Applications of Model Theory to Recursion Theory on Structures of strong Cofinality ω.[1]

by

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Introduction

One of the main aspects of generalized recursion theory is that of definability. Take for instance notions like invariant definability and inductive definability. Also when one is generalizing other parts of recursion theory - like computability - questions relating to definability arise.

Already the use of the term definability suggests that ties to model theory exist, and in fact this is one of the most important interfaces between model theory and generalized recursion theory. As so often happens in mathematics, when two separate theories meet, this is of benefit to both. As examples in this context it should be enough to mention the "Barwise Compactness Theorem" where model theory is benefitting from methods of recursion theory, while the paper "Omitting Types; Application to Recursion Theory." by Grilliot [8] is an example of the converse.

The success of these exchanges of methods has so far been limited to situations where the domains or the languages are countable. There exist examples which show that if these countability conditions are dropped, not only do many of the nice relationsships between recursion theory and model theory disappear, but there is also no reasonable way to repair that situation.

We have been interested in examples of uncountable situations where as many as possible of the interconnections of the countable are preserved. In our paper [11] we showed how some results from recursion theory can be applied to model theory to obtain new compactness theorems for a class of uncountable languages. In this paper we take the other approach, that is, we study extensions of $\in A$-logic and A-logic and their model theory, when A is uncountable of strong cofinality ω. We show that it is possible to obtain completeness theorems for these logics, and use this to obtain new proofs of some results on inductive definability. In our applications to inductive definability we try to make the analogy with the countable case as explicit as possible. In this way we are able to lift wellknown proofs from the countable theory (e.g. $s\text{-}\Pi_1^1$ is Σ inductive definable, Π_1^1 is first onrder inductive definable) to new proofs of results like $s\text{-}\Pi_1^1$ is $\Sigma(\mathcal{P})$ inductive and Π_1^1 is first order inductive over structures $\langle A,\in,R_1,\ldots,R_k\rangle$ when A is of strong cofinality ω.

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1. The Logics

The object of this section is to discuss the syntax and semantics of the logics known as $\in A$-logic and A-logic. We have chosen to use the terminology of Grilliot [8], since especially the term $\in A$-logic now seems to have become standard. Our approach to $\in A$-logic and A-logic is, however, closer to the presentation in Barwise [3] of what there is called M-logic. The reader should note that the way we (and Barwise) define A-logic will differ slightly from that of Grilliot.

In the following A will be some fixed transsitive set. A <u>language</u> L is a collection of relation, function and constant symbols. By a <u>structure</u> $\mathfrak{M}$ for L - for short an <u>L-structure</u> - we mean a pair $\langle M,f\rangle$ where f is a map with domain L and range relations, functions and constants over M such that f maps n-ary relationsymbols to n-ary relations etc. M is called the <u>domain</u> of $\mathfrak{M}$ and we will always assume that $A \subseteq M$.

We will require that the language L is atmost countable, and we will always assume that the binary relationsymbol ε is a member of L. Whenever we find it convenient we will use the standard notation, e.g. $\mathfrak{M} = \langle M,E,R\rangle$ is an L-structure $\langle M,f\rangle$, where $L = \{\varepsilon,\overline{R}\}$ and $f(\varepsilon) = E$, $f(\overline{R}) = R$.

Let L^+ be some language containing L, the unary relation symbol $\overline{A}$ and for each $a \in A$ the constant symbol $\overline{a}$ and possibly a countable number of other symbols.

1.1 Definition. Let $\mathfrak{A} = \langle A,f\rangle$ be an L-structure where ε is interpreted as $\in \cap A \times A$.

a) <u>An $\in A$-structure for L^+</u> is a structure $\mathfrak{B} = \langle B,g\rangle$

satisfying:

i) A is contained in the interpretation of $\overline{A}$ in $\mathfrak{b}$,

ii) the interpretation of $\overline{a}$ in $\mathfrak{b}$ is a for all $a \in A$, and

iii) the interpretation of ϵ in $\mathfrak{b}$ is such that $\langle B, g \upharpoonright L\rangle$ is an end-extension of $\langle A, f\rangle$.

b) <u>An A-structure for L^+</u> is an ϵA-structure $\mathfrak{b} = \langle B, g\rangle$ for L^+ where the interpretation of $\overline{A}$ in $\mathfrak{b}$ is A .

Note that our use of the terms "ϵA" and "A" in the above definition are slightly misleading, since the class of ϵA-structures and A-structures will depend on the whole of $\mathfrak{A} = \langle A, f\rangle$, not just on A . However, in any given context this should cause no confusion.

By $L_{\omega\omega}$ $(L^+_{\omega\omega})$ we mean the ordinary finitary first order language of L (L^+) . In any reference to formulas φ of L and L^+ we will assume that φ is a member of $L_{\omega\omega}$ and $L^+_{\omega\omega}$ respectively. The semantics of ϵA-logic (A-logic) will be the ordinary semantics of $L^+_{\omega\omega}$ restricted to the class of ϵA-structures (A-structures). We will use $\models_{\epsilon A}$ $(\models_{A})$ to denote the satisfaction relation of ϵA-logic (A-logic), and the corresponding semantic consequence relation $T \models_{\epsilon A} \varphi$ $(T \models_{A} \varphi)$ and validity $\models_{\epsilon A} \varphi$ $(\models_{A} \varphi)$, where $T \subseteq L^+_{\omega\omega}$ and $\varphi \in L^+_{\omega\omega}$.

When A is <u>countable</u> it is well known that one can introduce a <u>complete</u> notion of provability $\vdash_{\epsilon A}$ $(\vdash_{A})$ for ϵA-logic (A-logic), where we by complete refere to the fact that $\vdash_{\epsilon A}$ and $\vdash_{A}$ satisfy the completeness theorem:

1.2 Completeness theorems for $\in A$-logic and A-logic when A is countable.

Assume $T \subset L^+_{\omega\omega}$ and $\varphi \in L^+$ then:

$$T \models_{\in A} \varphi \ (T \models_A \varphi) \text{ if and only if } T \vdash_{\in A} \varphi \ (T \vdash_A \varphi) .$$

To give this theorem some content, we will define $\vdash_{\in A}$ and $\vdash_A$.

