FINITE SUBALGEBRAS OF A von NEUMANN ALGEBRA.*

Christian F. Skau

Abstract. If $\mathcal{M}$ is a von Neumann algebra on the Hilbert space H with a separating vector x in H we show that there is a 1 – 1 correspondence between the family of finite von Neumann subalgebras of $\mathcal{M}$ and certain projections $P(\mathcal{M},x)$ in $\mathcal{B}(H)$, which we explicitly characterize in terms of $\mathcal{M}$ and x. If $\mathcal{M}$ itself is finite with x a trace vector for $\mathcal{M}$ the family of projections $P(\mathcal{M},x)$ is closely related to the conditional expectations of $\mathcal{M}$ onto the various von Neumann subalgebras of $\mathcal{M}$ leaving the trace $\omega_x|_{\mathcal{M}}$ invariant.

1. INTRODUCTION.

Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert space H with a separating vector $x \in H$, i.e. $Mx = 0$ for $M \in \mathcal{M}$ implies $M = 0$. In [19] and [16] it was shown that there is a 1 – 1 correspondence between the family of abelian von Neumann subalgebras of $\mathcal{M}$ and a family of orthogonal projections $Q(\mathcal{M},x)$ in $\mathcal{B}(H)$ characterized by the property that $q \in Q(\mathcal{M},x)$ iff $x \in \text{range}(q)$ and $q\mathcal{M}'q$ is a commutative family of operators.

The program of this paper is to prove a more general result which also will give additional information about the correspondence alluded to above. Specifically, we establish a 1 – 1 order-preserving correspondence between the family of finite von Neumann

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subalgebras of $\mathcal{M}$ and a family of orthogonal projections $P(\mathcal{M},x)$ in $\mathcal{B}(H)$ characterized by the property that $p \in P(\mathcal{M},x)$ iff $x \in \text{range}(p)$ and $p\mathcal{M}'p$ generates a finite von Neumann algebra with a separating vector on $p(H)$ (Theorem 4.3). The crucial lemma in proving this result is stated in Theorem 4.1 and seems to be of some interest in itself. The projections $Q(\mathcal{M},x)$ referred to above is a closed subfamily of $P(\mathcal{M},x)$ in the strong-operator topology.

We proceed to show that in case $\mathcal{M}$ itself is a finite von Neumann algebra with x a separating trace vector for $\mathcal{M}$ the family of projections $\dot{P}(\mathcal{M},x)$, which then will be strong-operator closed, is closely related to the family $\Phi(\mathcal{M})$ of conditional expectations of $\mathcal{M}$ onto the various von Neumann subalgebras of $\mathcal{M}$ which is preserved by the faithful trace $\omega_x|_{\mathcal{M}}$. In fact, if we endow $\Phi(\mathcal{M})$ with the topology of pointwise convergence in the strong-operator topology it becomes homeomorphic with $P(\mathcal{M},x)$ in the strong-operator topology (Theorem 5.8). Furthermore, if we restrict ourselves to subalgebras containing the center of $\mathcal{M}$ this topology is independent of the particular trace vector x . Endowing $P(\mathcal{M},x)$ and the unitary group $\mathcal{U}$ in $\mathcal{M}$ with the strong-operator topology we organize $(\mathcal{U}, P(\mathcal{M},x))$, and hence $(\mathcal{U}, \Phi(\mathcal{M}))$, into a topological transformation group (Theorem 5.5). We also show that for $\mathcal{M}$ finite the projections $P(\mathcal{M},x)$ are closed under intersection for x any separating vector for $\mathcal{M}$ and we give a counterexample to show that this is not true if $\mathcal{M}$ is not finite.

In trying to prove the results quoted we are faced with the following (profound) question: For $\mathcal{R}$ a von Neumann algebra on H and x a vector in H does each vector z in $[\mathcal{R}x]$

have the form Tx , T being a closed densely-defined operator affiliated $\mathcal{R}$ (cf. Definition 2.1)? A partial and extremely useful answer was obtained by Murray-von Neumann [12; Lemma 9.2.1] - an arbitrary vector z in $[\mathcal{R}x]$ has the form z = BTx , where B is a (bounded) operator in $\mathcal{R}$ and T is a closed densely-defined operator affiliated $\mathcal{R}$. We call this result the "BT-theorem" and the first, when valid, the "T-theorem" (with respect to x) . It turns out that for finite von Neumann algebras the T-theorem holds, in fact, the validity of the T-theorem with respect to all y in $[\mathcal{R}x]$ is equivalent to $[\mathcal{R}x]$ being a finite projection in $\mathcal{R}'$, cf. [8].

The T-theorem for finite von Neumann algebras wil follow as a corollary of the BT-theorem and the theory of closed densely defined operators affiliated a finite von Neumann algebra as developed by Murray-von Neumann [12] (cf. also [18]). We will sketch the main features of that theory since we are going to make extensive use of it in proving our results. Besides this will make this paper more self-contained.

The elegant proof we give of Lemma 3.4, from which the BT-theorem is an immediate corollary, is due to R.V. Kadison (unpublished) and we present it here with his kind permission. We are also indebted to R.V. Kadison whose questions at the Functional Analysis Seminar at the University of Pennsylvania (Fall 1973) suggested ideas which led to considerable improvements in our original results.

2. NOTATION AND PRELIMINARIES.

By a von Neumann algebra $\mathcal{R}$ acting on the (complex) Hilbert space H we mean a self-adjoint algebra of operators $\mathcal{R}$ in $\mathcal{B}(H)$, the bounded operators on H , that contains the identity operator on H and is closed in the strong-operator topology. By the von Neumann Bicommutant Theorem we have $\mathcal{R} = \mathcal{R}''$, where we denote by $\mathcal{F}'$ the set of operators in $\mathcal{B}(H)$ commuting with a family $\mathcal{F}$ of operators in $\mathcal{B}(H)$.

We use the symbol $(\cdot,\cdot)$ to denote the inner product in H . By a projection in $\mathcal{B}(H)$ we will always mean an orthogonal projection. If p and q are projections in $\mathcal{B}(H)$ onto the closed subspaces E and F , respectively, we denote by $p \vee q (p \wedge q)$ the projection onto the subspace $E \oplus F (E \cap F)$, where $E \oplus F$ is the smallest closed subspace containing E and F .

If A is a subset of a topological space we denote by $\bar{A}$ the closure of A . Let X be a subset of the Hilbert space H and let $\mathcal{F}$ be a family of operators in $\mathcal{B}(H)$. Then we write $[\mathcal{F}X]$ for the closure of the linear span of $\{Fx | F \in \mathcal{F}, x \in X\}$. In particular, if X consists of one point x and $\mathcal{F}$ is a linear family of operators then $[\mathcal{F}x] = \overline{\{Fx | F \in \mathcal{F}\}}$. We will interchangeably use the notation $[\mathcal{F}X]$ for the orthogonal projection onto the closed linear subspace $[\mathcal{F}X]$. It will be clear from the context what we mean in each case. The range projection of an operator A in $\mathcal{B}(H)$ is the projection onto the subspace $[A(H)] = \overline{\text{range}(A)}$. If A is in the von Neumann algebra $\mathcal{R}$ then the range projection of A will be in $\mathcal{R}$.

If $x \in H$ we denote by ω_x the positive linear functional on $\mathcal{B}(H)$ defined by $A \to (Ax,x)$. We denote by $\omega_x|_{\mathcal{R}}$ the restriction of ω_x to $\mathcal{R}$.

Throughout this paper concepts and results from the theory of von Neumann algebras and C^*-algebras will be used quite freely. Our general references are the two books by Dixmier [3], [4].

We now introduce some terminology and definitions that will be useful in the next section. By the term "operator" we will mean a linear mapping T defined on a dense linear manifold $\mathcal{D}(T)$ of the Hilbert space H and with range $\mathcal{R}(T)$ in H. If T_1 and T_2 are two operators we write $T_1 \subset T_2$ if $\mathcal{D}(T_1) \subset \mathcal{D}(T_2)$ and T_1 and T_2 agree on $\mathcal{D}(T_1)$. We write $T_1 = T_2$ if $\mathcal{D}(T_1) = \mathcal{D}(T_2)$ and T_1 and T_2 agree on their common domain of definition.

If S and T are two operators with domains of definition $\mathcal{D}(S)$ and $\mathcal{D}(T)$, respectively, then by definition $\mathcal{D}(S+T) = \mathcal{D}(S) \cap \mathcal{D}(T)$ and $(S+T)x = Sx + Tx$ for $x \in \mathcal{D}(S+T)$. Also, $\mathcal{D}(ST) = \{x \in H \mid x \in \mathcal{D}(T), Tx \in \mathcal{D}(S)\}$ and $(ST)x = S(Tx)$ for $x \in \mathcal{D}(ST)$. It may well happen that $\mathcal{D}(S+T)$ (or $\mathcal{D}(ST)$) is not a dense set in H and hence $S+T$ (or ST) is not an operator according to the definition we have adopted. However, as we shall see in the next section these difficulties evaporates when we deal with operators affiliated a finite von Neumann algebra.

Definition 2.1. Let $\mathfrak{M}$ be a von Neumann algebra on H. Let T be an operator in H. Then T is affiliated $\mathfrak{M}$, in symbols $T \eta \mathfrak{M}$, if $M'T \subset TM'$ for every $M' \in \mathfrak{M}$. This is equivalent to $U'T = TU'$ for every unitary operator U' in $\mathfrak{M}$.

We notice that if T is bounded with $\mathcal{D}(T) = H$ then $T \eta \mathfrak{M}$ is equivalent to $T \in \mathfrak{M}$.

Definition 2.2. Let T be an operator in H. The graph of T is the linear subset of $H \times H$ defined by $\mathcal{G}(T) = \{(x, Tx) \mid x \in \mathcal{D}(T)\}$.

We say that T is <u>closed</u> if $\mathcal{G}(T)$ is a closed set in $H \times H$. We say that T is <u>closable</u> with closure $\overline{T}$ if the closure of $\mathcal{G}(T)$ defines an operator $\overline{T}$ in H.

<u>Remark.</u> It is a simple observation that T is closed if and only if $x_n \to x$, $Tx_n \to y$ for $\{x_n\}$ in $\mathcal{D}(T)$ implies $x \in \mathcal{D}(T)$ and $Tx = y$. Hence if T is closed and bounded then $\mathcal{D}(T) = H$ and so $T \in \mathcal{B}(H)$.

<u>Definition 2.3.</u> The operator T is said to be <u>symmetric</u> if $(Tx,y) = (x,Ty)$ for every pair of points x,y in $\mathcal{D}(T)$. A symmetric operator T is <u>positive</u> if $(Tx,x) \geq 0$ for all $x \in \mathcal{D}(T)$. If $T = T^*$ we say T is <u>self-adjoint</u>, where $\mathcal{D}(T^*)$ consists of those y in H such that $x \to (Tx,y)$, $x \in \mathcal{D}(T)$, is a bounded linear functional and T^*y is defined by the equation $(Tx,y) = (x,T^*y)$, $x \in \mathcal{D}(T)$.

<u>Remark.</u> It is easily seen that a self-adjoint operator is closed and has no proper symmetric extensions [7; Chapter XII].

We state the following well-known theorems about closed unbounded operators, referring to [7; Chapter XII] for proofs.

<u>Spectral Resolution.</u> Let T be a self-adjoint operator in H. Then there is a uniquely determined regular countably additive projection-valued measure E defined on the Borel sets of the real line and related to T by the equations

$$\mathcal{D}(T) = \{x \mid x \in H, \int_{-\infty}^{\infty} \lambda^2 (E(d\lambda)x,x) < \infty\}$$

and

$$Tx = \lim_{n \to \infty} \int_{-n}^{n} \lambda (E(d\lambda)x), \quad x \in \mathcal{D}(T).$$

Polar Decomposition. If T is a closed operator in H then T can be written in one and only one way as a product $T = PA$, where P is a partial isometry whose initial space is $\overline{\text{range}\ (T^*)}$ and whose final space is $\overline{\text{range}\ (T)}$, and A is a positive self-adjoint operator such that $\overline{\text{range}\ (A)} = \overline{\text{range}\ (T^*)}$.

Remarks. It is easy to verify that if T is a self-adjoint operator and $\mathfrak{M}$ is a von Neumann algebra then $T \eta \mathfrak{M}$ if and only if the range of the projection-valued measure E in the spectral resolution of T is in $\mathfrak{M}$ [7; Chapter XII]. We also have that if T is a closed operator with polar decomposition $T = PA$ then $T \eta \mathfrak{M}$ if and only if $P \in \mathfrak{M}$ and $A \eta \mathfrak{M}$ [12; Lemma 4.4.1]. From these two observations it follows that if T is a closed operator affiliated a von Neuman algebra $\mathfrak{M}$ and x is in $\mathfrak{D}(T)$ then we can find a sequence of (bounded) operators $\{M_n\}$ in $\mathfrak{M}$ such that $Tx = \lim_n M_n x$.

