_\beta(\bar{\mathfrak{g}}_4^\beta)$ are not conjugate (if $\alpha \neq \beta$), we conclude that the pairs $(\bar{\mathfrak{g}}_4,\mathfrak{g}_4)$ with different values of parameter α are not equivalent.

Consider homomorphisms

$$f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3,\mathbb{R}),\ i = 1,5,6,$$

where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i}\, x$ in the basis $\{e_1, e_2, u_1, u_2, u_3\}$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)$, $i = 1,5,6$, are not equivalent.

Thus the proof of the Proposition is complete.

Proposition 2.17. *Any pair $(\bar{\mathfrak{g}},\mathfrak{g})$ of type 2.17 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	$-u_2$	0	0	0

2.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	e_1
u_2	0	0	0	0	αe_2
u_3	$-u_1$	$-u_2$	$-e_1$	$-\alpha e_2$	0

3.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$-e_1$
u_2	0	0	0	0	αe_2
u_3	$-u_1$	$-u_2$	e_1	$-\alpha e_2$	0

4.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 - e_2$
u_2	0	0	0	0	$e_1 + \alpha e_2$
u_3	$-u_1$	$-u_2$	$e_2 - \alpha e_1$	$-e_1 - \alpha e_2$	0

, $\alpha \geqslant 0$

5.

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	e_1
u_3	$-u_1$	$-u_2$	0	$-e_1$	0

6.

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	e_1
u_2	0	0	0	0	e_1+e_2
u_3	$-u_1$	$-u_2$	$-e_1$	$-e_1-e_2$	0

7.

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$-e_1$
u_2	0	0	0	0	e_1-e_2
u_3	$-u_1$	$-u_2$	e_1	$-e_1+e_2$	0

8.

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	δe_1+u_1
u_2	0	0	0	0	$\beta e_2+\alpha u_2$
u_3	$-u_1$	$-u_2$	$-\delta e_1-u_1$	$-\beta e_2-\alpha u_2$	0

$, \quad -1<\alpha<1$

9.

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\delta e_1+e_2+u_1$
u_2	0	0	0	0	$\gamma e_1+\beta e_2+\alpha u_1$
u_3	$-u_1$	$-u_2$	$-\delta e_1-e_2-u_1$	$-\gamma e_1-\beta e_2-\alpha u_1$	0

$, \quad -1<\alpha<1$

10.

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	δe_1+u_1
u_2	0	0	0	0	$e_1+\beta e_2+\alpha u_2$
u_3	$-u_1$	$-u_2$	$-\delta e_1-u_1$	$-e_1-\beta e_2-\alpha u_2$	0

$, \quad -1<\alpha<1$

11.

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	e_2+u_1
u_3	$-u_1$	$-u_2$	0	$-e_2-u_1$	0

12.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	$-e_2+u_1$
u_3	$-u_1$	$-u_2$	0	e_2-u_1	0

13.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	e_1
u_2	0	0	0	0	αe_2+u_1
u_3	$-u_1$	$-u_2$	$-e_1$	$-\alpha e_2-u_1$	0

14.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$-e_1$
u_2	0	0	0	0	αe_2+u_1
u_3	$-u_1$	$-u_2$	e_1	$-\alpha e_2-u_1$	0

15.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	αe_1
u_2	0	0	0	0	$e_1+\alpha e_2+u_1$
u_3	$-u_1$	$-u_2$	$-\alpha e_1$	$-e_1-\alpha e_2-u_1$	0

16.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_1
u_3	$-u_1$	$-u_2$	0	$-u_1$	0

17.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	e_2
u_2	0	0	0	0	$\alpha e_1+\beta e_2+u_1$
u_3	$-u_1$	$-u_2$	$-e_2$	$-\alpha e_1-\beta e_2-u_1$	0

18.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	γe_2+u_1
u_2	0	0	0	0	$\alpha e_1+\beta e_2+u_1+u_2$
u_3	$-u_1$	$-u_2$	$-\gamma e_2-u_1$	$-\alpha e_1-\beta e_2-u_1-u_2$	0

19.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + u_1$
u_2	0	0	0	0	$\beta e_2 + u_1 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - u_1$	$-\beta e_2 - u_1 - u_2$	0

20.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + u_1$
u_2	0	0	0	0	$\beta e_1 + \alpha e_2 + u_1 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - u_1$	$-\beta e_1 - \alpha e_2 - u_1 - u_2$	0

21.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + u_1$
u_2	0	0	0	0	$e_1 + \alpha e_2 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - u_1$	$-e_1 - \alpha e_2 - u_2$	0

22.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 - \beta e_2 + u_1$
u_2	0	0	0	0	$\beta e_1 + \alpha e_2 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 + \beta e_2 - u_1$	$-\beta e_1 - \alpha e_2 - u_2$	0

, $\beta > 0$

23.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + u_1$
u_2	0	0	0	0	$\beta e_2 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - u_1$	$-\beta e_2 - u_2$	0

, $|\alpha| \leqslant |\beta|$

24.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\delta e_1 + \gamma e_2 + \alpha u_1 - u_2$
u_2	0	0	0	0	$\beta e_1 + \delta e_2 + u_1 + \alpha u_2$
u_3	$-u_1$	$-u_2$	$-\delta e_1 - \gamma e_2 - \alpha u_1 + u_2$	$-\beta e_1 - \delta e_2 - u_1 - \alpha u_2$	0

, $|\beta| \leqslant |\gamma|$

25.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + u_1$
u_2	0	0	0	0	$\beta e_2 - u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - u_1$	$-\beta e_2 + u_2$	0

, $|\alpha| \leqslant |\beta|$

26.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1+\beta e_2+u_1$
u_2	0	0	0	0	$e_1+\gamma e_2-u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1-\beta e_2-u_1$	$-e_1-\gamma e_2+u_2$	0

, $-1 \leqslant \beta \leqslant 1$

27.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	$2e_1$	e_2	u_1
e_2	0	0	e_2	0	u_2
u_1	$-2e_1$	$-e_2$	0	u_2	$2u_3$
u_2	$-e_2$	0	$-u_2$	0	0
u_3	$-u_1$	$-u_2$	$-2u_3$	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma 1. *Any virtual structure q on generalized module 2.17 is equivalent to one of the following:*

$$C_1(e_1) = \begin{pmatrix} p & s & 0 \\ r & t & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} s & v & 0 \\ t & w & 0 \end{pmatrix}$$

Proof. Put

$$C(e_i) = \begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} \\ c^i_{21} & c^i_{22} & c^i_{23} \end{pmatrix}, \quad i = 1, 2.$$

Let us check condition (6), Chapter II, for e_1, e_2:

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2)$$

We have:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & c^2_{11} \\ 0 & 0 & c^2_{21} \end{pmatrix} +$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & c^1_{12} \\ 0 & 0 & c^1_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain the system of linear equations:

$$\begin{cases} c^1_{12} = c^2_{11} \\ c^2_{21} = c^1_{22} \end{cases}$$

So, any virtual structure q on generalized module 2.17 has the form:

$$C(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^1_{12} & c^2_{12} & c^2_{13} \\ c^1_{22} & c^2_{22} & c^2_{23} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c^1_{13} & c^2_{13} & 0 \\ c^1_{23} & c^2_{23} & 0 \end{pmatrix}.$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & 0 \\ c^1_{21} & c^1_{22} & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} c^1_{12} & c^2_{12} & 0 \\ c^1_{22} & c^2_{22} & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

Thus it can be assumed that any virtual structure q on generalized module 2.17 has the form determined in Lemma 1. Then

$$\begin{aligned} &[e_1, e_2] = 0, \\ &[e_1, u_1] = pe_1 + re_2, \quad [e_2, u_1] = se_1 + te_2, \\ &[e_1, u_2] = se_1 + te_2, \quad [e_2, u_2] = ve_1 + we_2, \\ &[e_1, u_3] = u_1, \qquad\qquad [e_2, u_3] = u_2. \end{aligned}$$

Put

$$\begin{aligned} [u_1, u_2] &= a_1e_1 + a_2e_2 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3, \\ [u_1, u_3] &= b_1e_1 + b_2e_2 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3, \\ [u_2, u_3] &= c_1e_1 + c_2e_2 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3. \end{aligned}$$

