ON AXIOMATIZING RECURSION THEORY

Generalized recursion theory can be many different things. Starting from ordinary recursion theory one may e.g. move up in types over ω, or look to more general domains such as ordinals, admissible sets and acceptable structures. Alternatively, one may want to study in a more general setting one particular approach to ordinary recursion theory, thus e.g. try to develop a general theory based on schemes or fixed point operators, or work out a general theory of inductive definability, or develop in a suitable abstract setting the various model theoretic approaches such as representability in formal systems or invariant and implicit definability.

The approach of this paper is axiomatic. This is nothing new. Of previous axiomatic studies of recursion theory we mention Strong [12], Wagner [13], and Friedman [3]. Our interest in the axiomatics of generalized recursion theory was more directly inspired by Moschovakis [8], and any one familiar with his "Axioms for Computation Theories" will soon see our dependence upon his work.

Our objective is two-fold: First to contribute to the discussion and choice of the "correct" primitives for axiomatic recursion theory. Second to indicate new results, partly proved, partly conjectural, within the (modified) Moschovakis framework.

First one general remark on axiomatizing recursion theory. This may in itself be a worthy objective. Through an axiomatic analysis one may hope to get a satisfying classification and comparison of existing generalizations (technically through "representation" theorems and "imbedding" results). And one may, perhaps, also obtain a better insight into the "concret" examples on which the axiomatization is based. But it is not clear - and some disagree - that the field is at present ripe for axiomatization. Hence we are approaching our topic in a tentative manner.

As Moschovakis in his "Axioms..." [8] we take as our basic relation

$$\{a\}(\sigma) \simeq z \ ,$$

which asserts that the "computing device" named or coded by a acting on the input sequence $\sigma = (x_1,\ldots,x_n)$ gives z as output.

Let Θ denote the set of all computation triples (a,σ,z) such that the relation $\{a\}(\sigma) \simeq z$ obtains. It is possible to write down axioms for a computation set Θ which suffices to derive the most basic results of recursion theory, say up through the fixed-point or second recursion theorem.

However, many arguments seem to require an analysis not only of the computation tuples, but of the whole structure of "subcomputations" of a given computation tuple. Now computations, and hence subcomputations, can be many different things. And in an axiomatic analysis of the variety of approaches hinted at in the opening paragraph of the paper it would be rash to commit oneself at the outset to one specific idea of 'computation'.

In his "Axioms..." Moschovakis emphazised the fact that whatever computations may be, they have a well-defined length, which always is an ordinal, finite or infinite. Thus he proposed to add as a further primitive a map from the set Θ of computation tuples to the ordinals, denoting by $|a,\sigma,z|_\Theta$ the ordinal associated with the tuple $(a,\sigma,z) \in \Theta$.

In this paper we shall abstract another aspect of the notion of computation. We shall add as a further primitive a relation between computation tuples

$$(a',\sigma',z') < (a,\sigma,z) ,$$

which is intended to express that (a',σ',z') is a subcomputation of (a,σ,z), or, in other words, that the computation (a,σ,z) depends upon (a',σ',z'), the latter is necessary in order to establish the former. The basic axioms will state that the relation is transitive and wellfounded.

The rest of the paper will be divided in four sections. In section 1 we give the basic definition of a computation theory. In this we follow Moschovakis closely, making the modifications necessary due to our use of the subsomputation relation $<$ instead of the length concept.

In section 2 we list some basic facts about computation theories. This is in all essentials a repetition of material from "Axioms...", but is included for the convenience of the reader. In this part of the theory there does not seem to be much difference between the length concept and the subcomputation relation. The important thing is that both allows us to carry over to the abstract setting certain results proved in the "concrete" examples by transfinite induction on associated ordinals, or, alternatively, by a course-of-value induction on "subcomputations". The basic result here, due to Moschovakis in the axiomatic setting, is the first recursion theorem.

In section 3, on normality, we discuss axiomatic extensions of regularity. Of several possibilities we have chosen to emphazise two: One is the idea that a "computation" should be a finite object in the sense of the theory; the other is that the theory should satisfy the prewellordering property. The first we can express by requiring that for every $(a,\sigma,z) \in \Theta$, the set

$$\underset{\sim}{S}_{(a,\sigma,z)} = \{(a',\sigma',z') \mid (a',\sigma',z') < (a,\sigma,z)\}$$

is finite in the theory. One formulation of the other says that the set

$$\{(a',\sigma',z') \mid (a',\sigma',z') \in \Theta \wedge \|a',\sigma',z'\| \leq \|a,\sigma,z\|\}$$

is computable in the theory (where $\|a,\sigma,z\|$ is the ordinal of $\underset{\sim}{S}_{(a,\sigma,z)}$).

