THE KRULL ORDINAL, COPROF, AND NOETHERIAN LOCALIZATIONS OF LARGE POLYNOMIAL RINGS.

Tor H. Gulliksen.

Introduction.

In the following A will always denote a commutative, integral domain (with identity). In this paper we shall investigate a class of commutative, Noetherian, flat A-algebras which may be of interest since it is wide enough to include Noetherian rings of any given Krull ordinal. The Krull ordinal $\varkappa(R)$ of a Noetherian ring R will be used in the sense of Bass [1]. It coincides with cl.K - dim R as defined in Krause [5]. A definition of $\varkappa(R)$ is included in (1.5) below. Recall that $\varkappa(R)$ is an ordinal which coincides with the classical Krull dimension of R whenever one of them is finite.

Let A[X] be the polynomial ring in a set of transcendent elements. Let $\mathcal{M}$ be a family of finite subsets of X and let $A[X]_{\mathcal{M}}$ be the localization of A[X] with respect to the multiplicative set

$$A[X] \setminus \bigcup_{M \in \mathcal{M}} MA[X]$$

Let $\mathcal{P}(\mathcal{M})$ be the family consisting of all the subsets of all the members of $\mathcal{M}$. We will equip $\mathcal{P}(\mathcal{M})$ with a natural topology (2.1), and we shall see that there is an intimate connection between the topological spaces $\mathcal{P}(\mathcal{M})$ and Spec A[X] .

In § 1 we give some preliminaries on Krull ordinals. The

Krull ordinal $\dim \mathcal{P}$ of a Noetherian topological space $\mathcal{P}$ is introduced. In §2 we show that $\mathcal{P}(\mathcal{M})$ is a Noetherian topological space if and only if $\mathcal{P}(\mathcal{M})$ is a Noetherian ordered set with respace to inclusion; in which case $\dim \mathcal{P}(\mathcal{M})$ equals the Krull ordinal of the ordered set $\mathcal{P}(\mathcal{M})$. We also give an explicit construction of a Noetherian space $\mathcal{P}(\mathcal{M}_\alpha)$ of a given Krull ordinal α .

In §3 we show that if $\mathcal{P}(\mathcal{M})$ is Noetherian, then the canonical injection

$$\mathcal{P}(\mathcal{M}) \to \operatorname{Spec} A[X]_{\mathcal{M}}$$

sending P to $PA[X]_{\mathcal{M}}$ is a contineous map which restricts to a homeomorphism

$$\operatorname{Max}(\mathcal{M}) \sim \operatorname{Max}\operatorname{Spec} A[X]_{\mathcal{M}} \quad ,$$

$\operatorname{Max}(\mathcal{M})$ being the family of maximal members of $\mathcal{M}$.
§4 contains the main result: If $\mathcal{P}(\mathcal{M})$ is Noetherian, then $A[X]_{\mathcal{M}}$ is a Noetherian ring, and we have

$$\varkappa(A[X]_{\mathcal{M}}) = \dim \mathcal{P}(\mathcal{M})$$

In particular, if α is an ordinal, then there exists a Noetherian ring $A[X]_{\mathcal{M}_\alpha}$ such that

$$\varkappa(A[X]_{\mathcal{M}_\alpha}) = \alpha$$

Parts of this result has been obtained independently by Robert Gordon and J.C. Robson in a resent manuscript [4] §7. Using methods different from ours they show that if A is a field, if $X = \bigcup_{M \in \mathcal{M}} M$ and if $\mathcal{P}(\mathcal{M})$ has ascending chain condition with respect to inclusion, then $A[X]_{\mathcal{M}}$ is a Noetherian ring whose Krull ordinal is not less than the Krull ordinal of the ordered set $\mathcal{P}(\mathcal{M})$.

In §5 we discuss the function $\mathfrak{p} \mapsto \operatorname{coprof} R_{\mathfrak{p}}$ on $\operatorname{Spec} R$, R being Noetherian. We construct rings R for which the regular locus of R equals the Cohen-Macaulay locus of R without being a constructible set in $\operatorname{Spec} R$. We obtain a Noetherian domain R for which the function $\mathfrak{p} \mapsto \operatorname{coprof} R_{\mathfrak{p}}$ is not bounded on $\operatorname{Spec} R$. We also obtain a Noetherian domain of Krull dimension 2 which is not universally Cohen-Macaulay.

§ 1. Preliminaries on Krull ordinals.

1.1 Ordinal numbers. Ω will denote the class of ordinal numbers where we have adjoned the symbol -1 with the following conventions

(i) $-1 < \alpha$ for every ordinal α

(ii) $(-1)+1 = 0$

Whenever W is a set of ordinals, $\sup W$ will denote the least ordinal which is greater than or equal to every ordinal in W. Thus we define $\sup \emptyset = 0$.

1.2 Partially ordered sets. A partially ordered set will be called Noetherian if every subset has a maximal element. Let $\mathcal{P}$ be a non-empty Noetherian set. The function $\lambda : \mathcal{P} \to \Omega$ defined by

$$\lambda(P) = \sup\{\lambda(Q)+1 : P < Q\}$$

will briefly be called the ordinal map on $\mathcal{P}$. It is convenient to let λ be defined also outside $\mathcal{P}$, where it will be defined constantly equal to -1. In particular we have $\lambda(P) = 0$ if and only if P is a maximal element of $\mathcal{P}$. We let $\varkappa(\mathcal{P})$ denote the Krull ordinal of $\mathcal{P}$, see [1]. It is eqsily seen to be related to the ordinal map as follows:

$$\varkappa(\mathcal{P}) = \sup_{P \in \mathcal{P}} \lambda(P)$$

We define $\varkappa(\emptyset) = -1$.

