

CANONICAL QUANTUM FIELDS IN
TWO SPACE-TIME DIMENSIONS*

by

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ABSTRACT

We consider the restriction μ of the physical vacuum measure of the Wightman models with weak polynomial or exponential interactions in two space-time dimensions to the functions of the time zero fields. μ is $S(\mathbb{R})$ -quasi invariant, strictly positive and it determines a strongly continuous unitary representation of the Weyl commutation relations in $L_2(d\mu)$, with the function 1 cyclic for the fields and analytic for the conjugate momenta. μ defines two Dirichlet forms $(f, Hf) = \int \left| \frac{\delta f}{\delta \xi(x)} \right|^2 dx d\mu$ and $(f, Af) = \int x \left| \frac{\delta f}{\delta \xi(x)} \right|^2 dx d\mu$, which coincide on a dense domain of $L_2(d\mu)$ with the restrictions of the physical Hamiltonian respectively the physical Lorentz boost. The self-adjoint operator given by H generates a homogeneous Markov process on $S'(\mathbb{R})$ which solves a stochastic diffusion equation with osmotic velocity determined by μ . Self-adjoint operators associated with the above Dirichlet forms satisfy, together with the infinitesimal generator of space translations, the commutation relations of the generators of the Lie algebra of the inhomogeneous Lorentz group.

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1. Introduction

In this paper we consider, as in section 4 of [1], [2], the weakly coupled $P(\varphi)_2$ models ([3]) and the exponential interaction models ([4],[5]) of Bose quantum fields in two space-time dimensions.¹⁾ Such models satisfy in particular all Wightman axioms and their physical Hamiltonian has a mass gap at the lower end of its spectrum. It is an open question whether these models are canonical in the sense that the physical vacuum is a cyclic vector for the time zero fields or, equivalently ([6]), the contraction semigroup generated by the physical Hamiltonian is a Markov semigroup. For some discussions of these questions see [6] - [12]. In this paper we study the restriction of the above models to the subspace generated by the time zero fields, in a spirit related to Araki's Hamiltonian formalism [13]. Previous results were obtained by us in [1], [2], to which we refer for further references related to the subject.

Let μ^* be the measure correspondent to the physical vacuum for any of the models mentioned above. It is known that the time zero fields exist as multiplication operators in $L_p(d\mu^*)$, $1 \leq p < \infty$ with the physical vacuum as an analytic vector in $L_2(d\mu^*)$ ([14])²⁾ resp. ([5]). Let μ be the probability measure on the real space $S'(R)$ defined as the restriction of μ^* to the functions of the time zero fields.

In [1], [2] we established results on μ which we shall now recall in part. μ was proven to be a quasi invariant probability measure with respect to the nuclear rigging

$$S(R) \subset L_2(d\mu) \subset S'(R) \quad (\text{real spaces}),$$

so that μ defines a unitary strongly continuous representation

$\varphi \rightarrow U(\varphi), V(\varphi)$ of the Weyl commutation relations on $L_2(d\mu)$, with

$$(U(\varphi)f)(\xi) = e^{i\langle \xi, \varphi \rangle} f(\xi), \quad (V(\varphi)f)(\xi) = \sqrt{\frac{d\mu(\xi+\varphi)}{d\mu(\xi)}} f(\xi+\varphi) \quad (1.1)$$

and $f \in L_2(d\mu)$. Let $\pi(\varphi)$ be the infinitesimal generator of the unitary group $V(t\varphi)$, i.e. $\pi(\varphi)$ is the canonical momentum, conjugate to the canonical field $\langle \xi, \varphi \rangle$. We proved in [2] that the function 1 in $L_2(d\mu)$ (i.e. the time zero vacuum) is an analytic vector for $\pi(\varphi)$. Let FC_2 be the dense domain in $L_2(d\mu)$ consisting of functions on $S'(R)$ which are finitely based and C^n on their base, so that $f(\xi) = f(P_f \xi)$ for some projection P_f with finite dimensional range in $S(R)$ and such that the restriction f^* of f to the range of P_f is n -times continuously differentiable. It was proven in [2] that μ is strictly positive, in the sense that

$$\int_{S'(R)} f d\mu = \int_{R^n} f^*(x_1, \dots, x_n) \rho(x_1, \dots, x_n) dx_1 \dots dx_n, \quad (1.2)$$

where the density ρ is bounded away from zero, uniformly on compacts.

In [1] and in section 4 of [2] we considered moreover the Dirichlet form

$$\int \nabla \bar{f} \nabla f d\mu \equiv \int \left| \frac{\delta f}{\delta \xi(x)} \right|^2 dx d\mu(\xi), \quad (1.3)$$

obtained by closure from its restriction to FC_2 , where the gradient ∇ is naturally defined of FC_2 . The unique self-adjoint operator H associated with the Dirichlet form, called diffusion operator, is the Friedrichs extension of its restriction to FC_2 and, on FC_2 ,

$$H = -\Delta - \beta \cdot \nabla, \quad (1.4)$$

with the natural definition of the Laplacian Δ , and with

$\beta \cdot \nabla f = \sum_{j=1}^n (\beta \cdot \varphi_j)(\varphi_j \cdot \nabla) f$, where $f \in FC^2$, $\varphi_1, \dots, \varphi_n$ is an orthonormal base in the range of P_f and $\beta \cdot \varphi_j = 2i\pi(\varphi_j) \cdot 1$. β was called in [1], [2] the osmotic velocity corresponding to the measure μ .³⁾

H was proven in [2] to be positivity preserving⁴⁾ so that e^{-tH} , $t \geq 0$ is a Markov semigroup, and we studied the correspondent homogeneous symmetric diffusion process $\xi(t, x)$ on $S'(R)$. The relation between H and the physical Hamiltonian H_{ph} of the Wightman models of Ref. [3], [5] is, as proven in [1], [2],

$$(f, J_0 H_{ph} J_0 g) = (f, Hg), \quad (1.5)$$

for any f, g in FC_2 , where J_0 is the natural embedding of $L_2(d\mu)$ in $L_2(d\mu^*)$.

We come now to the main results and the distribution of the topics in this paper.

In section 2 we consider general Dirichlet forms associated with the nuclear rigging $S(R^n) \subset L_2(R^n) \subset S'(R^n)$, of the type

$$D_h(f, f) = \int h(x) \left| \frac{\delta f}{\delta \xi(x)} \right|^2 dx d\mu,$$

with $f \in FC_2$ and notation generalized from (1.3). h is any function in the space $\mathcal{O}_M(R^n)$ of multipliers on $S'(R^n)$. The quasi invariant measures μ are assumed to be such that 1 is in the domain of $\pi(\varphi)$, for all $\varphi \in S(R^n)$, where $\pi(\varphi)$ is the infinitesimal generator of the correspondent unitary group $V(t\varphi)$. The form D_h is the form of a symmetric operator on FC_2 , hence closable. The operator being real it has self-adjoint extensions, and for $h \geq 0$ we may take $H(h)$ to be the Friedrichs extension. Under some additional conditions we prove then that, on FC_4 , the commutator $[H(h_1), H(h_2)]$ is a vector field over $S'(R^n)$, with

components given by the kernel of the bounded linear map $[H(h_1), H(h_2)] \langle \xi, \varphi \rangle$ from $S(\mathbb{R}^n)$ to $L_2(d\mu)$.

