CANONICAL RELATIVISTIC QUANTUM FIELDS IN TWO SPACE - TIME DIMENSIONS*

by

Sergio Albeverio and Raphael Høegh-Krohn

Institute of Mathematics
University of Oslo
Blindern, Oslo 3
(Norway)

ABSTRACT

We construct models of local relativistic quantum fields in two space-time dimensions with weak polynomial or exponential interactions, which are canonical in the sense of the canonical Hamiltonian formalism. They are thus given in terms of the time zero vacuum, which determines a unitary strongly continuous representation of the canonical commutation relations for the time zero fields (for which the vacuum is cyclic) and their conjugate momenta, as well as a unitary strongly continuous representation of the inhomogeneous Lorentz group. The infinitesimal generators of time translations and Lorentz transformations are given by Dirichlet forms associated with μ. The infinitesimal generator of time translations generates a homogeneous Markov process solving a stochastic diffusion equation with osmotic velocity given by μ. The models satisfy conditions for Euclidean Markov fields discussed by Nelson and Simon. The measure μ is the restriction of the physical vacuum for the previously constructed Wightman models (with the same interactions) to the functions of the time zero fields.

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1. Introduction

In this paper we construct models of local relativistic quantum fields which are canonical, in the sense that they are given completely in terms of a probability measure μ , the time zero vacuum, which determines a representation of the canonical commutation relations for the time zero fields and their conjugate momenta, as well as the infinitesimal generators of the inhomogeneous Loretz group, unitarily strongly continuously represented in $L_2(d\mu)$, with 1 as invariant vector. We discussed problems related to the ones tackled in this paper in two previous papers [1] , [2] , and we refer to these also for references concerning previous work related to the subject.

In this paper we consider more particularly[1]) the weakly coupled $P(\varphi)_2$ models ([3]) and the exponential interactions models ([4], [5]) of Bose quantum fields in two space time dimensions. Such models satisfy in particular all the Wightman axioms and their physical Hamiltonian has a mass gap at the lower end of its spectrum. It is an open question whether these models are canonical in the sense that the physical vacuum is a cyclic vector for the time zero fields or , equivalently, ([6]), the contraction semigroup generated by the physical Hamiltonian is a Markov semigroup. In more technical terms it is open whether the models satisfy Nelson's axioms [7], [8] or the related Simon's ones [9] , see also Ref. [4], Ch.IV. For some discussions of these questions see [6] - [14].

In this paper we show that in any case one can use the models mentioned to construct canonical ones. We shall now briefly describe this construction.

Let μ^* be the measure correspondent to the physical vacuum for any of the models mentioned above.It is known that the time zero fields exist as multiplication operators in $L_p\ (d\mu^*), 1 \leq p < \infty$ with the physical vacuum as an analytic vector in $L_2(d\mu^*)$ ([13]^2) resp. [5]). Let be the probability measure on the real space S'(R) defined as the restriction of μ^* to the time zero fields, so that

$$\int e^{i\langle \xi,\varphi\rangle} d\mu = \lim_{n\to\infty} \int e^{i\langle \xi^*,\varphi\otimes\chi_n\rangle} d\mu^* \qquad (1.1)$$

for any $\xi \in S'(R)$, $\xi^* \in S'(R^2)$, $\varphi \in S(R)$, and any δ-sequence of functions χ_n out of S(R).

In [1],[2] we established results on μ which we shall now recall, at least partially. μ was first proven to be a quasi invariant probability measure with respect to the nuclear rigging

$S(R) \subset L_2(d\mu) \subset S'(R)$ (real spaces),

so that μ defines a unitary strongly continuous representation $\varphi \longrightarrow U(\varphi), V(\varphi)$ of the Weyl commutation relations on $L_2\ (d\mu)$, with

$$(U(\varphi)f)(\xi) = e^{i\langle \xi,\varphi\rangle} f(\xi), \quad (V(\varphi)f)(\xi) = \sqrt{\frac{d\mu(\xi+\varphi)}{d\mu(\xi)}}\, f(\xi+\varphi) \qquad (1.2)$$

and $f \in L_2(d\mu)$. Let $\pi(\varphi)$ be the infinitesimal generator of the unitary group $V(t\varphi)$,i.e. $\pi(\varphi)$ is the canonical momentum, conju gate to the canonical field $\langle\xi,\varphi\rangle$. We proved in [2] that the function 1 in $L_2(d\mu)$ (i.e. the time zero vacuum) is an analytic vector for $\pi(\varphi)$. Let FC_2 be the dense omain in $L_2\ (d\mu)$ consisting of functions on S' (R) which are finitely based and C^n on their base, so that $f(\xi) = f(P_f\xi)$ for some projection P_f with finite dimen sional range in S(R) and such that the restriction f^* of f to the range of P_f is n-times continuously differentiable. It was proven in 2 that μ is strictly positive, in the sense that

$$\int_{S'(R)} f\, d\mu = \int_{R^n} f^*(x_1,\ldots,x_n)\rho(x_1,\ldots,x_n)dx_1 \ldots dx_n \tag{1.3}$$

where the density ρ is bounded away from zero, uniformly on compacts.

In [1] and in section 4 of [2] we considered moreover the Dirichlet form

$$\int \nabla \bar{f}\, \nabla f\, d\mu \equiv \int \left|\frac{\delta f}{\delta \xi(x)}\right|^2 dx\, d\mu(\xi)\,, \tag{1.4}$$

obtained by closure from its restriction to FC_2 ,where the gradient ∇ is naturally defined of FC_2. The unique self-adjoint operator H associated with the Dirichlet form, called diffusion operator, is the Friedrichs extension of its restriction to FC_2 and on FC_2

$$H = -\Delta - \beta\cdot\nabla\,, \tag{1.5}$$

with the natural definition of the Laplacian Δ , and with $\beta\cdot\nabla f = \sum_{j=1}^{n} (\beta\cdot\varphi_j)(\varphi_j\cdot\nabla) f$, where $f \in FC^2$, $\varphi_1,\ldots\varphi_n$ is an orthonormal base in the range of P_f and $\beta\cdot\varphi_j = 2i\pi(\varphi_j)\cdot 1$. β was called in [1], [2] the osmotic velocity corresponding to the measure μ.[3])

The relation between H and the physical Hamiltonian H_{ph} of the Wightman models of Ref. [3] , [5] is, as proven in [1], [2] ,

$$(f, J_o H_{ph} J_o g) = (f, Hg)\,, \tag{1.6}$$

for any f,g in FC_2 , where J_o is the embedding of $L_2(d\mu)$ in $L_2(d\mu^*)$.

We come now to the main results and the distribution of the topics in this paper.