1.3 Lemma Let $\mathcal{P}$ be a partially ordered, Noetherian set, and let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be a finite covering of $\mathcal{P}$ of non-empty subsets having the following property: For each P, Q in $\mathcal{P}$ and $1 \leq i \leq n$

we have

$$(P < Q \text{ and } Q \in \mathscr{P}_i) \Longrightarrow P \in \mathscr{P}_i .$$

Then

$$\varkappa(\mathscr{P}) = \max_{i=1,\dots,n} \varkappa(\mathscr{P}_i)$$

<u>Proof</u> It suffices to prove the lemma for $n = 2$.
Let λ_1 and λ_2 be the ordinal maps on $\mathscr{P}_1$ and $\mathscr{P}_2$ respectively. By the convention in (2.1) we have $\lambda_i(P) = -1$ for $P \in \mathscr{P} \setminus \mathscr{P}_i$ $(i = 1,2)$.
Let λ be the ordinal map on $\mathscr{P}$. It suffices to prove that

(1) $$\lambda(P) = \max\{\lambda_1(P), \lambda_2(P)\}$$

for all $P \in \mathscr{P}$. We will prove (1) by induction on $\lambda(P)$.
We have

(2) $$\lambda(P) = \sup\{\lambda(Q)+1 : P < Q\}$$

If $\lambda(P) = 0$ then (1) is obviously satisfied. Let α be a non-zero ordinal, and assume that (1) is satisfied whenever $\lambda(P) < \alpha$. Now put $\lambda(P) = \alpha$. If $P \in \mathscr{P}_i \setminus \mathscr{P}_1 \cap \mathscr{P}_2$, then the condition on $\mathscr{P}_1$ and $\mathscr{P}_2$ gives $\lambda(P) = \lambda_i(P)$. Hence we may assume $P \in \mathscr{P}_1 \cap \mathscr{P}_2$.
We have

$$\begin{aligned}\lambda(P) &= \sup\{\lambda(Q)+1 : P < Q\}\\ &= \max_{i=1,2} \sup\{\lambda(Q)+1 : P < Q \text{ and } \lambda(Q) = \lambda_i(Q)\}\\ &= \max_{i=1,2} \sup\{\lambda_i(Q)+1 : P < Q \text{ and } Q \in \mathscr{P}_i\}\\ &= \max_{i=1,2} \lambda_i(P) .\end{aligned}$$

∎

1.4 <u>The Krull ordinal of a Noetherian topological space.</u> Let $\mathcal{P}$ be a non-empty Noetherian topological space. Let $\mathcal{I}(\mathcal{P})$ denote the family of all irreducible, non-empty, closed subsets of $\mathcal{P}$. We give $\mathcal{I}(\mathcal{P})$ the following ordering. For members I_1 and I_2 in $\mathcal{I}(\mathcal{P})$ we put $I_1 < I_2$ if and only if $I_1 \underset{\neq}{\supset} I_2$. Clearly $\mathcal{I}(\mathcal{P})$ becomes a Noetherian partially ordered set. We can now define the <u>Krull ordinal</u> of $\mathcal{P}$, notation $\dim \mathcal{P}$, as follows

$$\dim \mathcal{P} := \varkappa(\mathcal{I}(\mathcal{P})) \text{ .} \qquad \text{We put } \dim \emptyset = -1 \text{ .}$$

The <u>combinatorial dimension</u> of $\mathcal{P}$ is defined to be the supremum of all integers n for which there exists a chain

$$I_0 \underset{\neq}{\supset} I_1 \underset{\neq}{\supset} \cdots \underset{\neq}{\supset} I_n \quad \text{in } \mathcal{I}(\mathcal{P}) \text{ .}$$

Observe that it coincides with the Krull ordinal, $\dim \mathcal{P}$, whenever one of them is finite and non-negative.

1.5 <u>The Krull ordinal of a Noetherian ring.</u> Let R be a commutative, Noetherian ring. Then $\operatorname{Spec} R$ has a Krull ordinal, $\dim \operatorname{Spec} R$, as a Noetherian, topological space. It also has a Krull ordinal $\varkappa(\operatorname{Spec} R)$ as a set, partially ordered by inclusion. Clearly we have

$$\dim \operatorname{Spec} R = \varkappa(\operatorname{Spec} R)$$

This common value is called the <u>Krull ordinal</u> of R and will be denoted by $\varkappa(R)$ as in [1].

§ 2. A class of Noetherian topological spaces.

2.1 The space $\mathcal{P}(\mathcal{M})$. Let X be a fixed set, and let $\mathcal{M}$ be a family of finite subsets of X. If $\mathcal{M}$ is non-empty, let $\mathcal{P}(\mathcal{M})$ be the family of all subsets of the members of $\mathcal{M}$, and let $\mathrm{Max}(\mathcal{M})$ be the family of all the maximal members of $\mathcal{M}$. If $\mathcal{M}$ is empty, it is convenient to define $\mathcal{P}(\mathcal{M}) = \mathrm{Max}(\mathcal{M}) = \{\emptyset\}$. $\mathcal{P}(\mathcal{M})$ will always be ordered by inclusion. To each $P \in \mathcal{P}(\mathcal{M})$ we define

$$\mathcal{V}(P) = \{Q \in \mathcal{P}(\mathcal{M}) : P \subseteq Q\}$$

If P consists of a single element x we will write $\mathcal{V}(x)$ instead of $\mathcal{V}(\{x\})$. The topological space $\mathcal{P}(\mathcal{M})$ will be the set $\mathcal{P}(\mathcal{M})$ equipped with the weakest topology for which every set $\mathcal{V}(P)$ is closed. We will briefly say that $\mathcal{P}(\mathcal{M})$ is Noetherian if one of the equivalent conditions in the following proposition is satisfied.