In section 3 we consider the case where μ is the time zero vacuum measure of the quantum fields with exponential or polynomial interactions in two space-time dimensions. We first prove that, with $H(1) \equiv H$ and $H(\lambda) \equiv \Lambda$, where λ is the function $\lambda(x) = x$, we have for all $\varphi \in S(\mathbb{R})$

$$i[\pi(\varphi), H] = \langle \xi, (-\frac{d^2}{dx^2} + m^2) \varphi \rangle + \int_{-\infty}^{\infty} :v'(\xi(x)):\varphi(x)dx$$

and

$$i[\pi(\varphi), \Lambda] = \langle \xi, (-\frac{d}{dx} x \frac{d}{dx} + m^2 x) \varphi \rangle + \int_{-\infty}^{\infty} x :v'(\xi(x)):\varphi(x)dx$$

as bilinear forms on $FC_2 \times FC_2$, where v' is the derivative of the function v which gives the interaction. Λ coincides as a bilinear form on $FC_2 \times FC_2$ with the physical Lorentz boost. Applying the results of section 2 we get that any self-adjoint extensions of H and Λ satisfy, together with the infinitesimal generator of the space translations, the commutation relations of the generators of the Lie algebra of the inhomogeneous Lorentz group, in fact we have, on FC_2 , $e^{iaP} H e^{-iaP} = H$ and $e^{iaP} \Lambda e^{-iaP} = \Lambda + aH$, for all $a \in \mathbb{R}$. These results can be seen as a partial realization, for the models considered, of the canonical program discussed by Araki [13]. We expect that one has a unitary representation of the inhomogeneous Lorentz group itself in $L_2(d\mu)$.

2. Diffusion operators on the space of tempered distributions.

Consider the nuclear rigging

$$S(\mathbb{R}^n) \subset L_2(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \quad (2.1)$$

where $S(\mathbb{R}^n)$ is the Schwartz space and $S'(\mathbb{R}^n)$ its dual i.e. the space of tempered distributions. Let μ be an S -quasi invariant probability measure on S' , i.e. $d\mu(\xi)$ and $d\mu(\xi+\varphi)$ are equivalent measures for any φ in S . Such a measure μ gives rise to a unitary representation (U, V) on $L_2(d\mu)$ of the Weyl commutation relations on S . Namely, for $f \in L_2(d\mu)$, $(U(\varphi)f)(\xi) = e^{i\langle\varphi, \xi\rangle} f(\xi)$ and $(V(\varphi)f)(\xi) = \alpha(\xi, \varphi)f(\xi)$, where $\alpha(\xi, \varphi) = \left(\frac{d\mu(\xi+\varphi)}{d\mu(\xi)}\right)^{\frac{1}{2}}$. Let $\pi(\varphi)$ be the infinitesimal generator for the unitary group $V(t\varphi)$. We say that $\mu \in \mathcal{F}_n(S')$ if the function 1 is in the domain of $\pi(\varphi_1) \dots \pi(\varphi_n)$ for any n elements in S . For further details see [1], [2].

In what follows we shall always assume that $\mu \in \mathcal{F}_1(S')$. Let FC_k be the subspace of $L_2(d\mu)$ consisting of bounded finitely based and k -times differentiable functions i.e. $f \in FC_k$ iff there is an $f^* \in C_k(\mathbb{R}^1)$ and $\varphi_1, \dots, \varphi_k$ in S such that $f(\xi) = f^*(\langle\varphi_1, \xi\rangle, \dots, \langle\varphi_k, \xi\rangle)$. For any $f \in FC_1$ we define

$$\frac{\delta f}{\delta \xi(x)}(\xi) = \sum_{j=1}^1 f_j^*(\langle\varphi_1, \xi\rangle, \dots, \langle\varphi_1, \xi\rangle) \varphi_j(x) \quad (2.2)$$

where f_j^* are the partial derivatives of f^* . We see that $\frac{\delta f}{\delta \xi}$ is a continuous map from S' into S with finite dimensional range. Since $1 \in D(\pi(\varphi))$ for any $\varphi \in S$, we get that $\varphi \rightarrow \pi(\varphi)1$ is a linear mapping from S into $L_2(d\mu)$. Moreover, since S is a complete metric space, we have that $\varphi \rightarrow \pi(\varphi) \cdot 1$ is bounded, and then, by using that S is nuclear, we get that this mapping has a kernel which we denote $\frac{1}{21} \beta(x)$. β is a measurable mapping of

S' into S' called the osmotic velocity, and we have

$$\pi(\varphi)1 = \frac{1}{21} \int \beta(x)\varphi(x)dx. \quad (2.3)$$

For a proof of these facts see prop. 2.5 [1].

Let $h \in S'(R^n)$, then we define, for $f \in FC_1$, the Dirichlet form

$$D_h(f,f) = \frac{1}{2} \iint h(x) \left| \frac{\delta f}{\delta \xi(x)} \right|^2 dx d\mu(\xi). \quad (2.4)$$

This is well defined since $\frac{\delta f}{\delta \xi}$ is a continuous mapping from S' into a finite dimensional subspace of S , and by (2.2)

$\int h(x) \left| \frac{\delta f}{\delta \xi(x)} \right|^2 dx$ is uniformly bounded and continuous in ξ . Let $\mathcal{O}_M(R^n)$ be the space of multipliers for $S'(R^n)$, i.e. if $h \in \mathcal{O}_M(R^n)$ then $T(x) \rightarrow h(x)T(x)$ is a bounded linear transformation on S' . If $h \in \mathcal{O}_M(R^n)$ then the Dirichlet form (2.4) restricted to FC_2 is closable. It is namely given by a symmetric operator in $L_2(d\mu)$

$$D_h(f,f) = (f, H(h)f), \quad (2.5)$$

where

$$H(h)f = -\frac{1}{2} \int h(x) \left(\frac{\delta^2 f}{\delta \xi(x)^2} + \beta(x) \frac{\delta f}{\delta \xi(x)} \right) dx. \quad (2.6)$$

For details see theorem 2.6 [1] and the proof of it.

Let now $h \in \mathcal{O}_M(R^n)$ be non negative. Then $D_h(f,f)$ is non negative and closable, thus its closure defines a unique self-adjoint operator on $L_2(d\mu)$ which we shall also denote by $H(h)$. Since $h \geq 0 \Rightarrow H(h) \geq 0$ by (2.4), we have that $H(h)$ is monotone in h and since monotone convergence of semibounded forms implies resolvent convergence we have that, if $0 \leq h_n \nearrow h$, then $(1+H(h_n))^{-1}$ converges strongly to $(1+H(h))^{-1}$. As an integral in h , $D_h(f,f)$ is absolutely continuous with respect to the Lebesgue measure in R^n . Hence, by monotone convergence, $H(h)$ may be extended to all $h \geq 0$ in $L_\infty(R^n)$.

