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OPERATOR ALGEBRAS ASSOCIATED WITH HNN-EXTENSIONS

by

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Introduction

In [1], we studied some properties of the reduced group C^*-algebra and of the group von Neumann algebra associated with free products of groups with amalgamation. One of the princpal tools in obtaining these results was the implicit use of the natural underlying bipolar structure of free products of groups with amalgamation. A theorem of Stallings [8], in the generalized version of Lyndon-Schupp [5,p.210], states that a group has a bipolar structure if and only if it is either a non-trivial free product with amalgamation (possibly an ordinary free product) or an HNN-extension. So, as conjectured to us by P. de la Harpe, it should be natural to expect that our work in [1] could be pushed to include HNN-extensions.

The main purpose of this note is to establish the following:

Theorem: Let $B = \langle H,t;t^{-1}At=B,\Phi\rangle$ be an HNN-extension and suppose that H has an element z such that:

$$(\star) \qquad zAz^{-1} \cap A = \{1\} \quad \text{and} \quad z \notin B$$

Then $C_r^*(G)$ is simple with a unique tracial state and $U(G)$ is a II_1-factor which does not possess property Γ of Murray and von Neumann.

One should note that the conclusion of the theorem does not need to hold for all HNN-extensions: for example the group $G = \langle s,t;t^{-1}st=s\rangle$ is abelian and so $C_r^*(G)$ is certainly not simple, neither is $U(G)$ a factor. However, the conclusion of the theorem is true when G is a group having a presentation with at least 3 generators and a single defining relation; this follows because, as shown in [5; §IV.5 and p.293-294], one can always view such a G as an HNN-extension of a one relator group H which possess an element z satisfying a slightly stronger condition than (*).

In a recent work [2], de la Harpe introduces the notion of a Powers group as a possibly more natural definition than the one of a "group satisfying Powers property". We conclude this note by indicating how the groups considered in this note and in [1] can be seen to be Powers groups.

For notation not specified in the sequel, we refer to [1], which we also refer to for further references and some more information on the subject. See also [2].

We are very much indebted to P. de la Harpe for his suggestions and for sending us a preliminary version of [2].

Preliminaries

Our basic reference about HNN-extensions is [5], from which we quote here some definitions and results.

Let H be a group and let A and B be subgroups of H with $\Phi: A \to B$ an isomorphism. The HNN-extension of H relative to A, B and Φ is the group G given by

$$G = \langle H, t;\ t^{-1}at = \Phi(a),\ a \in A \rangle,$$

which we denote by $G = \langle H, t;\ t^{-1}At = B, \Phi \rangle$.

In the note, the letter h (or k), with or without subscripts, will denote an element of H. If h is thought of as a word, it is a word on the generators of H; that is h contains no occurences of $t^{\pm 1}$. The letter ε (or δ), with or without subscripts, will denote 1 or -1.

A sequence $h_0, t^{\varepsilon_1}, h_1, \ldots, t^{\varepsilon_n}, h_n$ $(n \geq 0)$ is said to be reduced if there is no consecutive subsequence t^{-1}, h_i, t with $h_i \in A$ or t, h_j, t^{-1} with $h_j \in B$.

One way to state the Normal Form Theorem for HNN-extensions is the following:

i) The group H is embedded in G by the map $h \to h$.

ii) If $h_0 t^{\varepsilon_1} \ldots t^{\varepsilon_n} h_n = 1$ in G where $n \geq 1$, then $h_0, t^{\varepsilon_1}, \ldots, t^{\varepsilon_n}, h_n$ is not reduced.

It is usual to be rather sloppy in formally distinguishing between a sequence $h_0, t^{\varepsilon_1}, \ldots, t^{\varepsilon_n}, h_n$ and the product $h_0 t^{\varepsilon_1} \ldots t^{\varepsilon_n} h_n$. So, if w is an element of G, a normal form of w is any sequence $h_0 t^{\varepsilon_1} \ldots t^{\varepsilon_n} h_n = w$ such that $h_0, t^{\varepsilon_1}, \ldots, t^{\varepsilon_n}, h_n$ is reduced.