2.2 Proposition. The following statements are equivalent :

(i) $\mathcal{P}(\mathcal{M})$ is Noetherian as an ordered set.

(ii) $\mathcal{P}(\mathcal{M})$ is Noetherian as a topological space.

Moreover, if (i) or (ii) is satisfied, then the non-empty irreducible, closed sets in $\mathcal{P}(\mathcal{M})$ are just the sets $\mathcal{V}(P)$. In particular we have

$$\dim \mathcal{P}(\mathcal{M}) = \varkappa(\mathcal{P}(\mathcal{M}))$$

Proof The implication (ii) ⟹ (i) is obvious in view of the fact that we have

(1) $$P_1 \subsetneq P_2 \iff \mathcal{V}(P_1) \supsetneq \mathcal{V}(P_2)$$

for all P_1 and P_2 in $\mathcal{P}(\mathcal{M})$. We will now show (i) ==> (ii). Let $\mathcal{F}$ be the collection consisting of the empty set and all finite unions of sets $\mathcal{V}(P)$ for $P \in \mathcal{P}(\mathcal{M})$. Clearly $\mathcal{F}$ is closed with respect to finite unions. Let us now assume that $\mathcal{P}(\mathcal{M})$ is Noetherian with respact to $\subset$.
It follows from (1) that any descending chain

$$\mathcal{V}(P_1) \supset \mathcal{V}(P_2) \supset \dots$$

is stationary. From this one can show that $\mathcal{F}$ has descending chain condition with respect to inclusion. Hence $\mathcal{F}$ is closed with respect to arbitrary intersections. This shows that $\mathcal{F}$ is the collection of closed sets in $\mathcal{P}(\mathcal{M})$, and hence $\mathcal{P}(\mathcal{M})$ is a Noetherian space. Clearly, the non-empty, irreducible colsed sets are the sets $\mathcal{V}(P)$ for $P \in \mathcal{P}(\mathcal{M})$. ∎

2.3 <u>Lemma</u> Let X and $\mathcal{P}(\mathcal{M})$ be as in (2.1). Assume that $\mathcal{P}(\mathcal{V}(x))$ is Noetherian for each $x \in X$. Then $\mathcal{P}(\mathcal{M})$ is Noetherian. Moreover

$$\varkappa(\mathcal{P}(\mathcal{M})) = \sup_{x \in X} \varkappa(\mathcal{P}(\mathcal{V}(x)))$$

<u>Proof</u> In proving (2.3) we will consider $\mathcal{P}(\mathcal{M})$ as ordered by inclusion. Then clearly $\mathcal{P}(\mathcal{M})$ is Noetherian. Letting λ be the ordinal map on $\mathcal{P}(\mathcal{M})$ we have:

$$\varkappa(\mathcal{P}(\mathcal{M})) = \sup_{x \in X} \lambda(\{x\})+1 = \sup_{x \in X} \varkappa(\mathcal{V}(x))+1 \leq \sup_{x \in X} \varkappa(\mathcal{P}(\mathcal{V}(x)))$$
$$\leq \varkappa(\mathcal{P}(\mathcal{M})) .$$ ∎

2.4 <u>Definition</u> Let x be a symbol. We define

$$\mathcal{M}[x] = \{M \cup \{x\} : M \in \mathcal{M}\}$$

2.5 <u>Lemma</u> $\mathcal{P}(\mathcal{M}[x])$ is Noetherian if and only if $\mathcal{P}(\mathcal{M})$ is, in which case we have

$$\kappa(\mathcal{P}(\mathcal{M}[x])) = \kappa(\mathcal{P}(\mathcal{M})) + 1 ,$$

provided that $x \notin \bigcup_{\mathcal{M}} M$.

<u>Proof</u> The lemma is easily verified and we omit the proof.

2.6 <u>The construction of a Noetherian topological space $\mathcal{P}(\mathcal{M}_\alpha)$ of a given Krull ordinal</u> α. Let α be an arbitrary ordinal. We shall construct partially ordered sets X_α as follows. If $\alpha = 0$ we put $X_\alpha = \emptyset$. If $\alpha > 0$, assume that X_β has been constructed for every $\beta < \alpha$. If $\alpha = \sup\{\beta : \beta < \alpha\}$, then we let X_α be the disjoint union of the sets X_β for $\beta < \alpha$. X_α will be ordered by letting each X_β keep its given ordering, and letting elements of X_{β_1} and X_{β_2} be incomparable if $\beta_1 < \beta_2 < \alpha$. If there exists an ordinal β such that $\alpha = \beta+1$, then we put $X_\alpha = X_\beta \cup \{x\}$ where x is a selected element not in X_β. We let X_α be ordered by letting X_β keep its given ordering, and by letting x be greater than every element in X_β.
In each case we let $\mathcal{M}_\alpha$ be the family of maximal linearly ordered subsets of X_α. Using (2.3) and (2.5) it is easily shown by transfinite induction that $\mathcal{P}(\mathcal{M}_\alpha)$ is Noetherian with respect to $\subset$, and that $\kappa(\mathcal{P}(\mathcal{M}_\alpha)) = \alpha$. By (2.2) $\mathcal{P}(\mathcal{M}_\alpha)$ is a Noetherian topological space with

$$\dim \mathcal{P}(\mathcal{M}_\alpha) = \alpha .$$

§ 3 Combinatorial localizations of polynomial rings.