1. COMPUTATION THEORIES: BASIC DEFINITIONS.

In this section we give the basic definitions using the subcomputation relation as primitive notion.

<u>Definition 1</u>. A COMPUTATION DOMAIN is a structure

$$\mathfrak{A} = \langle A,C,N,s,M,K,L\rangle ,$$

where A is the universe, $N \subseteq C \subseteq A$ and $\langle N, s \restriction N\rangle$ is isomorphic to the non-negative integers. C is called the set of codes.

M is a pairing function on C, i.e.

$$a,b \in C \text{ iff } M(a,b) \in C,$$

and

$$M(a,b) = M(a',b') \in C \ .\!\rightarrow.\ a = a' \wedge b = b'.$$

K and L are inverses to M, i.e. they map C into C and

$$c = M(a,b) \in C \text{ iff } a = K(c) \wedge b = L(c).$$

To facilitate the presentation we introduce some notational convention (following Moschovakis [8]). We use

$x,y,z,\ldots$	for elements in A.
$a,b,c,\ldots$	for elements in C.
$i,j,k,\ldots$	for elements in N.
$\sigma,\tau,\ldots$	for finite sequences from A.

σ,τ or (σ,τ) denotes the concatenation of sequences. And as usual $lh(\sigma)$ = the length of the sequence σ. A computation tuple is any sequence (a,σ,z) such that $a \in C$ and $lh(a,\sigma,z) \geq 2$.

Definition 2.

The system $\langle \Theta, < \rangle$ is called a COMPUTATION STRUCTURE on the domain $\mathfrak{A}$ if $<$ is a transitive relation on the set of computation tuples and Θ is the wellfounded part of $<$.

Thus $(a,\sigma,z) \in \Theta$ iff the set

$$\underset{\sim}{S}_{(a,\sigma,z)} = \{(a',\sigma',z') \mid (a',\sigma',z') < (a,\sigma,z)\}$$

is wellfounded with respect to the relation $<$. Note that if $(a,\sigma,z) \in \Theta$ and $(a',\sigma',z') < (a,\sigma,z)$, then $(a',\sigma',z') \in \Theta$.

NOTE: We have built into our definition the convention that something which looks like a computation, i.e. an arbitrary computation tuple (a,σ,z), is not a computation if and only if its subcomputation tree contains an infinite descending path. In practice this may not always be so, but if an attempt at a computation stops after a finite number of steps without giving a <u>bona fide</u> computation, we can always start repeating ourselves in some suitable way so as to obtain an infinite descending path. It will be seen that this convention simplifies matters in section 3.

As in "Axioms..." we shall make use of the notions of partial multiple-valued (pmv) function and functional. We recall some notations:

$f(\sigma) \to z$	iff.	$z \in f(\sigma)$.
	iff.	z is one value of f at σ.
$f(\sigma) = g(\sigma)$	iff.	$\forall z[f(\sigma) \to z$ iff $g(\sigma) \to z]$.
$f(\sigma) = z$	iff.	$f(\sigma) = \{z\}$.
	iff.	$f(\sigma) \to z \wedge \forall u[f(\sigma) \to u \to u = z]$.
$f \subseteq g$	iff.	$\forall \sigma \, \forall z[f(\sigma) \to z \to g(\sigma) \to z]$.

A mapping is a total, single-valued function.

Let $\langle \Theta, < \rangle$ be a computation structure on $\mathfrak{A}$. To every $a \in C$ and every natural number n we can associate a pmv function $\{a\}^n_\Theta$ in the following way:

$$\{a\}^n_\Theta(\sigma) \to z \text{ iff } lh(\sigma) = n \wedge (a,\sigma,z) \in \Theta .$$

<u>Definition 3</u>.

Let $\langle \Theta, < \rangle$ be a computation structure on $\mathfrak{A}$. A pmv function f on A is Θ-computable if for some $\hat{f} \in C$

$$f(\sigma) \to z \quad \text{iff} \quad (\hat{f},\sigma,z) \in \Theta \ .$$

We call $\hat{f}$ a Θ-code of f and write $f = \{\hat{f}\}_{\Theta}^{n}$, where n is the number of arguments of f.

A pmv functional on A

$$\varphi(\underset{\sim}{f},\sigma) = \varphi(f_1,\ldots,f_l,\ x_1,\ldots,x_n)$$

maps pmv functions on A and elements of A into subsets of A (including the empty subset, $\emptyset$). φ is called continuous if

$$f_1 \subseteq g_1,\ldots,\ f_l \subseteq g_l\ ,\ \varphi(\underset{\sim}{f},\sigma) \to z \ \to\ \varphi(\underset{\sim}{g},\sigma) \to z.$$

<u>Definition 4</u>.

