

GROUPS WITH HAUSDORFF UNITARY DUAL SPACE

ABSTRACT We show that for any non-compact, connected, semisimple Lie group  $G$  the unitary dual space  $\hat{G}$  endowed with the hull-kernel topology is non-Hausdorff. This result gives a structure theorem for the class of all connected, locally compact groups with Hausdorff unitary dual space.

1. INTRODUCTION Let  $G$  be a non-compact, connected, semisimple Lie group with Lie algebra  $\mathfrak{G}$ , and  $\hat{G}$  its unitary dual space consisting of all equivalence classes of irreducible unitary representations of  $G$ , equipped with the hull-kernel topology. We prove here that  $\hat{G}$  is non-Hausdorff. The idea of the proof is as follows. According to Kostant [2] the (class one) complementary series of  $G$  is parametrized by those elements  $\lambda$  in the closure of a certain convex set called the critical interval, CI, such that  $\lambda - \rho$  and  $-\lambda + \rho$  are conjugate under the Weyl group  $W$ , where  $\rho(x) = \frac{1}{2}\text{tr}(\text{ad}(x)|_{\mathfrak{n}})$   $\mathfrak{n}$  = the nilpotent part of the Iwasawa decomposition of  $\mathfrak{G}$ . To a boundary point of CI is associated a reducible representation of which can be unitarized and contains at least two non-equivalent irreducible components. We show then, applying results of Miličič [6], that the above mentioned parametrization is continuous at a boundary point. Thus there exists a sequence of irreducible complementary series representations with at least two different limit points.

Our main motivation for writing this article is the following. Combining the above mentioned result for semisimple groups with earlier work of Liukkonen and Mosak [5], and Peters [7], we obtain a structure theorem for the family of all connected, locally compact

groups possessing Hausdorff unitary dual space. In fact, as shown in [5] the primitive ideal space,  $\text{Prim } G$ , of the group  $C^*$ -algebra with the hull-kernel topology is a Hausdorff space if  $G$  is  $\sigma$ -compact and all of its conjugacy classes are precompact. The converse result was established in [7] for unimodular, amenable groups.

Now in the connected case a group  $G$  possessing precompact conjugacy classes is an extension of a vector group by a compact normal subgroup (see e.g. [5]), and such groups are of type I (an application of the Mackey theory). Hence  $\text{Prim } G$  and  $\hat{G}$  are homeomorphic, and so for amenable, unimodular groups we have

$\hat{G}$  is Hausdorff  $\Leftrightarrow \text{Prim } G$  is Hausdorff  $\Leftrightarrow G$  is an extension of a vector group by a compact normal subgroup.

Further,  $\hat{G}$  Hausdorff  $\Rightarrow G$  is CCR  $\Rightarrow G$  is unimodular, [2]. Hence unimodularity is automatic in our situation.

From the results of the present article (Proposition 3) the semisimple part in the Levi decomposition of a connected Lie group  $G$  with a Hausdorff unitary dual space must be compact. In other words, the solvable radical of  $G$  is cocompact. Accordingly  $G$  is amenable and the above remarks apply:  $G$  contains a compact, normal covector subgroup  $K$ . Now if  $G$  is an arbitrary connected l.c. group and  $\hat{G}$  is Hausdorff,  $G$  is a projective limit of Lie groups,  $G = \text{projlim}_i(G_i)$ , and it follows that each  $\hat{G}_i$  is Hausdorff, being a closed subspace of  $\hat{G}$ . We have proved the following.

Theorem 1. Let  $G$  be a connected, locally compact group. Its unitary dual space  $\hat{G}$ , endowed with the hull-kernel topology, is a Hausdorff space if and only if  $G$  is an extension of a vector group by a compact, connected group.

