The scattering matrix for some non-polynomial interactions I

by

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A B S T R A C T

We continue the study of quantum field theoretical models in n dimensional space-time with interaction densities which are bounded functions of an ultraviolet cut-off boson field. For the scattering matrix of the space cut-off interaction, constructed in terms of asymptotic fiesds, we prove analyticity in the coupling constant λ and convergence of the linked cluster expansion for sufficiently small λ. The correlation functions and imaginary time Wightman functions for the infinite volume limit constructed in a previous paper are also proved to have a linked cluster expansion, convergent for sufficiently small values of λ. This is then used, together with the results on the space cut-off S-matrix, to establish the existence and analyticity in λ of the infinite volume scattering functions and to prove reduction formulae for the infinite volume Wightman functions.

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1. Introduction

The basic quantities for the description of scattering processes for systems of infinitely many particles have been proven to exist only for certain classes of quantum field theoretical models [1], all of them breaking in one way or the other at least some of the Wightman axioms for a satisfactory local covariant quantum field theory. In particular only for few models with translation invariant interaction the existence of an S-matrix has been established. [2] In this paper we begin the study of the S-matrix for a class of models of self-coupled bosons in n dimensions, with translation invariant non-polynomial, ultraviolet cut-off interactions. These models have vacuum problems (vacuum polarization) as well as one-particle problems (translation invariance and pure creation terms in the interaction). They have an interaction of the form:

$$\lambda \int e^{is\varphi_\varepsilon(\vec{x})} d\nu(s) d\vec{x} \ ,$$

where $\varphi_\varepsilon(\vec{x})$ is an ultraviolet cut-off time zero Boson field in $(n-1)$ space dimensions [3] and $d\nu(s)$ is a finite measure of bounded support (with $d\nu(-s) = \overline{d\nu(s)}$, - meaning complex conjugate).

In a preceding paper [5][4] we proved, in particular, the existence, uniqueness and analyticity in the coupling constant λ of the vacuum in the infinite volume limit and of the corresponding imaginary time Wightman functions for this class of models, for all $|\lambda| < \lambda_0$, $\lambda_0 > 0$.

In this paper we start (section 2) from the construction of the scattering matrix for the space cut-off interaction in terms of asymptotic fields. An asymptotic expansion of this S-matrix

(or, more precisely, of its generating functional) in powers of λ is then derived. The following sections will then establish, through the identification of this expansion with a linked cluster expansion, the actual convergence of the series for all $|\lambda| < \lambda_0$. In section 3 we prove the linked cluster expansions, as convergent series for $|\lambda| < \lambda_0$, of the infinite volume correlation functions introduced in [5], of the corresponding imaginary time Wightman functions and of the corresponding truncated quantities. In section 4 we prove the linked cluster expansion, as asymptotic series for small values of $|\lambda|$, of the space cut-off S-matrix, starting from the asymptotic expansion derived in section 2. Moreover, we show that the S-matrix is given in terms of scattering functions which have a formal power series expansion in which every term has analytic continuation from the positive real axis in the time variables to the positive imaginary axis.
In section 5 we start by proving that the scattering functions are the analytic continuation of the correlation functions from positive times to positive imaginary time. This is done by first establishing the joint analyticity in the time variables, in the right half plane, and in the coupling constant λ, for $|\lambda| < \lambda_0$, of the correlation functions for the space cut-off interaction. This together with their linked cluster expansion (proven to converge for $|\lambda| < \lambda_0$ by the methods of section 3), yields the identification term by term and then, due to the convergence, globally of the scattering functions with the analytic continuation of the correlation functions. Moreover, this implies the convergence of the linked cluster expansion of the scattering matrix for the space cut-off interaction and its analyticity in λ, for $|\lambda| < \lambda_0$.

The S-matrix as defined originally in terms of the asymptotic fields is proven to be the sum of this expansion, for all complex λ with $|\lambda|$ sufficiently small.

In section 6 we prove that the scattering functions have unique limits when the space cut-off is taken away, for $-\lambda_0 < \lambda < \lambda_0$. The infinite volume scattering functions are uniformly bounded in all space time variables and analytic in time differences in the lower half plane. Moreover, they are the analytic continuation of the correlation functions and yield reduction formulae for the Wightman functions (in the same way as the correlation functions give reduction formulae for the imaginary time Wightman functions). Finally the finite volume scattering amplitude for given ingoing and outgoing momenta are expressed through the Fourier transforms of the scattering functions and the existence of the off-shell scattering amplitudes in the infinite volume limit is remarked. The discussion will be pursued in a forthcoming paper.
Throughout this paper we shall always use the same notations as in our previous discussion [5] of the models under consideration.

2. The scattering matrix for the space cut-off interaction

In the previous paper [5] we considered self-interacting boson fields with Hamiltonian of the form

$$H_1 = H_o + \lambda \int_{|\vec{x}|\leq 1} v(\varphi_\varepsilon(\vec{x}))d\vec{x} \,, \tag{2.1}$$

where H_o is the free energy of a free time zero boson field $\varphi(\vec{x})$ of mass $m > 0$, and $\varphi_\varepsilon(\vec{x}) = \int \chi_\varepsilon(\vec{x}-\vec{y})\varphi(\vec{y})d\vec{y}$, with $\vec{x} \in \mathbb{R}^{n-1}$, n being the number of dimensions of space-time, and $\chi_\varepsilon(\vec{x}) \in C_o^\infty(\mathbb{R}^{n-1})$, $\chi_\varepsilon(\vec{x}) \geq 0$, $\chi_\varepsilon(\vec{x}) = \chi_\varepsilon(-\vec{x})$.
$v(\alpha)$ is a real valued function of the form $v(\alpha) = \int e^{i\alpha s}d\nu(s)$, where $d\nu(s)$ is a bounded measure of bounded support on the real line. λ is a real number (the coupling constant). H_1 is then a self-adjoint operator, bounded from below, with the same domain as H_o in the Fock space $\mathscr{F}$ of the free boson field $\varphi(\vec{x})$ [6]. Interactions of the form (2.1) have also been considered in [7], where it was proven that the asymptotic fields exist as strong limits and the scattering matrix was then given in terms of these.

In [6] it was proven that H_1 has a simple lowest eigenvalue E_1 with the corresponding normalized eigenvector Ω_1 which can be chosen so that $(\Omega_1,\Omega_o) > 0$, where Ω_o is the Fock vacuum.

For any operator A on $\mathscr{F}$ we define:

$$A_1 = e^{-itH_1}\, e^{itH_o}\, A\, e^{-itH_o}\, e^{itH_1} \,. \tag{2.2}$$

The free time zero field $\varphi(\vec{x})$ is given in terms of the annihilation and creation operators $a(\vec{p})$ and $a^*(\vec{p})$ by

$$\varphi(\vec{x}) = 2^{-\frac{1}{2}}(2\pi)^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} e^{i\vec{p}\cdot\vec{x}}(a^*(-\vec{p}) + a(\vec{p}))\frac{d\vec{p}}{\mu(\vec{p})^{\frac{1}{2}}} \tag{2.3}$$

where $\mu(\vec{p}) = (\vec{p}^2 + m^2)^{\frac{1}{2}}$. Let D be the domain of H_o and $D_{\frac{1}{2}}$

the domain of $H_o^{\frac{1}{2}}$, and

$$a^{\#}(h) = \int a^{\#}(\vec{p})h(\vec{p})d\vec{p} \ , \tag{2.4}$$

where $a^{\#}$ stands for a or a^* . The following theorem was proven in [7].

Theorem 2.1

a) Let $h \in L_2(\mathbb{R}^{n-1})$ and $\Psi \in D_{\frac{1}{2}}$. Then $a_t^{\#}(h)\Psi$ converge strongly as t tends to $\pm\infty$. The limit operators $a_{\pm}^{\#}(h)$ are closable operators defined on $D_{\frac{1}{2}}$, and $a_{\pm}^{\#}(h)$ map $D_{\frac{1}{2}}$ into $\mathcal{F}$, uniformly boundedly in h with respect to the natural norm in $D_{\frac{1}{2}}$. If we denote the closure of $a_{\pm}^{\#}(h)$ also by $a_{\pm}^{\#}(h)$, then $a_{\pm}^*(\bar{h})$ and $a_{\pm}(h)$ are the adjoints of each other.

b) Let g and h be in $L_2(\mathbb{R}^{n-1})$. Then $a_{\pm}^{\#}(h)$ map D_o into the domain of $a_{\pm}^{\#}(g)$, and $a_{\pm}^{\#}(g)a_{\pm}^{\#}(h)$ map D_o into $\mathcal{F}$, uniformly boundedly in g and h with respect to the natural norm on D_o . $a_{\pm}^{\#}(h)$ satisfy the same commutation relations on D_o as do $a^{\#}(h)$ on D_o . H_1 and $a_{\pm}^{\#}(h)$ satisfy the same commutation relations as do H_o and $a^{\#}(h)$, in the sense that $e^{itH_1}\, a_{\pm}^{\#}(h)\, e^{-itH_1} = a_{\pm}^{\#}(h_{\pm t})$ on $D_{\frac{1}{2}}$, where $h_t(\vec{p}) = e^{i\mu(\vec{p})t}\, h(\vec{p})$ and h_{+t} goes with $a_{\pm}^*$ and h_{-t} with $a_{\pm}$.

c) $a_{\pm}(h)\Omega_1 = 0$ for all $h \in L_2(\mathbb{R}^{n-1})$. Let $\mathcal{F}_{\pm}$ be the Fock spaces constructed with $a_{\pm}^{\#}(h)$ as annihilation-creation operators and Ω_1 as Fock vacuums. Then $\mathcal{F}$ decomposes relatively to the asymptotic operators $a_{\pm}^{\#}(h)$ as a tensor product $\mathcal{F} = \mathcal{F}_{\pm} \otimes V_{\pm}^o$, where $\Omega_1 \otimes V_{\pm}^o$ is the subspace of $\mathcal{F}$ annihilated by $a_{\pm}(h)$ for all $h \in L_2(\mathbb{R}^{n-1})$. With respact to this decomposition, $H_1 - E_1$ decomposes as

$$H_1 - E_1 = H_o^{\pm} \otimes 1 + 1 \otimes H_{\pm}^o \ ,$$

where $H_o^\pm$ is the free energy operator (to the mass m) in $\mathcal{F}_\pm$, and $1 \otimes H_\pm^o$ is the restriction of H to the invariant subspace $\Omega_1 \otimes V_\pm^o$. ∎

The S-matrix element between n outgoing particles with momentum distributions given by $h_1,\ldots,h_n$ in L_2 and m incoming particles with momentum distributions given by $g_1,\ldots,g_m$ in L_2 is given by

$$S^1_{n,m}(h_1,\ldots,h_n;g_1,\ldots,g_m) = (a^*_-(h_1)\ldots a^*_-(h_n)\Omega_1, a^*_+(g_1)\ldots a^*(g_m)\Omega_1). \quad (2.5)$$

This gives the relation between the asymptotic fields and the S-matrix.

We now define the wave operators $W^1_\pm$ by

$$W^1_\pm a^*(h_1)\ldots a^*(h_n)\Omega_o = a^*_\pm(h_1)\ldots a^*_\pm(h_n)\Omega_1 . \quad (2.6)$$

Then $W_\pm$ are isometries mapping $\mathcal{F}$ onto $\mathcal{F}_\pm$, and by (2.5) the scattering operator S is given by

$$S^1 = W^{1*}_- W^1_+ . \quad (2.7)$$

Since $W^1_\pm$ are isometries, we have $\|S^1\| \leq 1$ ($\|\ \|$ denoting the norm in $\mathcal{F}$), and from the commutation relations for H and $a^*_\pm(h)$ we get that

$$[S^1,H_o] = 0 . \quad (2.8)$$

Let

$$\varphi(f) = \int\varphi(\vec{x})f(\vec{x})d\vec{x} , \quad (2.9)$$

with f real and in $\mathcal{H}^{-\frac{1}{2}}_{n-1}$, where $\mathcal{H}^{-\frac{1}{2}}_{n-1}$ is the Sobolev space of order $(-\frac{1}{2})$ (see [5]). Then $\varphi(f)$ is essentially self-adjoint on D. Let $\varphi_t(f)$ be given by (2.2) and $\varphi_\pm(f)$ be defined by (2.9) and (2.3), with $a^\#_\pm$ substituted for $a^\#$. It follows then from Theorem 2.1 that $\varphi_{\pm}(f)$ is also essentially self-adjoint on D . This implies that $e^{i\varphi_t(f)}$ converges strongly to $e^{i\varphi_{\pm}(f)}$ as $t \to \pm\infty$.

From (2.2) it follows easily that, for any bounded operator A :

$$A_t = A - i\lambda \int_0^t ds e^{-sH_1} e^{isH_o} [V_1(s), A] e^{-isH_o} e^{isH_1} , \qquad (2.10)$$

with $V_1 = \int_{|\vec{x}|\le 1} v(\varphi_\varepsilon(\vec{x}))d\vec{x}$ and $V_1(s) = e^{-isH_o} V e^{isH_o}$.