Let us check the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1, 2$, $1 \leqslant j < k \leqslant 3$ and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_1, [u_2, e_1]] + [u_2, [e_1, u_1]] = 0$
$\alpha_1pe_1 + \alpha_1re_2 + \alpha_2se_1 + \alpha_2te_2 + \alpha_3u_1 +$
$+spe_1 + sre_2 + ste_1 + t^2e_2 - pse_1 - pte_2 - rve_1 - rwe_2 = 0$
1. $\alpha_1p + \alpha_2s + ts - rv = 0$
2. $\alpha_1r + \alpha_2t + sr + t^2 - pt - rw = 0$
3. $\alpha_3 = 0$

2. $[e_1, [u_1, u_3]] + [u_1, [u_3, e_1]] + [u_3, [e_1, u_1]] = 0$
$\beta_1pe_1 + \beta_1re_2 + \beta_2se_1 + \beta_2te_2 + \beta_3u_1 - pu_1 - ru_2 = 0$
4. $\beta_1p + \beta_2s = 0$
5. $\beta_2t = 0$
6. $\beta_3 = p$
7. $r = 0$

3. $[e_1, [u_2, u_3]] + [u_2, [u_3, e_1]] + [u_3, [e_1, u_2]] = 0$

$\gamma_1 pe_1 + \gamma_2 se_1 + \gamma_2 te_2 + \gamma_3 u_1 + a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2 - su_1 - tu_2 = 0$

8. $\gamma_1 p + \gamma_2 s + a_1 = 0$
9. $\gamma_2 t + a_2 = 0$
10. $\gamma_3 + \alpha_1 = s$
11. $\alpha_2 = t$

4. $[e_2,[u_1,u_2]] + [u_1,[u_2,e_2]] + [u_2,[e_2,u_1]] = 0$

$\alpha_1 se_1 + \alpha_1 te_2 + \alpha_2 ve_1 + \alpha_2 we_2 +$
$+ vpe_1 + swe_1 + twe_2 - s^2 e_1 - ste_2 - tve_1 - twe_2 = 0$

12. $\alpha_1 s + \alpha_2 v + vp + ws - s^2 - tv = 0$
13. $\alpha_1 t + \alpha_2 w - st = 0$

5. $[e_2,[u_1,u_3]] + [u_1,[u_3,e_2]] + [u_3,[e_2,u_1]] = 0$

$\beta_1 se_1 + \beta_1 te_2 + \beta_2 ve_1 + \beta_2 we_2 + \beta_3 u_2 - (a_1 e_1 + a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2) -$
$- su_1 - tu_2 = 0$

14. $\beta_1 s + \beta_2 v - a_1 = 0$
15. $\beta_1 t + \beta_2 w - a_2 = 0$
16. $\beta_3 - \alpha_2 - t = 0$
17. $\alpha_1 + s = 0$

6. $[e_2,[u_2,u_3]] + [u_2,[u_3,e_2]] + [u_3,[e_2,u_2]] = 0$

$\gamma_1 se_1 + \gamma_1 te_2 + \gamma_2 ve_1 + \gamma_2 we_2 + \gamma_2 u_2 - vu_1 - wu_2 = 0$

18. $\gamma_1 s + \gamma_2 v = 0$
19. $\gamma_1 t + \gamma_2 w = 0$
20. $\gamma_3 = w$
21. $v = 0$

7. $[u_1,[u_2,u_3]] + [u_2,[u_3,u_1]] + [u_3,[u_1,u_2]] = 0$

$-c_1 pe_1 - c_2(se_1 + te_2) + \gamma_2[u_1,u_2] + \gamma_3[u_1,u_3] + b_1(se_1 + te_2) + b_2(ve_1 + we_2) +$
$+ \beta_1[u_1,u_2] - \beta_3[u_2,u_3] - a_1 u_1 - a_2 u_2 - \alpha_1[u_1,u_3] - \alpha_2[u_2,u_3] = 0$

22. $-c_1 p - c_2 s + \gamma_2 a_1 + \gamma_3 b_1 + b_1 s + b_2 v + \beta_1 a_1 - \beta_3 c_1 - \alpha_1 b_1 - \alpha_2 c_1 = 0$
23. $-c_2 t + \gamma_2 a_2 + \gamma_3 b_2 + b_1 t + b_2 w + \beta_1 a_2 - \beta_3 c_2 - \alpha_1 b_2 - \alpha_2 c_2 = 0$
24. $\gamma_2 \alpha_1 + \gamma_3 \beta_1 - \gamma_1 \alpha_2 - \beta_3 \gamma_1 - a_1 = 0$
25. $\gamma_2 \alpha_2 + \gamma_3 \beta_2 + \beta_1 \alpha_2 - \gamma_2 \beta_3 - \beta_2 \alpha_1 - a_2 - \gamma_2 \alpha_2 = 0$
26. $-\alpha_1 \beta_3 - \alpha_2 \gamma_3 + \gamma_2 \alpha_3 + \beta_1 \alpha_3 = 0$

Consider the following cases:

1°. $p^2 + s^2 + t^2 + w^2 \neq 0$.

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	$2te_1$	se_1+te_2	u_1
e_2	0	0	se_1+te_2	$2se_2$	u_2
u_1	$-2te_1$	$-te_2-se_1$	0	tu_2-su_1	$b_1e_1+b_2e_2+2tu_3$
u_2	$-se_1-te_2$	$-2se_2$	su_1-tu_2	0	$c_1e_1+c_2e_2+2su_3$
u_3	$-u_1$	$-u_2$	$-b_1e_1-b_2e_2-2tu_3$	$-c_1e_1-c_2e_2-2su_3$	0

,

where $p = 2t$, $w = 2s$. It follows that $s^2 + t^2 \neq 0$. The mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, such that

$$\pi(e_1) = e_2, \quad \pi(u_1) = u_2,$$
$$\pi(e_2) = e_1, \quad \pi(u_2) = u_1,$$
$$\pi(u_3) = u_3,$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}',\mathfrak{g}')$, where the latter has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	$2se_1$	te_1+se_2	u_1
e_2	0	0	te_1+se_2	$2te_2$	u_2
u_1	$-2se_1$	$-se_2-te_1$	0	su_2-tu_1	$b'_1e_1+b'_2e_2+2su_3$
u_2	$-te_1-se_2$	$-2te_2$	tu_1-su_2	0	$c'_1e_1+c'_2e_2+2tu_3$
u_3	$-u_1$	$-u_2$	$-b'_1e_1-b'_2e_2-2su_3$	$-c'_1e_1-c'_2e_2-2tu_3$	0

Thus, without loss of generality, it can be assumed that $t \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{27}, \mathfrak{g}_{27})$ by means of the mapping $\pi' : \bar{\mathfrak{g}}_{27} \to \bar{\mathfrak{g}}'$, where

$$\pi'(e_1) = \frac{1}{t}e_1,$$
$$\pi'(e_2) = e_2 - \frac{s}{t}e_1,$$
$$\pi'(u_1) = \frac{1}{t}u_1,$$
$$\pi'(u_2) = u_2 - \frac{s}{t}u_1,$$
$$\pi'(u_3) = u_3 + \frac{b'_1}{4t}e_1 + \frac{b'_2}{3t}e_2 - \frac{b'_2 s}{12t^2}e_1,$$

(we have $c'_1 = c'_2 = 0$.)

2°. $p = s = t = w = 0$.

The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$b_1e_1+b_2e_2+\beta_1u_1+\beta_2u_2$
u_2	0	0	0	0	$c_1e_1+c_2e_2+\gamma_1u_1+\gamma_2u_2$
u_3	$-u_1$	$-u_2$	$-b_1e_1-b_2e_2-\beta_1u_1-\beta_2u_2$	$-c_1e_1-c_2e_2-\gamma_1u_1-\gamma_2u_2$	0

(*)

There is a one-to-one correspondence between the set of pairs (*) and the set of pairs (A, B), $A, B \in \mathfrak{gl}(2,\mathbb{R})$, where

$$A = \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}, \quad B = \begin{pmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}.$$

Lemma 2. *Suppose $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$ are the pairs corresponding to pairs (A, B) and (A', B') of matrices respectively. Then the pairs are equivalent if and only if there exist $\lambda \in \mathbb{R}^*$, and $S \in \mathrm{GL}(2,\mathbb{R})$ such that*

$$\begin{cases} A' = \lambda^2 SAS^{-1} \\ B' = \lambda SBS^{-1} \end{cases} \qquad (**)$$

Proof. Suppose condition (**) is satisfied. Then the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$ are equivalent by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where the matrix of π has the form:

$$\begin{pmatrix} S & 0 & 0 \\ 0 & \lambda S & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Suppose $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. Then there exists $\pi' : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that $\pi'(\mathfrak{g}') = \mathfrak{g}$. Its matrix has the form:

$$A = \begin{pmatrix} S & 0 & * \\ 0 & \lambda S & * \\ 0 & 0 & \lambda \end{pmatrix}, \quad S \in \mathrm{GL}(2, \mathbb{R}), \quad \lambda \in \mathbb{R}^*,$$

in terms of the basis fixed before. This immediately implies condition (**).