Remark. Actually one can say somewhat more about the structure of such groups. For let  $G$  be a Lie group with a compact, connected, normal subgroup  $K$ , and  $G/K \approx \mathbb{R}^n$  a vector group. Now  $K$  is isomorphic to the direct product of a semisimple (compact) group  $S$  and a  $k$ -torus  $\mathbb{T}^k$ ,  $K \approx S \times \mathbb{T}^k$ ; and by Levi's theorem  $S$  must occur as a direct factor in  $G$  since  $G/K$  is solvable. Further the automorphism group of  $\mathbb{T}^k$  is discrete so the connected group  $G/K$  acts trivially on  $\mathbb{T}^k$  by automorphisms, and hence  $\mathbb{T}^k$  is central in  $G$ . Thus we have  $G = N \times S$  where  $S$  is compact, connected, and semisimple, and  $N$  is a 2-step nilpotent Lie group containing a central  $k$ -torus  $\mathbb{T}^k$  and with  $N/\mathbb{T}^k \approx \mathbb{R}^n$ . In other words,  $N$  satisfies a central, exact sequence of topological groups

$$1 \rightarrow \mathbb{T}^k \rightarrow N \rightarrow \mathbb{R}^n \rightarrow 1 \dots$$

Finally, in the non-Lie case,  $G$  is a projective limit of Lie groups  $N \times S$  of the above type.

As already mentioned, for a connected group  $G$ ,  $\hat{G}$  is Hausdorff if and only if  $\text{Prim} G$  is Hausdorff. Such an equivalence does not hold in general since any separable non-type I group in the class  $[\text{FC}]^-$  has a Hausdorff primitive dual space, [5]. A reasonable conjecture seems to be that  $\hat{G}$  is  $T_2$  if and only if  $G$  is of type I and  $G \in [\text{FC}]^-$ . We shall treat the nonconnected case later.

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2. NOTATION Let  $G$  be a connected semisimple Lie group with Iwasawa decomposition  $G = KAN$ , and  $B = MAN$  a minimal parabolic subgroup, i.e.  $M$  is the centralizer of  $A$  in  $K$ . We recall some concepts from [3]. If  $\mathcal{O}'_{\mathbb{C}} = \text{Hom}_{\mathbb{R}}(\mathcal{O}, \mathbb{C})$  is the complex dual to the Lie algebra  $\mathcal{O}$  of  $A$  then each  $\lambda \in \mathcal{O}'_{\mathbb{C}}$  defines a non unitary character  $b \mapsto b^{\lambda}$  of  $B$  such that  $b^{\lambda} = 1$  for  $b \in MN$ ,  $b^{\lambda} = \exp \lambda(x)$  for  $\exp x = b \in A$ .

Put  $X^{\lambda} =$  the space of all analytic  $K$ -finite functions  $f$  on  $G$  (i.e.  $K \cdot f$  spans a finite dimensional vector space, where  $(k \cdot f)(g) = f(k^{-1}g)$ ;  $k \in K, g \in G$ ) such that  $f(gb) = b^{-\lambda}f(g)$ ,  $g \in G$ ,  $b \in B$ . Then  $X^{\lambda}$  is a  $\mathcal{U}$ -module (not in general a  $G$ -module) where  $\mathcal{U}$  denotes the universal enveloping algebra of  $\mathfrak{g}$  over  $\mathbb{C}$ , and

$$(x \cdot f)(g) = \frac{d}{dt} f(\exp -tx \cdot g)|_{t=0}, \quad x \in \mathfrak{g}, g \in G, f \in X^{\lambda}.$$

[3;Thm.2] gives a necessary and sufficient condition for  $X^{\lambda}$  to be an irreducible  $\mathcal{U}$ -module (in the algebraic sense). Thereby arises a region called the critical strip, CS, in  $\mathcal{O}'_{\mathbb{C}}$  for which  $X^{\lambda}$  is always  $\mathcal{U}$ -irreducible, hence equal to  $Z^{\lambda} = \mathcal{U} \cdot 1_{\lambda}$  where  $1_{\lambda}$  is the unique function in  $X^{\lambda}$  which is identically 1 on  $K$ .

Let  $\Lambda \subset \mathcal{O}'$  be the set of roots for the action of  $\mathcal{O}$  on  $\mathfrak{g}$ , and for  $\varphi \in \Lambda$  let  $\mathfrak{g}^{\varphi} \subset \mathfrak{g}$  be the corresponding root space. The one dimensional space  $[\mathfrak{g}^{\varphi}, \mathfrak{g}^{-\varphi}] \cap \mathcal{O}$  ( $\varphi \in \Lambda$ ) is spanned by a unique element  $\omega_{\varphi}$  such that  $\varphi(\omega_{\varphi}) = 1$ .