If A_t converges strongly to $A_{\pm}$ as $t \to \pm\infty$, it follows from (2.10) that

$$A_t = A_- - i\lambda \int_{-\infty}^t ds e^{-isH_1} e^{isH_o} [V_1(s), A] e^{-isH_o} e^{isH_1} \qquad (2.11)$$

and

$$A_+ = A_- - i\lambda \int_{-\infty}^{+\infty} ds e^{-isH_1} e^{isH_o} [V_1(s), A] e^{-isH_o} e^{isH_1} , \qquad (2.12)$$

where the integrals are strongly convergent. Since $e^{i\varphi_t(f)}$ converges strongly, we get from (2.12)

$$e^{i\varphi_+(f)} = e^{i\varphi_-(f)} - i\lambda \int_{-\infty}^{+\infty} ds e^{-isH_1} e^{isH_o} [V_1(s), e^{i\varphi(f)}] e^{-isH_o} e^{isH_1} . \qquad (2.13)$$

Due to the form of V_1 and $v(\alpha)$, we see that $B_t = (e^{isH_o}[V_1(s), e^{i\varphi(f)}]e^{-isH_o})_t$ converges strongly as $t \to \pm\infty$. From (2.11) with A = B and t = 0 substituted in the integral in (2.13) we get

$$e^{i\varphi_+(f)} = e^{i\varphi_-(f)} - i\lambda \int_{-\infty}^{+\infty} ds e^{-isH_1} B_- e^{isH_1}$$
$$+ (-i\lambda)^2 \int_{-\infty}^{+\infty} ds \int_{-\infty}^{0} d\sigma e^{-i(s+\sigma)H_1} e^{i\sigma H_o} [V_1(\sigma), B] e^{-i\sigma H_o} e^{i(s+\sigma)H_1} . \qquad (2.14)$$

From Theorem 2.1 we have $B_- = e^{isH_1}[V_1(s)_-, e^{i\varphi_-(f)}]e^{-isH_1}$,
where $V_1(s)_-$ is $V_1(s)$ with $a_-^{\#}$ substituted for $a^{\#}$. Hence:

$$e^{i\varphi_+(f)} = e^{i\varphi_-(f)} + i\lambda \int_{-\infty}^{\infty} ds[V_1(s)_-, e^{i\varphi_-(f)}]$$

$$+ (-i\lambda)^2 \int_{-\infty}^{\infty} ds \int_{-\infty}^{s} d\sigma e^{-i\sigma H_1} e^{i\sigma H_o}[V_1(\sigma), V_1(s), e^{i\varphi(f)}]]e^{-i\sigma H_o} e^{i\sigma H_1} .$$

By iteration of this procedure we get

$$e^{i\varphi_+(f)} = \sum_{n=o}^{N} (-i\lambda)^n \int_{t_n \leq \dots \leq t_1} \int [V_1(t_n)_-, \dots [V_1(t_1)_-, e^{i\varphi_-(f)}] \dots]dt_1 \dots dt_n$$

$$+ (-i\lambda)^{N+1} \int_{\sigma \leq t_N \leq \dots \leq t_1} \cdots \int e^{-i\sigma H_1} e^{i\sigma H_o}[V_1(\sigma), [V(t_N), \dots, [V(t_1), e^{i\varphi(f)}] \dots]] e^{-i\sigma H_o} e^{i\sigma H_1} d\sigma dt_1 \dots dt_N , \tag{2.15}$$

where the integrals are strongly convergent.
We now define for f and g in $\mathcal{H}^{-\frac{1}{2}}_{n-1}$:

$$S_1(g;f) = (: e^{i\varphi_-(g)} : \Omega_1 , : e^{i\varphi_+(f)} : \Omega_1) , \tag{2.16}$$

where: : stands for the Wick product.[5)]
It follows from Theorem 2.1 that Ω_1 is an analytic vector with respect to $\varphi_\pm(f)$, so that $S_1(g;f)$ is infinitely differentiable with respect to f and g and we see from (2.5) that the derivatives determine the S-matrix elements $S_{n,m}$. From (2.15) we get the following asymptotic expansion in λ :

$$S_1(g;f) = \sum_{n=o}^{N} (-i\lambda)^n \int_{t_n \leq \dots \leq t_1} \dots \int (: e^{i\varphi_-(g)} : \Omega_1 , [V_1(t_n)_-, \dots, [V_1(t_1)_-, : e^{i\varphi_-(f)} :] \dots]\Omega_1) dt_1 \dots dt_n + O(|\lambda|^{N+1}), \tag{2.17}$$

where $|O(\alpha)| \leq$ const. $|\alpha|$, for small values of α .

Consider now the quantity

$$(:e^{i\varphi_-(g)}:\Omega_1,[V_1(t_n)_-,\dots,[V_1(t_1)_-,:e^{i\varphi_-(f)}:]\dots]\Omega_1) =$$

$$= (W_-:e^{i\varphi(g)}:\Omega_o, W_-[V_1(t_n),\dots[V_1(t_1),:e^{i\varphi(f)}:]\dots]\Omega_o) .$$

Since W_- is an isometry, this is equal to

$$(:e^{i\varphi(g)}:\Omega_o,[V_1(t_n),\dots,[V_1(t_1), :e^{i\varphi(f)}:]\dots]\Omega_o) .$$

Hence (2.17) takes the form

$$S_1(g;f) = \sum_{n=o}^{N} (-i\lambda)^n \int_{t_n\leq\dots\leq t_1} (\Omega_o, :e^{i\varphi(g)}:[V_1(t_n),\dots,[V_1(t_1),:e^{i\varphi(f)}:]\dots]\Omega_o)dt_1\dots dt_n$$
$$+ O(|\lambda|^{N+1}) . \tag{2.18}$$

This gives us an asymptotic expansion of $S_1(g;f)$ with respect to λ. We shall later on show (in section 5) that each term in this expansion can be rewritten in such a way as to obtain the linked cluster expansion. This will give us the connection with the correlation functions studied in [5] and we shall use this connection to prove that the series in (2.18) converges as $N \to \infty$ for all $|\lambda| < \lambda_o$, $\lambda_o > 0$, from which it follows that the S-matrix for the space cut-off interaction is analytic in λ for $|\lambda| < \lambda_o$ and given by the convergent linked cluster expansion. λ_o is equal to $C^{-1} e^{-2B-1}$, where C is defined in section 4 of [5] and B is defined in (4.10) of [5].

3. The linked cluster expansion for the infinite volume imaginary time Wightman functions

In [5] we proved that the infinite volume imaginary time Wightman functions exist as limits for $l \to \infty$ of the corresponding volume cut-off quantities for the models considered in the previous section. Moreover, we proved that they are analytic for $|\lambda| < \lambda_o$ and continuous in the time variables in the closed right hand half plane, and hence define the Wightman functions for the infinite volume models. From the formula (5.5) of [5] the imaginary time Wightman functions are given by

$$G^k(x_1,\ldots,x_k) = G_o^k(x_1,\ldots,x_k) + \sum_{r=1}^{k} \frac{1}{r!} \sum_{\substack{q\geq r,\, p\geq o\\ p+q=k}} \frac{1}{p!}$$

$$\sum_{\sigma\in S_k} G_o^p(x_{\sigma(1)},\ldots,x_{\sigma(p)}) \sum_{\substack{l_1+\ldots+l_r=q\\ l_m\geq 1}} \frac{1}{l_1!\ldots l_r!} \int\ldots\int \prod_{m=1}^{r} \tag{3.1}$$

$$[(is_m)^{l_m} \prod_{j=1}^{l_i} G_\epsilon(x_{\sigma(p+l_1+\ldots+l_m-j+1)}-y_m)]\rho^r(y_1s_1,\ldots,y_rs_r) \prod_{j=1}^{r} d\nu(s_j)dy_j \,.$$

The variables x_i and y_j are all space-time variables in $\mathbb{R}^n$. S_k is the set of permutations of $1,\ldots,k$ and $G_o^k(x_1,\ldots,x_k)$ are the free imaginary time Wightman functions, which are equal to zero if k is odd and are given, for $k = 2p$, by

$$G_o^k(x_1,\ldots,x_{2p}) = \frac{1}{2^p\; p!} \sum_{\sigma\in S_{2p}} G(x_{\sigma(1)}-x_{\sigma(2)})\ldots G(x_{\sigma(2p-1)}-x_{\sigma(2p)}), \tag{3.2}$$

with $G(x) = \int_{\mathbb{R}^n} \frac{e^{ipx}}{p^2+m^2}\, dp$.

$\rho^r(x_1s_1,\ldots,x_rs_r)$ is the infinite volume correlation function of Lemma 4.1 in [5], and

$$G_\varepsilon(x) = \int_{\mathbb{R}^n} \frac{e^{ipx}}{p^2+m^2} |\tilde{\chi}_\varepsilon(\vec{p})|^2 dp \ , \text{ with } \ \tilde{\chi}_\varepsilon(\tilde{p}) = \int_{\mathbb{R}^{n-1}} e^{i\vec{p}\vec{x}} \chi_\varepsilon(\vec{x}) d\vec{x} \ .$$

The infinite volume correlation function $\rho^r(x_1 s_1, \ldots, x_r s_r)$ is a limit, uniformly on compact sets, of the corresponding finite volume correlation functions $\rho^r_\Lambda(x_1 s_1, \ldots, x_r s_r)$, where Λ is a bounded domain in $\mathbb{R}^n$ (Lemma 4.1 of [5]). The ρ^r_Λ are defined by

$$\rho^k_\Lambda(x_1 s_1, \ldots, x_k s_k) = Z_\Lambda^{-1} \sum_{n=o}^{\infty} \frac{(-\lambda)^{n+k}}{n!} \int_{\Lambda^n} \cdots \int e^{-\sum\limits_{i<j}^{n+k} s_i s_j G_\varepsilon(x_i - x_j)} \prod_{j=k+1}^{n+k} d\mu(s_j) dx_j \ , \tag{3.3}$$

where

$$Z_\Lambda = \sum_{n=o}^{\infty} \frac{(-\lambda)^n}{n!} \int_{\Lambda^n} \cdots \int e^{-\sum\limits_{i<j} s_i s_j G_\varepsilon(x_i - x_j)} \prod_{j=1}^{n} d\mu(s_j) dx_j \ . \tag{3.4}$$

The apparent difference between (3.3) and (4.3) of [5] is due to the fact that $d\mu(s) = e^{-\frac{1}{2}s^2 G_\varepsilon(o)} d\nu(s)$, where $d\nu$ is the measure defined in Section 2 and used in (4.3) of [5]. Now

$$e^{-\sum\limits_{i<j}^{n+k} s_i s_j G_\varepsilon(x_i - x_j)} = \prod_{i<j}^{n+k} [(e^{-s_i s_j G_\varepsilon(x_i - x_j)} - 1) + 1] \ . \tag{3.5}$$

The product above runs over the set P of all $\frac{1}{2}(n+k)(n+k-1)$ unordered pairs (i,j) of different elements from the set $\{1,2,\ldots,n+k\}$. The product in (3.5) is therefore of the form $\prod\limits_{\ell \in P} (a_\ell + 1)$, with $a_\ell = e^{-s_i s_j G_\varepsilon(x_i - x_j)} - 1$ for $\ell = (i,j)$. One verifies easily that

$$\prod_{p \in P} (a_p + 1) = \sum_{Q \subset P} \prod_{\ell \in \Gamma} a_\ell \ , \tag{3.6}$$

where the sum runs over all subsets Γ of P , and by definition $\prod\limits_{\ell \in \emptyset} a_\ell = 1$, $\emptyset$ being the empty set.

A subset Γ of P is called a simple unoriented graph with labeled points, for short we will call it a graph. We shall say that i is a point of the graph Γ if there is a j such that $(i,j) \in \Gamma$. The unordered pairs in Γ are called the lines of the graph Γ . Note that Γ can also be the empty set or P itself. Combining (3.3) and (3.6) we get

$$\rho_\Lambda^k(x_1 s_1,\dots,x_k s_k) = Z_\Lambda^{-1} \sum_{n=0}^{\infty} \frac{(-\lambda)^{n+k}}{n!} \int_{\Lambda^n} \dots \int \sum_\Gamma \prod_{(i,j)\in\Gamma} (e^{-s_i s_j G_\epsilon(x_i-x_j)} - 1) \prod_{j=k+1}^{n+k} d\mu(s_j)dx_j, \tag{3.7}$$

where $\sum_\Gamma$ denotes the sum over all graphs with points from the set $\{1,\dots,n+k\}$. The points from the set $\{1,\dots,k\}$ will be called external points of Γ and the points from the set $\{k+1,\dots,k+n\}$ will be called internal points.

We say that two points i and j in Γ are connected if there is a sequence of points $i_1,\dots,i_s$ in Γ such that (i,i_1), $(i_1,i_2),\dots,(i_{s-1},i_s),(i_s,j)$ are lines in Γ . Let Γ_0 be the subgraph of Γ consisting of those internal points of Γ that are not connected with any external point of Γ , and let Γ_1 be the subgraph consisting of the points that are connected with external points. Γ_1 is called an externally connected graph. We define the product of two graphs $\Gamma' \times \Gamma''$ as the graph whose set of points is the union of the points of Γ' and the points of Γ'' and with lines which are those of Γ' and Γ'' . It follows from the definition of Γ_0 and Γ_1 that $\Gamma = \Gamma_0 \times \Gamma_1$. Therefore

$$\sum_\Gamma \prod_{\ell\in\Gamma} a_\ell = \sum_{\Gamma_0,\Gamma_1} \prod_{\ell\in\Gamma_0\times\Gamma_1} a_\ell = \sum_{\Gamma_0,\Gamma_1} \prod_{\ell\in\Gamma_0} a_\ell \prod_{\ell\in\Gamma_1} a_\ell , \tag{3.8}$$

where $\sum_{\Gamma_0,\Gamma_1}$ denotes the sum over all graphs Γ_0 with only internal points and over all graphs Γ_1 which are externally connected. Hence

$$\rho_\Lambda^k(x_1 s_1,\dots,x_k s_k) = Z_\Lambda^{-1} \sum_{n=0}^{\infty} \frac{(-\lambda)^{n+k}}{n!} \sum_{\Gamma_0,\Gamma_1} \int_{\Lambda^n} \dots \int \prod_{(i,j)\in\Gamma_0\times\Gamma_1} (e^{-s_i s_j G_\epsilon(x_i-x_j)} - 1) \prod_{j=k+1}^{k+n} d\mu(s_j)dx_j.$$

Any two terms which can be obtained from each other by permuting the internal points give equal contrubutions to the sum $\sum_{\Gamma_o,\Gamma_1}$, because of the symmetry of the integrand. Therefore we may take the ℓ points of Γ_o to be the ℓ last internal points $\{n+k-\ell+1,\dots,n+k\}$. Since we can pick ℓ points out of n points in $\binom{n}{\ell}$ different ways, we get

$$\rho_\Lambda^k(x_1 s_1,\dots,x_k s_k) = Z_\Lambda^{-1} \sum_{m,\ell=o}^{\infty} \frac{(-\lambda)^{m+k}}{m!} \frac{(-\lambda)^\ell}{\ell!} \sum_{\Gamma_1} \sum_{\Gamma_o} \int_{\Lambda^{m+\ell}}\!\!\dots\int \prod_{(i,j)\in\Gamma_o\times\Gamma_1} (e^{-s_i s_j G_\varepsilon(x_i-x_j)}-1) \prod_{j=k+1}^{k+m+\ell} d\mu(s_j)dx_j, \tag{3.9}$$

where $\sum_{\Gamma_1}$ runs over all externally connected graphs with points out of $\{1,\dots,k+m\}$ and $\sum_{\Gamma_o}$ over all graphs with only internal points from $\{k+m+1,\dots,k+m+\ell\}$.

