Multiplier representations of nilpotent Lie groups

(Preliminary report)

by

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In order to construct the unitary representations of a locally compact group by means of the Mackey machinery one often needs detailed information of multiplier representations for various kinds of groups. So far such information is available for relatively few groups, among them the abelian ones, [1]
We feel that this might be a good point to start a more systematic study of multiplier duals for nilpotent and solvable Lie groups. Thus in the present article we shall extend the Kirillov orbit theory to the case of cocycle representations of nilpotent Lie groups (1). Except for some obvious changes all our results are valid for exponential groups.

Let N be a simply connected, connected nilpotent Lie group, and let $\omega : N \times N \to T$ be a normalized analytic multiplier (cocycle) where T denotes the circle group. ω defines a central group extension

$$(1) \to T \to N(\omega) \to N \to (1)$$

where the multiplication rule of the nilpotent group $N(\omega)$ is given by

$$(s,m)(t,n) = (st\,\omega(m,n),\, m\cdot n);\ s,t \in T,\ m,n \in N.$$

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(1) The author would like to thank R. Høegh-Krohn for some enlightening conversations on the subject.

Corresponding to this is an exact sequence of Lie algebras

$$(0) \to \mathfrak{z} \to \mathfrak{n}(\omega) \to \mathfrak{n} \to (0)$$

with bracket operation

$$[(r,X),(s,Y)]_{\omega} = (B(X,Y),[X,Y]) ; \quad r,s \in \mathbb{R},\ X,Y \in \mathfrak{n},$$

where $[\cdot,\cdot]$ denotes the Lie product on $\mathfrak{n}$, and $B = B_{\omega}$ is a skew symmetric bilinear form on $\mathfrak{n} \times \mathfrak{n}$ satisfying the Jacobi-identity (see Parthasarathy [3] p. 35-36).

We next recall some facts concerning the ordinary irreducible representations of nilpotent Lie groups. Let $\mathfrak{n}^*$ be the space of all real valued linear functionals on $\mathfrak{n}$. If $\varphi \in \mathfrak{n}^*$ let $\mathfrak{a} \subset \mathfrak{n}$ be a subalgebra such that $\varphi[X,Y] = 0$, all $X,Y \in \mathfrak{a}$. Then, by the Campbell-Hausdorff formula, $\chi_{\varphi} = e^{i\varphi\circ\log}$ is a character of the subgroup $A = \exp \mathfrak{a}$. Such algebras $\mathfrak{a}$ are called subordinate to φ. By Kirillov [2], the induced representation $\pi_{\varphi} = \mathrm{Ind}_A^N(\chi_{\varphi})$ is irreducible iff $\mathfrak{a}$ is of maximal dimension among the subalgebras subordinate to φ. Moreover all $\pi \in \hat{N}$ may be obtained in this manner, i.e. $\pi = \pi_{\varphi}$ for some $\varphi \in \mathfrak{n}^*$. The group N acts on $\mathfrak{n}^*$ by the coadjoint representation ad^* :

$$ad^*(n)f(X) = f(Ad(n)X), \quad X \in \mathfrak{n}, \quad n \in N.$$

In the case of ordinary representations $\pi_{\varphi} \simeq \pi_{\psi}$ iff φ and ψ lies in the same orbit $\sigma(\varphi)$ under ad^*, and the map $\sigma(\varphi) \to \pi_{\varphi}$; $\mathfrak{n}^*/ad^*(N) \to \hat{N}$ is a bijection.

For multiplier representations analogous results hold as we shall see, however the action of N on $\mathfrak{n}^*$ will be more complicated. Naturally our proofs go by reduction to the case of ordinary representations by means of the extended group $N(\omega)$. This is possible since the ω-dual ${}_{\omega}\hat{N}$ may be identified to the subspace

$$\widehat{N(\omega)}_1 = \{\rho \in \widehat{N(\omega)} : \rho(t,1) = t^{-1}I \text{, all } t \in T\}$$

via the map

$$\pi \in {}_{\omega}\hat{N} \to \tilde{\tilde{\chi}}_1 \otimes \pi' \in \widehat{N(\omega)}_1 ,$$