If $h \in \mathcal{O}_M(R^n)$ and $h \geq 0$ then it is easily verified that

$H(h)$ is the limit in the strong resolvent sense of operators $H_m(h)$ such that $H_m(h)$ are given as direct integrals of forms which are Markov symmetric forms in the sense of Fukushima [18], as in theorem 2.7 of Ref. [1]. In this way we get the following theorem

Theorem 2.1

Let $h \in \mathcal{G}_M(\mathbb{R}^n)$ i.e. the space of multipliers on $S'(\mathbb{R}^n)$ such that $h \geq 0$. Then $e^{-tH(h)}$ is a conservative Markov semi-group i.e. for $f \in L_2(d\mu)$ such that $f \geq 0$ we have that $e^{-tH(h)}f \geq 0$ and $e^{-tH(h)}1 = 1$. Thus the corresponding Markov process $\xi_h(t)$ on $S'(\mathbb{R}^n)$ is a homogeneous Markov process with invariant measure μ . This process $\xi_h(x,t)$ on $S'(\mathbb{R}^n)$ satisfies the following stochastic differential equation

$$d\xi_h(x,t) = h(x)\beta(\xi_h(t))(x)dt + h(x)dW(x,t)$$

where $W(x,t)$ is the standard Wiener process on $S'(\mathbb{R}^n)$ given by the rigging $S(\mathbb{R}^n) \subset L_2(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$, and $\beta(\xi)(x) = \frac{1}{21}\pi(x) \cdot 1$ in the sense of (2.3). \square

We now remark that if h is not non negative the Dirichlet form $D_h(f,f)$ is no longer semibounded, hence there is in general no self-adjoint operator canonically associated to it. Since however $H(h)$, as defined on FC_2 by (2.6), is real as an operator in $L_2(d\mu)$, we know that, also in this case, $H(h)$ has at least one self-adjoint extension. In the following we shall denote, for h not non negative, by $H(h)$ any self-adjoint extension of the operator defined by (2.6) on the dense domain FC_2 in $L_2(d\mu)$. It is understood that, for $h \geq 0$, $H(h)$ still denotes the unique self-adjoint non negative operator of Theorem 2.1.

2.3.b

Let us now assume for some h and φ in $S(\mathbb{R}^n)$, that $\pi(\varphi) \cdot 1$ is in $D(H(h))$. Then $H(h)\pi(\varphi) \cdot 1$ is a bilinear map from $S \times S$ into $L_2(d\mu)$.

By the abstract kernel theorem we get in the same way as in prop.2.5 of [1] that there is a measurable mapping $U: \xi \rightarrow T(\xi)(x,y)$ from $S'(\mathbb{R}^n)$ into $S'(\mathbb{R}^n \times \mathbb{R}^n)$ such that, for h and φ in $S(\mathbb{R}^n)$, $\langle \varphi \times h, T(\xi) \rangle$ is in $L_2(d\mu)$ as a function of ξ and

$$\langle \varphi \times h, T(\xi) \rangle = \frac{1}{i} (H(h)\pi(\varphi) \cdot 1)(\xi). \quad (2.7)$$

Since obviously $H(h)\pi(\varphi) \cdot 1 = [H(h), \pi(\varphi)] \cdot 1$ and $[H(h), \pi(\varphi)]$ is a multiplication operator we also have

$$\langle \varphi \times h, T(\xi) \rangle = i[\pi(\varphi), H(h)]. \quad (2.8)$$

Remark:

In the case where the osmotic velocity $\beta(\xi)(x)$ is sufficiently smooth, we have that for $h \in S$

$$H(h) = \int h(y) \left(\frac{1}{2} \pi(y)^2 + V(y) \right) dx,$$

where $V(y) = \frac{1}{2} \frac{\delta \beta(y)}{\delta \xi(y)} + \frac{1}{4} \beta(y)^2$ (see section 2 Ref. [1]). In this case we see from (2.8) that

$$T(\xi)(x, y) = \frac{\delta V(y)}{\delta \xi(x)}. \quad (2.9)$$

Lemma 2.1

Let $\mu \in \mathcal{F}_2(S')$, then if for some $h_1 \in \mathcal{O}_M(\mathbb{R}^n)$ we have that $\pi(\varphi) \cdot 1 \in D(H(h_1))$ for any $\varphi \in S(\mathbb{R}^n)$ then, for any h_2 in $\mathcal{O}_M(\mathbb{R}^n)$, $H(h_2)$ maps FC_4 into the domain of $H(h_1)$.

Proof:

Let $f \in FC_4$ then

$$H(h_2)f = -\frac{1}{2} \int h_2(x) \frac{\delta^2 f}{\delta \xi(x)^2} dx - \frac{1}{2} \int h_2(x) \beta(x) \frac{\delta f}{\delta \xi(x)} dx. \quad (2.10)$$

The first term is obviously in FC_2 and since $\frac{\delta f}{\delta \xi(x)}$ as a function of x is in a fixed finite dimensional subspace of S , we may write the second term as

$$g = -i \int h_2(x) \frac{\delta f}{\delta \xi(x)} \pi(x) dx \cdot 1, \quad (2.11)$$

because $h_2(x) \frac{\delta f}{\delta \xi(x)}$ is again in S as a function of x . Since by assumption $\pi(\varphi) \cdot 1 \in D(H(h_1))$ for any $\varphi \in S$, we find that

$$\begin{aligned} H(h_1)g &= -\frac{i}{2} \iint h_1(y) h_2(x) \frac{\delta^3 f}{\delta \xi(y)^2 \delta \xi(x)} \cdot \pi(x) dx dy \cdot 1 \\ &\quad + \frac{i}{2} \iint h_1(y) h_2(x) \frac{\delta^2 f}{\delta \xi(y) \delta \xi(x)} \cdot \beta(y) \pi(x) dx dy \cdot 1 \\ &\quad - \frac{i}{2} \int h_2(x) \frac{\delta f}{\delta \xi(x)} H(h_1) \pi(x) dx \cdot 1. \end{aligned} \quad (2.12)$$

Now the first term in (3.12) is well defined since $1 \in D(\pi(\varphi))$ for any $\varphi \in S$ and

$$h_2(x) \int h_1(y) \frac{\delta^3 f}{\delta^2 \xi(y) \delta \xi(x)} dy \in S.$$

The second term is equal to

$$- \iint h_1(y) h_2(x) \frac{\delta^2 f}{\delta \xi(y) \delta \xi(x)} \pi(y) \pi(x) dx dy \cdot 1$$

which is in $L_2(d\mu)$ by the assumption that $\mu \in \mathcal{F}_2(S')$ and the abstract kernel theorem.

The third term is in $L_2(d\mu)$ by the assumption that $\pi(\varphi) \cdot 1$ is in $D(H(h_1))$ and again the abstract kernel theorem. This proves the lemma. \square

Let us now assume that $\mu \in \mathcal{F}_2(S')$ and that for some h_1 and h_2 in $\mathcal{O}_M(\mathbb{R}^n)$ we have that $\pi(\varphi) \cdot 1 \in D(H(h_i))$ for $i=1,2$ and all $\varphi \in S(\mathbb{R}^n)$. Then by the previous lemma $[H(h_1), H(h_2)]$ is defined on FC_4 . By (2.7), (2.10) and (2.12) we have that for $f \in FC_4$

$$\begin{aligned} 4 \cdot H(h_1)H(h_2)f &= \iint h_1(y)h_2(x) \frac{\delta^4 f}{\delta \xi(x)^2 \delta \xi(y)^2} dx dy \\ &- \iint h_1(y)h_2(x) \frac{\delta^3 f}{\delta \xi(x)^2 \delta \xi(y)} \cdot \beta(y) dx dy \\ &- \iint h_1(y)h_2(x) \frac{\delta^3 f}{\delta \xi(x) \delta \xi(y)^2} \beta(x) dx dy \\ &+ \iint h_1(y)h_2(x) \frac{\delta^2 f}{\delta \xi(y) \delta \xi(x)} \beta(x)\beta(y) dx dy \\ &+ 2 \iint h_1(y)h_2(x) \frac{\delta f}{\delta \xi(x)} T(\xi)(x,y) dx dy. \end{aligned} \tag{2.13}$$

We remark that by the assumption that $\pi(\varphi) \cdot 1 \in D(H(h_1))$ we have, by (2.7), that $T(\xi)(x,y)$ is defined on $h_1(y) \times h_2(x) \frac{\delta f}{\delta \xi(x)}$ for almost all ξ and the result is in $L_2(d\mu)$. By antisymmetrization

with respect to h_1 and h_2 we get that the four first terms in (2.13) fall out and the result is

$$[H(h_1), H(h_2)]f = \frac{1}{2} \iint (h_1(y)h_2(x) - h_1(x)h_2(y)) T(\xi)(x, y) \frac{\delta f}{\delta \xi(x)} dx dy \quad (2.14)$$