Since $\prod_{(i,j)\in\Gamma_o\times\Gamma_1} (e^{-s_i s_j G_\varepsilon(x_i-x_j)}-1) = \prod_{\Gamma_1}(e^{-s_i s_j G_\varepsilon(x_i-x_j)}-1)$. $\prod_{\Gamma_o}(e^{-s_i s_j G_\varepsilon(x_i-x_j)}-1)$, we see that the integral in (3.9) factors as a product, one factor being

$$\int_{\Lambda^\ell}\!\!\dots\int \prod_{\Gamma_o} (e^{-s_i s_j G_\varepsilon(x_i-x_j)}-1) \prod_{j=1}^{\ell} d\mu(s_j)dx_j \ .$$

Summing now over all graphs Γ_o with ℓ points and using (3.6), we get this equal to

$$\int_{\Lambda^\ell}\!\!\dots\int e^{-\sum_{i<j} s_i s_j G_\varepsilon(x_i-x_j)} \prod_{j=1}^{\ell} d\mu(s_j)dx_j \ .$$

If we now multiply by $\frac{(-\lambda)^\ell}{\ell!}$ and sum over ℓ we get Z_Λ by (3.4). Using this result we obtain then from (3.9)

$$\rho_\Lambda^k(x_1 s_1,\dots,x_k s_k) = \sum_{n=o}^{\infty} \frac{(-\lambda)^{n+k}}{n!} \sum_E \int_{\Lambda^n}\!\!\dots\int \prod_{(i,j)\in E} (e^{-s_i s_j G_\varepsilon(x_i-x_j)}-1) \prod_{k+1}^{k+n} d\mu(s_i)dx_i, \tag{3.10}$$

where E runs over all externally connected graphs. By Lemma 4.1 of [5] we know that ρ_Λ^k converges uniformly on compact subsets to the infinite volume correlation function $\rho^k(x_1s_1,\dots,x_ks_k)$, and moreover that ρ^k is analytic in λ for $|\lambda| < \lambda_o$. It follows from (3.10) that each term in the power series expansion for ρ^k_Λ converges as Λ increases to $\mathbb{R}^n$. By the fact that pointwise convergence of analytic functions in an open domain implies the convergence of their derivatives at a point, we get that

$$\rho^k(x_1s_1,\dots,x_ks_k) = \sum_{n=o}^{\infty} \frac{(-\lambda)^{n+k}}{n!} \sum_E \int\dots\int \prod_{(i,j)\in E} (e^{-s_is_jG_\varepsilon(x_i-x_j)}-1) \prod_{k+1}^{k+m} d\mu(s_i)dx_i\,, \tag{3.11}$$

and the series is convergent for $|\lambda| < \lambda_o$. This is the linked cluster ezpansion for the correlation function.

For later use we shall also introduce the truncated correlation functions $\rho^k_T(x_1s_1,\dots,x_ks_k)$. Let $X = \{(x_1s_1),\dots,(x_ks_k)\}$. We define

$$\rho(X) = \rho^k(x_1s_1,\dots,x_ks_k)\,. \tag{3.12}$$

The truncated correlation functions are then defined by

$$\rho^k_T(x_1s_1,\dots,x_ks_k) = \sum_{X=X_1\cup\dots\cup X_\ell} (-1)^{\ell-1}(\ell-1)!\,\rho(X_1)\dots\rho(X_\ell)\,, \tag{3.13}$$

where the sum is over all partitions of X into disjoint subsets $X_1,\dots,X_\ell$.

The inversion of (3.13) is given by the formula

$$\rho(X) = \sum_{X=X_1\cup\dots\cup X_\ell} \rho_T(X_1)\dots\rho_T(X_\ell)\,, \tag{3.14}$$

where $\rho_T(X)$ is defined according to (3.12).

We say that a graph Γ is connected if any two points of Γ are connected. It is obvious that any graph Γ is a product $\Gamma = C_1\times\dots\times C_\ell$ of its connected components C_i. For a fixed term in (3.11) we write E as the product of its connected components $E = C_1\times\dots\times C_\ell$. We then get that the integral in (3.11) is equal to

$$\prod_{m=1}^{\ell} \int \dots \int \prod_{(i,j)\in C_m} (e^{-s_i s_j G_\epsilon (x_i - x_j)} - 1) \prod_{i \in \operatorname{Int} C_m} d\mu(s_i) dx_i \ , \tag{3.15}$$

where $\operatorname{Int} C_m$ is the set of internal points of C_m. By the symmetry of the integrand in (3.15), we may permute the integration variables without changing the value of the integral. Let n_m be the number of points in $\operatorname{Int} C_m$, and let us permute the integration variables in such a way that the internal points in C_m become the points $\{k+n_1+\dots+n_{m-1}+1, \dots, k+n_1+\dots+n_m\}$. Since the number of ways one can divide $n = n_1+\dots+n_\ell$ points into groups containing $n_1,\dots,n_\ell$ points is $\frac{n!}{n_1!\dots n_\ell!}$, the summation over E in (3.11) gives $\frac{n!}{n_1!\dots n_\ell!}$ equal contributions of the form

$$\prod_{m=1}^{\ell} \int \dots \int \prod_{(i,j)\in C_m} (e^{-s_i^m s_j^m G_\epsilon (x_i^m - x_j^m)} - 1) \prod_{i=k_m+1}^{k_m+n_m} d\mu(s_i^m) dx_i^m \ ,$$

where $X_m = \{(x_1^m s_1^m), \dots, (x_{k_m}^m, s_{k_m}^m)\}$ is the subset of the set $X = \{(x_1 s_1), \dots, (x_k s_k)\}$ of external points which consists of the external points in C_m. Thus $k = k_1+\dots+k_\ell$ and $X = X_1 \cup \dots \cup X_\ell$ is a partition of X into disjoint subsets. Hence we get from (3.11) that

$$\rho(X) = \sum_{n=o}^{\infty} \sum_{\substack{n_1+\dots+n_\ell=n \\ n_1 \geq o,\dots,n_\ell \geq o}} \frac{(-\lambda)^{n_1+k_1}}{n_1!} \dots \frac{(-\lambda)^{n_\ell+k_\ell}}{n_\ell!} \sum_{C_1,\dots,C_\ell} \prod_{m=1}^{\ell}$$

$$\int \dots \int \prod_{(i,j)\in C_m} (e^{-s_i^m s_j^m G_\epsilon (x_i^m - x_j^m)} - 1) \prod_{i=k_m+1}^{k_m+n_m} d\mu(s_i^m) dx_i^m \ . \tag{3.16}$$

We now define the formal power series in λ

$$\sigma(X) = \sum_{n=o}^{\infty} \frac{(-\lambda)^{n+k}}{n!} \sum_{C} \int \dots \int \prod_{(i,j)\in C} (e^{-s_i s_j G_\epsilon(x_i - x_j)} - 1) \prod_{k+1}^{k+n} d\mu(s_i) dx_i, \quad (3.17)$$

where C runs over all connected graphs with external points. Then it follows from (3.16) that

$$\rho(X) = \sum_{X=X_1\cup\dots\cup X_n} \sigma(X_1)\dots\sigma(X_n) , \quad (3.18)$$

where the sum goes over all disjoint partitions of X and the equality is in the sense of formal power series. By utilizing that (3.13) is the algebraic inverse relation of (3.14), it follows that

$$\sigma(X) = \sum_{X=X_1\cup\dots\cup X_\ell} (-1)^{\ell-1}(\ell-1)!\rho(X_1)\dots\rho(X_\ell) , \quad (3.19)$$

where the equality is again to be understood in the sense of formal power series. But since, by (3.11), $\rho(X_m)$ has a power series expansion that converges for $|\lambda| < \lambda_o$, it follows from (3.19) that the power series (3.17) converges for $|\lambda| < \lambda_o$ and moreover it converges to $\rho_T(X)$, because of (3.19). We summarize this discussion in the following:

Lemma 3.1 The correlation function $\rho^k(x_1 s_1,\dots,x_k s_k)$ and the truncated correlation function $\rho_T^k(x_1 s_1,\dots,x_k s_k)$ are both given, for $|\lambda| < \lambda_o$ (where λ_o is given in Lemma 4.1 of [5]), by their convergent linked cluster expansions:

$$\rho^k(x_1 s_1,\dots,x_k s_k) =$$

$$= \sum_{n=o}^{\infty} \frac{(-\lambda)^{n+k}}{n!} \sum_{E} \int \dots \int \prod_{(i,j)\in E} (e^{-s_i s_j G_\epsilon(x_i - x_j)} - 1) \prod_{i=k+1}^{k+n} d\mu(s_i) dx_i$$

and

$$\rho_T^k(x_1 s_1,\dots,x_k s_k) =$$

$$= \sum_{n=o}^{\infty} \frac{(-\lambda)^{n+k}}{n!} \sum_{C} \int \dots \int \prod_{(i,j)\in C} (e^{-s_i s_j G_\epsilon(x_i - x_j)} - 1) \prod_{i=k+1}^{k+n} d\mu(s_i) dx_i ,$$

where E runs over all externally connected graphs and C runs over all connected graphs with external points. ■

Let now $\{\psi^k(x_1,\dots,x_k)\}_{k=1}^{\infty}$ be a sequence of symmetric functions. For any finite set $X = \{x_1,\dots,x_n\}$ we define $\psi(X) = \psi^n(x_1,\dots,x_n)$ and we define correspondingly the truncated functions $\psi_T^k(x_1,\dots,x_k)$ by (3.13). The inverse relation yielding ψ in terms of ψ_T is then given by (3.14). Let us now define the generating functional for the ψ-functions as the formal power series in z

$$\psi(zh) = 1 + \sum_{k=1}^{\infty} \frac{(iz)^k}{k!} \int\dots\int \psi^k(x_1,\dots,x_k)h(x_1)\dots h(x_k)dx_1\dots dx_k. \quad (3.20)$$

Correspondingly we define $\psi_T(zh)$ in the same way. Substituting (3.14) in (3.20) we get

$$1 + \sum_{k=1}^{\infty} \frac{(iz)^k}{k!} \int\dots\int \sum_{X=X_1\cup\dots\cup X_{\ell}} \psi_T(X_1)\dots\psi_T(X_{\ell})h(x_1)\dots h(x_k)dx_1\dots dx_k. \quad (3.21)$$

Because of symmetry reassons all terms with $|X_1| = k_1,\dots,|X_{\ell}| = k_{\ell}$ give, after integration, the same contribution. Here $|X|$ stands for the number of points in X. Therefore, after rearranging variables, we get that (3.21) can be written as

$$1 + \sum_{k=1}^{\infty} \frac{(iz)^k}{k!} \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{k_1+\dots+k_{\ell}=k} \frac{k!}{k_1!\dots k_{\ell}!} \int \psi_T(x_1,\dots,x_{k_1})h(x_1)\dots h(x_{k_1})dx_1\dots dx_{k_1}$$
$$\dots \int \psi_T(x_1,\dots,x_{k_{\ell}})h(x_1)\dots h(x_{k_{\ell}})dx_1\dots dx_{k_{\ell}}$$
$$= 1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \psi_T(zh)^{\ell},$$

where the equality is to be understood as the equality of formal power series. This then gives us the formal power series relation between the generating functionals for ψ and ψ_T:

$$\psi(zh) = e^{\psi_T(zh)}. \quad (3.22)$$

By inverting the power series of the exponent function, we get then

$$\psi_T(zh) = \log \psi(zh) , \tag{3.23}$$

also as a formal power series relation.

From (5.6) of [5] we have the generating functional $G(zh)$ for the imaginary time Wightman functions

$$G(zh) =$$

$$= e^{-\frac{z^2}{2}(h,h)_{-1}}[1+\sum_{r=1}^{\infty}\frac{1}{r!}\int\ldots\int\prod_{j=1}^{r}(e^{-s_j zh^{\varepsilon}(x_j)}-1)\ \rho^r(x_1s_1,\ldots,x_rs_r)\prod_{j=1}^{r}d\mu(s_j)dx_j] \tag{3.24}$$

and it follows from this formula that $G(zh)$ is actually a convergent power series for z in the whole complex plane. In (3.24) $(h,h)_{-1} = \int h(x)G(x-y)h(y)dxdy$ and $h^{\varepsilon}(x) = \int G_{\varepsilon}(x-y)h(y)dy$.

Intruoducing the generating functional $\rho(zh)$ for the correlation functions as the (convergent) power series

$$\rho(zh) = 1+\sum_{n=1}^{\infty}\frac{(iz)^n}{n!}\int\ldots\int\rho^n(x_1s_1,\ldots,x_ns_n)g(x_1,s_1)\ldots g(x_n,s_n)\prod_{j=1}^{n}d\mu(s_i)dx_i , \tag{3.25}$$

we may rewrite (3.24) as

$$G(zh) = e^{-\frac{z^2}{2}(h,h)_{-1}}\ \rho(-i(e^{-szh^{\varepsilon}}-1)) , \tag{3.26}$$

which by (3.22) is equal to

$$G(zh) = e^{-\frac{z^2}{2}(h,h)_{-1}}\ e^{\rho_T(-i(e^{-szh^{\varepsilon}}-1))} .$$

Using (3.23) for $G_T(zh)$ we finally get

$$G_T(zh) = -\frac{z^2}{2}(h,h)_{-1} + \rho_T(-i(e^{-szh^{\varepsilon}}-1)) . \tag{3.27}$$

We formulate now these results in a theorem.

<u>Theorem 3.1</u>

The linked cluster expansion for the infinite volume imaginary time Wightman functions and the corresponding truncated fundtions are given in terms of their generating functionals by (3.26) and (3.27) and the linked cluster expansions for the correlation functions and the truncated correlation functions in Lemma 3.1. The linked cluster expansions are all convergent for $|\lambda| < \lambda_0$. The linked cluster expansions for the imaginary time Wightman functions are given by (3.1) and Lemma 3.1 and for the truncated imaginary time Wightman functions they are given by Lemma 3.1 and

$$G_T^k(x_1,\dots,x_k) = \delta_{k2} G(x_1-x_2) + (i)^k \sum_{r=1}^{k} \frac{1}{r!} \sum_{\sigma\in S_k} \sum_{\substack{\ell_1+\dots+\ell_r=k \\ \ell_i \geq 1}}$$

$$\frac{1}{\ell_1!\dots\ell_r!} \int\dots\int \prod_{i=r}^{r} [(s_i)^{\ell_i} \prod_{j=1}^{\ell_i} G_\varepsilon(x_{\sigma(\ell_1+\dots+\ell_i-j+1)}-y_i)]$$

$$\rho_T^r(y_1 s_1,\dots,y_r s_r) \prod_{j=1}^{r} d\mu(s_j) dy_j ,$$

where $\delta_{k2} = 1$ for $k = 2$, and $\delta_{k2} = 0$ for $k \neq 2$.