In the next section we are going to consider finite von Neumann algebras, i.e. those von Neumann algebras where the only isometries are the unitary operator. This is equivalent to the existence of a faithful center-valued trace, cf. [3; Chapter III, § 8; 1].

3. OPERATORS AFFILIATED A FINITE von NEUMANN ALGEBRA.

This section will contain the basic material for proving our results. We first prove two simple lemmas that we shall need.

Lemma 3.1. Let T be an operator in H affiliated $\mathcal{M}$, where $\mathcal{M}$ is a von Neumann algebra acting on H. Assume T has a closure $\overline{T}$. Then $\overline{T} \eta \mathcal{M}$.

Proof. Let $y \in \mathcal{D}(\overline{T})$. Then there is a sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $x_n \to y$ and $Tx_n \to z = \overline{T}y$. Now let $M' \in \mathcal{M}'$. We have $M'Tx_n = TM'x_n$ for each n. Also $M'x_n \to M'y$ and $TM'x_n \to M'z$. Since $\{M'x_n\} \subset \mathcal{D}(T)$ we have $M'y \in \mathcal{D}(\overline{T})$ and $\overline{T}M'y = M'z = M'\overline{T}y$. This shows that $M'\overline{T} \subset \overline{T}M'$ and so $\overline{T} \eta \mathcal{M}$.

Lemma 3.2. Let T be a closed operator in H affiliated both $\mathcal{M}_1$ and $\mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras on H. Then $T \eta \{\mathcal{M}_1 \cap \mathcal{M}_2\}$.

Proof. We have $\{\mathcal{M}_1 \cap \mathcal{M}_2\}' = \{\mathcal{M}_1' \cup \mathcal{M}_2'\}''$. We observe that $AT \subset TA$ for any A in the *-algebra $\mathfrak{A}$ generated by $\mathcal{M}_1' \cup \mathcal{M}_2'$. Let $M' \in \{\mathcal{M}_1 \cap \mathcal{M}_2\}'$. Then there exists a net $\{A_\alpha\}$ in $\mathfrak{A}$ converging to M' in the strong-operator topology. Let $x \in \mathcal{D}(T)$. Then $A_\alpha Tx = TA_\alpha x$ for each α. Now $A_\alpha x \to M'x$ and $A_\alpha Tx \to M'Tx$. Since T is closed we get that $M'x \in \mathcal{D}(T)$ and $TM'x = M'Tx$. Hence $M'T \subset TM'$ and so $T \eta \{\mathcal{M}_1 \cap \mathcal{M}_2\}$.

Much of the difficulty in manipulations with unbounded operators lies in the unrelatedness of the domain and range of one such operator with the domain of another. When we know that these sets have "many" vectors in common, much of the difficulty disappears. It turns out that in the case $\mathcal{M}$ is a finite von Neumann algebra the closed operators affiliated $\mathcal{M}$ can be manipulated with in much the same way as bounded operators. The key lemma to this effect is Lemma 16.2.3 in [12], where finite factors are considered. By a slight generalization the non-factor case is settled in Theorem 3 in [18]. Specifically, Murray and von Neumann call a dense linear manifold K in the Hilbert space H "essentially dense" with respect to the von Neumann algebra $\mathfrak{M}$ if K is the union of an ascending sequence of closed linear manifolds whose projections belong to $\mathfrak{M}$. We state the basic result: Let $\mathfrak{M}$ be a finite von Neumann algebra on H and let K be essentially dense with respect to $\mathfrak{M}$. If T is a closed operator affiliated $\mathfrak{M}$ then the set $\{x \in H \mid x \in \mathfrak{D}(T), Tx \in K\}$ is essentially dense with respect to $\mathfrak{M}$. For the proof one utilizes the existence of a center-valued dimension function for $\mathfrak{M}$ (or a center-valued faithful trace for $\mathfrak{M}$) . Using this basic result one can show that closed operators affiliated $\mathfrak{M}$ can be added, multiplied and adjoints taken, and the resulting operators will have essentially dense domains of definition and be affiliated $\mathfrak{M}$. Besides they will be closable and their closures will by Lemma 3.1 again be affiliated $\mathfrak{M}$. Cf. Theorem 4 in [18].

We need one final result in dealing with closed operators affiliated a finite von Neumann algebra. The key lemma is Lemma 16.4.1 in [12] (or Theorem 5 in [18]) which effectively says that a closed symmetric operator affiliated a finite von Neumann algebra is self-adjoint (see Definition 2.3).

This is proved by application of von Neumann's elegant extension procedure for symmetric operators using the Cayley transform of an operator [14].

Applying the lemma cited above we can prove the following unique extension result, which is essentially the same as Lemma 16.4.2 in [12] and Corollary 5.1 in [18].

Lemma 3.3. Let $\mathcal{M}$ be a finite von Neumann algebra on H. Let S and T be two closed operators that are affiliated $\mathcal{M}$. If S and T agree on a dense set K in H, then $S = T$.

Proof. Form the operator $S - T$. By the discussion above $S - T$ will be closable (Theorem 4 [18]) and this closure must be equal to 0 by the hypothesis. Hence S and T agree on $\mathcal{D}(S) \cap \mathcal{D}(T)$ which is an essentially dense set with respect to $\mathcal{M}$ (Theorem 4 [18]). Set

$$V = S\big|_{\mathcal{D}(S) \cap \mathcal{D}(T)} = T\big|_{\mathcal{D}(S) \cap \mathcal{D}(T)} .$$

As S and T are closed operators, V is closable with closure $\overline{V}$. Clearly S and T are extensions of $\overline{V}$. It is easily verified that V is affiliated $\mathcal{M}$ and by Lemma 3.1 we get $\overline{V} \eta \mathcal{M}$.

Let $S = WB$ be the polar decomposition of S. Then $W \in \mathcal{M}$ is a partial isometry with initial space $\overline{\text{range}\ (B)}$ and final space $\overline{\text{range}\ (S)}$, and B is a positive selfadjoint operator affiliated $\mathcal{M}$. By associativity of multiplication of unbounded operators we get $W^*S = W^*(WB) = (W^*W)B = B$. Hence $S = WB = WW^*S$; and as S is an extension of $\overline{V}$ we also have $\overline{V} = WW^*\overline{V}$. Now $W^*S = B$ is self-adjoint, hence it is symmetric, and so $W^*\overline{V}$ is symmetric. Besides $W^*\overline{V}$ is closed, which follows immediately from the observation that W^* is a partial isometry with initial space $\overline{\text{range}\ (S)}$ and $\overline{\text{range}\ (\overline{V})} \subset \overline{\text{range}\ (S)}$. Also, $W^*\overline{V}$ has a dense domain of definition and is affiliated $\mathcal{M}$. By Lemma 16.4.1 in [12] referred to above $W^*\overline{V}$ is self-adjoint. Since $B = W^*S$ is a symmetric extension of $W^*\overline{V}$ it follows that $W^*S = W^*\overline{V}$, and hence $S = WW^*S = WW^*\overline{V} = \overline{V}$. Analogously we can show $T = \overline{V}$. This gives us $S = T$.

By using the results established above it is now straightforward to show that the family of closed unbounded operators affiliated a finite von Neumann algebra form a *-algebra, where the closures of $S+T$ and ST are defined to be the sum and product of S and T, respectively, in this algebra. The *-operation is the usual one. (Cf. Theorem XV, Chapter XVI [12] and Corollary 5.2 [18].)

The following lemma is a version of the BT-theorem that we alluded to in the Introduction. Taken together with the theory we have just outlined, we get as a corollary the T-theorem for finite von Neumann algebras.

<u>Lemma 3.4</u>. Let $\mathcal{R}$ be a von Neumann algebra acting on the Hilbert space H and let $x \in H$. If $z \in [\mathcal{R}x]$ there are operators V, B in $\mathcal{R}$, V being self-adjoint, and a vector y in $[\mathcal{R}x]$ orthogonal to the null space of V such that $Vy = x$, $By = z$.

Proof. Without loss of generality we may assume that $\|x\| = \|z\| = 1$. Since $z \in [\mathcal{R}x]$ there are operators $\{T_n\}$ in $\mathcal{R}$ such that $\sum_{n=0}^{\infty} T_n x = z$ and $\|T_n x\| \leq 4^{-n}$. If $H_n^2 = I + \sum_{n=0}^{n} 4^k T_k^* T_k$ then $\{H_n^2\}$ is a monotone increasing sequence of positive invertible operators in $\mathcal{R}$. H_n is the positive square root (in $\mathcal{R}$) of H_n^2. Recall that if J and K are positive invertible operators such that $J \leq K$ then $K^{-1} \leq J^{-1}$ [4; p.15]. Hence $\{H_n^{-2}\}$ is a monotone decreasing sequence of positive invertible operators tending in the strong-operator topology to some operator. Thus $\{H_n^{-1}\} = \{(H_n^{-2})^{\frac{1}{2}}\}$ tends strongly to some positive operator V in $\mathcal{R}$ since each real-valued continuous function f on the real line is strong-operator continuous on bounded sets of self-adjoint operators [11]. In this case $f(t)$ equals 0 for $t < 0$ and $\sqrt{t}$ for $t \geq 0$. Note that

$$\|H_n x\|^2 = (H_n^2 x,x) = (x,x) + \sum_{k=o}^{n} 4^k \|T_k x\|^2 \leq 1 + \sum_{k=o}^{n} 4^{-k} < 3 .$$

Thus the sequence $\{H_n x\}$ lies in the ball of radius $\sqrt{3}$ in H which is a weakly compact set in H. Hence some subnet $\{H_{n'} x\}$ converges weakly to a vector y' in H. Clearly $y' \in [\mathcal{R} x]$. We assert that $Vy' = x$. In fact, let $\epsilon > 0$ and a vector w in H be given. There is a positive integer N such that $\|H_n^{-1} w - Vw\| < \frac{\epsilon}{6}$ if $n \geq N$ and $|(H_{n'} x - y', Vw)| < \frac{\epsilon}{2}$ if $n' \geq N$. Then with $n' \geq N$, $|(H_{n'} x, Vw) - (H_{n'} x, H_{n'}^{-1} w)| = |(H_{n'} x, Vw) - (x,w)| \leq \|H_{n'} x\| \|Vw - H_{n'}^{-1} w\| \leq \frac{\epsilon}{6} \|H_{n'} x\| \leq \frac{\epsilon}{2}$. In addition, $|(y', Vw) - (H_{n'} x, Vw)| < \frac{\epsilon}{2}$ so that $|(y', Vw) - (x,w)| < \epsilon$, i.e. $|(Vy' - x, w)| < \epsilon$. Thus $(Vy' - x, w) = 0$ for all w and hence $Vy' = x$. For fixed n and $m \geq n$ we have

$$0 \leq 4^n H_m^{-1} T_n^* T_n H_m^{-1} \leq H_m^{-1} \left(\sum_{k=o}^{m} 4^k T_k^* T_k \right) H_m^{-1} =$$

$$= \left(I + \sum_{k=o}^{m} 4^k T_k^* T_k\right)^{-\frac{1}{2}} \left(\sum_{k=o}^{m} 4^k T_k^* T_k \right) \left(I + \sum_{k=o}^{m} 4^k T_k^* T_k\right)^{-\frac{1}{2}}$$

$$= \frac{\sum_{k=o}^{m} 4^k T_k^* T_k}{I + \sum_{k=o}^{m} 4^k T_k^* T_k} \leq I .$$

(We have used that if $A \leq B$ then $C^*AC \leq C^*BC$ for any operators A,B and C [4; p.14].)

Since $H_m^{-1} \to V$ in the strong-operator topology, we have $0 \leq 4^n V T_n^* T_n V \leq I$. Thus $2^n \|T_n V\| \leq 1$ and so $\|T_n V\| \leq \frac{1}{2^n}$.

Thus $\sum_{n=o}^{\infty} T_n V$ converges in the norm topology to an operator B in $\mathcal{R}$. Hence $z = \sum_{n=o}^{\infty} T_n x = \sum_{n=o}^{\infty} T_n V y' = By'$. Replacing y' by its projection y on the orthogonal complement of the null space of V we have $Vy = x$ and $By = \sum_{n=o}^{\infty} T_n V y = z$.

Since the projection q on $[\mathcal{R} x]$ is in $\mathcal{R}'$ and $y' \in [\mathcal{R} x]$, it follows that $y \in [\mathcal{R} x]$. In fact, let p be the projection onto the null space of V. Then $p \in \mathcal{R}$ and $y = (1-p)y' = (1-p)qy' = q(1-p)y' \in [\mathcal{R} x]$.

This concludes the proof.

Corollary 1 (BT-theorem). Let $\mathcal{R}$ be a von Neumann algebra on H and let x be a vector in H. If $z \in [\mathcal{R} x]$ there is an operator B in $\mathcal{R}$ and a closed (densely-defined) operator T affiliated $\mathcal{R}$ such that $x \in \mathcal{D}(T)$ and $z = BTx$.