For any $\mu \in \mathcal{F}_1(S')$ we say that $H = H(1)$ is the Dirichlet operator given by μ . We say that H is a harmonic oscillator on S' if μ is a non degenerate Gaussian measure i.e. its Fourier transform has the form $e^{-\frac{1}{2}(\varphi, B\varphi)}$, where B is a bounded positive operator on S with a bounded inverse on S . By Minlos theorem there is a unique μ corresponding to any B bounded and positive, and it is easily seen that if also B has a bounded inverse, then $\mu \in \mathcal{F}_1(S)$ with $\beta(\xi)(x) = -\int A(x, y)\xi(y)dy$, where $A(x, y)$ is the kernel of $A = B^{-1}$, which by assumption is a bounded map of S , hence, A being symmetric, it is also a bounded map of S' . So we see that harmonic oscillators have linear osmotic velocity fields. By a straight forward calculation we find that the mapping $T(\xi)$ in the case of a harmonic oscillator is given by

$$T(\xi)(x, y) = A(x, y) \int A(y, z)\xi(z)dz. \quad (2.15)$$

In this case we see from (2.15) that for $\varphi \in S(\mathbb{R}^n)$ and $h \in \mathcal{G}_M(\mathbb{R}^n)$ we have that $\varphi \times h$ is always in the domain of $T(\xi)$ and moreover

$$\langle \varphi \times h, T(\xi) \rangle = \langle hA\varphi, A\xi \rangle. \quad (2.16)$$

Since (2.16) is a continuous linear functional it is always in $L_2(d\mu)$ with respect to the Gaussian measure μ . We have therefore proved the following lemma

Lemma 2.2

If H is a harmonic oscillator on $S'(\mathbb{R}^n)$ then, for any $h \in \mathcal{G}_M(\mathbb{R}^n)$ and any $\varphi \in S(\mathbb{R}^n)$, $\pi(\varphi) \cdot 1 \in D(H(h))$. Moreover if

the corresponding osmotic velocity is

$$\beta(\xi)(x) = - \int A(x,y)\xi(y)dy$$

then the corresponding mapping $T(\xi)$ is given by

$$T(\xi)(x,y) = A(x,y) \int A(y,z)\xi(z)dz. \quad \square$$

Let us now return to the formula (2.14), and consider the expression

$$\iint h_1(y)h_2(x)T(\xi)(x,y)\frac{\delta f}{\delta \xi(x)} dx dy. \quad (2.17)$$

By the definition (2.7) this is equal to

$$-\frac{1}{2} \int \frac{\delta f}{\delta \xi(x)} \cdot h_2(x)H(h_1)\beta(x)dx. \quad (2.18)$$

Now by the definition (2.6) we have

$$-\frac{1}{2}h_2(x)\beta(x) = H(h_2)\xi(x), \quad (2.19)$$

where of course $H(h_2)\xi(x)$ is to be understood as a bounded linear map from S into $L_2(d\mu)$. That this map is bounded follows from the fact that $\mu \in \mathcal{F}_1^c(S')$.

In fact if we assume that we have $\pi(\varphi) \cdot 1 \in D(H(h_i))$ $i=1,2$ for any $\varphi \in S(\mathbb{R}^n)$, then (2.19) is a bounded mapping from $S(\mathbb{R}^n)$ into $D(H(h_1))$. Hence $H(h_1)H(h_2)\xi(x)$ is a bounded mapping from $S(\mathbb{R}^n)$ into $L_2(d\mu)$, so that (2.17) is equal to

$$\int \frac{\delta f}{\delta \xi(x)} : H(h_1)H(h_2)\xi(x) dx. \quad (2.20)$$

We have thus proven the following theorem

Theorem 2.2

If $\mu \in \mathcal{F}_2^c(S')$ and for some h_1 and h_2 in $\mathcal{O}(\mathbb{R}^n)$ we have that $\pi(\varphi) \cdot 1 \in D(H(h_i))$ $i=1,2$ for φ arbitrary in $S(\mathbb{R})$, then $H(h_i)$ maps FC_4 into $D(H(h_j))$. In particular, since

$\langle \varphi, \xi \rangle \in FC_4$, $H(h_i)$ maps $\langle \varphi, \xi \rangle$ into $D(H(h_j))$ so that $H(h_i)H(h_j)\langle \varphi, \xi \rangle$ is a bounded linear map from S into $L_2(d\mu)$, S being nuclear. Let $H(h_i)H(h_j)\xi(x)$ be the kernel of this map, then we have, for any $f \in FC_4$ that

$$[H(h_1), H(h_2)]f = \int \frac{\delta f}{\delta \xi(x)} [H(h_1), H(h_2)] \xi(x) dx.$$

We remark that this theorem shows that the commutant $[H(h_1), H(h_2)]$ is a first order derivation or a vector field over S' with components given by $[H(h_1), H(h_2)] \cdot \xi(x)$.

3. The diffusion operators of the local relativistic quantum fields in two space-time dimensions.

In this section we consider the cases where the measure μ is the restriction of the physical vacuum to the time zero fields for the models in which the infinite volume Schwinger functions exist and the corresponding energy operator has zero as an isolated, but not necessarily simple, eigenvalue. These models are the weak polynomial interactions [3], the strong polynomial interactions with Dirichlet boundary conditions [19] - [21] and the exponential interactions [5]. In all these cases we know that the restriction μ of the physical vacuum to the σ -algebra generated by the time zero fields is a measure on $S'(R)$. Thus we consider, as in [1], [2], the natural nuclear rigging

$$S(R) \subset L_2(R) \subset S'(R). \quad (3.1)$$

In Ref. [2] we proved that $\mu \in \mathcal{F}_n(S')$ for all n , in fact we proved that 1 is an analytic vector for $\pi(\varphi)$, for any $\varphi \in S$.

In all cases considered here the physical vacuum is given in terms of the Wightman functions $W_n(x_1, t_1, \dots, x_n, t_n)$ which are used to construct, by the Gelfand-Segal-Wightman construction, the physical Hilbert space \mathcal{H}_{ph} and the physical energy operator H_{ph} , which is the generator of the time translations in \mathcal{H}_{ph} . We have of course that $L_2(d\mu)$ is a closed subspace of \mathcal{H}_{ph} and one would naturally have liked to prove that \mathcal{H}_{ph} is identical with $L_2(d\mu)$ and H_{ph} is identical with the diffusion operator $H = H(1)$. This is however still an open question. What we have been able to prove is that for f and g in FC_2 we have that

$$(f, H_{ph} g) = (f, Hg) \quad (3.2)$$

For this result see Ref. [1],[2].

We consider in what follows polynomial or exponential interactions in two space time dimensions, i.e. infinite volume limits of volume cut-off interactions of the form

$$H_g = H_0 + \int g(x) : v(\xi(x)) : dx, \quad (3.3)$$

where H_g is the Hamiltonian for the free fields and $v(s)$ is the function giving the interaction density and $g \in S(R)$, $g \geq 0$.