1.3 Axioms for $\in A$-logic and A-logic. (Countable case.)

i) $\overline{A}(\overline{a})$ is an axiom of $\in A$-logic and A-logic for each $a \in A$. In addition, the sentence $\forall x \forall y(\overline{A}(y) \wedge x \varepsilon y \rightarrow \overline{A}(x))$ is an axiom of A-logic.

ii) Every atomic or negated atomic sentence of $L \cup \{\overline{a} | a \in A\}$ true in $\langle A,f\rangle$, is an axiom of $\in A$-logic and A-logic.

iii) The usual axioms of $L^+_{\omega\omega}$ are all axioms of $\in A$-logic and A-logic.

1.4 Definition: Let T be a set of formulas of L^+ . A finite formula $\varphi \in L^+_{\omega\omega}$ is a <u>consequence of T</u> in $\in A$-logic (A-logic), written $T \vdash_{\in A} \varphi$ ($T \vdash_A \varphi$) , if φ is contained in the smallest set of formulas containing T and the axioms of $\in A$-logic and A-logic and closed under the following rules:

i) (Modus ponens) If T "$\vdash$" φ and T "$\vdash$" $\varphi \rightarrow \psi$ then T "$\vdash$" ψ.

ii) (Generalisation) If T "$\vdash$" $(\varphi \rightarrow \psi(v_n))$ and v_n not free in φ then T "$\vdash$" $(\varphi \rightarrow \forall v_n \psi(v_n))$.

iii) ($\in A$-rule) Given $b \in A$. If $T \vdash_{\in A} \varphi(\overline{a}/v_n)$ for each $a \in b$ then $T \vdash_{\in A} \forall v_n(v_n \varepsilon \overline{b} \rightarrow \varphi(v_n))$.

v) (A-rule) If $T \vdash_A \varphi(\overline{a}/v_n)$ for every $a \in A$ then $T \vdash_A \forall v_n(\overline{A}(v_n) \rightarrow \varphi(v_n))$.

(The use of "$\vdash$" under i) and ii) is to indicate that the rule apply to both $\underset{\in A}{\vdash}$ and $\underset{A}{\vdash}$)

If $T = \emptyset$ we write $\underset{\in A}{\vdash} \varphi$ ($\underset{A}{\vdash} \varphi$) for $T \underset{\in A}{\vdash} \varphi$ ($T \underset{A}{\vdash} \varphi$).

Notice that the way things are set up, the $\in A$-rule is provable in A-logic using the A-rule. We could of course have dropped the requirement that $\overline{A} \in L^+$, clause i) of 1.1.a) and axiom i) of 1.3, for $\in A$-logic. We have included them purely for the sake of uniformity in the presentation of the two logics. Let us also point out that we are not requiring anything like proofs being members of A , in fact we have not even defined what a proof is, only the notion of provability is considered.

If A is uncountable, the situation is altered considerably. Not only will the completeness theorem for $\in A$-logic and A-logic fail, but for "most" A (for instance $A = H(\omega_1)$) there are no way of repairing the incompleteness by adding new definable sets of axioms or new definable rules. (By definable we are referring to some coding of formulas as members of A , and to first order definable sets of codes over $\langle A, f \rangle$.) As we shall see, however, there is a class of nicely behaved uncountable sets A for which the failure of the completeness theorems are less serious, and we will use the rest of this section to describe these sets and the modification that has to be made to $\underset{\in A}{\vdash}$ and $\underset{A}{\vdash}$, to obtain completeness.

1.5 Definition: We say that the set A has (is of) <u>strong cofinality</u> ω if $\omega \in A$ and

i) A is closed under power (i.e. if $a \in A$ then power (a) $\in A$),

ii) there exist a family $\{A_n \mid n \in \omega\}$ of transitive sets A_n, such that for each $n : A_n \in A$ and $A = \bigcup_{n \in \omega} A_n$.

The canonical examples of sets of strong cofinality ω are V_α (the set of sets of rank less that α) and $H(\beth_\alpha)(=\{x \mid \overline{\overline{TC(x)}} < \beth_\alpha\})$ for limes α of cofinality ω with $\alpha > \omega$. ($\beth_\alpha \underset{D}{=} \overline{\overline{V}}_\alpha$) .

1.6 Axioms for $\in$A-logic and A-logic when A has strong cofinality ω .

The axioms for $\in$A-logic and A-logic are the ones described under 1.3 and in addition:

iv) (Distributive law) For each $a \in A$ and $\varphi \in L^+_{\omega\omega}$ the following is an axiom of $\in$-A logic and A-logic:

$\exists z(z \epsilon \overline{\text{power}(a)} \wedge \forall y(y \epsilon z \Longleftrightarrow y \epsilon \overline{a} \wedge \varphi))$, where z does not occur free in φ .

v) (Cfω distribution) For each $a \in A$ and each $\varphi \epsilon L^+_{\omega\omega}$ the following is an axiom of A-logic.

$$\exists z[\overline{A}(z) \wedge \mathrm{Fnc}(z) \wedge \text{"}z : \overline{\omega} \to \overline{\text{power}(a)}\text{"} \wedge \text{"}\overline{a} = \bigcup\{z(i) \mid i \epsilon \overline{\omega}\}\text{"}$$
$$\wedge \forall k \epsilon \overline{\omega}(\forall x \epsilon z(k) \exists y(\overline{A}(y) \wedge \varphi(x,y)) \to$$
$$\to \exists v(\overline{A}(v) \wedge \forall x \epsilon z(\mathbf{k}) \exists y \epsilon v \varphi(x,y)))] ,$$

where z does not occur free in φ .

The definition of consequence is now exactly as under 1.4 except that iv) and v) above are included among the axioms. We keep the notation $\vdash_{\in A}$ and $\vdash_{A}$, it should be obvious in each case if it is the countable or cofinality ω version we have in mind.