4. The linked cluster expansion as the asymptotic series for the scattering matrix of the space cut-off interaction.

Let us from now on assume that the dimension of the space time is ≥ 4 .

In section 2 we derived an asymptotic expansion, (2.18), for the quantity $S_{\ell}(g;f)$ which generates the S-matrix for the space cut-off interaction. The term of order n in this expansion is

$$(-i\lambda)^n \int_{t_n \leq \dots \leq t_1} (\Omega_o, :e^{i\varphi(g)}:[V_{\ell}(t_n),\dots,[V_{\ell}(t_1), :e^{i\varphi(f)}]\dots]\Omega_o)dt_1\dots dt_n. \quad (4.1)$$

One finds easily that, for $t_n \leq \dots \leq t_1$,

$$[V_{\ell}(t_n),\dots,[V_{\ell}(t_1), :e^{i\varphi(f)}:]\dots] = \sum_{k=o}^{n} (-1)^k \sum_{\sigma} V_{\ell}(t_{i_n})\dots V_{\ell}(t_{i_{k+1}}) :e^{i\varphi(f)}: V_{\ell}(t_{i_1})\dots V_{\ell}(t_{i_k}),$$

where the $\sum_{\sigma}$ runs over all permutations $\sigma = \{i_1,\dots,i_n\}$ of $\{1,\dots,n\}$ such that $t_{i_n} \leq \dots \leq t_{i_{k+1}}$ and $t_{i_1} \geq \dots \geq t_{i_k}$. Since the integration in (4.1) is convergent, it is the limit of

$$(-i\lambda)^n \int \dots \int_{-a \leq t_n \leq \dots \leq t_1 \leq a} (\Omega_o, :e^{i\varphi(g)}: [V_{\ell}(t_n),\dots,[V_{\ell}(t_1), :e^{i\varphi(f)}:]\dots]\Omega_o)dt_1\dots dt_n$$

$$= (-i\lambda)^n \sum_{k=o}^{n} (-1) \sum_{\sigma} \int \dots \int_{-a \leq t_n \leq \dots \leq t_1 \leq a} (\Omega_o, :e^{i\varphi(g)}: V_{\ell}(t_{i_n})\dots V_{\ell}(t_{i_{k+1}}) :e^{i\varphi(f)}: V_{\ell}(t_{i_1})\dots V_{\ell}(t_{i_k})\Omega_o)dt_1\dots dt_n \quad (4.2)$$

$$= (-i\lambda)^n \sum_{r+k=n} (-1)^k \int \dots \int_{\substack{-a \leq t_1 \leq \dots \leq t_r \leq a \\ -a \leq \tilde{t}_1 \leq \dots \leq \tilde{t}_k \leq a}} (\Omega_o, :e^{i\varphi(g)}: V_{\ell}(t_1)\dots V_{\ell}(t_r) :e^{i\varphi(f)}: V_{\ell}(\tilde{t}_k)\dots V_{\ell}(\tilde{t}_1)\Omega_o)dt_1\dots dt_r d\tilde{t}_1\dots d\tilde{t}_k .$$

To compute the integrand in (4.2) we observe that

$$(\Omega_o, :e^{is_1\varphi_{\varepsilon}(t_1,\vec{x}_1)}::e^{is_2\varphi_{\varepsilon}(t_2,\vec{x}_2)}:\Omega_o) =$$

$$= (\Omega_o, :e^{is_1\varphi_\varepsilon(\vec{x})}: e^{i(t_1-t_2)H_o} :e^{is_2\varphi_\varepsilon(\vec{x}_2)}:\Omega_o) =$$

$$= e^{-s_1 s_2 G_\varepsilon(i(t_1-t_2),\vec{x}_1-\vec{x}_2))} ,$$

where $\varphi_\varepsilon(t,\vec{x}) = e^{-itH_o}\varphi_\varepsilon(\vec{x})e^{itH_o}$. An explicit calculation yields, for $x_o > 0$,

$$G_\varepsilon(x_o,\vec{x}) = \pi \int_{\mathbb{R}^{n-1}} e^{-\mu(\vec{p})x_o} e^{i\vec{p}\vec{x}} |\tilde{\chi}_\varepsilon(\vec{p})|^2 \frac{d\vec{p}}{\mu(\vec{p})} , \tag{4.3}$$

and $G_\varepsilon(it,\vec{x})$ is therefore given by

$$G_\varepsilon(it,\vec{x}) = \pi \int_{\mathbb{R}^{n-1}} e^{-i\mu(\vec{p})t} e^{i\vec{p}\vec{x}} |\tilde{\chi}_\varepsilon(\vec{p})|^2 \frac{d\vec{p}}{\mu(\vec{p})} .$$

We set $F_\varepsilon(t,\vec{x}) = G_\varepsilon(it,\vec{x})$. For a function $g(\vec{x})$ on $\mathbb{R}^{n-1}$ we define

$$g_\varepsilon(t,\vec{x}) = \int F_\varepsilon(t,\vec{x}-\vec{y})g(\vec{y})d\vec{y} .$$

By a direct computation we find that the integrand in (4.2) is equal to

$$e^{-(g,f)_{-\frac{1}{2}}} \int_{\substack{|\vec{x}_i|\le \ell \\ |\tilde{\vec{x}}_i|\le \ell}} \!\!\cdots\!\int \exp\Big\{-\sum_{i<j}^{r} s_i s_j F_\varepsilon(t_i-t_j,\vec{x}_i-\vec{x}_j) - \sum_{i>j}^{k} \tilde{s}_i\tilde{s}_j F_\varepsilon(\tilde{t}_i-\tilde{t}_j,\tilde{\vec{x}}_i-\tilde{\vec{x}}_j) - \tag{4.4}$$

$$- \sum_{i=1}^{r}\sum_{j=1}^{k} s_i\tilde{s}_j F_\varepsilon(t_i-\tilde{t}_j,\vec{x}_i-\tilde{\vec{x}}_j)\Big\} \cdot \exp\Big\{-\sum_{i=1}^{r} s_i f_\varepsilon(t_i,\vec{x}_i) - \sum_{j=1}^{k}\tilde{s}_j f_\varepsilon(-\tilde{t}_j,\tilde{\vec{x}}_j) -$$

$$- \sum_{i=1}^{r} s_i g_\varepsilon(-t_i,\vec{x}_i) - \sum_{j=1}^{k}\tilde{s}_j g_\varepsilon(-\tilde{t}_j,\tilde{\vec{x}}_j)\Big\} \cdot \prod_{i=1}^{r} d\mu(s_i)dx_i \prod_{j=1}^{k} d\mu(\tilde{s}_j)dx_j ,$$

where $d\mu(s) = e^{-\frac{1}{2}s^2 G_\varepsilon(o)} d\nu(s)$, $(g,f)_{-\frac{1}{2}}$ is the scalar product in the Sobolev space $\mathcal{H}^{-\frac{1}{2}}_{n-1}$ (see [5]). Let $A = \{1,\dots,r\}$ and $B = \{\tilde{1},\dots,\tilde{k}\}$, so that A and B are disjoint sets of labeled points. Let L be the set of all unordered pairs of different points in $A \cup B$. The elements in L will be called lines and subsets of L will be called graphs. We now define:

$$a_i = e^{-s_i f_\varepsilon(t_i,\vec{x}_i) - s_i g_\varepsilon(-t_i,\vec{x}_i)} - 1 \qquad \text{for } i \in A ,$$

$$b_{\tilde{j}} = e^{-\tilde{s}_j f_\varepsilon(-t_j,\tilde{\vec{x}}_j) - \tilde{s}_j g_\varepsilon(-t_j,\vec{x}_j)} - 1 \qquad \text{for } j \in B .$$

Furthermore we define

$$F_\ell = e^{-s_i s_j F_\varepsilon(-|t_i - t_j|,\vec{x}_i - \vec{x}_j)} - 1 \qquad \text{if } \ell = (i,j),\ i<j,\ i,j \in A ,$$

$$F_\ell = e^{-\tilde{s}_i \tilde{s}_j F_\varepsilon(|t_i - t_j|,\tilde{\vec{x}}_i - \tilde{\vec{x}}_j)} - 1 \qquad \text{if } \ell = (\tilde{i},\tilde{j}),\ \tilde{i}>\tilde{j},\ \tilde{i},\tilde{j} \in B ,$$

$$F_\ell = e^{-s_i \tilde{s}_j F_\varepsilon(t_i - t_j,\vec{x}_i - \tilde{\vec{x}}_j)} - 1 \qquad \text{if } \ell = (i,\tilde{j}),\ i \in A,\ \tilde{j} \in B .$$

Then the integrand in (4.4) may be written as

$$I_{r,k} = \prod_{\ell \in L} (F_\ell + 1) \prod_{i \in A} (a_i + 1) \prod_{\tilde{j} \in B} (b_{\tilde{j}} + 1) . \tag{4.5}$$

Expanding the products we get

$$I_{r,k} = \sum_{X \subset A} \sum_{Y \subset B} \sum_{\Gamma \subset L} \prod_{\ell \in \Gamma} F_\ell \prod_{i \in X} a_i \prod_{\tilde{j} \in Y} b_{\tilde{j}} . \tag{4.6}$$

By the definition of F_ℓ we see from (4.5) that $I_{r,k}$ is symmetric with respect to permutations of the points in A and similarly it is also symmetric with respect to permutations of the points in B. Hence (4.2) may be rewritten as

$$e^{-(g,f) - \frac{1}{2}} (-i\lambda)^n \sum_{r+k=n} \frac{(-1)^k}{r!k!} \int \dots \int_{\Lambda_a^{r+k}} I_{r,k} \prod_{j=1}^{r} d\mu(s_i) d\vec{x}_i dt_i \prod_{j=1}^{k} d\mu(\tilde{s}_j) d\tilde{\vec{x}}_j dt_j , \tag{4.7}$$

where $\Lambda_a = \{(t,\vec{x});\ |t| \leq a,\ |\vec{x}| \leq \ell \}$.

Since the integration over the t's in (4.2) is restricted to the bounded interval $[-a,a]$, the sum of (4.2) over n converges to

$$S_\ell^a(g;f) = (\Omega_0, :e^{i\varphi(g)}: e^{iaH_0} e^{-i2aH_\ell} e^{iaH_0} :e^{i\varphi(f)}: e^{-iaH_0} e^{i2aH_\ell} e^{-iaH_0} \Omega_0) \tag{4.8}$$

Therefore by summing (4.7) over n and using (4.6) we get

$$S_{\ell}^{a}(g;f) = e^{-(g,f)_{-\frac{1}{2}}} \sum_{|A|,|B|} \frac{(-i\lambda)^{|A|}}{|A|!} \frac{(i\lambda)^{|B|}}{|B|!} \sum_{\Gamma\subset L} \sum_{X\subset A} \sum_{Y\subset B} \int\dots\int_{\Lambda_a^{(|A|+|B|)}} \tag{4.9}$$

$$\prod_{\ell\in\Gamma} F_{\ell} \prod_{i\in X} a_i \prod_{\tilde{j}\in Y} b_{\tilde{j}} \prod_{i=1}^{|A|} d\mu(s_i)d\vec{x}_i dt_i \prod_{j=1}^{|B|} d\mu(\tilde{s}_j)d\tilde{\vec{x}}_j d\tilde{t}_j \ ,$$

where, for any finite set X , $|X|$ denotes the number of elements in X .

Let A_1 and B_1 be the elements in A and respectively in B that are points of Γ . Let X_1 respectively Y_1 be the elements in X respectively Y that are points of Γ , and let X_2 and Y_2 be their complements in X and Y . Let A_2 respectively B_2 be the complements of $A_1 \cup X_2$ respectively $B_1 \cup Y_2$ in A respectively B . Then $A = A_1 \cup X_2 \cup A_2$, $B = B_1 \cup Y_2 \cup B_2$ and $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$. All the unions are disjoint and $X_1 \subset A_1$, $Y_1 \subset B_1$. For a fixed term in the sum in (4.9), we permute the integration variables so that the points in A_1 and B_1 come first, and then come the points in X_2 and Y_2 and last the points in A_2 and B_2 . By the symmetry of $I_{r,k}$ we then get

$$\frac{|A|! \ |B|!}{|A_1|! \ |B_1|! \ |X_2|! \ |Y_2|! \ |A_2|! \ |B_2|!}$$

equal contributions to the sum. Hence (4.9) can be written as

$$S_{\ell}^{a}(g;f) = e^{-(g,f)_{-\frac{1}{2}}} \sum_{\substack{|A_1|,|B_1|,|X_2| \\ |Y_2|,|A_2|,|B_2|}} \frac{(-i\lambda)^{|A_1|+|X_2|+|A_2|}(i\lambda)^{|B_1|+|Y_2|+|B_2|}}{|A_1|! \ |B_1|! \ |X_2|! \ |Y_2|! \ |A_2|! \ |B_2|!} \tag{4.10}$$

$$\sum_{\Gamma\subset L} \sum_{\substack{X_1\subset A_1 \\ Y_1\subset B_1}} \int\dots\int_{\Lambda_a} \prod_{\ell\in\Gamma} F_{\ell} \prod_{i\in X_1} a_i \prod_{i\in X_2} a_i \prod_{\tilde{j}\in Y_1} b_{\tilde{j}} \prod_{\tilde{j}\in Y_2} b_{\tilde{j}}$$

$$\prod_i d\mu(s_i)d\vec{x}_i dt_i \prod_j d\mu(\tilde{s}_j)d\tilde{\vec{x}}_j d\tilde{t}_j \ .$$