Let $\langle \Theta, < \rangle$ be a computation structure on the domain $\mathfrak{A}$. A pmv continuous functional φ on A is called Θ-computable if there exists a $\hat{\varphi} \in C$ such that for all $e_1,\ldots,e_l \in C$ and all $\sigma = (x_1,\ldots,x_n)$ from A, we have:

<u>a</u>. $\varphi(\{e_1\}_{\Theta}^{n_1},\ldots,\ \{e_l\}_{\Theta}^{n_l},\sigma) \to z$ if $\{\hat{\varphi}\}_{\Theta}^{l+n}(e_1,\ldots,e_l,\sigma) \to z$.

<u>b</u>. If $\varphi(\{e_1\}_{\Theta}^{n_1},\ldots,\ \{e_l\}_{\Theta}^{n_l},\sigma) \to z$, then there exist pmv functions $g_1,\ldots,g_l$ such that

i. $g_1 \subseteq \{e_1\}_{\Theta}^{n_1},\ldots,\ g_l \subseteq \{e_l\}_{\Theta}^{n_l}$ and $\varphi(g_1,\ldots,g_l,\sigma) \to z$.

ii. For all $i = 1,\ldots,l$, if $g_i(t_1,\ldots,t_{n_i}) \to u$, then

$$(e_i,t_1,\ldots,t_{n_i},u) < (\hat{\varphi},e_1,\ldots,e_l,\sigma,z) \ .$$

NOTE: This is the first essential use of the subcomputation relation. For a motivation of the definition of Θ-computable functional, see "Axioms..." [8 , p. 209].

Definition 5.

Let $\langle\Theta,<\rangle$ be a computation structure on the domain $\mathfrak{A}$. $\langle\Theta,<\rangle$ is called a COMPUTATION THEORY on $\mathfrak{A}$ if there exist Θ-computable mappings $p_1,\ldots,p_{13}$ such that the following functions and functionals are Θ-computable with Θ-codes as indicated and such that the iteration property holds:

I. $f(\sigma) = y, \quad y \in C \quad f = \{p_1(n,y)\}^n_\Theta \ , \quad n = lh(\sigma).$

II-VIII. Similar to I and state that the following functions are Θ-computable: Identity function, the successor function s, the characteristic functions of C and N, the pairing function M and the inverses K and L.

IX. $\varphi(f,g,\sigma) = f(g(\sigma),\sigma) \quad \hat{\varphi} = p_9(n) \ , \quad n = lh(\sigma).$

X-XII. Similar to IX and state that the following functionals are Θ-computable: Primitive recursion, permutation of arguments, point evaluation.

XIII. Iteration property: For all n,m $p_{13}(n,m)$ is a Θ-code for a mapping $S^n_m(a,x_1,\ldots,x_n)$ such that for all $a,x_1,\ldots x_n \in C$ and all $y_1,\ldots y_m \in A$:

i. $\{a\}^{n+m}_\Theta (x_1,\ldots,x_n,y_1,\ldots y_m) = \{S^n_m(a,x_1,\ldots,x_m)\}(y_1,\ldots,y_m).$

ii. If $\{a\}^{n+m}_\Theta (x_1,\ldots,x_n,y_1,\ldots,y_m) \to z$, then

$$(a,x_1,\ldots,x_n,y_1,\ldots y_m) < (S^n_m(a,x_1,\ldots,x_n),y_1,\ldots,y_m,z) \ .$$

REMARKS. 1. The missing parts of the definition can be found in "Axioms..." [8 , p. 205-206].

2. If we drop the primitive $<$, the rest can be stated using Θ alone, and we arrive at Moschovakis' notion of a PRE-COMPUTATION

THEORY. This part seems to contain the basic core (i.e. "pre-Post" theory) of any systematization, including the fixed-point or second recursion theorem.

3. To every tuple $(a,\sigma,z) \in \Theta$ there is associated an ordinal $|a,\sigma,z|$ = the ordinal of the set $\underset{\sim}{S}_{(a,\sigma,z)}$. Θ with this ordinal assignment is a computation theory in the sense of Moschovakis.

2. COMPUTATION THEORIES : BASIC FACTS.

2.1. Inductive generation of theories and equivalence. Let $\mathfrak{A}$ be a computation domain and let

$$\underset{\sim}{f} = f_1,\dots f_l \quad \text{and} \quad \underset{\sim}{\varphi} = \varphi_1,\dots,\varphi_k$$

be sequences of pmv functions and continuous pmv functionals on A. It is possible to construct a theory $PR[\underset{\sim}{f},\underset{\sim}{\varphi}]$, the prime computation theory generated by $\underset{\sim}{f}$ and $\underset{\sim}{\varphi}$, which in the following precise sense is the least computation theory which makes all the functions $\underset{\sim}{f}$ and functionals $\underset{\sim}{\varphi}$ (uniformly) computable.