The formulation of axiom v) will perhaps need some remarks to explain the use of the terms "Fnc(z)", "$z:\overline{\omega} \to \overline{\text{power(a)}}$" and "$\overline{a} = \cup\{z(i) \mid i \in \overline{\omega}\}$". These should be taken as abbreviations for their usual definitions in terms of ϵ . If this is done properly one obtain that if

$$\underset{A}{\vdash} \text{"Fnc}(\overline{b})\text{"} \wedge \text{"}\overline{b}:\overline{\omega} \to \overline{\text{power(a)}}\text{"} \wedge \text{"}\overline{a} = \cup\{\overline{b}(i) \mid i \in \overline{\omega}\}\text{"},$$

then b is really a function mapping ω into the powerset of a , such that $a = \cup\{b(i) \mid i \in \omega\}$.

We end this section by claiming the sucess of these new axioms.

1.7 Completeness theorem for $\in A$-logic and A-logic when A has strong cofinality ω .

Let $\mathfrak{A} = \langle A, f \rangle$ be an L-structure where A has strong cofinality ω , and let $\underset{\in A}{\vdash}$ and $\underset{A}{\vdash}$ be as above.
If $T \subset L^+_{\omega\omega}$, $\varphi \in L^+_{\omega\omega}$ we then have:

$$T \underset{\in A}{\models} \varphi \; (T \underset{A}{\models} \varphi) \text{ if and only if } T \underset{\in A}{\vdash} \varphi \; (T \underset{A}{\vdash} \varphi) .$$

2. The Proofs

This section will contain nothing but the proofs of theorem 1.7 , and could for that reason have been postponded until after the section on applications. When it is put here, it is with the hope that the reader should not postpone reading these proofs indefinitely. We believe that there should be some novelity in the techniques involved.

We will start by listing some definitions, conventions and observations to be used throughout the proofs. Assume for the rest of this section that the L- structure $\mathfrak{A} = \langle A,f \rangle$ is fixed, where A is some set of strong cofinality ω .

2.1 Definition. Let T be a set of sentences of L^+ . T is said to be consistent in $\in$A-logic - for short $\in$A-consistent - (consistent in A-logic - A-consistent -) if for some sentence φ of L^+ it is not the case that $T \vdash_{\in A} \varphi \wedge \neg\varphi$ $(T \vdash_{A} \varphi \wedge \neg\varphi)$.

The completeness theorem can now be restated as:

2.2 T has an $\in$A-model (A-model) if and only if T is $\in$A-consistent (A-consistent).

For both instances of 2.2 the only if parts are almost immediate. The only way these differ from the ordinary countable versions are due to the axioms iv) and v).

That iv) is valid follows easily by using that A is closed under the powerset operation. (In the case of A-logic, iv) could have been replaced by the equivalent (in A-logic):

iv') $\exists x(\bar{A}(x) \wedge \forall y(y \varepsilon x \longleftrightarrow y \varepsilon \bar{a} \wedge \varphi))$.

While iv) just by the way it is formulated requires A to be closed under power, iv') is valid also when A is super transitive (i.e. $a \subseteq b \in A \Longrightarrow a \in A$) . Not much is gained by this observation, however, since closure under power is needed in order to make v) valid.)

To get a feeling for the content of axiom v) it is instructive to do the proof of its validity in some detail.

Assume $A = \bigcup_{i \in \omega} A_i$, where A_i is transitive and power $(A_i) \in A$. Let $a \in A$, $\varphi \in L^+_{\omega\omega}$ and the A-structure $\mathfrak{M} = \langle M,g \rangle$ for L^+ be given. Let $c = \{b \in a \mid \mathfrak{M} \models_A \exists y(\bar{A}(y) \wedge \varphi(\bar{b},y))\}$. Then c can be written as $\{b \in a \mid \text{for some } d \in A, \mathfrak{M} \models_A \varphi(\bar{b},\bar{d})\} = \bigcup_{i \in \omega} c_i$, where $c_i = \{b \in a \mid \text{for some } d \in A_i , \mathfrak{M} \models_A \varphi(\bar{b},\bar{d})\}$.

Define $f: \omega \to \text{power}(a)$ by

$$f(i) = \begin{cases} a-c & \text{for } i = 0 \\ c_{i-1} & \text{for } i > 0 . \end{cases}$$

Then $f \in A$, $\bigcup_{i \in \omega} f(i) = a$ and for all $i \in \omega$

$$\mathfrak{M} \models_A [\forall x \varepsilon \overline{f(i)}\, \exists y\, (\bar{A}(y) \wedge \varphi(x,y)) \to \forall x \varepsilon \overline{f(i)}\, \exists y \varepsilon \bar{A}_{i-1} \varphi(x,y)] .$$

Hence $\mathfrak{M}$ satisfy this particular instance of axiom v), and we conclude that axiom v) is valid.

The other halves of the completeness proofs are less immediate and we need to do some more preparatory work, most of which are common to both logics.

Let $L^o \subseteq L^+$ be $L^+ - \{\bar{a} \mid a \in A\}$. Notice that L^o is then countable.

From L^o we form L^o_{SK} - the Skolem language of L^o . (i.e. L^o_{SK} is the smallest language K containing L^o such that for every formula $\exists x\varphi(x,y_1,\ldots,y_n) \in K_{\omega\omega}$ there is a function symbol $t_\varphi(y_1,\ldots,y_n) \in K$.) Hence L^o_{SK} is still countable.

Let L^+_{SK} be $L^o_{SK} \cup \{\bar{a} \mid a \in A\}$. It should be noted that L^+_{SK} is not actually the Skolem language of L^+ . The Skolem language of L^+ would contain uncountably many (cardinality of A) new Skolem functions. There is of course a natural correspondance between the Skolem language of L^+ and L^+_{SK} , namely by mapping the Skolem function $t_{\varphi(\bar{a})}(x_1,\ldots,x_n)$ to the term $t_\varphi(\bar{a},x_1,\ldots,x_n)$ of L^+_{SK}. Hence L^+_{SK} will still play the role of providing witnesses for existential quantifiers of $L^+_{\omega\omega}$.

The Skolem axioms are all formulas of L^o_{SK} of the form:

$$\exists x\varphi(x,y_1,\ldots,y_n) \rightarrow \varphi({}^{t_\varphi(y_1,\ldots,y_n)}/x,y_1,\ldots,y_n)$$

If T is any set of sentences of L^+_{SK} let T_{SK} be T + the universal closure of all Skolem axioms.

