A CHANGE OF RING THEOREM WITH APPLICATIONS TO POINCARÉ SERIES AND INTERSECTION MULTIPLICITY.

Tor H. Gulliksen

INTRODUCTION

Let $x_1,\ldots,x_m$ be elements generating an ideal $\mathfrak{a}$ in a commutative ring R. Put $A := R/\mathfrak{a}$ and let M and N be A-modules. The purpose of this paper is to find a close relationship between $\mathrm{Tor}_\cdot^A(M,N)$ and $\mathrm{Tor}_\cdot^R(M,N)$. This is of course almost hopeless in general, so we make right away the assumption that the Koszul complex $K^R(x_1,\ldots,x_m)$ be acyclic, cf. (2.2).

Let $G = A[X_1,\ldots,X_m]$ be the polynomial ring in m variables of degree -2. The main idea is to turn $\mathrm{Tor}_\cdot^A(M,N)$ into a graded G-module in such a way that it becomes an artinian G-module whenever $\mathrm{Tor}_\cdot^R(M,N)$ is an artinian R-module. The main result (3.1) is formulated in terms of more general derived functors. In particular we obtain dual results for Ext. One of the consequences of (3.1) is the following (4.2):

Let $A,\mathfrak{m}$ be a local complete intersection with

$$m = \dim \mathfrak{m}/\mathfrak{m}^2 - \dim A$$

and let M and N be A-modules of finite type such that $M\otimes_A N$ has finite length $l(M\otimes_A N)$. Then there exists a polynomial $\pi_{M,N}^A(t)$ in $\mathbb{Z}[t]$ such that we have the following identity of powerseries

$$(*) \qquad \Sigma_p l(\mathrm{Tor}_p^A(M,N))t^p = (1-t^2)^{-m}\pi^A_{M,N}(t)$$

Moreover, if $R \to A$ is a surjective ringhomomorphism where R is regular, local of the same imbedding dimension as A then we have

$$\pi^A_{M,N}(-1) = \chi^R(M,N)$$

where

$$\chi^R(M,N) = \Sigma(-1)^p l(\mathrm{Tor}_p^R(M,N))$$

is the intersection multiplicity. Cf. [Se].

As an application of (*) we prove that if $A^* = A \ltimes M$ is the trivial extension of a local complete intersection A by a finitely generated A-module M, then the Poincaré series of A^* ;

$$\Sigma_p \dim \mathrm{Tor}_p^{A^*}(k,k)t^p$$

is a rational function (4.5), k being the residue field of A^* .

§§ 1 and 2 contain more or less well known lemmas. In § 3 we prove the main theorem (3.1), while § 4 contains the applications to intersection theory and the rationality problem for Poincaré series.

NOTATION

If H is a graded module, H_p $(p \in \mathbb{Z})$ will denote its homogeneous components, i.e.

$$H = \coprod_{p\in\mathbb{Z}} H_p$$

H is called positively graded if $H_p = 0$ for $p < 0$.
It is called negatively graded if $H_p = 0$ for $p > 0$.

If Y is a complex, its homology is denoted by $H.(Y)$.
We use the convention

$$H^{-p}(Y) = H_p(Y) \quad \text{for} \quad p \in \mathbb{Z} .$$

R and A will always be commutative rings with identity. Mod_R denotes the category of R-modules. Let

$$F : \mathrm{Mod}_R \to \mathrm{Mod}_A$$

be an additive functor. For $q \geq 0$ let L_qF and R^qF denote the $q^{\underline{th}}$ left derived functor, respectively the $q^{\underline{th}}$ right derived functor of F . It is convenient to introduce the following notation:

If F is covariant, put

$$D_pF := L_pF \quad \text{for} \quad p \geq 0$$
$$D_pF := 0 \quad \text{otherwise} .$$

If F is contravariant, put

$$D_pF := R^{-p}F \quad \text{for} \quad p \leq 0$$
$$D_pF := 0 \quad \text{otherwise} .$$

In both cases we put

$$D.F := \coprod D_pF \quad \text{where } p \text{ runs through } \mathbb{Z} .$$

§ 1. GRADED MODULES AND POINCARÉ SERIES.

1.1 DEFINITION. Let $G = \coprod G_i$ be a $\mathbb{Z}$-graded ring and let $H = \coprod H_p$ be a $\mathbb{Z}$-graded G-module. Let n be an integer. We define $H(n)$ to be the $\mathbb{Z}$-graded G-module

$$H(n) = \coprod H(n)_p$$

where $H(n)_p = H_{p-n}$.
If each H_p is a G_o-module of finite length $l(H_p)$, we define the Poincaré-series of H to be the formal powerseries

$$\chi_H(t) := \Sigma_p l(H_p) t^{|p|}$$

Observe that if $n \geq 0$ and if H is positively graded, then so is $H(n)$ and we have

$$\chi_{H(n)}(t) = t^n \chi_H(t)$$

whenever χ_H or $\chi_{H(n)}$ is defined.