From the Normal Form Theorem one obtains that, if u and v in G have normal forms $u = h_0 t^{\varepsilon_1} \ldots t^{\varepsilon_n} h_n$ and $v = k_0 t^{\delta_1} \ldots t^{\delta_m} k_m$ such that $u = v$ in G, then $n = m$ and $\varepsilon_i = \delta_i$, $i = 1, \ldots, n$. This allows to define the length of w, written $|w|$, for each element w of G, as the number of occurences of $t^{\pm 1}$ in any normal form of w. If $w \in H$, then $|w| = 0$.

At last, if u and v in G have normal forms $u = h_0 t^{\varepsilon_1} \ldots t^{\varepsilon_n} h_n$ and $v = k_0 t^{\delta_1} \ldots t^{\delta_m} k_m$, one says that there is cancellation in forming the product uv if either $\varepsilon_n = -1$, $h_n k_0 \in A$, and $\delta_1 = 1$, or if $\varepsilon_n = 1$, $h_n k_0 \in B$, and $\delta_1 = -1$.

Proof of the theorem

Let $G = \langle H,t;\ t^{-1}At=B,\Phi\rangle$ and suppose that H has an element z such that

$$(*) \qquad zAz^{-1} \cap A = \{1\} \quad \text{and} \quad z \notin B.$$

Observe first that $(*)$ implies that $z \notin A$. Indeed, if $z \in A$, then $zAz^{-1} \cap A = \{1\}$ implies that $A = \{1\}$, so $z = 1$ which is not compatible with $z \notin B$.

We now define $E \subseteq G$ by

$$E = \{g\in G-A \mid \text{if } g \text{ has a normal form } g = h_0 t^{\varepsilon_1}\dots t^{\varepsilon_n} h_n, \text{ then } h_0 \in H-A, \text{ or } h_0 \in A \text{ and } \varepsilon_1 = -1\},$$

and further, for $\ell = 1,2,\dots$, we define

$c_\ell \in G$ and $Z_\ell \subseteq G$ by

$$c_\ell = t(tz)^{-\ell} = t(z^{-1}t^{-1})\dots(z^{-1}t^{-1}) \quad \text{and}$$

$$Z_\ell = \{g\in G \mid c_\ell g \in E\}.$$

We are going to show that the Z_ℓ's are pairwise disjoint subsets of G and that the following is true:

$(**)$ For every finite subset F of $G-\{1\}$ and for every natural number $N \geq 1$ one has that $b_\ell f b_\ell^{-1} y \in Z_\ell$, for all $f \in F$, $y \in G-Z_\ell$, $\ell \in \{1,\dots,N\}$, where, for $j = 1+\max_{f\in F} |f|$, we have defined b_ℓ as

$$b_\ell = (tz)^\ell t^{-1}(zt)^j, \quad \ell = 1,2,\dots,N.$$

This will show that G possess Powers property ([1],[3]) which in turn will show the first assertion of the theorem.

Lemma 1: The Z_ℓ's are pairwise disjoint.

Proof: Suppose $\ell,\ell' \in N$, $\ell' = \ell+n$ where $n \in \mathbb{N}$ and $y \in Z_\ell$. Then $c_{\ell'} y = t(tz)^{-\ell'} y = t(tz)^{-n}t^{-1}t(tz)^{-\ell} y = t(tz)^{-n}t^{-1}c_\ell y$.

Since $y \in Z_\ell$, i.e. $c_\ell y \in E$ there can be no cancellation in forming the product of $t(tz)^{-n}t^{-1}$ and $c_\ell y$, so $c_{\ell'} y$ has a normal form $c_{\ell'} y = th_1 \ldots t^{\varepsilon_m} h_m$, i.e. $c_{\ell'} y \notin E$, i.e. $y \notin Z_{\ell'}$. .

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To ease our exposition we make the following definition:

Let $w \in G$. If w has a normal form of the following type:

(1) $zth_1 \ldots h_{n-1} t^{-1} z^{-1}$, $n \geq 2$.