Now the integrand is independent of the variables in A_2 and B_2, and $\prod_{\ell\in\Gamma} F_\ell$ is independent of the variables in X_2 and Y_2. Therefore we can sum over X_2, Y_2, A_2 and B_2 and obtain

$$S_\ell^a(g;f) = e^{-(g,f)_{-\frac{1}{2}}}$$

$$e^{-i\lambda|\Lambda|\,|\mu|}\, e^{i\lambda|\Lambda|\,|\mu|} \sum_{|A_1|,|B_1|} \frac{(-i\lambda)^{|A_1|}}{|A_1|!} \frac{(i\lambda)^{|B_1|}}{|B_1|!} \sum_{\Gamma\subset L} \sum_{X_1\subset A_1} \sum_{Y_1\subset B_1} \tag{4.11}$$

$$\int\cdots\int_{\Lambda_a^{(|A_1|+|B_1|)}} \prod_{\ell\in\Gamma} F_\ell \prod_{i\in X_1} a_i \prod_{\tilde{j}\in Y_1} b_{\tilde{j}} \prod_i^{|A_1|} d\mu(s_i)d\vec{x}_i dt_i \prod_j^{|B_1|} d\mu(\tilde{s}_j)d\tilde{\vec{x}}_j d\tilde{t}_j,$$

where $a(t_i,\vec{x}_i) = a_i$, $b(\tilde{t}_j,\tilde{\vec{x}}_j) = b_{\tilde{j}}$ and $|\Lambda|$ is the volume of Λ, $|\mu| = \int d\mu(s)$. The factors containing $|\Lambda|$ obviously cancel and in (4.11) $A_1 \cup B_1$ are the points of Γ.
We call the points in $X_1 \cup Y_1$, entering (4.11), the external points of Γ. As in the previous section, we have $\Gamma = \Gamma_0 \times \Gamma_3$, where Γ_0 is the subgraph of Γ consisting of the points in Γ which are not externally connected and Γ_3 is the subgraph of externally connected points.
Let A_0 and B_0 be the subsets of A_1 and B_1 that are points of Γ_0 and let A_3 and B_3 be the complements of A_0 and B_0 in A and B. Then $X_1 \subset A_3$ and $Y_1 \subset B_3$, since A_0 and B_0 are internal points. The sum in (4.11) can therefore be written as

$$\sum \frac{(-i\lambda)^{|A_1|}}{|A_1|!} \frac{(i\lambda)^{|B_1|}}{|B_1|!} \sum_{\Gamma_0,\Gamma_3} \sum_{X_1\subset A_3} \sum_{Y_1\subset B_3} \int\cdots\int_{\Lambda_a} \tag{4.12}$$

$$\prod_{\ell\in\Gamma} F_\ell \prod_{i\in X_1} a_i \prod_{\tilde{j}\in Y_1} b_{\tilde{j}} \prod_i d\mu(s_i)d\vec{x}_i dt_i \prod_j d\mu(\tilde{s}_j)d\tilde{\vec{x}}_j d\tilde{t}_j .$$

By rearranging the variables in such a way that those in A_3 and B_3 come first, we get $\frac{|A_1|!\ |B_1|!}{|A_0|!\ |B_0|!\ |A_3|!\ |B_3|!}$ equal contributions to the sum in (4.12). Hence (4.12) becomes the product of the two following series

$$\Sigma \frac{(-i\lambda)^{|A_0|}}{|A_0|!} \frac{(i\lambda)^{|B_0|}}{|B_0|!} \sum_{\Gamma_0} \int\ldots\int_{\Lambda} \prod_{\ell\in\Gamma_0} F_\ell \prod_i d\mu(s_i) d\vec{x}_i dt_i \prod_j d\mu(\tilde{s}_j) d\tilde{\vec{x}}_j dt_j, \qquad (4.13)$$

where Γ_0 runs over all graphs with points in $A_0 \cup B_0$, and

$$\Sigma \frac{(-i\lambda)^{|A_3|}}{|A_3|!} \frac{(i\lambda)^{|B_3|}}{|B_3|!} \sum_{X_1\subset A_3} \sum_{Y_1\subset B_3} \sum_{\Gamma_3} \int\ldots\int_{\Lambda} \prod_{\ell\in\Gamma_3} F_\ell \prod_{i\in X_1} a_i \qquad (4.14)$$

$$\prod_{\tilde{j}\in Y_1} b_{\tilde{j}} \prod_i d\mu(s_i) d\vec{x}_i dt_i \prod_j d\mu(\tilde{s}_j) d\tilde{\vec{x}}_j d\tilde{t}_j ,$$

where Γ_3 runs over all externally connected graphs. From (4.9) we see that (4.13) is nothing but $S^a_\ell(0,0)$, which is equal to $(\Omega_0,\Omega_0) = 1$.

We formulate these results in the following theorem:

<u>Theorem 4.1</u>

The linked cluster expansion for

$$S^a_\ell(g;f) = (\Omega_0, :e^{i\varphi(g)}: e^{iaH_0} e^{-2iaH_\ell} e^{iaH_0} :e^{i\varphi(f)}: e^{-iaH_0} e^{i2aH_\ell} e^{-iaH_0} \Omega_0)$$

is convergent for all λ and given by

$$S^a_\ell(g;f) = e^{-(g;f)_{-\frac{1}{2}}} \sum_{|A|,|B|} \frac{(-i\lambda)^{|A|}}{|A|!} \frac{(i\lambda)^{|B|}}{|B|!} \sum_{X\subset A} \sum_{Y\subset B} \sum_E \int\ldots\int_{\Lambda_a}$$

$$\prod_{\ell\in E} F_\ell \prod_{i\in X} a_i \prod_{\tilde{j}\in Y} b_{\tilde{j}} \prod_i d\mu(s_i) d\vec{x}_i dt_i \prod_j d\mu(\tilde{s}_j) d\tilde{\vec{x}}_j d\tilde{t}_j ,$$

where E runs over all externally connected graphs and

$$h(t,\vec{x}) = e^{-sf_\varepsilon(t,\vec{x})-sg_\varepsilon(-t,\vec{x})} - e^{-sf_\varepsilon(-t,\vec{x})-sg_\varepsilon(-t,\vec{x})} .$$

Defining moreover

$$S_{\ell}^{a}(g,f)_{T} = -(g,f)_{-\frac{1}{2}} + \sum_{|A|,|B|} \frac{(-i\lambda)^{|A|}}{|A|!} \frac{(i\lambda)^{|B|}}{|B|!} \sum_{X\subset A} \sum_{Y\subset B} \sum_{C} \int_{\Lambda_a}\cdots\int$$

$$\prod_{\ell\in C} F_{\ell} \prod_{i\in X} a_i \prod_{\tilde{j}\in Y} b_{\tilde{j}} \prod_i d\mu(s_i)d\vec{x}_i dt_i \prod_j d\mu(\tilde{s}_j)d\tilde{\vec{x}}_j d\tilde{t}_j ,$$

where C runs over all those externally connected graphs which are also connected, we have

$$S_{\ell}^{a}(g;f) = e^{S_{\ell}^{a}(g;f)_{T}} .$$

Proof: Only the last part of the theorem is not already proven above. The proof of this last part runs however entirely parallel to the proof of the last part of Lemma 3.1. ∎

We know by (2.18) that each term in the power series expansion of $S_{\ell}^{a}(g;f)$ converges as $a\to\infty$, and that the formal power series we get in the limit is the asymptotic expansion of $S_{\ell}(g;f)$. We shall now prove that each term in the linked cluster expansion of $S_{\ell}^{a}(g;f)$ converges as a tends to infinity, and the corresponding expansion is the linked cluster expansion of $S_{\ell}(g;f)$. It follows then that the linked cluster expansion of $S_{\ell}(g;f)$ is identical as formal power series in λ with the asymptotic expansion (2.18) of $S_{\ell}(g;f)$. Since one can always form exponentials of formal power series, it is enough to prove that each term in the power series expansion in λ of $S_{\ell}^{a}(g;f)_{T}$ converges as $a\to\infty$.

For fixed A, B and C in the linked cluster expansion of $S_{\ell}^{a}(g;f)_{T}$ in Theorem 4.1, consider now the integral

$$\int_{\Lambda_a}\cdots\int \prod_{\ell\in C} F_{\ell} \prod_{i\in X} a_i \prod_{\tilde{j}\in Y} b_{\tilde{j}} . \tag{4.15}$$

Since F_ℓ , a_i and $b_{\tilde{j}}$ fall off sufficiently rapidly in all the time variables, as solutions of the Klein-Gordon equation with smooth time zero values, we find that the integrand in (4.15) is absolutely integrable with respect to all the time variables in $\mathbb{R}^N$, where $N = |A| + |B|$. Because of the support properties of the Fourier transform, with respect to $\tilde{t}_j$, $j \in B$, of the integrand in (4.15), we see that the integral over $\mathbb{R}^k$ with respect to $\tilde{t}_1, \ldots, \tilde{t}_k$, $k = |B|$ is equal to zero. Hence as $a \to \infty$ all contributions, to the linked cluster expansion in Theorem 4.1, of terms of the form (4.15) with $B \neq \emptyset$ tend to zero. We formulate these results in the following theorem:

Theorem 4.2

For space-time dimension larger or equal to 4 we have that the linked cluster expansion of the scattering matrix for the space cut-off interaction is given by

$$S_\ell(g;f) \doteq e^{-(g,f)_{-\frac{1}{2}}} \sum_{|A|} \frac{(-i\lambda)^{|A|}}{|A|!} \sum_{X\subset A} \sum_{E} \int_{|\vec{x}_i|\leq \ell}\!\!\cdots\!\int \prod_{\ell\in E} F_\ell \prod_{i\in X} a_i \prod_{i=1}^{|A|} du(s_i)\,d\vec{x}_i\,dt_i \ ,$$

where E runs over all externally connected graphs with set of points A .

This linked cluster expansion of $S_\ell(g;f)$ gives also the asymptotic expansion of $S_\ell(g;f)$ in powers of λ .

If we define the corresponding truncated expression by the formal power series relation

$$S_\ell(g;f)_T = \log S_\ell(g;f) \ ,$$

then the formal power series for S_T is given by

$$S_{\ell}(g;f)_T = -(g,f)_{-\frac{1}{2}} + \sum_{|A|} \frac{(-i\lambda)^{|A|}}{|A|!} \sum_{X\subset A} \sum_{C} \int\cdots\int_{|\vec{x}_i|\leq\ell} \prod_{\ell\in C} F_{\ell} \prod_{i\in X} a_i \prod_{i=1}^{|A|} d\mu(s_i) d\vec{x}_i dt_i ,$$

where C runs over all the connected graphs which have external points. ∎

We see from the linked cluster expansion in Theorem 4.2 that, if we define the space cut-off scattering functions $\sigma_{\ell}^{k}(x_1 s_1,\dots,x_k s_k)$ as the formal power series

$$\sigma_{\ell}^{k}(x_1 s_1,\dots,x_k s_k) = \sum_{n=0}^{\infty} \frac{(-i\lambda)^{n+k}}{n!} \sum_{E} \int\cdots\int_{|\vec{x}_i|\leq} \prod_{\ell\in E} F_{\ell} \prod_{i=k+1}^{n+k} d\mu(s_i) d\vec{x}_i dt_i , \tag{4.16}$$

with $x_j = (t_j,\vec{x}_j)$, then $S_{\ell}(g;f)$ can be written as the following formal power series

$$S_{\ell}(g;f) = e^{-(g,f)_{-\frac{1}{2}}} \sigma_{\ell}(-ia) , \tag{4.17}$$

where $\sigma_{\ell}(a)$ is the generating functional for the scattering functions, defined by (3.20) and

$$a(t,\vec{x}) = e^{-sf_{\varepsilon}(t,\vec{x})-sg_{\varepsilon}(-t,\vec{x})} - 1 . \tag{4.18}$$

Using the symmetry properties of F_{ℓ} we can write (4.16) for $t_1 \leq\dots\leq t_k$ as

$$\sigma_{\ell}^{k}(x_1 s_1,\dots,x_k s_k) = \sum_{n=0}^{\infty} (-i\lambda)^{k+n} \sum_{E} \int\cdots\int_{\substack{|\vec{x}_i|\leq\ell \\ t_1\leq\dots\leq t_{n+k}}} \prod_{\ell\in E} F_{\ell} \prod_{i=k+1}^{k+n} d\mu(s_i) d\vec{x}_i dt_i . \tag{4.19}$$

The integrand in (4.19) is obviously only a function of the time variables $z_j = t_{j+1}-t_j$, $j = 1,\dots,k+n-1$. Since

$$F_{\ell} = e^{-s_i s_j G_{\varepsilon}(i(t_i-t_j),\vec{x}_i-\vec{x}_j)} - 1 \qquad \text{for } \ell = (i,j),\ i<j ,$$

we see that the integrand is analytic in $\operatorname{Im} z_j > 0$, and falls off as $|z_1|^{-3/2}\dots|z_{k+n-1}|^{-3/2}$ uniformly in $\operatorname{Im} z_j > 0$, $j=1,\dots,k+n-1$, for all space-time dimensions ≥ 4 .

If we introduce the new variables $z_j = t_{j+1} - t_j$ in (4.19) we will have the integration over the domain $z_j \geq 0$, $j = 1,\ldots,n$. Because of the uniform fall off, we may continue the path of integration from the positive real axis onto the positive imaginary axis. From this it follows that each term in the formal power series expansion (4.19) is analytic for $\mathrm{Im}\, t_i > 0$, $i = 1,2,\ldots,k$ and moreover we obtain the following formal power series relation

$$\sigma_{\ell}^{k}(\tilde{x}_1 s_1,\ldots,\tilde{x}_k s_k) = (-i)^k \sum_{n=0}^{\infty} \frac{(-\lambda)^{k+n}}{n!} \sum_{E} \int \ldots \int_{|\vec{x}_i| \leq \ell} \prod_{\ell \in E} (e^{-s_i s_j G_\epsilon(x_i - x_j)} - 1) \prod_{i=k+1}^{k+n} d\mu(s_i) dx_i , \quad (4.20)$$

where $\tilde{x}_j = (i(x_j)_o, \vec{x}_j)$, $x_j = ((x_j)_o, \vec{x}_j)$, $(x_j)_o$ being the time component of x_j.

<u>Lemma 4.1</u>

Let H_o be a self adjoint operator with zero as an isolated simple eigenvalue with eigenvector Ω_o, and the rest of its spectrum absolutely continuous. Let $H_\lambda = H_o + \lambda V$, where V is bounded and symmetric. Let P_o be the projection onto Ω_o and P_1 its orthogonal complement. Let $H_o^{-1} P_1$ be the inverse of H_o on the range of P_1.

If $\lambda_1 = \frac{1}{2} \| V H_o^{-1} P_1 \|^{-1}$ then for $|\lambda| < 2\lambda_1$ H_λ has a simple isolated eigenvalue E_λ with eigenvector Ω_λ, both depending analytically on λ.