Proof. Let B, V and y be as in lemma. Let p be the range projection of the operator V, i.e. p is the projection onto $H_1 = [V(H)]$. Then p is in $\mathcal{R}$ and we have $\text{null}(V) = \text{range}(V^*)^{\perp} = \text{range}(V)^{\perp} = H_1^{\perp}$. In particular, $y \in H_1$. V restricted to H_1 is a $1-1$ mapping of H_1 onto the dense linear manifold $V(H_1) = V(H)$ in H_1. Define T to be the operator in H with domain of definition $\mathcal{D}(T) = V(H_1) \oplus H_1^{\perp}$ and $T(Vh_1 \oplus h_2) = h_1$; $h_1 \in H_1$, $h_2 \in H_1^{\perp}$.

It is a routine matter to verify that T is a closed densely-defined operator in H that is affiliated $\mathcal{R}$. Since $Vy = x \in H_1$ we have $x \in \mathcal{D}(T)$ and $Tx = y$. Thus $BTx = By = z$.

Corollary 2 (T-theorem). Let $\mathcal{R}$ be a finite von Neumann algebra on H and let x be a vector in H. If $z \in [\mathcal{R} x]$ there exists a closed (densely-defined) operator T affiliated $\mathcal{R}$ such that $x \in \mathcal{D}(T)$ and $Tx = z$.

Proof. Immediate consequence of Corollary 1 and the theory outlined above of operators affiliated a finite von Neumann algebra. Specifically, the operator BT in Corollary 1 has a (densely-defined) closure which is affiliated $\mathcal{R}$.

We end this section by stating a theorem about finite von Neumann algebras referring to [13], [17; Proposition 2.9.2] and [3; Chapter I, § 6; 3 & Chapter III, § 1; 5] for proofs. We need the following definition.

Definition 3.5. A vector x in the Hilbert space H is said to be a trace vector for a von Neumann algebra $\mathcal{R}$ on H if x is separating for $\mathcal{R}$ and $(ABx,x) = (BAx,x)$ for all A and B in $\mathcal{R}$. In other words, $\omega_x|_{\mathcal{R}}$ is a faithful trace on $\mathcal{R}$.

Theorem 3.6. If $\mathcal{R}$ is a finite von Neumann algebra acting on H with a cyclic and separating vector in H then $\mathcal{R}'$ is finite, and there is a vector x in H which is a cyclic trace vector for $\mathcal{R}$. Then x is also a trace vector for $\mathcal{R}'$. For each A in $\mathcal{R}$ there is a unique A' in $\mathcal{R}'$ such that $Ax = A'x$. The mapping $A \to A'$ is a $*$-antiisomorphism of $\mathcal{R}$ onto $\mathcal{R}'$. The cyclic and separating vectors for $\mathcal{R}$ coincide.

4. MAIN RESULTS.

We state and prove a theorem that will be crucial in what follows and which seems to be of some independent interest.

<u>Theorem 4.1</u>. Let $\mathfrak{M}$ be a von Neumann algebra on H with a separating vector x in H. Let $\mathfrak{N}$ be a <u>finite</u> von Neumann subalgebra of $\mathfrak{M}$. If $M \in \mathfrak{M}$ and $Mx \in [\mathfrak{N}x]$ then $M \in \mathfrak{N}$.

Proof. By the T-theorem for finite von Neumann algebras (Cor. 2 to Lemma 3.4) we have $Mx = Tx$ for some closed operator T affiliated $\mathfrak{N}$. Now $N'T \subset TN'$ for each N' in $\mathfrak{N}'$ and so, in particular, we have since $\mathfrak{M}' \subset \mathfrak{N}'$

$$MM'x = M'Mx = M'Tx = TM'x$$

for each $M' \in \mathfrak{M}'$. Thus T and M agree on the set $\{\mathfrak{M}'x\}$ which is dense in H since x is separating for $\mathfrak{M}$, hence cyclic for $\mathfrak{M}'$ [3; p.6]. Since M is bounded and T is closed we must have $\mathcal{D}(T) = H$ and $M = T$. Also $T \eta \mathfrak{N}$ implies $T \in \mathfrak{N}$. So $M \in \mathfrak{N}$.

<u>Definition 4.2</u>. Let $\mathfrak{M}$ be a von Neumann algebra on the Hilbert space H with a separating vector x in H. Let p be a projection in $\mathcal{B}(H)$ (not necessarily in $\mathfrak{M}$ or $\mathfrak{M}'$!) such that

(i) $x \in \text{range}(p) = p(H)$.

(ii) $p\mathfrak{M}'p$ generates a finite von Neumann algebra $\mathcal{R}$ on $p(H)$ such that $\mathcal{R}$ has a separating vector.

Then we say p is a <u>finite projection associated with $\mathfrak{M}$ and x</u>. We denote these projections by $P(\mathfrak{M},x)$.

Remark 1. Let $p \in P(\mathcal{M},x)$ and let $\mathcal{R}$ be the von Neumann algebra generated by $p\mathcal{M}'p$ on $p(H)$. Since x is separating for $\mathcal{M}$, hence cyclic for $\mathcal{M}'$, it follows that x is cyclic for $\mathcal{R}$. Now the cyclic and separating vectors for $\mathcal{R}$ coincide since $\mathcal{R}$ is a finite von Neumann algebra (Theorem 3.6). Hence x is a separating vector for $\mathcal{R}$.

Remark 2. An important subfamily of $P(\mathcal{M},x)$ is $Q(\mathcal{M},x)$ characterized by the property that $q \in Q(\mathcal{M},x)$ iff $x \in \text{range}(q)$ and $q\mathcal{M}'q$ is an abelian family of operators. Indeed, if $q \in Q(\mathcal{M},x)$ then $q\mathcal{M}'q$ generates an abelian, hence finite, von Neumann algebra $\mathcal{C}$ on $q(H)$. Since x is a cyclic vector for $\mathcal{C}$ it follows that $\mathcal{C}$ is maximal abelian in $\mathcal{B}(q(H))$, i.e. $\mathcal{C} = \mathcal{C}'$ [3; Chapter I, § 6; 3]. So x is separating for $\mathcal{C}$.

Theorem 4.3. Let $\mathcal{M}$ be a von Neumann algebra on H with a separating vector x in H. Then there is a 1-1 correspondence between the family $F(\mathcal{M})$ of finite von Neumann subalgebras of $\mathcal{M}$ and the family of projections $P(\mathcal{M},x)$ defined in Definition 4.2. Specifically, if $\mathcal{N} \in F(\mathcal{M})$ then $\mathcal{N}$ corresponds to the projection $p_{\mathcal{N}}$ in $P(\mathcal{M},x)$ whose range is $[\mathcal{N}x]$ and we have $\mathcal{N} = \{\mathcal{M}' \cup p_{\mathcal{N}}\}'$. Also, $\mathcal{N}$ is *-antiisomorphic to the von Neumann algebra generated by $p_{\mathcal{N}}\mathcal{M}'p_{\mathcal{N}}$ on $[\mathcal{N}x]$.
This correspondence preserves ordering, i.e. $\mathcal{N}_1 \subset \mathcal{N}_2$ for $\mathcal{N}_1, \mathcal{N}_2 \in F(\mathcal{M})$ if and only if $p_{\mathcal{N}_1} \leq p_{\mathcal{N}_2}$. Moreover, if $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N} \in F(\mathcal{M})$ such that $\mathcal{N}_1 \cup \mathcal{N}_2 \subset \mathcal{N}$ then $p_{\mathcal{N}_1 \cap \mathcal{N}_2} = p_{\mathcal{N}_1} \wedge p_{\mathcal{N}_2}$.

Proof. Let $\mathcal{N}$ be a finite von Neumann subalgebra of $\mathcal{M}$ and let $p_{\mathcal{N}}$ be the projection onto $[\mathcal{N}x]$. Then $p_{\mathcal{N}} \in \mathcal{N}'$ and the central carrier of $p_{\mathcal{N}}$ is the identity operator on H since x is cyclic for $\mathcal{M}'$, hence cyclic for $\mathcal{N}'$, and so $\mathcal{N}$ is

*-isomorphic to the von Neumann algebra $\mathcal{N}p_{\mathcal{N}}$ on $[\mathcal{N}x]$ (cf. [3; Chapter I, § 1 & § 2]). Now $x \in [\mathcal{N}x]$ is separating for $\mathcal{N}p_{\mathcal{N}}$. In fact, let $N \in \mathcal{N}$ and let $Np_{\mathcal{N}}x = Nx = 0$. Then $N = 0$ since x is separating for $\mathcal{M}$ and so $Np_{\mathcal{N}} = 0$. Clearly x is cyclic for $\mathcal{N}p_{\mathcal{N}}$ and so x is separating for $(\mathcal{N}p_{\mathcal{N}})'$. Now

$$(\mathcal{N}p_{\mathcal{N}})' = p_{\mathcal{N}}\mathcal{N}'p_{\mathcal{N}} \supset p_{\mathcal{N}}\mathcal{M}'p_{\mathcal{N}}$$

and since $(\mathcal{N}p_{\mathcal{N}})'$ is a finite von Neumann algebra (Theorem 3.6) it follows that $p_{\mathcal{N}}\mathcal{M}'p_{\mathcal{N}}$ generates a finite von Neumann algebra $\mathcal{R}$ on $\text{range}(p_{\mathcal{N}}) = [\mathcal{N}x]$. Clearly x is separating for $\mathcal{R}$. So $p_{\mathcal{N}} \in P(\mathcal{M},x)$. Note that x is cyclic for $\mathcal{R}$. We shall prove shortly that $\mathcal{R} = (\mathcal{N}p_{\mathcal{N}})'$ and so $\mathcal{N}$ is *-antiisomorphic to $\mathcal{R}$ by Theorem 3.6.

Now let $\mathcal{N}_1, \mathcal{N}_2 \in F(\mathcal{M})$. Then obviously $\mathcal{N}_1 \subset \mathcal{N}_2$ implies $p_{\mathcal{N}_1} \leq p_{\mathcal{N}_2}$. On the other hand, assume $p_{\mathcal{N}_1} \leq p_{\mathcal{N}_2}$. Then for $N_1 \in \mathcal{N}_1$ we have $N_1x \in [\mathcal{N}_1 x] \subset [\mathcal{N}_2 x]$ and so $N_1 \in \mathcal{N}_2$ by Theorem 4.1. Hence $\mathcal{N}_1 \subset \mathcal{N}_2$ if and only if $p_{\mathcal{N}_1} \leq p_{\mathcal{N}_2}$. This proves that the mapping $\mathcal{N} \to p_{\mathcal{N}} = [\mathcal{N}x]$ is a 1-1 order-isomorphic map of $F(\mathcal{M})$ into $P(\mathcal{M}, x)$. We also have that if $\mathcal{N} \in F(\mathcal{M})$ and $p_{\mathcal{N}}M = Mp_{\mathcal{N}}$ for $M \in \mathcal{M}$ then $M \in \mathcal{N}$. In fact, $Mx = Mp_{\mathcal{N}}x = p_{\mathcal{N}}Mx \in [\mathcal{N}x]$ and so $M \in \mathcal{N}$ by Theorem 4.1. Hence we have $\mathcal{N} = \mathcal{M} \cap \{p_{\mathcal{N}}\}' = \{\mathcal{M}' \cup p_{\mathcal{N}}\}'$.

Let us return to the situation above where $\mathcal{N} \in F(\mathcal{M})$ and $\mathcal{R}$ is the finite von Neumann algebra generated by $p_{\mathcal{N}}\mathcal{M}'p_{\mathcal{N}}$ on $\text{range}(p_{\mathcal{N}}) = [\mathcal{N}x]$. $\mathcal{R}$ is a subalgebra of the finite von Neumann algebra $(\mathcal{N}p_{\mathcal{N}})'$ on $[\mathcal{N}x]$ and both have the cyclic and separating vector $x \in [\mathcal{N}x]$. By what we have just proved this implies $\mathcal{R} = (\mathcal{N}p_{\mathcal{N}})'$ and so $\mathcal{N}$ is *-antiisomorphic to $\mathcal{R}$.

Next we prove that if $p \in P(\mathcal{M},x)$ then there exists $\mathcal{N} \in F(\mathcal{M})$

such that $p = p_{\mathcal{N}} = [\mathcal{N}x]$. Indeed, define $\mathcal{N} = \{\mathcal{M}' \cup p\}'$. Then $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and also $p \in \mathcal{N}'$. By [3; Chapter I, § 2; Proposition 1] we have that $p\mathcal{M}'p$ generates the von Neumann algebra $p\mathcal{N}'p = (\mathcal{N}p)'$ on $p(H)$. By Remark 1 preceding this theorem we conclude that $(\mathcal{N}p)'$ is a finite von Neumann algebra with $x \in p(H)$ a cyclic and separating vector for $(\mathcal{N}p)'$. Hence $\mathcal{N}p$ is a finite von Neumann algebra on $p(H)$ (Theorem 3.6). Now $\mathcal{N}$ is *-isomorphic to $\mathcal{N}p$ since the central carrier of $p \in \mathcal{N}'$ is the identity operator on H and so $\mathcal{N}$ is finite, i.e. $\mathcal{N} \in F(\mathcal{M})$. Also x is cyclic for $\mathcal{N}p$ and so $p(H) = [\mathcal{N}px] = [\mathcal{N}x] = \text{range}(p_{\mathcal{N}})$, hence $p = p_{\mathcal{N}}$. Thus the mapping $\mathcal{N} \to p_{\mathcal{N}}$ from $F(\mathcal{M})$ to $P(\mathcal{M},x)$ is onto.