3.1 The ring $A[X]_{\mathcal{M}}$. A will always denote a commutative integral domain, and $A[X]$ is the polynomial ring in a set X of indeterminates. Let $\mathcal{M}$ be a family of finite subsets of X . If $\mathcal{M}$ is non-empty, we let $A[X]_{\mathcal{M}}$ denote the localization of $A[X]$ with respect to the multiplicatively closed set

$$A[X] \setminus \bigcup_{M \in \mathcal{M}} M A[X]$$

If $\mathcal{M}$ is empty we define $A[X]_{\mathcal{M}}$ to be the field of fractions of $A[X]$.

Let $\mathcal{P}(\mathcal{M})$ and $\mathrm{Max}(\mathcal{M})$ be as in (2.1). Whenever $P \in \mathcal{P}(\mathcal{M})$ we let (P) denote the ideal $PA[X]$. In particular $(\emptyset)$ is the zero-ideal in $A[X]$.

If Y is a subset of $A[X]$ we define

$$\mathcal{V}(Y) := \{P \in \mathcal{P}(\mathcal{M}) \ : \ Y \subseteq (P)\}$$

3.2 Lemma Let P be an element of $\mathcal{P}(\mathcal{M})$, let $\mathcal{T}$ be a non-empty family contained in $\mathcal{V}(P)$ and assume that

$$\bigcap_{Q \in \mathcal{T}} (Q) \not\subseteq (P)$$

Then there exists a non-empty, finite subset $F \subseteq X$ with $F \cap P = \emptyset$ such that

$$(*) \qquad \mathcal{T} \subseteq \bigcup_{x \in F} \mathcal{V}(P \cup \{x\})$$

Proof. Choose an element a in $\bigcap_{\mathcal{T}} (Q)$ but not in (P) , and select elements $x_1, \ldots, x_n$ in X such that $a \in A[x_1, \ldots, x_n]$. Put

$$F := \{x_1, \ldots, x_n\} \setminus P$$

Then $F \neq \emptyset$. If every Q in $\mathcal{J}$ meets F then clearly (*) is satisfied. Assume to the contrary that there exists a member Q_o in $\mathcal{J}$ such that $Q_o \cap F = \emptyset$. Then we would have

$$a \in (Q_o) \cap A[x_1,\ldots,x_n] \subseteq (P)$$

which is absurd. ▩

3.3 <u>Corollary</u> Let Y be any subset of $A[X]$, containing a non-zero element and such that $\mathcal{V}(Y)$ is non-empty. Then there exist a finite, non-empty set $\{x_1,\ldots,x_n\} \subseteq X$ such that

$$\mathcal{V}(Y) \subseteq \bigcup_{i=1}^{n} \mathcal{V}(x_i)$$

<u>Proof</u> This follows from (3.2) by putting $\mathcal{J} := \mathcal{V}(Y)$ and $P := \emptyset$.▩

3.4 <u>Lemma</u> Let Y be any subset of $A[X]$. Assume that $\mathcal{P}(\mathcal{M})$ is Noetherian. Then $\mathcal{V}(Y)$ is a closed subset of $\mathcal{P}(\mathcal{M})$.

<u>Proof</u> Assume that $P = Y \cap X$ is a maximal member of $\mathcal{P}(\mathcal{M})$ such that $\mathcal{V}(Y)$ is not closed. If $(Y) = (P)$ then $\mathcal{V}(Y) = \mathcal{V}(P)$ which is closed. Hence we may assume that $(Y) \neq (P)$ so

$$\bigcap_{Q \in \mathcal{V}(Y)} (Q) \neq (P)$$

By (3.2) there exist $x_1,\ldots,x_n$ in $X \setminus P$ such that

(1) $$\mathcal{V}(Y) = \bigcup_{i=1}^{n} \mathcal{V}(Y \cup \{x_i\})$$

However, by the maximality of $Y \cap X$, each trem in the union (1) is either empty or closed. Hence $\mathcal{V}(Y)$ is closed, which is a contradiction. ▩

3.5 <u>Proposition</u>* The following statements are equivalent :

(i) Let S be a subset of $A[X]$ which is closed with respect to addition and multiplication, and which is contained in $\bigcup_{M \in \mathcal{M}} MA[X]$, then there exists a member $M \in \mathcal{M}$ such that $S \subseteq MA[X]$.

(ii) $\mathcal{P}(\mathcal{M})$ is Noetherian.

<u>Proof</u> (i) ⟹ (ii) . If $P_1 \subsetneq \ldots \subsetneq P_n \subsetneq \ldots$ is a strictly increasing chain in $\mathcal{P}(\mathcal{M})$ then the ideal generated by $\bigcup_n P_n$ is contained in the union $\bigcup_M MA[X]$ although not contained in any of the ideals $MA[X]$.

