ON THE LENGTH OF FAITHFUL MODULES OVER ARTINIAN LOCAL RINGS.

by

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Introduction.

Let R be an Artinian local ring with residue field $k = R/\mathfrak{m}$. Let M be any faithful R-module, i.e. $rM = 0$ implies $r = 0$ for all $r \in R$. Then for a large class of rings R one has the inequality

$$(*) \qquad \ell(M) \geq \ell(R)$$

ℓ denoting classical length. It is easily seen that the inequality is valid whenever R is self injective, that is when $\dim_k \mathrm{Hom}_R(k,R) = 1$; see (2.8) in [1]. The purpose of the present note is to generalize this fact by showing that (*) is valid for all faithful R-modules M whenever $\dim_k \mathrm{Hom}_R(k,R) \leq 3$. This result is in a way the best possible, in fact for each integer $s \geq 4$ we can give an example of a local ring R and a faithful R-module M such that

$$\ell(M) < \ell(R) \qquad \text{and} \qquad \dim_k \mathrm{Hom}_R(k,R) = s.$$

Notations and definitions.

R will always be an Artinian local ring with maximal ideal $\mathfrak{m}$. R-modules are assumed to be unitary and finitely generated. If M is an R-module we define the annihilator

$$an(M) = \{r \in R \mid rM = 0\}$$

and the socle

$$s(M) = \{x \in M \mid \mathfrak{m}x = 0\}.$$

Observe that $s(M) \approx Hom_R(R/\mathfrak{m}, M)$.

$\ell(M)$ denotes the length of M. If $an(M) = \mathfrak{m}$ then dimM will denote the dimension of M as a vectorspace over $R/\mathfrak{m}$. E denotes the injective hull of the R-module $R/\mathfrak{m}$. We let M^* denote the dual of M, that is

$$M^* = Hom_R(M,E).$$

Recall that the functor $Hom_R(-,E)$ defines a duality on the category of finitely generated R-modules, cf. [2]. Note that

$$an(M) = an(M^*), \qquad s(M^*) \approx M/\mathfrak{m}M.$$

M will be called a faithful R-module if $an(M) = 0$. Observe that E is, up to isomorphism, the only faithful R-module with one-dimensional socle.

<u>Lemma 1.</u> Let M be a faithful R-module. Suppose that M/N is not faithful for any submodule $N \neq 0$. Then $s(M) = s(R)M$.

<u>Proof.</u> Let N be a submodule of M such that

$$s(M) = s(R)M \oplus N.$$

We are going to show that $N = 0$. Suppose $N \neq 0$. Then by the minimality of M there exists an element $r \neq 0$ in R such that $rM \subset N$. We may as well assume that $r \in s(R)$. It follows that $rM \subset s(R)M \cap N = 0$. Hence $r = 0$, which is a contradiction.

<u>Lemma 2.</u> Let M be a faithful R-module. Assume that neither N nor M/N is faithful for any submodule N such that $0 \neq N \neq M$. Then we have

(i) $\dim M/\mathfrak{m} M \leq \dim s(R)$

(ii) $\dim s(M) \leq \dim s(R)$.

Moreover, if $M \neq R$ then at least one of the inequalities is strict.

<u>Proof.</u> We will first prove (i). Let $m = \dim M/\mathfrak{m} M$ and let $g_1, \cdots, g_m$ be a minimal set of generators for M. Since (i) is obvious if $m = 1$, we may assume that $m \geq 2$.

For $1 \leq i \leq m$ let M_i be the submodule generated by all $g_1, \cdots, g_m$ except g_i. Put $\mathfrak{C}_i = an(M_i)$. By the minimality of M we have $\mathfrak{C}_i \neq 0$ hence $\mathfrak{C}_i \cap s(R) \neq 0$ for all i. Choose one non-zero element u_i in $\mathfrak{C}_i \cap s(R)$ for each i. Since M is faithful, the elements u_i are clearly linearly independent over the field $R/\mathfrak{m}$. It follows that $m \leq \dim s(R)$.