1.2 LEMMA. Let H be a $\mathbb{Z}$-graded left module over a not necessarily commutative ring G . Assume that $H_p = 0$ for all p sufficiently small (resp. large). Let $X : H \to H$ be a homogeneous G-linear map of negative degree w . Assume that the graded G-module $\operatorname{Ker} X$ is artinian (resp. noetherian), then H is an artinian (resp. noetherian) module over the ring $G[X]$.

PROOF. The noetherian case is a version of the Hilbert basis-satz. We will only prove the artinian case.

Put $I = \operatorname{Ker} X$ and let

$$H = H^o \supseteq H^1 \supseteq H^2 \supseteq \ldots$$

be a descending sequence of graded submodules of H .

For each pair of non-negative integers p, q put

$$I^{p,q} = I \cap X^p H^q$$

Then $I^{p,q}$ are left H-modules satisfying

$$I^{p,q} \supseteq I^{p+1,q} + I^{p,q+1}$$

Since I is artinian we can pick an integer Q such that

$$I^{p,q} = I^{p,Q} \qquad \text{for } q \geq Q \text{ , } p \geq 0 \text{ .}$$

We will now show that

$$X^p H^q = X^p H^Q \qquad \text{for } q \geq Q \text{ , } p \geq 0 \text{ .}$$

It suffices to show that in each degree r the inclusion map

$$(X^p H^Q)_r \hookrightarrow (X^p H^q)_r$$

is an isomorphism. This can be shown by induction on r applying the five lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_r^{p,q} & \hookrightarrow & (X^p H^q)_r & \xrightarrow{X} & (X^{p+1} H^q)_{r+w} & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I_r^{p,Q} & \hookrightarrow & (X^p H^Q)_r & \xrightarrow{X} & (X^{p+1} H^Q)_{r+w} & \longrightarrow & 0 \end{array}$$

∎

1.3 LEMMA. Let A be a commutative ring and let $G = A[X_1,\dots,X_m]$ be the polynomial ring. Let $w_1,\dots,w_m$ be non-zero integers, and give G a grading by putting $G_o = A$ and letting X_i have degree w_i . Let H be a graded G-module which is either positively or negatively graded, and such that each homogeneous component H_p is an A-module of finite length. Assume moreover that H is either an artinian or a noetherian G-module. Then there exists a polynomial $\pi(t)$ in $\mathbb{Z}[t]$ such that the Poincaré-series of H has the form

$$\chi_H(t) = [(1-t^{|w_1|})\cdots(1-t^{|w_m|})]^{-1}\pi(t)$$

(For $m = 0$ the formula should be read $\chi_H(t) = \pi(t)$)

PROOF. The standard proof which goes by induction on m can be adapted to all cases. Let us just sketch the proof in the case where H is artinian and positively graded. For $m = 0$ we have $H_p = 0$ for almost all p , hence $\chi_H(t)$ is a polynomial. Now let $m > 0$. Let us first treat the case where $w_m < 0$. Multiplication by X_m gives rise to an exact sequence of graded G-modules

$$0 \longrightarrow N \hookrightarrow H \xrightarrow{X_m} H(-w_m) \longrightarrow C \longrightarrow 0 \qquad (1)$$

with homomorphisms of degree zero. It follows that

$$(1-t^{-w_m})\chi_H(t) = \chi_N(t) - \chi_C(t) \qquad (2)$$

cf. (1). Since N and C are killed by X_m , they are modules over $G/X_mG \approx A[X_1,\ldots,X_{m-1}]$ and the induction hypotesis applies to N and C . Hence the desired formula for $\chi_H(t)$ follows from (2).

In the case where $w_n > 0$ we just have to replace (1) by a sequence of the form

$$0 \longrightarrow N \hookrightarrow H(w_m) \xrightarrow{X_m} H \longrightarrow C \longrightarrow 0$$

and repeat the argument. ∎

§ 2. DIFFERENTIAL GRADED MODULES AND ALGEBRAS

2.1 DEFINITION. A pair (K,d) will be called a DG-__algebra__ over a ring R if K is an associative, strictly skew-commutative differential graded algebra over R, with differential d of degree -1, and unit element 1, such that

$$K_o = R\cdot 1 \quad \text{and} \quad K_p = 0 \quad \text{for} \quad p < 0 .$$

A differential graded module over a DG-algebra (K,d) will briefly be called a DG-__module__ over K.

A triple (K,d,ε) will be called a DGA-__algebra__ over R if

(i) (K,d) is a DG-algebra over R

(ii) ε is a surjective algebra homomorphism from K onto a residue class ring of R such that

$$\varepsilon d = 0 \quad \text{and} \quad \varepsilon(K_p) = 0 \quad \text{for} \quad p > 0 .$$

ε will be called the __augmentation__.