where $\chi_1(t) = t^{-1}$, $t \in T$, and $\tilde{\tilde{\chi}}_1(t,n) = t^{-1}$, all $(t,n) \in N(\omega)$. π' denotes the lift of π to $N(\omega)$: $\pi'(t,n) = \pi(n)$.
Now, if $\rho \in \widehat{N(\omega)}_1$, let $\varphi \in \mathfrak{n}(\omega)^*$ be a functional such that $\rho = \pi_\varphi$. Since

$$\rho(e^{ir},1) = e^{-ir}I = \chi_\varphi(e^{ir})I = e^{i\varphi(r,o)}I$$

we have $\varphi(r,o) = -r$, $r \in \mathbb{R}$, and φ is of the form

$$\varphi = \psi + f_1$$

where $f_1(r,X) = -r$ and $\psi(r,X) = \varphi(0,X)$, all $(r,X) \in \mathfrak{n}(\omega)$. Thus ψ lives on the factor algebra $\mathfrak{n}$.

The following result transforms subordinacy in $\mathfrak{n}(\omega)^*$ into "ω-subordinacy" in $\mathfrak{n}^*$.

Lemma 1. Let ω be a normalized analytic multiplier of N and $\varphi \in \mathfrak{n}^*$. Assume $\mathfrak{a} \subseteq \mathfrak{n}$ is a subalgebra such that

$$(*) \qquad B_\omega(X,Y) + \varphi([X,Y]) = 0 \text{, all } X,Y \in \mathfrak{a}.$$

Then $\chi_\varphi = e^{i\varphi\circ\log}$ is an ω-character of $A = \exp\mathfrak{a}$. In particular $\omega|_{A\times A}$ is a trivial multiplier.

Proof. Concider the central extension $\mathfrak{a}(\omega)$ of $\mathfrak{a}$ by $\mathbb{R}$ given by $\omega|_{A\times A}$, and put

$$\psi(r,X) = \varphi(X) - r$$

$$\chi_\omega(\exp(r,X)) = e^{i\psi(r,X)} \text{, all } (r,X) \in \mathfrak{a}(\omega).$$

If $(*)$ holds χ_ψ is a character of $A(\omega)$ since

$$\psi[(r,X),(s,Y)]_{\omega} = \psi(B_{\omega}(X,Y),[X,Y])$$

$$= \psi([X,Y]) + B_{\omega}(X,Y)\,, \quad \text{all} \quad (r,X),(s,Y) \in \mathfrak{a}(\omega)\,.$$

Also

$$\chi_{\psi} \circ \tilde{\tilde{\chi}}_1^{-1}(\exp(r,X)) = e^{i\psi(r,X)}e^{ir} = e^{i\varphi(X)}$$

$$= \chi_{\varphi}(\exp X),\ (r,X) \in \mathfrak{a}(\omega)\,.$$

Since χ_{ψ} is an ordinary character of $A(\omega)$ and $\tilde{\tilde{\chi}}_1^{-1}$ is an ω-character, we have $\chi_{\psi}\tilde{\tilde{\chi}}_1^{-1}$ is an ω-character, and so is χ_{φ}. Moreover

$$\omega(x,y) = \chi_{\varphi}(xy)\chi_{\varphi}(x)^{-1}\chi_{\varphi}(y)^{-1}\,, \quad \text{all} \quad x,y \in A\,,$$

so that $\omega|_{A\times A}$ is a trivial multiplier.

Q.E.D.

In view of the last lemma we make the following

<u>Definition</u>. Let ω be a multiplier of N, $\varphi \in \mathfrak{n}^*$. A subalgebra $\mathfrak{a}$ of $\mathfrak{n}$ is said to be <u>ω-subordinate</u> to φ if

$$B_{\omega}(X,Y) + \varphi([X,Y]) = 0\,, \quad \text{all} \quad X,Y \in \mathfrak{a}\,.$$

When ω is cohomologous to 1, i.e. when ω is trivial, we may w.l.o.g. take $B_{\omega} \equiv 0$, and we get the usual condition $\varphi[X,Y] = 0$, $X,Y \in \mathfrak{a}$, stating that $\mathfrak{a}$ is subordinate to φ.