Thus the proof of the Lemma is completed.

Let us remark that the classification of pairs (*) (up to equivalence) reduces to the classification of the pairs of matrices (A, B), $A, B \in \mathfrak{gl}(2, \mathbb{R})$ (viewed up to transformations (**)). As a result of the latter classification we obtain $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, \ldots, 26$.

Since the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, \ldots, 26$, are solvable and the pair $(\bar{\mathfrak{g}}_{27}, \mathfrak{g}_{27})$ is unsolvable, we see that no one of the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, \ldots, 26$, is equivalent to the pair $(\bar{\mathfrak{g}}_{27}, \mathfrak{g}_{27})$.

Thus the proof of the Proposition is complete.

Proposition 2.18. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.18 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1
e_2	$-e_1$	0	0	u_1	u_3
u_1	0	0	0	0	0
u_2	0	$-u_1$	0	0	0
u_3	$-u_1$	$-u_3$	0	0	0

2.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	u_1
e_2	$-e_1$	0	0	u_1	u_3
u_1	0	0	0	u_1	0
u_2	0	$-u_1$	$-u_1$	0	$-u_3$
u_3	$-u_1$	$-u_3$	0	u_3	0

3.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	0	$e_2 + u_1$
e_2	$-e_1$	0	0	u_1	u_3
u_1	0	0	0	$-u_1$	0
u_2	0	$-u_1$	u_1	0	0
u_3	$-e_2 - u_1$	$-u_3$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_2.

Lemma. *Any virtual structure q on generalized module 2.18 is equivalent to one of the following:*

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = 0.$$

Proof. Without loss of generality it can be assumed that q is primary. Then we have:

$$C(e_1) = \begin{pmatrix} c_{11}^1 & c_{12}^1 & 0 \\ 0 & 0 & c_{23}^1 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^2 & c_{22}^2 & 0 \end{pmatrix}.$$

Let us check condition (6), Chapter II. After some calculation we obtain:

$$C(e_1) = \begin{pmatrix} c_{11}^1 & c_{12}^1 & 0 \\ 0 & 0 & c_{23}^1 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -c_{11}^1 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ c_{11}^1 & c_{12}^1 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) - A(x)H + HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_1(e_2) = 0.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.18. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{aligned} &[e_1, e_2] = e_1, \\ &[e_1, u_1] = 0, && [e_2, u_1] = 0, \\ &[e_1, u_2] = 0, && [e_2, u_2] = u_1, \\ &[e_1, u_3] = pe_2 + u_1, && [e_2, u_3] = u_3. \end{aligned}$$

Since the virtual structure q is primary, we have:

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$

where

$$\begin{aligned}
\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) &= \mathbb{R}e_1, \\
\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) &= \mathbb{R}e_2 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2, \\
\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) &= \mathbb{R}u_3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
[u_1, u_2] &= a_2 e_2 + \alpha_1 u_1 + \alpha_2 u_2, \\
[u_1, u_3] &= \beta_3 u_3, \\
[u_2, u_3] &= \gamma_3 u_3.
\end{aligned}$$

Using the Jacobi identity we see that

$$\begin{aligned}
[u_1, u_2] &= -(p + \gamma_3) u_1, \\
[u_1, u_3] &= 0, \\
[u_2, u_3] &= \gamma_3 u_3,
\end{aligned}$$

where the coefficients α_1 and p satisfy the equation $\gamma_3 p = 0$.

Consider the following cases:

1°. $\gamma_3 = p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $\gamma_3 \neq 0$, $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(u_1) &= -\gamma_3 u_1, \\
\pi(u_2) &= -\gamma_3 u_2, \\
\pi(u_3) &= -\gamma_3 u_3.
\end{aligned}$$

3°. $\gamma_3 = 0$, $p \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(u_1) &= p u_1, \\
\pi(u_2) &= p u_2, \\
\pi(u_3) &= p u_3.
\end{aligned}$$

It is evident that the pairs determined in the Proposition are not equivalent to each other.

Proposition 2.19. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$(\lambda-1)e_1$	0	0	u_1
e_2	$(1-\lambda)e_1$	0	u_1	u_1+u_2	λu_3
u_1	0	$-u_1$	0	0	0
u_2	0	$-u_1-u_2$	0	0	0
u_3	$-u_1$	$-\lambda u_3$	0	0	0

2. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	0	0	u_1
e_2	e_1	0	u_1	u_1+u_2	0
u_1	0	$-u_1$	0	0	0
u_2	0	$-u_1-u_2$	0	0	e_1
u_3	$-u_1$	0	0	$-e_1$	0

3. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	0	0	u_1
e_2	e_1	0	u_1	u_1+u_2	0
u_1	0	$-u_1$	0	0	0
u_2	0	$-u_1-u_2$	0	0	$-e_1$
u_3	$-u_1$	0	0	e_1	0

4. $\lambda = 0$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	0	0	u_1
e_2	e_1	0	u_1	u_1+u_2	0
u_1	0	$-u_1$	0	0	u_1
u_2	0	$-u_1-u_2$	0	0	αe_1+u_2
u_3	$-u_1$	0	$-u_1$	$-\alpha e_1-u_2$	0

5. $\lambda = 1/2$

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-\frac{1}{2}e_1$	0	0	u_1
e_2	$\frac{1}{2}e_1$	0	u_1	u_1+u_2	$e_1+\frac{1}{2}u_3$
u_1	0	$-u_1$	0	0	0
u_2	0	$-u_1-u_2$	0	0	0
u_3	$-u_1$	$-e_1-\frac{1}{2}u_3$	0	0	0

Proof.
Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & \lambda-1 \\ 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_2.

Lemma. *Any virtual structure q on generalized module 2.19 is equivalent to one of the following:*

a) $\lambda \neq 0$, $\lambda \neq \frac{1}{2}$, *and* $\lambda \neq 2$

$$C_1(e_1) = C_1(e_2) = 0;$$

b) $\lambda = 0$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} -p & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad p \in \mathbb{R};$$

c) $\lambda = \frac{1}{2}$

$$C_3(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_3(e_2) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}, \quad p \in \mathbb{R};$$

d) $\lambda = 2$

$$C_4(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}, \quad C_4(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p \in \mathbb{R}.$$

Proof. Let q be a virtual structure on generalized module 2.19. Without loss of generality it can be assumed that q is primary. Consider the following cases:

a) $\lambda \neq 0$, $\lambda \neq \frac{1}{2}$, and $\lambda \neq 2$. Since

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(1-\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_1,$$
$$U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad U^{(\lambda)}(\mathfrak{h}) \supset \mathbb{R}u_3,$$

we have

$$C(e_1) = C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, the virtual structure q is trivial. Put $C_1 = C$.

b) $\lambda = 0$. Since

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}e_1,$$
$$U^{(1)}(\mathfrak{h}) = \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad U^{(0)}(\mathfrak{h}) = \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{11} & c^2_{12} & 0 \\ 0 & 0 & c^2_{23} \end{pmatrix}.$$

Let us check condition (6), Chapter II, for $x, y \in \mathcal{E}$.