To prove (ii) we just have to apply (i) to the dual M^*, observing that M^* satisfies the same minimality conditions as M. We get

$$\dim s(M) = \dim M^*/\mathfrak{m}M^* \leq \dim s(R).$$

We will now assume that we have equality in both (i) and (ii), and we assume that M is not isomorphic to R. We are going to show that this is impossible.

Since M is faithful, but not isomorphic to R, we have $\dim M/\mathfrak{m}M \geq 2$. Let $g_1,\cdots,g_m$ and $u_1,\cdots,u_m$ be as above. The equality in (i) gives that $u_1,\cdots,u_m$ is a basis for $s(R)$. Hence by lemma 1 we obtain

$$s(M) = (u_1,\cdots,u_m)(g_1,\cdots,g_m) = (u_1g_1,\ u_2g_2,\cdots,u_mg_m).$$

Let $\mathfrak{C}$ be the annihilator of the element $g_1+\cdots+g_m$. By mininality of M we have $\mathfrak{C}\neq 0$ and hence $\mathfrak{C}\cap s(R)\neq 0$. Let u be a non-zero element in $\mathfrak{C}\cap s(R)$. Let $r_1,\cdots,r_m$ be elements in R such that $u = \sum_{i=1}^{m} r_iu_i$. We have

$$0 = u(g_1+\cdots+g_m) = \sum_{i=1}^{m} r_iu_ig_i\ .$$

Since not all r_i are in $\mathfrak{m}$, the equation above shows that $\dim s(M) < m$ contradicting the equality in (ii).

<u>Corollary.</u> Let M be as in lemma 2 and suppose that $\dim s(R) \leq 2$. Then $M \approx R$ or $M \approx E$.

<u>Proof.</u> If $M \neq R$ then by lemma 2 we have $\dim s(M) = 1$, hence $M \approx E$.

Theorem 1. Let R be an Artinian local ring with $\dim_{R/\mathfrak{m}} \mathrm{Hom}_R(R/\mathfrak{m}, R) \leq 3$. Let M be a faithful R-module. Then we have $\ell(M) \geq \ell(R)$.

Proof. Clearly we may assume that M is a faithful module of minimal length, so that M as well as M^* satisfies the assumption in lemma 2. If $\dim s(R) \leq 2$ then the theorem follows from the above corollary. We may therefore assume that $\dim s(R) = 3$. Moreover we may assume that M is not isomorphic to R. Hence using lemma 2 and the relation

$$\dim M/\mathfrak{m}M = \dim s(M^*)$$

we have either

$$\dim s(M^*) \leq 2 \qquad \text{or} \qquad \dim s(M) \leq 2.$$

There is no loss of generality in assuming that $\dim s(M) \leq 2$. If $\dim s(M) = 1$ then $M \approx E$, and if $\dim M/\mathfrak{m}M = 1$ then $M \approx R$. Hence in the rest of the proof we may work under the following assumptions:

$$\dim s(R) = 3, \quad \dim s(M) = 2 \quad \text{and} \quad \dim M/\mathfrak{m}M \geq 2.$$

By the second of these assumptions we can find non-zero irreducible submodules M_1, M_2 in M such that $0 = M_1 \cap M_2$; see § 2 in [1]. Put $\mathfrak{a}_i = \mathrm{an}(M/M_i)$ for $i = 1,2$. We will first show that

$$(1) \qquad \ell(M/M_i) = \ell(R/\mathfrak{a}_i) \qquad \text{for } i = 1,2.$$

Since M_i is irreducible we have $\dim s(M/M_i) = 1$. It follows that $(M/M_i)^*$ is a homomorphic image of R. Moreover we have

$$\mathrm{an}((M/M_i)^*) = \mathrm{an}(M/M_i) = \mathfrak{a}_i$$

hence

$$(M/M_1)^* \approx R/\mathfrak{a}_1$$

so (1) follows.

