TRANSFORMATION GROUPS
ON COHOMOLOGY PRODUCT OF SPHERES

by

Per Tomter

Introduction.

After the work of P.A. Smith on prime periodic transformations on acyclic spaces and homology spheres, much work in this direction has been devoted to generalizations to spaces whose cohomology ring is a truncated polynomial algebra or isomorphic to the cohomology ring of a product of two spheres. For a general reference see Bredon (3). The latter case is quite rich in possibilities for the structure of the fixed point set, but it has not been known whether all these possibilities can actually occur. In this paper a G-manifold is constructed which helps to fill out the gap between theory and examples.

We adopt this notation: p is a prime or zero, G_p is the cyclic group of order p if $p \neq 0$; $G_0 = S^1$ is the circle group. K_p is the prime field of characteristic p . X is always a paracompact topological space. We use sheaf-theoretic cohomology, and assume $\dim_{K_p} X < \infty$. $X \sim_p Y$ means that $H^*(X;K_p)$ is isomorphic to $H^*(Y;K_p)$ as a ring. If the group G acts on X , $X_G = E_G \times_G X$ is the bundle associated to a universal bundle $E_G \to B_G$. $X \sim_p P^r(q)$ means $H^*(X,K_p) \cong {}^{K_p}/a^{r+1}$, where degree $a = q$.

Let $X \sim_p S^m \times S^n$, and let G_p act on X with fixed point set F. For $p = 2$ there is in Bredon (4) a list of all possible fixed point sets, with examples in each case. For $p \neq 2$ and X totally non-homologous to zero in X_G, there are the following possibilities (Bredon (4)):

(1) $F \sim_p S^q \times S^r$

(2) $F \sim_p P^3(q)$

(3) $F \sim_p pt + P^2(q)$ (disjoint union).

(4) $F \sim_p S^q + S^r$ (q and/or r may be zero).

In addition, there are a few extra possibilities when X is not totally non-homologous to zero, which are all known to occur for $p = 3$. Linear actions on spheres give examples of (1) and of (4) with $q = r$. A known example of (4) with $q = 0$, $r = 2$, is constructed by considering a linear action on $CP(2)$ with $F = pt + S^2$. Let X be the connected sum $CP(2) \# - CP(2) \sim_p S^2 \times S^2$, taken at a point of the S^2-component of F. There is then an action on X with $F = S^2 \# - S^2 + pt + pt = S^2 + pt + pt$. In Theorem 2 in section 1 we also give an exapmle of (4) with q and r different, but both non-zero.

In (9) Su gave an example of (2) for the case $p = 2$. It is easy to generalize this to arbitrary p: Let $S^3 \to S^7 \to S^4$ be the Hopf bundle, and $S^2 \to CP(3) \overset{\pi}{\to} S^4$ the bundle obtained by taking the quotient of S^7 by S^1. Let ξ be the corresponding 3-plane bundle; i.e. π is the unit sphere bundle of ξ. Let η be any (m-2)-plane bundle over S^4 such that $\xi \oplus \eta$ is trivial. Then the unit sphere bundle $S(\xi \oplus \eta)$ has total space $S^4 \times S^m$. $\widetilde{KU}(S^4) \cong \pi_3(U)$, $\widetilde{KO}(S^4) \cong \pi_3(O)$; and in the sequence

$\pi_3(U) \to \pi_3(O) \to \pi_3({}^{O}/U)$ the first map is surjective. There is then a bundle with complex structure in the stable equivalence class of η ; let S^1 act by complex multiplication on this bundle and trivially on ξ . The action of S^1 on the unit sphere bundle has fixed point set $CP(3)$; similarly for the cyclic subgroup of prime period p .

In this paper we give an example of (3) by constructing a manifold $X \sim_p S^4 \times S^4$ for $p \neq 2$, and an action of $G_o = S^1$ on X with $F = CP(2) + pt$. Restricting the action to the cyclic group of prime period p , we also get an action of G_p on X with the same fixed point set. The only known examples of such a phenomenon has been for $p = 2$: the involution on $SU(3) \sim_2 S^3 \times S^5$ given by matrix inversion, and the canonical involution on the symmetric space ${}^{SU(3)}/SO(3) \sim_2 S^2 \times S^3$. By work of Bredon it follows that an example of the type constructed here cannot occur for an actual product of spheres.