Thus $v(s)$ is a polynomial bounded below, in the case of polynomial interactions, and

$$v(s) = \int \cosh(\alpha s) d\nu(\alpha), \quad (3.4)$$

where ν is a bounded positive measure with compact support in $(-\frac{4}{\sqrt{\pi}}, \frac{4}{\sqrt{\pi}})$, in the case of exponential interactions. $::$ denotes the Wick ordering. It is well known that if ν is of the form (3.4) ([5]) or ν is a polynomial bounded below with coefficients that are small compared with the free mass ([3]), then the infinite volume limit exists. We introduce the notation

$$:u(\xi):(\varphi) = \int_R :u(\xi(x)): \varphi(x) dx, \quad (3.5)$$

for any $\varphi \in S(R)$ and u any polynomial or exponential function of the form

$$u(s) = \int e^{\alpha s} d\rho(s), \quad (3.6)$$

where ρ is a bounded signed measure with compact support in $(-\sqrt{2\pi}, \sqrt{2\pi})$.

It is known from Theorem 2, section I.2 of [22] that if u is any polynomial, then $:u(\xi):(\varphi)$ is in $L_2(d\mu)$, for the case of polynomial interactions. We shall now see that for any exponential

function u of the form (3.6) we have $:u(\xi):(\varphi)$ in $L_2(d\mu)$, in the case of exponential interactions.

Let $\psi_n \in S(R)$ such that $\psi_n \geq 0$ and $\psi_n \rightarrow \delta$ in $S'(R)$, and consider the function

$$:u(\xi * \psi_n):(\varphi) = \iint :e^{\alpha \xi * \psi_n(x)} : d\rho(\alpha) \varphi(x) dx, \quad (3.8)$$

which is obviously μ -measurable. Now let $G_n = \frac{1}{2} \psi_n * (-\Delta + m^2)^{-\frac{1}{2}} \psi_n$, then

$$:e^{\alpha \xi * \psi_n(x)} : = e^{-\frac{1}{2} \alpha^2 G_n(0)} e^{\alpha \xi * \psi_n(x)} \quad (3.9)$$

so that

$$:e^{\alpha \xi * \psi_n(x)} : : e^{\beta \xi * \psi_n(y)} : = e^{-\frac{1}{2} (\alpha^2 + \beta^2) G_n(0)} e^{\xi * (\alpha \psi_n(x) + \beta \psi_n(y))}. \quad (3.10)$$

Let μ_0 be the free vacuum restricted to the time zero fields i.e. the μ in the case $v = 0$. We proved in theorem 6.1 of ref. [5] that, for the exponential interactions, the Schwinger functions are bounded by the free Schwinger functions. From this result we immediately get that, for $\varphi \in S(R)$ and $\varphi \geq 0$,

$$\int_{S'} e^{\langle \varphi, \xi \rangle} d\mu \leq \int_{S'} e^{\langle \varphi, \xi \rangle} d\mu_0. \quad (3.11)$$

From (3.10) and (3.11) we get

$$\int_{S'} | :u(\xi * \psi_n):(\varphi) |^2 d\mu \leq \iiint \int e^{\alpha \beta G_n(x-y)} |\varphi(x) \varphi(y)| d|\rho|(\alpha) d|\rho|(\beta) dx dy, \quad (3.12)$$

which by the assumption on ρ is uniformly bounded in n .

Moreover if $\| \cdot \|$ is the $L_2(d\mu)$ norm we have

$$\begin{aligned} \| :u(\xi * \psi_m):(\varphi) - :u(\xi * \psi_n):(\varphi) \| &\leq \iiint \int |\varphi(x)| |\varphi(y)| \\ & \left| e^{\alpha \beta G_{nn}(x-y)} + e^{\alpha \beta G_{mm}(x-y)} - 2e^{\alpha \beta G_{nm}(x-y)} \right| dx dy d|\rho|(\alpha) d|\rho|(\beta), \end{aligned} \quad (3.13)$$

where $G_{nm} = \frac{1}{2} \psi_n * (-\Delta + m^2)^{-\frac{1}{2}} \psi_m$. So by the assumption on ρ

$:u(\xi * \psi_n)(\varphi):$ is an $L_2(d\mu)$ convergent sequence. We denote the limit of course by $:u(\xi):(\varphi)$, which is then a function in $L_2(d\mu)$. We have thus the following lemma.

Lemma 3.1

In the case of polynomial interactions we have, for an arbitrary polynomial u , that $:u(\xi):(\varphi) \in L_2(d\mu)$ for any $\varphi \in S(R)$. In the case of exponential interactions we have $:u(\xi):(\varphi) \in L_2(d\mu)$ for any $\varphi \in S(R)$ and any exponential function

$$u(s) = \int e^{\alpha s} d\rho(\alpha), \quad (3.14)$$

where ρ is any bounded complex measure with compact support in $(-\sqrt{2\pi}, \sqrt{2\pi})$. \square

Let now $\pi(\varphi)$ be the infinitesimal generator of the unitary group of translations by $t\varphi$, $t \in R$, and consider

$$H_g^{t\varphi} = e^{it\pi(\varphi)} H_g e^{-it\pi(\varphi)}. \quad \text{Since}$$

$$H_g = H_0 + :v(\xi):(g), \quad (3.15)$$

we have that

$$H_g^{t\varphi} = H_0 + t\langle \xi, (-\Delta + m^2)\varphi \rangle + \frac{t^2}{2}\langle \varphi, (-\Delta + m^2)\varphi \rangle + :v(\xi + t\varphi):(g), \quad (3.16)$$

as operators on FC_2 in $L_2(d\mu_0)$. In fact however (3.16) is also true on FC_2 in $L_2(d\mu_g)$, where $d\mu_g$ is the measure given by the vacuum of H_g , because $\frac{d\mu_g}{d\mu_0}$ is in $L_p(d\mu_0)$ for all $p < \infty$ for the polynomial interactions, and for the exponential functions it follows as in lemma 3.1 from the fact that the Schwinger functions are decreasing functions of g . It follows from (3.16) that, for any $f \in FC_2$, we have that $H_g^{t\varphi} f$ is strongly n -times differentiable in $L_2(d\mu_g)$ for any n . Especially we get that the first

derivative is given by

$$H'_g = \langle \xi, (-\Delta + m^2)\varphi \rangle + t \langle \varphi, (-\Delta + m^2)\varphi \rangle + :v'(\xi + t\varphi):(\varphi g) \quad (3.17)$$

and the second derivative is given by

$$H''_g = \langle \varphi, (-\Delta + m^2)\varphi \rangle + :v''(\xi + t\varphi):(\varphi^2 g). \quad (3.18)$$

We thus remark that we also have the following strengthened version of lemma 3.1.

Lemma 3.2

For the case of polynomial interactions let $\varphi \in S(\mathbb{R})$ and u a fixed polynomial, then $:u(\xi):(\varphi) \in L_2(d\mu_g)$ and $\|:u(\xi):(\varphi)\|_g$ is uniformly bounded in g , for $g \in S(\mathbb{R})$ and $0 \leq g \leq 1$.

For the case of exponential interactions let $\varphi \in S(\mathbb{R})$ and u a fixed exponential function of the same form as in lemma 3.1, then $:u(\xi):(\varphi) \in L_2(d\mu_g)$ and $\|:u(\xi):(\varphi)\|_g$ is uniformly bounded in g for $g \in S(\mathbb{R})$ and $0 \leq g \leq 1$.