Let (K,d,ε) be a DGA-algebra over R with augmentation $\varepsilon : K \to A$. By a DGA-__module__ over K we mean a triple (L,d,η) where (L,d') is a DG-module over K and η is an R-linear map from L to an A-module, such that

$$\eta d' = 0$$

and

$$\eta(xl) = \varepsilon(x)\eta(l) \quad \text{for} \quad x \in K_o , \ l \in L_o .$$

2.2 EXAMPLE. Let $x_1,\ldots,x_m$ be a sequence of elements in a commutative ring R. Let

$$K = K^R(x_1,\ldots,x_m)$$

be the Koszul complex generated over R by $x_1,\ldots,x_m$, cf. [Se] ch. IV no.2. K is a DG-algebra over R. Observe that $K_o = R$.

Equipped with the augmentation induced by the canonical map

$$R \to R/(x_1,\ldots,x_m)$$

K becomes a DGA-algebra over R. By the augmentation ideal in K we will always mean the kernel of this augmentation. Recall that if $x_1,\ldots,x_m$ is a regular sequence, then K is acyclic.

2.3 DEFINITION. Let K be any DGA-algebra over R and let L be a DGA-module over K with augmentation $\eta : L \to M$. Let w be a non-negative integer, and let $(x_\alpha)_{\alpha\in I}$ be a set of homogeneous cycles in $\operatorname{Ker}\eta$, of degree w. By the symbol combination

$$L\{\ldots,T_\alpha,\ldots;dT_\alpha = x_\alpha\}$$

we shall mean the DGA-module L' over K, uniquely determined by (i) - (iii) below:

(i) As a graded K-module, L' is the direct sum of L and the free K-module with basis $(T_\alpha)_{\alpha\in I}$, each T_α being a homogeneous element of degree $w+1$.

(ii) The differential d on L' is defined as follows: By (i) every element in L' can be expressed uniquely in the form

$$l + \Sigma_\alpha k_\alpha T_\alpha$$

where l and k_α are homogeneous elements in L and K respectively, k_α being zero for almost all α.
Letting d_K and d_L denote the differential on K and L respectively, we can now define d as follows

$$d(l+\Sigma k_\alpha T_\alpha) := d_L(l) + \Sigma_\alpha(d_K(k_\alpha)T_\alpha + (-1)^{[k_\alpha]}k_\alpha x_\alpha)$$

where $[k_\alpha]$ denotes the degree of k_α.

(iii) We equip L' with the augmentation induced by the augmentation on L.

It is now straight forward to check that L' is a DGA-module over K .

2.4 LEMMA. Let K be a DGA-algebra over R , with augmentation $\epsilon : K \to A$, A being a factor ring of R . Let M be an A-module. Then there exists an acyclic DGA-module L over K with augmentation onto M , and such that L is free as a K-module.

PROOF. We shall obtain L as the union of an ascending chain of DGA-modules over K

$$L^o \subseteq L' \subseteq \dots \subseteq L^n \subseteq \dots$$

We will define L^n inductively. For $n = 0$, choose a set of generators $(m_\alpha)_{\alpha \in I^o}$ for the A-module M . Let L^o be the free DG-module over K generated by a set of generators $(T_\alpha^o)_{\alpha \in I^o}$ of degree zero. Now we equip L^o with the unique augmentation $\eta : L^o \to M$ sending T_α^o to m_α for all α . Now let $n \geq 0$ and assume that L^n has been constructed. Let $(x_\alpha)_{\alpha \in I}$ be a set of generators for the R-module

$$Z_n(L^n) \cap \operatorname{Ker} \eta / B_n(L^n) .$$

Here $Z_n(L^n)$ and $B_n(L^n)$ denote the set of n-cycles and n-boundaries in L^n . η denotes the augmentation on L^n . If I is empty, put $L^{n+1} \coloneqq L^n$, otherwise define

$$L^{n+1} \coloneqq L^n\{\dots, T_\alpha, \dots; dT_\alpha = x_\alpha\} .$$

Finally put

$$L \coloneqq \bigcup_{n \geq o} L^n$$

It is easily seen that L is an acyclic DGA-module over K , with augmentation onto M . It is also clear that L is free as a graded K-module. ∎

2.5 REMARK. The construction of L above can be made canonical in following way: Each time a set of generators is to be chosen, one can select the maximal one.

2.6 LEMMA. Let K be an acyclic DGA-algebra with augmentation-ideal I. Let L be a DGA-module over K, which is free as a graded K-module. Then the canonical map $L \to L/IL$ induces an isomorphism

$$H.(L) \approx H.(L/IL)$$

PROOF. If L is generated by elements of degree zero, then as a complex we have $L \approx K$ hence $IL \approx I$ so $H.(IL) = 0$. Now let L^n be the sub-DGA-module of L generated by the elements of degree $\leq n$. We have an exact sequence of complexes

$$0 \longrightarrow IL^n \xrightarrow{i} IL^{n+1} \longrightarrow \text{Coker } i \longrightarrow 0$$

where Coker i, as a complex, is isomorphic to a direct sum of copies of the complex I.
Hence $H.(\text{Coker } i) = 0$ so