We also discuss the relationship of this example with the work of Wu-Yi Hsiang. In (6) Hsiang proved a general theorem, which shows that the ideal of relations between a set of generators for $H^*(X_{G_o}, K_o)$ has a finite number of zeroes in one-to-one correspondence with the connected components of F . If $X \sim_o S^m \times S^n$; m,n even, this ideal can be generated by two parabolas. In section 2 we compute explicitly the equivariant cohomology of the example in section 1, and show that in addition to the possibilities listed in (6), those parabolas may have one transversal intersection point and another point of tangency with intersection number three.

1. Let Q denote the quaternions with the usual basis $(1,i,j,k)$, so $Q = C + jC$. $Q(n+1)$ is the $(n+1)$-dimensional right vector space over Q, and the quotient under the right action of Q is the quaternion projective n-space, $QP(n)$. In (8) Hsiang and Su study torus actions on $QP(n)$, we need some modifications of this. Let G be a torus group and $\{\omega_i\}_{i=1,\dots,s}$ a set of distinct weight vectors for G. If $Q(n+1) = Q(k_1+1) \oplus \dots \oplus Q(k_s+1)$, $\sum_{i=1}^{s} (k_i+1) = n+1$, the left action of G given by $g \cdot (x_1,\dots,x_s)$ $= (\exp(2\pi i\langle\omega_1,g\rangle)x_1,\dots,\exp(2\pi i\langle\omega_s,g\rangle)x_s)$, $x_i \in Q(k_i+1)$, induces a "linear" action on $QP(n)$. If $\omega_i = 0$, $QP(k_i)$ is one component of the fixed point set. If $\omega_i \neq 0$, it is clear that on $C(k_i+1)$ and $jC(k_i+1)$ G acts through right multiplication by $\exp(2\pi i\langle\omega_i,g\rangle)$ and $\exp(-2\pi i\langle\omega_i,g\rangle)$ respectively. It follows that up to equivariant complex automorphism we may assume $\omega_i \neq -\omega_j$ for $i \neq j$; and in this case $CP(k_i)$ is the component of the fixed point set corresponding to ω_i. Let p_i be the line through $(0,0,\dots,1_i,\dots,0)$ in $Q(k_1+1) \oplus \dots \oplus Q(k_s+1)$, where 1_i represents the point $(1,0,\dots,0)$ in $Q(k_i+1)$; then p_i is a fixed point in $QP(n)$.

Proposition 1. The local representation of G around p_i in $QP(n)$ is given by the following weights with multiplicities: $\{(\pm\omega_j-\omega_i;k_j+1)$ for $j \neq i$, $(0;k_i)$, $(-2\omega_i,k_i)\}$.

(Notice that the local representation around p_i is here identified as a complex representation; this is of some importance since we need to keep track of orientations.)

Proof. This is an easy exercise using the usual projective coordinates around p_i. The linear action on the tangent space is given by $g \cdot (x_1,\dots,x_s) = (\exp(2\pi i\langle\omega_1,g\rangle x_1 \exp(-2\pi i\langle\omega_i,g\rangle),\dots$ $\dots,\exp(2\pi i\langle\omega_s,g\rangle x_s \exp(-2\pi i\langle\omega_i,g\rangle))$ where $x_j \in Q(k_j+1)$ for $j \neq i$, $x_i \in Q(k_i)$. Restricting to the subspaces C^n and jC^n we get the weight systems $\{(\omega_j-\omega_i;k_j+1),\ j\neq i,\ (0;k_i)\}$ and $\{(-\omega_j-\omega_i;k_j+1),\ j\neq i,\ (-2\omega_i;k_i)\}$ respectively.

Now, let $G_o = U(1)$ be the circle group, and let ω be the weight vector of the standard representation. Consider the following linear actions of G_o on $QP(2)$:

1) Weight system $\{\omega\}$, $k_1 = 2$. The fixed point set $F_1 = CP(2)$ and the local representation has weight system $(0;2),(-2\omega;2)$.

