Homogeneous spaces
B. Komrakov seminar

SUBALGEBRAS OF LOW CODIMENSION IN REDUCTIVE LIE ALGEBRAS

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Foreword

This preprint is a result of the first part of the work devoted to classification of three- and four-dimensional homogeneous spaces.

We consider classification of lower-dimensional homogeneous spaces as an immediate continuation and global version of classification results obtained by Sophus Lie. Two-dimensional homogeneous spaces were classified locally over the field $\mathbb{C}$ by Sophus Lie [L1] and globally by G.D. Mostow [M]. (The complete classification of two-dimensional homogeneous spaces can also be found in [KTD].) Sophus Lie also obtained the classification of maximal effective subalgebras of codimension 3 and classifications of various classes of subalgebras in $\mathfrak{gl}(3,\mathbb{C})$ and $\mathfrak{gl}(4,\mathbb{C})$ [L2].

The problem of classification of three- and four-dimensional homogeneous spaces (even in the simplest case—locally over $\mathbb{C}$) is an extremely difficult one. Quite a large subclass of three-dimensional homogeneous spaces was described in [KT].

The classification of maximal (and moreover, of primitive and almost primitive) effective subalgebras in Lie algebras over $\mathbb{R}$ and $\mathbb{C}$ was completed by B. Komrakov [K1–K4]. There one can also find the history of the question. All maximal subalgebras in reductive Lie algebras can be extracted from these works. But since we are interested in all subalgebras of low codimension (which, certainly, include all maximal ones), we develop the techniques introduced in works of B. Komrakov so that they are suitable to the solution of our problem. We describe these methods from the very beginning. Thus the present work can be read independently.

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We shall make use of the following notation:

k is a field of zero characteristic;

$\mathfrak{g}$ a nonzero finite-dimensional Lie algebra over k;

$\mathrm{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$;

$\mathfrak{r}(\mathfrak{g})$ the radical (or equivalently, the largest solvable ideal) of $\mathfrak{g}$;

$\mathfrak{s}(\mathfrak{g})$ the nilpotent radical of $\mathfrak{g}$, that is the intersection of kernels of all finite-dimensional representations of $\mathfrak{g}$: $\mathfrak{s}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{r}(\mathfrak{g})] = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r}(\mathfrak{g})$; let us remark that $\mathfrak{s}(\mathfrak{g})$ is equal to the intersection of all maximal ideals in $\mathfrak{g}$;

$\mathcal{M}(\mathfrak{g})$ the set of all maximal subalgebras of $\mathfrak{g}$;

$\varphi(\mathfrak{g})$ the Frattini ideal of $\mathfrak{g}$, i.e., the largest ideal in $\mathfrak{g}$ that is contained in each maximal subalgebra of $\mathfrak{g}$;

$\mathcal{A}(\mathfrak{g}) = \{\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g}) | \mathfrak{s}(\mathfrak{g}) \not\subset \mathfrak{g}_0\} = \{\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g}) | \mathfrak{s}(\mathfrak{g}) + \mathfrak{g}_0 = \mathfrak{g}\}$;

$\mathcal{L}(\mathfrak{g})$ the set of all maximal submodules of the $\mathfrak{g}$-module $\mathfrak{s}(\mathfrak{g})$;

$\mathcal{B}(\mathfrak{g}) = \{\mathfrak{m} \in \mathcal{L}(\mathfrak{g}) | [\mathfrak{g}, \mathfrak{s}(\mathfrak{g})] \not\subset \mathfrak{m}\} = \{\mathfrak{m} \in \mathcal{L}(\mathfrak{g}) | [\mathfrak{g}, \mathfrak{s}(\mathfrak{g})] + \mathfrak{m} = \mathfrak{s}(\mathfrak{g})\}$;

$Z(\mathfrak{g}_0)$ the centralizer of a subalgebra $\mathfrak{g}_0$ in $\mathfrak{g}$;

$N(\mathfrak{g}_0)$ the normalizer of a subalgebra $\mathfrak{g}_0$ in $\mathfrak{g}$;

$\mu(\mathfrak{g}_0) = \mu(\mathfrak{g}, \mathfrak{g}_0)$ the largest ideal in $\mathfrak{g}$ contained in the subalgebra $\mathfrak{g}_0$.

All the references to be made are to the book "*Lie Groups and Lie Algebras*" by N. Bourbaki [Bou].

Let us recall some definitions. A *special automorphism* of $\mathfrak{g}$ is an automorphism of $\mathfrak{g}$ of the form $\exp \operatorname{ad} x$, where $x \in \mathfrak{s}(\mathfrak{g})$.

Any subalgebra of $\mathfrak{g}$ complementary to the radical $\mathfrak{r}(\mathfrak{g})$ is called a *Levi subalgebra* of $\mathfrak{g}$. Any two Levi subalgebras of $\mathfrak{g}$ can be mapped into each other by means of special automorphisms (Chapter I, §6, no. 8, Theorem 5).

A *Cartan subalgebra* is a nilpotent subalgebra that coincides with its own normalizer. Any two Cartan subalgebras of a solvable Lie algebra can be mapped into each other by means of special automorphisms (Chapter VII, §3, no. 4, Theorem 3). If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{h} + [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (Chapter VII, §2, no. 1, Corollary 3 to Proposition 4).

Let $\mathfrak{p}$ be a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of the Lie algebra $Z(\mathfrak{p}) \cap \mathfrak{r}(\mathfrak{g})$. The subalgebra $\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{h}$ is called a *Levi–Cartan subalgebra* of $\mathfrak{g}$. Any two Levi–Cartan subalgebras of $\mathfrak{g}$ can be mapped into each other by means of special automorphisms. This immediately follows from the similar property for Levi subalgebras and Cartan subalgebras of solvable Lie algebras.

Suppose $\mathfrak{h}$ is a finite-dimensional nilpotent Lie algebra over k and V is a finite-dimensional $\mathfrak{h}$-module. Set

$$V^0(\mathfrak{h}) = \{v \in V \mid h^i.v = 0 \quad \text{for all } h \in \mathfrak{h} \text{ and sufficiently large } i \in \mathbb{N}\},$$

$$V^+(\mathfrak{h}) = \sum_{h \in \mathfrak{h}} \Big(\bigcap_{i \in \mathbb{N}} h^i.V \Big).$$

The $\mathfrak{h}$-module V is a direct sum of the submodules $V^0(\mathfrak{h})$ and $V^+(\mathfrak{h})$, and moreover

$$\mathfrak{h}.V^+(\mathfrak{h}) = V^+(\mathfrak{h}).$$

The decomposition

$$V = V^0(\mathfrak{h}) \oplus V^+(\mathfrak{h})$$

is called the *Fitting decomposition* of the $\mathfrak{h}$-module V (Chapter VII, §1, no. 1, Corollary 2 to Theorem 1).

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then

$$\mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$$

(Chapter VII, §2, no. 1, Proposition 4).

1.1. Lemma. *Suppose* $\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{h}$ *is a Levi–Cartan subalgebra of* $\mathfrak{g}$*; then*

$$Z(\mathfrak{p}) \subset \mathfrak{r}(\mathfrak{g});$$

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h} \oplus \big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})];$$

$$\mathfrak{r}(\mathfrak{g}) = \mathfrak{h} \oplus \big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})];$$

$$\mathfrak{s}(\mathfrak{g}) = \big([\mathfrak{h}, \mathfrak{h}] + \mathfrak{h} \cap [\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})]\big) \oplus \big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})].$$

If $\mathfrak{a}$ *is a solvable ideal in* $\mathfrak{g}$*, then*

$$\mathfrak{a} = (\mathfrak{h} \cap \mathfrak{a}) \oplus \big(\big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) \cap \mathfrak{a}\big) \oplus \big([\mathfrak{p}, \mathfrak{r}(\mathfrak{g})] \cap \mathfrak{a}\big).$$

Proof. a) The $\mathfrak{p}$-module $\mathfrak{g}$ is semisimple (Chapter I, §6, no. 8, Corollary 2 to Theorem 5, and no. 2, Theorem 2), and $\mathfrak{a} = \big(Z(\mathfrak{p}) \cap \mathfrak{a}\big) \oplus [\mathfrak{p}, \mathfrak{a}]$ (Chapter I, §3, no. 5, Proposition 6), and also

$$[\mathfrak{p}, \mathfrak{a}] = \big[\mathfrak{p}, [\mathfrak{p}, \mathfrak{a}]\big],$$

$$\big[Z(\mathfrak{p}), [\mathfrak{p}, \mathfrak{a}]\big] = \big[\mathfrak{p}, [Z(\mathfrak{p}), \mathfrak{a}]\big] \subset [\mathfrak{p}, \mathfrak{a}],$$

$$\mathfrak{g} = Z(\mathfrak{p}) \oplus [\mathfrak{p}, \mathfrak{g}] = Z(\mathfrak{p}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})] \oplus [\mathfrak{p}, \mathfrak{p}] = \big(Z(\mathfrak{p}) \cap \mathfrak{r}(\mathfrak{g})\big) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})] \oplus \mathfrak{g}.$$

Consequently $Z(\mathfrak{p}) \subset \mathfrak{r}(\mathfrak{g})$ and $\mathfrak{a} = \big(Z(\mathfrak{p}) \cap \mathfrak{a}\big) \oplus \big([\mathfrak{p}, \mathfrak{r}(\mathfrak{g})] \cap \mathfrak{a}\big)$. Furthermore

$$\big(Z(\mathfrak{p}) \cap \mathfrak{a}\big)^0(\mathfrak{h}) = \big(Z(\mathfrak{p})\big)^0(\mathfrak{h}) \cap \mathfrak{a} = \mathfrak{h} \cap \mathfrak{a},$$

$$\big(Z(\mathfrak{p}) \cap \mathfrak{a}\big)^+(\mathfrak{h}) = \big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) \cap \mathfrak{a}$$

(Chapter VII, §1, no. 1, Corollary 2 to Theorem 1), which implies

$$Z(\mathfrak{p}) \cap \mathfrak{a} = (\mathfrak{h} \cap \mathfrak{a}) \oplus \big(\big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) \cap \mathfrak{a}\big).$$

b) $$\mathfrak{s}(\mathfrak{g}) = \big[\mathfrak{p} \oplus Z(\mathfrak{p}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})], Z(\mathfrak{p}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})]\big] = [Z(\mathfrak{p}), Z(\mathfrak{p})] + [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})] + [\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})],$$

$$[Z(\mathfrak{p}),\ Z(\mathfrak{p})] = \big[\mathfrak{h} \oplus \big(Z(\mathfrak{p})\big)^+(\mathfrak{h}), \mathfrak{h} \oplus \big(Z(\mathfrak{p})\big)^+(\mathfrak{h})\big] = [\mathfrak{h}, \mathfrak{h}] + \big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) + [\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})],$$

(since $\big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) = \big[\mathfrak{h}, \big(Z(\mathfrak{p})(\mathfrak{h})\big)\big] \subset \mathfrak{s}(\mathfrak{g})$),

$$\mathfrak{s}(\mathfrak{g}) = \big([\mathfrak{h}, \mathfrak{h}] \oplus (Z(\mathfrak{p})^+(\mathfrak{h}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})]\big) + [\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})] = \\ = \big([\mathfrak{h}, \mathfrak{h}] + \mathfrak{h} \cap [\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})]\big) \oplus \big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})]$$

(for $\big[\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})\big] \subset \big(\mathfrak{h} \cap [\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})]\big) \oplus \big(Z(\mathfrak{p})\big)^+(\mathfrak{h}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})]\big)$.

1.2. Lemma. *Suppose* $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$; *then* $\mu(\mathfrak{g}_0) = \{0\}$ *if and only if one of the following conditions holds:*

(*i*) $\mathfrak{g}$ *is simple and* $\mathfrak{g}_0$ *is a maximal subalgebra of* $\mathfrak{g}$;
(*ii*) $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ *and* $\mathfrak{g}_0 = \{a + \varphi(a) \mid a \in \mathfrak{a}\}$, *where* $\varphi\colon \mathfrak{a} \to \mathfrak{b}$ *is an isomorphism of simple ideals of* $\mathfrak{g}$;
(*iii*) $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}_0$, $\mathfrak{a}$ *being a minimal commutative ideal in* $\mathfrak{g}$ *which is, at the same time, a faithful simple* $\mathfrak{g}_0$*-module*

If $\mathfrak{g}_0 \in \mathcal{A}(\mathfrak{g})$ *and* $\mu(\mathfrak{g}_0) = \{0\}$, *then condition* (*iii*) *is satisfied, and also* $\mathfrak{g}_0 \neq \{0\}$, $\mathfrak{s}(\mathfrak{g}) = \mathfrak{a}$, *and* $\mathfrak{g}_0$ *is a Levi–Cartan subalgebra of* $\mathfrak{g}$.

Proof. It is clear that subalgebras $\mathfrak{g}_0$ specified in the lemma are maximal in $\mathfrak{g}$ and $\mu(\mathfrak{g}_0) = \{0\}$.