Moreover for $|\lambda| < \lambda_1$

$$\operatorname*{weak\,lim}_{t \to \pm\infty} e^{it(H_\lambda - E_\lambda)} \Omega_o = (\Omega_\lambda, \Omega_o) \Omega_\lambda .$$

Proof: The first part of the lemma is well known from regular perturbation theory. The moreover part is proved as follows. We expand in powers of λ and get

$$e^{iaH_\lambda} \Omega_o = \sum_{n=0}^{\infty} (i\lambda)^n \int_{o \leq t_1 \leq \ldots \leq t_n \leq a} \ldots \int V(t_1) \ldots V(t_n) \Omega_o dt_1 \ldots dt_n = \sum_{n=0}^{\infty} (i\lambda)^n \int_{o \leq t_1 \leq \ldots \leq t_n \leq a} \ldots \int e^{-it_1 H_o} V$$

$$e^{-i(t_2 - t_1)H_o} V \ldots e^{-i(t_n - t_{n-1})H_o} V \Omega_o dt_1 \ldots dt_n = \sum_{n=0}^{\infty} (i\lambda)^n \int_o^a \ldots \int_o^a e^{-is_1 H_o} V e^{-is_2 H_o} V \ldots e^{-is_n H_o} V$$

$$\Omega_o ds_1 \ldots ds_n = \sum_{n=0}^{\infty} (i\lambda)^n \sum_{\epsilon_1 \ldots \epsilon_n} \int_o^a \ldots \int_o^a e^{-is_1 H_o} P_{\epsilon_1} V \ldots e^{-is_n H_o} P_{\epsilon_n} V \Omega_o ds_1 \ldots ds_n = (\Omega_o, e^{iaH_\lambda} \Omega_o)$$

$$\sum_{n=0}^{\infty} (i\lambda)^n \int_o^a \ldots \int_o^a e^{-is_1 H_o} P_1 V \ldots e^{-is_n H_o} P_1 V \Omega_o ds_1 \ldots ds_n .$$

By integrating we find that

$$(\Omega_o, e^{isH_\lambda}\Omega_o)^{-1} e^{iaH_\lambda}\Omega_o = \sum_{n=o}^{\infty} \lambda^n H_o^{-1} P_1 (1-e^{-iaH_o}) V \ldots H_o^{-1} P_1 (1-e^{-iaH_o}) V \Omega_o .$$

For $|\lambda| < \lambda_1$ this series is strongly convergent, uniformly in a, and term by term it converges weakly to the Rayleigh-Schrødinger expansion for Ω_λ. This implies, by the convergence of the Rayleigh-Schrødinger expansion, that $(\Omega_o, e^{iaH_\lambda}\Omega_o)^{-1} e^{iaH_\lambda}\Omega_o \to (\Omega_o, \Omega_\lambda)^{-1}\Omega_\lambda$ weakly. By taking the inner product with Ω_λ this gives that $e^{iaE_\lambda}(\Omega_o, e^{iaH_\lambda}\Omega_o)^{-1}$ converges to $|(\Omega_o, \Omega_\lambda)|^{-2}$. This, together with the weak convergence above, proves the lemma.

Remark: This proof has relations to methods developed by De Witt and Lanford and extended by Hepp (Theorem 2.5 of [8]).

Theorem 4.3.

Let $|\lambda| < \lambda_1$, where λ_1 is given in lemma 4.1, and $A = :e^{i\varphi(g)}:$, $B = :e^{i\varphi(f)}:$, then

$$(\Omega_o, e^{-2itH_\ell}\Omega_o)^{-1} (e^{itH_\ell} e^{-itH_o} A\Omega_o, e^{-itH_\ell} e^{itH_o} B\Omega_o)$$

converges to $(A_-\Omega_\ell, B_+\Omega_\ell)$ as $a \to \infty$, where A_- is the norm limit of A_t as $t \to -\infty$ and B_+ is the norm limit of B_t as $t \to \infty$, with $A_t = e^{-itH_\ell} e^{itH_o} A e^{-itH_o} e^{itH_\ell}$ and similarly for B.

Proof: $(\Omega_o, e^{-i2aH_\ell}\Omega_o)^{-1} (e^{iaH_\ell} e^{-iaH_o} A\Omega_o, e^{-iaH_\ell} e^{iaH_o} B\Omega_o)$

$$= (\Omega_o, e^{-i2aH_\ell}\Omega_o)^{-1} (A_{-a} e^{iaH_\ell}\Omega_o, B_a e^{-iaH_\ell}\Omega_o) .$$

One verifies easily that $\|[V_\ell(s), :e^{i\varphi(f)}:]\| \leq c(|s|+1)^{-3/2}$, from which it follows from (2.10) that A_{-a} and B_a converge in norm to A_- and B_+. Hence it is enough to prove that $(\Omega_o, e^{-i2aH_\ell}\Omega_o)^{-1} (e^{iaH_\ell}\Omega_o, A_-^* B_+ e^{-iaH_\ell}\Omega_o)$ converges as $a \to \infty$. Since the finite dimensional operators are dense, by uniform boundedness it is enough to prove that the expression converges with a finite dimensional operator C replacing $A_-^* B_+$. By linearity it is then enough to prove that for any pair of vectors Φ and Ψ we have convergence of

$$(\Omega_o, e^{-i2aH_\ell}\Omega_o)^{-1} (e^{iaH_\ell}\Omega_o, \Phi)(\Psi, e^{-iaH_\ell}\Omega_o) .$$

By lemma 4.1 this however converges to $(\Omega_\ell, \Phi)(\Psi, \Omega_\ell)$, which completes the proof of the theorem.

5. The convergence of the linked cluster expansion for the space cut-off scattering matrix.

Let $\Lambda \subset \mathbb{R}^n$ be a bounded domain. Define the quantities F_Λ, Z_Λ, f_x as in section 4 of [5] and set $\Phi_\epsilon(x) = \Phi(f_x)$, where $\Phi = \Phi_{\mathcal{H}_n^{-1}}$ is the generalized Gaussian stochastic process indexed by the Sobolev space $\mathcal{H}_n^{-1}$ defined in section 3 of [5]. We then have from section 4 of [5]:

$$(-\lambda)^k Z_\Lambda^{-1} F_\Lambda\left(\sum_{i=1}^k s_i f_{x_i}\right) = (-\lambda)^k Z_\Lambda^{-1} \mathbb{E}\left(e^{i\sum_{j=1}^k s_j \Phi_\epsilon(x_j)} e^{-\lambda\int_\Lambda v(\Phi_\epsilon(x))dx}\right) , \qquad (5.1)$$

where $\mathbb{E}$ is the expectation in the probability space for Φ (see [5]). Expanding with respect to λ we get this equal to

$$(-\lambda)^k Z_\Lambda^{-1} \sum_{n=o}^{\infty} \frac{(-\lambda)^n}{n!} \int\cdots\int_{\Lambda^n} \mathbb{E}\left(e^{i\sum_{i=1}^k s_i \Phi_\epsilon(x_i)} v(\Phi_\epsilon(x_{k+1}))\cdots v(\Phi_\epsilon(x_{k+n}))\right) \prod_{j=k+1}^{k+n} dx_j$$

$$= Z_\Lambda^{-1} \sum_{n=o}^{\infty} \frac{(-\lambda)^{k+n}}{n!} \int\cdots\int_{\Lambda^n} \mathbb{E}\left(e^{i\sum_{i=1}^{k+n} s_i \Phi_\epsilon(x_i)}\right) \prod_{j=k+1}^{k+n} d\nu(s_j)dx_j = Z_\Lambda^{-1} \sum_{n=o}^{\infty} \frac{(-\lambda)^{k+n}}{n!} \int\cdots\int_{\Lambda^n}$$

$$e^{-\frac{1}{2}\sum_{i,j=1}^{k+n} s_i s_j G_\epsilon(x_i - x_j)} \prod_{j=k+1}^{k+n} d\nu(s_j)dx_j ,$$

which by (4.3) of [5] is equal to $\rho_\Lambda^k(x_1 s_1,\ldots,x_k s_k)$.

Hence we have proven the formula:

$$\rho_\Lambda^k(x_1 s_1,\ldots,x_k s_k) = (-\lambda)^k [\mathbb{E}(e^{-\lambda\int_\Lambda v(\Phi_\epsilon(x)dx)})]^{-1} \mathbb{E}\left(e^{i\sum_{i=1}^k s_i \Phi_\epsilon(x_i)} e^{-\lambda\int_\Lambda v(\Phi_\epsilon(x))dx}\right). \quad (5.2)$$

Choose $\Lambda_{a,\ell} = \{(x_o,\vec{x}); |x_o| \le a, |\vec{x}| \le \ell\}$. Let $\rho_\ell^k(x_1 s_1,\ldots,x_k s_k)$ be the limit of $\rho^k_{\Lambda_{a,\ell}}$ for $a \to \infty$, which exists by (5.2) and the Lemma 3.4 of [5]. Moreover we have the following:

<u>Lemma 5.1</u>

The limit $\rho_\ell^k(x_1 s_1,\ldots,x_k s_k)$ of $\rho^k_{\Lambda_{a,\ell}}(x_1 s_1,\ldots,x_k s_k)$ as

$a \to \infty$ exists and the convergence is uniform on compact subsets. Moreover:

$$\rho_\ell^k(x_1 s_1,\ldots,x_k s_k) = (-\lambda)^k (\Omega_\ell, e^{is_1\varphi_\epsilon(\vec{x}_1)} e^{-(t_2-t_1)\bar{H}_\ell} e^{is_2\varphi_\epsilon(\vec{x}_2)} \ldots e^{-i(t_k-t_{k-1})\bar{H}_\ell} e^{is_k\varphi_\epsilon(\vec{x}_k)} \Omega_\ell),$$

for $(x_i)_o = t_i$, $t_1 \leq \ldots \leq t_k$, $\bar{H}_\ell = H_\ell - E_\ell$, E_ℓ being the eigenvalue of Ω_ℓ .

Proof: The uniform convergence on compact subsets follows as in the proof of Lemma 4.1 of [5] and the formula for the limit follows from Lemma 3.4 of [5]. ∎

Lemma 5.2

The space cut-off correlation functions $\rho_\ell^k(x_1 s_1,\ldots,x_k s_k)$ and the corresponding truncated functions $\rho_{\ell,T}^k(x_1 s_1,\ldots,x_k s_k)$ are given, for $|\lambda| < \lambda_o$, by their convergent linked cluster expansions

$$\rho_\ell^k(x_1 s_1,\ldots,x_k s_k) = \sum_{n=o}^{\infty} \frac{(-\lambda)^{k+n}}{n!} \sum_E \int\limits_{|\vec{x}_i|\leq\ell} \cdots \int \prod_{(i,j)\in E} (e^{-s_i s_j G_\epsilon(x_i-x_j)} - 1) \prod_{i=k+1}^{k+n} d\mu(s_i) dx_i$$

and

$$\rho_{\ell,T}^k(x_1 s_1,\ldots,x_k s_k) = \sum_{n=o}^{\infty} \frac{(-\lambda)^{k+n}}{n!} \sum_C \int\limits_{|\vec{x}_i|\leq\ell} \cdots \int \prod_{(i,j)\in C} (e^{-s_i s_j G_\epsilon(x_i-x_j)} - 1) \prod_{i=k+1}^{k+n} d\mu(s_i) dx_i ,$$

where E runs over all externally connected graphs and C over all connected graphs with external points.

Proof: The proof is given by the one of Lemma 3.1. ∎

Lemma 5.3

The space cut-off correlation functions $\rho_\ell^k(x_1 s_1,\ldots,x_k s_k)$ are translation invariant with respect to the time variables and, as functions of $z_i = (x_{i+1})_o - (x_i)_o$, $i = 1,\ldots,k-1$ they are analytic in the domain $\mathrm{Re}\, z_i > 0$, $i = 1,\dots,k-1$ and continuous in $\mathrm{Re}\, z_i \geq 0$.

Moreover they satisfy, for $t_1 \leq \dots \leq t_k$,

$$\rho_\ell^k(\tilde{x}_1 s_1,\dots,\tilde{x}_k s_k) = (-\lambda)^k (\Omega_\ell, e^{is_1\varphi_\varepsilon(\vec{x}_1)} e^{-i(t_2-t_1)\bar{H}_\ell} e^{is_2\varphi_\varepsilon(\vec{x}_2)} \dots e^{-i(t_k-t_{k-1})\bar{H}_\ell} e^{is_k\varphi_\varepsilon(\vec{x}_k)} \Omega_\ell) ,$$

where $\tilde{x}_j = (it_j, \vec{x}_j)$, and $\rho_\ell^k(\tilde{x}_1 s_1,\dots,\tilde{x}_k s_k)$ are analytic in λ for $|\lambda| < \lambda_o$ and symmetric in $\tilde{x}_1 s_1,\dots,\tilde{x}_k s_k$.

Proof: The translation invariance, continuity and analyticity in the time variables follow immediately from Lemma 5.1. From the analyticity in λ of the correlation functions for real $(x_j)_o$, $j = 1,\dots,k$ (Lemma 5.2), the analyticity of the state $\omega_\ell(A) = (\Omega_\ell, A\Omega_\ell)$ follows as for the infinite volume limit in Theorem 6.3 of [5]. The norm analyticity of e^{itH_ℓ} in λ for all values of λ follows from the norm boundedness of the space cut-off interaction. Since $(\Omega_\ell, e^{itH_\ell}\Omega_\ell) = e^{itE_\ell}$ it follows first that E_ℓ is analytic for $|\lambda| < \lambda_o$ and then that $e^{it\bar{H}_\ell}$ is norm-analytic for $|\lambda| < \lambda_o$. From the formula in Lemma 5.3 it follows then that $\rho_\ell^k(\tilde{x}_1 s_1,\dots,\tilde{x}_k s_k)$ is analytic in λ for $|\lambda| < \lambda_o$. ∎

From Lemma 5.2 and (4.20) we have that

$$\rho_\ell^k(x_1 s_1,\dots,x_k s_k) = (-i)^k \sigma_\ell^k(\tilde{x}_1 s_1,\dots,\tilde{x}_k s_k) , \qquad (5.3)$$

with $\tilde{x} = (-ix_o, \vec{x})$, in the sense of formal power series. Recalling now that the formal power series in (4.20) was obtained by termwise analytic continuations in the time variables from the formal power series for $\sigma_\ell^k(x_1 s_1,\dots,x_k s_k)$, it follows from (5.3) that

$$\sigma_\ell^k(x_1 s_1,\dots,x_k s_k) = i^k \rho_\ell^k(\tilde{x}_1 s_1,\dots,\tilde{x}_k s_k) , \qquad (5.4)$$

with $\tilde{x} = (ix_o, \vec{x})$, in the sense of formal power series.