2) Weight system $\{0,2\omega\}$, $k_1 = 1$, $k_2 = 0$. The fixed point set $F_2 = QP(1) + pt = S^4 + pt = F_2^1 + F_2^2$; and the local representation around a fixed point in S^4 has weight system $(0;2)$, $(2\omega;1)$, $(-2\omega;1)$.

Proposition 2. Let p_1 and p_2 be points in F_1 and F_2^1 respectively. Then there are disc neigbourhoods V_1 of p_1 and V_2 of p_2 in $QP(2)$ and a G_o-equivariant diffeomorphism f from V_1 to V_2 preserving orientation.

Proof. It is sufficient to check for the linear action of G_o aroud p_1 and p_2. The one-dimensional complex representations of G_o corresponding to the weights 2ω and -2ω are equivalent under an orientation-reversing isomorphism. Combine this with a reflection in the subspace corresponding to the zero-weight.

q.e.d.

Theorem 1.

There exists a manifold X such that $X \sim_p S^4 \times S^4$ for $p \neq 2$ and an action of G_p on X such that the fixed point set

$F = CP(2) + pt = F^1 + F^2$, (p any prime or zero).

Proof. Let X_1 and X_2 be copies of $QP(2)$ with the above G_o-actions 1) and 2) respectively. By switching the orientation of X_2 , the above diffeomorphism f becomes orientation-reversing. Let X be the equivariant connected sum $X_1 \#_f (-X_2)$. We have the quaternion line bundles $Q(3) = \bar{X}_1 \to X_1$ and $Q(3) = \bar{X}_2 \to X_2$, let e_1 and e_2 be their respective symplectic characteristic classes, which generate $H^*(X_1, \mathbb{Z})$ and $H^*(X_2, \mathbb{Z})$. By a standard argument with Mayer-Vietoris sequences $H^4(X,\mathbb{Z}) \cong H^4(X_1,\mathbb{Z}) \oplus H^4(X_2,\mathbb{Z})$; i.e. there are corresponding elements ξ_1 and ξ_2 in $H^4(X,\mathbb{Z})$, and $\langle \xi_1^2, [X] \rangle = \langle e_1^2, [X_1] \rangle = 1$, $\langle \xi_2^2, [X] \rangle = \langle e_2^2 [-X_2] \rangle = -1$. So $\xi_1^2 = -\xi_2^2$ is the fundamental cohomology class of X , and $H^*(X,\mathbb{Z})$ is the algebra generated by ξ_1, ξ_2 with the relations $\xi_1^2 = -\xi_2^2$, $\xi_1 \xi_2 = 0$. With $x = \xi_1 + \xi_2$, $y = \xi_1 - \xi_2$ we have $x^2 = y^2 = 0$, $xy \neq 0$; and it is clear that $H^*(X;K_p) \cong H^*(S^4 \times S^4, K_p)$ for $p \neq 2$.
By restricting f to $V_1 \cap F_1$ and $V_2 \cap F_2^1$, we get an orientation-reversing equivariant diffeomorphism h (here we keep the natural orientations on F_1 and F_2^1). Hence $F = F_1 \#_h F_2 = CP(2) \#_h S^4 + pt = CP(2) + pt$.

Theorem 2.

There exists a manifold Y such that $Y \sim_p S^4 \times S^4$ for $p \neq 2$ and an action of G_p on Y such that the fixed point set $F^1 = S^4 + S^2$, (p any prime or zero).

Proof. Consider two actions on $QP(2)$:

1) Weights $\{0, 2\omega\}$, $k_1 = 1$, $k_2 = 0$. $F_1 = QP(1) + pt = S^4 + pt$. The local representation around the isolated fixed point has weight system $(-2\omega; 4)$. (Use Prop. 1.)

2) Weights $\{2\omega,0\}$, $k_1 = 1$, $k_2 = 0$. $F_2 = pt + CP(1) = pt + S^2$. The local representation around the isolated fixed point has weight system $(2\omega;2)$, $(-2\omega;2)$.