Definition 6.

Let $\langle\Theta,<\rangle$ and $\langle\Theta',<'\rangle$ be computation theories on the same domain $\mathfrak{A}$. We say that

$$\Theta \leq \Theta'$$

(dropping, as we usual do, the explicit reference to $<$ and $<'$) if there exists a Θ'-computable mapping $p(a,n)$ such that $p(C,N) \subseteq N$ and such that for all n-tuples σ, all $a \in C$ and $z \in A$

i. $(a,\sigma,z) \in \Theta$ iff $(p(a,n),\sigma,z) \in \Theta'$

ii. If $(a,\sigma,z),(a',\sigma',z') \in \Theta$ and $(a,\sigma,z) < (a',\sigma',z')$, then $(p(a,n),\sigma,z) <' (p(a',n'),\sigma',z')$.

If $\Theta' \leq \Theta$ and $\Theta \leq \Theta'$, we say that Θ and Θ' are equivalent and write $\Theta \sim \Theta'$.

REMARK. It seems that Moschovakis' motivation for his version of the notion of equivalence (see [8 , p. 217-218]) is even more appropriate for the present version.

As in "Axioms..." [8 p, 218] we have the following result which justifies the claim made above.

i. Let $\langle\Theta,<\rangle$ be a computation theory on $\mathfrak{A}$, and let $\underset{\sim}{f}$ and $\underset{\sim}{\varphi}$ be sequences of pmv functions and continuous functionals on $\mathfrak{A}$. Then

$$\Theta \leq \Theta[\underset{\sim}{f},\underset{\sim}{\varphi}] ,$$

and if H is any other computation theory on $\mathfrak{A}$ such that $\Theta \leq H$ and f are H-computable and φ are uniformly H-computable, then

$$\Theta[\underset{\sim}{f},\underset{\sim}{\varphi}] \leq H .$$

REMARK. There are some difficulties in carrying over (iii) [8 , p.219] to the present frame. We mention this since this is the only example of a result in [8 , § 1-8] which has not had an immediate counterpart. (The difficulty is that if we pass from Θ to H via a map p and then back to Θ via a map q, the ordinal of (a,σ,z) is less than the ordinal of $(q(p(a,n),n),\sigma,z)$, but the former is not necessarily a subcomputation of the latter.

2.2. The first recursion theorem. The theorem was proved by Moschovakis in the axiomatic setting. The proof carries immediately over to the present set-up.

THEOREM. Let $\langle \Theta, < \rangle$ be a computation theory on $\mathfrak{A}$. Let $\varphi(f,x)$ be a Θ-computable continuous pmv functional over A. Let f^* be the least solution of

$$\forall x \in A : \quad \varphi(f,x) = f(x) .$$

Then f^* is Θ-computable.

The theorem is particularily important in discussing the relationship between recursion theory and inductive definability. It corresponds to the fact, and can sometimes be used to show, that Σ_1 inductive definitions has a Σ_1 minimal solution.

2.3. Selection operators. We first give the definition of Θ-semicomputable and Θ-computable relations.

Definition 7.

The relation $R(\sigma)$ is Θ-semicomputable if there is a Θ-computable pmv function f such that

$$R(\sigma) \quad \text{iff.} \quad f(\sigma) \to 0 .$$

The relation $R(\sigma)$ is Θ-computable if there is a Θ-computable mapping f such that

$$R(\sigma) \quad \text{iff.} \quad f(\sigma) = 0 .$$

The existence of a selection operator seems to be necessary in order to prove some of the basic facts about Θ-semicomputable and Θ-computable relations, such as closure of Θ-semicomputable relations under $\exists$-quantification and disjunction, and also to prove that a relation R is Θ-computable iff R and $\neg R$ are Θ-semicomputable.

Definition 8.

Let $\langle\Theta,<\rangle$ be a computation theory on $\mathfrak{A}$. An n-ary selection operator for $\langle\Theta,<\rangle$ is an n+1-ary Θ-computable pmv function $q(a,\sigma)$ with Θ-code $\hat{q}$ such that

i. If there is an x such that $\{a\}_\Theta(x,\sigma) \to 0$, then $q(a,\sigma)$ is defined and $\forall x[q(a,\sigma) \to x \to \{a\}(x,\sigma) \to 0]$.

ii. If $\{a\}(x,\sigma) \to 0$ and $q(a,\sigma) \to x$, then

$$(a,x,\sigma,0) < (\hat{q},a,\sigma,x).$$

REMARKS. 1. For a motivation of ii, equally valid in the present case, see [8 , p. 225].