It remains to show that if $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N} \in F(\mathcal{M})$ such that $\mathcal{N}_1 \cup \mathcal{N}_2 \subset \mathcal{N}$ then $p_{\mathcal{N}_1 \cap \mathcal{N}_2} = p_{\mathcal{N}_1} \wedge p_{\mathcal{N}_2}$. Clearly $p_{\mathcal{N}_1 \cap \mathcal{N}_2} \leq p_{\mathcal{N}_1} \wedge p_{\mathcal{N}_2}$. Now let $y \in \text{range}(p_{\mathcal{N}_1} \wedge p_{\mathcal{N}_2}) = [\mathcal{N}_1 x] \cap [\mathcal{N}_2 x]$. By the T-theorem for finite von Neumann algebras (Corollary 2 to Lemma 3.4) there exist closed operators T_1 and T_2 in H such that $T_1 \eta \mathcal{N}_1$, $T_2 \eta \mathcal{N}_2$, $x \in \mathcal{D}(T_1) \cap \mathcal{D}(T_2)$ and $y = T_1 x = T_2 x$. Clearly $T_1 \eta \mathcal{N}$, $T_2 \eta \mathcal{N}$ since $\mathcal{N}_1 \cup \mathcal{N}_2 \subset \mathcal{N}$. Let $N' \in \mathcal{N}'$. Then

$$T_1 N' x = N' T_1 x = N' T_2 x = T_2 N' x \; .$$

Hence T_1 and T_2 agree on the dense set $\{\mathcal{N}'x\}$ $(\supset \{\mathcal{M}'x\})$ in H and so $T_1 = T_2$ by Lemma 3.3. Let $T = T_1 = T_2$. Then $T \eta \mathcal{N}_1$ and $T \eta \mathcal{N}_2$ and so by Lemma 3.2 $T \eta (\mathcal{N}_1 \cap \mathcal{N}_2)$. By the remarks at the end of Section 2 we conclude that $y = Tx \in [(\mathcal{N}_1 \cap \mathcal{N}_2)x]$. Hence $\text{range}(p_{\mathcal{N}_1} \wedge p_{\mathcal{N}_2}) \subset \text{range}(p_{\mathcal{N}_1 \cap \mathcal{N}_2})$ and so $p_{\mathcal{N}_1 \cap \mathcal{N}_2} = p_{\mathcal{N}_1} \wedge p_{\mathcal{N}_2}$.

This concludes the proof of the theorem.

Corollary 1. Let $\mathcal{M}$ be a von Neumann algebra on H with a separating vector x in H. Then the mapping $\mathcal{B} \to p_{\mathcal{B}} = [\mathcal{B}x]$ induces a 1-1 order-isomorphic correspondence between the family $A(\mathcal{M})$ of abelian von Neumann subalgebras $\{\mathcal{B}\}$ of $\mathcal{M}$ and the family of projections $Q(\mathcal{M},x)$ defined above (Remark 2 to Definition 2.2). $Q(\mathcal{M},x)$ is closed in the strong-operator topology. $\mathcal{B} \in A(\mathcal{M})$ is maximal abelian in $\mathcal{M}$ if and only if $p_{\mathcal{B}} \in \{\mathcal{M}' \cup \mathcal{B}\}''$.

Proof. It is an immediate consequence of the theorem that the restriction of the mapping $\mathcal{N} \in F(\mathcal{M}) \to p_{\mathcal{N}} \in P(\mathcal{M},x)$ to $A(\mathcal{M})$ is onto $Q(\mathcal{M},x)$.

Now let $\{q_\alpha\} \subset Q(\mathcal{M},x)$ and $q_\alpha \to q$ in the strong-operator topology. Then clearly q is a projection and $x \in \text{range}(q)$. Let $M_1', M_2' \in \mathcal{M}'$. Then $(qM_1'q)(qM_2'q) = \lim_\alpha (q_\alpha M_1' q_\alpha)(q_\alpha M_2' q_\alpha) = \lim_\alpha (q_\alpha M_2' q_\alpha)(q_\alpha M_1' q_\alpha) = (qM_2'q)(qM_1'q)$. So $q\mathcal{M}'q$ is a commutative family of operators and so $q \in Q(\mathcal{M},x)$. Hence $Q(\mathcal{M},x)$ is closed in the strong-operator topology.

Assume $\mathcal{B} \in A(\mathcal{M})$ is maximal abelian in $\mathcal{M}$. Then $\mathcal{M} \cap \mathcal{B}' = \mathcal{B}$ and so $p_{\mathcal{B}} = [\mathcal{B}x] \in \mathcal{B}' = \{\mathcal{M} \cap \mathcal{B}'\}' = \{\mathcal{M}' \cup \mathcal{B}\}''$. Conversely, assume $p_{\mathcal{B}} \in \{\mathcal{M}' \cup \mathcal{B}\}'' = \{\mathcal{M} \cap \mathcal{B}'\}'$. Since $\mathcal{M}' \subset \{\mathcal{M}' \cup \mathcal{B}\}''$ we get $\{\mathcal{M}' \cup p_{\mathcal{B}}\}'' \subset \{\mathcal{M} \cap \mathcal{B}'\}'$. Taking commutants we have $\mathcal{M} \cap \mathcal{B}' \subset \{\mathcal{M}' \cup p_{\mathcal{B}}\}'$. By the theorem we have $\mathcal{B} = \{\mathcal{M}' \cup p_{\mathcal{B}}\}'$ and so $\mathcal{B} \subset \mathcal{M} \cap \mathcal{B}' \subset \{\mathcal{M}' \cup p_{\mathcal{B}}\}' = \mathcal{B}$. Hence $\mathcal{B} = \mathcal{M} \cap \mathcal{B}'$ and so $\mathcal{B}$ is maximal abelian in $\mathcal{M}$.

Corollary 2. Let $\mathcal{M}$ be a finite von Neumann algebra on H with a cyclic and separating vector. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ with a cyclic and separating vector. Then $\mathcal{N} = \mathcal{M}$.

Proof. Let $x \in H$ be a separating vector for $\mathcal{M}$. Then clearly x is separating for $\mathcal{N}$ and so is cyclic for $\mathcal{N}$ by Theorem 3.6.

Hence x is a joint cyclic and separating vector for $\mathcal{M}$ and $\mathcal{N}$ and so $[\mathcal{M}x] = [\mathcal{N}x]$ = the identity operator on H. By the theorem we have $\mathcal{N} = \mathcal{M}$.

Remarks. In [10] Kadison proved Corollary 2 using a different approach and he showed that the hypothesis of finiteness is essential. In fact, in the same paper he constructs an example of a type I factor $\mathcal{M}$ on a separable Hilbert space H with a cyclic and separating vector x and a proper type I subfactor $\mathcal{N}$ with the same cyclic and separating vector x. Indeed, let K be a Hilbert space of dimension $\aleph_0$. Let $H = K \otimes K \otimes K$ and let $\mathcal{M} = \mathcal{B}(K) \otimes \mathcal{B}(K) \otimes I_K$, $\mathcal{N} = \mathcal{B}(K) \otimes I_K \otimes I_K$, where I_K is the identity operator on K. Then $\mathcal{N}$ is a proper type I subfactor of the type I factor $\mathcal{M}$. It is easy to see that $\mathcal{M}, \mathcal{M}', \mathcal{N}$ and $\mathcal{N}'$ each have cyclic vectors in H. By [6] $\mathcal{M}, \mathcal{M}', \mathcal{N}$ and $\mathcal{N}'$ have a joint cyclic vector x which accordingly will be jointly cyclic and separating for $\mathcal{M}$ and $\mathcal{N}$.

Let K, H and $\mathcal{M}$ be as above and let $\mathcal{N}_1 = \mathfrak{A} \otimes \mathcal{B}(K) \otimes I_K$, where $\mathfrak{A}$ is a maximal abelian von Neumann subalgebra of $\mathcal{B}(K)$. Then $\mathcal{N}_1$ is a type I (non-factor) proper von Neumann subalgebra of $\mathcal{M}$. By [6] there exists a joint cyclic and separating vector x for $\mathcal{M}$ and $\mathcal{N}_1$. Moreover, every maximal abelian von Neumann subalgebra of $\mathcal{N}_1$ is maximal abelian in $\mathcal{M}$, a fact which is readily verified.

Example. In Theorem 4.3 we proved that $p_{\mathcal{N}_1 \cap \mathcal{N}_2} = p_{\mathcal{N}_1} \wedge p_{\mathcal{N}_2}$ for $\mathcal{N}_1, \mathcal{N}_2 \in F(\mathcal{M})$ if $\mathcal{N}_1 \cup \mathcal{N}_2$ generates a finite von Neumann subalgebra of $\mathcal{M}$. In particular, if $\mathcal{M}$ itself is finite this is always so. However, if $\mathcal{M}$ is infinite it is not true in general that $p_{\mathcal{N}_1 \cap \mathcal{N}_2} = p_{\mathcal{N}_1} \wedge p_{\mathcal{N}_2}$ for $\mathcal{N}_1$ and $\mathcal{N}_2$ in $F(\mathcal{M})$. We shall construct an example illustrating this (Proposition 4.4). First some preliminary discussion.

Let H be a Hilbert space of dimension $\aleph_0$. Then we can find two distinct self-adjoint unbounded operators that agree on a dense set in H. For example, let $H = L^2([0,1])$ with Lebesgue measure on $[0,1]$. Let T_1 be the operator in H with

$$\mathcal{D}(T_1) = \{f \in L^2([0,1]) \mid f(x) = \int_0^x g(t)dt + \text{constant}\,,$$
$$g \in L^2([0,1])\,,\ f(0) = f(1)\}$$

and

$$(T_1 f)(x) = -if'(x) \quad \text{for} \quad f \in \mathcal{D}(T_1)\,.$$

Let T_2 be the operator in H with

$$\mathcal{D}(T_2) = \{f \in L^2([0,1]) \mid f(x) = \int_0^x g(t)dt + \text{constant}\,,$$
$$g \in L^2([0,1])\,,\ f(0) = e^{-i\lambda} f(1)\,,\ \lambda \in \langle 0,1\rangle\}$$

and

$$(T_2 f)(x) = -if'(x) \quad \text{for} \quad f \in \mathcal{D}(T_2)\,.$$

Then T_1 and T_2 are self-adjoint and agree on the set

$$\{f \in L^2([0,1]) \mid f(x) = \int_0^x g(t)dt,\ g \in L^2([0,1])\,,$$
$$f(0) = f(1) = 0\}$$

which is dense in $L^2([0,1])$; cf. [15; § 119]. Clearly $T_1 \neq T_2$ since $\mathcal{D}(T_1) \neq \mathcal{D}(T_2)$.

<u>Proposition 4.4.</u> There exists a type I_∞ factor $\mathcal{M}$, acting on a separable Hilbert space H with a separating vector x, and two abelian von Neumann subalgebras $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathcal{M}$ such that

$p_{\mathcal{C}_1} \wedge p_{\mathcal{C}_2} \neq p_{\mathcal{C}_1 \cap \mathcal{C}_2}$, where $p_{\mathcal{C}_1} = [\mathcal{C}_1 x]$, $p_{\mathcal{C}_2} = [\mathcal{C}_2 x]$ and $p_{\mathcal{C}_1 \cap \mathcal{C}_2} = [(\mathcal{C}_1 \cap \mathcal{C}_2)x]$.

Proof. Let K be a Hilbert space of dimension $\aleph_0$ and let T_1 and T_2 be two distinct unbounded self-adjoint operators that agree on a dense set in K . Let $\{y_n\}$ be a countable dense set in K contained in $\mathcal{D}(T_1) \cap \mathcal{D}(T_2)$ such that $T_1 y_n = T_2 y_n$ for each n .