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have:

$$\begin{pmatrix} 0 & 0 & -c^1_{13} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -c^2_{23} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & c^2_{11} \\ 0 & 0 & 0 \end{pmatrix} -$$

$$-\begin{pmatrix} 0 & 0 & c_{13}^1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $c_{11}^2 + c_{23}^2 = 0$, and the virtual structure q has the form:

$$C(e_1) = \begin{pmatrix} 0 & 0 & c_{13}^1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} -c_{23}^2 & c_{12}^2 & 0 \\ 0 & 0 & c_{23}^2 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c_{12}^2 & 0 & 0 \\ 0 & 0 & c_{13}^1 - c_{12}^2 \end{pmatrix},$$

and $C_2(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} -c_{23}^2 & 0 & 0 \\ 0 & 0 & c_{23}^2 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_2 are equivalent.

c) $\lambda = \frac{1}{2}$. Since

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(1/2)}(\mathfrak{h}) = \mathbb{R}e_1,$$
$$U^{(1)}(\mathfrak{h}) = \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad U^{(1/2)}(\mathfrak{h}) = \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now put $C_3 = C$.

d) $\lambda = 2$. Since

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_1,$$
$$U^{(1)}(\mathfrak{h}) = \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad U^{(2)}(\mathfrak{h}) = \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^1 & c_{22}^1 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us check condition (6), Chapter II, for e_1 and e_2. We have

$$\begin{pmatrix} 0 & 0 & 0 \\ c_{21}^1 & c_{22}^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^1 & c_{22}^1 + c_{21}^1 & 0 \end{pmatrix}.$$

It follows that $c_{21}^1 = 0$, and the virtual structure q has the form:

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22}^1 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now put $C_4 = C$.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.19. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\lambda \neq 0$, $\lambda \neq \frac{1}{2}$, and $\lambda \neq 2$. Then

$$\begin{aligned} &[e_1, e_2] = (\lambda - 1)e_1, && \\ &[e_1, u_1] = 0, && [e_2, u_1] = u_1, \\ &[e_1, u_2] = 0, && [e_2, u_2] = u_1 + u_2, \\ &[e_1, u_3] = u_1, && [e_2, u_3] = \lambda u_3. \end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*.$$

Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(1-\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_1,$$
$$U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad U^{(\lambda)}(\mathfrak{h}) \supset \mathbb{R}u_3,$$

and

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(1+\lambda)}(\mathfrak{h}), \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(1+\lambda)}(\mathfrak{h}). \end{aligned}$$

1.1°. Suppose $\lambda = -1$. Then

$$\begin{aligned} &[u_1, u_2] = a_1 e_1, \\ &[u_1, u_3] = b_2 e_2, \\ &[u_2, u_3] = c_2 e_2. \end{aligned}$$

Let us check the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1, 2$; $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3):

1. $[e_1, [u_1, u_3]] + [u_3, [e_1, u_1]] + [u_1, [u_3, e_1]] = 0$
$-2b_2 e_2 = 0$
1. $b_2 = 0$

2. $[e_1, [u_2, u_3]] + [u_3, [e_1, u_2]] + [u_2, [u_3, e_1]] = 0$
$-2c_2 e_2 + a_1 e_1 = 0$
2. $c_2 = 0$
3. $a_1 = 0$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

1.2°. Suppose $\lambda \neq -1$. Then

$$\begin{aligned} &[u_1, u_2] = 0, \\ &[u_1, u_3] = 0, \\ &[u_2, u_3] = 0. \end{aligned}$$

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $\lambda = 0$. Then

$$\begin{aligned} &[e_1, e_2] = -e_1, \\ &[e_1, u_1] = 0, \qquad [e_2, u_1] = u_1 - pe_1, \\ &[e_1, u_2] = 0, \qquad [e_2, u_2] = u_1 + u_2, \\ &[e_1, u_3] = u_1, \qquad [e_2, u_3] = pe_2. \end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}u_3, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2.$$

Therefore

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \end{aligned}$$

and

$$\begin{aligned} &[u_1, u_2] = 0, \\ &[u_1, u_3] = b_1 e_1 + \beta_1 u_1 + \beta_2 u_2, \\ &[u_2, u_3] = c_1 e_1 + \gamma_1 u_1 + \gamma_2 u_2. \end{aligned}$$

Using the Jacobi identity we obtain:

$$\begin{cases} p = b_1 = \beta_2 = 0, \\ \gamma_2 = \beta_1. \end{cases}$$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	0	0	u_1
e_2	e_1	0	u_1	$u_1 + u_2$	0
u_1	0	$-u_1$	0	0	$\beta_1 u_1$
u_2	0	$-u_1 - u_2$	0	0	$c_1 e_1 + \gamma_1 u_1 + \beta_1 u_2$
u_3	$-u_1$	0	$-\beta u_1$	$-c_1 e_1 - \gamma_1 u_1 - \beta_1 u_2$	0

2.1°. Suppose $\beta_1 = 0$.

2.1.1°. $c_1 = 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, \\ &\pi(u_1) = u_1, \\ &\pi(u_2) = u_2 - \gamma_1 e_1, \\ &\pi(u_3) = u_3. \end{aligned}$$

2.1.2°. $c_1 > 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(u_1) &= \frac{1}{\sqrt{c_1}} u_1, \\
\pi(u_2) &= \frac{1}{\sqrt{c_1}} (u_2 - \gamma_1 e_1), \\
\pi(u_3) &= \frac{1}{\sqrt{c_1}} u_3.
\end{aligned}$$

2.1.3°. $c_1 < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(u_1) &= \frac{1}{\sqrt{-c_1}} u_1, \\
\pi(u_2) &= \frac{1}{\sqrt{-c_1}} (u_2 - \gamma_1 e_1), \\
\pi(u_3) &= \frac{1}{\sqrt{-c_1}} u_3.
\end{aligned}$$

2.2°. Suppose $\beta_1 \neq 0$. Then the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned}
\pi(e_1) &= \beta_1 e_1, \\
\pi(e_2) &= e_2, \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= u_2 - \gamma_1 e_1, \\
\pi(u_3) &= \frac{1}{\beta_1} u_3,
\end{aligned}$$

establishes the equivalence of $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$. Here $\alpha = \frac{1}{\beta_1^2}(c_1 + \beta_1 \gamma_1)$.

It remains to show that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, 2, 3, 4$, are not equivalent to each other.

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{R})$, where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i}$, $i = 1, \ldots, 4$, in the basis $\{u_1, e_1, u_2\}$. We have:

$$f_1(\bar{\mathfrak{g}}_1) = \left\{ \begin{pmatrix} x & y & x \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\},$$

$$f_2(\bar{\mathfrak{g}}_2) = \left\{ \begin{pmatrix} x & y & x \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\},$$

$$f_3(\bar{\mathfrak{g}}_3) = \left\{ \begin{pmatrix} x & y & x \\ 0 & x & -y \\ 0 & 0 & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\},$$

$$f_4(\bar{\mathfrak{g}}_4) = \left\{ \begin{pmatrix} x+y & y & x+y \\ 0 & x & \alpha y \\ 0 & 0 & x+y \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}.$$

Since the subalgebras $f_1(\bar{\mathfrak{g}}_1)$, $f_2(\bar{\mathfrak{g}}_2)$, $f_3(\bar{\mathfrak{g}}_3)$, and $f_4(\bar{\mathfrak{g}}_4)$ of the Lie algebra $\mathfrak{gl}(3, \mathbb{R})$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$, $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$, and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ are not equivalent to each other.

3°. $\lambda = \frac{1}{2}$. Then

$$\begin{aligned} &[e_1, e_2] = -\frac{1}{2} e_1, && \\ &[e_1, u_1] = 0, && [e_2, u_1] = u_1, \\ &[e_1, u_2] = 0, && [e_2, u_2] = u_1 + u_2, \\ &[e_1, u_3] = u_1, && [e_2, u_3] = p e_1 + \frac{1}{2} u_3. \end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(1/2)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_3.$$

Therefore

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(3/2)}(\mathfrak{h}), \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(3/2)}(\mathfrak{h}), \end{aligned}$$

and

$$\begin{aligned} &[u_1, u_2] = 0, \\ &[u_1, u_3] = 0, \\ &[u_2, u_3] = 0. \end{aligned}$$

3.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

3.2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(e_2) &= e_2, \\ \pi(u_1) &= \frac{1}{p} u_1, \\ \pi(u_2) &= \frac{1}{p} u_2, \\ \pi(u_3) &= \frac{1}{p} u_3. \end{aligned}$$

Since the virtual structures $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ and $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$ for any $i = 1, 2, 3, 4$ are not isomorphic, we see that the pairs determined in the Proposition are not equivalent to each other.

4°. $\lambda = 2$. Then

$$\begin{aligned} &[e_1, e_2] = e_1, \\ &[e_1, u_1] = 0, \qquad [e_2, u_1] = u_1, \\ &[e_1, u_2] = pe_2, \quad [e_2, u_2] = u_1 + u_2, \\ &[e_1, u_3] = u_1, \quad [e_2, u_3] = 2u_3. \end{aligned}$$

Since the virtual structure q is primary, we have:

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_1 \oplus \mathbb{R}u_2,$$
$$\bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{R}u_3, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_1.$$

Therefore

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(3)}(\mathfrak{h}), \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(3)}(\mathfrak{h}), \end{aligned}$$

and

$$\begin{aligned} &[u_1, u_2] = \alpha_3 u_3, \\ &[u_1, u_3] = 0, \\ &[u_2, u_3] = 0. \end{aligned}$$

Using the Jacobi identity we obtain $p = \alpha_3 = 0$. It is clear that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

This completes the proof of the Proposition.