Conversely, suppose that $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$, $\mu(\mathfrak{g}_0) = \{0\}$, and $\mathfrak{a}$ is a minimal ideal in $\mathfrak{g}$. Then $\mathfrak{a} + \mathfrak{g}_0 = \mathfrak{g}$ and either $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$ or $[\mathfrak{a}, \mathfrak{a}] = \{0\}$.

a) If $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$, then the ideal $\mathfrak{a}$ is simple (for it cannot contain proper characteristic ideals of $\mathfrak{g}$), and its centralizer $\mathfrak{b}$ is an ideal complementary to $\mathfrak{a}$: $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ (Chapter I, §5, no. 5, Corollary 2 to Proposition 5; §1, no. 4, Proposition 2 and Proposition 3; §6, no. 2, Proposition 2 and no. 1, Corollary 3 to Proposition 1).

If $\mathfrak{g}_0 \cap \mathfrak{a} \neq \{0\}$, then $N(\mathfrak{g}_0 \cap \mathfrak{a})$ is different from $\mathfrak{g}$, and contains $\mathfrak{g}_0 + \mathfrak{b}$; therefore $\mathfrak{b} = \{0\}$ and $\mathfrak{g} = \mathfrak{a}$.

If $\mathfrak{g}_0 \cap \mathfrak{a} = \{0\}$, then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{b}$, $\mathfrak{g}_0 \neq \{0\}$ (since $\dim \mathfrak{g} \geqslant 3$), $\mathfrak{b} \neq \{0\}$, and for every $b \in \mathfrak{b}$ there exists a unique $\psi(b) \in \mathfrak{a}$ such that $\psi(b) + b \in \mathfrak{g}_0$. It is clear that the mapping $\psi\colon \mathfrak{b} \to \mathfrak{a}$ is a homomorphism of Lie algebras. It is injective (since $\ker \psi \subset \mu(\mathfrak{g}_0)$) and surjective (since $\mathfrak{g}_0 \subset \psi(\mathfrak{b}) \oplus \mathfrak{b}$, $\mathfrak{g}_0 \neq \psi(\mathfrak{b}) \oplus \mathfrak{b}$, and therefore $\psi(\mathfrak{b}) \oplus \mathfrak{b} = \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$). Let $\varphi = \psi^{-1}$; then $\mathfrak{g}_0 = \{a + \varphi(a) \mid a \in \mathfrak{a}\}$.

b) If $[\mathfrak{a}, \mathfrak{a}] = \{0\}$, then $\mathfrak{a} + \mathfrak{g}_0 \subset N(\mathfrak{g}_0 \cap \mathfrak{a})$, $\mathfrak{g}_0 \cap \mathfrak{a} \subset \mu(\mathfrak{g}_0)$, and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}_0$. Since the kernel

$$\{x \in \mathfrak{g}_0 \mid [x, \mathfrak{a}] = \{0\}\}$$

of the $\mathfrak{g}_0$-module $\mathfrak{a}$ lies in $\mu(\mathfrak{g}_0)$, we see that the module is simple and faithful.

c) If $\mathfrak{g}_0 \in \mathcal{A}(\mathfrak{g})$ and $\mu(\mathfrak{g}_0) = \{0\}$, then $\mathfrak{s}(\mathfrak{g}) \neq \{0\}$, which is possible only if condition (*iii*) is satisfied. If $\mathfrak{g}_0 = \{0\}$, then $\mathfrak{g} = \mathfrak{a}$ and $\mathfrak{s}(\mathfrak{g}) = \{0\}$, which leads to a contradiction. Thus $\mathfrak{g}_0$ is a nonzero reductive Lie algebra (Chapter I, §6, no. 4, Proposition 5). Consequently $\mathfrak{r}(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{r}(\mathfrak{g}_0)$ and $\mathfrak{s}(\mathfrak{g}) = [\mathfrak{a} \oplus \mathfrak{g}_0, \mathfrak{a} \oplus \mathfrak{r}(\mathfrak{g}_0)] = [\mathfrak{g}_0, \mathfrak{a}] = \mathfrak{a}$.

Let $\mathfrak{p}$ be a Levi subalgebra of $\mathfrak{g}_0$, then

$$\mathfrak{g}_0 = \mathfrak{p} \oplus \mathfrak{r}(\mathfrak{g}_0),$$

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{r}(\mathfrak{g}_0) \oplus \mathfrak{a} = \mathfrak{p} \oplus \mathfrak{r}(\mathfrak{g}),$$

that is $\mathfrak{p}$ is a Levi subalgebra of $\mathfrak{g}$. If $\mathfrak{p} \neq \{0\}$ then $[\mathfrak{p}, \mathfrak{r}(\mathfrak{g})] = \mathfrak{a} = \mathfrak{s}(\mathfrak{g})$ and $Z(\mathfrak{p}) = \mathfrak{r}(\mathfrak{g}_0)$, but if $\mathfrak{p} = \{0\}$, then $Z(\mathfrak{p}) = \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}^+(\mathfrak{g}_0)$. Therefore, $\mathfrak{g}_0$ is a Levi–Cartan subalgebra of $\mathfrak{g}$.

1.3. Lemma.

(*i*) $[\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})] \subset \varphi(\mathfrak{g}) \subset \mathfrak{s}(\mathfrak{g})$.
(*ii*) $\mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_0 = \mathfrak{s}(\mathfrak{g}) \cap \mu(\mathfrak{g}_0)$ *for all* $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$.

(*iii*) *If* $\mathfrak{g}_0 \in \mathcal{A}(\mathfrak{g})$, *then* $\mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_0 \in \mathcal{B}(\mathfrak{g})$.

(*iv*) *Suppose* $\mathfrak{m} \in \mathcal{B}(\mathfrak{g})$, $\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{h}$ *is a Levi–Cartan subalgebra of* $\mathfrak{g}$, *and* $\mathfrak{g}_0 = \mathfrak{q} + \mathfrak{m}$. *Then* $\mathfrak{g}_0 \in \mathcal{A}(\mathfrak{g})$, $\mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_0 = \mathfrak{m}$, $\mathfrak{q} \cap \mathfrak{m} = [\mathfrak{h}, \mathfrak{h}] + \mathfrak{h} \cap [\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})]$, *and* $\mathfrak{q}$ *is a Levi–Cartan subalgebra of* $\mathfrak{g}_0$.

(*v*) *If* $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}(\mathfrak{g})$ *and* $\mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_1 = \mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_2$, *then the subalgebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *can be mapped into each other by means of special automorphisms.*

(*vi*) *Two subalgebras* $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}(\mathfrak{g})$ *are conjugate with respect to* $\operatorname{Aut}(\mathfrak{g})$ *if and only if so are the ideals* $\mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_1$, $\mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_2 \in \mathcal{B}(\mathfrak{g})$.

(*vii*) *If* $\mathcal{B}(\mathfrak{g}) \neq \emptyset$, *then*

$$\varphi(\mathfrak{g}) = \bigcap_{\mathfrak{m} \in \mathcal{B}(\mathfrak{g})} \mathfrak{m}.$$

Proof. a) It is clear that $\varphi(\mathfrak{g})$ lies in each commutative ideal of $\mathfrak{g}$. Consequently $\varphi(\mathfrak{g}) \subset \mathfrak{s}(\mathfrak{g})$.

If $\mathfrak{g}_0 \in \mathcal{A}(\mathfrak{g})$, $\mathfrak{g}' = \mathfrak{g}/\mu(\mathfrak{g}_0)$, and $\mathfrak{g}_0' = \mathfrak{g}_0/\mu(\mathfrak{g}_0)$, then

$\mathfrak{g}_0' \in \mathcal{A}(\mathfrak{g}')$,
$\mu(\mathfrak{g}_0') = \{0\}$,
$\mathfrak{s}(\mathfrak{g}')$ is a minimal ideal in $\mathfrak{g}'$,
$[\mathfrak{g}', \mathfrak{s}(\mathfrak{g}')] = \mathfrak{s}(\mathfrak{g}')$

(Chapter I, §6, no. 2, Corollary 2 to Proposition 2). Therefore

$[\mathfrak{g}, \mathfrak{s}(\mathfrak{g})] \not\subset \mu(\mathfrak{g}_0)$,
$\mathfrak{s}(\mathfrak{g}) \cap \mu(\mathfrak{g}_0) \in \mathcal{B}(\mathfrak{g})$,
$[\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})] \subset \mu(\mathfrak{g}_0)$.

Further

$$[\mathfrak{s}(\mathfrak{g}) + \mathfrak{g}_0, \mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_0] \subset [\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})] + \mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_0 = \mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_0,$$

and therefore $\mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_0 = \mathfrak{s}(\mathfrak{g}) \cap \mu(\mathfrak{g}_0)$.

b) Let $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}(\mathfrak{g})$, $\mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_1 = \mathfrak{s}(\mathfrak{g}) \cap \mathfrak{g}_2 = \mathfrak{m}$, $\mathfrak{g}' = \mathfrak{g}/\mathfrak{m}$, $\mathfrak{g}_1' = \mathfrak{g}_1/\mathfrak{m}$, and $\mathfrak{g}_2' = \mathfrak{g}_2/\mathfrak{m}$. Then $\mathfrak{g}_1'$ and $\mathfrak{g}_2'$ are Levi–Cartan subalgebras of $\mathfrak{g}'$. Suppose x' is an element of $\mathfrak{s}(\mathfrak{g}')$ such that

$$\exp \operatorname{ad} x'(\mathfrak{g}_1') = \mathfrak{g}_2',$$

and $x \in \mathfrak{s}(\mathfrak{g}) \cap x'$; then

$$\exp \operatorname{ad} x(\mathfrak{g}_1) = \mathfrak{g}_2.$$

c) Let $\mathfrak{m} \in \mathcal{B}(\mathfrak{g})$ and let $\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{h}$ be a Levi–Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{g}' = \mathfrak{g}/[\mathfrak{s}(\mathfrak{g}), \mathfrak{s}(\mathfrak{g})]$, and $\pi\colon \mathfrak{g} \to \mathfrak{g}'$ a canonical surjection. Set $\mathfrak{m}' = \pi(\mathfrak{m})$, $\mathfrak{p}' = \pi(\mathfrak{p})$, $\mathfrak{h}' = \pi(\mathfrak{h})$, and $\mathfrak{q}' = \pi(\mathfrak{q})$. Then $\mathfrak{m}' \in \mathcal{B}(\mathfrak{g}')$, and since $\mathfrak{p}' + \mathfrak{r}(\mathfrak{g}') = \mathfrak{g}'$ and $\mathfrak{p}'$ is isomorphic to $\mathfrak{p}$, we see that $\mathfrak{p}'$ is a Levi subalgebra of $\mathfrak{g}'$. Since $\pi(Z(\mathfrak{p})) \subset Z(\mathfrak{p}')$, $\pi([\mathfrak{p}, \mathfrak{r}(\mathfrak{g})]) = [\mathfrak{p}', \mathfrak{r}(\mathfrak{g}')]$, and $\pi\big(Z(\mathfrak{p}) \oplus [\mathfrak{p}, \mathfrak{r}(\mathfrak{g})]\big) = Z(\mathfrak{p}') \oplus [\mathfrak{p}', \mathfrak{r}(\mathfrak{g}')]$, we conclude that $Z(\mathfrak{p}') = \pi\big(Z(\mathfrak{p})\big)$. Furthermore, $\mathfrak{h}'$ is a Levi subalgebra of the Lie algebra $Z(\mathfrak{p}')$ (Chapter VII, §2, no. 1, Corollary 2 to Proposition 4) and, consequently, $\mathfrak{q}' = \mathfrak{p}' \oplus \mathfrak{h}'$ is a Levi–Cartan subalgebra of the Lie algebra $\mathfrak{g}'$. It is clear that

$$\mathfrak{s}(\mathfrak{g}') = [\mathfrak{h}', \mathfrak{h}'] \oplus \big(Z(\mathfrak{p}')\big)^+(\mathfrak{h}') \oplus [\mathfrak{p}', \mathfrak{r}(\mathfrak{g}')],$$

$$[\mathfrak{g}', \mathfrak{s}(\mathfrak{g}')] = [\mathfrak{h}', [\mathfrak{h}', \mathfrak{h}']] \oplus \big(Z(\mathfrak{p}')\big)^+(\mathfrak{h}') \oplus [\mathfrak{p}', \mathfrak{r}(\mathfrak{g}')],$$

$$\mathfrak{m}' = ([\mathfrak{h}',\mathfrak{h}']\cap\mathfrak{m}') \oplus \big((Z(\mathfrak{p}'))^+(\mathfrak{h}')\cap\mathfrak{m}'\big) \oplus ([\mathfrak{p}',\mathfrak{r}(\mathfrak{g}')]\cap\mathfrak{m}'),$$

and therefore

$$[\mathfrak{h}',\mathfrak{h}']\cap\mathfrak{m}' + [\mathfrak{h}',[\mathfrak{h}',\mathfrak{h}']] = [\mathfrak{h}',\mathfrak{h}'].$$