(ii) ⟹ (i) . Assume that $\mathcal{P}(\mathcal{M})$ is Noetherian. Let S be as in (i) . We are going to show that $\mathcal{V}(S)$ is non-empty. Let F be a variabel, running through all finite subsets of X . Then we have

$$\mathcal{V}(S) = \bigcap_F \mathcal{V}(S \cap A[F] \tag{1}$$

Since $\mathcal{P}(\mathcal{M})$ is a Noetherian space by (2.2) and since each term $\mathcal{V}(S \cap A[F])$ is closed by (3.4), the intersection (1) reduces to a finite intersection. Hence there exists a finite subset F_* of X such that

$$\mathcal{V}(S) = \mathcal{V}(S \cap A[F_*]) \tag{2}$$

We have

$$S \cap A[F_*] \subseteq \bigcup_{\mathcal{M}} (M \cap F_*)A[X]$$

Since the right hand side reduces to a finite union, there exists

*) The essential content of (3.5) has been independently established in the proof of Theorem 7.13 in [4].

a member $M_* \in \mathcal{M}$ such that

(3) $$S \cap A[F_*] \subseteq (M_* \cap F_*)A[X]$$

By (2) and (3) we have $M_* \in \mathcal{V}(S)$. ∎

3.6 Theorem Let $A[X]$ be the polynomial ring over an integral domain A . Let $\mathcal{M}$ be a family of finite subsets of X , and let $\mathcal{P}(\mathcal{M})$ be equipped with the natural topology. Assume that $\mathcal{P}(\mathcal{M})$ is Noetherian. Then the map

$$\varphi : \mathcal{P}(\mathcal{M}) \to \operatorname{Spec} A[X]_{\mathcal{M}}$$

sending P to $PA[X]_{\mathcal{M}}$ is a contineous injection which restricts to a homeomorphism

$$\bar{\varphi} : \operatorname{Max}(\mathcal{M}) \sim \operatorname{MaxSpec} A[X]_{\mathcal{M}}$$

Proof By (3.4) φ is contineous. $\bar{\varphi}$ is closed, and (3.5) shows that φ restricts to a bijection $\operatorname{Max}(\mathcal{M}) \to \operatorname{MaxSpec} A[X]_{\mathcal{M}}$. ∎

§ 4 The main theorem.

4.1 Theorem Let A be a commutative integral domain, and let $A[X]$ be the polynomial ring in a set of transcendent elements. Let $\mathcal{M}$ be a family of finite subsets of X . Then the following statements are equivalent :

(i) $A[X]_{\mathcal{M}}$ is a Noetherian ring.

(ii) $\mathcal{P}(\mathcal{M})$ is Noetherian with respect to inclusion.

Moreover, if (i) or (ii) is satisfied, then $\mathcal{P}(\mathcal{M})$ is a Noetherian topological space and we have

$$\varkappa(A[X]_{\mathcal{M}}) = \dim \mathcal{P}(\mathcal{M}) .$$

4.2 Corollary Let A be a commutative integral domain; let α be an ordinal, and let X_α and $\mathcal{M}_\alpha$ be as in (2.6). Then $A[X_\alpha]_{\mathcal{M}_\alpha}$ is a Noetherian, commutative integral domain with Krull ordinal α.

4.3 Remark (4.2) disproves the conjecture (2.9) in [1] suggesting that Krull ordinals of commutative, Noetherian rings have a countable bound.

The proof of (4.1) goes by induction on $\dim \mathcal{P}(\mathcal{M})$. Before entering the proof we need some lemmas concerning change of the family $\mathcal{M}$. As before, let $\mathcal{V}(x)$ be the family $\{P \in \mathcal{P}(\mathcal{M}) : x \in P\}$.

4.4 Lemma Assume that $A[X]_{\mathcal{V}(x)}$ is Noetherian for every x in X. Then

(i) $A[X]_{\mathcal{M}}$ is Noetherian.

(ii) $\varkappa(A[X]_{\mathcal{M}}) = \sup_{x \in X} \varkappa(A[X]_{\mathcal{V}(x)})$.

Proof (i) Let us first observe that $\mathcal{P}(\mathcal{M})$ is Noetherian. Indeed, for each $x \in X$ we have a canonical orderpreserving injection

$$\mathcal{P}(\mathcal{V}(x)) \to \operatorname{Spec} A[X]_{\mathcal{V}(x)}$$

Hence $\mathcal{P}(\mathcal{V}(x))$ is Noetherian for each x, so $\mathcal{P}(\mathcal{M})$ is Noetherian. Let $\mathfrak{A}$ be a non-zero ideal in $A[X]$ and let $\mathfrak{A}_{\mathcal{M}}$ (resp. $\mathfrak{A}_{\mathcal{V}(x)}$) denote the extension of $\mathfrak{A}$ to $A[X]$ (resp. $A[X]_{\mathcal{V}(x)}$). We are going to show that $\mathfrak{A}_{\mathcal{M}}$ is finitely generated. We may assume that $\mathfrak{A} = \mathfrak{A}_{\mathcal{M}} \cap A[X]$. Let a be a non-zero element in $\mathfrak{A}$. By (3.3) there exist elements $x_1, \ldots, x_n$ in X such that

$$\mathfrak{V}(a) \subseteq \bigcup_{i=1}^{n} \mathfrak{V}(x_i)$$

For each i $(1 \leq i \leq n)$ $A[X]_{\mathfrak{V}(x_i)}$ is Noetherian and we can choose a finitely generated ideal $\mathfrak{a}^i$ in $A[X]$ such that

$$\mathfrak{a}^i \subseteq \mathfrak{a} \quad \text{and} \quad \mathfrak{a}^i_{\mathfrak{V}(x_i)} = \mathfrak{a}_{\mathfrak{V}(x_i)}$$

Put $\mathfrak{a}^* = (a) + \sum_{i=1}^{n} \mathfrak{a}^i$. We have $\mathfrak{a}^* \subseteq \mathfrak{a}$.