Insection 2 we make general considerations about Diriclet forms associated with the nuclear rigging

$$S(R^n) \subset L_2(R^n) \subset S'(R^n).$$

They are of the type

$$D_h(f,f) \equiv \int h(x) \left|\frac{\delta f}{\delta \xi(x)}\right|^2 dx\, d\mu ,$$

with $f \in FC_2$ and notation generalized from (1.4). h is any function in the space $\mathcal{O}(R^n)$ of multipliers on $S'(R^n)$. The quasi invariant measures μ are assumed to be such that 1 is in the domain of $\pi(\varphi)$, for all $\varphi \in S(R^n)$, where $\pi(\varphi)$ is defined in the same way as above. The form D_h is shown to be closable so that its closure defines uniquely a self-adjoint operator $H(h)$. Under some additional conditions we prove that, on FC_4, the commutator $[H(h_1), H(h_2)]$ is a vector field over $S'(R^n)$, with components given by the kernel of the bounded linear map $[H(h_1), H(h_2)] < \langle \xi,\varphi \rangle$ from $S(R^n)$ to $L_2(d\mu)$.

In section 3 the results are then applied to the case where μ is the time zero vacuum measure of quantum fields with exponential or polynomial interactions. In this case the above kernel is simply equal to $(h_1 \nabla h_2 - h_2 \nabla h_1) \nabla \xi(x)$.

For $h_1 \equiv 1$ we have $H(h_1) = H$, the diffusion operator associated with μ. For $h_2(x) \equiv x$, setting $H(h_2) \equiv \Lambda$, we then have

$[\Lambda H] = iP$, with P the infinitesimal generator of space translations, naturally induced in $L_2(d\mu)$ by the space translations acting on the fields $\langle \xi,\varphi \rangle$. Moreover we find that Λ, P, H have on FC_4 all the correct commutation relations of the infinitesimal generators of the inhomogeneous Loretz group. Moreover we prove that we have indeed a unitary strongly continuous representation of the inhomogeneous Lorentz group on the canonical space $L_2(d\mu)$, generated by Λ, P, H.

We thus see that the measure μ has given us local relativistic canonical models. These results carry through , for the models considered, the program discussed by Araki [17].

We expect of course that the canonical models constructed in the present paper from the restriction of the physical vaccum of the Euclidean models to the time zero fields coincide with the usual models [3] ,[5] constructed by analytic continuation from the Euclidean models.However we have not yet been able to prove this. In any case the canonical models of this paper are models that have the Markov property with respect to half planes and thus satisfy the conditions for Euclidean Markov fields discussed by Nelson [7] and Simon [9] .

2. Diffusion operators on the space of tempered distributions.

Consider the nuclear rigging

$$S(R^n) \subset L_2(R^n) \subset S'(R^n) \tag{2.1}$$

where $S(R^n)$ is the Schwartz space and $S'(R^n)$ its dual i.e. the space of tempered distributions. Let μ be an S-quasi invariant probability measure on S', i.e. $d\mu(\xi)$ and $d\mu(\xi+\varphi)$ are equivalent measures for any φ in S. Such a measure μ gives rise to a unitary representation (U,V) on $L_2(d\mu)$ of the Weyl commutation relations on S. Namely, for $f \in L_2(d\mu)$, $(U(\varphi)f)(\xi) = e^{i\langle\varphi,\xi\rangle}f(\xi)$ and $(V(\varphi)f)(\xi) = \alpha(\xi,\varphi)f(\xi)$, where $\alpha(\xi,\varphi) = \left(\frac{d\mu(\xi+\varphi)}{d\mu(\xi)}\right)^{\frac{1}{2}}$.
Let $\pi(\varphi)$ be the infinitesimal generator for the unitary group $V(t\varphi)$. We say that $\mu \in \mathcal{F}_n(S')$ if the function 1 is in the domain of $\pi(\varphi_1)\dots\pi(\varphi_n)$ for any n elements in S. For further details see [1], [2].

In what follows we shall always assume that $\mu \in \mathcal{F}_1(S')$.
Let FC_k be the subspace of $L_2(d\mu)$ consisting of bounded finitely based and k-times differentiable functions i.e. $f \in FC_k$ iff there is an $f^* \in C_k(R^l)$ and $\varphi_1,\dots,\varphi_l$ in S such that $f(\xi) = f^*(\langle\varphi_1,\xi\rangle,\dots,\langle\varphi_l,\xi\rangle)$. For any $f \in FC_1$ we define

$$\frac{\delta f}{\delta\xi(x)}(\xi) = \sum_{j=1}^{l} f_j^*(\langle\varphi_1,\xi\rangle,\dots,\langle\varphi_l,\xi\rangle)\varphi_j(x) \tag{2.2}$$

where f_j^* are the partial derivatives of f^*. We see that $\frac{\delta f}{\delta\xi}$ is a continuous map from S' into S with finite dimensional range. Since $1 \in D(\pi(\varphi))$ for any $\varphi \in S$, we get that $\varphi \to \pi(\varphi)1$ is a linear mapping from S into $L_2(d\mu)$. Moreover, since S is a complete metric space, we have that $\varphi \to \pi(\varphi)\cdot 1$ is bounded, and then, by using that S is nuclear, we get that this mapping has a kernel which we denote $\frac{1}{2i}\beta(x)$. β is a measurable mapping of

S' into S' called the osmotic velocity, and we have

$$\pi(\varphi)1 = \frac{1}{2i}\int \beta(x)\varphi(x)dx . \qquad (2.3)$$

For a proof of these facts see prop. 2.5 [1].

Let $h \in S'(R^n)$, then we define, for $f \in FC_1$, the Dirichlet form

$$D_h(f,f) = \frac{1}{2}\iint h(x)\left|\frac{\delta f}{\delta\xi(x)}\right|^2 dx\, d\mu(\xi) . \qquad (2.4)$$

This is well defined since $\frac{\delta f}{\delta\xi}$ is a continuous mapping from S' into a finite dimensional subspace of S, and by (2.2) $\int h(x)\left|\frac{\delta f}{\delta\xi(x)}\right|^2 dx$ is uniformly bounded and continuous in ξ. Let $\mathcal{O}_M(R^n)$ be the space of multipliers for $S'(R^n)$, i.e. if $h \in \mathcal{O}_M(R^n)$ then $T(x) \to h(x)T(x)$ is a bounded linear transformation on S'. If $h \in \mathcal{O}_M(R^n)$ then the Dirichlet form (2.4) restricted to FC_2 is closable. It is namely given by a symmetric operator in $L_2(d\mu)$

$$D_h(f,f) = (f,H(h)f) , \qquad (2.5)$$

where

$$H(h)f = -\tfrac{1}{2}\int h(x)\left(\frac{\delta^2 f}{\delta\xi(x)^2} + \beta(x)\frac{\delta f}{\delta\xi(x)}\right)dx . \qquad (2.6)$$

For details see theorem 2.6 [1] and the proof of it.