So the two local representations are equivalent under an orientation-preserving isomorphism. Taking equivariant connected sum at the isolated fixed points and proceeding as in Theorem 1, we obtain: $Y \sim_p S^4 \times S^4$ for $p \neq 2$ and $F^1 = S^4 + S^2$. q.e.d.

By restricting the bundle $Q(3) \to QP(2)$ to $F_1 = CP(2)$ this splits as a complex bundle into the Whitney sum of the two complex line bundles $C \oplus C \oplus C \to CP(2)$ and $Cj \oplus Cj \oplus Cj \to CP(2)$. The first is the Hopf bundle, whose Chern class c generates $H^*(F_1;\mathbb{Z})$; since right multiplication by j is antilinear, the second is the dual of the Hopf bundle. If j_1 is the inclusion of F_1 in X_1, then determining $j_1^*(e_1)$ amounts to finding the symplectic characteristic class of this restricted bundle. Since its second Chern class is $-c^2$, it follows (from Borel-Hirzebruch (2), p.488) that $j_1^*(e_1) = c^2$. Since $H^2(F;\mathbb{Z}) \cong H^2(F_1;\mathbb{Z}) \oplus H^2(F_2,\mathbb{Z}) = H^2(F_1;\mathbb{Z})$, c corresponds uniquely to an element ζ of $H^2(F;\mathbb{Z})$, and ζ^2 is the fundamentsl cohomology class of F^1. In the next section, we use the following observations (from now on we need only cohomology with rational coefficients).

Proposition 3. Let j be the inclusion of F in X.
Then $j^*(\xi_1) = j^*(\xi_2) = \zeta^2$.

Proof. We may identify V_1 and V_2 (via f) with an eight-disc D^8, the boundaries with S^7, $D^8 \cap F_1$ with D^4 with boundary S^3. Let $X_i^o = \overline{X_i - D^8}$, $F_i^o = \overline{F_i - D^4}$, $i = 1,2$, From the Mayer-Vietoris sequence of the pairs (X_1^o,S^7) and (D^8,S^7) it follows that

$H^4(X_1,S^7) \cong H^4(X_1^o,S^7) \oplus H^4(D^8,S^7) = H^4(X_1^o,S^7)$, similarly $H^4(F_1,S^3) \cong H^4(F_1^o,S^3) \oplus H^4(D^4,S^3)$. Also, from the long exact sequence of the pair (X_1,S^7) : $H^4(X_1,S^7) \cong H^4(X_1)$, and we have the commutative diagram

$$\begin{array}{ccccccc} H^4(X_1) & \cong & H^4(X_1,S^7) & \cong & H^4(X_1^o,S^7) & \oplus & H^4(D^8,S^7) \\ j_1^*\downarrow & & j_1^*\downarrow & & j_1^*\downarrow & & \downarrow \\ H^4(F_1) & \leftarrow & H^4(F_1,S^3) & \cong & H^4(F_1^o,S^3) & \oplus & H^4(D^4,S^3) \end{array}$$

Thus e_1 corresponds uniquely to an element ξ_1' in $H^4(X_1^o,S^7)$, and $j_1^*(\xi_1')$ is the orientation class of $H^4(F_1^o,S^3)$. By the Mayer-Vietoris sequence for pairs we also have the commutative diagram:

$$\begin{array}{ccccc} H^4(X,S^7) & \cong & H^4(X_1^o,S^7) & \oplus & H^4(X_2^o,S^7) \\ \downarrow & & \downarrow & & \downarrow \\ H^4(F,S^3) & \cong & H^4(F_1^o,S^3) & \oplus & H^4(F_2^o,S^3) \end{array}$$

Combining with the commutative diagram:

$$\begin{array}{ccccc} 0 & \rightarrow & H^4(X,S^7) & \rightarrow & H^4(X) \\ \downarrow & & \downarrow & & \downarrow \\ H^3(S^3) & \rightarrow & H^4(F,S^3) & \rightarrow & H^4(F) \end{array}$$

it follows that $j^*(\xi_1)$ is the fundamental cohomology class of F^1. For ξ_2 the argument is similar, but easier; it is obvious that $j_2^*(e_2)$ is the symplectic characteristic class of the Hopf bundle over $F_2^1 = QP(1)$.

q.e.d.