$$H.(IL^n) = H.(IL^{n+1})$$

Hence by induction

$$H.(IL^n) = 0 \qquad \text{for all } n \geq 0 .$$

It follows that $H.(IL) = 0$, whence the map

$$H.(L) \to H.(L/IL)$$

is an isomorphism. ∎

2.7 LEMMA. Consider the following diagram of complexes:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow & - & \downarrow & + & \downarrow & & \\
0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0 \\
 & & \downarrow & + & \downarrow & + & \downarrow & & \\
0 & \longrightarrow & Z' & \longrightarrow & Z & \longrightarrow & Z'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with exact rows and columns, $+$ indicating that the square is commutative, $-$ indicating an anticommutative square. Then the exact homology sequences yield a commutative diagram

$$
\begin{array}{ccccc}
H.(Y) & \longrightarrow & H.(Y'') & \longrightarrow & H.(Y') \\
\downarrow & & \downarrow & & \downarrow \\
H.(Z) & \longrightarrow & H.(Z'') & \longrightarrow & H.(Z') \\
\downarrow & & \downarrow & & \downarrow \\
H.(X) & \longrightarrow & H.(X'') & \longrightarrow & H.(X')
\end{array}
$$

PROOF. This follows easily from Prop. 2.1, §2 Chap. IV in [C.E] by forcing the upper diagram to be commutative by changing sign of the upper, left vertical map. ∎

§ 3 THE MAIN THEOREM

From now on let $x_1,\ldots,x_m$ be elements in a commutative ring R generating an ideal $\mathfrak{a}$. Let $K := K^R(x_1,\ldots,x_m)$ be the Koszul complex. Cf. (2.2). Put $A := R/\mathfrak{a}$ and consider the functor

$$T : \mathrm{Mod}_R \to \mathrm{Mod}_A$$

defined by $T(M) = M/\mathfrak{a}M$. Let

$$\bar{F} : \mathrm{Mod}_A \to \mathrm{Mod}_A$$

be a given A-linear functor and consider the composition $F := \bar{F}\circ T$. We will consider Mod_A as a subcategory of Mod_R in the obvious way. Observe that $\bar{F}$ is the restriction of F to Mod_A . In the following

$$G = A[X_1,\ldots,X_m]$$

denotes the polynomial ring, negatively graded by giving each X_i the weight -2 . The main result in this paper is the following:

3.1 THEOREM. Assume that the Koszul complex $K = K^R(x_1,\ldots,x_m)$ is exact. Then for each A-module M , the graded A-module

$$D.\bar{F}(M) = \coprod D_q\bar{F}(M)$$

can be given structure of a graded G-module in such a way that:

(i) $D.\bar{F}$ becomes a functor from the category of A-modules to the category of $\mathbb{Z}$- graded G-modules.

(ii) If F is covariant (resp. contravariant) and if $D.F(M)$ is an artinian (resp. noetherian) R-module, then $D.\bar{F}(M)$ is an artinian, positively graded (resp. noetherian, negatively graded) G-module.

(iii) If $D_qF(M)$ is an A-module of finite length for all q

in $\mathbb{Z}$, then so is $D_q\bar{F}(M)$.

(iv) If $D.F(M)$ is an A-module of finite length, then there exists a polynomial $\pi(t)$ in $\mathbb{Z}[t]$ such that

$$\Sigma_q l(D_q\bar{F}(M))t^{|q|} = (1-t^2)^{-m}\pi(t) .$$

Moreover we have

$$\pi(-1) = \Sigma_q(-1)^q l(D_qF(M)) .$$

Before entering the proof of the theorem we shall prove the following lemma:

3.2 LEMMA. Let L be a DG-module over the DG-algebra $K = K^R(x_1,\dots,x_m)$. Assume that L is free as a graded K-module. Let I be the augmentation ideal in K . Then

(i) $H.(F(L/IL))$ has a structure of a $\mathbb{Z}$-graded G-module with the following properties:

(ii) If F is covariant (resp. contravariant) and if the graded A-module $H.(F(L))$ is artinian (resp. noetherian), then the graded G-module $H.(F(L/IL))$ is positively graded and artinian (resp. negatively graded and noetherian).

(iii) If $H_q(F(L))$ is an A-module of finite length for each q , then so is $H_q(F(L/IL))$.