By Lemma 5.3 however, the power series on the right hand side of (5.4) converges for $|\lambda| < \lambda_0$, hence the formal power series for σ_ℓ^k, as defined by (4.19), converges for $|\lambda| < \lambda_0$. Under the assumption of space-time dimension ≥ 4 we have:

Lemma 5.4

The scattering functions $\sigma_\ell^k(x_1 s_1, \ldots, x_k s_k)$ are analytic in λ for complex λ with $|\lambda| < \lambda_0$ and given by the convergent linked cluster expansion, which converges to:

$$\sigma_\ell^k(t_1\vec{x}_1 s_1, \ldots, t_k\vec{x}_k s_k) = (-i\lambda)^k (\Omega_\ell, e^{is_1\varphi_\epsilon(x_1)} e^{-i(t_2-t_1)\bar{H}_\ell} \ldots e^{-i(t_k-t_{k-1})\bar{H}_\ell} e^{is_k\varphi_\epsilon(x_k)} \Omega_\ell).$$

Hence for real λ they are uniformly bounded continuous functions of all their variables. Moreover, there is a $\lambda_2 > 0$ such that for all complex λ with $|\lambda| < \lambda_2$ we have the following uniform estimate

$$|\sigma_\ell^k(x_1 s_1, \ldots, x_k s_k)| \leq \alpha(|\lambda|)\beta^k$$

for $t_1 \leq \ldots \leq t_k$, where $\alpha(|\lambda|)$ and β^k depend only on ϵ, ℓ.

Proof: The first part of the lemma follows from the previous lemma and from (5.3) and the remarks above. The proof of the rest goes as follows. Let us define the time ordered scattering functions by

$$\check{\sigma}_\ell^k(x_1 s_1, \ldots, x_k s_k) = \sigma_\ell^k(x_1 s_1, \ldots, x_k s_k) \tag{5.5}$$

for $t_1 \leq \ldots \leq t_k$ and the requirement that $\check{\sigma}_\ell^k(x_1 s_1, \ldots, x_k s_k)$ is symmetric under permutation of its variables. It follows then from the linked cluster expansion (4.19) for the scattering functions that the time-ordered scattering functions have the follwing linked cluster expansion:

$$\check{\sigma}^k_\ell(x_1s_1,\dots,x_ks_k) = \sum_{n=o}^{\infty} \frac{(-i\lambda)^{n+k}}{n!} \sum_E \int \dots \int_{|\vec{x}_i| \le \ell} \prod_{\ell\in E} F^T_\ell \prod_{i=k+1}^{k+n} d\mu(s_i)d\vec{x}_i dt_i \ , \tag{5.6}$$

where $F^T_\ell = \exp[-s_js_kG_\varepsilon(-i|t_j-t_k|,\vec{x}_j-\vec{x}_k)]-1$ for $\ell = (j,k)$ with $j < k$, and E runs over all externally connected graphs. It follwws from the first part of this lemma that (4.19) is convergent and this implies that (5.6) is convergent for $|\lambda| < \lambda_1 \le \lambda_o$. Hence $\check{\sigma}^k_\ell(x_1s_1,\dots,x_ks_k)$ are actually the correlation functions with respect to the interaction potential

$$\sum_{i<j} s_is_j\, F^T_\varepsilon(x_i-x_j)\ , \tag{5.7}$$

in the sense of [5], where $F^T_\varepsilon(t,\vec{x}) = G_\varepsilon(-i|t|,\vec{x})$. Now

$$F^T_\varepsilon(t,\vec{x}) = \pi \int_{\mathbb{R}^{n-1}} e^{-i|t|\mu(\vec{p})}\, e^{i\vec{p}\vec{x}} |\chi_\varepsilon(\vec{p})|^2 \frac{d\vec{p}}{\mu(\vec{p})}\ , \tag{5.8}$$

so that, using that $\chi_\varepsilon(\vec{p}) = \chi_\varepsilon(-\vec{p})$, we get

$$F^T_\varepsilon(t,\vec{x}) = \pi\int_{\mathbb{R}^{n-1}} \cos t\mu(\vec{p}) \cos \vec{p}\cdot\vec{x}\, |\chi_\varepsilon(\vec{p})|^2 \frac{d\vec{p}}{\mu(\vec{p})} + i\pi\int_{\mathbb{R}^{n-1}} \sin|t|\mu(\vec{p}) \cos \vec{p}\cdot\vec{x} |\chi_\varepsilon(\vec{p})|^2 \frac{d\vec{p}}{\mu(\vec{p})}\ . \tag{5.9}$$

From (5.9) we observe that $|F^T_\varepsilon(t,\vec{x})| \le \text{const}\cdot(|t|+1)^{-3/2}$ for all $\vec{x}$ in $|\vec{x}| \le \ell$ and all t, if the dimension of space-time is larger or equal to 4, as we have assumed. This implies that

$$C = \sup_{s_i} \int_{|\vec{x}_j| \le \ell} |e^{-s_is_jF^T_\varepsilon(x_i-x_j)} - 1|\, d\mu(s_j)d\vec{x}_j dt_j \tag{5.10}$$

is finite, so that the interaction is regular, in the sense of [5]. Moreover, from (5.9) we see that the real part of (5.9) is positive definite and hence

$$\operatorname{Re} \sum_{i<j}^{m} s_is_j\, F^T_\varepsilon(x_i-x_j) \ge -Bm\ . \tag{5.11}$$

This is the proper form of a stability condition which can be

used similarly as in [5], to estimate $e^{-\sum_{j=2}^{m} s_i s_j F_\varepsilon^T(x_i-x_j)}$.
We see namely that in such an estimate the imaginary time of F_ε^T plays no role. Therefore by (5.11) we can use the methods of [5] not only to prove that the time cut-off scattering functions converge as the time cut-off is taken away, which we knew already, but also to get the estimate, uniform in the variables $x_1 s_1, \ldots, x_k s_k$, for all complex $|\lambda| < \lambda_2$, $\lambda_2 = C^{-1} e^{-2B-1}$:

$$|\check{\sigma}_{\ell}^{k}(x_1 s_1, \ldots, x_k s_k)| \leq C^{-k} \frac{|\lambda|}{1-|\lambda| C e^{2B+1}} . \qquad (5.12)$$

This then completes the proof of the lemma.

Theorem 5.1

There is a $\hat{\lambda}_o > 0$ such that the generating functional $S_{\ell}(g;f)$ for the space cut-off scattering matrix is analytic for complex λ with $|\lambda| < \hat{\lambda}_o$, and $\hat{\lambda}_o = \min(\lambda_o, \lambda_1, \lambda_2)$, where λ_1 is given in theorem 4.3 and λ_2 in lemma 5.4.
Moreover, for $|\lambda| < \hat{\lambda}_o$,

$$S_{\ell}(g;f) = e^{-(g,f)_{-\frac{1}{2}}} \check{\sigma}_{\ell}(-ia)$$

and

$$S_{\ell}(g;f)_T = -(g,f)_{-\frac{1}{2}} + \check{\sigma}_{\ell,T}(-ia) ,$$

where

$$a(t,\vec{x},s) = \exp(-sf_\varepsilon(t,\vec{x}) - sg_\varepsilon(-t,\vec{x})) - 1 ,$$

$f_\varepsilon(t,\vec{x}) = \int G_\varepsilon(it,\vec{x}-\vec{y}) f(\vec{y}) d\vec{y}$, and $\check{\sigma}_{\ell}(-ia)$ is the generating functional for the time ordered scattering functions $\check{\sigma}_{\ell}^{k}(t_1\vec{x}_1 s_1 \ldots t_k\vec{x}_k s_k)$ given by

$$\check{\sigma}_{\ell}(-ia) = \sum_{n=o}^{\infty} \frac{1}{k!} \int_{|\vec{x}_i| \leq \ell} \cdots \int \check{\sigma}_{\ell}^{k}(t_1\vec{x}_1 s_1 \ldots t_k\vec{x}_k s_k)$$
$$a(t_1\vec{x}_1 s_1) \ldots a(t_k\vec{x}_k s_k) \prod_{i=1}^{k} d\mu(s_i) d\vec{x}_i dt_i \ . \qquad (5.13)$$

The scattering functions $\sigma_{\ell}^{k}(t_1\vec{x}_1 s_1, \ldots, t_k\vec{x}_k s_k)$ are analytic in λ for $|\lambda| < \lambda_o$, with λ_o independent of the space cut-off, and are given by the reduction formula

$$\sigma_{\ell}^{k}(t_1\vec{x}_1 s_1, \ldots, t_k\vec{x}_k s_k) =$$
$$(-i\lambda)^k (\Omega_{\ell}, e^{is_1\varphi_\varepsilon(\vec{x}_1)} e^{-i(t_2-t_1)\bar{H}_{\ell}} \cdots e^{-i(t_k-t_{k-1})\bar{H}_{\ell}} e^{is_k\varphi_\varepsilon(x_k)} \Omega_{\ell}) \qquad (5.14)$$

and also by the linked cluster expansion, convergent for $|\lambda| < \lambda_o$:

$$\sigma_{\ell}^{k}(x_1 s_1, \ldots, x_k s_k) =$$
$$\sum_{n=o}^{\infty} (-i\lambda)^{n+k} \sum_{E} \int_{t_{k+1} \leq \cdots \leq t_{k+n}} \cdots \int \prod_{(i,j)\in E} (e^{-s_i s_j G_\varepsilon(\tilde{x}_i - \tilde{x}_j)} - 1) \prod_{i=k+1}^{k+n} d\mu(s_i) dx_i \ , \qquad (5.15)$$

where E runs over all externally connected graphs and $\tilde{x}_j = (i(x_j)_o, \vec{x}_j)$. Similarly the truncated scattering functions $\sigma_{\ell,T}$ are given by their connected graph expansion, which is also convergent for $|\lambda| < \lambda_o$:

$$\sigma_{\ell,T}^{k}(x_1 s_1, \ldots, x_k s_k) =$$
$$\sum_{n=o}^{\infty} (-i\lambda)^{n+k} \sum_{C} \int_{t_{k+1} \leq \cdots \leq t_{k+n}} \cdots \int \prod_{(i,j)\in C} (e^{-s_i s_j G_\varepsilon(\tilde{x}_i - \tilde{x}_j)} - 1) \prod_{i=k+1}^{k+n} d\mu(s_i) dx_i \ . \qquad (5.16)$$

Proof: By (5.4) and Lemma 5.3 we know that $\sigma_{\ell}^{k}(x_1 s_1, \ldots, x_k s_k)$ is analytic and given by its linked cluster expansion for $|\lambda| < \lambda_o$. This proves (5.14) and (5.15). (5.16) follows by direct algebraic calculations from (5.15). From lemma 5.4 we know that

$\check{\sigma}^k_{\ell}(x_1s_1,\ldots,x_ks_k)$ is uniformly bounded by $\alpha(|\lambda|)\beta^k$ for complex λ with $|\lambda| < \lambda_2$. By the integrability of the function a this gives that the sum defining $\check{\sigma}_{\ell}(-ia)$ is convergent also for complex λ with $|\lambda| < \lambda_2$, and therefore that $\check{\sigma}_{\ell}(-ia)$ is analytic in λ for complex λ with $|\lambda| \leq \lambda_2$.

This implies, by section 4, that the asymptotic expansion of $S_{\ell}(g;f)$ is actually a convergent powerseries for $|\lambda| < \lambda_2$.

Consider now the expression

$$S_{\ell,t}(g;f) = (\Omega_o, e^{-2itH_{\ell}}\Omega_o)^{-1}(e^{itH_{\ell}}e^{-itH_o}A\Omega_o, e^{-itH_{\ell}}e^{itH_o}B\Omega_o) \tag{5.17}$$

of theorem 4.3. Expanding (5.17) in powers of λ we find easily that we get the linked cluster expansion for $S_{\ell,t}(g,f)$ in terms of the time cut-off time ordered scattering functions $\check{\sigma}_{\ell,t}$. It is

$$S_{\ell,t}(g;f) = e^{-(g,f)_{-\frac{1}{2}}}\check{\sigma}_{\ell,t}(-ia) , \tag{5.18}$$

where the linked cluster expansion for $\check{\sigma}_{\ell,t}$ is obtained as the linked cluster expansion for $\check{\sigma}_{\ell}$ with all time integrations restricted to the intervall $[-t,t]$. From the proof of lemma 5.4 it follows that

$$|\check{\sigma}^k_{\ell,t}(x_1s_1,\ldots,x_ks_k)| \leq \alpha(|\lambda|)\beta^k \tag{5.18}$$

and converges uniformly on compact subsets to $\check{\sigma}^k_{\ell}(x_1s_1,\ldots,x_ks_k)$. This implies that $\check{\sigma}_{\ell,t}(-ia)$ converges to $\check{\sigma}_{\ell}(-ia)$ as $t \to \infty$. From (5.18) we then get that $S_{\ell,t}(g;f)$ converges as $t \to \infty$ to the sum of the linked cluster expansion for the scattering matrix. If we have $|\lambda| < \lambda_1$, where λ_1 is given in theorem 4.3, then we get by theorem 4.3 and (5.17) that $S_{\ell,t}(g,f)$ converges to $S_{\ell}(g;f)$ as $t \to \infty$. This identifies $S_{\ell}(g,f)$ with the sum of the linked cluster expansion and proves the theorem. [6)]

6. The infinite volume scattering functions

In this section we assume again that the dimension of space-time is larger or equal to 4 . The finite volume scattering functions $\sigma_{\ell}^{k}(x_1 s_1,\dots,x_k s_k)$ are, by (5.14), given by the following expression:

$$\sigma_{\ell}^{k}(t_1\vec{x}_1 s_1,\dots,t_k\vec{x}_k s_k) = (-i\lambda)^k(\Omega_{\ell},\alpha_{t_1}^{\ell}(e^{is\varphi_\varepsilon(\vec{x}_1)})\dots\alpha_{t_k}^{\ell}(e^{is\varphi_\varepsilon(\vec{x}_k)})\Omega_{\ell}), \quad (6.1)$$

where $\alpha_t^{\ell}(A) = e^{-itH_{\ell}}A\,e^{itH_{\ell}}$ is the automorphism considered in section 6 of [5]. By the theorem 6.1 of [5] we know that $\alpha_t^{\ell}(e^{is\varphi_\varepsilon(\vec{x})})$ converges in norm to $\alpha_t(e^{is\varphi_\varepsilon(\vec{x})})$ as $\ell\to\infty$, where $\alpha_t(A) = e^{-itH}A\,e^{itH}$ and H is the infinite volume Hamiltonian of the theorem 6.3 of [5]. Moreover, from theorem 6.3 of [5] we also have that ω_{ℓ} converges weakly to ω as $\ell\to\infty$ for $|\lambda| < \lambda_o$, where $\omega_{\ell}(A) = (\Omega_{\ell},A\Omega_{\ell})$, and ω is analytic in λ for $|\lambda| < \lambda_o$.