2. In this section we assume $p = 0$; i.e. $G = S^1$ and all cohomology is understood with rational coefficients. The equivariant cohomology of X is $H^*(X_G)$, we study the structure of this as an algebra over $H^*(B_G)$. We may assume that the odd-dimensional cohomology of X vanishes, it is then clear from the Leray spectral sequence that X is totally non-homologous to zero in the fibre bundle $X_G \to B_G$. It follows by an application of the Vietoris-Begle theorem (see Borel (1), p.53) that the homomorphism j_G^* in the long exact sequence $\to H^*(X_G) \overset{j_G^*}{\to} H^*(F_G) \to$ $\to H_c^*((X-F)_G) \to \cdots$ is a monomorphism with finite-dimensional cokernel. Let R be the quotient field of $H^*(B_G)$ and A the localization of $H^*(X_G)$ as an $H^*(B_G)$-module at the zero ideal. The theorem of Wu-Yi Hsiang may now be stated as follows:

Theorem 3 (Hsiang (6,7)).

Let $x_1,\ldots,x_r$ be a system of generators for the R-algebra A and I the ideal of defining relations; i.e. $0 \to I \to R[x_1,\ldots,x_r]$ $\to A \to 0$ is an exact sequence of R-modules. Then:

(i) The radical of I decomposes into the intersection of s maximal ideals $M_j = M(P_j)$., whose variety is the point $P_j \in R^r$.

(ii) There is a one-to-one correspondence between the connected components of the fixed point set $F = F' + \ldots + F^s$ and the above s maximal ideals such that $H^*(F^j) \oplus R \cong A/I_j$, where $I_j = A \cap I_{P_j}$ (localization of I at P_j).

(iii) $I = I_1 \cap \ldots \cap I_s = I_1 \ldots I_s$. $I_j + I_1 \ldots I_{j-1} \cdot I_{j+1} \ldots I_s = 1$.

Remark. The algebraic set $V(I)$ consists of the points P_i , $i = 1,\ldots,s$ and $P_i = (a_i^1,\ldots,a_i^r)$ is determined by $j_i^*(x_k) =$ $b_i^k + a_i^k$ where j_i is the inclusion of F_G^i in X_G , $a_i^k \in H^*(B_G)$

and $b_i^k \in \tilde{H}^*(F^i) \oplus H^*(B_G)$. Also $\ker j_i^* = I_j$.

Since I is an ideal with a finite number of zeroes, there is a standard isomorphism J of $R[x_1,\dots,x_r]/I$ with $\prod_{i=1}^{s} \mathcal{O}_i/I\mathcal{O}_i$. See Fulton (5), p.55. Here $\mathcal{O}_i$ is the local ring of R^r at P_i . The kernel of J composed with the projection on the i-th factor is $I_{P_i} \cap R[x_1,\dots,x_r]/I = I_i/I$, so $\mathcal{O}_i/I\mathcal{O}_i \cong R[x_1,\dots,x_r]/I_i \cong H^*(F^i) \otimes R$. In one case, when $X \sim_0 S^m \times S^n$; m, n even, it is easily seen that for a suitable choice of generators x_1, x_2 we have $A = R[x_1,x_2]/I$ with I generated by two parabolas $x_1^2 = c_1x_2 + d_1$, $x_2^2 = c_2x_1 + d_2$; $c_i, d_i \in H^*(B_G)$, $i = 1,2$. (Hsiang (6)). In this case the intersection number of these parabolas at P_i is given by $\dim_R(\mathcal{O}_i/I\mathcal{O}_i)$ (Fulton (5)), which is $\dim H^*(F^i)$ = the Euler characteristic of F^i . Hence we have:

Proposition 4. The localization of $j_G^* : H^*(X_G) \to H^*(F_G) = \prod_{i=1}^{s} H^*(F^i) \otimes H^*(B_G)$ at the zero ideal of $H^*(B_G)$ is the standard isomorphism J . In particular, if $X \sim_0 S^m \times S^n$; m, n even, the Euler characteristic of the component F^i of the fixed point set F equals the intersection number at the corresponding intersection point P_i of the above parabolas.