Suppose that $\mathfrak{h}'' = \mathfrak{h}'/([\mathfrak{h}',\mathfrak{h}']\cap\mathfrak{m}')$; then $[\mathfrak{h}'',[\mathfrak{h}'',\mathfrak{h}'']] = [\mathfrak{h}'',\mathfrak{h}'']$. From that the Lie algebra $\mathfrak{h}''$ is nilpotent it follows that $[\mathfrak{h}'',\mathfrak{h}''] = \{0\}$ and $[\mathfrak{h}',\mathfrak{h}'] \subset [\mathfrak{h}',\mathfrak{h}']\cap\mathfrak{m}'$. Therefore $[\mathfrak{h}',\mathfrak{h}']\cap\mathfrak{m}' = [\mathfrak{h}',\mathfrak{h}']$ and either $(Z(\mathfrak{p}'))^+(\mathfrak{h}') \not\subset \mathfrak{m}'$ or $[\mathfrak{p}',\mathfrak{r}(\mathfrak{g}')] \not\subset \mathfrak{m}'$. Consequently the subalgebra

$$\mathfrak{g}_0' = \mathfrak{q}' + \mathfrak{m}' = \mathfrak{p}' \oplus \mathfrak{h}' \oplus \big((Z(\mathfrak{p}'))^+(\mathfrak{h}')\cap\mathfrak{m}'\big) \oplus ([\mathfrak{p}',\mathfrak{r}(\mathfrak{g}')]\cap\mathfrak{m}')$$

lies in $\mathcal{A}(\mathfrak{g}')$ and $\mathfrak{s}(\mathfrak{g}')\cap\mathfrak{g}_0' = \mathfrak{m}'$. It is obvious that $\mathfrak{g}_0 = \pi^{-1}(\mathfrak{g}_0') \in \mathcal{A}(\mathfrak{g})$, $\mathfrak{s}(\mathfrak{g})\cap\mathfrak{g}_0 = \mathfrak{m}$, and $\mathfrak{q} \subset \mathfrak{g}_0$. Moreover $\mathfrak{q}'\cap\mathfrak{m}' = [\mathfrak{h}',\mathfrak{h}']$ and $\mathfrak{q}\cap\mathfrak{m} = \mathfrak{h}\cap\mathfrak{m} = [\mathfrak{h},\mathfrak{h}] + \mathfrak{h}\cap[\mathfrak{s}(\mathfrak{g}),\mathfrak{s}(\mathfrak{g})]$. Notice that the centralizer of $\mathfrak{p}$ in $\mathfrak{g}_0$ is equal to $\mathfrak{h} \oplus \big((Z(\mathfrak{p}))^+(\mathfrak{h})\cap\mathfrak{m}\big)$ and also $[\mathfrak{h}, (Z(\mathfrak{p}))^+(\mathfrak{h})\cap\mathfrak{m}] = (Z(\mathfrak{p}))^+(\mathfrak{h})\cap\mathfrak{m}$. Therefore $\mathfrak{h}$ is a Cartan subalgebra of the just mentioned centralizer.

1.4. Consider the case where $\mathfrak{g}$ is a semisimple Lie algebra.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\bar{k}$ the algebraic closure of the field k. Then the Lie algebra $\bar{\mathfrak{g}} = \bar{k}\otimes_k \mathfrak{g}$ is semisimple (Chapter I, §6, no. 10), and its subalgebra $\bar{\mathfrak{h}} = \bar{k}\oplus_k\mathfrak{h}$ is its Cartan subalgebra (Chapter VII, §2, no. 1, Proposition 3); moreover, every derivation $\operatorname{ad} h$ ($h \in \bar{\mathfrak{h}}$) is diagonalizable (Chapter VII, §2, no. 4, Theorem 2). For a linear form α on $\bar{\mathfrak{h}}$ put

$$\bar{\mathfrak{g}}^\alpha(\bar{\mathfrak{h}}) = \{x \in \bar{\mathfrak{g}} \mid [h,x] = \alpha(h)x \quad \text{for all} \quad h \in \bar{\mathfrak{h}}\}.$$

If $\alpha \neq 0$ and $\bar{\mathfrak{g}}^\alpha(\bar{\mathfrak{h}}) \neq \{0\}$, then the form α is called a *root of the Lie algebra* $\bar{\mathfrak{g}}$ *with respect to the Cartan subalgebra* $\bar{\mathfrak{h}}$. The set $R = R(\bar{\mathfrak{g}},\bar{\mathfrak{h}})$ of all such roots is called the *root system of* $\bar{\mathfrak{g}}$ *with respect to* $\bar{\mathfrak{h}}$. We have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{h}} \bigoplus \Big(\bigoplus_{\alpha\in R} \bar{\mathfrak{g}}^\alpha(\bar{\mathfrak{h}})\Big) \quad \text{and} \quad \bar{\mathfrak{g}}^0(\bar{\mathfrak{h}}) = \bar{\mathfrak{h}}$$

(Chapter VII, §1, no. 1, Proposition 3 and Theorem 1). The set $R = R(\bar{\mathfrak{g}},\bar{\mathfrak{h}})$ is a reduced root system of the space $\bar{\mathfrak{h}}^*$; in other words, 1) R generates the whole of $\bar{\mathfrak{h}}^*$, 2) $\frac{1}{2}R\cap R = \emptyset$, 3) if, for $\alpha\in R$, H_α is an element of $\bar{\mathfrak{h}}$ such that $\alpha(H_\alpha) = 2$, then the mapping $s_\alpha\colon \lambda \mapsto \lambda - \lambda(H_\alpha)\alpha$ ($\lambda \in \bar{\mathfrak{h}}^*$) maps R into itself, and also 4) $R(H_\alpha) \subset \mathbb{Z}$ (Chapter VII, §2, no. 2, Theorem 2). Each vector subspace $\bar{\mathfrak{g}}^\alpha(\bar{\mathfrak{h}})$ ($\alpha \in R$) is one-dimensional, and moreover

$$[\bar{\mathfrak{g}}^\alpha(\bar{\mathfrak{h}}), \bar{\mathfrak{g}}^{-\alpha}(\bar{\mathfrak{h}})] = \bar{k}H_\alpha \quad (\alpha \in R),$$

$$[\bar{\mathfrak{g}}^\alpha(\bar{\mathfrak{h}}), \bar{\mathfrak{g}}^{\beta}(\bar{\mathfrak{h}})] = \bar{\mathfrak{g}}^{\alpha+\beta}(\bar{\mathfrak{h}}) \quad (\alpha,\beta \in R,\ \alpha+\beta\neq 0)$$

(Chapter VIII, §2, no. 2, Theorem 1 and Proposition 4). A subset

$$B = \{\alpha_1,\dots,\alpha_l\} \subset R$$

is called a *base of the root system* R, if B is a basis of $\bar{\mathfrak{h}}^*$ and every root $\alpha \in R$ can be written as

$$\alpha = \pm(n_1\alpha_1 + \cdots + n_l\alpha_l), \quad \text{where} \quad (n_1, \dots, n_l) \in \mathbb{Z}_+^l.$$

Suppose $\widetilde{B}$ is the subset of R that contains B together with all roots $\alpha \in R$ such that $\alpha - \alpha_i \notin R$ for all $i = 1, 2, \dots, l$. Then $\widetilde{B}$ is called an *extended base* of R. To each base (extended base) of R one can assign a Dynkin diagram (extended Dynkin diagram) (Chapter VI, §4, no. 2). For a subset $B_0 \subset \widetilde{B}$, by $R(B_0)$ denote the following subset of R:

$$\left\{ \sum_{\beta \in B_0} n_\beta \beta \in R \;\middle|\; n_\beta \in \mathbb{Z} \text{ for all } \beta \in B_0 \right\}.$$

If $B_0 \subset B$, by $P(B_0)$ denote the following subset of R:

$$R(B_0) \cup \left\{ \sum_{i=0}^{l} n_i \alpha_i \in R \;\middle|\; n_i \in \mathbb{Z}_+ \text{ for all } i = 1, \dots, l \right\}.$$

It is obvious that the subspaces

$$\bar{\mathfrak{g}}(B_0) = \bar{\mathfrak{h}} \bigoplus \Big(\bigoplus_{\alpha \in R(B_0)} \bar{\mathfrak{g}}^\alpha(\bar{\mathfrak{h}}) \Big),$$

$$\bar{\mathfrak{p}}(B_0) = \bar{\mathfrak{h}} \bigoplus \Big(\bigoplus_{\alpha \in P(B_0)} \bar{\mathfrak{g}}^\alpha(\bar{\mathfrak{h}}) \Big)$$

are subalgebras of the Lie algebra $\bar{\mathfrak{g}}$. Moreover, the subalgebra $\bar{\mathfrak{g}}(B_0)$ is reductive in $\bar{\mathfrak{g}}$ (Chapter VIII, §3, no. 1, Proposition 2), and is a Levi–Cartan subalgebra of $\bar{\mathfrak{p}}(B_0)$. The subspace

$$\bar{\mathfrak{s}}(B_0) = \bigoplus_{\alpha \in P(B_0) \setminus R(B_0)} \bar{\mathfrak{g}}^\alpha(\bar{\mathfrak{h}})$$

is the nilpotent radical of the Lie algebra $\bar{\mathfrak{p}}(B_0)$. The subalgebra $\bar{\mathfrak{p}}(B_0)$ is said to be the *parabolic subalgebra of* $\bar{\mathfrak{g}}$ *associated with* $\bar{\mathfrak{h}}$, B *and* $B_0 \subset B$. A subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ is called *parabolic* if there exist a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a base B of the root system $R = R(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$, and a subset $B_0 \subset B$ such that

$$\bar{\mathfrak{g}}_0 = \bar{\mathfrak{p}}(B_0),$$

i.e., if $\bar{\mathfrak{g}}_0$ is a parabolic subalgebra of $\bar{\mathfrak{g}}$ associated with $\bar{\mathfrak{h}}$, B, and $B_0 \subset B$.

It is known that a maximal subalgebra of a semisimple Lie algebra $\mathfrak{g}$ is either reductive or parabolic in $\mathfrak{g}$ (Chapter VIII, §10, no. 1, Corollary 1 to Theorem 2).

Suppose $k = \mathbb{C}$ and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ is a mapping satisfying the following conditions:

(*i*) $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in \mathfrak{g}$,
(*ii*) $\tau(\alpha x) = \bar{\alpha}\tau(x)$ for all $\alpha \in \mathbb{C}$, $x \in \mathfrak{g}$,
(*iii*) $\tau([x, y]) = [\tau(x), \tau(y)]$ for all $x, y \in \mathfrak{g}$,
(*iv*) $\tau^2 = \mathrm{id}_{\mathfrak{g}}$.

Then τ is called an *anti-involution* of $\mathfrak{g}$, while the subset

$$\mathfrak{g}^\tau = \{x \in \mathfrak{g} | \tau(x) = x\}$$

is called the *real form of* $\mathfrak{g}$ *associated with* τ. The $\mathfrak{g}^\tau$ is, of course, a real Lie algebra, and

$$\mathfrak{g} = \mathfrak{g}^\tau \oplus \sqrt{-1}\,\mathfrak{g}^\tau \quad \text{(over } \mathbb{R}\text{)}.$$

Let now $k = \mathbb{R}$ and $\bar{k} = \mathbb{C}$. The Lie algebra $\mathfrak{g}$ can be identified with the subset $1 \otimes \mathfrak{g}$, and moreover

$$\bar{\mathfrak{g}} = \mathfrak{g} \oplus \sqrt{-1}\,\mathfrak{g} \quad \text{(over } \mathbb{R}\text{)}.$$

The mapping $\sigma\colon \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ defined by

$$\sigma(x + \sqrt{-1}\,y) = x - \sqrt{-1}\,y \quad \text{for all} \quad x, y \in \mathfrak{g}$$

is an anti-involution of $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{g}}^\sigma = \mathfrak{g}$. An endomorphism J of $\mathfrak{g}$ such that

$$\begin{aligned} &J([x,y]) = [x, J(y)] \quad \text{for all} \quad x, y \in \mathfrak{g}; \\ &J^2 = -\operatorname{id}_{\mathfrak{g}}, \end{aligned}$$

is called a *complex structure on* $\mathfrak{g}$. If a real Lie algebra $\mathfrak{g}$ possesses a complex structure J, then by $\mathfrak{g}(J)$ we denote the complex Lie algebra defined on the set $\mathfrak{g}$ by extension of the base field:

$$(a + \sqrt{-1}\,b)x = ax + bJ(x) \quad \text{for all} \quad x \in \mathfrak{g}.$$

If, in addition, τ is an anti-involution of $\mathfrak{g}(J)$, then τ is an involutorial automorphism of $\mathfrak{g}$, and the complexification $\bar{\mathfrak{g}}$ of $\mathfrak{g}$ can be written as a direct sum of the following two ideals:

$$\bar{\mathfrak{g}}^\pm = \{x \pm \sqrt{-1}J(x) \mid x \in \mathfrak{g}\}.$$

These two ideals are mapped into each other by σ and by the involution $\bar{\tau}$, where $\bar{\tau}$ is the complexification of τ. Furthermore, each of them is isomorphic to the Lie algebra $\mathfrak{g}(J)$:

$$\mathfrak{g}(J) \to \bar{\mathfrak{g}}^-,\ x \mapsto \frac{1}{2}\big(x - \sqrt{-1}J(x)\big) \quad \text{for all} \quad x \in \mathfrak{g}(J),$$

and is an eigensubspace the endomorphism $\overline{J}$, where $\overline{J}$ is the complexification of J.