2. By an oversight Moschovakis [8 , p.225] only required $\{a\}(q(a,\sigma),\sigma) \to 0$ rather than $\forall x[q(a,\sigma) \to x \to \{a\}(x,\sigma) \to 0]$, which is necessary when working with pmv objects.

2.4. The notion of finiteness in general computation theories.

Any good general approach to recursion theory must embody a suitable notion of "finiteness". We repeat the basic definitions from "Axioms..." [8 , p.230-233].

Definition 9.

A computation theory $\langle\Theta,<\rangle$ on $\mathfrak{A}$ is called regular if

i. $C = A$.

ii. Equality "$x=y$" is Θ-computable.

iii. $\langle\Theta,<\rangle$ has selection operators.

<u>Definition 10.</u>

Let $\langle \Theta, < \rangle$ be a regular theory on $\mathfrak{A}$, and let $B \subseteq A$. By the B-quantifier we understand the continuous pmv functional $\underset{\sim}{E}_B(f)$ defined by

$$\underset{\sim}{E}_B(f) \to \begin{cases} 0 & \text{if } \exists x \in B\ [f(x) \to 0] . \\ 1 & \text{if } \forall x \in B\ [f(x) \to 1] . \end{cases}$$

The set B is called Θ-finite with Θ-canonical code e, if the B-quantifier $\underset{\sim}{E}_B$ is Θ-computable with Θ-code e.

We refer the reader to [8] for a list of five properties of this particular notion of finiteness which may justify the claim that it is "natural". But it would be premature to conclude from this that the present version gives all the properties of ordinary (i.e. true) finiteness necessary for the combinatorial arguments of e.g. degree theory.

3. ON NORMALITY.

In this section we will discuss the problem of how to extend regularity. Of several possibilities we have chosen to emphazise two.

The idea that a "computation" is a finite object in the sense of the theory seems to play an important rôle in many arguments of recursion theory. "Computation" is not a primitive of our system, but a perhaps satisfactory approximation consists in requiring that a computation tuple depends on only a finite number of other tuples, i.e. for every $(a,\sigma,z) \in \Theta$, the set

$$\underset{\sim}{S}_{(a,\sigma,z)} = \{(a',\sigma',z') \mid (a',\sigma',z') < (a,\sigma,z)\}$$

is finite (uniformly in (a,σ,z)) in the sense for section 2.4.

Note: A more complete technical statement would require that there is a Θ-computable mapping $p(n)$ such that for each $(a,\sigma,z) \in \Theta$ $\{p(n)\}_\Theta(a,\sigma,z)$ is a Θ-canonical code for $\underset{\sim}{S}_{(a,\sigma,z)}$. And since we are dealing with sequences of arbitrary finite length, we assume some suitable coding convention.

REMARK. A computation theory in the sense of Moschovakis is a pair $\langle \Theta, |\,|_\Theta \rangle$, where $|\,|_\Theta$ is a map from Θ into the ordinals. Using a length-function it seems to be difficult to capture the idea that a "computation" should be a finite object in the sense of the theory.

As a first approximation one could consider the set

$$\{(a',\sigma',z') \mid |a',\sigma',z'|_\Theta \leq |a,\sigma,z|_\Theta\}.$$

But one cannot outright say that this set, which is the set of <u>all</u> computations with smaller length, is a finite object in the theory. Indeed, if this requirement is made, the set of natural numbers necessarily will be finite in Θ.

Some restriction must therefore be added and Moschovakis proposed to compare computations of equal length, i.e. he required the finiteness of the set

$$\{(a',\sigma',z') \mid lh(\sigma') = lh(\sigma) \wedge |a',\sigma',z'|_\Theta \leq |a,\sigma,z|_\Theta\}.$$

It is possible to proceed on this basis, and it may have some advantage later on (see section 4), but it is not in my opinion a satisfactory conceptual analysis of the motivation behind normality (see "Axioms..." [8,p. 233]).

The prewellordering property is another important tool in general recursion theory. One formulation is as follows. Let $\|a,\sigma,z\|$ denote the ordinal of the set $\underset{\sim}{S}_{(a,\sigma,z)}$, $(a,\sigma,z) \in \Theta$. The theory $\langle \Theta, < \rangle$ has the prewellordering property if there exists a Θ-computable function $p(x,y)$ such that if either $x \in \Theta$ or $y \in \Theta$ then $p(x,y)$ is defined and single-valued, and whenever $y \in \Theta$, then

$$p(x,y) = 0 \quad \text{iff} \quad x \in \Theta \wedge \|x\| \leq \|y\|,$$

in other words, the set $\{x \mid x \in \Theta \wedge \|x\| \leq \|y\|\}$ is Θ-computable, uniformly in y.