Now, return to the example of Theorem 1. Via the transgression in the universal bundle $G \to E_G \to B_G$ we may identify the weight $\omega \in H^1(G)$ with a unique element in $H^2(B_G)$ (also denoted by ω). We have the inclusion maps: $j_{1G} : F_{1G} \to X_{1G}$, $j_{2G}^1 : F_{2G}^1 \to X_{2G}$, $j_{2G}^2 : F_{2G}^2 \to X_{2G}$, $j_G : F_G \to X_G$. Also, the inclusion of the fibres in the total spaces: $i_1 : X_1 \to X_{1G}$, $i_2 : X_2 \to X_{2G}$, $i : X \to X_G$. Let $\bar{e}_1$ and $\bar{e}_2$ be the symplectic characteristic classes of the quaternion line bundles $\bar{\alpha}_1 : \bar{X}_{1G} \to X_{1G}$ and $\bar{\alpha}_2 : \bar{X}_{2G} \to X_{2G}$ respectively; then $i_1^*(\bar{e}_1) = e_1$, $i_2^*(\bar{e}_2) = e_2$.

Proposition 5. $j_{1G}^*(\bar{e}_1) = c^2 + 2c \otimes \omega + \omega^2$. $j_{2G}^{1*}(\bar{e}_2) = \eta$

$j_{2G}^{2*}(\bar{e}_2) = 4\omega^2$,

(Here η denotes the symplectic characteristic class of $F_2^1 = QP(1)$.)

Proof. This amounts to studying the restrictions of these bundles; e.g. $\bar{\alpha}_2 \mid F_{2G}^1$ is the tensor product of the quaternion Hopf bundle over $F_2^1 = QP(1)$ with the trivial line bundle over B_G , so $j_{2G}^{1*}(\bar{e}_2) = \eta$. $\bar{\alpha}_2 \mid F_{2G}^2$ corresponds to the bundle $E_G \times_G Q \to B_G$ where the G-action on Q is given by the weight 2ω . As a complex bundle this decomposes as $E_G \times_G (C+jC) \to B_G$, and the complex weights are $\pm 2\omega$. So the second Chern class is $-4\omega^2$, the first symplectic characteristic class is $4\omega^2$, and $j_{2G}^{2*}(\bar{e}_2) = 4\omega^2$. Similarly, $\bar{\alpha}_1 \mid F_{1G}$ as a complex bundle is the Whitney sum of the bundles given by (i) $E_G \times_G (C+C+C) \to B_G \times F_1 = F_{1G}$ and (ii) $E_G \times_G (Cj+Cj+Cj) \to F_{1G}$. (i) is the tensor product of the Hopf bundle on $F_1 = CP(2)$ and the bundle $E_G \times_G C \to B_G$ given by the complex weight ω on G , (ii) is the tensor product of the dual of the Hopf bundle with the corresponding bundle given by the weight $-\omega$. The second Chern class of $\bar{\alpha}_1 \mid F_{1G}$ is $(c+\omega)(-c-\omega)$, the first symplectic characteristic class is $(c+\omega)^2$, and $j_{1G}^*(e_1) = c^2 + 2c \otimes \omega + \omega^2$.

q.e.d.

We have: $X = X_1^o \cup X_2^o$, $X_i = X_i^o \cup D^8$, $X_i \cap D^8 = X_1^o \cap X_2^o = S^7$.
$F^1 + F^2 = F = F_1^o \cup F_2^o$, $F_i = F_i^o \cup D^4$, $F_i^o \cap D^4 = F_1^o \cap F_2^o = S^3, i = 1,2.$

From the Mayer-Vietoris sequence for equivariant cohomology we have: $H^p(S^7_G) \to H^{p+1}(X_G) \to H^{p+1}(X^o_{1G}) \oplus H^{p+1}(X^o_{2G}) \xrightarrow{T} H^{p+1}(S^7_G)$

and : $H^p(S^7_G) \to H^{p+1}(X_{iG}) \to H^{p+1}(X^o_{iG}) \oplus H^{p+1}(D^8_G) \xrightarrow{T_i} H^{p+1}(S^7_G) \to$,

$i = 1,2$, where the homomorphisms are $H^*(B_G)$-maps (up to sign). Then $H^4(X_{iG}) \cong \ker T_i \cong H^4(X^o_{iG})$ and $H^4(X_G) \cong \ker T$. The $\bar{e}_i$'s determine unique elements $\bar{\xi}_i$ in $H^4(X_G)$, $i = 1,2$, Let 1_i be the generator of $H^o(F^i)$, $i = 1,2$.