Consider the von Neumann algebra $\mathcal{M} = \mathcal{B}(K) \otimes I_{K_\infty}$ on the separable Hilbert space $H = K \otimes K_\infty$, where K_∞ is the standard Hilbert space of dimension $\aleph_0$. $\mathcal{M}$ is in Dixmier's terminology an "ampliation" of $\mathcal{B}(K)$ and is isomorphic to $\mathcal{B}(K)$, hence a type I_∞ factor [3; Chapter I, § 2; 4] . It consists of copies of operators in $\mathcal{B}(K)$ "along the diagonal", where we are having in mind the standard unsymmetric realization of $K \otimes K_\infty$ as the Hilbert sum of K a countable number of times, cf. [3; Chapter I, § 2; 3] . Accordingly we will denote elements in $H = K \otimes K_\infty$ by $\{z_n\}$ where $z_n \in K$ and $\sum_{n=1}^{\infty} \|z_n\|^2 < \infty$. We will denote elements in $\mathcal{M}$ by $\tilde{M}$, where $\tilde{M} = M \otimes I_{K_\infty}$ $(M \in \mathcal{B}(K))$; in other words, $\tilde{M}$ is a copy of M along the diagonal. Let $\tilde{T}_1 = T_1 \otimes I_{K_\infty}$, $\tilde{T}_2 = T_2 \otimes I_{K_\infty}$ be the copies of T_1 and T_2 , respectively, along the diagonal. It is an easy observation that $\tilde{T}_1$ and $\tilde{T}_2$ are self-adjoint operators in $K \otimes K_\infty$ with domains of definition

$$\mathcal{D}(\tilde{T}_i) = \{\{x_n\} \in K \otimes K_\infty \mid x_n \in \mathcal{D}(T_i) \text{ for all } n ,$$

$$\sum_{n=1}^{\infty} \|T_i x_n\|^2 < \infty\} \quad (i = 1,2) .$$

Choose a sequence $\{\lambda_n\}$ of non-zero real numbers such that

$$\sum_{n=1}^{\infty} |\lambda_n|^2 \|y_n\|^2 < \infty ,$$

$$\sum_{n=1}^{\infty} |\lambda_n|^2 \|T_1 y_n\|^2 < \infty \quad \text{and}$$

$$\sum_{n=1}^{\infty} |\lambda_n|^2 \|T_2 y_n\|^2 < \infty .$$

Then $x = \{\lambda_n y_n\}$ is a separating vector for $\mathcal{M}$ in $H = K \otimes K_\infty$. Besides $x \in \mathcal{D}(\tilde{T}_1) \cap \mathcal{D}(\tilde{T}_2)$ and $\tilde{T}_1 x = \tilde{T}_2 x$. Since $T_1 \neq T_2$ we have $\tilde{T}_1 \neq \tilde{T}_2$.

Let $T_1 = \int_{-\infty}^{\infty} \lambda E(d\lambda)$, $T_2 = \int_{-\infty}^{\infty} \lambda F(d\lambda)$ be the spectral resolutions of T_1 and T_2, respectively. Then it is easily seen that $\tilde{T}_1 = \int_{-\infty}^{\infty} \lambda \tilde{E}(d\lambda)$, $\tilde{T}_2 = \int_{-\infty}^{\infty} \lambda \tilde{F}(d\lambda)$ are the spectral resolutions of $\tilde{T}_1$ and $\tilde{T}_2$, respectively.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the abelian von Neumann subalgebras of $\mathcal{M}$ generated by $\{\tilde{E}(\Omega) | \Omega$ Borel set in $\mathbb{R}\}$ and $\{\tilde{F}(\Omega) | \Omega$ Borel set in $\mathbb{R}\}$, respectively. Then clearly $\tilde{T}_1 \eta \mathcal{C}_1$ and $\tilde{T}_2 \eta \mathcal{C}_2$. By remarks at the end of Section 2 we have $\tilde{T}_1 x \in [\mathcal{C}_1 x]$ and $\tilde{T}_2 x \in [\mathcal{C}_2 x]$. Hence

$$\tilde{T}_1 x = \tilde{T}_2 x \in [\mathcal{C}_1 x] \cap [\mathcal{C}_2 x] .$$

We want to show that $\tilde{T}_1 x \notin [(\mathcal{C}_1 \cap \mathcal{C}_2) x]$. Assume to the contrary that $\tilde{T}_1 x \in [(\mathcal{C}_1 \cap \mathcal{C}_2) x]$. Since $\mathcal{C}_1 \cap \mathcal{C}_2$, being abelian, is a finite von Neumann algebra on H with separating vector x, there is by Corollary 2 to Lemma 3.4 an operator S in H affiliated $\mathcal{C}_1 \cap \mathcal{C}_2$ with $x \in \mathcal{D}(S)$ such that $\tilde{T}_1 x = Sx$. Since obviously $S \eta \mathcal{C}_1$ we get that $\tilde{T}_1$ and S agree on the set $\{\mathcal{C}_1' x\}$, which is dense in H since x is cyclic for $\mathcal{M}' (\subset \mathcal{C}_1')$. By Lemma 3.3 we conclude that $S = \tilde{T}_1$. Analogously we have $S = \tilde{T}_2$ and hence $\tilde{T}_1 = \tilde{T}_2$, a contradiction. So we have $\tilde{T}_1 x \notin [(\mathcal{C}_1 \cap \mathcal{C}_2) x]$.

This shows that $p_{\mathcal{C}_1 \cap \mathcal{C}_2} \neq p_{\mathcal{C}_1} \wedge p_{\mathcal{C}_2}$.

Remark. A natural question to ask in connection with Theorem 4.3 is the following: if $\mathcal{N}_1$ and $\mathcal{N}_2$ are two finite von Neumann subalgebras of $\mathcal{M}$, i.e. $\mathcal{N}_1, \mathcal{N}_2 \in F(\mathcal{M})$, such that $\mathcal{N}_1 \cup \mathcal{N}_2 \subset \mathcal{N}$ for some $\mathcal{N} \in F(\mathcal{M})$, will then $p_{\mathcal{N}_3} = p_{\mathcal{N}_1} \vee p_{\mathcal{N}_2}$, where $\mathcal{N}_3$ is the (finite) von Neumann subalgebra of $\mathcal{M}$ generated by $\mathcal{N}_1$ and $\mathcal{N}_2$? The answer is no in general as the following simple example shows.

Let $\mathcal{M}$ be a maximal abelian von Neumann algebra on a four-dimensional Hilbert space. By choosing an appropriate basis we may assume that

$$\mathcal{M} = \left\{ \begin{pmatrix} \alpha & & & 0 \\ & \beta & & \\ & & \delta & \\ 0 & & & \gamma \end{pmatrix} \middle| \alpha, \beta, \delta, \gamma \in \mathbb{C} \right\} .$$

Clearly, the vector $x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is separating for $\mathcal{M}$.

Let

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} \alpha & & & 0 \\ & \beta & & \\ & & \alpha & \\ 0 & & & \beta \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C} \right\}$$

and

$$\mathcal{C}_2 = \left\{ \begin{pmatrix} \delta & & & 0 \\ & \delta & & \\ & & \gamma & \\ 0 & & & \gamma \end{pmatrix} \middle| \delta, \gamma \in \mathbb{C} \right\}$$

be two abelian subalgebras of $\mathcal{M}$ and let $\mathcal{C}_3$ be the (abelian) subalgebra of $\mathcal{M}$ generated by $\mathcal{C}_1$ and $\mathcal{C}_2$. We want to show that $p_{\mathcal{C}_3} \neq p_{\mathcal{C}_1} \vee p_{\mathcal{C}_2}$, i.e. $[\mathcal{C}_3 x] \neq [\mathcal{C}_1 x] \vee [\mathcal{C}_2 x]$. Now

$$\operatorname{range}([\mathcal{C}_1 x] \vee [\mathcal{C}_2 x]) = \{C_1 x + C_2 x \mid C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2\} ,$$

since in a finite-dimensional space all linear subspaces are closed.

Let

$$C_1 = \begin{pmatrix} 1 & & & 0 \\ & 2 & & \\ & & 1 & \\ 0 & & & 2 \end{pmatrix} \in \mathcal{C}_1 \quad \text{and} \quad C_2 = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} \in \mathcal{C}_2 .$$

Then

$$C_1 C_2 = \begin{pmatrix} 1 & & & 0 \\ & 2 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} \in \mathcal{C}_3 = \{\mathcal{C}_1 \cup \mathcal{C}_2\}'' .$$

If $[\mathscr{C}_3 x] = [\mathscr{C}_1 x] \vee [\mathscr{C}_2 x]$ we must have $(C_1 C_2)x = D_1 x + D_2 x$ for some $D_1 \in \mathscr{C}_1$ and $D_2 \in \mathscr{C}_2$; since x is separating for $\mathfrak{M}$ it follows that $C_1 C_2 = D_1 + D_2$. Hence we must have

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \alpha & \beta \end{pmatrix} + \begin{pmatrix} \delta & \delta & 0 \\ 0 & \gamma & \gamma \end{pmatrix}$$

for some $\alpha, \beta, \delta, \gamma \in \mathbb{C}$. But this is readily verified to be impossible. Hence we have $[\mathscr{C}_3 x] \neq [\mathscr{C}_1 x] \vee [\mathscr{C}_2 x]$.

5. GROUP ACTION ON $P(\mathcal{M},x)$ AND CONDITIONAL EXPECTATIONS.

Let $\mathcal{M}$ be a von Neumann algebra on H with a separating vector x and let $\mathcal{U}$ denote the unitary group in $\mathcal{M}$. We want to study the mapping $\mathcal{U} \times P(\mathcal{M},x) \to P(\mathcal{M},x)$ defined by $(U,p_{\mathcal{N}}) \to p_{U^*\mathcal{N}U}$, where $\mathcal{N}$ is in $F(\mathcal{M})$, i.e. the family of finite von Neumann subalgebras of $\mathcal{M}$ (cf. Thm. 4.3). Under this mapping $(\mathcal{U},P(\mathcal{M},x))$ becomes a (right) transformation group and we would like to investigate the continuity properties of the said mapping when we endow $\mathcal{U}$ and $P(\mathcal{M},x)$ with the (relativized) strong-operator topology. [Recall at this stage that the strong-operator and weak-operator topologies coincide on the unitary operators in $\mathcal{B}(H)$, cf. [9; p.51]. From the observation that $p-(I-p) = 2p - I$ is a unitary operator for each projection p in $\mathcal{B}(H)$ it follows that the strong-operator and weak-operator topologies also coincide on the projections in $\mathcal{B}(H)$].

We first show (Proposition 5.2) that for $\mathcal{N} \in F(\mathcal{M})$ fixed the mapping $U \to p_{U^*\mathcal{N}U}$ is continuous from $\mathcal{U}$ into $P(\mathcal{M},x)$. We then proceed to show in Theorem 5.5 that if x is a trace vector for $\mathcal{M}$, in particular, $\mathcal{M}$ is finite, the mapping $(U,p_{\mathcal{N}}) \to p_{U^*\mathcal{N}U}$ is jointly continuous from $\mathcal{U} \times P(\mathcal{M},x)$ onto $P(\mathcal{M},x)$, in other words, $(\mathcal{U},P(\mathcal{M},x))$ is a topological transformation group, cf. Definition 5.4. Finally we establish in Theorem 5.8 the close corresondence that exists between $P(\mathcal{M},x)$ and the family $\Phi(\mathcal{M})$ of conditional expectations (cf. Definition 5.6) of $\mathcal{M}$ onto its various von Neumann subalgebras which is preserved by the faithful trace $\omega_x|_{\mathcal{M}}$, i.e. $\omega_x|_{\mathcal{M}} \circ \varphi = \omega_x|_{\mathcal{M}}$ for $\varphi \in \Phi(\mathcal{M})$.

We start our proofs by showing the following lemma which we shall need to prove the ensuing proposition.

Lemma 5.1. Let $\mathcal{M}$ be a finite von Neumann algebra on the Hilbert space H and let $x \in H$ be a separating vector for $\mathcal{M}$. Let $\{x_n\}$ be a sequence of vectors in H converging to x . Then the sequence $\{p_n\}$ of projections in $\mathcal{B}(H)$ converges in the strong-operator topology to the projection p in $\mathcal{B}(H)$, where $p_n = [\mathcal{M}x_n]$, $p = [\mathcal{M}x]$.

Proof. By the remarks above it is sufficient to prove that $\{p_n\}$ converges to p in the weak-operator topology. Since the unit ball of $\mathcal{B}(H)$ is compact in the weak-operator topology it is therefore enough to prove that any weak-operator convergent subnet of $\{p_n\}$ converges to p in the weak-operator topology. So let $\{p_{n'}\}$ be a subnet of $\{p_n\}$ converging to $c (\geq 0)$ in the weak-operator topology. We are through if we can show that $c = p$.