Since $D_h(f,f)$ is closable its closure defines a self-adjoint operator on $L_2(d\mu)$ which we shall also denote by $H(h)$. Since $h \geq 0 \Rightarrow H(h) \geq 0$ by (2.4), we have that $H(h)$ is monotone in h and since monotone convergence of semibounded forms implies resolvent convergence we have that, if $0 \leq h_n \nearrow h$, then $(1+H(h_n))^{-1}$ converges strongly to $(1+H(h))^{-1}$. As an integral in h, $D_h(f,f)$ is absolutely continuous with respect to the Lebesgue measure in R^n. Hence, by monotone convergence, $H(h)$ may be extended to all h in $L_\infty(R^n)$.

If $h \in \mathcal{O}_M(R^n)$ and $h \geq 0$ then it is easily verified that

$H(h)$ is the limit in the strong resolvent sense of operators $H_m(h)$ such that $H_m(h)$ are given as direct integrals of forms which are Markov symmetric forms in the sense of Fukushima [17], as in theorem 2.7 of Ref. [1]. In this way we get the following theorem

Theorem 2.1

Let $h \in \mathcal{O}_M(R^n)$ i.e. the space of multipliers on $S'(R^n)$ such that $h \geq 0$. Then $e^{-tH(h)}$ is a conservative Markov semi-group i.e. for $f \in L_2(d\mu)$ such that $f \geq 0$ we have that $e^{-tH(h)}f \geq 0$ and $e^{-tH(h)}1 = 1$. Thus the corresponding Markov process $\xi_h(t)$ on $S'(R^n)$ is a homogeneous Markov process with invariant measure μ. This process $\xi_h(x,t)$ on $S'(R^n)$ satisfies the following stochastic differential equation

$$d\xi_h(x,t) = h(x)\beta(\xi_h(t))(x)dt + h(x)dW(x,t)$$

where $W(x,t)$ is the standard Wiener process on $S'(R^n)$ given by the rigging $S(R^n) \subset L_2(R^n) \subset S'(R^n)$, and $\beta(\xi)(x) = \frac{1}{2i}\pi(x)\cdot 1$ in the sense of (2.3). □

Let us now assume for some h and φ in $S(R^n)$, that $\pi(\varphi)\cdot 1$ is in $D(H(h))$. Then $H(h)\pi(\varphi)\cdot 1$ is a bilinear map from $S\times S$ into $L_2(d\mu)$.

By the abstract kernel theorem we get in the same way as in prop.2.5 of [1] that there is a measurable mapping $U: \xi \to T(\xi)(x,y)$ from $S'(R^n)$ into $S'(R^n\times R^n)$ such that, for h and φ in $S(R^n)$, $\langle\varphi\times h, T(\xi)\rangle$ is in $L_2(d\mu)$ as a function of ξ and

$$\langle\varphi\times h,\ T(\xi)\rangle = \frac{1}{i}(H(h)\pi(\varphi)\cdot 1)(\xi)\,. \tag{2.7}$$

Since obviously $H(h)\pi(\varphi)\cdot 1 = [H(h),\pi(\varphi)]\cdot 1$ and $[H(h),\pi(\varphi)]$ is a multiplication operator we also have

$$\langle\varphi\times h, T(\xi)\rangle = i[\pi(\varphi),H(h)]\,. \tag{2.8}$$

Remark:

In the case where the osmotic velocity $\beta(\xi)(x)$ is sufficiently smooth, we have that for $h \in S$

$$H(h) = \int h(y)\left(\tfrac{1}{2}\pi(y)^2 + V(y)\right)dx ,$$

where $V(y) = \frac{1}{2}\frac{\delta\beta(y)}{\delta\xi(y)} + \frac{1}{4}\beta(y)^2$ (see section 2 Ref. [1]). In this case we see from (2.8) that

$$T(\xi)(x,y) = \frac{\delta V(y)}{\delta\xi(x)} . \qquad (2.9)$$

Lemma 2.1

Let $\mu \in \mathcal{P}_2(S')$, then if for some $h_1 \in \mathcal{O}_M(R^n)$ we have that $\pi(\varphi)\cdot 1 \in D(H(h_1))$ for any $\varphi \in S(R^n)$ then, for any h_2 in $\mathcal{O}_M(R^n)$, $H(h_2)$ maps FC_4 into the domain of $H(h_1)$.

Proof:

Let $f \in FC_4$ then

$$H(h_2)f = -\frac{1}{2}\int h_2(x)\frac{\delta^2 f}{\delta\xi(x)^2}dx - \frac{1}{2}\int h_2(x)\,\beta(x)\frac{\delta f}{\delta\xi(x)}dx . \qquad (2.10)$$

The first term is obviously in FC_2 and since $\frac{\delta f}{\delta\xi(x)}$ as a function of x is in a fixed finite dimensional subspace of S, we may write the second term as

$$g = -i\int h_2(x)\frac{\delta f}{\delta\xi(x)}\,\pi(x)dx\cdot 1 , \qquad (2.11)$$

because $h_2(x)\frac{\delta f}{\delta\xi(x)}$ is again in S as a function of x. Since by assumption $\pi(\varphi)\cdot 1 \in D(H(h_1))$ for any $\varphi \in S$, we find that

$$\begin{aligned} H(h_1)g = & -\frac{i}{2}\iint h_1(y)h_2(x)\frac{\delta^3 f}{\delta\xi(y)^2\delta\xi(x)}\cdot\pi(x)dxdy\cdot 1 \\ & +\frac{i}{2}\iint h_1(y)h_2(x)\frac{\delta^2 f}{\delta\xi(y)\delta\xi(x)}\cdot\beta(y)\pi(x)dxdy\cdot 1 \\ & -\frac{i}{2}\int h_2(x)\frac{\delta f}{\delta\xi(x)}H(h_1)\,\pi(x)dx\cdot 1 . \end{aligned} \qquad (2.12)$$

Now the first term in (3.12) is well defined since $1 \in D(\pi(\varphi))$ for any $\varphi \in S$ and

$$h_2(x) \int h_1(y) \frac{\delta^3 f}{\delta^2 \xi(y) \delta \xi(x)} dy \in S .$$

The second term is equal to

$$- \iint h_1(y) h_2(x) \frac{\delta^2 f}{\delta \xi(y) \delta \xi(x)} \pi(y) \pi(x) dx dy \cdot 1$$

which is in $L_2(d\mu)$ by the assumption that $\mu \in \mathcal{P}_2(S')$ and the abstract kernel theorem.
The third term is in $L_2(d\mu)$ by the assumption that $\pi(\varphi) \cdot 1$ is in $D(H(h_1))$ and again the abstract kernel theorem. This proves the lemma. □