Does finiteness of the sets $\underset{\sim}{S}_{(a,\sigma,z)}$ for $(a,\sigma,z) \in \Theta$ imply the prewellordering property? It immediately gives something weaker, we can define a Θ-computable pmv function $q(x,y)$ such that if

$$x,y \in \Theta \text{ , then } q(x,y) = 0 \quad \text{iff} \quad \|x\| \leq \|y\|.$$

This is so since we can computably quantify over finite sets, hence have the following recursion equation

$$\|x\| \leq \|y\| \quad \text{iff} \quad \forall x' \in \underset{\sim}{S}_x \, \exists y' \in \underset{\sim}{S}_y \, \|x'\| \leq \|y'\|.$$

But the general prewellordering theorem seems to require some extra assumption. We have one result which seems to include some examples of theories over infinite domains.

3.1. Let $\langle \Theta, < \rangle$ be a regular theory satisfying the following two properties,

(i) the relation $<$ is Θ-semicomputable;

(ii) $\underset{\sim}{S}_{(a,\sigma,z)}$ is Θ-finite uniformly in (a,σ,z).

Then $\langle \Theta, < \rangle$ has the prewellordering property.

In this case we have the following recursion equations for the function $p(x,y)$

a. $p(x,y) = 0$ if $\forall x' \in \underset{\sim}{S}_x \, \exists y' \, [y' < y \wedge p(x',y') = 0]$.

b. $p(x,y) = 1$ if $\exists x' \, [x' < x \wedge \forall y' \in \underset{\sim}{S}_y \, p(x',y') = 1]$.

Note, that since $<$ is defined for all tuples of the form (a,σ,z) and Θ is the well-founded part of $<$, assumption (i) is rather problematic. However, the following argument may add some plausibility. An arbitrary computation tuple (a,σ,z) may or may not represent a "true" computation. But as soon as we are given a set of instructions a and an input sequence σ we should be able to start "generating" the "subcomputations", and this is what (i.) is intended to express.

The results of the next section will require that a regular theory $\langle \Theta, < \rangle$ has at least one of the properties discussed in this section: the prewellordering property, or the (uniform) Θ-finiteness of the sets $\underset{\sim}{S}_{(a,\sigma,z)}$. We will call a regular theory with both of these properties <u>normal</u>.

REMARK. "Normality" was already used by Moschovakis in [8] in a somewhat different technical sense. For lack of suitable words we use the same one. But we emphazise that our use of it is provisional. Further work may lead to a better definition of "normality".

4. BEYOND NORMALITY.

Moving beyond normality there is one fundamental distinction to make: Either a theory $\langle \Theta, < \rangle$ has a domain which is finite in the sense of the theory, or its domain is infinite.

4.1. Thories over finite domains. In analogy with Moschovakis [8] we call a normal theory $\langle \Theta, < \rangle$ on a domain $\mathfrak{A}$ a Spector theory if the domain A is Θ-finite. For such theories one can prove a great number of results which were originally established for hyperarithmetic and hyperprojective theory (see Moschovakis [7] and [8], - a detailed exposition within the axiomatic set up can be found in Vegem [14]).

Hyperarithmetic theory over ω is the theory of recursion in ${}^2E$, hyperprojective theory is a generalization of this to more general domains. How different is an arbitrary Spector theory from recursion in some functional over the domain, i.e. what kind of representation theorems do we have for Spector theories?

We state some results. But first a few terminological remarks. In many cases it is convenient to assume that the search operator ν is computable in Θ, where $\nu(f) = \{x \mid f(x) \to 0\}$. It is known from [8, p. 226] that if Θ has a selection operator, then Θ is weakly equivalent to $\Theta[\nu]$, hence, for reasons detailed there, we may as well work with $\Theta[\nu]$, which allows us greater freedom in defining pmv objects.

For convenience we also introduce the following notations:

sc(Θ) = the set of all Θ-computable relations on $\mathfrak{A}$.

en(Θ) = the set of all Θ-semicomputable relations on $\mathfrak{A}$.

(The notation en(Θ), the envelope of Θ, is taken from Moschovakis [9].)