From now on we will assume that the sequence $\langle A_n \mid n \in \omega\rangle$ is chosen such that $A = \bigcup_{i\in\omega} A_i$, each A_i is transitive, $\omega \in A_o$ and power $(A_i) \in A_{i+1}$. (This is possible by our definition of strong cofinality ω .)

Let $\{\varphi_i \mid i \in \omega\}$ be some fixed enumeration of the formulas of L^o_{SK} .

2.3 Definition. To each formula φ_i of L^o_{SK} we define the formula θ^i_n of L^+_{SK} as follows:

$$\theta_n^i(z) \underset{D}{\longleftrightarrow} \forall x_1,\dots,x_{l_i}(\langle x_1,\dots,x_{l_i}\rangle \varepsilon z \longleftrightarrow$$

$$x_1 \varepsilon \overline{A}_n \wedge \dots \wedge x_{l_i} \varepsilon \overline{A}_n \wedge \varphi_i(x_1,\dots,x_{l_i})) ,$$

where l_i is the number of free variables of φ_i and z is chosen such that z is not free in φ_i . If φ_i has no free variables, let $\theta_n^i(z)$ be:

$$\theta_n^i(z) \underset{D}{\longleftrightarrow} \forall x(x\varepsilon z \leftrightarrow x\varepsilon\overline{A}_n \wedge \varphi_i)$$

A central point in what will follow is contained in the next observation.

2.4 If T is an ϵA-consistent (A-consistent) set of sentences of L_{SK}^+ , then there is some $c \in A$ such that $T \cup \{\theta_n^i(\overline{c})\}$ is ϵA-consistent (A-consistent).

This is seen by applying axiom iv) the appropriate number of times together with the ϵA-rule (A-rule). (If φ_i has only one free variable, then <u>one</u> application of axiom iv) followed by <u>one</u> application of the appropriate rule is enough, etc.)

We should now be sufficiently prepaired for the remaining half of the completeness theorem for ϵA-logic:

Assume T is an ϵA-consistent set of sentences of L^+ . We will successively choose elements c_n^i of A $(i \leq n,\ n \in \omega)$, such that the set

(1) $\qquad T_{SK} \cup \{\wedge\{\theta_m^i(\overline{c}_m^i) \mid i \leq m,\ m \leq n\}\}\ (= T_{SK} \cup \{\theta_n\})$

is ϵA-consistent.

This can be done inductively as follows. Suppose c_m^j has been chosen for all $j \leq m$ and $m < n$, and for $m = n$ and $j < i \leq n$, and such that $T_{SK} \cup \{\bigwedge\{\theta_m^j(\bar{c}_m^j) \mid j \leq m, m < n\}\} \cup \{\bigwedge\{\theta_n^j(\bar{c}_n^j) \mid j < i\}\}$ $(= T_{SK} \cup \{\theta'\})$ is consistent. Then at <u>stage (n,i)</u> of the construction choose some element c_n^i of A such that $T_{SK} \cup \{\theta'\} \cup \{\theta_n^i(\bar{c}_n^i)\}$ is consistent. (This is possible by 2.4)

Assuming this construction has been carried out, we asscciate with each c_n^i a set of sentences s_n^i of L_{SK}^+ :

(2) $$s_n^i \underset{D}{=} \{\varphi_i(\bar{a}_1,\dots,\bar{a}_{l_i}) \mid \langle a_1,\dots,a_{l_i}\rangle \in c_n^i\} \cup \{\neg\varphi_i(\bar{a}_1,\dots,\bar{a}_{l_i}) \mid \langle a_1,\dots,a_{l_i}\rangle \in A_n^{(l_i)} \smallsetminus c_n^i\} ,$$

where $A_n^{(m)} \underset{D}{=} \{\langle a_1,\dots,a_m\rangle \mid a_i \in A_n \ (i=1,\dots,m)\}$.

If φ_i has no free variables, let s_i^n be $\{\varphi_i\}$ if $c_n^i \neq \emptyset$, $\{\neg\varphi_i\}$ if $c_n^i = \emptyset$.

(3) Let $s_n \underset{D}{=} \bigcup_{i \leq n} s_n^i$ and $S \underset{D}{=} \bigcup_{n \in \omega} s_n$.

2.5 Claim: Suppose φ and ψ are sentences of L_{SK}^+ , then

I not both φ and $\neg\varphi$ are members of S ,

II either φ or $\neg\varphi$ is a member of S ,

III $T_{SK} \subseteq S$,

IV i) if $\varphi \wedge \psi \in S$ then $\varphi \in S$ and $\psi \in S$,

ii) if $\exists x\varphi \in S$ then for some constant term t $\varphi({}^t/x) \in S$,

V if $t\varepsilon\bar{a} \in S$ for some constant term t , then there is a $b \in A$ such that $t = \bar{b} \in S$.

Before we enter a proof of 2.5, observe that it is now standard to show that the term model constructed from S is a model of T, and by V in fact an $\in A$-model. Hence the completeness theorem of $\in A$-logic follows.

Proof of 2.5.

I and II: Assume the sentence φ of L^+_{SK} is given. Then by the definition of L^+_{SK}, φ has to be of the form $\varphi_i(\bar{a})$ for some φ_i of L^o_{SK}, $a \in A$. (We take the case where φ_i has one free variable, the general case is just a notational variant of this.)

For some $n \in \omega$ we will have $a \in A_n$ and we can without loss of generality assume that $n \geq i$. This means that at stage (n,i) φ_i was considered, and either $\varphi_i(\bar{a})$ or $\neg\varphi_i(\bar{a})$ was put into s^i_n (since $s^i_n = \{\varphi_i(\bar{a}) \mid a \in c^i_n\} \cup \{\neg\varphi_i(\bar{a}) \mid a \in A_n - c^i_n\}$). This proves II.

Assume that $\varphi_i(\bar{a})$ was put in at this stage. That means in particular that $T_{SK} \cup \{\theta_n\} \underset{\in A}{\vdash} \varphi_i(\bar{a})$. If $\neg\varphi$ $(=\varphi_j(\bar{a}))$ was put at some stage (m,j) we could then conclude that $T_{SK} \cup \{\theta_m\}$ $\underset{\in A}{\vdash} \varphi_j(\bar{a})$ and hence for $l = \max\{m,n\}$ obtain that $T_{SK} \cup \{\theta_l\}$ $\underset{\in A}{\vdash} \varphi \wedge \neg\varphi$, contradicting the $\in A$-consistency of $T_S \cup \{\theta_l\}$.