Let us now assume that $\mu \in \mathcal{P}_2(S')$ and that for some h_1 and h_2 in $\mathcal{O}_M(R^n)$ we have that $\pi(\varphi) \cdot 1 \in D(H(h_i))$ for $i = 1,2$ and all $\varphi \in S(R^n)$. Then by the previous lemma $[H(h_1), H(h_2)]$ is defined on FC_4. By (2.7), (2.10) and (2.12) we have that for $f \in FC_4$

$$\begin{aligned} 4 \cdot H(h_1)H(h_2)f &= \iint h_1(y) h_2(x) \frac{\delta^4 f}{\delta \xi(x)^2 \delta \xi(y)^2} dx\, dy \\ &- \iint h_1(y) h_2(x) \frac{\delta^3 f}{\delta \xi(x)^2 \delta \xi(y)} \cdot \beta(y) dx\, dy \\ &- \iint h_1(y) h_2(x) \frac{\delta^3 f}{\delta \xi(x) \delta \xi(y)^2} \beta(x) dx\, dy \qquad (2.13) \\ &+ \iint h_1(y) h_2(x) \frac{\delta^2 f}{\delta \xi(y) \delta \xi(x)} \beta(x) \beta(y) dx\, dy \\ &+ 2 \iint h_1(y) h_2(x) \frac{\delta f}{\delta \xi(x)} T(\xi)(x,y) dx\, dy . \end{aligned}$$

We remark that by the assumption that $\pi(\varphi) \cdot 1 \in D(H(h_1))$ we have, by (2.7), that $T(\xi)(x,y)$ is defined on $h_1(y) \times h_2(x) \frac{\delta f}{\delta \xi(x)}$ for almost all ξ and the result is in $L_2(d\mu)$. By antisymmetrization with respect to h_1 and h_2 we get that the four first terms in (2.13) fall out and the result is

$$[H(h_1),H(h_2)]f = \frac{1}{2}\iint (h_1(y)h_2(x) - h_1(x)h_2(y))T(\xi)(x,y)\frac{\delta f}{\delta\xi(x)}dx\,dy \quad (2.14)$$

For any $\mu \in \mathcal{F}_1(S')$ we say that $H = H(1)$ is the Dirichlet operator given by μ. We say that H is an harmonic oscillator on S' if μ is a non degenerate Gaussian measure i.e. its Fourier transform has the form $e^{-\frac{1}{4}(\varphi,B\varphi)}$, where B is a bounded positive operator on S with a bounded universe on S. By Minlos theorem there is a unique μ corresponding to any B bounded and positive, and it is easily seen that if also B has a bounded universe, then $\mu \in \mathcal{F}_1(S)$ with $\beta(\xi)(x) = -\int A(x,y)\xi(y)dy$, where $A(x,y)$ is the kernel of $A = B^{-1}$, which by assumption is a bounded map of S, hence, A being symmetric, it is also a bounded map of S'. So we see that harmonic oscillators have linear osmotic velocity fields. By a straight forward calculation we find that the mapping $T(\xi)$ in the case of a harmonic oscillator is given by

$$T(\xi)(x,y) = A(x,y)\int A(y,z)\xi(z)dz. \quad (2.15)$$

In this case we see from (2.15) that for $\varphi \in S(R^n)$ and $h \in \mathcal{O}_M(R^n)$ we have that $\varphi \times h$ is always in the domain of $T(\xi)$ and moreover

$$\langle \varphi \times h, T(\xi)\rangle = \langle hA\varphi, A\xi\rangle. \quad (2.16)$$

Since (2.16) is a continuous linear functional it is always in $L_2(d\mu)$ with respect to the Gaussian measure μ. We have therefore proved the following lemma

Lemma 2.2

If H is a harmonic oscillator on $S'(R^n)$ then, for any $h \in \mathcal{O}_M(R^n)$ and any $\varphi \in S(R^n)$, $\pi(\varphi)\cdot 1 \in D(H(h))$. Moreover if the corresponding osmotic velocity is

$$\beta(\xi)(x) = -\int A(x,y)\xi(y)dy$$

then the corresponding mapping $T(\xi)$ is given by

$$T(\xi)(x,y) = A(x,y)\int A(y,z)\xi(z)dz . \qquad \square$$

Let us now return to the formula (2.14), and consider the expression

$$\iint h_1(y)h_2(x)T(\xi)(x,y)\frac{\delta f}{\delta\xi(x)}\,dx\,dy . \tag{2.17}$$

By the definition (2.7) this is equal to

$$-\frac{1}{2}\int \frac{\delta f}{\delta\xi(x)} \cdot h_2(x)H(h_1)\beta(x)dx . \tag{2.18}$$

Now by the definition (2.6) we have

$$-\frac{1}{2}h_2(x)\,\beta(x) = H(h_2)\xi(x) , \tag{2.19}$$

where of course $H(h_2)\ \xi(x)$ is to be understood as a bounded linear map from S into $L_2(d\mu)$. That this map is bounded follows from the fact that $\mu \in \mathscr{F}_1(S')$.
In fact if we assume that we have $\pi(\varphi)\cdot 1 \in D(H(h_i))$ $i=1,2$ for any $\varphi \in S(R^n)$, then (2.19) is a bounded mapping from $S(R^n)$ into $D(H(h_1))$. Hence $H(h_1)H(h_2)\xi(x)$ is a bounded mapping from $S(R^n)$ into $L_2(d\mu)$, so that (2.17) is equal to

$$\int\frac{\delta f}{\delta\xi(x)} : H(h_1)\,H(h_2)\,\xi(x)\,dx . \tag{2.20}$$

We have thus proven the following theorem

<u>Theorem 2.2</u>

If $\mu \in \mathscr{F}_2(S')$ and for some h_1 and h_2 in $\mathscr{O}(R^n)$ we have that $\pi(\varphi)\cdot 1 \in D(H(h_i))$ $i=1,2$ for φ arbitrary in $S(R)$, then $H(h_i)$ maps FC_4 into $D(H(h_j))$. In particular, since

$\langle\varphi,\xi\rangle \in FC_4$, $H(h_i)$ maps $\langle\varphi,\xi\rangle$ into $D(H(h_j))$ so that $H(h_i)H(h_j)\langle\varphi,\xi\rangle$ is a bounded linear map from S into $L_2(d\mu)$, S being nuclear. Let $H(h_i)H(h_j)\,\xi(x)$ be the kernel of this map, then we have, for any $f \in FC_4$ that

$$[H(h_1),H(h_2)]f = \int \frac{\delta f}{\delta\xi(x)}\,[H(h_1),H(h_2)]\,\xi(x)\,dx\,.$$

We remark that this theorem shows that the commutant $[H(h_1),H(h_2)]$ is a first order derivation or a vector field over S' with components given by $[H(h_1),H(h_2)]\cdot\xi(x)$.