<u>Lemma 2</u>. Let $\varphi \in \mathfrak{n}^*$. $\mathfrak{a} \subseteq \mathfrak{n}$ is maximal among the ω-subordinate subalgebras of $\mathfrak{n}$ iff $\mathfrak{a}(\omega)$ (given by $B_{\omega}|_{\mathfrak{a}\times\mathfrak{a}}$) is maximal subordinate to $\psi = \varphi + f_1$ in $\mathfrak{n}(\omega)^*$.

<u>Proof</u>. Let $\mathfrak{a} \subseteq \mathfrak{n}$ be maximal ω-subordinate. $\mathfrak{B} \supseteq \mathfrak{a}(\omega)$ is subordinate to $\varphi + f_1$ iff $\mathfrak{B}/\mathfrak{z}$ is ω-subordinate to φ since

$$\psi[(s,X),(t,Y)]_{\omega} = \varphi([X,Y]) + B_{\omega}(X,Y)\,.$$

But $\mathcal{B}/_{\mathfrak{z}} \supseteq \mathfrak{a}$, so that $\mathcal{B}/_{\mathfrak{z}} = \mathfrak{a}$ by maximality. Hence $\mathcal{B} = \mathfrak{a}(\omega)$ and $\mathfrak{a}(\omega)$ is maximal.

The converse is similar.

Q.E.D.

Theorem 3. Let ω be a normalized analytic multiplier of the simply connected and connected nilpotent Lie group N, and let B_ω be the corresponding skew symmetric bilinear form on the Lie algebra $\mathfrak{n}$. Assume $\varphi \in \mathfrak{n}^*$ and let $\mathfrak{a} \subset \mathfrak{n}$ be a subalgebra ω-subordinate to φ, and $A = \exp \mathfrak{a}$. Then we have

(1) $\pi_\varphi = \omega - \mathrm{Ind}_A^N(\chi_\varphi)$ is irreducible iff $\mathfrak{a}$ is of maximal dimension among the algebras ω-subordinate to φ.

(2) Every $\pi \in {}_\omega\hat{N}$ is of the form $\pi = \omega - \mathrm{Ind}_A^N(\chi_\varphi)$ for some $\varphi \in \mathfrak{n}^*$ and $\mathfrak{a} \subseteq \mathfrak{n}$ maximal ω-subordinate to φ.

(3) The group N acts on $\mathfrak{n}^*$ by

$$(n,\varphi) \to n \cdot \varphi = \mathrm{ad}^*(n)\varphi + B_\omega\left(\log n, \frac{e^{\mathrm{ad}(\log n)} - I}{\mathrm{ad}(\log n)}(\cdot)\right)$$

(4) Let $\mathfrak{a}$ and $\mathfrak{a}'$ be ω-subordinate to φ and φ' respectively. Then $\pi_\varphi \simeq \pi_{\varphi'}$ iff the orbits $N \cdot \varphi$ and $N \circ \varphi'$ are equal.

(5) The correspondence $N \cdot \varphi \mapsto \pi_\varphi$, $\mathfrak{n}^*/N \to {}_\omega\hat{N}$ is a bijection.

Proof.

(1): Let $\varphi \in \mathfrak{n}^*$, and let $\mathfrak{a} \subseteq \mathfrak{n}$ be a subalgebra. If $\mathfrak{a}$ is maximal ω-subordinate to φ then $\mathfrak{a}(\omega)$ is maximal subordinate to $\psi = \varphi + f_1$ (Lemma 2), so that

$$\pi_\psi = \mathrm{Ind}_{A(\omega)}^{N(\omega)}(\chi_\psi)$$

is irreducible. Now, let $\chi_1(t) = t^{-1}$, $t \in T$, and let $\tilde{\chi}_1(t,a) = t^{-1}$, $(t,a) \in A(\omega)$, $\tilde{\tilde{\chi}}_1(t,n) = t^{-1}$, $(t,n) \in \mathfrak{n}(\omega)$. Then

$$\chi_\psi = \tilde{\chi}_1 \otimes \chi'_\varphi$$

so that

$$\mathrm{Ind}_{A(\omega)}^{N(\omega)}(\chi_\psi) = \mathrm{Ind}_{A(\omega)}^{N(\omega)}(\tilde{\chi}_1 \otimes \chi'_\varphi)$$

$$= \tilde{\tilde{\chi}}_1 \otimes (\omega\text{-}\mathrm{Ind}_{A(\omega)}^{N(\omega)}(\chi'_\varphi))$$

$$= \tilde{\tilde{\chi}}_1 \otimes (\omega\text{-}\mathrm{Ind}_A^N(\chi_\varphi))'$$

and it follows that

$$\pi_\varphi = \omega - \mathrm{Ind}_A^N(\chi_\varphi)$$

is irreducible. The converse follows by reversing the argument.