4.1.1. For any normal $\langle \Theta, < \rangle$ on a domain $\mathfrak{A}$, there exists a <u>partial</u> functional F on $\mathfrak{A}$ such that

$$\Theta \sim \mathrm{PR}[F].$$

This result is jointly due to J. Moldestad and D. Normann, and the proof uses the uniform finiteness of the sets $\underset{\sim}{S}_{(a,\sigma,z)}$. D. Normann has also adapted the main result of Sachs [10] to show

4.1.2. For any Spector theory over ω there exists a <u>total</u> functional F such that

$$\mathrm{sc}(\Theta) = {}_1\mathrm{sc}(F)$$

$$(\text{and} \quad {}_1\mathrm{en}(F) \subseteq \mathrm{en}(\Theta)).$$

In Moschovakis [9] we find a counterexample to show that 4.1.2 cannot be lifted from sections to envelopes:

4.1.3. There exists a Spector theory Θ over ω such that $\mathrm{en}(\Theta) \neq {}_1\mathrm{en}(F)$, for all total type-2 functionals over ω.

The problem remains to characterize those Spector theories which are equivalent to prime recursion is some total type-2 functional over the domain. This is, of course, only one step toward a full classification of Spector theories.

In this connection a Gandy-Spector theorem may be of interest. It is not at all clear how to formulate the theorem in a sufficiently general form. Our proposal for a "weak" version:

<u>Definition</u>. Let $P(X,\sigma)$ be a second order relation over $\mathfrak{A}$. $P(X,\sigma)$ is called Θ-computable with index p if whenever $\{e\}_\Theta$ is the characteristic function of some set $X_e \in \mathrm{sc}(\Theta)$, then

$$\{p\}_{\Theta}(e,\sigma) \to \begin{cases} 0, & \text{if } P(X_e,\sigma) \\ 1, & \text{ow} \end{cases}$$

We say that the theory Θ has the weak Gandy-Spector property if whenever $R \in en(\Theta)$, there is a Θ-computable P such that

$$R(\sigma) \text{ iff. } (\exists X \in sc(\Theta))P(X,\sigma).$$

If there is a weak version, there must also be a strong one, e.g. we may replace the requirement that P is Θ-computable with the requirement that P is first order with respect to the language adequate to describe the domain $\mathfrak{A}$. The original Gandy-Spector theorem is of this absolute type, but there are, of course, Spector theories which do not have this strong Gandy-Spector theorem. A problem remains: Does every Spector theory have the weak Gandy-Spector property?

We shall comment on one more topic. The connection between inductive definability and hyperprojectivity was investigated by Grilliot [4]. His results was adapted to the present frame by Moldestad [6]. Led $Ind(\Delta)$ denote the class of relations which are inductive in some operator of class Δ, i.e. reducible to some Γ^{∞}, where $\Gamma \in \Delta$.

4.14. Let Θ be a Spector theory on $\mathfrak{A}$ and $\underset{\sim}{R}$ a sequence of Θ-computable relations on $\mathfrak{A}$.

i. $Ind(\Sigma_2(\underset{\sim}{R})) \subseteq en(\Theta)$.

ii. $\Theta \sim PR[\underset{\sim}{R},=,\vee,E]$, if and only if $Ind(\Sigma_2(\underset{\sim}{R})) = en(\Theta)$.

Here the implication from right to left in ii. goes beyond Grilliot [4], and gives a certain characterization of the "minimal" Spector theory on a domain $\mathfrak{A}$.

4.2. Theories over infinite domains. In Moschovakis "Axioms..." the case of infinite domains is particularily satisfying. Any Friedberg theory in his sense is a recursion theory generated in a "natural" way from an admissible prewellordering of the domain.

In our case the situation is more complicated. Our definition of normality does not automatically give us a prewellordering of the domain. In order for us to proceed we must add extra assumptions on the domain. The goal will be to abstract the "natural" or "minimal" recursion theory associated with an admissible set (possibly with urelements,- for this concept see Barwise [1]).

REMARK. The more complicated situation in our set up is perhaps not a too serious disadvantage. There seems to be recursion theories over infinite domains (i.e. infinite in the sense of the theory) on which there is no natural associated prewellordering (see [5]). And there seems to be admissible sets where the prewellordering associated with the rank function is not admissible in the sense of "Axioms..." [8]. Such examples should not be denied their proper existence in an axiomatic analysis of computation theories.

We admit at once that we do not yet claim to have a "good" definition of computation theories over infinite domains. We shall make some preliminary suggestions, and hope that further work will lead to a "correct" analysis.

Let $\langle \Theta, < \rangle$ on $\mathfrak{A}$ be normal in the sense of section 3. We shall assume that there is a Θ-computable partial ordering (po) $\preceq$ of the domain A such that the initial segments of this po are well-founded and (uniformly) Θ-finite.