3. The diffusion operators of the local relativistic quantum fields in two space-time dimensions.

In this section we consider the cases where the measure μ is the restriction of the physical vacuum to the time zero fields for the models in which the infinite volume Schwinger functions exist and the corresponding energy operator has zero as an isolated, but not necessarily simple, eigenvalue. These models are the weak polynomial interactions [3], the strong polynomial interactions with Dirichlet boundary conditions [19] - [21] and the exponential interactions [5]. In all these cases we know that the restriction μ of the physical vacuum to the time zero fields is a measure on $S'(R)$. Thus we consider, as in [1], [2], the natural nuclear rigging

$$S(R) \subset L_2(R) \subset S'(R). \tag{3.1}$$

In Ref. [2] we proved that $\mu \in \mathcal{P}_n(S')$ for all n, in fact we proved that 1 is an analytic vector for $\pi(\varphi)$, for any $\varphi \in S$. Let $H(h)$ be the corresponding diffusion operators in the sense of the previous section and let $H = H(1)$.

In all cases considered here the physical vacuum is given in terms of the Wightman functions $W_n(x_1,t_1,\ldots,x_n,t_n)$ which are used to construct, by the Gelfand-Segal-Wightman construction, the physical Hilbert space $\mathcal{H}_{ph}$ and the physical energy operator H_{ph}, which is the generator of the time translations in $\mathcal{H}_{ph}$. We have of course that $L_2(d\mu)$ is a closed subspace of $\mathcal{H}_{ph}$ and one would naturally have liked to prove that $\mathcal{H}_{ph}$ is identical with $L_2(d\mu)$ and H_{ph} is identical with the diffusion operator $H = H(1)$. This is however still an open question. What we have been able to prove is that for f and g in FC_2 we have that

$$(f, H_{ph}\, g) = (f, Hg). \tag{3.2}$$

For this result see Ref. [1], [2].

The proof that $\mu \in \mathcal{F}_1(S')$ in Ref. [2], which gives the existence of the diffusion operator H, and also the proof of the analyticity of μ, i.e. that 1 is an analytic vector of all $\pi(\varphi)$, were based on the following formula, proven in [1], [2],

$$i[\pi(\varphi), H] = \langle \varphi, (-\Delta + m^2)\xi \rangle + :v'(\xi):(\varphi), \tag{3.3}$$

where both sides are to be considered as bilinear forms on $FC_2 \times FC_2$. Here $:v'(\xi):(\varphi) = \int :v'(\xi(x)):\varphi(x)dx$, and v' is the derivative of the function giving the interaction i.e. the volume cut-off interaction is of the form

$$H_1 = H_o + \int_{|x| \leq 1} :v(\xi(x)):dx, \tag{3.4}$$

where H_o is the free energy and v is a polynomial bounded below for the polynomial interactions and $v(s) = \int \cosh(\alpha s) d\nu(\alpha)$ with ν any positive measure with compact support in $(-\frac{4}{\sqrt{\pi}}, \frac{4}{\sqrt{\pi}})$ for the exponential interactions, $: \, :$ denoting the Wick ordering. The formula (3.3) was proved by using the expression (3.4) for the corresponding space cut-off Hamiltonian H_1, the fact that, on FC_2, H_1 coincides with the diffusion operator given by the corresponding space cut-off vacuum μ_1, together with the weak convergence of μ_1.

Let now $h(x) > 0$ and $h(x) \in \mathcal{O}_M(R)$. Then consider the modified Wightman functions

$$W_n^h(x_1, t_1, \ldots, x_n, t_n) = W_n(x_1, h(x_1)t_1, \ldots, x_n, h(x_n)t_n). \tag{3.5}$$

We see that W_n^h is invariant under time translations and satisfy the same positivity conditions and the same analyticity conditions in the time differences as do the original Wightman functions.

Moreover, since $h(x) > 0$, the Hilbert space constructed by the Gelfand-Segal construction from W_n^h is in a natural way identical with $\mathcal{H}_{ph}$. The corresponding generator of the time translations $H_{ph}(h)$ is of course different from H_{ph}. We get however by the same proof as for (3.2) that, for f and g in $FC_2 \subset L_2(d\mu)$,

$$(f,H_{ph}(h)g) = (f,H(h)g) . \tag{3.6}$$

We also immediately get from (3.5), or from the corresponding expression for the Schwinger functions, that $H_{ph}(h)$ is the infinite volume limit of the corresponding finite volume Hamiltionans

$$H_l(h) = H_o(h) + \int_{|x|\leq l} H(x):v(\xi(x)):dx , \tag{3.7}$$

in exactly the same way as H_{ph} is the limit of H_l. $H_o(h)$ in (3.7) is given by (3.5) with the free Wightman functions on the right hand side. However in the free case it is immediate to see that the operators $H_o(h)$ constructed from (3.5) with the free Wightman functions actually leave invariant the functions of the time zero fields. In fact one has that $H_o(h)$ is essentially self adjoint on $FC_2 \subset L_2(d\mu)$ in the same way as in the proof of Theorem 4.0 in Ref. [1]. Hence from the correspondent of (3.6) for the free case we get that $H_o(h)$ actually coincides with the corresponding diffusion operator given by the free vacuum.

Now we find by (2.8), lemma 2.2 and a simple calculation that

$$i[\pi(\varphi),H_o(h)] = \langle\varphi,(-\nabla h\nabla+hm^2)\xi\rangle . \tag{3.8}$$

Hence in the same way as we proved formula (3.3) above in section 4 of Ref. [1], [2] we now get by (3.7) that

$$i[(\varphi),H(h)] = \langle\varphi,(-\nabla h\nabla+hm^2)\xi\rangle + :v'(\xi):(h\varphi) . \tag{3.9}$$

(3.3) and (3.9) hold only in the sense of bilinear forms on $FC_2 \times FC_2$, Up to now we have assumed that $h(x) > 0$. However since as bilinear forms on $FC_2 \times FC_2$ both sides of (3.9) are obviously linear and continuous as functions of h, $\mathcal{O}_M(R)$ being given the topology of the bounded operators from $S'(R)$ into $S'(R)$, it follows that (3.9) holds as an identity between bilinear forms on $FC_2 \times FC_2$ for all $h \in \mathcal{O}_M(R)$. We now need the following lemma.

<u>Lemma 3.1</u>

For any $\varphi \in S(R)$ we have $:v'(\xi):(\varphi) \in L_2(d\mu)$, in the cases of exponential interactions or respectively weak polynomial interactions i.e. if

$$v(s) = \int \cosh(s\alpha)d\nu(\alpha)$$

where ν is any positive measure with compact support in $(-\frac{4}{\sqrt{\pi}}, \frac{4}{\sqrt{\pi}})$, or respectively $v(s) = \sum_{k=0}^{2n} a_k s^k$ where $a_{2n} > 0$ and $a_k, k = 0, \dots, 2n$ are sufficiently small.