(2): If $\pi \in {}_\omega\hat{N}$ then the irreducible representation $\rho = \tilde{\tilde{\chi}}_1 \otimes \pi'$ of $N(\omega)$ is induced from some character χ_ψ, $\psi \in \mathfrak{n}(\omega)^*$, of a subalgebra $\mathfrak{a}(\omega)$ maximal subordinate to ψ:

$$\rho = \mathrm{Ind}_{A(\omega)}^{N(\omega)}\chi_\psi .$$

Put $\mathfrak{a} = \mathfrak{a}(\omega)/\mathfrak{z}$, and $\varphi = \psi - f_1$. Then φ can be regarded as a functional of $\mathfrak{n}$. Arguing as in the proof of (1) we have

$$\rho = (\omega\text{-}\mathrm{Ind}_A^N(\chi_\varphi))' \otimes \tilde{\tilde{\chi}}_1 = \pi' \otimes \tilde{\tilde{\chi}}_1$$

so that

$$\omega - \mathrm{Ind}_A^N \chi_\varphi = \pi ,$$

and π is obtained by inducing χ_φ from $A = \exp(\mathfrak{a})$ where $\mathfrak{a}$ is maximal ω-subordinate to φ (Lemma 2). This completes the proof of (2).

(3): We compute the adjoint action in the extended algebra $\mathfrak{n}(\omega)$.

$$\mathrm{Ad}(\exp(s,X))(t,Y) = \exp[(s,X),(t,Y)]_\omega$$

$$= \exp(B_\omega(X,Y),[X,Y]) , \quad (s,X),(t,Y) \in \mathfrak{n}(\omega) .$$

Letting

$$T_X(t,Y) = (B_\omega(X,Y),[X,Y]) = \begin{pmatrix} B_\omega(X,\cdot) & 0 \\ [X,\cdot] & 0 \end{pmatrix} \begin{pmatrix} s \\ X \end{pmatrix}$$

we have

$$(\exp T_X)(t,Y) = \Big(\sum_{n=0}^{\infty} \frac{1}{n!} T_X^n\Big)(t,Y)$$

$$= (t,Y) + \Big(\frac{1}{2!} B_\omega(X,Y) + \frac{1}{3!} B_\omega(X,\mathrm{ad}(X)Y) + \ldots + B_\omega(X,\mathrm{ad}^n(X)Y)$$

$$+ \ldots, [X,Y] + \frac{1}{2!}[X,[X,Y]] + \ldots + \frac{1}{n!}\mathrm{ad}^n(X)Y + \ldots \Big)$$

$$= \Big(B_\omega\big(X,\frac{e^{\mathrm{ad}X} - I}{\mathrm{ad}X} Y\big) + t\ ,\ e^{\mathrm{ad}(X)}Y\Big)$$

$$= \Big(B_\omega\big(X,\frac{e^{\mathrm{ad}X} - I}{\mathrm{ad}X} Y\big) + t\ ,\ \mathrm{Ad}(\exp X)Y\Big) .$$

Hence

$$\mathrm{Ad}(\exp(s,X))(t,Y) = B_\omega\big(X,\frac{e^{\mathrm{ad}X} - I}{\mathrm{ad}X} Y\big) + t\ ,\ \mathrm{Ad}(\exp X)Y)$$

which is independent of the first coordinate s.
Thus, for $\psi \in \mathfrak{n}^*$, we can write

$$(\exp X)\cdot\psi = \mathrm{ad}^*(\exp X)\psi + B_\omega\big(X,\frac{e^{\mathrm{ad}X} - I}{\mathrm{ad}X}(\cdot)\big) .$$