Proposition 6. $j^*_G(\bar{\xi}_1) = \zeta^2 + 2\zeta \otimes \omega + 1_1 \otimes \omega^2 + 1_2 \otimes \omega^2$.

$j^*_G(\xi_2) = \zeta^2 + 1_2 \otimes 4\omega^2$.

Proof.

$$\begin{array}{ccccccc}
H^p(S^7_G) & \to & H^{p+1}(X_G) & \to & H^{p+1}(X^o_{1G}) \oplus H^{p+1}(X^o_{2G}) & \to \\
\downarrow & & \downarrow & & \downarrow \qquad\qquad \downarrow & \\
H^p(S^7) & \to & H^{p+1}(X) & \to & H^{p+1}(X^o_1) \oplus H^{p+1}(X^o_2) & \to
\end{array}$$

From commutative diagrams it follows that $i^*(\bar{\xi}_1) = \xi_1$, $i^*(\bar{\xi}_2) = \xi_2$. Clearly we can write $j^*(\bar{\xi}_1) = \alpha_1\zeta^2 + \zeta \otimes \alpha_2 + 1_1 \otimes \alpha_3 + 1_2 \otimes \alpha_4$, $\alpha_i = H^*(B_G)$, $i = 1,\dots,4$. From Proposition 3 and the commutative diagram

$$\begin{array}{ccc}
H^*(X_G) & \to & H^*(X) \\
\downarrow & & \downarrow \\
H^*(F_G) & \to & H^*(F)
\end{array}$$

it follows that $\alpha_1 = 1$.

We have the commutative diagrams:

$$\begin{array}{ccccccccc}
H^3(S^7_G) & \to & H^4(X_G) & \xrightarrow{V} & H^4(X^o_{1G}) \oplus H^4(X^o_{2G}) & \xrightarrow{T} & H^4(S^7_G) & \to \\
\downarrow & & j^*_G\downarrow & & \downarrow \qquad\qquad \downarrow & & \downarrow & \\
H^3(S^3_G) & \to & H^4(F_G) & \xrightarrow{V_F} & H^4(F^o_{1G}) \oplus H^4(F^o_{2G}) & \xrightarrow{T_F} & H^4(S^3_G) & \to
\end{array}$$

and

$$\begin{array}{ccccccccc}
H^3(S^7_G) & \to & H^4(X_{1G}) & \overset{V_1}{\to} & H^4(X^o_{1G}) \oplus H^4(D^8_G) & \to & H^4(S^7_G) & \to \\
\downarrow & & \downarrow & & \downarrow \qquad\quad \downarrow & & \downarrow & \\
H^3(S^3_G) & \to & H^4(F_{1G}) & \overset{V_{1F}}{\to} & H^4(F^o_{1G}) \oplus H^4(D^4_G) & \to & H^4(S^3_G) & \to
\end{array}$$

Combining with Proposition 5 it follows that the image of $V_1(\bar{e}_1)$ in $H^4(F^o_{1G})$ may be identified with $c \otimes 2\omega + \omega^2$. Since $V(\bar{\xi}_1) \in \ker T$, the image in $H^4(F^o_{1G}) \oplus H^4(F^o_{2G})$ must be in $\ker T_F$, and must then have a component $\omega^2 \oplus \omega^2$ in $H^4(F^o_{2G}) = H^4((F^1_2)^o_G) \oplus H^4(F^2_{2G})$. It follows that $\alpha_2 = 2\omega$, $\alpha_3 = \alpha_4 = \omega^2$. Similar proof for $\bar{\xi}_2$.

q.e.d.

Theorem 4.