Proof: The proof follows easily from the proof of lemma 3.1. \square

Consider now, for any $\varphi \in S(\mathbb{R})$, the measure $d\mu_\varphi$ defined by $d\mu_\varphi(\xi) = d\mu(\xi + \varphi)$. By [1] and [2], $d\mu_\varphi$ is equivalent to $d\mu$. Let $h \in \mathcal{O}_M(\mathbb{R})$ and consider the operator $\tilde{H}^\varphi(h)$ defined on $FC_2 \subset L_2(d\mu_\varphi)$ by

$$(\tilde{H}^\varphi(h)f)(\xi) = -\frac{1}{2} \int h(x) \left(\frac{\delta^2 f}{\delta \xi(x)^2} + \beta_\varphi(x) \frac{\delta f}{\delta \xi(x)} \right) dx, \quad (3.19)$$

where β_φ is the osmotic velocity (in the sense of [1],[2]) for $d\mu_\varphi$. It follows easily that $\beta_\varphi(\psi) \in L_2(d\mu_\varphi)$, in fact $\|\beta_\varphi(\psi)\|_\varphi = \|\beta(\psi)\|_\varphi$, since $\beta_\varphi(\psi)(\xi)$ is just the translate by φ of $\beta(\psi)(\xi)$. Hence $\tilde{H}^\varphi(h)$ is a densely defined operator on $L_2(d\mu_\varphi)$.

Now, by the equivalence of $d\mu_\varphi$ and $d\mu$, we may consider $\tilde{H}^\varphi(h)$ as a densely defined operator in $L_2(d\mu)$. We shall now consider its representative $H^\varphi(h)$ in $L_2(d\mu)$. If $(,)_\varphi$ is the inner product in $L_2(d\mu_\varphi)$ we have by (3.19) that, for any $f \in FC_2$,

$$(f, \tilde{H}^\varphi(h)f)_\varphi = \frac{1}{2} \iint h(x) \left| \frac{\delta f}{\delta \xi(x)} \right|^2 dx d\mu_\varphi(\xi). \quad (3.20)$$

Since the equivalence between $L_2(d\mu)$ and $L_2(d\mu_\varphi)$ is given by multiplication by $(\frac{d\mu_\varphi}{d\mu})^{\frac{1}{2}}$, we have that

$$(f, \tilde{H}^\varphi(h)f)_\varphi = \left(\left(\frac{d\mu_\varphi}{d\mu} \right)^{\frac{1}{2}} f, H^\varphi(h) \left(\frac{d\mu_\varphi}{d\mu} \right)^{\frac{1}{2}} f \right), \quad (3.21)$$

where $(,)$ is the inner product in $L_2(d\mu)$. Let now $V(\varphi)$ be the unitary group of translations in $L_2(d\mu)$ i.e. $(V(\varphi)f)(\xi) = \left(\frac{d\mu_\varphi}{d\mu} \right)^{\frac{1}{2}} f(\xi + \varphi)$ and let $f_\varphi = f(\xi + \varphi)$. Then we get from (3.21) that

$$(f_\varphi, \tilde{H}^\varphi(h)f_\varphi)_\varphi = (V(\varphi)f, H^\varphi(h)V(\varphi)f), \quad (3.22)$$

on the other hand by (3.20) we have that the left hand side of (3.22) is simply $(f, H(h)f)$. Observing now that FC_2 is invariant under $f \rightarrow f_\varphi$ for any $\varphi \in S(R)$, we have proven that

$$H(h) = V(\varphi)H^\varphi(h)V(\varphi) \quad (3.23)$$

on FC_2 .

Let now $d\mu^*$ be as before the measure on $S'(R^2)$ given by the interacting Euclidean fields corresponding to the interaction given by the function v . Then $d\mu$ is the restriction of $d\mu^*$ to the Boolean σ -algebra generated by the linear functions of the form $\int \xi(x,0)\varphi(x)dx$, where $\xi(x,t) \in S'(R^2)$. For any $\varphi \in S(R)$, let $d\mu_\varphi^*$ be the image of $d\mu^*$ under the mapping $\xi(x,t) \rightarrow \xi(x,t) + \varphi(x)$ in $S'(R^2)$.

It follows then that $d\mu_\varphi$ is the restriction of $d\mu_\varphi^*$ to the time zero fields i.e. the σ -algebra generated by the linear functions of the form $\int \xi(x,0)\varphi(x)dx$. The measure $d\mu_\varphi^*$ is obviously invariant under time translations and therefore the time translations generate a strongly continuous one parameter group in $L_2(d\mu_\varphi^*)$, whose infinitesimal generator will be denoted by \tilde{H}_{ph}^φ . By introducing now a space cut-off in the interaction for the Euclidean measure $d\mu_\varphi^*$ we get, in the same way as in the proof of theorem 4.2 of Ref. [1], that, for f in FC_2 of the time zero fields,

$$(f, \tilde{H}_{ph}^\varphi f)_\varphi = \iint \left| \frac{\delta f}{\delta \xi(x)} \right|^2 dx d\mu_\varphi, \quad (3.24)$$

which by (3.20) is equal to $(f, \tilde{H}^\varphi f)_\varphi$, where $\tilde{H}^\varphi = \tilde{H}^\varphi(1)$ and the inner product on the left hand side of (3.24) is the inner product in $L_2(d\mu_\varphi^*)$. By introducing a time cut-off in the interaction for $d\mu_\varphi^*$ we see that

$$d\mu_\varphi^* = \lim_{T \rightarrow \infty} \left[\int e^{-U_T} d\mu^* \right]^{-1} e^{-U_T} d\mu^*, \quad (3.25)$$

where

$$U_T(\xi) = \int_{-T}^T \int_{-\infty}^{\infty} [\xi(x,t)(-\Delta+m^2)\varphi(x) + \frac{1}{2}(\varphi, (-\Delta+m^2)\varphi) + :v(\xi(x,t)+\varphi(x)) - v(\xi(x,t)):] dx dt \quad (3.26)$$

and the limit is in the sense of weak limits of measures, and

$\Delta = \frac{d^2}{dx^2}$. Hence \tilde{H}_{ph}^φ is by the Trotter-Kato formula a perturbation of H_{ph}^φ by the L_2 -function of the time zero field given by

$$U^\varphi(\xi) = \langle \xi, (-\Delta+m^2)\varphi \rangle + \langle \varphi, (-\Delta+m^2)\varphi \rangle + \int_{-\infty}^{\infty} :v(\xi(x)+\varphi(x)) - v(\xi(x)) : dx. \quad (3.27)$$

If H_{ph}^φ is the representative of \tilde{H}_{ph}^φ in $L_2(d\mu^*)$, then we have that, on FC_2 ,

$$H_{ph}^\varphi - H_{ph} = U^\varphi(\xi) \quad (3.28)$$

We verify that $U^\varphi(\xi)$ is in $L_2(d\mu)$ by observing that if φ is of compact support this follows immediately from lemma 3.1 and the fact that $:v(\xi(x) + \varphi(x)) - v(\xi(x)):$ is zero outside the support of φ . For $\varphi \in S(\mathbb{R})$ it follows by using a decomposition of the identity. Hence we have, for f_1 and f_2 in FC_2 , that

$$(f_1, (V(\varphi)HV(-\varphi) - H)f_2) = (f_1, U^\varphi f_2) \quad (3.29)$$

the inner products being the inner products in $L_2(d\mu)$, because of (3.28), (3.2), (3.24) and (3.23). Since U^φ by (3.27) is differentiable with respect to φ in the strong $L_2(d\mu)$ topology it follows by taking derivatives in (3.29) that

$$i[\pi(\varphi), H] = \langle \xi, (-\Delta + m^2)\varphi \rangle + \int_{-\infty}^{\infty} :v'(\xi(x)):\varphi(x)dx \quad (3.30)$$

as bilinear forms on $FC_2 \times FC_2$ in $L_2(d\mu)$.

Hence we have the following lemma.