Since M is faithful we have $\mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$. Since $\dim s(R) = 3$, at least one of the two vectorspaces $s(\mathfrak{a}_1)$ and $s(\mathfrak{a}_2)$ is one-dimensional.

We will assume that $\dim s(\mathfrak{a}_1) = 1$.

In view of (1) it now suffices to show that $\ell(M_1) \geq \ell(\mathfrak{a}_1)$. Since $\mathfrak{a}_1 M \subsetneq M_1$ it will be sufficient to prove the following:

(2) $\qquad \ell(\mathfrak{a}_1 M) \geq \ell(\mathfrak{a}_1)$.

Let $g_1, g_2, \cdots, g_m$ be a minimal set of generators for M. Put $\mathfrak{b}_i = \mathrm{an}(g_i)$ for $1 \leq i \leq m$. Then $\bigcap_{i=1}^{m} \mathfrak{b}_i = 0$. Hence one of the $\mathfrak{b}_i$, say $\mathfrak{b}_1$, does not contain $s(\mathfrak{a}_1)$. Since $\dim s(\mathfrak{a}_1) = 1$ we conclude that $\mathfrak{a}_1 \cap \mathfrak{b}_1 = 0$. We obtain $\mathfrak{a}_1 M \supset \mathfrak{a}_1 g_1 \approx \mathfrak{a}_1(R/\mathfrak{b}_1) \approx \mathfrak{a}_1/\mathfrak{a}_1 \cap \mathfrak{b}_1 = \mathfrak{a}_1$ which yields (2).

<u>Theorem 2.</u> Let $s \geq 4$ be an integer. Then there exists a local Artinian ring R and a faithful R-module M such that

(i) $\qquad \dim_{R/\mathfrak{m}} \mathrm{Hom}_R(R/\mathfrak{m}, R) = s$

(ii) $\qquad \ell(M) < \ell(R)$.

<u>Proof.</u> Let $m \geq 2$ be an integer and let k be an arbitrary field. Let R_m be the k-algebra of $(m+2)\times(m+2)$-matrices of the form

$$(3)\qquad \left(\begin{array}{ccc|c} & \lambda I_{m,m} & & O_{m,2} \\ \hline a_1 & \cdots & a_m & \\ b_1 & \cdots & b_m & \lambda I_{2,2} \end{array}\right)$$

where $\lambda, a_1,\cdots,a_m, b_1,\cdots,b_m$ run through k and $I_{p,q}$ and $O_{p,q}$ denotes the identity matrix and the zero-matrix of size $p \times q$. Clearly R_m is a commutative local Artinian ring of length $\ell(R_m) = 2m+1$. In fact the socle of R_m coincides with the maximal ideal which consists of all matrices of the form (3) in which $\lambda = 0$. Hence $\dim s(R_m) = 2m$.

Now let M be the k-vectorspace k^{m+2}. Clearly M becomes a faithful R_m-module in the obvious way. We have

$$\ell(M) = \dim_k M = m+2 < 2m+1 = \ell(R_m).$$

This proves the theorem in the case where s is even.

Let us now assume that s i odd. Write $s = 2m-1$ where $m \geq 3$. Consider R_m and M as before. Let R_m' be the subring consisting of all matrices of the form (3) in which $a_m = 0$. Clearly R_m' is a local ring of length $2m$ and $\dim s(R_m') = 2m-1 = s$. Moreover M is a faithful R_m'-module with

$$\ell(M) = \dim_k M = m+2 < 2m = \ell(R_m') .$$

The proof is now complete.

<u>Remark.</u> Let $R = \mathbf{C}[X,Y]/(X,Y)^4$. It can be shown that $\ell(M) \geq \ell(R)$ for any faithful R-module, inspite of the fact that $\dim s(R) = 4$.

REFERENCES

[1] Bass, H.: On the ubiquity of Gorenstein rings. Math. Zeithschr.82, 8-28 (1963).

[2] Matlis, E.: Injective modules over noetherian rings. Pacific J. Math. 8, 511-528 (1958).