Proposition 2.20. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.20 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	0
e_2	0	0	0	0	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	0	$-u_1$	0	0	0

2.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	0
e_2	0	0	0	0	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1
u_3	0	$-u_1$	0	$-e_1$	0

3.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$e_1 + u_1$	0
e_2	0	0	0	e_2	u_1
u_1	0	0	0	$2u_1$	0
u_2	$-e_1 - u_1$	$-e_2$	$-2u_1$	0	$e_2 - u_3$
u_3	0	$-u_1$	0	$-e_2 + u_3$	0

4.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	0
e_2	0	0	0	e_2	u_1
u_1	0	0	0	$2u_1$	0
u_2	$-e_1-u_1$	$-e_2$	$-2u_1$	0	e_1-u_3
u_3	0	$-u_1$	0	$-e_1+u_3$	0

5.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	0
e_2	0	0	0	e_2	u_1
u_1	0	0	0	u_1	0
u_2	$-e_1-u_1$	$-e_2$	$-u_1$	0	0
u_3	0	$-u_1$	0	0	0

6.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	0
e_2	0	0	0	e_1	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	$-e_1$	0	0	e_2
u_3	0	$-u_1$	0	$-e_2$	0

7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	0
e_2	0	0	0	e_1	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	$-e_1$	0	0	0
u_3	0	$-u_1$	0	0	0

8.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	0
e_2	0	0	0	e_1+e_2	u_1
u_1	0	0	0	$2u_1$	0
u_2	$-e_1-u_1$	$-e_1-e_2$	$-2u_1$	0	e_2-u_3
u_3	0	$-u_1$	0	$-e_2+u_3$	0

9.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	αe_1
e_2	0	0	0	0	$u_1+(\alpha+1)e_2$
u_1	0	0	0	0	$2\alpha u_1$
u_2	$-u_1$	0	0	0	$e_1+\alpha u_2$
u_3	$-\alpha e_1$	$-u_1-(\alpha+1)e_2$	$-2\alpha u_1$	$-e_1-\alpha u_2$	0

10.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	αe_1
e_2	0	0	0	0	$u_1+(\alpha+1)e_2$
u_1	0	0	0	0	$(2\alpha+1)u_1$
u_2	$-u_1$	0	0	0	$e_2+(\alpha+1)u_2$
u_3	$-\alpha e_1$	$-u_1-(\alpha+1)e_2$	$-(2\alpha+1)u_1$	$-e_2-(\alpha+1)u_2$	0

11.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-e_1$
e_2	0	0	0	e_1	u_1
u_1	0	0	0	$-e_1$	$-u_1$
u_2	$-u_1$	$-e_1$	e_1	0	$-e_2$
u_3	e_1	$-u_1$	u_1	e_2	0

12.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-2e_1$
e_2	0	0	0	e_1	$-e_2 + u_1$
u_1	0	0	0	0	$-3u_1$
u_2	$-u_1$	$-e_1$	0	0	$e_2 - u_2$
u_3	$2e_1$	$e_2 - u_1$	$3u_1$	$u_2 - e_2$	0

13.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-e_1$
e_2	0	0	0	$-e_1$	u_1
u_1	0	0	0	e_1	$-u_1$
u_2	$-u_1$	e_1	$-e_1$	0	e_2
u_3	e_1	$-u_1$	u_1	$-e_2$	0

14.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-2e_1$
e_2	0	0	0	$-e_1$	$-e_2 + u_1$
u_1	0	0	0	0	$-3u_1$
u_2	$-u_1$	e_1	0	0	$e_2 - u_2$
u_3	$2e_1$	$e_2 - u_1$	$3u_1$	$u_2 - e_2$	0

15.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$u_1 + e_1$	e_2
e_2	0	0	0	e_2	u_1
u_1	0	0	0	u_1	0
u_2	$-u_1 - e_1$	$-e_2$	$-u_1$	0	0
u_3	$-e_2$	$-u_1$	0	0	0

16.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$u_1 - e_1$	e_2
e_2	0	0	0	$-e_2$	u_1
u_1	0	0	0	$-u_1$	0
u_2	$-u_1 + e_1$	e_2	u_1	0	0
u_3	$-e_2$	$-u_1$	0	0	0

17.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$u_1 + \alpha e_1$	$e_1 + e_2$
e_2	0	0	0	αe_2	$u_1 + e_2$
u_1	0	0	0	αu_1	u_1
u_2	$-u_1 - \alpha e_1$	$-\alpha e_2$	$-\alpha u_1$	0	0
u_3	$-e_1 - e_2$	$-u_1 - e_2$	$-u_1$	0	0

18.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	αe_1
e_2	0	0	0	0	$u_1 + \alpha e_2$
u_1	0	0	0	0	$(\alpha + 1)u_1$
u_2	$-u_1$	0	0	0	u_2
u_3	$-\alpha e_1$	$-u_1 - \alpha e_2$	$-(\alpha + 1)u_1$	$-u_2$	0

19.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$(\beta+1)e_1$
e_2	0	0	0	0	$u_1+\beta e_2$
u_1	0	0	0	0	$(\alpha+\beta)u_1$
u_2	$-u_1$	0	0	0	$(\alpha-1)u_2$
u_3	$-(\beta+1)e_1$	$-u_1-\beta e_2$	$-(\alpha+\beta)u_1$	$(1-\alpha)u_2$	0

20.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1+e_1	$(\beta+1)e_1$
e_2	0	0	0	e_2	$u_1+\beta e_2$
u_1	0	0	0	u_1	$(\beta+1)u_1$
u_2	$-u_1-e_1$	$-e_2$	$-u_1$	0	0
u_3	$-(\beta+1)e_1$	$-u_1-\beta e_2$	$-(\beta+1)u_1$	0	0

21.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$u_1+\beta e_1$	αe_1
e_2	0	0	0	$e_1+\beta e_2$	$u_1+(\alpha+1)e_2$
u_1	0	0	0	βu_1-e_1	αu_1
u_2	$-u_1-\beta e_1$	$-e_1-\beta e_2$	$-\beta u_1+e_1$	0	0
u_3	$-\alpha e_1$	$-u_1-(\alpha+1)e_2$	$-\alpha u_1$	0	0

, $\beta \geqslant 0$

22.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	αe_1
e_2	0	0	0	e_1	$u_1+(\alpha+1)e_2$
u_1	0	0	0	0	$(\alpha-1)u_1$
u_2	$-u_1$	$-e_1$	0	0	$-u_2$
u_3	$-\alpha e_1$	$-u_1-(\alpha+1)e_2$	$(1-\alpha)u_1$	u_2	0

23.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	αe_1+u_1	βe_1
e_2	0	0	0	αe_2-e_1	$u_1+(\beta+1)e_2$
u_1	0	0	0	$e_1+\alpha u_1$	βu_1
u_2	$-\alpha e_1-u_1$	$-\alpha e_2+e_1$	$-e_1-\alpha u_1$	0	0
u_3	$-\beta e_1$	$-u_1-(\beta+1)e_2$	$-\beta u_1$	0	0

, $\alpha \geqslant 0$

24.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	αe_1
e_2	0	0	0	$-e_1$	$u_1+(\alpha+1)e_2$
u_1	0	0	0	0	$(\alpha-1)u_1$
u_2	$-u_1$	e_1	0	0	$-u_2$
u_3	$-\alpha e_1$	$-u_1-(\alpha+1)e_2$	$(1-\alpha)u_1$	u_2	0

25.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$u_1 - \alpha e_1$	$\frac{\alpha-1}{3} e_1$
e_2	0	0	0	$-e_1 - \alpha e_2$	$u_1 + \frac{\alpha+2}{3} e_2$
u_1	0	0	0	$\frac{3}{2} e_1 - e_2 - \frac{3+2\alpha}{2} u_1$	$\frac{2\alpha+1}{6} u_1 - \frac{1}{2} e_1$
u_2	$-u_1 + \alpha e_1$	$e_1 + \alpha e_2$	$-\frac{3}{2} e_1 + e_2 + \frac{3+2\alpha}{2} u_1$	0	$\frac{1}{2}(u_2 + 3u_3)$
u_3	$\frac{1-\alpha}{3} e_1$	$-u_1 - \frac{\alpha+2}{3} e_2$	$\frac{1}{2} e_1 - \frac{2\alpha+1}{6} u_1$	$-\frac{1}{2}(u_2 + 3u_3)$	0

26.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$e_2 + \alpha e_1$
e_2	0	0	0	0	$u_1 + \alpha e_2$
u_1	0	0	0	0	$(\alpha + 1) u_1$
u_2	$-u_1$	0	0	0	u_2
u_3	$-e_2 - \alpha e_1$	$-u_1 - \alpha e_2$	$-(\alpha + 1) u_1$	$-u_2$	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of the Lie algebra $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A(e_1) = A(e_2) = 0$ and for $x \in \mathfrak{g}$, the matrix $B(x)$ is identified with x.