III: If $\varphi \in T_{SK} - S$ then by II $\neg\varphi \in s_n \subseteq S$ for some n, but this would imply that $T_{SK} \cup \{\neg\varphi\}$ had to be $\in A$-consistent, which is absurd.

IV: The proof of i) is straight forward, while ii) is taken care of by the Skolem axioms being contained in S.

V: At first sight it seems perhaps a little surprising that we are able to obtain the "omitting types" result of V just by carelessly choosing the c^i_n's with $\in A$-consistency as the only requirement. We shall see, however, that since the A_i's were choosen to be transitive, there is in some sense a standard omitting types argument involved.

By the term t we mean the constant term $t(\bar{a}_1,\ldots,\bar{a}_l)$ obtained from $t(x_1,\ldots,x_l)$ of L^o_{SK} by substituting a_i for x_i $(i = 1,\ldots,l)$. The possibility of the list $x_1,\ldots,x_l$ being empty is allowed.

Hence t might be one of the terms of L^+ or one of the new terms of L^+_{SK} , or a composition of both. What will be used is that the formula $t(\bar{a}_1,\ldots,\bar{a}_l) \varepsilon \bar{a}$ is an instance of the formula $t(x_1,\ldots,x_l) \varepsilon x_{l+1}$ $(=\varphi_i(x_1,\ldots,x_{l+1}))$ of L^o_{SK} .

Assume $t(\bar{a}_1,\ldots,\bar{a}_l)\varepsilon\bar{a} \in S$. Then it must have been put in at some stage (n,i) . Hence at this stage the formula θ^i_n was considered and we have that for some sentence θ of L^+_{SK} the following is $\in$A-consistent:

$$T_{SK} \cup \{\theta\} \cup \{\forall x_1,\ldots,x_{l+1}(\langle x_1,\ldots,x_{l+1}\rangle \varepsilon \bar{c}^i_n \Longleftrightarrow x_1 \varepsilon \bar{A}_n \wedge \ldots \wedge x_{l+1} \varepsilon \bar{A}_n \wedge t(x_1,\ldots,x_l)\varepsilon x_{l+1})\} ,$$

where $\langle a_1,\ldots,a_l,a\rangle \in c^i_n$.

In particular it follows that

$$T_{SK} \cup \{\theta_n\} \underset{\in A}{\vdash} t(\bar{a}_1,\ldots,\bar{a}_l)\varepsilon\bar{a} \wedge \bar{a}\varepsilon\bar{A}_n .$$

By applying the fact that A_n is transitive together with the $\in$A-rule, we get:

(1) $$T_{SK} \cup \{\theta_n\} \underset{\in A}{\vdash} t(\bar{a}_1,\ldots,\bar{a}_l)\varepsilon\bar{A}_n .$$

At some stage (m,j) the formula $t(x_1,\ldots,x_l) = x_{l+1}$ $(=\varphi_j(x_1,\ldots,x_{l+1}))$ was considered where $m = \max\{n,j\}$, and c^j_m was choosen such that for some θ' of L^+_{SK}

(2) $$T_{SK} \cup \{\theta'\} \cup \{\forall x_1,\ldots,x_{l+1}(\langle x_1,\ldots,x_{l+1}\rangle \varepsilon \bar{c}^j_m \Longleftrightarrow x_1 \in \bar{A}_m \wedge \ldots \wedge x_{l+1} \in \bar{A}_m \wedge t(x_1,\ldots,x_l) = x_{l+1})\}$$

is consistent.

For some $b \in A_m$ we must then have:

(3) $\langle a_1,\ldots,a_l,b\rangle \in c_m^j$.

If not, we could use the ϵA-rule to obtain $T_{SK} \cup \{\theta_m\} \vdash_{\epsilon A} \neg t(\bar{a}_1,\ldots,\bar{a}_l)\epsilon\bar{A}_m$ which again by the transitivity of A_m and the ϵA-rule would contradict (1).

From (2) and (3) we are now able to conclude that $t(\bar{a}_1,\ldots,\bar{a}_l) = \bar{b} \in s_m^j \subseteq S$ which proves V . —|

For many details of the completeness proof for A-logic, we will be able to refer to the completeness proof for ϵA-logic. It will therefore be convenient **first** to give an outline of the A-logic proof by a comparision with that of ϵA-logic.

As in the previous proof, our main object will be to construct a set S of sentences with the properties of 2.5 except that V will have to be replaced by:

V': If $\bar{A}(t) \in S$ for some constant term t , then there is a $b \in A$ such that $t = \bar{b} \in S$.

Also the definition of S will be essentially the same as before. That is; we will construct elements c_n^i of A by appealing to the consistency of $\theta_n^i(\bar{c}_n^i)$ (of course this time the A-consistency), and define the sets s_n^i from c_n^i exactly as for ϵA-logic, to obtain S as we did earlier.

There is, however, one nontrivial difference hidden in this description due to the fact that we cannot afford to be as generous in our choice of the c_n^i's as in the previous proof. That is , we will have to add some requirements on the c_n^i's in addition to the A-consistency of $\theta_n^i(\bar{c}_n^i)$ at that particular stage of the construction. The way this will be achieved is by the introduction of some auxiliary constants at each stage which will be required to satisfy some new boundedness conditions. This will have the effect of restricting the possible A-consistent choices of the c_n^i's .

Assume as before that $\{\varphi_i \mid i \in \omega\}$ is an enumeration of the formulas of L^o_{SK} . In addition let $\{t_i \mid i \in \omega\}$ be some enumeration of the function symbols of L^o_{SK} .

For each number i and n , we define the formulas $\eta_n^i(z)$ and $\nu^i(u,v)$.