The basic intention is now to express that there is a suitable correspondence between the complexity of the domain and the complexity of computations in Θ, and which is lacking in the case of theories over finite domains. More precisely, we assume the existence of a "coding" and "decoding" process:

4.2.1. There is a Θ-computable mapping $p_c(n)$ such that if $(a,\sigma,z) \in \Theta$ then $\{p_c(n)\}_\Theta(a,\sigma,z)$ is defined and single-valued and $\{p_c(n)\} \upharpoonright \underset{\sim}{S}_{(a,\sigma,z)}$ is an order isomorphism of $\underset{\sim}{S}_{(a,\sigma,z)}$ into the segment $\{w \in A | w \precsim \{p_c(n)\}(a,\sigma,z)\}$.

4.2.2. There is a Θ-computable mapping $p_d(n)$ such that $\{p_d(n)\}_\Theta(a,\sigma,z,w) = 0$ iff $(a,\sigma,z) \in \Theta \wedge \{p_c(n)\}(a,\sigma,z) = W$.

Note that 4.2.2 implies that the domain must be Θ-infinite, otherwise Θ would be Θ-computable.

We mention two other assumptions which establishes the correspondence between Θ and A in a somewhat different way:

4.2.3. $|\precsim| = \sup\{\|a,\sigma,z\| \mid (a,\sigma,z) \in \Theta\}$,

where $|\precsim|$ is the ordinal of the po $\precsim$, i.e. the sup of the ordinals of initial segments.

4.24. There exists a Θ-computable mapping $p(n)$ such that $\{p(n)\}_\Theta(a,\sigma,z,w) = 0$ iff $(a,\sigma,z) \in \Theta \wedge \|a,\sigma,z\| = |w|$, where $|w|$ is the ordinal of the segment $\{w' \in A | w' \prec w\}$.

Here 4.2.1 and 4.2.2 postulates an effective coding/decoding process, and gives a kind of "local" correspondence, each $\underset{\sim}{S}_{(a,\sigma,z)}$ can be coded into an initial segment of the po.
In contrast 4.2.3 and 4.2.4 states a kind of "global" corre-

spondence between associated ordinals. This is in some respects parallel to the difference between the prewellordering property and the finiteness of the sets $\underset{\sim}{S}_{(a,\sigma,z)}$ as discussed in section 3.

A topic of central importance is the relationship between theories over finite and infinite domains. The basic example here is the relationship between hyperarithmetic and meta-recursion (or L_{ω_1}-recursion) theory. This was generalized in Barwise, Gandy, Moschovakis [2] (- see also Barwise [1]).

In our context we are looking for a theorem which states that finite theories can be imbedded into infinite ones. The idea behind is simply this. A computation theory on an infinite domain can be a "good" recursion theory, in particular, if we have a suitable correspondence (coding/decoding) between the domain and computations over the domain, and if the semi-computable relations are exactly the Σ_1-definable relations over the domain. The imbedding theorem should say that we can "enlarge" a finite theory to a "good" infinite theory. And, as a possible application, we would expect that fine structure results for, say, the semi-computable relations of the given finite theory could be obtained by "pull-back" from the enlargement. The motivating example is again various results for Π^1_1 sets obtained <u>via</u> meta-recursion theory.

4.2.5. Let $\langle \Theta, < \rangle$ be a Spector theory on a domain $\mathfrak{A}$. It is then possible to construct.

i. a domain $\langle \mathfrak{A}^*, \precsim \rangle$ where $\mathfrak{A}^*$ extends $\mathfrak{A}$ and $\precsim$ is a well-founded partial ordering on $\mathfrak{A}^*$, and

ii. an "infinite" theory $\langle \Theta^*, <^* \rangle$ on $\langle \mathfrak{A}^*, \preceq \rangle$ such that

<u>a</u>. $\preceq$ is Θ^*-computable and initial segments of $\preceq$ are (uniformly) Θ^*-finite;

<u>b</u>. Θ^*-semicomputability equals Σ_1-definibility over $\mathfrak{A}^*$;

<u>c</u>. a subset $R \subseteq A$ is Θ-computable iff R is Σ_1 over $\mathfrak{A}^*$ iff R is Θ^*-semicomputable.

REMARK. The result and method of proof is clearly inspired by Barwise, Gandy, Moschovakis [2]. There is also some recent unpublished works of P. Aczel and Y. Moschovakis which overlap with the present result.- We have deliberately used the word "infinite theory" to describe Θ^*, since it is a bit unclear at the moment how strong properties we can enforce on Θ^*. We also expect that <u>c</u>. can be strengthened to assert the equivalence between Θ and $\Theta^* \upharpoonright \mathfrak{A}$.

4.3. One further goal is to push the analysis of "computation" so far as to establish the domain of validity for the priority arguments. Only then can we claim to have a reasonably complete axiomatic analysis of generalized recursion theory. This is very much an open field. (But note the interesting suggestions in Simpson [11].)

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