Let  $\Lambda^{\pm} = \{\varphi \in \Lambda : \varphi/2 \text{ is not a root}\}$ , and denote by  $\Lambda^{\pm}_+$  the positive elements in  $\Lambda^{\pm}$  w.r.t. a lexicographical ordering of  $\mathcal{O}'$ .

For any  $\varphi \in \Lambda^{\pm}_+$  let  $T_{\varphi}$  be the open interval

$$T_{\varphi} = \{t \in \mathbb{R} : |t| < (\dim \mathfrak{g}^{\varphi})/2 \text{ if } 2\varphi \text{ is not a root,} \\ \text{and } |t| < (\dim \mathfrak{g}^{\varphi})/2 + 1 \text{ if } 2\varphi \text{ is a root}\}.$$

Now define  $\rho(x) = \frac{1}{2}\text{tr}(\text{adx}|_{\mathfrak{n}})$ , for all  $x \in \mathfrak{a}$ , where  $\mathfrak{n}$  = nilpotent part in the Iwasawa decomposition of  $\mathfrak{g}$ . For  $\lambda$  in a certain subset  $\mathfrak{a}^*$  of  $\mathfrak{a}'_{\mathbb{C}}$  it is possible to associate a unique unitary, irreducible representation  $\pi^\lambda$  of  $G$  whose differential induces the given  $\mathcal{U}$ -module structure on  $Z^\lambda$ . The (class one) complementary series is defined as the family of all such representations  $\pi^\lambda$  where  $\lambda$  ( $\in \mathfrak{a}^*$ ) is in the closure of the critical interval  $CI = \{\lambda \in \mathfrak{a}' : (\lambda - \rho)(\omega_\varphi) \in T_\varphi \text{ for all } \varphi \in \Lambda_+^1\}$ , [3, § 7.3].

3. THE SEMISIMPLE CASE In this section  $G$  will denote a connected semisimple Lie group. Notations will be as in Section 2.

Lemma 2. Let  $G$  be a connected, semisimple Lie group. If  $\lambda_0$  is a boundary point of the convex set  $CI$ , then the  $\mathcal{U}$ -module  $X^{\lambda_0}$  is reducible. In particular the corresponding (non-unitary) representation  $\pi^0$  of  $G$  is reducible.

Proof. By assumption  $\lambda_0$  is a boundary point of  $CI$ , hence there is a functional  $\varphi \in \Lambda_+^1$  such that

$$(1) \quad |(\lambda_0 - \rho)(\omega_\varphi)| = \begin{cases} (\dim \mathfrak{g}^\varphi)/2 & , \text{ if } 2\varphi \text{ is not a root} \\ (\dim \mathfrak{g}^\varphi)/2 + 1 & , \text{ if } 2\varphi \text{ is a root.} \end{cases}$$

a)  $2\varphi$  is not a root (i.e.  $\dim \mathfrak{g}^{2\varphi} = 0$ ). Then by (1)

$$(\lambda_0 - \rho)(\omega_\varphi) + (\dim \mathfrak{g}^\varphi)/2 = \begin{cases} 0 & , \text{ if } (\lambda_0 - \rho)(\omega_\varphi) < 0 \\ \dim \mathfrak{g}^\varphi & , \text{ if } (\lambda_0 - \rho)(\omega_\varphi) > 0 \end{cases}$$

which in both cases is  $0 \pmod{\mathbb{Z}}$ . Hence [3, Theorem 2] gives that  $X^{\lambda_0}$  is reducible.

b)  $2\varphi$  is a root (i.e.  $\dim \mathfrak{g}^{2\varphi} \in \{1,3,7\}$  by say [8, pp. 31-32]).  
 By [8, Lemma 2, p. 33]  $\dim \mathfrak{g}^\varphi$  is an even integer. Hence  
 $m_\varphi = (\dim \mathfrak{g}^\varphi)/2 + \dim \mathfrak{g}^{2\varphi}$  is an integer. We wish to apply [3, Theorem 2]  
 again, and have by the identity (1) above,

$$\begin{aligned} (\lambda_0 - \rho)(\omega_\varphi) + m_\varphi &= (\lambda_0 - \rho)(\omega_\varphi) + (\dim \mathfrak{g}^\varphi)/2 + \dim \mathfrak{g}^{2\varphi} \\ &= \begin{cases} \dim(\mathfrak{g}^{2\varphi}) - 1, & \text{if } (\lambda_0 - \rho)(\omega_\varphi) < 0 \\ \dim(\mathfrak{g}^\varphi) + \dim(\mathfrak{g}^{2\varphi}) + 1, & \text{if } (\lambda_0 - \rho)(\omega_\varphi) > 0, \end{cases} \end{aligned}$$

which, in both cases, is an integer. Hence again  $X^{\lambda_0}$  is irreducible. So the proof is complete.