Since $x_{n'} \to x$ it is easily seen that $p_{n'}p \to p$ in the strong-operator topology and, consequently, $p_{n'}p \to p$ in the weak-operator topology. Now $p_{n'}p \to cp$ in the weak-operator topology and so $p = cp$. Taking adjoints we have $p = cp = pc$. Hence

$$c = pcp + (I-p)cp + pc(I-p) + (I-p)c(I-p) = p + (I-p)c(I-p) \geq p .$$

We observe that p and the $p_{n'}$'s are projections in $\mathcal{M}'$ and so, in particular, c is in $\mathcal{M}'$. Now $[\mathcal{M}'x] = H$ since x is separating for $\mathcal{M}$ and so $p_{n'} \prec p$ for all n' , where $\prec$ denotes the partial ordering of projections in $\mathcal{M}'$ [3; Chapter III, § 1; Thm.2] . So there exists for each n' a partial isometry $v_{n'}$ in $\mathcal{M}'$ such that $v_{n'}^* v_{n'} = p_{n'}$, $v_{n'}v_{n'}^* \leq p$. Now $p = [\mathcal{M}x]$ is a finite projection in $\mathcal{M}'$ since $\mathcal{M}$ is a finite von Neumann algebra [17; Proposition 2.9.5]. Hence there exists a faithful family $\{\tau_j | j \in J\}$ of normal semifinite traces on $(\mathcal{M}')^+$ such that $\tau_j(p) < \infty$ for all $j \in J$ [17; Lemma 2.5.3]. We have

$\tau_j(p_{n'}) = \tau_j(v_{n'}^* v_{n'}) = \tau_j(v_{n'} v_{n'}^*) \leq \tau_j(p)$ and $\tau_j(c)$ $\leq \liminf_{n'} \tau_j(p_{n'}) \leq \tau_j(p)$ for all $j \in J$ [3; Chapter I, § 6; 1]. In conjunction with $c \geq p$ this gives us $\tau_j(c-p) = 0$ for all $j \in J$ and so $c = p$. This completes the proof.

Remark. The assumption that $\mathcal{M}$ is finite in the above lemma is crucial. In fact, if $\mathcal{M}$ is infinite there exists a separating vector x for $\mathcal{M}$ and a sequence $\{x_n\}$ of vectors converging to x such that the sequence of projections $\{[\mathcal{M}x_n]\}$ does not converge to $[\mathcal{M}x]$ in the strong-operator topology. We indicate shortly how we show this. Decomposing $\mathcal{M}$ by a central projection into a direct sum of its finite part and its properly infinite part (cf. [3; Chapter I, § 6; 7]) we may reduce the situation to $\mathcal{M}$ being properly infinite with a separating and cyclic vector. By [3; Chapter I, § 2; Prop. 5 & Chapter III, § 8; Cor. 2] $\mathcal{M}$ is spatially isomorphic to $\mathcal{M}_1 \otimes \mathcal{B}(H_\infty)$ acting on $H_1 \otimes H_\infty$, where H_∞ has dimension $\aleph_0$ and $\mathcal{M}_1$ is a von Neumann algebra on H_1 with a separating and cyclic vector. From this we see easily that $\mathcal{M}$ has a separating vector x which is not cyclic for $\mathcal{M}$. By [6] the cyclic vectors for $\mathcal{M}$ are dense and so we can find a sequence $\{x_n\}$ of cyclic vectors converging to x. Since $[\mathcal{M}x_n] = I$ for each n the sequence of projections $\{[\mathcal{M}x_n]\}$ does not converge in the strong-operator topology to $[\mathcal{M}x] \neq I$.

Proposition 5.2. Let $\mathcal{M}$ be a von Neumann algebra on H with a separating vector x and let $\mathcal{N}$ be a finite von Neumann subalgebra of $\mathcal{M}$. Then the mapping $U \to p_{U^*\mathcal{N}U}$ from the unitaries $\mathcal{U}$ in $\mathcal{M}$ into $P(\mathcal{M},x)$ is strong-operator continuous.

Proof. For $U \in \mathcal{U}$ we have $p_{U^*\mathcal{N}U} = [(U^*\mathcal{N}U)x]$, i.e. $p_{U^*\mathcal{N}U}$ is the projection onto the closed subspace $U^*([\mathcal{N}(Ux)])$.

Let q_U denote the projection onto $[\mathcal{N}(Ux)]$. Then U^*q_U is a partial isometry with final space $U^*([\mathcal{N}(Ux)])$ and so $p_{U^*\mathcal{N}U} = U^*q_U(U^*q_U)^* = U^*q_UU$. Now let $U_\alpha \xrightarrow[\alpha]{} U_o$ in the strong-operator topology, where all the U's are in $\mathcal{U}$. Then $U_\alpha x \xrightarrow[\alpha]{} U_o x$ and $U_o x$ is a separating vector for $\mathcal{N}$. By Lemma 5.1 we conclude that $q_{U_\alpha} \xrightarrow[\alpha]{} q_{U_o}$ in the strong-operator topology. Also $U_\alpha^* \xrightarrow[\alpha]{} U_o^*$ in the strong-operator topology since this topology coincides with the weak-operator topology on the unitaries and the *-operation is weak-operator continuous. Hence $U_\alpha^* q_{U_\alpha} U_\alpha \xrightarrow[\alpha]{} U_o^* q_{U_o} U_o$ in the strong-operator topology, multiplication being jointly strong-operator continuous on bounded sets of operators. This shows that $U \to p_{U^*\mathcal{N}U}$ is strong-operator continuous at $U_o \in \mathcal{U}$ and the proof is complete.

Remark. We do not know whether the mapping $(U, p_{\mathcal{N}}) \to p_{U^*\mathcal{N}U}$ is jointly continuous from $\mathcal{U} \times P(\mathcal{M},x)$ onto $P(\mathcal{M},x)$ in general. However, if x is a trace vector for $\mathcal{M}$ (cf. Definition 3.5) this is so as we shall presently establish. First we give a characterization of a trace vector x for $\mathcal{M}$ in terms of the family $P(\mathcal{M},x)$, repectively $P(\mathcal{M}',x)$, of finite projections associated with $\mathcal{M}$ and x , respectively $\mathcal{M}'$ and x .

Proposition 5.3. Let $\mathcal{M}$ be a von Neumann algebra on H and let x be a cyclic and separating vector for $\mathcal{M}$. Then x is a trace vector for $\mathcal{M}$ (in particular, $\mathcal{M}$ is finite) if and only if $P(\mathcal{M},x) = P(\mathcal{M}',x)$. Also, if x is a trace vector for $\mathcal{M}$ then $P(\mathcal{M},x)$ is closed in the strong-operator topology.

Proof. Assume x is a trace vector for $\mathcal{M}$. Since $\omega_x|_{\mathcal{M}}$ is a faithful trace on $\mathcal{M}$ we conclude that $\mathcal{M}$ is finite and by Theorem 3.6 the mapping $\psi : \mathcal{M} \to \mathcal{M}'$ is a *-antiisomorphism, where $\psi(M)$ for M in $\mathcal{M}$ is the unique operator M' in $\mathcal{M}'$ such that $Mx = M'x$. Thus ψ gives rise to a 1-1 mapping $\tilde{\psi}$ of the family $F(\mathcal{M})$ of (finite) von Neumann subalgebras of $\mathcal{M}$ onto the corresponding family $F(\mathcal{M}')$ for $\mathcal{M}'$, where $\tilde{\psi}(\mathcal{N}) = \{\psi(N) \mid N \in \mathcal{N}\}$ for $\mathcal{N} \in F(\mathcal{M})$. Clearly $[\mathcal{N}x] = [\tilde{\psi}(\mathcal{N})x]$ for each $\mathcal{N} \in F(\mathcal{M})$ and so $P(\mathcal{M},x) = P(\mathcal{M}',x)$ by Theorem 4.3.

Next we assume that $P(\mathcal{M},x) = P(\mathcal{M}',x)$. Let $\mathcal{C} \in F(\mathcal{M})$ be an abelian von Neumann subalgebra of $\mathcal{M}$. Then by assumption there exists $\mathcal{C}_1 \in F(\mathcal{M}')$ such that $p_{\mathcal{C}} = [\mathcal{C}x] = [\mathcal{C}_1 x] = p_{\mathcal{C}_1}$. From the proof of Theorem 4.3 we have that $p_{\mathcal{C}}\mathcal{M}'p_{\mathcal{C}}$ generates the von Neumann algebra $p_{\mathcal{C}}\mathcal{C}'p_{\mathcal{C}}$ on $[\mathcal{C}x]$. Now $p_{\mathcal{C}}\mathcal{C}'p_{\mathcal{C}} = (\mathcal{C}p_{\mathcal{C}})' = \mathcal{C}p_{\mathcal{C}}$ since $\mathcal{C}p_{\mathcal{C}}$ is maximal abelian on $[\mathcal{C}x]$, x being a cyclic vector for $\mathcal{C}p_{\mathcal{C}}$. We also have

$$\mathcal{C}_1 p_{\mathcal{C}} = p_{\mathcal{C}}\mathcal{C}_1 p_{\mathcal{C}} \subset p_{\mathcal{C}}\mathcal{M}'p_{\mathcal{C}} \subset \mathcal{C}p_{\mathcal{C}} .$$

We conclude that $\mathcal{C}_1 p_{\mathcal{C}} = \mathcal{C}p_{\mathcal{C}}$ since x is a cyclic vector for $\mathcal{C}_1 p_{\mathcal{C}}$ on $[\mathcal{C}x] = [\mathcal{C}_1 x]$ and hence $\mathcal{C}_1 p_{\mathcal{C}}$ is maximal abelian. Thus $\mathcal{C}_1$ is an abelian von Neumann subalgebra of $\mathcal{M}'$ since $\mathcal{C}_1$ is *-isomorphic to $\mathcal{C}_1 p_{\mathcal{C}_1}$, the central carrier of $p_{\mathcal{C}_1}$ being I.

Now let A and B be operators in $\mathcal{M}$, B being normal, i.e. $BB^* = B^*B$. Then B is contained in an abelian von Neumann subalgebra $\mathcal{C}$ of $\mathcal{M}$. Let $\mathcal{C}_1 \in F(\mathcal{M}')$ such that $[\mathcal{C}_1 x] = [\mathcal{C}x]$. We established above that $\mathcal{C}_1$ is abelian, thus $\mathcal{C}_1$ consists of normal operators. Let $\{C_n\}$ be a sequence of operators in $\mathcal{C}_1$ such that $C_n x \to Bx$. Now $C_n - B$ is a normal operator for each n since C_n and B commute. Thus $\|(C_n - B)y\| = \|(C_n^* - B^*)y\|$ for each y in H. Hence $C_n^* x \to B^* x$. We have $(ABx,x) = \lim_n (AC_n x,x) = \lim_n (C_n A_x, x) = \lim_n (Ax, C_n^* x) = (Ax, B^* x) = (BAx,x)$. Since each operator in $\mathcal{M}$ is the sum of two normal operators, we conclude that x is a trace vector for $\mathcal{M}$.

To prove the remaining part of the proposition let $\{p_\alpha\}$ be a net in $P(\mathcal{M},x)$ such that $p_\alpha \to p$ in the strong-operator topology. Then p is a projection such that $x \in \text{range}(p) = p(H)$. For arbitrary M',N' in $\mathcal{M}'$ and any α we have $(p_\alpha M'p_\alpha p_\alpha N'p_\alpha x,x) = (p_\alpha N'p_\alpha p_\alpha M'p_\alpha x,x)$ since x is a trace vector for the von Neumann algebra generated by $p_\alpha \mathcal{M}'p_\alpha$ on $p_\alpha(H)$, cf. Theorem 3.6 and Theorem 4.3. Thus we have $(pM'ppN'px,x) = (pN'ppM'px,x)$. From this we see easily that $(R_1R_2x,x) = (R_2R_1x,x)$ for R_1,R_2 in the von Neumann algebra $\mathcal{R}$ generated by $p\mathcal{M}'p$ on $p(H)$. According to Definition 4.2 the proof is complete if we can show that x is separating for $\mathcal{R}$ because this entails that $\mathcal{R}$ is finite. Assume therefore $R \in \mathcal{R}$, $Rx = 0$. For all R_1,R_2 in $\mathcal{R}$ we have $(RR_1x,R_2x) = (R_2^*RR_1x,x) = (R_1R_2^*Rx,x) = 0$. Thus $R = 0$ since x is cyclic for $\mathcal{R}$. So x is separating for $\mathcal{R}$.

<u>Remark</u>. If x is a trace vector for the von Neumann algebra $\mathcal{M}$ on the Hilbert space H we see from the proof of the above proposition that we have the following alternative characterization of the family $P(\mathcal{M},x)$, namely, $p \in P(\mathcal{M},x)$ iff $x \in \text{range}(p)$ and $(pM_1'ppM_2'px,x) = (pM_2'ppM_1'px,x)$ for any $M_1',M_2' \in \mathcal{M}'$ (compare this with Remark 2 to Definition 4.2).

Before we prove the next theorem we give the following definition, cf. [9; pp. 38-39].

<u>Definition 5.4</u>. A (right) <u>topological</u> <u>transformation</u> <u>group</u> (G,X) is a topological group G together with a topological space X and a continuous map $(g,x) \to xg$ of $G \times X$ into X such that $x(gh) = (xg)h$, and if e is the identity of G, $xe = x$ for all g,h in G and x in X. (G,X) is <u>polonais</u> if G and X are polonais, i.e. they are separable and metrizable by a complete

metric. Similarly we say that (G,X) is Hausdorff if both G and X have that property.