Lemma.

(*i*) *If* $\bar{k} = k$ *and* $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$, *then the subalgebra* $\mathfrak{g}_0$ *is either semisimple or parabolic in* $\mathfrak{g}$.

(*ii*) *If* $k = \mathbb{R}$, $\bar{k} = \mathbb{C}$, $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$, *but* $\bar{\mathfrak{g}}_0 \notin \mathcal{M}(\bar{\mathfrak{g}})$, *then there exists a subalgebra* $\tilde{\mathfrak{g}}_0 \in \mathcal{M}(\bar{\mathfrak{g}})$ *such that* $\bar{\mathfrak{g}}_0 = \tilde{\mathfrak{g}}_0 \cap \sigma(\tilde{\mathfrak{g}}_0)$.

(*iii*) *If* $k = \mathbb{R}$, $\bar{k} = \mathbb{C}$, *and* $\mathfrak{g}$ *is simple and possesses a complex structure* J, *then the set* $\mathcal{M}(\mathfrak{g})$ *is equal to the union of the set* $\mathcal{M}\big(\mathfrak{g}(J)\big)$ *and the set of all real forms of* $\mathfrak{g}(J)$.

Proof. (*i*) Suppose that $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$ and the center $\mathfrak{z}_0$ of $\mathfrak{g}_0$ is nonzero. Then $\mathfrak{g}_0 = Z(\mathfrak{z}_0)$ and there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}_0$ and $\mathfrak{z}_0 \subset \mathfrak{h}$ (Chapter VII, §2, no. 3, Proposition 10). Set

$$R_0 = \{\alpha \in R(\mathfrak{g}, \mathfrak{h}) \mid \alpha(\mathfrak{z}_0) = \{0\}\}.$$

Then

$$\mathfrak{g}_0 = \mathfrak{h} \bigoplus \Big(\bigoplus_{\alpha \in R_0} \mathfrak{g}^{\alpha}(\mathfrak{h}) \Big),$$

$$\mathfrak{z}_0 = \{h \in \mathfrak{h} | R_0(h) = \{0\}\}.$$

Let z be a nonzero element of the set

$$\mathfrak{z}_0 \cap \Big(\sum_{\alpha \in R(\mathfrak{g},\mathfrak{h})} \mathbb{Z} H_\alpha \Big);$$

then the subspace

$$\mathfrak{h} \bigoplus \Big(\bigoplus_{\alpha(z) \geqslant 0} \mathfrak{g}^{\alpha}(\mathfrak{h}) \Big)$$

is a parabolic subalgebra of $\mathfrak{g}$ that contains $\mathfrak{g}_0$ and is different from $\mathfrak{g}$ (Chapter VI, §1, no. 7, Definition 4 and Proposition 20 when $P = \{\alpha \in R \mid \alpha(z) \geqslant 0\}$). We are led to a contradiction.

(*ii*) This statement is clear, since there exists a subalgebra $\mathfrak{g}_1 \subset \mathfrak{g}$ such that $\bar{\mathfrak{g}}_1 = \tilde{\mathfrak{g}}_0 \cap \sigma(\tilde{g}_0)$.

(*iii*) Let $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$ and let $\bar{\mathfrak{g}}_0^{\pm}$ be a projection of $\bar{\mathfrak{g}}_0$ to $\bar{\mathfrak{g}}^{\pm}$ parallel to $\bar{\mathfrak{g}}^{\mp}$.

a) If $\bar{\mathfrak{g}}_0^{+} \neq \bar{\mathfrak{g}}^{+}$, then

$$\bar{\mathfrak{g}}_0 = (\bar{\mathfrak{g}}_0^{+} \oplus \bar{\mathfrak{g}}^{-}) \cap \sigma(\bar{\mathfrak{g}}_0^{+} \oplus \bar{\mathfrak{g}}^{-}) = (\bar{\mathfrak{g}}_0^{+} \oplus \bar{\mathfrak{g}}^{-}) \cap (\bar{\mathfrak{g}}^{+} \oplus \sigma(\bar{\mathfrak{g}}_0^{+})) = \bar{\mathfrak{g}}_0^{+} \oplus \sigma(\bar{\mathfrak{g}}_0^{+})$$

and

$$\bar{\mathfrak{g}}_0^{-} = \sigma(\bar{\mathfrak{g}}_0^{+}) \neq \bar{\mathfrak{g}}^{-}.$$

It is clear that

$$\bar{\mathfrak{g}}_0^{\pm} \in \mathcal{M}(\bar{\mathfrak{g}}^{\pm}), \qquad \overline{J}(\bar{\mathfrak{g}}_0) = \bar{\mathfrak{g}}_0,$$

$$J(\mathfrak{g}_0) = \overline{J}(\bar{\mathfrak{g}}_0 \cap \mathfrak{g}) = \bar{J}(\bar{\mathfrak{g}}_0) \cap J(\mathfrak{g}) = \bar{\mathfrak{g}}_0 \cap \mathfrak{g} = \mathfrak{g}_0,$$

and also $\mathfrak{g}_0(J|\mathfrak{g}_0) \in \mathcal{M}\big(\mathfrak{g}(J)\big)$.

b) We now show that if $\bar{\mathfrak{g}}_0^{+} = \bar{\mathfrak{g}}^{+}$, then $\bar{\mathfrak{g}}^{+} \cap \bar{\mathfrak{g}}_0 = \{0\}$. We have

$$[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^{+} \cap \bar{\mathfrak{g}}_0] = [\bar{\mathfrak{g}}_0 + \bar{\mathfrak{g}}^{-}, \bar{\mathfrak{g}}^{+} \cap \bar{\mathfrak{g}}_0] = [\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}^{+} \cap \bar{\mathfrak{g}}_0] \subset \bar{\mathfrak{g}}^{+} \cap \bar{\mathfrak{g}}_0.$$

If $\bar{\mathfrak{g}}^{+} \cap \bar{\mathfrak{g}}_0 \neq \{0\}$, then $\bar{\mathfrak{g}}^{+} \cap \bar{\mathfrak{g}}_0 = \bar{\mathfrak{g}}^{+}$, $\bar{\mathfrak{g}}^{+} \subset \bar{\mathfrak{g}}_0$, and $\bar{\mathfrak{g}}^{-} = \sigma(\bar{\mathfrak{g}}^{+}) \subset \sigma(\bar{\mathfrak{g}}_0) = \bar{\mathfrak{g}}_0$, i.e., $\bar{\mathfrak{g}}_0 = \bar{\mathfrak{g}}$ and $\mathfrak{g}_0 = \bar{\mathfrak{g}}$, which yields a contradiction. Thus $\bar{\mathfrak{g}}^{+} \cap \bar{\mathfrak{g}}_0 = \{0\}$ and $\bar{\mathfrak{g}}^{-} \cap \bar{\mathfrak{g}}_0 = \sigma(\bar{\mathfrak{g}}^{+} \cap \bar{\mathfrak{g}}_0) = \{0\}$. Therefore, for any $x \in \bar{\mathfrak{g}}^{+}$ there exists a unique $\varphi(x) \in \bar{\mathfrak{g}}^{-}$ such that $x + \varphi(x) \in \bar{\mathfrak{g}}_0$; in addition, the mapping $\varphi\colon \bar{\mathfrak{g}}^{+} \to \bar{\mathfrak{g}}^{-}$ is an isomorphism of Lie algebras and $\bar{\mathfrak{g}}_0 = \{x + \varphi(x) \mid x \in \bar{\mathfrak{g}}^{+}\}$. It remains to note that $\bar{\mathfrak{g}}_0 + \overline{J}(\bar{\mathfrak{g}}_0) = \bar{\mathfrak{g}}$ and $\bar{\mathfrak{g}}_0 \cap \overline{J}(\bar{\mathfrak{g}}_0) = \{0\}$; therefore $\mathfrak{g} = \mathfrak{g}_0 \oplus J(\mathfrak{g}_0)$ (since if $\mathfrak{g}_0 \cap J(\mathfrak{g}_0) \neq \{0\}$ then $\bar{\mathfrak{g}}_0 \cap \overline{J}(\bar{\mathfrak{g}}_0) \neq \{0\}$, and besides, $\mathfrak{g}_0 + J(\mathfrak{g}_0)$ is a subalgebra of $\mathfrak{g}$).

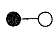
c) If $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g}(J)) \backslash \mathcal{M}(\mathfrak{g})$, then let $\mathfrak{g}_0' \in \mathcal{M}(\mathfrak{g})$ be a subalgebra containing $\mathfrak{g}_0$. Clearly $\mathfrak{g}_0' \notin \mathcal{M}(\mathfrak{g}(J))$; but then $\mathfrak{g} = \mathfrak{g}_0' \oplus J(\mathfrak{g}_0')$, which contradicts the conditions $\mathfrak{g}_0 \subset \mathfrak{g}_0' \cap J(\mathfrak{g}_0')$ and $\mathfrak{g}_0 \neq \{0\}$.

d) If $\mathfrak{g}_0$ is a real form of the Lie algebra $\mathfrak{g}(J)$ such that $\mathfrak{g}_0 \notin \mathcal{M}(\mathfrak{g})$, then let $\mathfrak{g}_0' \in \mathcal{M}(\mathfrak{g})$ denote a subalgebra containing $\mathfrak{g}_0$. It is clear that $\mathfrak{g}_0'$ cannot be a real form of $\mathfrak{g}(J)$; but then $J(\mathfrak{g}_0') = \mathfrak{g}_0'$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus J(\mathfrak{g}_0) \subset \mathfrak{g}_0'$, and we are again led to a contradiction.

1.5. Lemma. *Suppose $k = \mathbb{C}$, $\mathfrak{g}$ is simple, $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$, and $\operatorname{codim} \mathfrak{g}_0 \leqslant 4$. Then there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ lying in $\mathfrak{g}_0$, and the root system $R = R(\mathfrak{g}, \mathfrak{h})$ has a base B such that*

(*i*) *if $\mathfrak{g}_0$ is parabolic, then $\mathfrak{g}_0 = \mathfrak{p}(B_0)$ for some subset $B_0 \subset B$, and the pair $(\mathfrak{g}, \mathfrak{g}_0)$ can be given by one of the following Dynkin diagrams:*

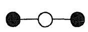
 $\operatorname{codim} \mathfrak{g}_0 = 1;$

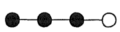
 $\operatorname{codim} \mathfrak{g}_0 = 2;$

 $\operatorname{codim} \mathfrak{g}_0 = 3;$

 $\operatorname{codim} \mathfrak{g}_0 = 3;$

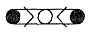
 $\operatorname{codim} \mathfrak{g}_0 = 3;$

 $\operatorname{codim} \mathfrak{g}_0 = 4;$

 $\operatorname{codim} \mathfrak{g}_0 = 4;$

(black vertices on the diagram of B correspond to roots lying in B_0);

(*ii*) *if $\mathfrak{g}_0$ is semisimple, then $\mathfrak{g}_0 = \mathfrak{g}(B_0)$ for some subset $B_0 \subset \widetilde{B}$, and the pair $(\mathfrak{g}, \mathfrak{g}_0)$ can be given by the following Dynkin diagram of the extended base $\widetilde{B}$:*

 $\operatorname{codim} \mathfrak{g}_0 = 4,$

where the black vertices correspond to the roots lying in B_0.

Proof. (*i*) Let $\mathfrak{g}_0 = \mathfrak{p}(B_0)$ be a parabolic subalgebra of $\mathfrak{g}$, associated with certain $\mathfrak{h}$, B, and B_0 (see section 1.4). Then $B = B_0 \cup \{\alpha\}$ for some $\alpha \in B$ and the $\mathfrak{g}(B_0)$-module $\mathfrak{g}$ is a direct sum of the submodules $\mathfrak{g}(B_0)$, $\mathfrak{s}(B_0)$, and

$$\mathfrak{s}_*(B_0) = \bigoplus_{\alpha \in P(B_0) \backslash R(B_0)} \mathfrak{g}^{-\alpha}(\mathfrak{h}).$$

It is known that there exists an automorphism θ of the Lie algebra $\mathfrak{g}$ such that $\theta(\mathfrak{h}) = \mathfrak{h}$ and $\theta(\mathfrak{g}^{\alpha}(\mathfrak{h})) = \mathfrak{g}^{-\alpha}(\mathfrak{h})$ for all $\alpha \in R$ (Chapter VIII, §4, no. 4, Proposition 5). It is clear that $\mathfrak{s}_*(B_0) = \theta(\mathfrak{s}(B_0))$ and $\theta(\mathfrak{g}(B_0)) = \mathfrak{g}(B_0)$. Therefore the kernel of the $\mathfrak{g}(B_0)$-module $\mathfrak{s}_*(B_0)$ is also the kernel of the $\mathfrak{g}(B_0)$-module $\mathfrak{s}(B_0)$ and, consequently, an ideal in $\mathfrak{g}$. So the $\mathfrak{g}(B_0)$-module $\mathfrak{s}_*(B_0)$ is faithful and $\operatorname{codim} \mathfrak{g}_0 = \dim \mathfrak{s}_*(B_0)$. Therefore $\dim \mathfrak{g}(B_0) \leqslant 4^2 = 16$ and $\dim \mathfrak{g} \leqslant 16+4+4 = 24$.