Then

$$\text{(I)}\quad \mathrm{ad}^*(\exp(s,X))(\psi+f_1) = \mathrm{ad}^*(\exp X)\psi + B_\omega\big(X,\frac{e^{\mathrm{ad}X} - I}{\mathrm{ad}X}(\cdot)\big) + f_1$$

$$= (\exp X)\psi + f_1\ ,\ (s,X) \in \mathfrak{n}(\omega) ,$$

where $f_1(s,X) = -s$. From this relation it follows at once that N acts on $\mathfrak{n}^*$ by $(\exp X,\psi) \to (\exp X)\cdot\psi$ since

$$(\exp X)((\exp Y)\psi) + f_1 = \mathrm{ad}^*\exp(0,X)[(\exp Y)\psi + f_1]$$

$$= \mathrm{ad}^*\exp(0,X)[\mathrm{ad}^*\exp(0,Y)(\psi+f_1)]$$

$$= \mathrm{ad}^*(\exp(0,X)\exp(0,Y))(\psi+f_1)$$

$$= \mathrm{ad}^*(\exp((0,X) + (0,Y) + \tfrac{1}{2}(B_\omega(X,Y),[X,Y]) + \dots)(\psi+f_1)$$

$$= \mathrm{ad}^*(\exp(\tfrac{1}{2}B_\omega(X,Y) + \dots\ ,\ X + Y + \tfrac{1}{2}[X,Y] + \dots)(\psi+f_1)$$

$$= \exp(X + Y + \tfrac{1}{2}[X,Y] + \dots)\psi + f_1$$

$$= (\exp X\ \exp Y)\psi + f_1\ , \qquad X,Y \in \mathfrak{n},\ \ \psi \in \mathfrak{n}^*,$$

where we used the Campbell-Hausdorff formula. This proves (3).

To verify (4) we only have to note that $\psi,\varphi \in \mathfrak{n}^*$ give equivalent ω-representations $\iff$ $\psi + f_1$ and $\varphi + f_1$ give equivalent representations of $N(\omega)$ $\iff$ there is $\exp(s,X) \in N(\omega)$ such that $\mathrm{ad}^*(\exp(s,X))(\psi+f_1) = \varphi + f_1$ $\iff$ there is $\exp X \in N$ such that $(\exp X) \circ \psi = \varphi$ (using (I)).

(5): Let $\mathfrak{n}(\omega)^*_1$ be the image in $\mathfrak{n}(\omega)^*$ of $\mathfrak{n}^*$ under the map $\varphi \mapsto \varphi + f_1$. From (I) above it is clear that the map $F: N\circ\varphi \mapsto N\circ\varphi+f_1$ is a bijection between $\mathfrak{n}^*/_N$ and $\mathfrak{n}(\omega)^*_1/\mathrm{ad}^*N(\omega)$. If k denotes the Kirillov correspondence of $\mathfrak{n}(\omega)/\mathrm{ad}^*N(\omega)$ onto $\widehat{N(\omega)}$ and $m:\ \pi \in {}_\omega\hat{N} \mapsto \tilde{\tilde{\chi}}_1 \otimes \pi' \in \widehat{N(\omega)}_1$, then the following diagram commutes

$$\begin{array}{ccc} \mathfrak{n}^*/N & \xrightarrow{\ k_\omega\ } & {}_\omega\hat{N} \\ F\downarrow & & m\downarrow \\ \mathfrak{n}^*(\omega)_1/\mathrm{ad}^*N(\omega) & \xrightarrow{\ k\ } & \widehat{N(\omega)}_1 \end{array}$$

since

$$N \circ \varphi \overset{F}{\longmapsto} N \circ \varphi + f_1 \overset{k}{\longmapsto} \mathrm{Ind}_{A(\omega)}^{N(\omega)}(\chi'_\varphi \otimes \chi'_1) = \pi'_\varphi \otimes \tilde{\tilde{\chi}}_1$$

and

$$N \circ \varphi \overset{k_\omega}{\longmapsto} \omega - \mathrm{Ind}_A^N \chi_\varphi = \pi_\varphi \overset{m}{\longmapsto} \pi'_\varphi \otimes \tilde{\tilde{\chi}}_1\,.$$

Now F, k, and m are bijections, hence so is k_ω. This completes the proof of (5).

References

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