Recall that $F(\mathcal{M})$ denotes the finite von Neumann subalgebras of $\mathcal{M}$ and $\mathcal{U}$ denotes the unitary operators in $\mathcal{M}$.

<u>Theorem 5.5</u>. Let $\mathcal{M}$ be a finite von Neumann algebra on the Hilbert space H with a separating and cyclic vector and let x be a trace vector for $\mathcal{M}$. Then $(\mathcal{U},P(\mathcal{M},x))$ is a Hausdorff (right) topological transformation group under the mapping $(U,p_{\mathcal{N}}) \to p_{U^*\mathcal{N}U}$, where $\mathcal{N} \in F(\mathcal{M})$, $U \in \mathcal{U}$ and where $\mathcal{U}$ and $P(\mathcal{M},x)$ are given the strong-operator topology. If H is separable $(\mathcal{U},P(\mathcal{M},x))$ is polonais.

Finally, let x_1 and x_2 in H be two trace vectors for $\mathcal{M}$ and let $\{\mathcal{N}_\alpha\}$ be a net in $F(\mathcal{M})$ such that each $\mathcal{N}_\alpha$ contains the center $\mathcal{Z}$ of $\mathcal{M}$. Let $\mathcal{N} \in F(\mathcal{M})$. Then we have $p^{(1)}_{\mathcal{N}_\alpha} \xrightarrow[\alpha]{} p^{(1)}_{\mathcal{N}}$ if and only if $p^{(2)}_{\mathcal{N}_\alpha} \xrightarrow[\alpha]{} p^{(2)}_{\mathcal{N}}$, where $p^{(i)}_{\mathcal{N}_\alpha} = [\mathcal{N}_\alpha x_i] \in P(\mathcal{M},x_i)$, $p^{(i)}_{\mathcal{N}} = [\mathcal{N}x_i] \in P(\mathcal{M},x_i)$, $i = 1,2$; convergence being in the strong-operator topology.

Proof. Let $U_\beta \xrightarrow[\beta]{} U_o$, $p_{\mathcal{N}_\gamma} \xrightarrow[\gamma]{} p_{\mathcal{N}_o}$ in the strong-operator topology, where the U's are in $\mathcal{U}$ and the $\mathcal{N}$'s are in $F(\mathcal{M})$. For each U_β there is a unique $U'_\beta \in \mathcal{M}'$ such that $U_\beta x = U'_\beta x$ and $U'_\beta \xrightarrow[\beta]{} U'_o$ in the strong-operator topology, where $U'_o \in \mathcal{M}'$ such that $U_o x = U'_o x$ (Theorem 3.6). Now $p_{U^*_\beta \mathcal{N}_\gamma U_\beta}$ is the projection onto the subspace

$$[(U^*_\beta \mathcal{N}_\gamma U_\beta)x] = U^*_\beta([\mathcal{N}_\gamma(U_\beta x)]) = U^*_\beta([\mathcal{N}_\gamma(U'_\beta x)]) = U^*_\beta U'_\beta([\mathcal{N}_\gamma x]) .$$

The operator $U^*_\beta U'_\beta p_{\mathcal{N}_\gamma}$ is a partial isometry with final space $U^*_\beta U'_\beta([\mathcal{N}_\gamma x])$ and so $p_{U^*_\beta \mathcal{N}_\gamma U_\beta} = (U^*_\beta U'_\beta p_{\mathcal{N}_\gamma})(U^*_\beta U'_\beta p_{\mathcal{N}_\gamma})^* =$

$$U_\beta^* U_\beta' p_{\mathcal{N}_\gamma} U_\beta'^* U_\beta \underset{\beta,\gamma}{\longrightarrow} U_o^* U_o' p_{\mathcal{N}_o} U_o'^* U_o = p_{U_o^* \mathcal{N}_o U_o} .$$

Thus the mapping $(U, p_{\mathcal{N}}) \to p_{U^*\mathcal{N}U}$ is (jointly) continuous in the strong-operator topology. This completes the non-trivial part of the proof that $(\mathcal{U}, P(\mathcal{M},x))$ is a Hausdorff (right) topological transformation group.

If H is separable, the unit ball of $\mathcal{B}(H)$ is polonais in the strong-operator topology [3; Chapter I, § 3; 1] . By proposition 5.3 $P(\mathcal{M},x)$ is closed in the strong-operator topology and so $P(\mathcal{M},x)$ is polonais. Also $\mathcal{U}$ is a G_δ subset of the unit ball of $\mathcal{B}(H)$ in the strong-operator topology ([5; Lemma 4]) and so is polonais ([2; Chapter IX, § 6; Theorem 1]). Thus $(\mathcal{U}, P(\mathcal{M},x))$ is polonais.

To prove the remaining part of the theorem we first observe that with $\{\mathcal{N}_\alpha\}$ and $\mathcal{N}$ given as stated we have that $\mathcal{N}$ contains the center $\mathcal{Z}$ of $\mathcal{M}$. This is so since either $p_{\mathcal{Z}}^{(1)} \leq p_{\mathcal{N}}^{(1)}$ or $p_{\mathcal{Z}}^{(2)} \leq p_{\mathcal{N}}^{(2)}$ and so by Theorem 4.3 we have $\mathcal{Z} \subset \mathcal{N}$. Assume for the moment that we can find a cyclic vector y for $\mathcal{M}$ such that $[\mathcal{Z} x_1] = [\mathcal{Z} y]$ and $(M y, y) = (M x_2, x_2)$ for all M in $\mathcal{M}$. Then there exists a unitary operator U' in $\mathcal{M}'$ such that $U'y = x_2$ [4; Proposition 2.4.1]. Then we have $[\mathcal{N}_\alpha x_2] = [\mathcal{N}_\alpha (U'y)] = U'([\mathcal{N}_\alpha y]) = U'([\mathcal{N}_\alpha x_1])$ for all α . Now $U' p_{\mathcal{N}_\alpha}^{(1)}$ is a partial isometry with final space $U'([\mathcal{N}_\alpha x_1])$ and so $p_{\mathcal{N}_\alpha}^{(2)} = U' p_{\mathcal{N}_\alpha}^{(1)} (U' p_{\mathcal{N}_\alpha}^{(1)})^* = U' p_{\mathcal{N}_\alpha}^{(1)} U'^*$ for all α . Analogously we have $p_{\mathcal{N}}^{(2)} = U' p_{\mathcal{N}}^{(1)} U'^*$. We infer that $p_{\mathcal{N}_\alpha}^{(1)} \underset{\alpha}{\longrightarrow} p^{(1)}$ if and only if $p_{\mathcal{N}_\alpha}^{(2)} \underset{\alpha}{\longrightarrow} p_{\mathcal{N}}^{(2)}$, convergence being in the strong-operator topology.

It remains to establish that we can find a vector y in H with the desired properties. By [3; Chapter III, § 4; Lemma 1 &

Proposition 4] there exists an increasing net $\{C_\beta\}$ of positive operators in $\mathcal{Z}$ (not necessarily bounded above) such that

$$(Mx_2,x_2) = \omega_{x_2}(M) = \lim_\beta \omega_{x_1}(C_\beta M) = \lim_\beta(C_\beta Mx_1,x_1) = \lim_\beta(MC_\beta^{\frac{1}{2}}x_1,C_\beta^{\frac{1}{2}}x_1)$$

for every positive operator M in $\mathcal{M}$. We want to prove that $\lim_\beta C_\beta^{\frac{1}{2}}x_1 = y$ exists. Indeed, $\{C_\beta^{\frac{1}{2}}\}$ is an increasing net of operators in $\mathcal{Z}$ and so for $\beta_1 \prec \beta_2$

$$\|C_{\beta_2}^{\frac{1}{2}}x_1 - C_{\beta_1}^{\frac{1}{2}}x_1\|^2 = (C_{\beta_2}x_1,x_1) + (C_{\beta_1}x_1,x_1) - 2(C_{\beta_1}^{\frac{1}{2}}C_{\beta_2}^{\frac{1}{2}}x_1,x_1)$$

$$\leq 2(C_{\beta_2}x_1,x_1) - 2(C_{\beta_1}x_1,x_1) .$$

We conclude from this that $\{C_\beta^{\frac{1}{2}}x_1\}$ is a Cauchy net since for any β' we have $(C_{\beta'}x_1,x_1) \leq \lim_\beta(C_\beta x_1,x_1) = (x_2,x_2)$. So $\lim_\beta C_\beta^{\frac{1}{2}}x_1 = y$ exists and we have $(Mx_2,x_2) = (My,y)$ for every positive M in $\mathcal{M}$ and hence for all M in $\mathcal{M}$. Clearly y is separating for $\mathcal{M}$ since $\omega_y|_{\mathcal{M}}$ is faithful, and so y is cyclic for $\mathcal{M}$ by Theorem 3.6. Obviously $y \in [\mathcal{Z}x_1]$ and also y is separating for $\mathcal{Z}p_{\mathcal{Z}}$ on $p_{\mathcal{Z}}(H) = [\mathcal{Z}x_1]$. Hence y is cyclic for $\mathcal{Z}p_{\mathcal{Z}}$ since $\mathcal{Z}p_{\mathcal{Z}}$ is maximal abelian on $[\mathcal{Z}x_1]$. This shows that $x_1 \in [\mathcal{Z}y]$, thus rendering $[\mathcal{Z}x_1] = [\mathcal{Z}y]$.

This completes the proof of the theorem.

<u>Definition 5.6</u>. Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. A <u>conditional expectation</u> on $\mathcal{N}$ is a positive linear map φ of $\mathcal{M}$ into $\mathcal{N}$ such that $\varphi(I) = I$ and $\varphi(NM) = N\varphi(M)$; $N \in \mathcal{N}$, $M \in \mathcal{M}$. By taking adjoints, it follows immediately that $\varphi(MN) = \varphi(M)N$; $N \in \mathcal{N}$, $M \in \mathcal{M}$. We say φ is <u>faithful</u> if $\varphi(M) = 0$ for $M \in \mathcal{M}^+$ implies $M = 0$; and φ is <u>normal</u> if $M_\alpha \uparrow M$ implies $\varphi(M_\alpha) \uparrow \varphi(M)$.

We list some elementary properties of conditional expectations in the following proposition and refer to [1; Proposition 6.1.1] for proofs.

Proposition 5.7. Let $\mathcal{M}$ be a von Neumann algebra and let φ be a conditional expectation on the von Neumann subalgebra $\mathcal{N}$. Then

(i) $\varphi(M)^*\varphi(M) \leq \varphi(M^*M)$; $M \in \mathcal{M}$

(ii) φ is a projection onto $\mathcal{N}$, i.e. φ is an idempotent and $\varphi(\mathcal{M}) = \mathcal{N}$.

(iii) Let ρ be a faithful state on $\mathcal{M}$ which preserves φ, i.e. $\rho\circ\varphi = \rho$. Then for every $M \in \mathcal{M}$, $\varphi(M)$ is that (unique) element in $\mathcal{N}$ which best approximates M in the norm $\|T\|_\rho = \rho(T^*T)^{\frac{1}{2}}$; $T \in \mathcal{M}$.

(iv) Let ρ be any state which preserves φ. If ρ is faithful, then so is φ ; if ρ is faithful and normal, then φ is normal.

Remarks. The concept of conditional expectation is very important in probability theory. Umegaki [21] showed that for a von Neumann algebra having a faithful normal finite trace, though non-commutative, one can define the conditional expectation on a given von Neumann subalgebra in a most natural way; when the subalgebra is the center, for example, this construction yields the center-valued trace. Tomiyama [20] showed that a conditional expectation on $\mathcal{N}$ can be characterized as an idempotent linear mapping of $\mathcal{M}$ on $\mathcal{N}$ having norm 1, and which leaves the identity fixed.

Let x be a trace vector for the finite von Neumann algebra $\mathcal{M}$ acting on the Hilbert space H. It turns out that there is a close relationship between the family of projections $P(\mathfrak{M},x)$ and conditional expectations in $\mathfrak{M}$. We are going to make this precise in the following theorem where we also construct the conditional expectations in question. First some preliminary remarks. Let $\Phi(\mathfrak{M})$ denote the family of conditional expectations of $\mathfrak{M}$ onto its various von Neumann subalgebras which leaves the trace $\omega_x|_{\mathfrak{M}}$ invariant, i.e. $\omega_x|_{\mathfrak{M}} \circ \varphi = \omega_x|_{\mathfrak{M}}$ for $\varphi \in \Phi(\mathfrak{M})$. For $\mathfrak{N}$ in $F(\mathfrak{M})$, the family of (finite) von Neumann subalgebras of $\mathfrak{M}$, let $\varphi_{\mathfrak{N}}$ in $\Phi(\mathfrak{M})$ denote the conditional expectation on $\mathfrak{N}$. By Proposition 5.7 (iii) $\varphi_{\mathfrak{N}}$ is uniquely determined by $\mathfrak{N}$. $\Phi(\mathfrak{M})$ is a subset of the locally convex Hausdorff space $\mathcal{B}(\mathfrak{M},\mathfrak{M})$ of bounded linear mappings of $\mathfrak{M}$ into $\mathfrak{M}$ with the topology of pointwise convergence in the strong-operator topology, i.e. $L_\alpha \xrightarrow[\alpha]{} L$ in $\mathcal{B}(\mathfrak{M},\mathfrak{M})$ iff $L_\alpha(M) \xrightarrow[\alpha]{} L(M)$ in the strong-operator topology for all M in $\mathfrak{M}$. We assume $\Phi(\mathfrak{M})$ is endowed with the relativized topology.