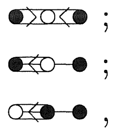
We now list Dynkin diagrams for complex Lie algebras of dimension $\leqslant 24$ and for their maximal parabolic subalgebras:

○	$\dim \mathfrak{g} = 3:$	○	$\dim \mathfrak{g}_0 = 2;$
○–○	$\dim \mathfrak{g} = 8:$	●–○	$\dim \mathfrak{g}_0 = 6;$
○⇐○	$\dim \mathfrak{g} = 10:$	●⇐○	$\dim \mathfrak{g}_0 = 7;$
		○⇐●	$\dim \mathfrak{g}_0 = 7;$
○⇚○	$\dim \mathfrak{g} = 14:$	●⇚○	$\dim \mathfrak{g}_0 = 9;$
		○⇚○	$\dim \mathfrak{g}_0 = 9;$
○–○–○	$\dim \mathfrak{g} = 15:$	●–○–●	$\dim \mathfrak{g}_0 = 11;$
		●–●–○	$\dim \mathfrak{g}_0 = 12;$
○–○⇐○	$\dim \mathfrak{g} = 21:$	●–○⇐●	$\dim \mathfrak{g}_0 = 14;$
		●–●⇐○	$\dim \mathfrak{g}_0 = 15;$
		○–●⇐●	$\dim \mathfrak{g}_0 = 16;$
○–○⇒○	$\dim \mathfrak{g} = 21:$	●–○⇒●	$\dim \mathfrak{g}_0 = 14;$
		○–○⇒○	$\dim \mathfrak{g}_0 = 15;$
		○–○⇒○	$\dim \mathfrak{g}_0 = 16;$
○–○–○–○	$\dim \mathfrak{g} = 24:$	●–●–○–●	$\dim \mathfrak{g}_0 = 18;$
		●–●–●–○	$\dim \mathfrak{g}_0 = 20.$

(*ii*) Let V be a submodule of the $\mathfrak{g}_0$-module $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus V$. It is clear that V is faithful. Therefore $\mathfrak{g}_0$ is isomorphic to some subalgebra of the Lie algebra $\mathfrak{sl}(V)$, and also $\dim \mathfrak{g}_0 \leqslant 4^2 - 1 = 15$ and $\dim \mathfrak{g} \leqslant 19$. Note that the rank of $\mathfrak{g}_0$ does not exceed that of $\mathfrak{g}$ (Chapter VII, §2, no. 2, Definition 2; no. 3, Corollary 1 to Theorem 1 and Proposition 10). Below we list Dynkin diagrams for semisimple complex Lie algebras $\mathfrak{p}$ of dimension $\leqslant 15$ and rank $\leqslant 3$ together with minimal dimensions m of faithful $\mathfrak{p}$-modules:

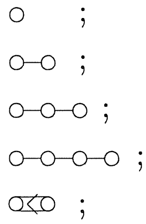
○	$\dim \mathfrak{p} = 3,\ m = 2;$
○ ○	$\dim \mathfrak{p} = 6,\ m = 4$ (○⇐○);
○–○	$\dim \mathfrak{p} = 8,\ m = 3;$
○ ○ ○	$\dim \mathfrak{p} = 9,\ m = 6;$
○⇐○	$\dim \mathfrak{p} = 10,\ m = 4$ (○⇚○);
○–○ ○	$\dim \mathfrak{p} = 11,\ m = 5;$
○⇐○ ○	$\dim \mathfrak{p} = 13,\ m = 6;$
○⇚○	$\dim \mathfrak{p} = 14,\ m = 7;$
○–○–○	$\dim \mathfrak{p} = 15,\ m = 4.$

In parentheses we indicate Dynkin diagrams of simple complex Lie algebras $\tilde{\mathfrak{g}}$ satisfying the conditions $\operatorname{rang} \tilde{\mathfrak{g}} \geqslant \operatorname{rang} \mathfrak{p}$ and $\dim \mathfrak{p} + m \leqslant \dim \tilde{\mathfrak{g}} \leqslant \dim \mathfrak{p} + 4$. It remains to list Dynkin diagrams for proper semisimple maximal rank subalgebras of these Lie algebras:



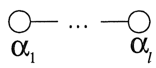
where the Dynkin diagram of the subalgebra corresponds to the subdiagram of the extended Dynkin diagram of $\tilde{\mathfrak{g}}$ formed by black vertices.

1.6. Suppose that $k = \mathbb{C}$ and that $\mathfrak{g}$ is simple and has one of the following Dynkin diagrams:



Let us describe the sets $\mathcal{M}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{g})$, and also the set $\mathbb{R}(\mathfrak{g})$ of anti-involutions of $\mathfrak{g}$.

A_l) Let V be a complex vector space of dimension $l+1$ ($l = 1, 2, 3, 4$); then the Lie algebra $\mathfrak{sl}(V)$ is a simple Lie algebra of type A_l, i.e., its Dynkin diagram has the form:



The mapping $\mathfrak{sl}(V) \to \mathfrak{sl}(V^*)$ defined by

$$x \mapsto -{}^t x \quad \text{for all} \quad x \in \mathfrak{sl}(V),$$

is a canonical isomorphism of Lie algebras

1°. By $\mathbb{R}(V)$ denote the set of all anti-involutions of V. For each $\tau \in \mathbb{R}(V)$ the subset

$$V^\tau = \{v \in V \,|\, \tau(v) = v\}$$

is a real vector space of dimension $l+1$ and

$$V = V^\tau \oplus \sqrt{-1}\, V^\tau \quad (\text{over } \mathbb{R}).$$

The mapping $\tau^* \colon V^* \to V^*$ defined by

$$\langle v, \tau^*(v^*) \rangle = \overline{\langle \tau^{-1}(v), v^* \rangle} \quad \text{for all} \quad v \in V,\ v^* \in V^*$$

lies in $\mathbb{R}(V^*)$, and moreover $(V^\tau)^*$ can be naturally identified with $(V^*)^{\tau^*}$:

$$v^* \equiv (v \mapsto \langle v, v^* \rangle) \quad \text{for all} \quad v \in V^\tau,\ v^* \in (V^*)^{\tau^*}.$$

The mapping $\mathring{\tau} \colon \mathfrak{sl}(V) \to \mathfrak{sl}(V)$ such that

$$\mathring{\tau}(x) = \tau \circ x \circ \tau^{-1} \quad \text{for all} \quad x \in \mathfrak{sl}(V)$$

is an anti-involution of $\mathfrak{sl}(V)$. We shall denote the real form $\big(\mathfrak{sl}(V)\big)^{\mathring{\tau}}$ by $\mathfrak{sl}(V,\tau)$. It is obvious that $\mathfrak{sl}(V,\tau)$ is isomorphic to the Lie algebra $\mathfrak{sl}(V^\tau)$. Notice that if $\tau_1, \tau_2 \in \mathbb{R}(V)$, then

$$\mathring{\tau}_1 = \mathring{\tau}_2 \Longleftrightarrow \tau_1 = \alpha\tau_2 \quad \text{for some} \quad \alpha \in \mathbb{T} = \{a \in \mathbb{C} \mid a\bar{a} = 1\}.$$

2°. Let $\mathbb{H}(V)$ denote the set of antilinear mappings $\mu: V \to V$ such that $\mu^2 = -\operatorname{id}_V$. It is clear that $\mathbb{H}(V) \neq \emptyset$ if and only if V is even-dimensional, since for every $\mu \in \mathbb{H}(V)$ the space V is made into the right quaternion space $V(\mu)$ of regularity $\frac{1}{2}(l+1)$. Indeed, it is convenient to consider the quaternion field as the subring

$$\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}$$

of the ring of all complex 2×2 matrices. We put

$$v \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = av + \mu(bv) \quad \text{for all} \quad a, b \in \mathbb{C},\ v \in V.$$

The mapping $\mu^*: V^* \to V^*$ defined by

$$\langle v, \mu^*(v^*)\rangle = \overline{\langle \mu^{-1}(v), v^*\rangle} \quad \text{for all} \quad v \in V,\ v^* \in V^*,$$

lies in $\mathbb{H}(V^*)$, and the mapping $\mathring{\mu}: \mathfrak{sl}(V) \to \mathfrak{sl}(V)$, such that

$$\mathring{\mu}(x) = \mu \circ x \circ \mu^{-1} \quad \text{for all} \quad x \in \mathfrak{sl}(V),$$

is an anti-involution of $\mathfrak{sl}(V)$. The real form $\big(\mathfrak{sl}(V)\big)^{\mathring{\mu}}$ is denoted by $\mathfrak{sl}(V,\mu)$. It is clear that $\mathfrak{sl}(V,\mu)$ consists of those endomorphisms of the quaternion space $V(\mu)$ whose reduced trace is equal to zero. (Recall that the reduced trace of an endomorphism x of $V(\mu)$ is the number $\frac{1}{2}\operatorname{tr} x$, x being considered as a real endomorphism of the space obtained from V by reduction of the base field to $\mathbb{R}$. It is equal to $\operatorname{tr} x$, if x is considered as a complex endomorphism of V.) Note that if $\mu_1, \mu_2 \in \mathbb{H}(V)$, then

$$\mathring{\mu}_1 = \mathring{\mu}_2 \Longleftrightarrow \mu_1 = \alpha\mu_2 \quad \text{for some} \quad \alpha \in \mathbb{T}.$$

3°. Let $\mathbb{E}(V)$ denote the set of all nondegenerate Hermitian forms on V. We identify a form $\xi \in \mathbb{E}(V)$ with the antilinear isomorphism $V \to V^*$ such that

$$\xi(v_1, v_2) = \langle v_1, \xi(v_2)\rangle \quad \text{for all} \quad v_1, v_2 \in V;$$

the mapping $\xi^*: V^* \times V^* \to \mathbb{C}$ defined by

$$\xi^*(v_1^*, v_2^*) = \overline{\xi\big(\xi^{-1}(v_1^*), \xi^{-1}(v_2^*)\big)} \quad \text{for all} \quad v_1^*, v_2^* \in V^*$$

lies in $\mathbb{E}(V^*)$, and the signature of the form ξ^* coincides with that of ξ. The mapping $\mathring{\xi}: \mathfrak{sl}(V) \to \mathfrak{sl}(V)$ such that

$$\mathring{\xi}(x) = -{}^t(\xi \circ x \circ \xi^{-1}) \quad \text{for all} \quad x \in \mathfrak{sl}(V),$$

is an anti-involution of $\mathfrak{sl}(V)$. The real form $\big(\mathfrak{sl}(V)\big)^{\overset{\circ}{\xi}}$ is denoted by $\mathfrak{sl}(V,\xi)$. Clearly

$$\mathfrak{sl}(V,\xi) = \big\{x \in \mathfrak{sl}(V) \mid \xi\big(x(v_1), v_2\big) + \xi\big(v_1, x(v_2)\big) = 0 \quad \text{for all} \quad v_1, v_2 \in V\big\}.$$

Note that if $\xi_1, \xi_2 \in \mathbb{E}(V)$, then

$$\overset{\circ}{\xi}_1 = \overset{\circ}{\xi}_2 \Longleftrightarrow \xi_1 = \alpha\xi_2 \quad \text{for some} \quad \alpha \in \mathbb{R}^*.$$

4°. For each $\varphi \in \mathrm{GL}(V)$ the mapping $\overset{\circ}{\varphi}: \mathfrak{sl}(V) \to \mathfrak{sl}(V)$ defined by

$$\overset{\circ}{\varphi}(x) = \varphi \circ x \circ \varphi^{-1} \quad \text{for all} \quad x \in \mathfrak{sl}(V)$$

is an automorphism of the Lie algebra $\mathfrak{sl}(V)$, and the mapping $\varphi^*: V^* \to V^*$ defined by

$$\langle v, \varphi^*(v^*)\rangle = \langle \varphi^{-1}(v), v^*\rangle \quad \text{for all} \quad v \in V, v^* \in V^*$$

lies in $\mathrm{GL}(V^*)$. The mapping $\varphi \mapsto \varphi^*$ is an isomorphism of the groups $\mathrm{GL}(V)$ and $\mathrm{GL}(V^*)$. Note that if $\varphi_1, \varphi_2 \in \mathrm{GL}(V)$, then

$$\overset{\circ}{\varphi}_1 = \overset{\circ}{\varphi}_2 \Longleftrightarrow \varphi_1 = \alpha\varphi_2 \quad \text{for some} \quad \alpha \in \mathbb{C}^*.$$

4.1°. If $\varphi \in \mathrm{GL}(V)$ and $\tau \in \mathbb{R}(V)$, then

$$\overset{\circ}{\varphi} \circ \overset{\circ}{\tau} = \overset{\circ}{\tau} \circ \overset{\circ}{\varphi} \Longleftrightarrow \tau \circ \varphi \circ \tau^{-1} = \alpha\varphi \quad \text{for some} \quad \alpha \in \mathbb{T}.$$

Moreover there exists a $\varphi' \in \mathrm{GL}(V)$ such that

$$\overset{\circ}{\varphi'} = \overset{\circ}{\varphi} \quad \text{and} \quad \tau \circ \varphi' \circ \tau^{-1} = \varphi'.$$