The equivariant cohomology of X is the quotient of the polynomial algebra $H^*(B_G)[\bar{x},\bar{y}]$ by the ideal generated by the two parabolas $\bar{x}^2 = -2\omega^2\bar{y} + 3\omega^4$, $\bar{y}^2 = 2\omega^2\bar{x} + 3\omega^4$. These parabolas have one intersection point $(-\omega^2,\omega^2)$ with intersection number three and another transversal intersection point $(3\omega^2,-3\omega^2)$, corresponding to the components $F^1 = CP(2)$ and $F^2 = pt$ of the fixed point set respectively.

Proof. From the Leray-Hirsch theorem it follows immediately that $H^*(X_G)$ is generated by $\bar{\xi}_1$ and $\bar{\xi}_2$ as an $H^*(B_G)$-algebra. Change the generators to $\bar{\eta}_1 = \bar{\xi}_1 - \omega^2$, $\bar{\eta}_2 = \bar{\xi}_2 - 2\omega^2$. Then $j_G^*(\bar{\eta}_1^2) = 4\zeta^2 \otimes \omega^2$, $j_G^*(\bar{\eta}^2{}_2) = -4\zeta^2 \otimes \omega^2 + 1_1 \otimes 4\omega^4 + 1_2 \otimes 4\omega^4$, and by the injectivity of j_G^* we have:

(1) $\bar{\eta}_2^2 + \bar{\eta}_1^2 = 4\omega^4$.

$j_G^*(\bar{\eta}_1,\bar{\eta}_2) = -\zeta^2 \otimes 2\omega^2 - \zeta \otimes 4\omega^3$, and hence

(2) $\bar{\eta}_1(\bar{\eta}_2+2\omega^2) = 0$.

To normalize to the parabolas as in Hsiang (6), let $\bar{x} = \bar{\eta}_1+\bar{\eta}_2+\omega^2$, $\bar{y} = \bar{\eta}_1 - \bar{\eta}_2\omega^2$. The relations are then (a) $\bar{x}^2 + 2\omega^2\bar{y} - 3\omega^4 = 0$ and (b) $\bar{y}^2 - 2\omega^2\bar{x} - 3\omega^4 = 0$, two parabolas with intersection points $P_1 = (-\omega^2,\omega^2)$ and $P_2 = (3\omega^2,-3\omega^2)$. According to the remarks after Theorem 3 these are easily seen to correspond to the components F^1 and F^2 respectively. P_2 is a transversal intersection point with intersection number one. The ideal I spanned by the polynomials in (a) and (b) contains the polynomial $\bar{x}^4 - 6\omega^4\bar{x} - 8\omega^6\bar{x} - 3\omega^8 = (\bar{x}+\omega^2)^3(\bar{x}-3\omega^2)$. So $(\bar{x}+\omega^2)^3 \in I\Theta_1$, and the dimension of the local ring Θ_1 over $I\Theta_1$ is three; i.e. the intersection number of P_1 is three.

References.

1. A. Borel et al. "Seminar on Transformation Groups". Ann. of Math. Studies 46, Princeton University Press.

2. A. Borel and F. Hirzebruch. "Characteristic classes and homogeneous spaces, I". Amer.J.Math. 80 (1958), 458-538.

3. G.E. Bredon. "Cohomological aspects of transformation groups" Proc. Conf. Transformation Groups. New Orleans 1967. pp. 245-280. Springer Verlag, 1968.

4. G.E. Bredon. "Introduction to Compact Transformation Groups". Academic Press, 1972.

5. W. Fulton. "Algebraic Curves". Benjamin, 1969.

6. W.Y. Hsiang. "On some fundamental theorems in cohomology theory of topological transformation groups". Taita (Nat. Taiwan Univ.) J.Math. 2 (1970), 61-87.

7. W.Y. Hsiang. "Some fundamental theorems in cohomology theory of topological transformation groups". Bull. Amer. Math. Soc. No.6, Nov. 1971, 1094-1098.

8. W.Y. Hsiang and J.C. Su. "On the geometric weight system of topological actions on cohomology quaternionic projective spaces I", (preprint).

9. J.C. Su. "An example". Proc. Conf. Transformation Groups, New Orleans, 1967, p. 351.