Theorem 5.8. Let $\mathfrak{M}$ be a finite von Neumann algebra on H and let x be a trace vector for $\mathfrak{M}$. Let $p_{\mathfrak{N}} \in P(\mathfrak{M},x)$ for some $\mathfrak{N} \in F(\mathfrak{M})$. Then $\mathfrak{N}p_{\mathfrak{N}} = p_{\mathfrak{N}}\mathfrak{M}p_{\mathfrak{N}}$ and the mapping $M \to p_{\mathfrak{N}}Mp_{\mathfrak{N}}$ of $\mathfrak{M}$ onto $\mathfrak{N}p_{\mathfrak{N}}$ gives rise to the conditional expectation $\varphi_{\mathfrak{N}}$ on $\mathfrak{N}$ such that $\omega_x|_{\mathfrak{M}} \circ \varphi_{\mathfrak{N}} = \omega_x|_{\mathfrak{M}}$, i.e. $\varphi_{\mathfrak{N}} \in \Phi(\mathfrak{M})$. In particular, $\varphi_{\mathfrak{Z}}$ is the center-valued trace Tr, where $\mathfrak{Z}$ is the center of $\mathfrak{M}$.

The correspondence between $P(\mathfrak{M},x)$ and $\Phi(\mathfrak{M})$ given by $p_{\mathfrak{N}} \to \varphi_{\mathfrak{N}}$ $(\mathfrak{N} \in F(\mathfrak{M}))$ is a homeomorphism, where $P(\mathfrak{M},x)$ is given the strong-operator topology and $\Phi(\mathfrak{M})$ the topology of pointwise convergence in the strong-operator topology. Furthermore, if $\mathfrak{N} \in F(\mathfrak{M})$ contains the center $\mathfrak{Z}$ of $\mathfrak{M}$ then $Tr \circ \varphi_{\mathfrak{N}} = Tr$.

Proof. We may assume without loss of generality that x is cyclic for $\mathcal{M}$. Let $\mathcal{N} \in F(\mathcal{M})$ and let $p_{\mathcal{N}} \in P(\mathcal{M},x)$ be the projection onto $[\mathcal{N}x]$. By proposition 5.3 there exists $\mathcal{N}_1 \in F(\mathcal{M}')$ such that $p_{\mathcal{N}_1} = p_{\mathcal{N}}$. Now

$$\mathcal{N}p_{\mathcal{N}} = p_{\mathcal{N}}\mathcal{N}p_{\mathcal{N}} \subset p_{\mathcal{N}}\mathcal{M}p_{\mathcal{N}} \subset p_{\mathcal{N}}\mathcal{N}_1'p_{\mathcal{N}} = (\mathcal{N}_1p_{\mathcal{N}})' .$$

Since $(\mathcal{N}_1p_{\mathcal{N}})'$ is a finite von Neumann algebra on $[\mathcal{N}x]$ with a cyclic and separating vector x, and the same is true for $\mathcal{N}p_{\mathcal{N}}$ (cf. Definition 4.2), we conclude by Theorem 4.3 that $\mathcal{N}p_{\mathcal{N}} = (\mathcal{N}_1p_{\mathcal{N}})'$ and so $\mathcal{N}p_{\mathcal{N}} = p_{\mathcal{N}}\mathcal{M}p_{\mathcal{N}}$.

The mapping $Np_{\mathcal{N}} \to N$ $(N \in \mathcal{N})$ is a *-isomorphism between $\mathcal{N}p_{\mathcal{N}}$ and $\mathcal{N}$ since the central carrier of $p_{\mathcal{N}}$ in $\mathcal{N}'$ is I, x being cyclic for $\mathcal{N}'$ [3; Chapter I, § 2; Proposition 2]. Composing this map with the mapping of $\mathcal{M}$ onto $\mathcal{N}p_{\mathcal{N}} = p_{\mathcal{N}}\mathcal{M}p_{\mathcal{N}}$ defined by $M \to p_{\mathcal{N}}Mp_{\mathcal{N}}$ $(M \in \mathcal{M})$ we get a mapping $\varphi_{\mathcal{N}}$ of $\mathcal{M}$ onto $\mathcal{N}$ characterized by $\varphi_{\mathcal{N}}(M)p_{\mathcal{N}} = p_{\mathcal{N}}Mp_{\mathcal{N}}$ for all $M \in \mathcal{M}$. Obviously $\varphi_{\mathcal{N}}$ is a positive linear map of $\mathcal{M}$ onto $\mathcal{N}$ and $\varphi_{\mathcal{N}}(I) = I$. Let $M \in \mathcal{M}$, $N \in \mathcal{N}$. Then $p_{\mathcal{N}}NMp_{\mathcal{N}} = (Np_{\mathcal{N}})(p_{\mathcal{N}}Mp_{\mathcal{N}})$ since $p_{\mathcal{N}} \in \mathcal{N}'$. Hence $\varphi_{\mathcal{N}}(NM) = N\varphi_{\mathcal{N}}(M)$, establishing that $\varphi_{\mathcal{N}}$ is a conditonal expectation on $\mathcal{N}$. Now for $M \in \mathcal{M}$ we have

$$(\varphi_{\mathcal{N}}(M)x,x) = (\varphi_{\mathcal{N}}(M)p_{\mathcal{N}}x,x) = (p_{\mathcal{N}}Mp_{\mathcal{N}}x,x) = (Mx,x) ,$$

and so $\omega_x|_{\mathcal{M}} \circ \varphi_{\mathcal{N}} = \omega_x|_{\mathcal{M}}$. Hence $\varphi_{\mathcal{N}} \in \Phi(\mathcal{M})$ by the definition of $\Phi(\mathcal{M})$. (Observe that by Proposition 5.7 (iv) $\varphi_{\mathcal{N}}$ is both faithful and normal).

With $\mathcal{Z}$ the center of $\mathcal{M}$ we have for $M_1, M_2 \in \mathcal{M}$ and $Z_1, Z_2 \in \mathcal{Z}$

$$\begin{aligned}(p_{\mathcal{Z}}M_1M_2p_{\mathcal{Z}}Z_1x,Z_2x) &= (Z_2^*Z_1M_1M_2x,x) = (M_2Z_2^*Z_1M_1x,x) \\ &= (M_2M_1Z_1x,Z_2x) = (p_{\mathcal{Z}}M_2M_1p_{\mathcal{Z}}Z_1x,Z_2x) ,\end{aligned}$$

using that x is a trace vector for $\mathcal{M}$ and that $p_{\mathcal{Z}} = [\mathcal{Z}x] \in \mathcal{Z}'$. Hence $p_{\mathcal{Z}}M_1M_2p_{\mathcal{Z}} = p_{\mathcal{Z}}M_2M_1p_{\mathcal{Z}}$, proving that $\varphi_{\mathcal{Z}}(M_1M_2) = \varphi_{\mathcal{Z}}(M_2M_1)$. By uniqueness of the center-valued trace we have $Tr = \varphi_{\mathcal{Z}}$ [3; Chapter III, § 5; 1].

Assume next that $\mathcal{N} \in F(\mathcal{M})$ and $\mathcal{N} \supset \mathcal{Z}$. Then $p_{\mathcal{N}} \geq p_{\mathcal{Z}}$ and so for $M \in \mathcal{M}$ we have

$$\{\varphi_{\mathcal{Z}}(\varphi_{\mathcal{N}}(M))\}p_{\mathcal{Z}} = p_{\mathcal{Z}}\varphi_{\mathcal{N}}(M)p_{\mathcal{Z}} = p_{\mathcal{Z}}(\varphi_{\mathcal{N}}(M)p_{\mathcal{N}})p_{\mathcal{Z}}$$
$$= p_{\mathcal{Z}}(p_{\mathcal{N}}Mp_{\mathcal{N}})p_{\mathcal{Z}} = p_{\mathcal{Z}}Mp_{\mathcal{Z}} = \varphi_{\mathcal{Z}}(M)p_{\mathcal{Z}} .$$

Hence $\varphi_{\mathcal{Z}} \circ \varphi_{\mathcal{N}} = \varphi_{\mathcal{Z}}$ and so $Tr \circ \varphi_{\mathcal{N}} = Tr$ since $Tr = \varphi_{\mathcal{Z}}$.

The mapping $p_{\mathcal{N}} \to \varphi_{\mathcal{N}}$ for $\mathcal{N} \in F(\mathcal{M})$ is a bijection between $P(\mathcal{M}, x)$ and $\Phi(\mathcal{M})$ (cf. Theorem 4.3). Let $p_{\mathcal{N}_\alpha} \underset{\alpha}{\to} p_{\mathcal{N}_0}$ in the strong-operator topology, where the $\mathcal{N}$'s are in $F(\mathcal{M})$, and let $M \in \mathcal{M}$. For $M' \in \mathcal{M}'$ we have

$$\varphi_{\mathcal{N}_\alpha}(M)(M'x) = (M'\varphi_{\mathcal{N}_\alpha}(M))(p_{\mathcal{N}_\alpha}x) = (M'p_{\mathcal{N}_\alpha}Mp_{\mathcal{N}_\alpha})x$$
$$\underset{\alpha}{\to} (M'p_{\mathcal{N}_0}Mp_{\mathcal{N}_0})x = (M'\varphi_{\mathcal{N}_0}(M)p_{\mathcal{N}_0})x = \varphi_{\mathcal{N}_0}(M)(M'x) .$$

Since $\|\varphi_{\mathcal{N}}(M)\| \leq \|M\|$ for $\mathcal{N} \in F(\mathcal{M})$ and $\{\mathcal{M}'x\}$ is dense in H we conclude that $\varphi_{\mathcal{N}_\alpha}(M) \underset{\alpha}{\to} \varphi_{\mathcal{N}_0}(M)$ in the strong-operator topology.

Conversely, let $\varphi_{\mathcal{N}_\alpha} \underset{\alpha}{\to} \varphi_{\mathcal{N}_0}$ in the topology of pointwise strong-operator convergence, where the $\mathcal{N}$'s are in $F(\mathcal{M})$. Let $M \in \mathcal{M}$. Then

$$p_{\mathcal{N}_\alpha}(Mx) = (p_{\mathcal{N}_\alpha}Mp_{\mathcal{N}_\alpha})x = (\varphi_{\mathcal{N}_\alpha}(M)p_{\mathcal{N}_\alpha})x = \varphi_{\mathcal{N}_\alpha}(M)x$$
$$\underset{\alpha}{\to} \varphi_{\mathcal{N}_0}(M)x = p_{\mathcal{N}_0}(Mx) .$$

Since $\{\mathcal{M}x\}$ is dense in H we conclude that $p_{\mathcal{N}_\alpha} \underset{\alpha}{\to} p_{\mathcal{N}_0}$ in the strong-operator topology.

Remark 1. By the above theorem and Theorem 5.5 we can organize $(\mathcal{U}, \Phi(\mathcal{M}))$ into a topological transformation group by the mapping $(U, \varphi_{\mathcal{N}}) \to \varphi_{U^*\mathcal{N}U}$, where $U \in \mathcal{U}$, $\mathcal{N} \in F(\mathcal{M})$.

Remark 2. By the above theorem we may draw as a corollary the result proved in Theorem 5.5 using a different approach, namely that for x_1 and x_2 two trace vectors for $\mathcal{M}$, convergence in $P(\mathcal{M}, x_1)$ coincides with convergence in $P(\mathcal{M}, x_2)$ if we adhere to von Neumann subalgebras of $\mathcal{M}$ containing the center of $\mathcal{M}$ (cf. Theorem 5.5). In fact, $\mathcal{N} \in F(\mathcal{M})$ and $\mathcal{N} \supset \mathcal{Z}$ implies $Tr \circ \varphi_{\mathcal{N}} = Tr$. This uniquely determines $\varphi_{\mathcal{N}}$ (regardless of the trace vector x), a fact which is readily verified. Together with the topological homeomorphism between $P(\mathcal{M}, x)$ and $\Phi(\mathcal{M})$ proved in the above theorem this immediately gives us the result quoted. (Notice that we do not have to assume that x is a cyclic trace vector to get this; we could also have avoided that assumption in the proof of Theorem 5.5).

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