2.6 Definition.

$$\eta_n^i(z) \underset{D}{\longleftrightarrow} \text{"}Fnc(z) \wedge z:\overline{\omega} \to \overline{\text{Power } A_n} \wedge \overline{A}_n = \cup \{z(k) \mid k \in \overline{\omega}\}\text{"} \wedge \forall s \in \overline{\omega}\, \exists v[\overline{A}(v) \wedge \forall x_1,\dots,x_{l_i} \in z(s)(\neg \overline{A}(t_i(x_1,\dots,x_{l_i}) \vee t(x_1,\dots,x_{l_i}) \in v)] .$$

$$\nu^i(u,v) \underset{D}{\longleftrightarrow} \forall x_1,\dots,x_{l_i} \in u(\neg\overline{A}(t_i(x_1,\dots,x_{l_i}) \vee t_i(x_1,\dots,x_{l_i}) \in v)$$

If t_i has no argument places (i.e. t_i is a constant term of L^o_{SK}) the definition of η_n^i and ν^i shall be as above except that the quantifiers $\forall x_1,\dots,x_{l_i} \in z(s)(\in u)$ are omitted.

2.7 Observation. Assume T is some A-consistent set of sentences. Then for some $f \in A$: $f:\omega \to \text{power } A_n$, $A_n = \bigcup_{i\in\omega} f(i)$ and $T \cup \{\eta_n^i(\overline{f})\}$ is A-consistent.

Given that $T \cup \{\eta_n^i(\overline{f})\}$ is A-consistent there is for each $j \in \omega$ some $d \in A$ such that

$$T \cup \{\eta_n^i(\overline{f})\} \cup \{\nu^i(\overline{f(j)},\overline{d})\} \text{ is A-consistent.}$$

Proof: The proof is just a simple exercise in how to apply the A-rule to axiom v) of 1.6. The instances of axiom v) that should be considered, is for formulas like $\neg A(t(x)) \vee t(x) = y$.

Assume now that T is an A-consistent set of sentences of A-logic and let T_{SK} be as before. Our aim will be for each $n \in \omega$, to choose elements f_m^i , $d_m^{i,j}$ and c_m^i of A for $i \leq m \leq n$, $j \leq n$ such that the following set of sentences is A-consistent:

$$(*) \quad \begin{aligned} &T_{SK} \cup \{\bigwedge\{\eta_m^i(\overline{f}_m^i) \mid i \leq m, m \leq n\}\} \cup \{\bigwedge\{\nu^i(\overline{f_m^i(j)}, \overline{d_m^{i,j}}) \mid i \leq m, m \leq n, j \leq n\}\} \\ &\cup \{\bigwedge\{\theta_m^i(\overline{c}_m^i) \mid i \leq m \leq n\}\} \quad (\underset{D}{\equiv} T_{SK} \cup \{\eta_n\} \cup \{\nu_n\} \cup \{\theta_n\} \underset{D}{\equiv} T_{SK} \cup \{\chi_n\}) \end{aligned}$$

To see that this actually is possible, assume that choices are made for each $m < n$ such that $T_{SK} \cup \{\chi_{n-1}\}$ is A-consistent. Then start successively for each $i \leq n$ to choose f_n^i such that $T_{SK} \cup \{\chi_{n-1}\} \cup \{\eta_n\}$ becomes A-consistent. (This is possible by observation 2.7.) Then start with the $d_n^{i,j}$'s , say, by for each $i < n$ construct $d_n^{i,j}$ for all $j \leq n$ etc. Finally after ν_n has been constructed such that $T_{SK} \cup \{\chi_{n-1}\} \cup \{\eta_n\} \cup \{\nu_n\}$ is consistent, construct the c_n^i's as for ϵA-logic except that the consistency now of course will refer to the A-consistency with $T_{SK} \cup \{\chi_{n-1}\} \cup \{\eta_n\} \cup \{\nu_n\}$.

The order in which this construction is carried out is immaterial (except that obviously f_n^i will have to be defined before $d_n^{i,j}$) what matters is that η_n, ν_n and θ_n are defined for each n in such a way that (*) becomes A-consistent.

From this point on we construct s_n^i from c_n^i and S from s_n^i exactly as for ϵA-logic, and we make the following claim.

2.8 Claim: Suppose φ and ψ are sentences of L^{+}_{SK}, then S satisfies the properties I to IV of 2.2, and in addition

V': If $A(t) \in S$ for some constant term t, then for some $b \in A$, $t = \overline{b} \in S$.

As previously remarked, I - IV yield that the term model constructed from S will be a model of T, but by V' it has to be an A-model. Hence what remains is to prove the claim.

Proof of 2.8: The proofs of I - IV are almost carbon copies of the corresponding proofs for $\in$A-logic. The fact that the c^i_n's this time are chosen more carefully does not alter anything.

V': Assume $\overline{A}(t(\overline{a})) \in S$ where t is some function symbol of L^{o}_{SK}, say, t is $t_i(x)$. (We assume for simplicity that t has only one variable place.) Suppose now that $\overline{A}(t_i(x))$ is the j'th formula in the enumeration of L^{o}_{SK}, and that $a \in A_n$ for some $n \geq \max\{i,j\}$. From these assumptions it now follows that $\overline{A}(t(\overline{a})) \in s^j_n$ (otherwise $\neg\overline{A}(t_i(\overline{a})) \in s^j_n$, contradicting $\overline{A}(t_i(\overline{a})) \in S$).

Hence we can conclude that

$$T_{SK} \cup \{\chi_n\} \underset{A}{\vdash} \overline{A}(t_i(\overline{a})) .$$

In constructing χ_n the function f^i_n was introduced, and from the formalized facts about the constant symbol $\overline{f}^i_n$ in A-logic (i.e. $T_{SK} \cup \{\chi_n\} \vdash \eta^i_n(\overline{f}^i_n))$ it follows that f^i_n actually maps ω into power (A_n) in such a way that $A_n = \bigcup_{k\in\omega} f^i_n(k)$. In particular this gives us some $k \in \omega$ such that $a \in f^i_n(k)$.

At some later stage $m = \max\{n,k\}$, we will then have that $T \cup \{\chi_m\} \underset{A}{\vdash} t_i(\overline{a}) \epsilon \overline{d}^{i,k}_n$. This, by the definition of ν_n.

Hence for some $l \in \omega$ we get that $d_n^{i,k} \in A_l$, and by applying the A-rule and the fact that A_l is transitive, it follows that $T \cup \{\chi_m\} \vdash_A t_i(\bar{a}) \epsilon \bar{A}_l$.