(resp. (2) $(zt)^p = zt\, zt \ldots zt$ for a $p \in \mathbb{N}$),

(resp. (3) $(zt)^{-p} = t^{-1}z^{-1}t^{-1}z^{-1} \ldots t^{-1}z^{-1}$ for a $p \in \mathbb{N}$),

(resp. (4) $(zt)^p h = zt \ldots zt\, h$ for a $p \in \mathbb{N}$ and a $h \in H-A$),

(resp. (5) $h(zt)^{-p} = h\, t^{-1}z^{-1} \ldots t^{-1}z^{-1}$ for a $p \in \mathbb{N}$ and a $h \in H-A$),

(resp. (6) h, where $h \in H-A$),

then we say that

w is of <u>type 1</u>, (resp. <u>type 2</u>), (resp. <u>type 3</u>), (resp. <u>type 4</u>), (resp. <u>type 5</u>), (resp. <u>type 6</u>).

<u>Lemma 2</u>: <u>Let $w \in G$. If w is of one of the types 1-6, then $w(zt)^{-1}$ (resp. $(zt)w$) is of one of the types 1-6 unless $w = zt$ (resp. $w=(zt)^{-1}$).</u>

<u>Proof</u>: If w is of type 1 (resp. type 3), (resp. type 5), then $w(zt)^{-1}$, then w is clearly of type 1 (resp. type 3), (resp. type 5).

If w is of type 6, then $w(zt)^{-1}$ is of type 5.

If w is of type 4, i.e. $w = (zt)^p h = (zt) \ldots (zt)h$ for a $p \in \mathbb{N}$ and a $h \in H-A$, then

$$w(zt)^{-1} = (zt) \ldots (zt)h\, t^{-1}z^{-1}.$$

If $h \in H-B$, then $w(zt)^{-1}$ is of type 1.
Otherwise, if $h \in B-\{1\}$, let $\tilde{h} = \Phi^{-1}(h) = tht^{-1} \in A-\{1\}$. Then (*) implies that $z\tilde{h}z^{-1} \in H-A$, which gives that $w(zt)^{-1}$ is of type 6 if $p = 1$ or of type 4 otherwise.
If w is of type 2, then $w(zt)^{-1}$ is of type 2 unless $w = zt$.
We can proceed in the same way for $(zt)w$.

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Lemma 3: Let $r = zt \in G$. For $m \in \mathbb{N} \cup \{0\}$, let $P(m)$ be the following assertion: for all $g \in G-\{1\}$, such that $|g| = m$, one has that $r^{m+1} g\, r^{-(m+1)}$ is of one of the types 1-6. Then $P(m)$ is true for all $m \in \mathbb{N} \cup \{0\}$.

Proof:

i) Let $g \in H-\{1\}$. Then $rgr^{-1} = ztgt^{-1}z^{-1}$ is of type 1 if $g \in H-B$. If $g \in B$, then set $\tilde{g} = \Phi^{-1}(g) = tgt^{-1} \in A-\{1\}$; (*) implies that $z\tilde{g}z^{-1} \in H-A$, which gives that rgr^{-1} is of type 6. Thus $P(0)$ is true.

ii) Suppose $g \in G-\{1\}$ has a normal form $g = k_0 t\, k_1$, so

$$r^2gr^{-2} = zt\ zt\ \underline{k_0 t\ k_1} t^{-1} z^{-1} t^{-1} z^{-1}.$$

If $k_1 \in H-B$, then r^2gr^{-2} is clearly of type 1.
Suppose so that $k_1 \in B$ and set $\tilde{k}_1 = \Phi^{-1}(K_1) = tk_1t^{-1} \in A$.
Define then $g' = k_0 t\ k_1 t^{-1} z^{-1} = k_0 \tilde{k}_1 z^{-1} \in H$.
If $g' = 1$ then $r^2gr^{-2} = zt$, i.e. of type 2, else as in i). we obtain that $zt\ g't^{-1}z^{-1}$ is of type 1 or of type 6 which gives that $r^2gr^{-2} = ztzt\ g't^{-1}z^{-1}$ is of type 1 or of type 4.
If we suppose so that $g \in G-\{1\}$ has a normal form $g = k_0 t^{-1} k_1$, we can proceed in the same way and obtain that r^2gr^{-2} is either of type 1, type 3 of type 5.
Thus $P(1)$ is true.

iii) Suppose $P(m)$ is true for $m \in \mathbb{N}$ and let $g \in G-\{1\}$, $|g| = m+1$, have a normal form $g = K_0 t^{\varepsilon_1} k_1 \ldots k_m t^{\varepsilon_{m+1}} k_{m+1}$. We consider

$$r^{m+2} g\, r^{-(m+2)} = r^m ztzt\, \overbrace{k_0 t^{\varepsilon_1} k_1 \cdots k_m t^{\varepsilon_{m+1}} k_{m+1}}^{g}\, t^{-1} z^{-1} t^{-1} z^{-1} r^{-m}.$$

If there is no cancellation in forming the product $(zt)g$ and in forming the product $g(t^{-1}z^{-1})$, then $r^{m+2} g\, r^{-(m+2)}$ is clearly of type 1.