Set

$$\mathrm{GL}(V,\tau) = \{\varphi \in \mathrm{GL}(V) \mid \tau \circ \varphi \circ \tau^{-1} = \varphi\}.$$

The group $\mathrm{GL}(V,\tau)$ can be identified with the group $\mathrm{GL}(V^\tau)$ in an obvious way. Note that if $\varphi_1, \varphi_2 \in \mathrm{GL}(V,\tau)$, then

$$\overset{\circ}{\varphi}_1 = \overset{\circ}{\varphi}_2 \Longleftrightarrow \varphi_1 = \alpha\varphi_2 \quad \text{for some} \quad \alpha \in \mathbb{R}^*.$$

4.2°. If $\varphi \in \mathrm{GL}(V)$ and $\mu \in \mathbb{H}(V)$, then

$$\overset{\circ}{\varphi} \circ \overset{\circ}{\mu} = \overset{\circ}{\mu} \circ \overset{\circ}{\varphi} \Longleftrightarrow \mu \circ \varphi \circ \mu^{-1} = \alpha\varphi \quad \text{for some} \quad \alpha \in \mathbb{T}.$$

Moreover there exists a $\varphi' \in \mathrm{GL}(V)$ such that

$$\overset{\circ}{\varphi'} = \overset{\circ}{\varphi} \quad \text{and} \quad \mu \circ \varphi' \circ \mu^{-1} = \varphi'.$$

Set

$$\mathrm{GL}(V,\mu) = \{\varphi \in \mathrm{GL}(V) \mid \mu \circ \varphi \circ \mu^{-1} = \varphi\}.$$

The group $\mathrm{GL}(V,\mu)$ consists of automorphisms of the quaternion space $V(\mu)$. Note that if $\varphi_1,\varphi_2\in\mathrm{GL}(V,\mu)$, then

$$\mathring{\varphi}_1=\mathring{\varphi}_2 \Longleftrightarrow \varphi_1=\alpha\varphi_2 \quad \text{for some} \quad \alpha\in\mathbb{R}^*.$$

4.3°. If $\varphi\in\mathrm{GL}(V)$ and $\xi\in\mathbb{E}(V)$, then

$$\mathring{\varphi}\circ\mathring{\xi}=\mathring{\xi}\circ\mathring{\varphi}\Longleftrightarrow \xi\circ\varphi\circ\xi^{-1}=\alpha\varphi^* \quad \text{for some} \quad \alpha\in\mathbb{R}^*.$$

Moreover there exists a $\varphi'\in\mathrm{GL}(V)$ such that

$$\mathring{\varphi}'=\mathring{\varphi} \quad \text{and} \quad \xi\circ\varphi'\circ\xi^{-1}=\pm(\varphi')^*,$$

where the sign of $(\varphi')^*$ is uniquely determined by the sign of α. Set

$$\mathrm{GL}^{\pm}(V,\xi)=\{\varphi\in\mathrm{GL}(V)|\,\xi\circ\varphi\circ\xi^{-1}=\pm\varphi^*\}$$

and

$$\mathrm{GL}(V,\xi)=\mathrm{GL}^{+}(V,\xi)\cup\mathrm{GL}^{-}(V,\xi).$$

It can be easily seen that

$$\mathrm{GL}^{\pm}(V,\xi)=\{\varphi\in\mathrm{GL}(V)\mid \xi\big(\varphi(v_1),\varphi(v_2)\big)=\pm\xi(v_1,v_2) \quad \text{for all } v_1,v_2\in V\};$$

besides, $\mathrm{GL}^{-}(V,\xi)\neq\emptyset$ if and only if the space V is even-dimensional and the signature of the form ξ is equal to 0. Note that if $\varphi_1,\varphi_2\in\mathrm{GL}(V,\xi)$, then

$$\mathring{\varphi}_1=\mathring{\varphi}_2 \Longleftrightarrow \varphi_1=\alpha\varphi_2 \quad \text{for some} \quad \alpha\in\mathbb{T}.$$

5°. By $\mathrm{IL}(V)$ denote the set of all isomorphisms $V\to V^*$. We identify a $\psi\in\mathrm{IL}(V)$ with a nondegenerate bilinear form on V:

$$\psi(v_1,v_2)=\langle v_1,\psi(v_2)\rangle \quad \text{for all} \quad v_1,v_2\in V.$$

The mapping $\mathring{\psi}\colon\mathfrak{sl}(V)\to\mathfrak{sl}(V)$ defined by

$$\mathring{\psi}(x)=-{}^t(\psi\circ x\circ\psi^{-1}) \quad \text{for all} \quad x\in\mathfrak{sl}(V)$$

is an automorphism of the Lie algebra $\mathfrak{sl}(V)$. Note that if $\psi_1,\psi_2\in\mathrm{IL}(V)$, then

$$\mathring{\psi}_1=\mathring{\psi}_2 \Longleftrightarrow \psi_1=\alpha\psi_2 \quad \text{for some} \quad \alpha\in\mathbb{C}^*.$$

Let $\mathbf{S}(V)$ be the set of nondegenerate symmetric bilinear forms on V and let $\mathbf{\Lambda}(V)$ be the set of nondegenerate skew-symmetric bilinear forms on V. It is clear that $\mathbf{\Lambda}(V)\neq\emptyset$ if and only if V is even-dimensional.

The sets $\mathbf{S}(V)$ and $\mathbf{\Lambda}(V)$ can be identified with subsets of $\mathrm{IL}(V)$. If $\psi \in \mathrm{IL}(V)$, then

$$\mathring{\psi}^2 = \mathrm{id}_{\mathfrak{sl}(V)} \Longleftrightarrow \psi \in \mathbf{S}(V) \cup \mathbf{\Lambda}(V).$$

By $\mathfrak{sl}(V, \psi)$ denote the subalgebra of fixed points of the automorphism $\mathring{\psi}$. It can be easily seen that

$$\mathfrak{sl}(V,\psi) = \{x \in \mathfrak{sl}(V) \mid \psi(x(v_1), v_2) + \psi(v_1, x(v_2)) = 0 \quad \text{for all } v_1, v_2 \in V\}.$$

Note that if $\dim V \geqslant 3$ and $\psi \in \mathbf{S}(V) \cup \mathbf{\Lambda}(V)$, then $\mathfrak{sl}(V,\psi)$ is a semisimple maximal subalgebra in $\mathfrak{sl}(V)$.

5.1°. If $\psi \in \mathrm{IL}(V)$ and $\tau \in \mathbb{R}(V)$, then

$$\mathring{\psi} \circ \mathring{\tau} = \mathring{\tau} \circ \mathring{\psi} \Longleftrightarrow \tau^* \circ \psi \circ \tau^{-1} = \alpha\psi \quad \text{for some} \quad \alpha \in \mathbb{T}.$$

Moreover there extists a $\psi' \in \mathrm{IL}(V)$ such that

$$\mathring{\psi}' = \mathring{\psi} \quad \text{and} \quad \tau^* \circ \psi' \circ \tau^{-1} = \psi'.$$

Set

$$\mathrm{IL}(V,\tau) = \{\psi \in \mathrm{IL}(V) \mid \tau^* \circ \psi \circ \tau^{-1} = \psi\}.$$

The set $\mathrm{IL}(V,\tau)$ can be naturally identified with the set $\mathrm{IL}(V^\tau)$. Note that if $\psi_1, \psi_2 \in \mathrm{IL}(V,\tau)$, then

$$\mathring{\varphi}_1 = \mathring{\varphi}_2 \Longleftrightarrow \varphi_1 = \alpha\varphi_2 \quad \text{for some} \quad \alpha \in \mathbb{R}^*.$$

Put

$$\mathbf{S}(V,\tau) = \mathbf{S}(V) \cap \mathrm{IL}(V,\tau)$$

and

$$\mathbf{\Lambda}(V,\tau) = \mathbf{\Lambda}(V) \cap \mathrm{IL}(V,\tau).$$

If $\psi \in \mathbf{S}(V) \cup \mathbf{\Lambda}(V)$ and $\tau \in \mathbb{R}(V)$, then the subalgebra $\mathfrak{sl}(V,\psi)$ is stable under $\mathring{\tau}$ if and only if

$$\alpha\psi \in \mathbf{S}(V,\tau) \cup \mathbf{\Lambda}(V,\tau) \quad \text{for some} \quad \alpha \in \mathbb{C}^*.$$

In this case the subalgebra $\mathfrak{sl}(V,\psi) \cap \mathfrak{sl}(V,\tau)$ corresponds to $\mathfrak{sl}(V^\tau, \alpha\psi)$.

5.2°. If $\psi \in \mathrm{IL}(V)$ and $\mu \in \mathbb{H}(V)$, then

$$\mathring{\psi} \circ \mathring{\mu} = \mathring{\mu} \circ \mathring{\psi} \Longleftrightarrow \mu^* \circ \psi \circ \mu^{-1} = \alpha\psi \quad \text{for some} \quad \alpha \in \mathbb{T}.$$

Moreover there exists a $\psi' \in \mathrm{IL}(V)$ such that

$$\mathring{\psi}' = \mathring{\psi} \quad \text{and} \quad \mu^* \circ \psi' \circ \mu^{-1} = \psi'.$$

Set

$$\mathrm{IL}(V,\mu) = \{\psi \in \mathrm{IL}(V) \mid \mu^* \circ \psi \circ \mu^{-1} = \psi\}.$$

It is clear that the set $\mathrm{IL}(V,\mu)$ consists of isomorphisms of quaternion spaces $V(\mu) \to V^*(\mu^*)$. Note that if $\psi_1, \psi_2 \in \mathrm{IL}(V,\mu)$, then

$$\mathring{\psi}_1 = \mathring{\psi}_2 \Longleftrightarrow \psi_1 = \alpha\psi_2 \quad \text{for some} \quad \alpha \in \mathbb{R}^*.$$

Now set

$$\mathbf{S}(V,\mu) = \mathbf{S}(V) \cap \mathrm{IL}(V,\mu),$$
$$\mathbf{\Lambda}(V,\mu) = \mathbf{\Lambda}(V) \cap \mathrm{IL}(V,\mu).$$

If $\psi \in \mathbf{S}(V) \cup \mathbf{\Lambda}(V)$ and $\mu \in \mathbb{H}(V)$, then the subalgebra $\mathfrak{sl}(V,\psi)$ is stable under $\mathring{\mu}$ if and only if

$$\alpha\psi \in \mathbf{S}(V,\mu) \cup \mathbf{\Lambda}(V,\mu) \quad \text{for some} \quad \alpha \in \mathbb{C}^*.$$

5.3°. If $\psi \in \mathrm{IL}(V)$ and $\xi \in \mathbb{E}(V)$, then

$$\mathring{\psi} \circ \mathring{\xi} = \mathring{\xi} \circ \mathring{\psi} \Longleftrightarrow \xi^*\big(\psi(v_1), \psi(v_2)\big) = \alpha\xi(v_1, v_2) \quad \text{for some } \alpha \in \mathbb{R}^*$$

and all $v_1, v_2 \in V$. Moreover there exists a $\psi' \in \mathrm{IL}(V)$ such that

$$\mathring{\psi}' = \mathring{\psi} \quad \text{and} \quad \xi^*\big(\psi(v_1), \psi(v_2)\big) = \pm\xi(v_1, v_2) \quad \text{for all } v_1, v_2 \in V,$$

where the sign of $\xi(\cdot,\cdot)$ is uniquely determined by that of α. Set

$$\mathrm{IL}^{\pm}(V,\xi) = \{\psi \in \mathrm{IL}(V) \mid \xi^*(\psi(v_1), \psi(v_2)) = \pm\xi(v_1, v_2) \quad \text{for all } v_1, v_2 \in V\},$$
$$\mathrm{IL}(V,\xi) = \mathrm{IL}^+(V,\xi) \cup \mathrm{IL}^-(V,\xi).$$

It is clear that $\mathrm{IL}^-(V,\xi) \neq \emptyset$ if and only if V is even-dimensional and the signature of ξ equals 0. Note that if $\psi_1, \psi_2 \in \mathrm{IL}(V,\xi)$, then

$$\mathring{\psi}_1 = \mathring{\psi}_2 \Longleftrightarrow \psi_1 = \alpha\psi_2 \quad \text{for some} \quad \alpha \in \mathbb{T}.$$

Now set

$$\mathbf{S}^{\pm}(V,\xi) = \mathbf{S}(V) \cap \mathrm{IL}^{\pm}(V,\xi),$$
$$\mathbf{\Lambda}^{\pm}(V,\xi) = \mathbf{\Lambda}(V) \cap \mathrm{IL}^{\pm}(V,\xi),$$
$$\mathbf{S}(V,\xi) = \mathbf{S}^+(V,\xi) \cup \mathbf{S}^-(V,\xi),$$
$$\mathbf{\Lambda}(V,\xi) = \mathbf{\Lambda}^+(V,\xi) \cup \mathbf{\Lambda}^-(V,\xi).$$

Suppose $\psi \in \mathbf{S}(V) \cup \mathbf{\Lambda}(V)$ and $\mu \in \mathbb{E}(V)$; then the subalgebra $\mathfrak{sl}(V,\psi)$ is stable under $\mathring{\xi}$ if and only if

$$\alpha\psi \in \mathbf{S}(V,\xi) \cup \mathbf{\Lambda}(V,\xi) \quad \text{for some} \quad \alpha \in \mathbb{C}^*.$$

Let $\xi \in \mathbb{E}(V)$ and $\psi \in \mathbf{S}(V,\xi) \cup \mathbf{\Lambda}(V,\xi)$.