This corresponds to the point in the proof of 2.5 where (1) of that proof was obtained, and from this point on we can make use of the details of that proof, obtaining as the final conclusion that for some $b \in A$, $t_i(\bar{a}) = \bar{b} \in S$. $\dashv$

To end this section we will make some remarks on the role of axiom v) in A-logic.

Suppose we would try to prove the completeness theorem for A-logic the way it was done for ϵA-logic (except that ϵA-consistency should be replaced by A-consistency in the choices of the c_n^i's). Also in that case there would be implicit a construction of functions $f : \omega \to \text{power}(A_n)$. This, by the construction of c_k^i's for formulas like $t(x) = y$ (i.e. $f(k) = \{a \in A_n \mid \langle a,b \rangle \in c_k^i$ for some $b \in A_k\}$).

This stepwise construction of functions would, however, lead to serious trouble at the point of our proof of V' where we could conclude that the given $a \in A_n$ was a member of $f(k)$ for some k. This, because we at any particular stage would have constructed only a finite part of f , say $f \restriction m$, and there is no reason to believe that $A_n = \bigcup_{i=o}^{m} f(i)$ or that $a \in f(k)$ for some $k \leq m$.

The feature of axiom v) is exactly that it solves this problem by enabling us to choose the <u>whole</u> of f in <u>one</u> step.

We refer the reader to the papers by C. Karp [10] and J. Green [6], where they in their construction of consistency properties are entering the same problem. It should be fair to say, however, that their solution is a little less satisfactory since they must require that the sequence $\langle A_n \mid n \in \omega \rangle$ is definable in the logic considered. This leads to an unnecessary loss of generality in the formulation of their main results.

3. The Applications

Let the transitive set A and the L-structure $\mathfrak{A} = \langle A,f\rangle$ be fixed. We recall that a relation R over A is said to be $s\text{-}\underset{\sim}{\Pi}_1^1$ definable on $\mathfrak{A}$ (or just $s\text{-}\underset{\sim}{\Pi}_1^1$ on $\mathfrak{A}$) if there is a Σ formula $\psi(x_1,\dots,x_n,S_1,\dots,S_n)$ of $L \cup \{\bar{a} \mid a \in A\} \cup \{S_1,\dots,S_m\}$ such that:

$$R(a_1,\dots,a_n) \iff \mathfrak{A} \models \forall S_1 \dots S_m \psi(\bar{a}_1,\dots,\bar{a}_n) .$$

Let L^+ be as in section 1 and assume that L^+ contains a countable list of new relation symbols, including $S_1,\dots,S_m$. We can now use the fact that Σ formulas persist under end-extensions to obtain that

$R(a_1,\dots,a_n) \iff \langle N,h\rangle \models \psi(\bar{a}_1,\dots,\bar{a}_n)$ for all L^+-structures $\langle N,h\rangle$ such that $\langle N,h\restriction L\rangle$ is an end-extension of $\mathfrak{A}$.

If we reformulate this using the terminology of section 1 , we get

(1) $$R(a_1,\dots,a_n) \iff \models_{\in A} \psi(\bar{a}_1,\dots,\bar{a}_n) .$$

If A is countable or A is of strong cofinality ω we can use the completeness of $\in A$-logic to obtain

(2) $$R(a_1,\dots,a_n) \iff \vdash_{\in A} \psi(\bar{a}_1,\dots,\bar{a}_n) .$$

Suppose the formulas of $L^+_{\omega\omega}$ is coded in some way as elements of A . If for instance A is closed under ordinary pairing it is standard to show that the coding can be carried out such that the predicate $Ax_0(y) \underset{D}{\iff}$ "y is a code of one of the axioms i) - iii) of 1.3", is $\underset{\sim}{\Delta}$ definable over $\mathfrak{A}$. If in addition A is closed under the power set operation, the predicate $Ax_1(y) \underset{D}{\iff}$ "y is a code of one of the axioms of i) - iv) of 1.6 " can be given a $\underset{\sim}{\Delta}$ definition over the structure $(\mathfrak{A};\mathcal{P})$ - for short a $\underset{\sim}{\Delta}(\mathcal{P})$ definition - where $\mathcal{P}$ is the graph of the power set relation on A (i.e. $\mathcal{P}(a,b) \leftrightarrow a = \text{power}(b)$).

If we use this together with the definition of $\underset{\in A}{\vdash}$ we obtain that the set Val of codes of valid formulas of $\in$A-logic can be given an inductive definition Γ as follows:

$$(3) \quad x \in \Gamma(S) \underset{D}{\leftrightarrow} Ax(x) \vee \exists y(\ulcorner y \to x \urcorner \in S \wedge y \in S)$$
$$\vee \exists y(y = \ulcorner \varphi \to \psi(v) \urcorner \wedge y \in S \wedge x = \ulcorner \varphi \to \forall v \psi(v) \urcorner)$$
$$\vee \exists y(\forall z \epsilon y \ulcorner \varphi(\overline{z}/v) \urcorner \in S \wedge x = \ulcorner \forall v(v \epsilon \overline{y} \to \varphi(v)) \urcorner) .$$

Where Ax is either Ax_0 or Ax_1 depending on A being countable or of strong cofinality ω.

Hence we get that $Val(x) \leftrightarrow x \in I_\Gamma$. (For terminologi concerning inductive definability, see for instance our paper [11].)

If we apply this result to (2), we obtain:

$$(4) \qquad R(a_1,\ldots,a_n) \iff \psi(\overline{a}_1,\ldots,\overline{a}_n) \in I_\Gamma .$$

We summarize what is obtained so far in the next theorem.

3.1 Theorem. a) Assume A is countable, transitive and sufficiently closed under pairing (closure under set theoretic pairing is more than enough) then every $s\text{-}\underset{\sim}{\Pi}^1_1$ relation on $\mathfrak{A} = \langle A,f\rangle$ is $\underset{\sim}{\Sigma}$ inductively definable over $\mathfrak{A}$.

b) Assume A is of strong cofinality ω, then every $s\text{-}\underset{\sim}{\Pi}^1_1$ relation on $\mathfrak{A}$ (in fact on $\langle \mathfrak{A};\mathcal{P}\rangle$) is $\underset{\sim}{\Sigma}(\mathcal{P})$ inductively definable over $\mathfrak{A}$.