- If there is cancellation in forming $(zt)g$, that is we have $\varepsilon_1 = -1$ and $k_0 \in B$, then set $\tilde{k}_0 = \Phi^{-1}(k_0) = tk_0t^{-1} \in A$ and define $k' = ztk_0t^{-1}k_1 = z\tilde{k}_0 k_1 \in H$. Furthermore, define $g' = k' t^{\varepsilon_2} \ldots t^{\varepsilon_{m+1}} k_{m+1} = (zt)g \in G$, so $|g'| = m$ and $g' \neq 1$. Now we can use that $P(m)$ is supposed to be true to obtain that $r^{m+1} g' r^{-(m+1)}$ is of one of the types 1-6. Since $|g| \geqslant 2$ and $r^{m+2} g\, r^{-(m+2)} = (r^{m+1} g' r^{-(m+1)})(zt)^{-1}$, it follows from Lemma 2 that $r^{m+2} g\, r^{-(m+2)}$ is of one of the types 1-6.
- If there is cancellation in forming $g(t^{-1}z^{-1})$, we can define $g'' = g(t^{-1}z^{-1})$ and proceed in the same way.

Thus we have shown that $P(m+1)$ is true and the proof of the lemma is achieved by induction.

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<u>Lemma 4</u>: <u>Let F be a finite subset of $G-\{1\}$ and let $j = 1+\max|f|$. Then $(zt)^j f (zt)^{-j}$ is of one of the types 1-6, for all $f \in F$.</u>

<u>Proof</u>: If $f = (zt)^p$ (resp. $f=(zt)^{-p}$) $\in F$ for a $p \in \mathbb{N}$, then $(zt)^j f (zt)^{-j}$ is obviously of type 2 (resp. type 3). Otherwise, the result follows easily from Lemma 3 and repeated use of Lemma 2.

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Let now F be a finite subset of $G-\{1\}$ and $N \in \mathbb{N}$. Define $j = \max_{f \in F} |f|$ and $b_\ell = (tz)^\ell z^{-1}(st)^j$ for all $\ell \in \{1,\dots,N\}$.

Let so $y \in G-S_\ell$ for an $\ell \in \{1,\dots,N\}$ and $f \in F$. We have that

$$c_\ell(b_\ell f b_\ell^{-1} y) = (st)^j f (st)^{-j} c_\ell y.$$

Since $y \in G-S_\ell$ then either $c_\ell y \in A$ or $c_\ell y$ has a normal form $c_\ell y = h_0 t k_1 \dots t^{\varepsilon_n} k_n$ where $k_n \in A$. On the other hand, Lemma 4 states that $(st)^j f (st)^{-j}$ is of one of the types 1–6. So in forming the product $((st)^j f (st)^{-j})(c_\ell y)$ there are twelve possibilities and for each it is elementary to check that there can be no cancellation. This gives that $c_\ell(b_\ell f b_\ell^{-1} y)$ has a normal form

$$c_\ell(b_\ell f b_\ell^{-1} y) = h_0' t^{\delta_1} \dots t^{\delta_m} h_m' \quad \text{where } h_0' \in H-A.$$

Thus $c_\ell(b_\ell f b_\ell^{-1} y) \in E$, i.e. $b_\ell f b_\ell^{-1} y \in S_\ell$.
That means we have shown that (**) is satisfied.

To prove the second assertion of the theorem, we first observe that G is I.C.C., which follows as in [1]. Next, by [6], it is enough to show that the following two statements are true:

i) $E \cup (z^{-1}Ez) = G-\{1\}$.

ii) E, $\alpha E \alpha^{-1}$ and $\beta E \beta^{-1}$ are pairwise disjoint where $\alpha = tzt^{-1}$, $\beta = tzsz^{-1}t^{-1} \in G$.