<u>Proof</u>: We first consider the case of exponential interactions.

Let $\psi_n \in S(R)$ such that $\psi_n \geq 0$ and $\psi_n \to \delta$ in $S'(R)$, and consider

$$:v'(\xi * \psi_n):(\varphi) = \iint \alpha :\sinh(\alpha \cdot \xi * \psi_n(x)): d\nu(\alpha)\varphi(x)dx\,, \quad (3.10)$$

where $\xi * \psi_n$ is the convolution of ξ with ψ_n.

Let now $G_n = \frac{1}{2}\psi_n * (-\Delta + m^2)^{-\frac{1}{2}}\psi_n$, then

$$:e^{\alpha\xi*\psi_n(x)}: = e^{-\frac{1}{2}\alpha^2 G_n(o)} e^{\alpha\xi*\psi_n(x)} \quad (3.11)$$

so that

$$:e^{\alpha\xi*\psi_n(x)}: :e^{\beta\xi*\psi_n(y)}: = e^{-\frac{1}{2}(\alpha^2+\beta^2)G_n(o)} e^{\xi*(\alpha\psi_n(x)+\beta\psi_n(y))}. \quad (3.12)$$

Let μ_o be the free vacuum restricted to the time zero fields, i.e. the μ in the case $v = 0$. We proved in theorem 6.1

of Ref. [5] that, for the exponential interaction, the Schwinger functions are bounded by the free Schwinger functions. From this result we immediately get that for $\varphi \in S(R)$ and $\varphi \geq 0$ we have

$$\int_{S'} e^{\langle\varphi,\xi\rangle} d\mu \leq \int_{S'} e^{\langle\varphi,\xi\rangle} d\mu_o . \qquad (3.13)$$

From (3.12 and (3.10) we then get

$$\int_{S'} |:v'(\xi*\psi_n):(\varphi)|^2 d\mu \leq \iiiint |\alpha||\beta| e^{\alpha\beta G_n(x-y)} |\varphi(x)\varphi(y)| \, d\nu(\alpha)d\nu(\beta)dxdy. \qquad (3.14)$$

It is proved in [4], [5] that (3.14) is bounded uniformly in n. Moreover if $\| \;\|$ is the norm in $L_2(d\mu)$ then

$$\|:v'(\xi*\psi_m):(\varphi) - :v'(\xi*\psi_n):\varphi\|^2 \leq \iiiint |\alpha||\beta| \, |\varphi(x)\varphi(y)| \qquad (3.15)$$
$$|e^{\alpha\beta G_{nn}(x-y)} + e^{\alpha\beta G_{mm}(x-y)} - 2e^{\alpha\beta G_{nm}(x-y)}| \, dxdyd\nu(\alpha)d\nu(\beta) ,$$

where $G_{nm} = \frac{1}{2}\psi_n * (-\Delta+m^2)^{-\frac{1}{2}}\psi_m$. So by the assumption on ν above $:v'(\xi*\psi_n):(\varphi)$ is an $L_2(d\mu)$ convergent sequence. Introduce now the momentum and space cut-off Hamiltonian

$$H_l^n(h) = H_o(h) + \int_{|x|\leq l} h(x):v(\xi*\psi_n):dx . \qquad (3.16)$$

We have that

$$i[\pi(\varphi),H_l^n(h)] = i[\pi(\varphi),H_o(h)] + :v'(\xi*\psi_n):(h\cdot\chi_l\cdot\psi_n*\varphi) \qquad (3.17)$$

where χ_l is the characteristic function for the interval $[-l,l]$. By what was said above the right hand side of (3.17) converges, as l and n tend to infinity, strongly in $L_2(d\mu)$. On the other hand we have by the construction of the space cut-off exponential interaction, Ref. [4], that the left hand side converges weakly, in the sense of forms defined on a fixed dense domain, to $i[\pi(\varphi),H_l(h)]$. However we have in the same way as in (3.9) that

$$i[\pi(\varphi),H_l(h)] = i[\pi(\varphi),H_o(h)] + :v'(\xi):(\chi_l h\varphi) , \qquad (3.18)$$

which obviously converges to the right hand side of (3.9) if φ has compact support. Hence if φ has compact suppotrt we get from (3.14) and what is said above that

$$\int_{S'} |:v'(\xi):(\varphi)|^2 d\mu \leq \iiiint |\alpha||\beta| e^{\alpha\beta G(x-y)} |\varphi(x)\varphi(y)| d\nu(\alpha) d\nu(\beta) dx dy . \tag{3.19}$$

On the other hand (3.19) is obviously a continuous bilinear form on $S \times S$ so it extends easely by continuity to all $\varphi \in S(R)$. Hence the lemma is proved for the case of exponential interactions.

For the case of weak polynomial interactions it is an immediate consequence of theorem 2, section I.2 of Ref. [22]. This proves the lemma. □

Let us now assume that we have an interaction such that the conclusion of lemma 3.1 holds. Then it follows from (3.9), linear functions being in $L_2(d\mu)$, that $\pi(\varphi)\cdot 1 \in D(H(h))$ for any $\varphi \in S(R)$ and any $h \in \mathcal{O}_M(R)$. Therefore the conclusions of theorem (2.2) hold for any pair of h_1 and h_2 in $\mathcal{O}_M(R)$.

From (3.9) however we see that the mapping $H(h_1)H(h_2)\xi(x)$ considered in theorem 2.2 is, in the present case of local relativistic fields, given by

$$H(h_1)H(h_2)\xi(x) = H(h_1)\frac{1}{i}\pi(x)\cdot 1 = i[\pi(x),H(h_1)] . \tag{3.20}$$

So that

$$H(h_1)H(h_2)\xi(x) = h_2(x)(-\nabla h_1\nabla + h_1 m^2)\xi(x) + h_1\cdot h_2(x):v'(\xi(x)): , \tag{3.21}$$

from which it follows

$$[H(h_1),H(h_2)]\xi(x) = (h_1(\nabla h_2) - h_2(\nabla h_1))\cdot\nabla\xi(x) . \tag{3.22}$$