Proof: Immediate by the previous remarks. Just observe that (3) is a $\underset{\sim}{\Sigma}$ $(\underset{\sim}{\Sigma}(\mathcal{P}))$ definition of Γ.

3.1 a) is due to P. Aczel [1] and the proof we just gave is identical to his proof. 3.2 b) is due to Ph.W. Grant [6], but his proof is different as he employs a game theoretic argument.

Notice that if the relation R is $s\text{-}\underset{\sim}{\Pi}_1^1$ on $\langle A,f\rangle$ <u>in</u> some relation T (i.e. $R(a_1,\dots,a_n) \iff \langle A,f;T\rangle \models \Psi(a_1,\dots,a_n,T_+)$) where T occurs positive in the $s\text{-}\Pi_1^1$-formula Ψ), then the previous proof is easily modified to yield that T might be chosen to occur positive in the inductive definition Γ .

This can be achieved by replacing the predicate $Ax(x)$ in (3) by $Ax(x) \vee T_+(x)$ when $T_+(x) \underset{D}{\iff} x \in \{\ulcorner T(\bar{a}_1,\dots,\bar{a}_l)\urcorner \mid (a_1,\dots,a_l) \in T\}$. (i.e. T_+ is the codes of the positive diagram of T.) The observation to be made is that:

$$R(a_1,\dots,a_n) \iff \{T(\bar{a},\dots,\bar{a}_l) \mid a_1,\dots,a_l \in T\} \underset{\in A}{\vdash} \psi(\bar{a}_1,\dots,\bar{a}_n)$$

for the appropriate ψ .

By appealing to our lemma 2.7 of [11] we obtain, using the terminology of Definition 1.1 of that paper, that if $\langle A,f\rangle$ is countable admissible (A has strong cofinality ω and $\langle A,f,\mathcal{P}\rangle$ is admissible) then $\langle A,f\rangle$ ($\langle A,f,\mathcal{P}\rangle$) is <u>uniformly Σ_1-complete</u>.

Hence by our theorem 1.2 of [11] we get:

3.2 Theorem.

a) (The Barwise Compactness Theorem) If $\langle A,f\rangle$ is countable admissible, then $\langle A,f\rangle$ is $\underset{\sim}{\Sigma}_1$-compact.

b) (The Barwise-Karp Compactness Theorem) If A has strong cofinality ω and $\langle A,f\rangle$ is power set admissible (i.e. $\langle A,f,\mathcal{P}\rangle$ is admissible), then $\langle A,f\rangle$ is $\underset{\sim}{\Sigma}_1(\mathcal{P})$ compact.

The proof we outlined for 3.2 a) is similar to the proof described by Aczel in [2]. Our proof of b) is new since it involves the $\in A$-completeness, however, given the $\in A$-completeness theorem, the proof of b) is of course well known.

The relationship that exists between $s\text{-}\underset{\sim}{\Pi}_1^1$ and $\in A$-logic carries over to the similar relationship between $\underset{\sim}{\Pi}_1^1$ and A-logic. Since the arguments are similar to the ones just given in the beginning of this section, we restrict our selves to give the conclusion:

If R is $\underset{\sim}{\Pi}_1^1$ on $\langle A,f\rangle$, then for some first order formula ψ of L^+:

$$R(a_1,\dots,a_n) \iff \models_A \psi(\bar{a}_1,\dots,\bar{a}_n)$$

Again we can for sufficiently nice A code the formulas of L^+ and let Ax be the codes of the axioms; i) - iii) if A is countable and i) - v) if A is of strong cofinality ω. The inductive definition Γ' of the codes of valid formulas of A-logic is the same as Γ of $\in A$-logic, except that the disjunct relating to the $\in A$-rule is replaced by the A-rule:

$$\forall z(\ulcorner\varphi(\bar{z}/v)\urcorner \in S \wedge x = \ulcorner\forall v(\bar{A}(v) \to \varphi(v))\urcorner)$$

This has the effect that Γ' not is $\underset{\sim}{\Sigma}$ definable, but we obtain:

3.3 (The Abstract Suslin-Kleene Theorem)

a) Assume A is countable and closed under pairing, then every $\underset{\sim}{\Pi}_1^1$-relation over $\langle A,f\rangle$ is first order inductivly definable over $\langle A,f\rangle$.

b) If A is of strong cofinality ω, then every $\underset{\sim}{\Pi}_1^1$-relation over $\langle A,f\rangle$ is first order inductivly definable over $\langle A,f\rangle$.

(The power set relation does not enter into the formulation of b) since A is closed under power and hence the power set relation is first order definable, in fact Π_1^0 definable, over $\langle A,\in\rangle$.)

It should not be nessecary to repeat the history of a), let us just mention that the proof we have given here is implicit in Barwise [3].

The result 3.2 b) is due to Chang-Moschovakis [5] even if it there is phrased a little less general. Our proof is new, however, the Chang-Moschovakis proof involves a game argument.

The previous comments regarding s-$\underset{\sim}{\Pi}_1^1$ in some extra relations apply equally well to $\underset{\sim}{\Pi}_1^1$. Thus by adding codes of the positive diagrams of the relations involved, we obtain, with the terminology of our paper [11], that $\langle A,f\rangle$ is a <u>uniform Kleene structure</u> where A is as in 3.3 a) or b).

As we showed in [11] also this result can be used to obtain Σ_1-compactness theorems, but this requires some more effort than the ones in 3.2 , and is the main content of that paper.

We have during the preparation of this paper had hopes that it should be possible to prove a stronger omitting types theorem than the ones implicit in the completeness theorems for $\in$A-logic and A-logic. The aim would be to do some analogs of the results in Grilliot [8]. At present we have serious doubts that this is possible, but with the hope that someone should find a counter example (or a proof) we will be more explicit and state one open problem.

Does the Gandy-Kreisel-Tait Theorem hold for structures $\langle A,\in\rangle$ when A has strong cofinality ω ? (i.e. is the Δ_1^1-relations of $\langle A, \in \rangle$ exactly the relations definable in all A-models?) A negative answer to this would set a limit for how general omitting types theorems it is possible to get in this setting, while a positive answer should indicate the existence of more general omitting types theorems.

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