(iv) If $H.(F(L))$ is an A-module of finite length, then there exists a polynomial $\pi(t)$ in $\mathbb{Z}[t]$ such that we have the following identity of powerseries:

$$\Sigma_q l(H_q(F(L/IL)))t^{|q|} = (1-t^2)^{-m}\pi(t) .$$

Moreover we have

$$\pi(-1) = \Sigma_q(-1)^q l(H_q(F(L))) .$$

PROOF. Let $T_1,\dots,T_m$ be a set of algebra generators for K such that $dT_i = x_i$ for $i = 1,\dots,m$. Let $\mathfrak{a}$ be the ideal generated by $x_1,\dots,x_m$ and put $Y := L/\mathfrak{a}L$. Clearly we have an identity of complexes

$$F(L) = F(Y) \tag{1}$$

Let $[1,m]$ denote the set $\{1,\dots,m\}$. In the following we shall let S denote an arbitrary subset of $[1,m]$, and I_S denotes the ideal in the algebra K which is generated by T_i for each $i \in S$. We consider the following DG-module over K :

$$Y^S := Y/I_S Y$$

We put $Y^{\emptyset} := Y$. Observe that

$$L/IL = Y^{[1,m]} \tag{2}$$

Since Y^S is in particular a complex of R-modules, we may apply F and obtain a complex $F(Y^S)$ of A-modules.
From (1) and (2) we have

$$H.(F(L)) = H.(F(Y^{\emptyset})) \tag{3}$$

$$H.(F(L/IL)) = H.(F(Y^{[1,m]})) \tag{4}$$

(i). We shall now equip the graded A-module $H.(F(Y^S))$ with a structure of a graded G-module. Let us start by defining the action of X_i on $H.(F(Y^S))$ for an arbitrary i in $[1,m]$. If i is not in S , then we let X_i act as the zero-map. Let us now assume that $i \in S$. Consider the homogeneous map $f_i : Y \to Y$ of degree 1 , defined by

$$f_i(y) = (-1)^p T_i y \qquad \text{for every } y \in Y_p$$

One sees that f_i is a K-linear map which commutes with the differential. Put

$$S_i := S \setminus \{i\} .$$

One easily checks that f_i induces a DG-homomorphism

$$Y^{S_i} \to Y^{S_i}$$

whose kernel equals $T_i Y^{S_i}$, which in turn equals the kernel of the canonical map of degree zero

$$g_i : Y^{S_i} \to Y^S .$$

Hence f_i induces an injective map $Y^S \to Y^{S_i}$, which by abuse of notation will be denoted by T_i, regardless of S. Thus we have an exact sequence of complexes over A

$$0 \longrightarrow Y^S \xrightarrow{T_i} Y^{S_i} \xrightarrow{g_i} Y^S \longrightarrow 0 \qquad (5)$$

From now on we will assume that F is covariant. The proof in the contravariant case is similar and will be left to the reader. (5) splits as a sequence of A-modules. Hence we obtain an exact sequence of A-modules

$$0 \longrightarrow F(Y^S) \longrightarrow F(Y^{S_i}) \longrightarrow F(Y^S) \longrightarrow 0 \qquad (6)$$

Let ∂^i denote the connection homomorphism in the homology sequence associated to (6). Now we define the action of X_i on $H.(F(Y^S))$ as follows

$$X_i h := (-1)^p \partial^i(h) \qquad \text{where } h \in H_p(F(Y^S)) .$$

In this way X_i may be considered as a homogeneous map of degree -2. The reason for the factor $(-1)^p$ is that we want certain maps arising later to be G-linear.

In order to have an action of G on $H.(F(Y^S))$ it remains to show that X_i and X_j are commuting operators on $H.(F(Y^S))$ for $i,j \in [1,m]$.

If i or j is not in S, then this is obviously the case. Hence there is no loss of generality assuming that i and j are distinct elements in S. Put

$$S_{ij} := S \setminus \{i,j\}$$

Using exact sequences of the type (5) we obtain a diagram of complexes with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^S & \xrightarrow{T_j} & Y^{S_j} & \longrightarrow & Y^S & \longrightarrow & 0 \\
 & & \downarrow T_i & & \downarrow T_i & & \downarrow T_i & & \\
0 & \longrightarrow & Y^{S_i} & \xrightarrow{T_j} & Y^{S_{ij}} & \longrightarrow & Y^{S_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^S & \xrightarrow{T_j} & Y^{S_j} & \longrightarrow & Y^S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which every square is commutative, except the left upper square which is anti-commutative. Recalling that every short exact sequence in the diagram splits, we apply the functor F and obtain the following diagram by considering the associated homology sequence:

$$\begin{array}{ccc}
H.(F(Y^S)) & \xrightarrow{\partial^j} & H.(F(Y^S)) \\
\downarrow & & \downarrow \\
H.(F(Y^{S_i})) & \xrightarrow{\partial^j} & H.(F(Y^{S_i})) \\
\downarrow & & \downarrow \\
H.(F(Y^S)) & \xrightarrow{\partial^j} & H.(F(Y^S)) \\
\downarrow \partial^i & & \downarrow \partial^i \\
H.(F(Y^S)) & \xrightarrow{\partial^j} & H.(F(Y^S))
\end{array} \tag{7}$$

By (2.7) the lower square is commutative, hence X_i and X_j commute. Before proving (ii) we shall make an additional remark which will be referred to later. It is easily seen that the middle square is also commutative, while the upper wquare is anti-commutative. However, replacing ∂^i and ∂^j in (7) by X_i and X_j respectively, we obtain a commutative diagram. Hence every map in the homology triangle

$$\begin{array}{ccc} H.(F(Y^S)) & \xrightarrow{X_i} & H.(F(Y^S)) \\ & H.(F(Y^{S_i})) & \end{array} \qquad (8)$$

associated to (6) is G-linear.