Let $g \in G-\{1\}$, $g \notin E$.
If $g \in A-\{1\}$ then $zgz^{-1} \in H-A$ by (*), so $zgz^{-1} \in E$.
i.e. $g \in (z^{-1}Ez)$.
Else if g has a normal form $g = h_0 t^{\varepsilon_1} \dots t^{\varepsilon_n} h_n$ where $h_0 \in A$ and $\varepsilon_1 = 1$, then zgz^{-1} has a normal form $zgz^{-1} = (zh_0) t^{\varepsilon_1} \dots t^{\varepsilon_n}(h_n z^{-1})$, so $zgz^{-1} \in E$ since $zh_0 \in H-A$, i.e. $g \in (z^{-1}Ez)$.
Thus i) is proved.

Let now $g \in E$, so g has a normal form $g = h_0 t^{\varepsilon_1} \dots t^{\varepsilon_n} h_n$ where $h_0 \in H-A$, or $h_0 \in A$ and $\varepsilon_1 = -1$.

We want to show that

$$\alpha g \alpha^{-1},\ \beta g \beta^{-1},\ \beta^{-1} \alpha g \alpha^{-1} \beta \notin E$$

which will imply that

$$(\alpha E \alpha^{-1}) \cap E = \emptyset,\ (\beta E \beta^{-1}) \cap E = \emptyset \text{ and } (\alpha E \alpha^{-1}) \cap (\beta E \beta^{-1}) = \emptyset,$$

i.e. ii) is true.

The clue here is to observe that, since α, β and $\beta^{-1}\alpha = (tzt^{-1}z^{-1}t^{-1})(tzt^{-1}) = tzt^{-1}t^{-1}$ all three "end" with t^{-1}, there can be no cancellation in forming αg, βg or $(\beta^{-1}\alpha)g$. Furthermore, the cancellations that may occur in $\alpha g \alpha^{-1}$, $\beta g \beta^{-1}$ or $\beta^{-1}\alpha g \alpha^{-1}\beta$ will always stop before "eating up" the whole thing. Since α, β and $\beta^{-1}\alpha$ all three "begin" with t, so will $\alpha g \alpha^{-1}$, $\beta g \beta^{-1}$ and $\beta^{-1}\alpha g \alpha^{-1}\beta$ and we are done.

As a sample, we show that $\beta g \beta^{-1} \notin E$.

If $g = h_0 \in H-A$, then $\beta g \beta^{-1}$ has a normal form $\beta g \beta^{-1} = tztz^{-1}t^{-1}h_0 tzt^{-1}z^{-1}t^{-1}$, so $\beta g \beta^{-1} \notin E$.

Suppose $|g| = n \geq 1$.

We have that

$$\beta g \beta^{-1} = tztz^{-1}t^{-1}h_0 t^{\varepsilon_1} h_1 \dots h_{n-1} t^{\varepsilon_n} h_n tzt^{-1}z^{-1}t^{-1}.$$

As pointed out above, since $h_0 \in H-A$, or $h_0 \in A$ and $\varepsilon_1 = -1$, there can be no cancellation in forming βg.

If there is no cancellation in forming $g\beta^{-1}$ either, then clearly $\beta g \beta^{-1} \in E$.

Otherwise, we must have that $\varepsilon_n = -1$ and $h_n \in A$, so let $k'_n = \phi(h_n) = t^{-1}h_n t$ and $h'_n = h_{n-1}k'_n z$.

[Before going further it may be helpful to have the following picture in mind:

<u>n=1</u> $\quad tztz^{-1}t^{-1}\,\overbrace{h_0 t^{\varepsilon_1} h_1}\, tzt^{-1}z^{-1}t^{-1}$

h_0'

<u>n=2</u> $\quad tztz^{-1}t^{-1}\,\overbrace{h_0 t^{\varepsilon_1} h_1 t^{\varepsilon_2} h_2}\, tzt^{-1}z^{-1}t^{-1}$

h_1'

h_0'

<u>n≥3</u> $\quad tztz^{-1}t^{-1}\,\overbrace{h_0 t^{\varepsilon_1} h_1 \cdots h_{n-3} t^{\varepsilon_{n-2}} h_{n-2} t^{\varepsilon_{n-1}} h_{n-1} t^{\varepsilon_n} h_n}\, tzt^{-1}z^{-1}t^{-1}$

h'_{n-1}

h'_{n-2}

h'_{n-3}

The arcs are ment to indicate what can be drawn together if there is cancellation at the actual step.]