Theorem 6.1

The finite volume scattering functions $\sigma_{\ell}^{k}(x_1 s_1,\dots,x_k s_k)$ are uniformly bounded for real λ in ℓ and $x_1 s_1,\dots,x_k s_k$ and converge pointwise for $-\lambda_o<\lambda<\lambda_o$ to the infinite volume scattering functions $\sigma^{k}(x_1 s_1,\dots,x_k s_k)$, which are given by

$$\sigma^{k}(t_1\vec{x}_1 s_1,\dots,t_k\vec{x}_k s_k) = (-i\lambda)^k(\Omega,e^{is_1\varphi_\varepsilon(\vec{x}_1)}e^{-i(t_2-t_1)H}\dots\dots e^{-i(t_k-t_{k-1})H}e^{is_k\varphi_\varepsilon(\vec{x}_k)}\Omega),$$

where Ω is the unique infinite volume vacuum, $(\Omega,A\Omega)=\omega(A)$. Moreover, there exists a $\lambda_1>0$ such that $\sigma^{k}(t_1\vec{x}_1 s_1,\dots,t_k\vec{x}_k s_k)$ is analytic for complex λ , with $|\lambda|<\lambda_o$, if $-\lambda_1<t_{i+1}-t_i<\lambda_1$, $i = 1,2,\dots,k-1$.

<u>Proof</u>: The first part of the Lemma follows from (6.1). The moreover part is proven by observing that

$$\sigma^k(t_1\vec{x}_1 s_1,\dots,t_k\vec{x}_k s_k) = (-i\lambda)^k \omega(\alpha_{t_1}(e^{is_1\varphi_\varepsilon(\vec{x}_1)})\dots\alpha_{t_n}(e^{is_n\varphi_\varepsilon(\vec{x}_n)})) \quad (6.2)$$

and using that ω is analytic in λ for $|\lambda| < \lambda_o$ and $\alpha_t(e^{is\varphi_\varepsilon(\vec{x})})$ is norm analytic in λ for $|\lambda| < \lambda_o$ if $-\lambda_1 < t < \lambda_1$, the latter following from the proofs of [8], where it is shown that $\alpha_t(e^{is\varphi_\varepsilon(\vec{x})})$ is norm analytic for complex λ satisfying $|\lambda t| < d$. ∎

From (5.5) we know that $\sigma_\ell^k(t_1\vec{x}_1 s_1,\dots,t_k\vec{x}_k s_k)$ is analytic in $t_1,\dots,t_n$ and uniformly bounded for $\mathrm{Im}(t_{i+1}-t_i) < 0$, and from (5.3) we have that σ_ℓ^k is given, for $\mathrm{Re}(t_i) = 0$, $i = 1,\dots,k$, by the finite volume correlation functions. From Lemma 6.1 we know that σ_ℓ^k is uniformly bounded and converges if the time variables are kept on the real axis. This then implies that σ_ℓ^k will converge for $\mathrm{Im}(t_{i+1}-t_i) \le 0$. From this and (5.3) we get the following Theorem:

<u>Theorem 6.2</u>

For $-\lambda_o < \lambda < \lambda_o$ the infinite volume scattering functions are uniformly bounded and continuous in $t_1\vec{x}_1 s_1,\dots,t_k\vec{x}_k s_k$. They are analytic in $t_1,\dots,t_k$ for $\mathrm{Im}(t_{i+1}-t_i) < 0$, and related to the infinite volume correlation functions by

$$\rho^k(\tilde{x}_1 s_1,\dots,\tilde{x}_k s_k) = (-i)^k \sigma^k(x_1 s_1,\dots,x_k s_k), \quad (6.3)$$

where $\tilde{x} = (ix_o,\vec{x})$.

Moreover, the infinite volume correlation functions are given, for $t_1 \le \dots \le t_k$ and $-\lambda_o < \lambda < \lambda_o$, by

$$\rho^k(t_1\vec{x}_1 s_1,\dots,t_k\vec{x}_k s_k) = (-\lambda)^k(\Omega, e^{is_1\varphi_\varepsilon(\vec{x}_1)} e^{-(t_2-t_1)H} \cdots e^{-(t_k-t_{k-1})H} e^{is_k\varphi_\varepsilon(\vec{x}_k)}\Omega). \tag{6.4}$$

■

<u>Remark</u> The formula (6.3) gives an improvement, for real λ, on the bounds for ρ^k in Lemma 4.1 of [5], in as much as we have now

$$|\rho^k(x_1 s_1,\dots,x_k s_k)| \le |\lambda|^k$$

for $-\lambda_o < \lambda < \lambda_o$.

We can now use Theorem 6.1 and Theorem 6.2 to improve the Theorem 6.4 of [5]. By performing analytic continuations in the time variables occurring in the expression (3.1) for the imaginary time infinite volume Wightman functions we get the relation between the real time Wightman functions and the infinite volume scattering functions. These analytic continuations are possible, as a consequence of (3.1), the theorems 6.1 and 6.2 and the Theorem 6.4 of [5]. We express this in the following theorem:

<u>Theorem 6.3</u>

For $-\lambda_o < \lambda < \lambda_o$, the infinite volume Wightman functions $W^k(x_1,\dots,x_k)$ are given in terms of the infinite volume scattering functions by the reduction formula

$$W^k(x_1,\dots,x_k) = W_o^k(x_1,\dots,x_k) + \sum_{r=1}^{k} \frac{1}{r!} \sum_{\substack{q\ge r,\, p\ge o\\ p+q=k}} \frac{1}{p!} \sum_{\pi\in S_k} W_o^p(x_{\pi(1)},\dots,x_{\pi(p)})$$

$$\sum_{\substack{\ell_1+\dots+\ell_r=q\\ \ell_m\ge 1}} \frac{1}{\ell_1!\dots\ell_r!} \int\dots\int \prod_{m=1}^{r} [(is_m)^{\ell_m} \prod_{j=1}^{\ell_m} F_\varepsilon(x_{\pi(p+\ell_1+\dots+\ell_m-j+1)}-y_m)]\,\sigma^r(y_1s_1,\dots,y_rs_r) \prod_{j=1}^{r} d\mu(s_j)dy_j ,$$

where $W_o^k(x_1,\dots,x_k)$ are the Wightman functions for the free field of mass m, $F_\varepsilon(t,\vec{x}) = G_\varepsilon(it,\vec{x})$ and the identity is in the sense of tempered distributions. ■

We shall now consider the finite volume scattering matrix $S^{\ell}_{n,m}(h_1,\ldots,h_n;g_1,\ldots,g_m)$ defined by (2.5) for $h_1,\ldots,h_n$ and $g_1,\ldots,g_m$ in $L_2(\mathbb{R}^{n-1})$. Let us define the finite volume scattering amplitudes $S^{\ell}_{n,m}(\vec{p}_1,\ldots,\vec{p}_n;\vec{q}_1,\ldots,\vec{q}_m)$ by

$$S^{\ell}_{n,m}(h_1,\ldots,h_n;g_1,\ldots,g_m) = \tag{6.5}$$
$$= \int S^{\ell}_{n,m}(\vec{p}_1,\ldots,\vec{p}_n;\vec{q}_1,\ldots,\vec{q}_m)h_1(\vec{p}_1)\ldots h_n(\vec{p}_n)g_1(\vec{q}_1)\ldots g_m(\vec{q}_m)d\vec{p}_1\ldots d\vec{p}_n d\vec{q}_1\ldots d\vec{q}_m$$

Introducing the truncated scattering amplitudes by the formula corresponding to (3.13), we get the relation between the above scattering amplitudes, which we denote now by $S^{\ell}(P;Q)$, and the truncated ones in the form

$$S^{\ell}(P;Q) = \sum_{\substack{P=P_1\cup\ldots\cup P_k \\ Q=Q_1\cup\ldots\cup Q_k}} S^{\ell,T}(P_1,Q_1)\ldots S^{\ell,T}(P_k,Q_k) \tag{6.6}$$

where the sum runs over all disjoint partitions of $P=\{\vec{p}_1,\ldots,\vec{p}_n\}$, $Q=\{\vec{q}_1,\ldots,\vec{q}_m\}$.

In Theorem 5.1 we have an explicit expression for the generating functional $S_{\ell,T}(g;f)$ in terms of the scattering functions. This, together with the definition (2.16) of the generating functional $S_{\ell}(g;f)$ for the scattering matrix, enables us to find an explicit expression for the finite volume scattering amplitude:

<u>Theorem 6.4</u>

The finite volume scattering amplitude $S^{\ell}_{n,m}(\vec{p}_1,\ldots,\vec{p}_n;\vec{q}_1,\ldots\vec{q}_m)$ is given in terms of the corresponding truncated amplitudes by (6.6), and the finite volume truncated scattering amplitudes are given by the reduction formula

$$S^{\ell,T}_{n,m}(\vec{p}_1,\dots,\vec{p}_n;\vec{q}_1,\dots,\vec{q}_m) = \sum_{r=1}^{\min(n,m)} \frac{1}{r!} \sum_{\substack{k_1+\dots+k_r=n\\ \ell_1+\dots+\ell_r=m}} \frac{(is_1)^{k_1+\ell_1}\dots(is_r)^{k_r+\ell_r}}{k_1!\,\ell_1!\dots k_r!\,\ell_r!}$$

$$\prod_{i=1}^{n} |\chi_\varepsilon(\vec{p}_1)|^2 \prod_{j=1}^{m} |\chi_\varepsilon(\vec{q}_j)|^2 \cdot \sum_{\substack{\pi\in S_n\\ \sigma\in S_m}} \tilde{\sigma}^r_{\ell,T}\Big(\sum_1^{k_1} p_{\pi(i)} - \sum_1^{\ell_1} q_{\sigma(j)}, \sum_{k_1+1}^{k_1+k_2} p_{\pi(i)} -$$

$$- \sum_{\ell_1+1}^{\ell_1+\ell_2} q_{\sigma(j)},\dots, \sum_{k_1+\dots+k_{r-1}}^{n} p_{\pi(i)} - \sum_{\ell_1+\dots+\ell_r+1}^{m} q_{\sigma(j)}; s_1,\dots,s_r\Big) \prod_{j=1}^{r} d\mu(s_j) +$$

$$+ \delta_{n_1,1}\delta_{m_1,1}\delta(\vec{p}_1-\vec{q}_1)\,.$$

Here S_n is the set of all permutations of $1,\dots,n$, the symbol $\delta_{n,1}$ means 1 for $n = 1$ and zero for $n \neq 1$, and $p_i^0 = \mu(\vec{p}_i)$, $q_j^0 = \mu(\vec{q}_j)$. $\tilde{\sigma}^r_{\ell,T}(p_1,\dots,p_r;s_1,\dots,s_r)$ are the Fourier transforms of the functions $\check{\sigma}^r_{\ell,T}(x_1s_1,\dots,x_rs_r)$ in the sense of tempered distributions, where $\check{\sigma}^r_{\ell,T}(x_1s_1,\dots,x_rs_r)$ is equal to the truncated scattering functions $\sigma^r_{\ell,T}(x_1s_1,\dots,x_rs_r)$ for $t_1 \leq \dots \leq t_r$, and is symmetric with respect to the variables $x_1s_1,\dots,x_rs_r$. ∎

Remark: The fact that it is possible to restrict the variables in the arguments of $\tilde{\sigma}^r_{\ell,T}$ to the mass shell $p^0 = \mu(\vec{p})$ in the reduction formula of the Theorem 6.4 is a simple consequence of the existence of the truncated generating functional $S_{\ell,T}(g;f)$ and of the Theorem 5.1.

Although the infinite volume scattering functions exist by Theorem 6.1, we may not yet conclude that the limit of the finite volume scattering amplitudes exists as $\ell \to \infty$. Nevertheless, by Theorem 6.4 we have already that the infinite volume off shell scattering amplitudes exist as limit for $\ell \to \infty$ in the sense of tempered distributions of the off shell amplitudes for the space cut-off interaction, the latter being defined by the right hand side of the formula in Theorem 6.4 without the restrictions $(p_i)^o = \mu(\vec{p}_i)$ and $(q_i)^o = \mu(\vec{q}_i)$. We will discuss this problem in a forthcoming paper.

Footnotes

1) See e.g. the introduction of [1] and the references given therein.

2) Besides Lee-type models and quadratic interactions (see the corresponding references in [1]), we should mention the case of fermion interactions with ultraviolet cut-off and no pure creation or annihilation terms in the interaction [2], as well as the case of Nelson's type models (simplified Yukawa interaction): [3], [1], [4].

3) For technical reasons most results are stated and proved only for the case of a number of space-time dimensions $n \geq 4$.

4) See also the references given in [5] concerning previous work on this class of models.

5) We use in general the notation: : for the Wick product. Thus:

$$:e^{i\varphi(h)}: = e^{i\varphi^{(+)}(h)}e^{i\varphi^{(-)}(h)} ,$$

where

$$\varphi^{(+)}(h) = 2^{-\frac{1}{2}}(2\pi)^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} e^{i\vec{p}\vec{x}} a^*(-\vec{p}) \frac{d\vec{p}}{\mu(\vec{p})^{\frac{1}{2}}}$$

$$\varphi^{(-)}(h) = 2^{-\frac{1}{2}}(2\pi)^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} e^{i\vec{p}\vec{x}} a(\vec{p}) \frac{d\vec{p}}{\mu(\vec{p})^{\frac{1}{2}}}$$

The Wick product $:e^{i\varphi_\pm(h)}:$ for the corresponding asymptotic in and out fields is defined in the same way, with $a^\#_\pm$ instead of $a^\#$.

6) We have incidentally that the space cut-off scattering matrix is non trivial (i.e. different from the identity and from zero). This follows already from the fact that it has a non trivial, explicitely given asymptotic expansion.

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