Hence we have proved the following theorem

Theorem 3.1

If μ is the time zero vacuum for a local relativistic interaction such that $:v'(\xi):(\varphi) \in L_2(d\mu)$ for any $\varphi \in S(R)$, which is for instance the case for exponential interactions or weak polynomial interactions, we have that the conclusions of theorem 2.2 hold, and moreover that, for any $f \in FC_4$,

$$[H(h_1),H(h_2)]f = \int (h_1 \nabla h_2 - \nabla_2 \nabla h_1) \cdot \nabla \xi(x) \cdot \frac{\delta f}{\delta \xi(x)} dx$$

for any pair h_1 and h_2 in $\mathcal{O}_M(R)$. □

Take now $h_1(x) = x$ and $h_2(x) = 1$. Then, with the notations $\Lambda = H(h_1)$ and $H = H(h_2)$, for such h_1, h_2, we have, for any $f \in FC_4$,

$$[\Lambda,H]f = -\int \nabla \xi(x) \cdot \frac{\delta f(x)}{\delta \xi(x)} dx . \tag{3.23}$$

The one parameter group of space translations $\xi \to \xi_a$ where $\xi_a(x) = \xi(x-a)$ induces transformations in $S'(R)$ which leave μ invariant. Thus these transformations induce a one parameter unitary group in $L_2(d\mu)$. Let P the self-adjoint infinitesimal generator of this group, then we see that the right hand side of (3.23) is simply iPf. So we have proved that, for $f \in FC_4$,

$$[\Lambda,H]f = iPf . \tag{3.24}$$

P being the infinitesimal generator of a translation group which leaves invariant the dense domain FC_4 of $L_2(d\mu)$, we have that P is essentially self-adjoint on FC_4. Since obviously

$$e^{iaP} H(h) e^{-iaP} = H(h_{-a}) , \tag{3.25}$$

where $h_{-a}(x) = h(x+a)$, we have that

$$e^{iaP}\, H\, e^{-iaP} = H \tag{3.26}$$

while

$$e^{iaP}\, \Lambda\, e^{-iaP} = \Lambda + aH\,. \tag{3.26}$$

From this we then have the following lemma

Lemma 3.2

On FC_4 we have the following commutation relations

$$[\Lambda,H] = iP\,,\ [H,P] = 0\,,\ [\Lambda,P] = iH\,. \quad \square \tag{3.28}$$

Now since P maps FC_4 into FC_3 we obviously have that $PFC_4 \subset FC_2 \subset D(\Lambda)$. On the other hand since $[\Lambda,P] = iH$ on FC_4 we also get that Λ maps FC_4 into the domain of P. By this and (3.28) we get that

$$[\Lambda,[\Lambda,H]] = -H \tag{3.29}$$

on FC_4. From (3.28) and (3.29) it follows that $e^{i\lambda\Lambda} H e^{-i\lambda\Lambda}$ is strongly analytic in λ on FC_4 for all values of Λ and

$$e^{i\lambda\Lambda} H e^{-i\lambda\Lambda} = \cosh\lambda \cdot H - \sinh\lambda\, P\,. \tag{3.30}$$

Moreover we get in the same way

$$e^{i\lambda\Lambda} P e^{-i\lambda\Lambda} = \cosh\lambda\, P - \sinh\lambda\, H\,. \tag{3.31}$$

We may now write (3.30) in the following way

$$H - \mathrm{tgh}\,\lambda\, P = \frac{1}{\cosh\lambda}\, e^{i\lambda\Lambda} H e^{-i\lambda\Lambda} \tag{3.32}$$

which shows that for any $|a| < 1$ we have $H - aP \geq 0$. This implies that $(f,(H+1)f) - \mathrm{tgh}\,\lambda\,(f,Pf)$ and $(f,(H+1)f)$ are equivalent norms on FC_4. From the fact that μ is a strictly positive measure, Theorems 4.1, 4.2 of Ref. [2], it follows immediately by standard finite dimensional mollifier techniques that one can approximate any $g \in FC_2$ by elements in FC_4 in the sense of the Dirichlet norm $(f,(H+1)f)$. Hence H, which is the Friedrichs extension of H restricted to FC_2, is the same as the Friedrichs extension of H restricted to FC_4. The closure of FC_4 in the Dirichlet norm $(f,(H+1)f)$ is therefore the domain of $H^{\frac{1}{2}}$. By what we have above the closure of FC_4 in the norm $(e^{-i\lambda\Lambda}f,(H+1)e^{-i\lambda\Lambda}f)$ is the same as for the Dirichlet norm. This gives us that $e^{-i\lambda\Lambda}$ leaves $D(H^{\frac{1}{2}})$ invariant, and (3.30) and (3.31) hold in the sense of forms on $D(H^{\frac{1}{2}})$.

Since H is translation invariant, (3.26), the spectral measures of H and P commute, and we can consider the joint spectral resolution of P and H:

$$L_2(d\mu) = \int_{R^2} \mathcal{H}_p \, d\nu(p) . \tag{3.33}$$

Since (3.30) and (3.31) hold in the sense of forms on $D(H^{\frac{1}{2}})$ we see that $d\nu(p)$ must be quasi invariant under the action of the homogeneous Lorentz group in R^2, and, for any Lorentz transformation $L(\lambda)$, $\mathcal{H}_p$ and $\mathcal{H}_{L(\lambda)p}$ must be identical for ν-almost all p. So that $\mathcal{H}_p$ depends only on the orbits of the homogeneous Lorentz group. Moreover since H is nonnegative with zero as an isolated eigenvalue, we see that ν has support in the forward cone. Since ν is quasi-invariant under the Lorentz group, we may just as well take it to be invariant since there is always an invariant equivalent measure. We may of course also choose the identification of $L(\lambda)(p_o,o)$ with $\mathcal{H}_{(p_o,o)}$ as the one given by the unitary operator $e^{i\lambda\Lambda}$ if $p_o \neq 0$, since there is a one-to-one correspondence between the elements of the orbits different from $(0,0)$ and λ. Let us now observe that if $Hf = 0$ then $f \in D(H(h))$ and $H(h)f = 0$ for any $h \in \mathcal{O}_M(R)$. This follows from the fact noted in section 2 that the form $(f,H(h)f)$ for fixed f is an absolutely continuous positive measure in h, from which it follows that $(f,Hf) = (f,H(1)f)$ is the L_1-norm of the density, If $Hf = 0$ the L_1-norm is zero and the density is zero almost everywhere. Hence $(f,H(h)f) = 0$ which implies for $h \geq 0$ that $f \in D(H(h)^{\frac{1}{2}})$ and $H(h)^{\frac{1}{2}}f = 0$, hence that $f \in D(H(h))$ and $H(h)f = 0$. Now if h is not positive we write it as a difference of two positive functions.

From this we get that $e^{i\lambda\Lambda}$ is the identity on $\mathcal{H}_{(o,o)}$. Hence we have that $e^{i\lambda\Lambda}$ is induced by the action of the homogeneous Lorentz group $L(\lambda)$ in the spectral plane R^2 of (3.33). Since ν is Lorentz invariant this proves that we have a strongly continuous unitary representation of the inhomogeneous Lorentz group on $L_2(d\mu)$. We formulate this in the following theorem, in which we state the results only for the case of a unique vacuum.