(++) If $\psi \in \mathbf{S}^+(V,\xi)$, then $\tau = \xi^{-1} \circ \psi \in \mathbb{R}(V)$ and there exists a $\psi_0 \in \mathbf{S}(V^\tau)$ such that the restrictions of ψ and ξ to V^τ coincide with ψ_0.

(−−) If $\psi \in \mathbf{\Lambda}^-(V,\xi)$, then $\tau = \xi^{-1} \circ \psi \in \mathbb{R}(V)$ and there exists a $\psi_0 \in \mathbf{\Lambda}(V^\tau)$ such that the restrictions of ψ and ξ to V^τ coincide with $\sqrt{-1}\psi_0$.

(+−) If $\psi \in \mathbf{S}^-(V,\xi)$, then $\mu = \xi^{-1} \circ \psi \in \mathbb{H}(V)$ and there exists a subspace W in V such that $V = W \oplus \mu(W)$, $\xi\big(W, \mu(W)\big) = \{0\}$, and the restriction of ξ to W is positive definite, while the restriction of ξ to $\mu(W)$ is negative definite. Moreover

$$\xi\big(\mu(v_1), \mu(v_2)\big) = -\overline{\xi(v_1, v_2)} \qquad \text{and}$$

$$\psi(v_1, v_2) = \xi(v_1, \mu(v_2))$$

for all $v_1, v_2 \in V$.

(−+) If $\psi \in \mathbf{\Lambda}^+(V,\xi)$, then $\mu = \xi^{-1} \circ \psi \in \mathbb{H}(V)$ and there exists a subspace W in V such that $V = W \oplus \mu(W)$, $\xi\big(W, \mu(W)\big) = \{0\}$, and the restrictions of the form ξ to W and to $\mu(W)$ are nondegenerate and have the same signature, which is half that of ξ. Moreover

$$\xi\big(\mu(v_1), \mu(v_2)\big) = \overline{\xi(v_1, v_2)} \qquad \text{and}$$

$$\psi(v_1, v_2) = \xi(v_1, \mu(v_2))$$

for all $v_1, v_2 \in V$.

Note that in any of the cases just described

$$\mathring{\psi} \circ \mathring{\xi} = \mathring{\xi} \circ \mathring{\psi} = (\xi^{-1} \circ \psi)^{\circ}.$$

6°. For every proper subspace V_0 in V the subalgebra

$$\mathfrak{sl}(V, V_0) = \{x \in \mathfrak{sl}(V) \mid x(V_0) \subset V_0\}$$

is parabolic and is maximal in $\mathfrak{sl}(V)$. The canonical isomorphism $\mathfrak{sl}(V) \to \mathfrak{sl}(V^*)$ maps it into the subalgebra $\mathfrak{sl}(V^*, {}^tV_0)$, where

$${}^tV_0 = \big\{ v^* \in V^* \mid \langle V_0, v^* \rangle = \{0\} \big\}.$$

Notice that V_0 is the only proper subspace of V which is stable with respect to $\mathfrak{sl}(V, V_0)$.

6.1°. Suppose V_0 is a proper subspace of V and $\tau \in \mathbb{R}(V)$. Then $\mathfrak{sl}(V, V_0)$ is stable under $\mathring{\tau}$ if and only if V_0 is stable under τ. It can be easily seen that the subalgebra $\mathfrak{sl}(V, V_0) \cap \mathfrak{sl}(V, \tau)$ of the Lie algebra $\mathfrak{sl}(V, \tau)$ corresponds to the subalgebra $\mathfrak{sl}(V^\tau, V_0^\tau)$ of $\mathfrak{sl}(V^\tau)$, where $V_0^\tau = V_0 \cap V^\tau$.

6.2°. Let V_0 be a proper subspace of V and $\mu \in \mathbb{H}(V)$. Then the subalgebra $\mathfrak{sl}(V, V_0)$ is stable under $\mathring{\mu}$ if and only if V_0 is stable under μ. In this case V_0 is even-dimensional, since it is a subspace of the quaternion space $V(\mu)$.

6.3°. Let again V_0 be a proper subspace of V and $\xi \in \mathbb{E}(V)$. Then the subalgebra $\mathfrak{sl}(V, V_0)$ is stable under $\overset{\circ}{\xi}$ if and only if the restriction of ξ to V_0 equals 0 and $V_0^\perp = V_0$, where

$$V_0^\perp = \{v \in V | \xi(V_0, v) = \{0\}\}.$$

This is possible only if the space V is even-dimensional, the signature of ξ is equal to 0, and $\dim V_0 = \frac{1}{2} \dim V$.

7°. Let $\tau \in \mathbb{R}(V)$ and let $\mathfrak{g}_0$ be a maximal subalgebra of the Lie algebra $\mathfrak{sl}(V, \tau)$. Assume that the $\mathfrak{g}_0$-module V^τ is simple but there exists a proper subspace V_0 in V such that $\mathfrak{g}_0 \subset \mathfrak{sl}(V, V_0)$. Then $V = V_0 \oplus \tau(V_0)$ and there exists a unique $J \in \mathrm{GL}(V, \tau)$ such that

$$J(v) = \sqrt{-1}\, v \quad \text{for all} \quad v \in V_0.$$

Moreover $J^2 = -\mathrm{id}_V,\ \ \mu = \tau \circ J = J \circ \tau \in \mathbb{H}(V)$ and

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{sl}(V, V_0) \cap \mathfrak{sl}(V, \tau) \\ &= \mathfrak{sl}(V, J) \cap \mathfrak{sl}(V, \tau) \\ &= \mathfrak{sl}(V, \mu) \cap \mathfrak{sl}(V, \tau), \end{aligned}$$

where $\mathfrak{sl}(V, J)$ is the subalgebra of fixed points of the second-order automorphism $\overset{\circ}{J}$.

8°. Let $\mu \in \mathbb{H}(V)$ and let $\mathfrak{g}_0$ be a maximal subalgebra of the Lie algebra $\mathfrak{sl}(V, \mu)$. Assume that the $\mathfrak{g}_0$-module $V(\mu)$ is simple but there exists a proper subset V_0 in V such that $\mathfrak{g}_0 \subset \mathfrak{sl}(V, V_0)$. Then $V = V_0 \oplus \mu(V_0)$ and there exists a unique $J \in \mathrm{GL}(V, \mu)$ such that

$$J(v) = \sqrt{-1}\, v \quad \text{for all} \quad v \in V_0.$$

Moreover $J^2 = -\mathrm{id}_V,\ \ \tau = \mu \circ J = J \circ \mu \in \mathbb{H}(V)$ and

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{sl}(V, V_0) \cap \mathfrak{sl}(V, \mu) \\ &= \mathfrak{sl}(V, J) \cap \mathfrak{sl}(V, \mu) \\ &= \mathfrak{sl}(V, \tau) \cap \mathfrak{sl}(V, \mu). \end{aligned}$$

7°–8°. Let $\mathfrak{g}_0$, J, and τ be define as in 7° or 8°. Then

$$\mathfrak{g}_0 = \mathfrak{sl}(V, J) \cap \mathfrak{sl}(V, \tau).$$

Suppose J^τ is the restriction of J to V^τ and $V^\tau(J^\tau)$ is the corresponding complex space. Then $\mathfrak{g}_0$ consists of linear extensions to V of those endomorphisms $y \in \mathfrak{gl}\big(V^\tau(J^\tau)\big)$ that satisfy $\operatorname{tr} y + \overline{\operatorname{tr} y} = 0$.

9°. Suppose $\xi \in \mathbb{E}(V)$, $\mathfrak{g}_0$ is a maximal subalgebra of the Lie algebra $\mathfrak{sl}(V, \xi)$, and $\mathfrak{g}_0 \subset \mathfrak{sl}(V, V_0)$ for some proper subspace V_0 in V. Assume that V_0 is a minimal subspace with this property. It is clear that

$$\mathfrak{g}_0 = \mathfrak{sl}(V, V_0) \cap \mathfrak{sl}(V, \xi) = \mathfrak{sl}(V, V_0^\perp) \cap \mathfrak{sl}(V, \xi).$$

Therefore the restriction of ξ to V_0 is either zero or nondegenerate. In the former case, $\mathfrak{g}_0$ is parabolic in $\mathfrak{sl}(V,\xi)$; in the latter case, $V = V_0 \oplus V_0^{\perp}$, $\mathfrak{g}_0$ is reductive in $\mathfrak{sl}(V,\xi)$, and in addition

$$\mathfrak{g}_0 = \mathfrak{sl}(V,L) \cap \mathfrak{sl}(V,\xi),$$

where $L \in \mathrm{GL}^{+}(V,\xi)$ and

$$L(v+u) = v - u \quad \text{for all} \quad v \in V_0,\ u \in V_0^{\perp}.$$

So we have described the sets $\mathcal{M}(\mathfrak{g})$, $\mathrm{Aut}(\mathfrak{g})$, $\mathbb{R}(\mathfrak{g})$, and $\mathcal{M}(\mathfrak{g}^{\tau})$, $\mathrm{Aut}(\mathfrak{g}^{\tau})$ for each $\tau \in \mathbb{R}(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{sl}(V)$.

C_2) Let V be a complex vector space of dimension 4 and $\psi \in \mathbf{\Lambda}(V)$. Then $\mathfrak{sl}(V,\psi)$ is a simple Lie algebra of type C_2, i.e., its Dynkin diagram has the form:

Recall that

$$\mathfrak{sl}(V,\psi) = \big\{\, x \in \mathfrak{gl}(V) \mid \psi\big(x(v_1), v_2\big) + \psi\big(v_1, x(v_2)\big) = 0 \quad \text{for all } v_1, v_2 \in V \,\big\}.$$

1°. Set

$$\mathrm{GL}(V,\psi) = \{\varphi \in \mathrm{GL}(V) \mid \psi \circ \varphi \circ \psi^{-1} = \varphi^{*}\}.$$

Every automorphism of $\mathfrak{sl}(V,\psi)$ has the form:

$$\check{\varphi} = \overset{\circ}{\varphi}\,|_{\mathfrak{sl}(V,\psi)}$$

for some $\varphi \in \mathrm{GL}(V,\psi)$. It is obvious that

$$\mathrm{GL}(V,\psi) = \{\varphi \in \mathrm{GL}(V) \mid \psi\big(\varphi(v_1), \varphi(v_2)\big) = \psi(v_1, v_2) \quad \text{for all } v_1, v_2 \in V\}.$$

Note that if $\varphi_1, \varphi_2 \in \mathrm{GL}(V,\psi)$, then

$$\check{\varphi}_1 = \check{\varphi}_2 \Longleftrightarrow \varphi_1 = \pm\varphi_2.$$

2°. Now set

$$\mathbb{R}(V,\psi) = \{\tau \in \mathbb{R}(V) \mid \psi \circ \tau \circ \psi^{-1} = \tau^{*}\},$$
$$\mathbb{H}(V,\psi) = \{\mu \in \mathbb{H}(V) \mid \psi \circ \mu \circ \psi^{-1} = \mu^{*}\}.$$

It is easily seen that

$$\mathbb{R}(V,\psi) = \{\tau \in \mathbb{R}(V) \mid \psi\big(\tau(v_1), \tau(v_2)\big) = \overline{\psi(v_1, v_2)} \quad \text{for all } v_1, v_2 \in V\},$$

$$\mathbb{H}(V,\psi) = \{\mu \in \mathbb{H}(V) \mid \psi\big(\mu(v_1), \mu(v_2)\big) = \overline{\psi(v_1, v_2)} \quad \text{for all } v_1, v_2 \in V\}.$$

If $\tau \in \mathbb{R}(V,\psi)$, then the restriction of ψ to V^{τ} lies in $\mathbf{\Lambda}(V^{\tau})$.

If $\mu \in \mathbb{H}(V, \psi)$, then $\xi = \psi \circ \mu \in \mathbb{E}(V)$, and

$$\xi\big(\mu(v_1), \mu(v_2)\big) = \overline{\xi(v_1, v_2)},$$

$$\psi(v_1, v_2) = -\xi\big(v_1, \mu(v_2)\big)$$

for all $v_1, v_2 \in V$. In addition, there exists a subspace W in V such that $V = W \oplus \mu(W)$, $\xi\big(W, \mu(W)\big) = \{0\}$, and the restrictions of ξ to W and to $\mu(W)$ are nondegenerate and have the same signature, which is half that of ξ.