(ii). Let us now assume that $H.(F(L))$ is an artinian graded A-module. We will prove that $H.(F(L/IL))$ is an artinian graded G-module. For each $S \subseteq [1,m]$ consider the following homogeneous subring of G

$$G^S := A[T_{i_1},\ldots,T_{i_s}] \quad \text{where} \quad S = \{i_1,\ldots,i_s\} .$$

Put $G^{\emptyset} := A$. By (4) it suffices to prove that $H.(F(Y^S))$ is an artinian graded G^S-module for all S . This will be done by induction on s , the number of elements in S . If $s = 0$ then $S = \emptyset$, so it is true by assumption, because of (3). Now let r be a positive integer, and suppose that $H.(F(Y^S))$ is an artinian G^S-module whenever S has less than r elements. Now assume that S has exactly r elements. Choose an element i in S . By (8) we have an exact sequence of G^{S_i}-modules

$$H.(F(Y^{S_i})) \longrightarrow H.(F(Y^S)) \xrightarrow{X_i} H.(F(Y^S))$$

By the induction hypothesis $H.(F(Y^{S_i}))$ is an artinian G^{S_i}-module. Since the map X_i is of negative degree, it follows from (1.2) that $H.(F(Y^S))$ is an artinian G^S-module, which was to be shown.

(iii). Assume that $H_q(F(Y^{S_i}))$ is an A-module of finite length for all q. Using the exactness of (8) and induction on p, one easily shows that $H_p(F(Y^S))$ is an A-module of finite length for all p.

Now (iii) easily follows, using (3) and (4).

(iv). Let us now assume that $H.(F(L))$ is an A-module of finite length. In particular we have

$$H_q(F(L)) = 0 \quad \text{for all } q \text{ sufficiently large.}$$

By (ii) $H.(F(L/IL))$ is an artinian graded module over $G = A[X_1,\dots,X_m]$. Hence by (1.3) there exists a polynomial $\pi(t)$ such that

$$\Sigma_q l(H_q(F(L/IL))t^q = (1-t^2)^{-m}\pi(t)$$

It remains to show that

$$\pi(-1) = \Sigma_q(-1)^q l(H_q(F(L))) \tag{9}$$

Consider the exact triangle (8). To simplify the notation, put

$$H := H.(F(Y^{S_i}))$$

$$\bar{H} := H.(F(Y^S))$$

and let $\chi_H(t)$ and $\chi_{\bar{H}}(t)$ be the corresponding Poincaré-series, cf. (1.1). In the proof of (ii) we have seen that $\bar{H}$ is an artinian graded G^S-module. Hence by (1.3) there exists a polynomial $\bar{g}(t)$ in $\mathbb{Z}[t]$ such that

$$\chi_{\bar{H}}(t) = (1-t^2)^{-s}\,\bar{g}(t) \tag{10}$$

s being the cardinality of S. Similarly there exists a polynomial $g(t)$ such that

$$\chi_H(t) = (1-t^2)^{-s+1}g(t) \tag{11}$$

To prove (9) it clearly suffices to prove the following

$$\bar{g}(-1) = g(-1) \tag{12}$$

Let U be the kernel of the homogeneous map $H \to \bar{H}$ in (8). Then for all p we obtain from (8) an exact sequence of A-modules

$$0 \longrightarrow U_p \hookrightarrow H_p \longrightarrow \bar{H}_p \longrightarrow \bar{H}_{p-2} \longrightarrow U_{p-1} \longrightarrow 0$$

Hence we have an exact sequence of positively graded modules and homogeneous maps of degree zero

$$0 \longrightarrow U \longrightarrow H \longrightarrow \bar{H} \longrightarrow \bar{H}(2) \longrightarrow U(1) \longrightarrow 0$$

Looking at the corresponding Poincaré-series we get

$$(1-t^2)\chi_{\bar{H}} = \chi_H - (1+t)\chi_U \tag{13}$$

As a submodule of H , U is an artinian graded module over G^{S_i}. Hence by (1.3) there exists a polynomial $u(t)$ such that

$$\chi_U(t) = (1-t^2)^{-s+1}u(t) \tag{14}$$

Multiplying both sides of (13) by $(1-t^2)^{s-1}$ and substituting (10), (11) and (14) we obtain

$$\bar{g}(t) = g(t) - (1+t)u(t)$$

which yields (12). ∎

PROOF OF (3.1). The theorem will be proved only in the case where F is covariant. The contravariant case can be proved similarly and will be left to the reader.