Lemma 1. *Any virtual structure q on the generalized module 2.20 has the form:*

$$C(e_1) = \begin{pmatrix} 0 & c_{12}^1 & c_{13}^1 \\ 0 & c_{22}^1 & c_{23}^1 \end{pmatrix}, \qquad C(e_2) = \begin{pmatrix} 0 & c_{12}^2 & c_{13}^2 \\ 0 & c_{22}^2 & c_{23}^2 \end{pmatrix}.$$

Proof. Put

$$C(e_i) = \left(c_{jk}^i\right)_{\substack{1 \leqslant j \leqslant 2 \\ 1 \leqslant k \leqslant 3}}, \quad i = 1, 2.$$

Checking condition (6), Chapter II, for e_1 and e_2, we obtain

$$c_{11}^1 = c_{21}^1 = c_{11}^2 = c_{21}^2 = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.20. Then the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by a certain virtual structure on the generalized module 2.20. Lemma 1 implies that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$c_{12}^1 e_1 + c_{22}^1 e_2 + u_1$	$c_{13}^1 e_1 + c_{23}^1 e_2$
e_2	0	0	0	$c_{12}^2 e_1 + c_{22}^2 e_2$	$c_{13}^2 e_1 + c_{23}^2 e_2 + u_1$
u_1	0	0	0	x	y
u_2	$-c_{12}^1 e_1 - c_{22}^1 e_2 - u_1$	$-c_{12}^2 e_1 - c_{22}^2 e_2$	$-x$	0	z
u_3	$-c_{13}^1 e_1 - c_{23}^1 e_2$	$-c_{13}^2 e_1 - c_{23}^2 e_2 - u_1$	$-y$	$-z$	0

,

where $x, y, z \in \bar{\mathfrak{g}}$.

Therefore $\mathfrak{a} = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}u_1$ is a commutative subalgebra of $\bar{\mathfrak{g}}$. Note that $\mathfrak{a} = Z_{\bar{\mathfrak{g}}}(\mathfrak{g})$.

Now prove that $\mathfrak{a}$ is an ideal in $\bar{\mathfrak{g}}$. It is clear that $[e_1, x] \in \mathfrak{a}$ and $[e_2, x] \in \mathfrak{a}$ for all $x \in \bar{\mathfrak{g}}$. Assume that there exists $x \in \bar{\mathfrak{g}}$ such that $[u_1, x] \notin \mathfrak{a}$. Then there exists $y \in \mathfrak{g}$ such that $\big[y, [u_1, x]\big] \neq 0$. But $\big[u_1, [x, y]\big] = 0$ (since $[x, y] \in \mathfrak{a}$) and $\big[x, [y, u_1]\big] = 0$ (since $[y, u_1] = 0$). Therefore the Jacobi identity for the triple (x, y, u_1)

$$\big[y, [u_1, x]\big] + \big[u_1, [x, y]\big] + \big[x, [y, u_1]\big] = 0$$

does not hold. Thus, our assumption leads to a contradiction, and therefore $\mathfrak{a}$ is a commutative ideal in $\bar{\mathfrak{g}}$.

Consider two cases:

I. The Lie algebra contains no subalgebras complementary to $\mathfrak{a}$.

Lemma 2. *Any virtual structure on the generalized module 2.20 is equivalent to one and only one of the following:*

a) $C_1(e_1) = C_1(e_2) = 0$;

b) $C_2(e_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C_2(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$;

c) $C_3(e_1) = 0$, $C_3(e_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

d) $C_4(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $C_4(e_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

e) $C_5(e_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C_5(e_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$;

f) $C_6(e_1) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $C_6(e_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$;

g) $C_7(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}$, $C_7(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p+1 \end{pmatrix}$;

h) $C_8(e_1) = \begin{pmatrix} 0 & 1 & p \\ 0 & 0 & 0 \end{pmatrix}$, $C_8(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & p+1 \end{pmatrix}$;

i) $C_9(e_1) = \begin{pmatrix} 0 & p & q \\ 0 & 0 & 0 \end{pmatrix}$, $C_9(e_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & p & q+1 \end{pmatrix}$, $p \geqslant 0$;

j) $C_9(e_1) = \begin{pmatrix} 0 & p & q \\ 0 & 0 & 0 \end{pmatrix}$, $C_9(e_2) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & p & q+1 \end{pmatrix}$, $p \geqslant 0$.

Proof. The Lemma is immediate from the classification of the virtual pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.20 ($\lambda = \mu = 0$) (Proposition 3.20).

Now suppose that the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the virtual structure a) from Lemma 2. Then

$$\begin{aligned} &[e_1, e_2] = 0, \\ &[e_1, u_1] = 0, \quad [e_2, u_1] = 0, \\ &[e_1, u_2] = u_1, \quad [e_2, u_2] = 0, \\ &[e_1, u_3] = 0, \quad [e_2, u_3] = u_1. \end{aligned}$$

Since $\mathfrak{a}$ is an ideal, it can be assumed that

$$[u_1, u_2] = a_1 e_1 + a_2 e_2 + \alpha_1 u_1,$$
$$[u_1, u_3] = b_1 e_1 + b_2 e_2 + \beta_1 u_1,$$
$$[u_2, u_3] = c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.$$

Using the Jacobi identity, we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	0
e_2	0	0	0	0	u_1
u_1	0	0	0	$-\gamma_3 u_1$	$\gamma_2 u_1$
u_2	$-u_1$	0	$\gamma_3 u_1$	0	x
u_3	0	$-u_1$	$-\gamma_2 u_1$	$-x$	0 ,

where $x = c_1 e_1 + c_2 e_2 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3$.

Put $u_2' = u_2 + x_1 e_1 + x_2 e_2 + x_3 u_1$ and $u_3' = u_3 + y_1 e_1 + y_2 e_2 + y_3 u_1$. The subspace $\mathbb{R}u_2' \oplus \mathbb{R}u_3'$ is not a subalgebra complementary to $\mathfrak{a}$ if and only if the equation

$$[u_2', u_3'] = s u_2' + t u_3' \qquad (1)$$

is unsolvable (with respect to s, t, x_i, y_j, $i, j = 1, 2, 3$). From (1) it follows that $s = \gamma_2$, $t = \gamma_3$, and that the variables x_i, y_j, $i, j = 1, 2, 3$, satisfy the following system of linear equations:

$$\begin{cases} \gamma_2 x_1 + \gamma_3 y_1 & = c_1, \\ \gamma_2 x_2 + \gamma_3 y_2 & = c_2, \\ y_1 - x_2 & = \gamma_1. \end{cases}$$

The system is inconsistent if and only if $\gamma_2 = \gamma_3 = 0$ and $c_1^2 + c_2^2 \neq 0$.

There is no restriction of generality in assuming $c_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi \colon \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = c_1 e_1 + c_2 e_2, \qquad \pi(e_2) = e_2,$$
$$\pi(u_1) = u_1, \quad \pi(u_2) = \tfrac{1}{c_1} u_2 - \tfrac{\gamma_1}{c_1} e_2, \quad \pi(u_3) = c_1 u_3 - c_2 u_2.$$

In a similar way we consider the cases b)–j) of Lemma 2 and obtain the following pairs:

b) 2.20.3, 2.20.4;
c) 2.20.6;
d) the desired pairs do not exist
e) 2.20.8;
f) the desired pairs do not exist
g) 2.20.9, 2.20.10;
h) the desired pairs do not exist
i) 2.20.11, 2.20.12;
j) 2.20.13, 2.20.14.

The pairs obtained in different cases are not equivalent, since the corresponding virtual pairs are not isomorphic.

Since the algebra

$$[\bar{\mathfrak{g}}_3, \mathcal{D}^2\bar{\mathfrak{g}}_3] + \mathcal{D}^2\bar{\mathfrak{g}}_3$$

is non-commutative and the algebra

$$[\bar{\mathfrak{g}}_4, \mathcal{D}^2\bar{\mathfrak{g}}_4] + \mathcal{D}^2\bar{\mathfrak{g}}_4$$

is commutative, we see that the pairs $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ are not equivalent.