Lemma 3.3

For the interactions considered we have that

$$i[\pi(\varphi), H] = \langle \xi, (-\Delta + m^2)\varphi \rangle + \int_{-\infty}^{\infty} :v'(\xi(x)):\varphi(x)dx$$

as bilinear forms on $FC_2 \times FC_2$ in $L_2(d\mu)$. □

Consider now the Euclidean quantum field $\xi(x, t)$ corresponding to the interaction given by the function v . Consider also the analytic transformation of the (x, t) plane given by $(x, t) \rightarrow (u, \tau)$ where $w = u + i\tau$ is given by $w = \ln z$ with $z = x + it$ so that

$$u = \ln \sqrt{x^2 + t^2} \quad \text{and} \quad \tau = \operatorname{arctg} \frac{t}{x}$$

or

$$x = e^u \cos \tau \quad \text{and} \quad t = e^u \sin \tau.$$

We have then

$$\iint_{\mathbb{R}^2} (|\frac{\partial \psi}{\partial t}|^2 + |\frac{\partial \psi}{\partial x}|^2) dx dt = \int_{-\infty}^{\infty} \int_0^{2\pi} (|\frac{\partial \psi}{\partial \tau}|^2 + |\frac{\partial \psi}{\partial u}|^2) du d\tau \quad (3.32)$$

and

$$dt dx = e^{2u} d\tau du. \quad (3.33)$$

We introduce now for $(u, \tau) \in \mathbb{R} \times [0, 2\pi]$ the random field

$$\eta(u, \tau) = \xi(x(u, \tau), t(u, \tau)) \quad (3.34)$$

where the functions $x(u, \tau)$ and $t(u, \tau)$ are given in (3.31). By introducing a rotational invariant space-time cut-off for the Euclidean interaction one easily sees that $\eta(u, \tau)$ is a random field on $\mathbb{R} \times [0, 2\pi]$ i.e. actually on $\mathbb{R} \times C$, where C is the unit circle, which is invariant under the action of the circle group on C . Moreover, the corresponding probability measure $d\mu_\eta^*$ on $S'(\mathbb{R} \times C)$ is the weak limit of

$$\left[\int e^{-V_T} d\mu_\eta^0 \right]^{-1} e^{-V_T} d\mu_\eta^0 \quad (3.35)$$

as $T \rightarrow \infty$, where μ_η^0 is the Gaussian random field with expectation zero and covariance $E(\langle \psi, \eta \rangle^2)$ given by $(\psi A \psi)$ where A^{-1} is given by the form

$$(\psi, A^{-1} \psi) = \int \int_{\mathbb{R}C} (|\frac{\partial \psi}{\partial \tau}|^2 + |\frac{\partial \psi}{\partial u}|^2 + e^{2u} m^2 |\psi|^2) du d\tau. \quad (3.36)$$

The function V_T is given by

$$V_T(\eta) = \int_{-\infty}^T \int_0^{2\pi} :v(\eta(u, \tau)) : e^{2u} du d\tau, \quad (3.37)$$

where $::$ is the Wick ordering with respect to the Gaussian measure $d\mu_\eta^0$.

That the limit of (3.35) exists as $T \rightarrow \infty$ and is equal to $d\mu_\eta^*$ follows immediately by expressing (3.35) in terms of ξ , since it then is only the expression for the space-time cut-off Euclidean measure with a rotational invariant cut-off.

Let now $d\mu_\eta$ be the restriction of $d\mu_\eta^*$ to the subalgebra generated by the "time zero" fields $\eta_0(u) = \eta(u,0)$. It then follows that $d\mu_\eta$ is the image of $d\mu$ by the mapping

$$\eta_0(u) = \xi(x(u,0),0) = \xi_0(e^u). \quad (3.38)$$

So that for instance for $f \in FC_2$ we have

$$\iint \left| \frac{\delta f}{\delta \eta(u)} \right|^2 du d\mu_\eta = \iint_0^\infty x \left| \frac{\delta f}{\delta \xi(x)} \right|^2 dx d\mu \quad (3.39)$$

since $\frac{\delta f}{\delta \eta(u)} = \frac{\xi f}{\delta \xi(x)} \frac{dx}{du}$, where $x = e^u$. We see that the right hand side coincides with $(f, H(\lambda)f)$, where $\lambda(x) = x$ for $x \geq 0$ and $\lambda(x) = 0$ for $x \leq 0$.

We shall say that $f \in FC_2^+$ if f is the image of an element in FC_2 by the mapping induced by $u \rightarrow e^u = x$. It then follows from (3.39) that for $f \in FC_2^+$ we have that $H(\lambda)f$ is the diffusion operator for the η -field, applied to f . In the same way as we proved (3.2) we therefore get that, for f and g in FC_2^+ ,

$$(f, H(\lambda)g) = (f, \Lambda_{ph} g), \quad (3.40)$$

where Λ_{ph} is the infinitesimal generator for the strongly continuous unitary group of Euclidean rotations in $L_2(d\mu^*)$. In fact we get (3.40) by approximating the right hand side in (3.39) by using (3.35) and the well known form of the Euclidean rotations for the free field.

Let now φ be of compact support in $(0, \infty)$. Then by (3.23) we have that

$$H(\lambda) = V(-\varphi)H^\varphi(\lambda)V(\varphi), \quad (3.41)$$

where, by (3.20) and (3.39),

$$(f, \tilde{H}^\varphi(\lambda)f)_\varphi = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\delta f}{\delta \eta(u)} \right|^2 du d\mu_\eta^\varphi \quad (3.42)$$

$d\mu_\eta^\varphi$ being the image of $d\mu_\eta$ under the mapping $\eta_0(u) \rightarrow \eta_0(u) + \varphi(e^u)$.

In correspondence with the construction of $d\mu_\varphi^*$ we now introduce

$d\mu_{\eta, \varphi}^*$ as the image of $d\mu_\eta^*$ by the mapping $\eta(u, \tau) \rightarrow \eta(u, \tau) + \varphi(e^u)$.

We get in the same way as we got (3.27) and (3.28), but with the η -field instead of the ξ -field, that

$$\Lambda_{ph}^\varphi - \Lambda_{ph} = V^\varphi = \langle (\xi + \varphi), (-\frac{d}{dx} x \frac{d}{dx} + xm^2)\varphi \rangle + \int_0^\infty x : v(\xi(x) + \varphi(x)) - v(\xi(x)) : dx, \quad (3.43)$$

where we have expressed finally V^φ in terms of ξ_0 instead of η_0 by the relation (3.38), and the Wick ordering $::$ is therefore with respect to the free vacuum for the ξ -field.

Since φ is assumed to have compact support in $(0, \infty)$ we get by lemma 3.1 that V^φ is in $L_2(d\mu)$.

Identifying now, as for the ξ -field, Λ_{ph}^φ with $H^\varphi(\lambda)$ of (3.41) on $FC_2^+ \times FC_2^+$ we get

$$(f_1, (V(\varphi)H(\lambda)V(-\varphi) - H(\lambda))f_2) = (f_1, V^\varphi f_2), \quad (3.44)$$

for f_1 and f_2 in $FC_2^+ \times FC_2^+$. From (3.43) we see that V^φ is strongly differentiable in φ with respect to the $L_2(d\mu)$ topology and we thus have the following lemma.