To each A-module M we select an acyclic DGA-module L over K , which is free as a graded K-module, and whose augmentation maps L onto M . This can be done in view of (2.4) and (2.5). Let I be the augmentation ideal in K . Since K is acyclic, it follows from (2.6) that L/IL is an A-free resolution of M . Hence

$$D_q\bar{F}(M) \approx H_q(\bar{F}(L/IL)) = H_q(F(L/IL))$$

By (3.2) $D.\bar{F}(M) = \coprod_q D_q\bar{F}(M)$ is a graded G-module.

To prove (i) let $\varphi : M \to M'$ be a homomorphism of A-modules. We are going to show that the induced map $D.\bar{F}(\varphi)$ is G-linear. Let L and L' be selected acyclic DGA-modules over K with augmentations η and η' onto M and M' respectively. Since L is K-free and L' is acyclic we have a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\Phi} & L' \\ \downarrow \eta & & \downarrow \eta \\ M & \xrightarrow{\varphi} & M' \end{array}$$

where Φ is a homomorphism of DG-modules over K .
Φ induces an A-homomorphism $\bar{\Phi} : L/IL \to L'/IL'$ and we have

$$D.\bar{F}(\varphi) = H.(\bar{\Phi}) .$$

Put $S := \{1,\dots,m\}$ and let $i \in S$. Using the notation in the proof of (3.2) we have

$$L/IL = Y^S$$

noreover we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y^S & \xrightarrow{T_i} & Y^{S_i} & \longrightarrow & Y^S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y^S & \xrightarrow{T_i} & Y^{S_i} & \longrightarrow & Y^S & \longrightarrow & 0 \end{array}$$

where the rows are split-exact (cf. (5) in the proof of (3.2)) and the vertical maps are induced by Φ . From this diagram we obtain a commutative diagram

$$\begin{array}{ccc} H.(F(Y^S)) & \xrightarrow{X_i} & H.(F(Y^S)) \\ \downarrow D.\bar{F}(\varphi) & & \downarrow D.\bar{F}(\varphi) \\ H.(F(Y^S)) & \xrightarrow{X_i} & H.(F(Y^S)) \end{array}$$

showing that $D.\bar{F}(\varphi)$ is G-linear. Since

$$D.F(M) = H.(F(L)) ,$$

(ii) - (iv) follows immediately from (ii) - (iv) in (3.2). ∎

§ 4 APPLICATIONS OF (3.1) TO POINCARÉ SERIES AND INTERSECTION MULTIPLICITY .

4.1 COROLLARY. Let R be a noetherian ring and let $\mathfrak{a}$ be an ideal in R which is generated by a regular sequence $x_1,\dots,x_m$. Put $A := R/\mathfrak{a}$. Let M and N be A-modules of finite type such that $l(M \otimes N) < \infty$.

(i) If $\mathrm{Ext}_R^p(M,N) = 0$ for p sufficiently large, then there exists a polynomial $f(t)$ in $\mathbb{Z}[t]$ such that

$$\Sigma_p l(\mathrm{Ext}_A^p(M,N))t^p = (1-t^2)^{-m}f(t)$$

(ii) If $\mathrm{Tor}_p^R(M,N) = 0$ for p sufficiently large, then there exists a polynomial $g(t)$ in $\mathbb{Z}[t]$ such that

$$\Sigma_p l(\mathrm{Tor}^A{}_p(M,N))t^p = (1-t^2)^{-m}g(t)$$

PROOF. The condition $l(M \otimes N) < \infty$ yields that $\mathrm{Tor}_p^R(M,N)$ and $\mathrm{Ext}_R^p(M,N)$ as well as $\mathrm{Tor}_p^A(M,N)$ and $\mathrm{Ext}_A^p(M,N)$ have finite

length for all p.
Since $x_1,\ldots,x_m$ is a regular sequence, the Koszul complex $K^R(x_1,\ldots,x_m)$ is acyclic. Now everything follows immediately from (3.1). ∎

Recall that a <u>local complete intersection</u> is a local ring whose completion is the quotient of a regular local ring modulo a regular sequence.

4.2 COROLLARY. Let M and N be modules of finite type over a local complete intersection $A, \mathfrak{m}$. Assume that $M \otimes_A N$ has finite length. Then there exists a polynomial $\pi^A_{M,N}(t)$ in $\mathbb{Z}[t]$ only depending on A, M and N such that

(i) $\Sigma_p l(\mathrm{Tor}^A_p(M,N))t^p = (1-t^2)^{-m}\pi^A_{M,N}(t)$

where $m = \dim \mathfrak{m}/\mathfrak{m}^2 - \dim A$.