If $n = 1$ then $\beta g\beta^{-1}$ has a normal form

$\beta g\beta^{-1} = tztz^{-1}t^{-1}h_0't^{-1}z^{-1}t^{-1}$.

Suppose now $n \geq 2$.

If there is no cancellation in forming $(t^{\varepsilon_{n-1}}h'_{n-1})t^{-1}$ then

$\beta g\beta^{-1}$ has a normal form

$\beta g\beta^{-1} = tztz^{-1}t^{-1}h_0 t^{\varepsilon_1}\ldots t^{\varepsilon_{n-1}}h'_{n-1}t^{-1}z^{-1}t^{-1}$. Otherwise we must have that $\varepsilon_{n-1} = 1$ and $h'_{n-1} \in B$, so let $k'_{n-1} = \phi^{-1}(h'_{n-1}) = th'_{n-1}t^{-1}$ and $h'_{n-1} = h_{n-2}k'_{n-1}z^{-1}$.

If $n = 2$ then $\beta g\beta^{-1}$ has a normal form $\beta g\beta^{-1} = tztz^{-1}t^{-1}h_0't^{-1}$.

Suppose at last $n \geq 3$.

If there is no cancellation in forming $(t^{\varepsilon_{n-2}}h'_{n-2})t^{-1}$ then $\beta g\beta^{-1}$ has a normal form

$$\beta g\beta^{-1} = tztz^{-1}t^{-1}h_0 t^{\varepsilon_1}\ldots t^{\varepsilon_{n-2}}h'_{n-2}t^{-1}.$$

Otherwise we must have $\varepsilon_{n-2} = 1$ and $h'_{n-2} \in B$, so let $k'_{n-2} = \phi^{-1}(h'_{n-2}) = th'_{n-2}t^{-1}$ and $h'_{n-3} = h_{n-3}k'_{n-2}$. Then $\beta g\beta^{-1}$ has a normal form

$$\beta g\beta^{-1} = tztz^{-1}t^{-1}h'_0 \quad \text{if} \quad n = 3 \quad \text{or}$$

$$\beta g\beta^{-1} = tztz^{-1}t^{-1}h_0t^{\varepsilon_1}\cdots t^{\varepsilon_{n-3}}h'_{n-3} \quad \text{if} \quad n > 3.$$

Thus in all the possible cases that may occur, we see that $\beta g\beta^{-1} \notin E$.

[The corresponding pictures for $\alpha g\alpha^{-1}$ and $\beta^{-1}\alpha g\alpha^{-1}\beta$ are as follows:

For $\alpha g\alpha^{-1}$:

<u>n=1</u>

$$tzt^{-1}\, h_0t^{\varepsilon_1}h_1\, tz^{-1}t^{-1}$$

<u>n≥2</u>

$$tzt^{-1}\, h_0t^{\varepsilon_1}\cdots h_{n-2}t^{\varepsilon_{n-1}}h_{n-1}t^{\varepsilon_n}h_n\, tz^{-1}t^{-1}\ .$$

For $\beta^{-1}\alpha g\alpha^{-1}\beta$:

<u>n=1</u>

$$tzt^{-1}t^{-1}\, h_0t^{\varepsilon_1}h_1\, ttz^{-1}t^{-1}$$

<u>n=2</u>

$$tzt^{-1}t^{-1}\, h_0t^{\varepsilon_1}h_1t^{\varepsilon_2}h_2\, ttz^{-1}t^{-1}$$

<u>n≥3</u>

$$tzt^{-1}t^{-1}\, h_0t^{\varepsilon_1}\cdots h_{n-3}t^{\varepsilon_{n-2}}h_{n-2}t^{\varepsilon_{n-1}}h_{n-1}t^{\varepsilon_n}h_n\, ttz^{-1}t^{-1}\ \Big].$$

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(End of the proof of the theorem)

Remarks:

1) We are also able to prove that the cooncIusion of the theorem is true if we replace $(*)$ by

$(*')$ $(zAz^{-1}) \cap B = \{1\}$, $z \notin A$ and $z \notin B$.