It is easily seen that $\mathfrak{a}^*_{\mathfrak{m}} = \mathfrak{a}_{\mathfrak{m}}$ for every prime ideal $\mathfrak{m}$ in $A[X]_{\mathcal{M}}$ of the form $\mathfrak{m} = (M)_{\mathcal{M}}$ where $M \in \mathcal{M}$. However, by (3.6), these primeideals include all maximal ideals of $A[X]_{\mathcal{M}}$. It follows that $\mathfrak{a}^*_{\mathcal{M}} = \mathfrak{a}_{\mathcal{M}}$ so $\mathfrak{a}_{\mathcal{M}}$ is finitely generated.

(ii) Clearly we have

$$\kappa(A[X]_{\mathfrak{V}(x)}) \leq \kappa(A[X]_{\mathcal{M}}) \qquad \text{for all} \quad x \in X$$

To prove the opposite inequality we may assume that the family has at least one non-empty member. In the following let $\mathfrak{p}$ denote an arbitrary non-zero prime ideal in $A[X]_{\mathcal{M}}$. Letting λ and λ' be the canonical maps on $\operatorname{Spec} A[X]_{\mathcal{M}}$ and $\operatorname{Spec} A[X]_{\mathfrak{V}(\mathfrak{p})}$ respectively, we have

(1) $$\kappa(A[X]_{\mathcal{M}}) = \sup_{\mathfrak{p}}(\lambda(\mathfrak{p}) + 1)$$

Moreover

(2) $$\lambda(\mathfrak{p}) + 1 = \lambda'(\mathfrak{p}A[X]_{\mathfrak{V}(\mathfrak{p})}) + 1 \leq \kappa(\operatorname{Spec} A[X]_{\mathfrak{V}(\mathfrak{p})})$$

To any such $\mathfrak{p} \neq 0$ there exist, by (3.3), elements $x_1, \ldots, x_n$ in X such that

$$\mathfrak{V}(\mathfrak{p}) \subseteq \mathfrak{V}(x_1) \cup \ldots \cup \mathfrak{V}(x_n)$$

Hence we have

$$\operatorname{Spec} A[X]_{\mathfrak{V}(\mathfrak{p})} \subseteq \bigcup_{i=1}^{n} \operatorname{Spec} A[X]_{\mathfrak{V}(x_i)}$$

Using (1.3) we obtain

$$\varkappa(\operatorname{Spec} A[X]_{\mathfrak{V}(\mathfrak{p})}) \leq \max_{i=1,\dots,n} \varkappa(\operatorname{Spec} A[X]_{\mathfrak{V}(x_i)})$$

Combining this with (2) we obtain

$$\lambda(\mathfrak{p}) + 1 \leq \sup_{x \in X} \varkappa(\operatorname{Spec} A[X]_{\mathfrak{V}(x)})$$

Hence, by (1), we obtain the desired unequality

$$\varkappa(A[X]_{\mathcal{M}}) \leq \sup_{x \in X} \varkappa(A[X]_{\mathfrak{V}(x)})$$

4.5 <u>Lemma</u> Let x be an element in $X \setminus \bigcup_{\mathcal{M}} M$.
Put $Y := X \setminus \{x\}$. Then we have

$$A[X]_{\mathcal{M}[x]} = (A[Y]_{\mathcal{M}})[x]_{1+(x)}$$

where the subscript $1+(x)$ means localization with respect to the multiplicative set $1+(x)$, (x) being the ideal in $(A[Y]_{\mathcal{M}})[x]$ generated by x.

The proof of 4.3 is straight forward and will be omitted.

4.6 <u>Lemma</u> Let R be a Noetherian ring, and let $R[x]_{1+(x)}$ be the polynomial ring in one variable, localized with respect to the multiplicative set $1+(x)$. Then we have

$$\varkappa(R[x]_{1+(x)}) = \varkappa(R) + 1$$

<u>Proof</u> The inequality $\leq$ follows from (2.8) in [1]. Put $R' = R[x]_{1+(x)}$. The canonical homomorphism $R' \to R$, sending x

to 0 induces an injection $\operatorname{Spec} R \to \operatorname{Spec} R'$ by which the image of (0) is not the zero-ideal in R'. This shows that $\varkappa(R') \geq \varkappa(R) + 1$. ▨

4.7 Proof of theorem 4.1. That (i) implies (ii) is trivial in view of the fact that the canonical map

$$\mathcal{P}(\mathcal{M}) \to \operatorname{Spex} A[X]_{\mathcal{M}}$$

is an order preserving injection. Let us now assume that $\mathcal{P}(\mathcal{M})$ is Noetherian, cf. (2.1). By (2.2) it suffices to show the following

(*) $A[X]_{\mathcal{M}}$ is a Noetherian ring and we have

$$\varkappa(A[X]_{\mathcal{M}}) = \varkappa(\mathcal{P}(\mathcal{M})) .$$

We are going to prove (*) by induction on $\varkappa := \varkappa(\mathcal{P}(\mathcal{M}))$. If $\varkappa = 0$, then either $\mathcal{M} = \emptyset$ or $\mathcal{M} = \{\emptyset\}$. In both cases (*) is obviously satisfied. Let α be a non-zero ordinal and let us assume that (*) is satisfied whenever $\varkappa < \alpha$. Now assume that $\varkappa = \alpha$. By (2.3) and (4.4) there is no loss of generality assuming that $\mathcal{M} = \mathcal{V}(x)$ for some $x \in X$. Consider the family

$$\mathcal{N} := \{M \setminus \{x\} : M \in \mathcal{M}\}$$

and the set $Y := X \setminus \{x\}$. We have $\mathcal{M} = \mathcal{N}[x]$, and by (2.5) we have

$$\varkappa(\mathcal{P}(\mathcal{M})) = \varkappa(\mathcal{P}(\mathcal{N})) + 1$$

Hence by the induction hypotesis $A[Y]_{\mathcal{N}}$ is Noetherian with Krull ordinal equal to $\varkappa(\mathcal{P}(\mathcal{N}))$. By (4.5) we have

$$A[X]_{\mathcal{M}} = (A[Y]_{\mathcal{N}})[x]_{1+(x)}$$

Hence by (4.6) $A[X]_{\mathcal{M}}$ is a Noetherian ring of Krull ordinal $\varkappa(\mathcal{P}(\mathcal{M}))$. ▨

§ 5 Examples where coprof $R_{\mathfrak{p}}$ bahaves badly on $\operatorname{Spec} R$.