Let $\pi : \bar{\mathfrak{g}}_i \to \bar{\mathfrak{a}}_i = \bar{\mathfrak{g}}_i/\mathcal{D}^2\bar{\mathfrak{g}}_i$, $i = 9, 10$, be a canonical surjection and $\mathfrak{a}_i = \pi\big(Z(\mathcal{D}\bar{\mathfrak{g}}_i)\big)$. Then the pairs $(\bar{\mathfrak{a}}_9, \mathfrak{a}_9)$ and $(\bar{\mathfrak{a}}_{10}, \mathfrak{a}_{10})$ are not equivalent. Therefore the pairs $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ and $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$ are also not equivalent.

The pairs $(\bar{\mathfrak{g}}_{11}, \mathfrak{g}_{11})$ and $(\bar{\mathfrak{g}}_{12}, \mathfrak{g}_{12})$ are not equivalent, since the corresponding virtual pairs are not isomorphic. Similarly we prove that the pairs $(\bar{\mathfrak{g}}_{13}, \mathfrak{g}_{13})$ and $(\bar{\mathfrak{g}}_{14}, \mathfrak{g}_{14})$ are also not equivalent.

II. The Lie algebra $\bar{\mathfrak{g}}$ does contain subalgebras $\mathfrak{b}$ complementary to the ideal $\mathfrak{a}$.

Lemma 3. *Let d be the projection of $\bar{\mathfrak{g}}$ onto $\mathfrak{a}$ associated with the decomposition $\bar{\mathfrak{g}} = \mathfrak{a} \oplus \mathfrak{b}$. Then $d \in \mathrm{Der}(\bar{\mathfrak{g}}, \mathfrak{g}) = \{\phi \in \mathrm{Der}(\bar{\mathfrak{g}}) \mid \phi(\mathfrak{g}) = \mathfrak{g}\}$.*

Proof. Since $\mathfrak{g} \in \mathfrak{a}$, we have $d(\mathfrak{g}) = \mathfrak{g}$. In order to prove the Lemma it is sufficient to show that the condition

$$d\big([x, y]\big) = \big[d(x), y\big] + \big[x, d(y)\big] \tag{2}$$

holds in the following cases:

1°. $x, y \in \mathfrak{a}$. Then both sides of equality (2) are equal to zero, since the ideal $\mathfrak{a}$ is commutative.

2°. $x \in \mathfrak{a}$, $y \in \mathfrak{b}$. Then $[x, y] \in \mathfrak{a}$ and $d\big([x, y]\big) = [x, y]$. On the other hand, since $d(x) = x$ and $d(y) = 0$, we obtain

$$\big[d(x), y\big] + \big[x, d(y)\big] = [x, y].$$

3°. $x, y \in \mathfrak{b}$. Then $[x, y] \in \mathfrak{b}$ and both sides of equality (2) are equal to zero.

The proof of Lemma 3 is complete.

For every complementary subalgebra $\mathfrak{b}$ let us construct a Lie algebra $\bar{\mathfrak{p}} = \bar{\mathfrak{p}}(\bar{\mathfrak{g}}, \mathfrak{b})$ in the following way:

$\bar{\mathfrak{p}} = \bar{\mathfrak{g}} \times \mathbb{R}$, where the bracket operation is defined by $\big[(x, a), (y, b)\big] = \big([x, y] + \mathrm{ad}(y) - bd(x), 0\big)$. Here d is the projection of $\bar{\mathfrak{g}}$ onto the ideal $\mathfrak{a}$. Let us remark that $(\bar{\mathfrak{p}}, \mathfrak{p})$ is an isotropically-faithful pair of type 3.20 ($\lambda = \mu = 0$). It is easily proved that the pairs $(\bar{\mathfrak{p}}, \mathfrak{p})$ corresponding to different complementary subalgebras $\mathfrak{b}$ are pairwise equivalent.

Since $\dim \bar{\mathfrak{p}} = \dim \bar{\mathfrak{g}} + 1$, we see that $\bar{\mathfrak{g}}$ is a maximal subalgebra of $\bar{\mathfrak{p}}$ such that $\bar{\mathfrak{g}} \supset \mathcal{D}\bar{\mathfrak{p}}$. Moreover, the Lie algebra $\bar{\mathfrak{g}}$ defines uniquely the subalgebra $\mathfrak{g}$, namely $\mathfrak{g} = \bar{\mathfrak{g}} \cap \mathfrak{p}$.

Therefore, in order to determine all the desired pairs it is sufficient

1°. for any pair $(\bar{\mathfrak{p}}, \mathfrak{p})$ of type 3.20 ($\lambda = \mu = 0$) to find (up to the action of the group $\mathrm{Aut}(\bar{\mathfrak{p}}, \mathfrak{p})$) all maximal subalgebras $\bar{\mathfrak{g}}$ in $\bar{\mathfrak{p}}$ of codimension 1 such that $\bar{\mathfrak{g}} \supset \mathcal{D}\bar{\mathfrak{p}}$;
2°. for any subalgebra $\bar{\mathfrak{g}}$ from 1° to construct the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\mathfrak{g} = \bar{\mathfrak{g}} \cap \mathfrak{p}$;
3°. to select all isotropically-faithful pairs of type 2.20 from the obtained ones.

Proposition 3.20 gives the classification of all pairs of type 3.20. In particular, the pairs of type 3.20 ($\lambda = \mu = 0$) are the following ones:
a) 3.20.1 ($\lambda = \mu = 0$);
b) 3.20.4 ($\lambda = \mu = 0$);
c) 3.20.9 ($\lambda = \mu = 0$);
d) 3.20.13;
e) 3.20.14;
f) 3.20.15;
g) 3.20.16;
h) 3.20.17;
i) 3.20.18.

For example, consider the pair 3.20.1 ($\lambda = \mu = 0$). It has the form:

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	0	0
u_3	0	0	$-u_1$	0	0	0

Direct computation shows that

$$\mathrm{Aut}(\bar{\mathfrak{p}}, \mathfrak{p}) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & \multicolumn{2}{c}{\multirow{2}{*}{$\mu\,{}^tA^{-1}$}} & 0 & 0 & 0 \\ b & & & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & c & d \\ 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{A}} \\ 0 & 0 & 0 & 0 & & \end{pmatrix} \middle| \begin{array}{l} \mu \in \mathbb{R}^*; \\ A \in \mathrm{GL}(2, \mathbb{R}), \ \begin{pmatrix} c \\ d \end{pmatrix} = -{}^tA \begin{pmatrix} a \\ b \end{pmatrix} \end{array} \right\}.$$

Up to the action of $\mathrm{Aut}(\bar{\mathfrak{p}}, \mathfrak{p})$, any maximal subalgebra $\bar{\mathfrak{g}}$ of $\bar{\mathfrak{p}}$ such that $\mathcal{D}\bar{\mathfrak{p}} \subset \bar{\mathfrak{g}}$ and $\dim(\bar{\mathfrak{g}} \cap \mathfrak{g}) = 2$ is isomorphic to one of the following algebras:
1°. $\bar{\mathfrak{g}} = \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}u_3$.
In this case $\mathfrak{g} = \bar{\mathfrak{g}} \cap \mathfrak{p} = \mathbb{R}e_2 \oplus \mathbb{R}e_3$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$. The equivalence is established by the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_{i+1}, \quad i = 1, 2;$$
$$\pi(u_j) = u_j, \quad j = 1, 2, 3.$$

2°. $\bar{\mathfrak{g}} = \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}u_1 \oplus \mathbb{R}(u_2 - e_1) \oplus \mathbb{R}u_3$.
In this case $\mathfrak{g} = \bar{\mathfrak{g}} \cap \mathfrak{p} = \mathbb{R}e_2 \oplus \mathbb{R}e_3$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_{i+1}, \quad i = 1, 2;$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2 - e_1,$$
$$\pi(u_3) = u_3.$$

Continuing in the same way, we obtain the pairs 2.20.7, 2.20.16–2.20.26.

Proposition 2.21. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 2.21 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	u_1	λu_2	$(2\lambda-1)u_3$
e_2	$(\lambda-1)e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0
u_3	$(1-2\lambda)u_3$	$-u_2$	0	0	0

2. $\lambda = 1/2$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	u_1	$\frac{1}{2}u_2$	0
e_2	$-\frac{1}{2}e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	$-\frac{1}{2}u_2$	$-u_1$	0	0	e_2
u_3	0	$-u_2$	0	$-e_2$	0

3. $\lambda = 1/2$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	u_1	$\frac{1}{2}u_2$	0
e_2	$-\frac{1}{2}e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	$-\frac{1}{2}u_2$	$-u_1$	0	0	$-e_2$
u_3	0	$-u_2$	0	e_2	0

4. $\lambda = 0$

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	$-u_3$
e_2	$-e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	u_1	u_2
u_2	0	$-u_1$	$-u_1$	0	u_3
u_3	u_3	$-u_2$	$-u_2$	$-u_3$	0

Proof.
Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 2\lambda - 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } \lambda \neq 1.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1-\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 \\ \lambda - 1 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ we denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 2.21 is equivalent to one of the following:*

a) $\lambda = 0$.

$$C(e_1) = C(e_2) = 0;$$

b) $\lambda = \frac{1}{2}$.

$$C(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & -\frac{1}{2}p & 0 \end{pmatrix}, \quad C(e_2) = 0, \quad p \in \mathbb{R};$$

c) $\lambda = \frac{2}{3}$.