Lemma 3.4

For the interactions considered and for $\varphi \in S(\mathbb{R})$ with support in $[0, \infty)$ we have that

$$i[\pi(\varphi), H(\lambda)] = \langle \xi, (-\frac{d}{dx} x \frac{d}{dx} + m^2 x) \varphi \rangle + \int_0^{\infty} x: v'(\xi(x)): \varphi(x) dx,$$

where the identity is to be understood in the sense of bilinear forms on $FC_2^+ \times FC_2^+$. \square

Let now f and g be in FC_2 , but so that they are dependent on functions only of compact support. Let $f \rightarrow f_a$ be the transformation induced in FC_2 by the translation $x \rightarrow x+a$ on the real line. Let us also assume that $\varphi \in S(\mathbb{R})$ with compact support and let $\Lambda = H(x)$, so that for any h in FC_2 we have

$$\Lambda h = -\frac{1}{2} \iint x \left(\frac{\delta^2 h}{\delta \xi(x)^2} + \beta(x) \frac{\delta h}{\delta \xi(x)} \right) dx d\mu(\xi). \quad (3.45)$$

Hence for $h \in FC_2^+$ we have that $\Lambda h = H(\lambda)h$. Let now $U(a)$ be the strongly continuous unitary group in $L_2(d\mu)$ which is induced by the translation $x \rightarrow x-a$, since $d\mu$ is translation invariant. Then obviously

$$U(a)\Lambda U(-a) = H(x+a) = H(x) + aH = \Lambda + aH \quad (3.46)$$

on FC_2 . So let now f and g in FC_2 be dependent on functions of compact support and let $\varphi \in S(\mathbb{R})$ be also of compact support.

Then

$$\begin{aligned} (f, [i\pi(\varphi), \Lambda]g) &= (U(a)f, U(a)[i\pi(\varphi), \Lambda]g) \\ &= (f_a, [i\pi(\varphi_a), \Lambda + aH]g_a). \end{aligned} \quad (3.47)$$

By taking a big enough we have that f_a and g_a are FC_2^+ and φ_a has support in $(0, \infty)$, but then (3.47) is given by

$$(f_a, [i\pi(\varphi_a), H(\lambda)]g_a) + a(f_a, [i\pi(\varphi_a), H]g_a), \quad (3.48)$$

which by lemma 3.3 and lemma 3.4 is equal to

$$\begin{aligned}
& (f_a, [\langle \xi, (-\frac{d}{dx}(x+a)\frac{d}{dx} + m^2(x+a))\varphi_a \rangle + \int_{-\infty}^{\infty} (x+a) :v'(\xi(x)):\varphi_a(x)dx]g_a) \\
& = (f, [\langle \xi, (-\frac{d}{dx}x\frac{d}{dx} + m^2x)\varphi + \int_{-\infty}^{\infty} x :v'(\xi(x)):\varphi(x)dx]g). \tag{3.49}
\end{aligned}$$

A continuity argument then gives us that the left hand side of (3.47) is equal to the right hand side of (3.49) also for arbitrary f and g in FC_2 and $\varphi \in S(R)$. Hence we have proven the following theorem.

Theorem 3.1

For the exponential interactions or the weak polynomial interactions we have, for any $\varphi \in S(R)$, that

$$i[\pi(\varphi), H] = \langle \xi, (-\frac{d^2}{dx^2} + m^2)\varphi \rangle + \int_{-\infty}^{\infty} :v'(\xi(x)):\varphi(x)dx$$

and

$$i[\pi(\varphi), \Lambda] = \langle \xi, (-\frac{d}{dx}x\frac{d}{dx} + m^2x)\varphi \rangle + \int_{-\infty}^{\infty} x :v'(\xi(x)):\varphi(x)dx$$

as bilinear forms on $FC_2 \times FC_2$ in $L_2(d\mu)$, where v' is the derivative of the function v which gives the interaction and $::$ is the Wick ordering with respect to the free field of mass m .

Moreover the right handsides of both expressions are in $L_2(d\mu)$. \square

It follows immediately from this theorem that for any $\varphi \in S(R)$ we have that $\pi(\varphi) \cdot 1$ is in $D(H) \cap D(\Lambda)$, if we, in agreement with the convention of section 2, denote by H also the Friedrichs extension of H on FC_2 and by Λ any self adjoint extension of Λ on FC_2 . This follows from the fact that both $[\pi(\varphi), H]$ and $[\pi(\varphi), \Lambda]$ are in $L_2(d\mu)$ while $H \cdot 1 = \Lambda \cdot 1 = 0$. So by Theorem 2.2 we have that both H and Λ map FC_4 into $D(H) \cap D(\Lambda)$, and for any

$f \in FC_4$ we have

$$[\Lambda, H]f = \int \frac{\delta f}{\delta \xi(x)} [\Lambda, H] \xi(x) dx. \quad (3.50)$$

Now we see that

$$H\xi(\varphi) = -i\pi(\varphi), \quad \Lambda\xi(\varphi) = -i\pi(x\varphi) \quad (3.51)$$

Combining this with the formulas of theorem 3.1 we get that, for any $f \in FC_4$,

$$[\Lambda, H]f = - \int \frac{\delta f}{\delta \xi(x)} \frac{d}{dx} \xi(x) dx, \quad (3.52)$$

where $\frac{d}{dx} \xi(x)$ is in the derivative in the sense of tempered distributions.

Let now P be the infinitesimal generator of the translation group $U(a)$ given by the translation $x \rightarrow x - a$. It follows immediately that FC_1 is in the domain of P and that

$$iPf = \int \frac{\delta f}{\delta \xi(x)} \frac{d}{dx} \xi(x) dx. \quad (3.53)$$

We have thus proven the following relation on FC^4

$$[\Lambda, H] = -iP. \quad (3.54)$$

By (3.46) we also have on FC_3

$$[\Lambda, P] = iH \quad (3.55)$$

We have now actually proven the following theorem.

Theorem 3.2

Let \mathfrak{d}_μ be the restriction to the σ -algebra of the time zero fields for an exponential or weak polynomial Euclidean interaction in two space time dimensions. Define for $f \in FC_2$

$$Hf = -\frac{1}{2} \int \left(\frac{\delta^2 f}{\delta \xi(x)^2} + \beta(x) \frac{\delta f}{\delta \xi(x)} \right) dx$$

and

$$\Lambda f = -\frac{1}{2} \int x \left(\frac{\delta^2 f}{\delta \xi(x)^2} + \beta(x) \frac{\delta f}{\delta \xi(x)} \right) dx.$$

Then H and Λ are densely defined symmetric operators in $L_2(d\mu)$ with self adjoint extensions.

Let P be the infinitesimal generator of the space translations, then

$$e^{iaP} H e^{-iaP} = H \quad \text{and} \quad e^{iaP} \Lambda e^{-iaP} = \Lambda + aH$$

on the domain FC_2 , which is invariant under space translations.

Let $\bar{\Lambda}$ and \bar{H} be the closures of H and Λ as defined above then both Λ and H map FC_4 into $D(\bar{\Lambda}) \cap D(\bar{H})$ and

$$[\bar{\Lambda}, \bar{H}] = -iP$$

on FC_4 . From what is said above we have

$$[\bar{H}, P] = 0 \quad \text{and} \quad [\bar{\Lambda}, P] = i\bar{H}$$

on FC_3 . \square

Footnotes

- 1) Some partial results are also obtained for the $P(\varphi)_2$ models with Dirichlet boundary conditions and isolated (but not necessarily unique) vacua. These models were also considered in [2], where references to the Euclidean theory for such models are also given.
- 2) Other proofs are in [12] and [6], Th. VIII. 33.
- 3) This concept and terminology has its roots in the work on stochastic mechanics and stochastic field theory, see [15],[16],[17] and references therein.
- 4) A correspondent result has recently been obtained, by other means, for Wightman theories satisfying certain conditions, by I. Herbst [23].

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