Theorem 3.2

Let μ be the vacuum for the exponential interactions or the weak polynomial interactions in two space time dimensions restricted to time zero fields. Then μ is analytic in the sense that $\mu \in \mathcal{F}_n(S')$ for all n and, in $L_2(d\mu)$, 1 is an analytic vector for the canonical momentum $\pi(\varphi)$ for any $\varphi \in S$.
Moreover μ is strictly positive in the sense that, for any $f \in FC$ with $f(\xi) = f^*(\langle\varphi_1,\xi\rangle,\ldots,\langle\varphi_n,\xi\rangle)$,

$$\int f\,d\mu = \int_{R^n} f^*(x_1,\ldots,x_n)\rho(x_1,\ldots,x_n)dx_1\ldots dx_n$$

where ρ is bounded below by a positive constant on any compact R^n. Moreover for any $h \in \mathcal{O}_M(R)$ the space of multipliers on $S(R)$, we have that the Dirichlet form

$$D_h(f,f) = \int h(x)\left|\frac{\delta f}{\delta\xi(x)}\right|^2 dx$$

is closable and the diffusion operator $H(h)$ is defined as the self-adjoint operator given by D_h restricted to FC_2. For $h \geq 0$ we have that $e^{-tH(h)}$ is a Markov semigroup with invariant measure μ. For the corresponding homogeneous Markov process $\xi(t)$ we have that $\xi(t)$ satisfies the stochastic differential equation

$$d\xi(x,t) = \beta(\xi(t))(x)dt + dw(x,t) ,$$

where $W(x,t)$ is the standard Wiener process on $S'(R)$ and $\beta(x) = 2i\pi(x)\cdot 1$. Moreover there is a unitary representation of the inhomogeneous Lorentz group on $L_2(d\mu)$ which leaves 1 invariant, such that its Lie algebra is spanned by $H = H(1)$, $\Lambda = H(x)$ and P, the infinitesimal generator of the space translations. □

For comments to these results we refer back to section 1.

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Footnotes

1) Some partial results are also obtained for the $P(\varphi)_2$ models with Dirichlet boundary conditions and isolated (but not necessarily unique) vacua. These models were also considered in [2] , where references to the Euclidean theory for such models are also given.

2) Other proofs are in [12] and [6] ,Th VIII.33.

3) This concept and terminology has its roots in the work on stochastic mechanics and stochastic field theory, see [14], [15],[16] and references therein.

References

[1] S.Albeverio - R.Høegh-Krohn, Quasi-invariant measures,symmetric diffusion processes and quantum fields, Oslo University Preprint, May 1975; to appear in the Proceedings of the International Colloquium on Mathematical Methods of Quantum Field Theory (Marseille June 1975),CNRS.

[2] S.Albeverio - R.Høegh-Krohn,Dirichlet forms and diffusion processes on rigged Hilbert spaces, Oslo University Preprint, November 1975.

[3] J.Glimm-A.Jaffe -T.Spencer, The particle structure of the weakly coupled P($)_2$ model and other applications of high temperature expansions, pp. 132-142, in Constructive Quantum Field Theory, Eds.G.Velo-A.Wightman,Berlin: Springer 1973.

[4] R.Høegh-Krohn, A general class of quantum fields without cutoffs in two space-time dimensions, Comm. Math. Phys. 21, 244-255 (1971).

[5] S.Albeverio - R.Høegh-Krohn, The Wightman axioms and the mass gap for strong interactions of exponential type in two space-time dimensions, J.Functional Analysis 16 ,39-82 (1974).

[6] B.Simon, The $P(\varphi)_2$ Euclidean (Quantum) Field Theory, Princeton: Princeton University Press 1974.

[7] E.Nelson, Construction of quantum fields from Markoff fields, J.Functional Analysis 12 ,97-112 (1973).

[8] E.Nelsom, Probability theory and Euclidean field theory, pp 94-124 in book under Ref. [3].

[9] B.Simon, Positivity of the Hamiltonian semigroup and the con struction of Euclidean region fields, Helv.Phys.Acta 46 , 686-696 (1973).

[10] C.Newman, The construction of stationary two-dimensional Markoff fields with an application to quantum field theory, J.Functional Analysisis 14 , 44- 61 (1973).

[11] J.Fröhlich, Schwinger functions and their generating functionals I, II , Helv.Phys.Acta 47 ,265-306 (1974); Adv.Math.

[12] J.Fröhlich, Verification of axioms for Euclidean and relativistic fields and Haag's Theorem in a class of $P(\varphi)_2$ -models, Ann.Inst.H.Poincare A, 21 , 271-317 (1974).

[13] J.Glimm-A.Jaffe, A $\lambda(\phi^4)_2$ quantum field theory without cutoffs, II. The field operators and the approximate vacuum, Ann.Math. 91 , 362- 401 (1970).

[14] E.Nelson, Dynamical Theories of Brownian Motion , Princeton: Princeton University Press 1967.

[15] F.Guerra, On stochastic field theory, in Proc.IInd Aix-en-Provence International Conference on Elementary Particles 1973, J.de Phys.Suppl.T.34, Fasc. 11-12,Colloque, C-1-95-98.

[16] S.Albeverio-R.Høegh-Krohn, A remark on the connection between stochastic mechanics and the heat equation, J. Math.Phys. 15 , 1745-1747 (1974).

[17] H.Araki, Hamiltonian formalism and the canonical commutation relations in quantum field theory, J.Math.Phys. 1 ,492-504 (1960)

[18] M.Fukushima, On the generation of Markov processes by symmetric forms, pp.46-79 in Proceedings of the Second Japan-USSR Symposium on Probability Theory, Berlin: Springer 1973.

[19] F.Guerra-L.Rosen-B.Simon, Correlation inequalities and the mass gap in $P(\phi)_2$,III.Mass gap for a class of strongly coupled theories with non zero external field, Comm.Math. Phys.41 , 19-32 (1975).

[20] R.L.Dobrushin-R.A.Minlos, Construction of a one-dimensional quantum field via a continuous Markov field, Funct.Anal. and Appl. 7, 324-325 (1973) (transl.); and Markov fields theory and Application to Quantum Boson Fields, to appear in the Proceedings of the XII-th Winter School in Karpacz.

[21] J.Glimm-A.Jaffe- T.Spencer, Phase transitions for φ_2^4 quantum fields, to appear in Comm.Math.Phys.

[22] J.P.Eckmann- H.Epstein- J.Fröhlich, Asymptotic perturbation expansion for the S-matrix and the definition of time ordered functions in relativistic quantum fields models, Genève - ETH Preprint, 1975.