(ii) $\pi^{\hat{A}}_{\hat{M},\hat{N}}(t) = \pi^A_{M,N}(t)$,

^ denoting $\mathfrak{m}$-adic completion

(iii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then

$$\pi^A_{M,N}(-1) = \pi^A_{M',N}(-1) + \pi^A_{M'',N}(-1)$$

(iv) If A is a homomorphic image of a regular local ring R of the same imbedding dimension as A, then considering M and N as R-modules we have

$$\pi^A_{M,N}(-1) = \chi^R(M,N)$$

where

$$\chi^R(M,N) = \Sigma_p(-1)^p l(\mathrm{Tor}^R_p(M,N)) .$$

PROOF. $\hat{M} \otimes_{\hat{A}} \hat{N}$ has finite length over $\hat{A}$, moreover we have

$$\mathrm{Tor}_p^{\hat{A}}(\hat{M},\hat{N}) = \mathrm{Tor}_p^{A}(M,N) \quad \text{for all } p \tag{1}$$

and

$$\dim \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2 - \dim \hat{A} = \dim \mathfrak{m}/\mathfrak{m}^2 - \dim A$$

Hence (ii) follows from (i). To prove (i) it suffices to assume that $A = \hat{A}$. Hence we may assume that A has the form $A = R/\mathfrak{a}$ where R is a regular local ring and $\mathfrak{a}$ is generated by an R-sequence $x_1,\ldots,x_m$,which may be chosen in the square of the maximal ideal $\tilde{\mathfrak{m}}$ in R . In that case we have

$$\begin{aligned} m = \dim R - \dim A &= \dim \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2 - \dim A \\ &= \dim \mathfrak{m}/\mathfrak{m}^2 - \dim A \end{aligned}$$

Now (i) follows from (ii) in (4.1). We will now prove (iv). Let A be of the form $A \approx R/\mathfrak{a}$ where R is regular of the same embedding dimension as A . Let $x_1,\ldots,x_m$ be a minimal set of generators for $\mathfrak{a}$. Then $x_1,\ldots,x_m$ is an R-sequence and we have

$$m = \dim \mathfrak{m}/\mathfrak{m}^2 - \dim A$$

It follows from (iv) in 3.1) that

$$\begin{aligned} \pi^A_{M,N}(-1) &= \Sigma_q(-1)^q l(\mathrm{Tor}_q^R(M,N)) \\ &= \chi^R(M,N) \ . \end{aligned}$$

To prove (iii) we just have to pass to the completion and apply (ii) and (iv) and use the additivity of $\chi^R(-,\hat{N})$. ∎

4.3 EXAMPLE. Let A be a local complete intersection with residue field k and imbedding dimension n . It follows from theorem 6 in [Ta] that $\pi^A_{k,k}(t) = (1+t)^n$.

4.4 REMARK. Let $A, \mathfrak{m}$ be a local (noetherian) ring, and let M and N be A-modules of finite type such that $M \otimes_A N$ has finite length. Let $R \to \hat{A}$ be any minimal surjective ring homomorphism from a regular local ring R onto the completion of A , minimal meaning that R and $\hat{A}$ have the same imbedding dimension. Under this assumption it can be shown that the intersection multiplicity $\chi^R(\hat{M},\hat{N})$ is an integer depending only of A, M and N . A reasonable notation for this would be $\chi^A(M,N)$. Clearly this generalizes the Serre intersection multiplicity to arbitrary local (noetherian) rings. (4.2) then shows that the "intersection multiplicity" $\chi^A(M,N)$ over a complete intersection A can be expressed intrinsicly in terms of the Poincaré series of $\mathrm{Tor}^A_\bullet(M,N)$ without reference to an "ambrient space".

4.5 COROLLARY. Let $A, \mathfrak{m}$ be a local complete intersection and let $A^* = A \ltimes M$ be the trivial extension of A by a finitely generated A-module M . Let k be the residue field of A^* . Then the powerseries

$$\Sigma_p l(\mathrm{Tor}_p^{A^*}(k,k))t^p$$

represents a rational runction.

PROOF: Put $n = \dim \mathfrak{m}/\mathfrak{m}^2$, $m = n - \dim A$. k may be identified with the residue field of A . By (4.3) we have

$$\Sigma_p l(\mathrm{Tor}_p^A(k,k))t^p = (1-t^2)^{-m}(1+t)^n$$

By (4.2) there exists a polynomial $\pi^A_{M,k}(t)$ such that

$$\Sigma_p l(\mathrm{Tor}_p^A(M,k))t^p = (1-t^2)^{-m}\pi^A_{M,k}(t) .$$

It follows from Theorem 2 in [Gu] that

$$\Sigma_p l(\mathrm{Tor}_p^{A^*}(\mathbf{k},k))t^p = [(1-t^2)^m - t\pi_{M,k}^A(t)]^{-1}(1+t)^n \; .$$

■

REFERENCES

[C.E] Cartan, H., Eilenberg, S.: Homological Algebra. Princeton University Press 1956.

[Gu] Gulliksen, T.H.: Massey Operations and the Poincaré Series of Certain Local Rings. J.of Algebra 22 (1972).

[Se] Serre, J.P.: Algèbre Locale Multiplicités. Springer Lecture Notes (1965).

[Ta] Tate, J.: Homology of Noetherian rings and local rings. Illinois J. of Math. 1 (1957), 14-25.