Assume now that  $G$  has finite center. Let  $D'(G)$  denote the space of distributions on  $G$  endowed with the weak topology,  $\text{tr} : \hat{G} \rightarrow D'(G)$  the injective map which assigns to each  $\pi$  in  $\hat{G}$  its distribution character  $\text{tr}(\pi)$ , and  $\Omega$  the closure of  $\text{tr}(\hat{G})$  in  $D'(G)$ . By Harish-Chandra's character formula [1, Theorem 2] the character of any irreducible complementary series representation  $\pi^\lambda$ ,  $\lambda \in \overline{\mathbb{C}I}$ , is given by integration against a continuous function  $\theta_\lambda$  on  $G$  which depends continuously on the parameter  $\lambda$ ,

$$\text{tr}(\pi^\lambda(f)) = \int_{G_1} \theta_\lambda(x) f(x) dx, \quad \text{all } f \in C_c(G),$$

where  $G_1$  is a certain closed subgroup of  $G$ . We prove next the main result of this section.

Proposition 3. Let  $G$  be a connected, noncompact, semisimple Lie group. Then its unitary dual space  $\hat{G}$  equipped with the hull-kernel topology is not a Hausdorff space.

Proof. We may clearly assume that  $G$  has finite center. Suppose  $\{\pi_n^\lambda\}_{n=1,2,\dots}$  is a sequence of irreducible complementary series

representations of  $G$ , where  $\lambda_n \in CI$  for each  $n = 1, 2, \dots$ , and  $\lambda_n - \rho$ ,  $-\lambda_n + \rho$  are conjugate under the Weyl group  $W$ , and that  $\lambda_n$  converges to a boundary point  $\lambda_0$  of  $CI$ . Then by Harish-Chandra's character formula [1, Theorem 2] and the Lebesgue convergence theorem

$$\text{tr}(\pi^{\lambda_n}(f)) = \int_{G_1} \theta_{\lambda_n}(x) f(x) dx \xrightarrow{n \rightarrow \infty} \int_{G_1} \theta_{\lambda_0}(x) f(x) dx = \tau_0(f),$$

for all  $f \in C_c(G)$ .

Hence  $\tau_0$  is a central distribution,  $\tau_0 \in \Omega$ , and  $\tau^0$  is the character of the representation  $\pi^0$  which is associated to the reducible  $\mathcal{U}$ -module  $X^{\lambda_0}$  (Lemma 2).

Now, by a result of Miličić, [6, Theorem 5.8],  $\tau_0$  is uniquely decomposable into the sum of finitely many distribution characters of elements of  $\hat{G}$ ,  $\tau_0 = \sum_{\pi \in \Gamma(\tau_0)} n_\pi \text{tr}(\pi)$ , where  $n_\pi$  is a positive integer, and  $\Gamma(\tau_0)$  is a unique finite subset of  $\hat{G}$  called the carrier of  $\tau_0$ . Moreover, by [6, Theorem 5.6] the sequence  $\{\pi^{\lambda_n}\}$  converges to  $\pi$  for each  $\pi \in \Gamma(\tau_0)$ , as  $\lambda_n \rightarrow \lambda_0$ . By Lemma 2  $\pi^0$  is reducible, and to show that  $\hat{G}$  is non-Hausdorff it suffices to prove that  $\Gamma(\tau_0)$  consists of more than one element. This follows from the fact that  $\pi^0$  contains a spherical component  $\pi_1$  with multiplicity one (the only  $K$ -fixed vectors are the functions constant on  $K$ ). Thus  $\pi^{\lambda_0}$  contains at least two non-equivalent components. The proof is complete.

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