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = 0, \quad p \in \mathbb{R};$$

d) $\lambda \notin \{0, \frac{1}{2}, \frac{2}{3}\}$.

$$C(e_1) = C(e_2) = 0.$$

Proof. By statement 9, Chapter II, without loss of generality it can be assumed that the corresponding virtual structure is primary. Then

$$\begin{aligned} &\mathfrak{g}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_1, && U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1, \\ &\mathfrak{g}^{(1-\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_1, && U^{(\lambda)}(\mathfrak{h}) \supset \mathbb{R}u_2, \\ & && U^{(2\lambda-1)}(\mathfrak{h}) \supset \mathbb{R}u_3, \end{aligned}$$

and for

$\lambda = 0$:

$$C(e_1) = \begin{pmatrix} 0 & c_2 & 0 \\ c_1 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_4 \\ 0 & c_3 & 0 \end{pmatrix};$$

$\lambda = \frac{1}{2}$:

$$C(e_1) = \begin{pmatrix} 0 & 0 & c_2 \\ 0 & c_1 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_3 \end{pmatrix};$$

$\lambda = \frac{2}{3}$:

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_1 \end{pmatrix}, \quad C(e_2) = 0.$$

Consider in detail the first case. Checking condition (6), Chapter II, for $x, y \in \mathcal{E}$, we obtain:

$$C(e_1) = 0, \qquad C(e_2) = \begin{pmatrix} 0 & 0 & c_4 \\ 0 & c_3 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & c_4 & 0 \\ c_4 - c_3 & 0 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) - A(x)H + HB(x)$ for $x \in \mathfrak{g}$. Then, we see that $C_1(e_1) = C_1(e_2) = 0$.

Similarly we obtain the other results of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.21. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider now the case, when $\lambda = \frac{1}{2}$.

Using the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1, 2$, $1 \leqslant j < k \leqslant 3$, we see that the pair has the form:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	u_1	$\frac{1}{2}u_2$	0
e_2	$-\frac{1}{2}e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	$\beta_1 u_1$
u_2	$-\frac{1}{2}u_2$	$-u_1$	0	0	$c_2e_2 + \beta_1 u_2$
u_3	0	$-u_2$	$-\beta_1 u_1$	$-c_2e_2 - \beta_1 u_2$	0

1.1°. $4c_2 + \beta_1^2 \neq 0$. The mapping $\pi : \bar{\mathfrak{g}}_{2,3} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(e_2) &= e_2, \\ \pi(u_1) &= \frac{1}{t}u_1, \\ \pi(u_2) &= \frac{1}{t}u_2 + \frac{1}{2}\beta_1 e_2, \\ \pi(u_3) &= \frac{1}{t}u_3 - \beta_1 e_1, \quad t = \frac{2}{\sqrt{|4c_2 + \beta_1^2|}} \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_{2,3}, \mathfrak{g}_{2,3})$.

1.2°. $4c_2 + \beta_1^2 = 0$. The mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(e_2) &= e_2, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2 + \frac{1}{2}\beta_1 e_2, \\ \pi(u_3) &= u_3 - \beta_1 e_1, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $\lambda = \frac{2}{3}$. Similarly we obtain:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$\frac{1}{3}e_2$	u_1	$\frac{2}{3}u_2$	$\frac{1}{3}u_3$
e_2	$-\frac{1}{3}e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	$-\frac{2}{3}u_2$	$-u_1$	0	0	$\gamma_1 u_1$
u_3	$-\frac{1}{3}u_3$	$-u_2$	0	$-\gamma_1 u_1$	0

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(u_1) &= u_1,\\
\pi(u_2) &= u_2,\\
\pi(u_3) &= u_3 - \gamma_1 e_2.
\end{aligned}$$

3°. $\lambda \notin \{\frac{1}{2}, \frac{2}{3}\}$. Continuing in the same way as in the cases 1° and 2°, we obtain:

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	u_1	λu_2	$(2\lambda-1)u_3$
e_2	$(\lambda-1)e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	$\alpha_1 u_1$	$\alpha_1 u_2$
u_2	$-\lambda u_2$	$-u_1$	$-\alpha_1 u_1$	0	$\alpha_1 u_3$
u_3	$(1-2\lambda)u_3$	$-u_2$	$-\alpha_1 u_2$	$-\alpha_1 u_3$	0

,

where the coefficients α_1 and λ satisfy the equation $\alpha_1\lambda = 0$.

4.1°. $\alpha_1 \neq 0$. The mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(u_1) &= \frac{1}{\alpha_1}u_1,\\
\pi(u_2) &= \frac{1}{\alpha_1}u_2,\\
\pi(u_3) &= \frac{1}{\alpha_1}u_3,
\end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$.

4.2°. $\alpha = 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{R})$, $i = 2, 3$, where $f_i(x)$ is the matrix of the mapping $\operatorname{ad} x|_{\mathcal{D}\bar{\mathfrak{g}}_i}$ in the basis $\{e_2, u_1, u_2\}$ of $\mathcal{D}\bar{\mathfrak{g}}_i$, $x \in \bar{\mathfrak{g}}_i$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 2, 3$, are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.
The proof of Proposition is complete.

Proposition 2.22. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.22 is trivial:*

[,]	e_1	e_2	u_1	u_2	u_3
e_1	0	e_1	0	u_1	u_2
e_2	$-e_1$	0	0	u_2	$2u_3$
u_1	0	0	0	0	0
u_2	$-u_1$	$-u_2$	0	0	0
u_3	$-u_2$	$-2u_2$	0	0	0

.

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$, the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$, spanned by e_2.

Lemma. *Any virtual structure q on generalized module 2.22 is equivalent to the following one:*

$$C_1(e_1) = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Let q be a virtual structure on generalized module 2.22. Without loss of generality it can be assumed that q is primary. Since

$$\mathfrak{g}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_1, \quad \mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{R}e_2,$$
$$U^{(0)}(\mathfrak{h}) = \mathbb{R}u_1, \quad U^{(1)}(\mathfrak{h}) = \mathbb{R}u_2, \; U^{(2)}(\mathfrak{h}) = \mathbb{R}u_3,$$

we have

$$C_1(e_1) = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us check condition (6), Chapter II, for $e_1, e_2 \in \mathcal{E}$:

$$c(e_1) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have

$$\begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \end{pmatrix} = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix} + \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so $q = s = 0$.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.22. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the virtual structure determined in the Lemma. Then

$$\begin{aligned}
&[e_1, e_2] = e_1, \\
&[e_1, u_1] = pe_1, \quad [e_2, u_1] = 0, \\
&[e_1, u_2] = u_1, \quad [e_2, u_2] = u_2, \\
&[e_1, u_3 = u_2, \quad [e_2, u_3] = 2u_3.
\end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \quad \text{for all} \quad \alpha \in \mathfrak{h}^*.$$

Thus

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_1, \quad \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}u_1,\\
&\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{R}u_3,
\end{aligned}$$

and

$$\begin{aligned}
[u_1, u_2] &= \alpha_2 u_2,\\
[u_1, u_3] &= \beta_3 u_3,\\
[u_2, u_3] &= 0.
\end{aligned}$$

Let us check the Jacobi identity for the triples (e_i, u_j, u_k) $i = 1, 2,$ $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3).

1. $[e_1, [u_1, u_2]] + [u_1, [u_2, e_1]] + [u_2, [e_1, u_1]] = 0$
$\alpha_2 u_1 - p u_1 = 0$
1. $\alpha_2 - p = 0$

2. $[e_1, [u_1, u_3]] + [u_1, [u_3, e_1]] + [u_3, [e_1, u_1]] = 0$
$\beta_3 u_2 - \alpha_2 u_2 - p u_2 = 0$
2. $\beta_3 - p - \alpha_2 = 0$

3. $[e_1, [u_2, u_3]] + [u_2, [u_3, e_1]] + [u_3, [e_1, u_2]] = 0$
$\beta_3 u_3 = 0$
3. $\beta_3 = 0$

This implies that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

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