Since the proof follows roughly the same lines as in the above, we just mention the following: Given a finite subset F of $G-\{1\}$, then one can show that $(zt)^j g(zt)^{-j}$ has a normal form of one of the types $zt\ldots t^{-1}z^{-1}$, $(zt)^p = zt\ldots zt$ or $(zt)^{-p} = t^{-1}z^{-1}\ldots t^{-1}z^{-1}$ where $j = 2+\max_{f \in F}|f|$; one then defines E, c_ℓ, Z_ℓ and b_ℓ as before (with $j=2+\max|f|$); for the last part one defines α and β as before and one shows that $G-\{1\} = E \cup ((ztz)^{-1}E(ztz))$.

2) A consequence of the theorem is that, invoquing [7,Prop.1.6], any group which may be described as in the theorem contains no non-trivial amenable normal subgroup. This generalizes a result of Karrass and Solitar in [4] where they show that a group having a presentation with at least 3 generators and a single defining relation, contains no non-trivial abelian normal subgroup.

3) A group G is called a Powers group in [2] if it satisfies the following property:
Given a finite subset $F \subset G-\{1\}$ and $N \in \mathbb{N}$, there exist a partition $G = Y \amalg Z$ and elements $b_1,\ldots,b_N$ in G such that

a) $fY \cap Y = \emptyset$ for all $f \in F$

b) $b_K Z \cap b_\ell Z = \emptyset$ for all $k,\ell = 1,\ldots,N$ with $k \neq \ell$.

Clearly a Powers group is a group possessing Powers property (back to old notation with $Z_\ell = b_\ell Z$).

Let now G be a group given as in the theorem. We indicate how G can be seen to be a Powers group. With the same notation as in the proof of the theorem, given a finite subset $F \subset G-\{1\}$ one defines $Z = \{g \in G \mid (zt)^j g \in E\}$ where $j = 1+\max_{f\in F}|f|$,

$Y = G-Z$ and $b_\ell = (tz)^\ell t^{-1}(zt)^j$ for $\ell = 1,2,\ldots$.

Then a) follows easily from Lemma 4.
Further, if $\ell = k+n$, where $\ell,k,n \in \mathbb{N}$, then $b_k^{-1}b_\ell = (zt)^{-j}t(tz)^n t^{-1}(zt)^j$, and thus $(zt)^j b_k^{-1}b_\ell g = t(tz)^n t^{-1}(zt)^j g$, $g \in G$. So, if $g \in Z$, i.e. $(zt)^j g \in E$, there is no cancellation in forming the product $t^{-1}((zt)^j g)$. This gives that $(zt)^j(b_k^{-1}b_\ell g) \notin E$, i.e. $b_k^{-1}b_\ell g \notin Z$, and b) follows.

4) If $G = H \underset{A}{*} K$ is an amalgam possessing a blocking pair for A in one of the factors of G, we can also show, using a result of [1], that G is a Powers group.
Suppose $\{x_1,x_2\}$ is a blocking pair for A in K and $\alpha \in H-A$. Set $r = \alpha x_1$ and $s = \alpha x_2$. Given a finite subset $F \subset G-\{1\}$, define $Z = \{g \in G \mid sr^j g$ has a normal form which begins with an element of $H-A\}$ where $j = 1+\max_{f\in F}|f|$, $Y = G-Z$ and $b_\ell = r^\ell s^2 r^j$, $\ell = 1,2,\ldots$.

Then a) follows now from [1, Lemma 2] without difficulty.
Further, if $\ell = k+n$, where $\ell,k,n \in N$, then $b_k^{-1}b_\ell g = (x_2^{-1}x_1)\alpha r^{n-1} s\, sr^j g$, $g \in G$. So, if $g \in Z$, i.e. g has a normal form $g = g_1 \ldots g_m$ where $g_1 \in H-A$, we see that $sr^j(b_k^{-1}b_\ell g)$ has a normal form which begins with $x_2^{-1} \in K-A$ (since $\{x_1,x_2\}$ is a blocking pair for A in K), so $b_k^{-1}b_\ell g \notin Z$ and b) follows.

5) Using a more geometrical approach, P. de la Harpe has obtained in [2] some results which are nearly related to those obtained in this note. He also gives other examples of Powers groups.

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