5.1 The map $\mathfrak{p} \mapsto \operatorname{coprof} R_{\mathfrak{p}}$. Let R be a commutative, Noetherian ring. Let $\operatorname{prof} R_{\mathfrak{p}}$ be the length of a maximal regular sequence in $\mathfrak{p}R_{\mathfrak{p}}$, and let $\dim R_{\mathfrak{p}}$ denote the Krull dimension of $R_{\mathfrak{p}}$. Recall the definition

$$\operatorname{coprof} R_{\mathfrak{p}} := \dim R_{\mathfrak{p}} - \operatorname{prof} R_{\mathfrak{p}} .$$

CM(R) (resp. Reg(R)) will denote the Cohen-Macaulay locus of R (resp. regular locus of R) i.e. the set of all points in $\operatorname{Spec} R$ where $R_{\mathfrak{p}}$ is Cohen-Macaulay (resp. regular).

If R is a homomorphic image of a regular ring, then by a theorem due to Auslander [EGA, IV, 6.11.2] the map $\mathfrak{p} \mapsto \operatorname{coprof} R_{\mathfrak{p}}$ is upper semicontineous on $\operatorname{Spec} R$. In particular this function is bounded, and CM(R) is an open set in $\operatorname{Spec} R$. It is known that CM(R) is not open in general. In [3] Ferrand and Raynaud have given an example of a local ring of dimension 3 whose Cohen-Macaulay locus is not an open set. The present section is devoted to the construction of a class of Noetherian domains showing that in general there is little connection between dim and prof as functions on $\operatorname{Spec} R$. In particular the function $\mathfrak{p} \mapsto \operatorname{coprof} R_{\mathfrak{p}}$ need not be bounded.

5.2 Lemma Let k be a field, and assume that ω is an element which is algebraic over k, but not contained in k. For integers $r \geq 0$ and $c \geq 1$ consider the polynomial ring in $r+c$ transcendent elements over $k(\omega)$

$$k(\omega)[y_1, \dots, y_r, x_1, \dots, x_c]$$

and the subring

$$A := k[y_1, \dots, y_r, x_1, \dots, x_c, \omega x_1, \dots, \omega x_c]$$

Let $\mathfrak{m}$ be the maximal ideal in A which is generated by y_i, x_j and ωx_j for $0 \leq i \leq r$, $1 \leq j \leq c$. Then we have

$$\operatorname{prof} A_{\mathfrak{m}} = r + 1$$

$$\dim A_{\mathfrak{m}} = r + c$$

Moreover, if $\mathfrak{p}$ is a prime ideal in A of height less than c, then $A_{\mathfrak{p}}$ is regular.

<u>Proof</u> Let ω be algebraic of degree $n > 0$, and let $\alpha_o, \ldots, \alpha_{n-1}$ be elements of k such that $\omega^n = \sum_{i=o}^{n-1} \alpha_i \omega^i$.
Since for $1 \leq j \leq c$ we have

$$(\omega x_j)^n = \sum_{i=o}^{n-1} \alpha_i \omega^i x_j^n = \sum_{i=o}^{n-1} \alpha_i (\omega x_j)^i x_j^{n-i} \in x_j A$$

we see that $y_1, \ldots, y_r, x_1, \ldots, x_c$ is a system of parameters for $A_{\mathfrak{m}}$, so $\dim A_{\mathfrak{m}} = r + c$. To prove that $\operatorname{prof} A_{\mathfrak{m}} = r + 1$, we will show that the A-regular sequence $y_1, \ldots, y_r, x_1$ is maximal in $A_{\mathfrak{m}}$. It suffices to show that every element in $\mathfrak{m} A_{\mathfrak{m}}$ is a zero-divisor for $A_{\mathfrak{m}}/\mathfrak{a}$ where $\mathfrak{a} := (y_1, \ldots, y_r, x_1) A_{\mathfrak{m}}$. We have

$$x_j(\omega x_1) = (\omega x_j) x_1 \in x_1 A \quad \text{for all } j$$

hence, since $y_1, \ldots, y_r, x_1, \ldots, x_c$ is a system of parameters, there exists an integer s such that

$$\mathfrak{m}^s A_{\mathfrak{m}} \, \omega x_1 \subset \mathfrak{a}$$

On the other hand ωx_1 is not in $\mathfrak{a}$.

Now let $\mathfrak{p}$ be a prime ideal in A of height less than c. Since ω must be an element of $A_{\mathfrak{p}}$, it is easily seen that A equals the localization of the regular ring

$$k(\omega)[y_1, \ldots, y_r, x_1, \ldots, x_c]$$

with respect to the multiplicative set $A \setminus \mathfrak{p}$. Hence $A_{\mathfrak{p}}$ is regular. ∎

5.3 Theorem Let $\mathbb{N}$ be the set of positive integers and let f and g be functions $\mathbb{N} \to \mathbb{N}$ such that

$$1 \leq f(n) \leq g(n) \qquad \text{for all } n \in \mathbb{N} .$$

Then there exists a Noetherian integral domain R and a bijection

$$\mathbb{N} \to \operatorname{Max} \operatorname{Spec} R$$

such that letting $\mathfrak{m}_n$ denote the image of n we have

(i) $\operatorname{prof} R_{\mathfrak{m}_n} = f(n)$

(ii) $\dim R_{\mathfrak{m}_n} = g(n)$

(iii) A proper subset of Max Spec R is closed if and only if it is finite.

Proof Let $p_1, p_2, \dots, p_n, \dots$ be the odd prime numbers ordered by size. For each n in $\mathbb{N}$ let ω_n be a primitive p_n^{th} root of 1 and consider the following extension of $\mathbb{Q}$,

$$\tilde{\mathbb{Q}} \coloneqq \mathbb{Q}(\omega_1, \dots, \omega_n, \dots)$$

For each n in $\mathbb{N}$ let us choose sets of transcendents over $\tilde{\mathbb{Q}}$

$$Y_n = \{y_{n1}, \dots, y_{nr}\} \ , \ X_n = \{x_{n1}, \dots, x_{nc}\}$$

where $r \coloneqq f(n) - 1$ and $c \coloneqq g(n) - f(n) + 1$. Let A (resp. $\tilde{A}$) denote the polynomial ring generated over $\mathbb{Q}$ (resp. $\tilde{\mathbb{Q}}$) by Y_n and X_n for all $n > 0$. Let $\omega_n X$ denote the set

$$\{\omega_n x_{n1}, \dots, \omega_n x_{nc}\}$$

and let A' be the ring between A and $\tilde{A}$ which is generated over $\mathbb{Q}$ by Y_n, X_n and $\omega_n X_n$ for all $n > 0$. Let M_n' be the ideal in A' which is generated by Y_n, X_n and $\omega_n X_n$. Let S be the multiplicative set

$$S := A' \setminus \bigcup_{n>0} M'_n$$

Put $R := A'_S$. I claim that R is the required example. We will first show that the maximal ideals in R are just the ideals $\mathfrak{m}_n := M'_n R$. For this it suffices to show the following:

(*) Let I be an ideal of A' which is contained in the union $\bigcup M'_n$. Then I is contained in at least one of the M'_n .

To prove (*), let M_n (resp. $\tilde{M}_n$) be the ideal in A (resp. $\tilde{A}$) generated by Y_n and X_n . Observe that I is contained in $\bigcup \tilde{M}_n$. Hence by (3.5) I is contained in some $\tilde{M}_n$. Thus it suffices to show that $A \cap \tilde{M}_n = M'_n$ for all n . But since the ideal M'_n is contained in the ideal $A \cap \tilde{M}_n$, and both of them are prime ideals lying over M_n , and the extension $A \to A'$ is integral, it follows that $M'_n = A \cap \tilde{M}_n$.

If a is a non-zero element in A' , then a is contained in only finitely many of the ideals M'_n , so (iii) follows.

Letting $\mathbb{Q}_n$ be the field generated over $\mathbb{Q}$ by every ω_m , Y_m and X_m for $m \neq n$, one easily shows that we have

$$R_{\mathfrak{m}_n} = \mathbb{Q}_n[Y_n, X_n, \omega_n X_n]_{(Y_n, X_n, \omega_n X_n)}$$

Since ω_n is not in $\mathbb{Q}_n$, (5.2) gives

$$\operatorname{prof} R_{\mathfrak{m}_n} = r + 1 = f(n)$$

$$\dim R_{\mathfrak{m}_n} = r + c = g(n)$$

That R is Noetherian follows from (E1.1) on page 203 in [6]. ∎

5.4 <u>Corollary</u> Let R be the ring constructed in (5.3) . If $1 = f(n) < g(n)$ for all n , then the sets $\operatorname{Reg}(R)$ and $\operatorname{CM}(R)$ coincide with the set of all non-maximal prime ideals, which is a non-constructible set in $\operatorname{Spec} R$ (By a constructible set we mean a finite union of sets of the form $U \cap F$ where U is open and F is closed).

5.5 <u>Example</u> Putting $f(n) = 1$ and $g(n) = n+1$ for all n, we obtain a Noetherian domain R for which the function

$$\mathfrak{p} \mapsto \operatorname{coprof} R_{\mathfrak{p}}$$

is not bounded on $\operatorname{Spec} R$.

5.6 <u>Example</u> Putting $f(n) = 1$ and $g(n) = 2$ for all n, we obtain a Noetherian domain of dimension 2 which is not universally Cohen Macaulay. This gives an answer to the question raised in [EGA, IV,6.11.9 (ii)].

References.

[1] H. Bass, Descending chains and the Krull ordinal of commutative Noetherian rings. J. of pure and appl. algebra. Vol 1, no 4 (1971).

[2] J. Dieudonné, A. Grothendieck, Éléments de géométrie algébrique (EGA). Publications de l'I.H.E.S. no.24.

[3] D. Ferrand, M. Raynaud, Fibres formelles d'un anneau local Noethérien. Ann.scient.Éc.Norm.Sup. 4e série, t.3(1970) pp. 295-311.

[4] R. Gordon, J.C. Robson, Krull dimension, critical modules and monoform modules. To appear.

[5] G. Krause, On the Krull-dimension of left Noetherian left Matlis-rings, Math.Z. 118 (1970).

[6] M. Nagata, Local rings (Interscience, 1962).