Each anti-involution of $\mathfrak{sl}(V, \psi)$ has the form:

$$\check{\eta} = \mathring{\eta}|_{\mathfrak{sl}(V,\psi)} \quad \text{for some} \quad \eta \in \mathbb{R}(V, \psi) \cup \mathbb{H}(V, \psi).$$

We shall denote the real form $\big(\mathfrak{sl}(V, \psi)\big)^{\check{\eta}}$ by $\mathfrak{sl}(V, \psi, \eta)$. Note that if $\eta_1, \eta_2 \in \mathbb{R}(V, \psi) \cup \mathbb{H}(V, \psi)$, then

$$\check{\eta}_1 = \check{\eta}_2 \iff \eta_1 = \pm\eta_2.$$

3°. Every maximal subalgebra of $\mathfrak{sl}(V, \psi)$ has the form:

$$\mathfrak{sl}(V, \psi, V_0) = \mathfrak{sl}(V, \psi) \cap \mathfrak{sl}(V, V_0)$$

where V_0 is a proper subspace of V such that the restriction of ψ to V_0 is either zero or nondegenerate. Set

$$V_0^\perp = \{v \in V \mid \psi(v, V_0) = \{0\}\};$$

then $\mathfrak{sl}(V, \psi, V_0^\perp) = \mathfrak{sl}(V, \psi, V_0)$. If $V_0 \subset V_0^\perp$, then $\mathfrak{sl}(V, \psi, V_0)$ is a parabolic subalgebra, but if $V = V_0 \oplus V_0^\perp$, then $\mathfrak{sl}(V, \psi, V_0)$ is semisimple. Notice that V_0 and $V_0^\perp$ are the only proper subsets of V stable with respect to $\mathfrak{sl}(V, \psi, V_0)$.

4°. Let V_0 be a proper subspace of V such that the restriction of ψ to V_0 is either zero or nondegenerate, and let $\eta \in \mathbb{R}(V, \psi) \cup \mathbb{H}(V, \psi)$. The subalgebra $\mathfrak{sl}(V, \psi, V_0)$ is stable under $\mathring{\eta}$ if and only if η preserves the set $\{V_0, V_0^\perp\}$.

5°. Suppose $\tau \in \mathbb{R}(V, \psi)$ and $\mathfrak{g}_0$ is a maximal subalgebra of the Lie algebra $\mathfrak{sl}(V, \psi, \tau)$. Now assume that the $\mathfrak{g}_0$-module V^τ is simple, but there exists a proper subspace V_0 in V such that $\mathfrak{g}_0 \subset \mathfrak{sl}(V, V_0)$. Then $V = V_0 \oplus \tau(V_0)$ and there exists a unique $J \in \mathrm{GL}(V, \tau)$ such that

$$J(v) = \sqrt{-1}\, v \quad \text{for all} \quad v \in V_0.$$

In addition, $J^2 = -\mathrm{id}_V$, $\mu = \tau \circ J = J \circ \tau \in \mathbb{H}(V)$, and

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{sl}(V, V_0) \cap \mathfrak{sl}(V, \psi, \tau) \\ &= \mathfrak{sl}(V, J) \cap \mathfrak{sl}(V, \psi, \tau) \\ &= \mathfrak{sl}(V, \mu) \cap \mathfrak{sl}(V, \psi, \tau). \end{aligned}$$

In a similar way we construct the $J^\perp \in \mathrm{GL}(V, \tau)$ such that

$$J^\perp(v) = \sqrt{-1}\, v \quad \text{for all} \quad v \in V_0^\perp.$$

It can be easily verified that

$$\psi\big(v_1, J^{\perp}(v_2)\big) = -\psi\big(J(v_1), v_2\big) \quad \text{for all} \quad v_1, v_2 \in V.$$

Therefore, $J+J^{\perp} \in \mathfrak{sl}(V,\psi,\tau)$ and $[J+J^{\perp},\mathfrak{g}_0] = \{0\}$, and consequently, $J+J^{\perp} \in \mathfrak{g}_0$ and V_0 is stable with respect to $J^{\perp}$, which implies that $J \circ J^{\perp} = J^{\perp} \circ J$ and $(J \circ J^{\perp})^2 = \mathrm{id}_V$. Note that the restriction of $J \circ J^{\perp}$ to V^{τ} is an automorphism of the $\mathfrak{g}_0$-module V^{τ}. It follows that $J \circ J^{\perp} = \pm\,\mathrm{id}_V$.

If $J \circ J^{\perp} = -\,\mathrm{id}_V$, then $J^{\perp} = J \in \mathrm{GL}(V,\psi)$, $V_0^{\perp} = V_0$, $\mu \in \mathbb{H}(V,\psi)$, and

$$\mathfrak{g}_0 = \mathfrak{sl}(V,\psi,\mu) \cap \mathfrak{sl}(V,\psi,\tau).$$

If $J \circ J^{\perp} = \mathrm{id}_V$, then $J^{\perp} = -J$, $\sqrt{-1}J \in \mathrm{GL}(V,\psi)$, $V_0^{\perp} = \tau(V_0)$, and

$$\mathfrak{g}_0 = \mathfrak{sl}(V,\sqrt{-1}J) \cap \mathfrak{sl}(V,\psi,\tau),$$

and also, the restriction of ψ to V_0 is nondegenerate.

6°. Let $\mu \in \mathbb{H}(V,\psi)$ and let $\mathfrak{g}_0$ be a maximal subalgebra of the Lie algebra $\mathfrak{sl}(V,\psi,\mu)$. Now assume that the $\mathfrak{g}_0$-module $V(\mu)$ is simple and there exists a proper subspace V_0 in V such that $\mathfrak{g}_0 \subset \mathfrak{sl}(V,V_0)$. Then $V = V_0 \oplus \mu(V_0)$ and there exists a unique $J \in \mathrm{GL}(V,\mu)$ such that

$$J(v) = \sqrt{-1}\,v \quad \text{for all} \quad v \in V_0.$$

In addition, $J^2 = -\,\mathrm{id}_V$, $\tau = \mu \circ J = J \circ \mu \in \mathbb{R}(V)$, and

$$\begin{aligned}\mathfrak{g}_0 &= \mathfrak{sl}(V,V_0) \cap \mathfrak{sl}(V,\psi,\mu)\\ &= \mathfrak{sl}(V,J) \cap \mathfrak{sl}(V,\psi,\mu)\\ &= \mathfrak{sl}(V,\tau) \cap \mathfrak{sl}(V,\psi,\mu).\end{aligned}$$

In a similar way we construct the $J^{\perp} \in \mathrm{GL}(V,\mu)$ such that

$$J^{\perp}(v) = \sqrt{-1}\,v \quad \text{for all} \quad v \in V_0^{\perp}.$$

It is easy to see that

$$\psi\big(v_1, J^{\perp}(v_2)\big) = -\psi\big(J(v_1), v_2\big) \quad \text{for all} \quad v_1, v_2 \in V.$$

Therefore $J + J^{\perp} \in \mathfrak{sl}(V,\psi,\mu)$ and $[J + J^{\perp},\mathfrak{g}_0] = \{0\}$. It follows that $J + J^{\perp} \in \mathfrak{g}_0$ and that V_0 is stable with respect to $J^{\perp}$, and so $J \circ J^{\perp} = J^{\perp} \circ J$ and $(J \circ J^{\perp})^2 = \mathrm{id}_V$. Note that $J \circ J^{\perp}$ is an automorphism of the $\mathfrak{g}_0$-module $V(\mu)$. Therefore, $J \circ J^{\perp} = \pm\,\mathrm{id}_V$.

If $J \circ J^{\perp} = -\,\mathrm{id}_V$, then $J^{\perp} = J \in \mathrm{GL}(V,\psi)$, $V_0^{\perp} = V_0$, $\tau \in \mathbb{R}(V,\psi)$, and

$$\mathfrak{g}_0 = \mathfrak{sl}(V,\psi,\tau) \cap \mathfrak{sl}(V,\psi,\mu).$$

If $J \circ J^{\perp} = \mathrm{id}_V$, then $J^{\perp} = -J$, $\sqrt{-1}J \in \mathrm{GL}(V, \psi)$, $V_0^{\perp} = \mu(V_0)$,

$$\mathfrak{g}_0 = \mathfrak{sl}(V, \sqrt{-1}J) \cap \mathfrak{sl}(V, \psi, \mu),$$

and the restriction of ψ to V_0 is nondegenerate.

5°–6°. Let now $\mathfrak{g}_0$, J, $J^{\perp}$, and τ be defined as in 5° or 6°. Then for $J = J^{\perp}$,

$$\mathfrak{g}_0 = \mathfrak{sl}(V, J) \cap \mathfrak{sl}(V, \psi, \tau).$$

Suppose J^{τ} is the restriction of J to V^{τ} and $V^{\tau}(J^{\tau})$ is the corresponding complex space. Then the mapping $\xi : V^{\tau}(J^{\tau}) \times V^{\tau}(J^{\tau}) \to \mathbb{C}$ defined by

$$\xi(v_1, v_2) = \psi\big(J(v_1), v_2\big) + \sqrt{-1}\,\psi(v_1, v_2) \quad \text{for all} \quad v_1, v_2 \in V^{\tau},$$

lies in $\mathbb{E}\big(V^{\tau}(J^{\tau})\big)$, and $\mathfrak{g}_0$ consists of linear extensions to V of those endomorphisms $y \in \mathfrak{gl}\big(V^{\tau}(J^{\tau})\big)$ which satisfy the condition

$$\xi\big(y(v_1), v_2\big) + \xi\big(v_1, y(v_2)\big) = 0 \quad \text{for all} \quad v_1, v_2 \in V^{\tau}.$$

7°. If $\eta \in \mathbb{R}(V, \psi) \cup \mathbb{H}(V, \psi)$ and $\varphi \in \mathrm{GL}(V, \psi)$, then

$$\check{\eta} \circ \check{\varphi} = \check{\varphi} \circ \check{\eta} \Longleftrightarrow \eta \circ \varphi \circ \eta^{-1} = \pm\varphi.$$

Put

$$\mathrm{GL}^{\pm}(V, \psi, \eta) = \{\varphi \in \mathrm{GL}(V, \psi) \mid \eta \circ \varphi \circ \eta^{-1} = \pm\varphi\},$$

and

$$\mathrm{GL}(V, \psi, \eta) = \mathrm{GL}^{+}(V, \psi, \eta) \cup \mathrm{GL}^{-}(V, \psi, \eta).$$

Each automorphism of the Lie algebra $\mathfrak{sl}(V, \psi, \eta)$ can be written as the restriction of some automorphism $\check{\varphi}$, where $\varphi \in \mathrm{GL}(V, \psi, \eta)$, to $\mathfrak{sl}(V, \psi, \eta)$.

1.7. Lemma. *Suppose $k = \mathbb{R}$, $\mathfrak{g}$ is simple, $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$, and $\operatorname{codim} \mathfrak{g}_0 \leqslant 4$. Then the pair $(\mathfrak{g}, \mathfrak{g}_0)$ can be written in one and only one of the forms:*

[1,1] $\quad \mathfrak{g} = \mathfrak{sl}(2; \mathbb{R})$,

$$\mathfrak{g}_0 = \mathfrak{st}(2; \mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} \middle| \, \alpha, \beta \in \mathbb{R} \right\};$$

[2,1] $\quad \mathfrak{g} = \mathfrak{sl}(2; \mathbb{R})$,

$$\mathfrak{g}_0 = \mathfrak{so}(2; \mathbb{R}) = \left\{ \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} \middle| \, \alpha \in \mathbb{R} \right\};$$

[2,2]

$$\mathfrak{g} = \mathfrak{su}(2) = \left\{ \begin{pmatrix} \sqrt{-1}\alpha & a \\ -\bar{a} & -\sqrt{-1}\alpha \end{pmatrix} \middle| \, \alpha \in \mathbb{R},\ a \in \mathbb{C} \right\},$$

$$\mathfrak{g}_0 = \mathfrak{su}_0(2) = \left\{ \begin{pmatrix} \sqrt{-1}\alpha & 0 \\ 0 & -\sqrt{-1}\alpha \end{pmatrix} \middle| \, \alpha \in \mathbb{R} \right\};$$

[2,3]
$$\mathfrak{g} = \mathfrak{sl}(2;\mathbb{C}),$$
$$\mathfrak{g}_0 = \mathfrak{st}(2;\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \middle| \, a,b \in \mathbb{C} \right\};$$

[2,4]
$$\mathfrak{g} = \mathfrak{sl}(3;\mathbb{R}),$$
$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \middle| \, A \in \mathfrak{gl}(2;\mathbb{R}),\ B \in \mathbb{R}^2 \right\};$$

[3,1]
$$\mathfrak{g} = \mathfrak{sl}(2;\mathbb{C}),$$
$$\mathfrak{g}_0 = \mathfrak{sl}(2;\mathbb{R});$$

[3,2]
$$\mathfrak{g} = \mathfrak{sl}(2;\mathbb{C}),$$
$$\mathfrak{g}_0 = \mathfrak{su}(2);$$

[3,3]
$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b & \sqrt{-1}\alpha \\ c & \bar{a}-a & -\bar{b} \\ \sqrt{-1}\beta & -\bar{c} & -\bar{a} \end{pmatrix} \middle| \, \alpha,\beta \in \mathbb{R};\ a,b,c \in \mathbb{C} \right\},$$
$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & b & \sqrt{-1}\alpha \\ 0 & \bar{a}-a & -\bar{b} \\ 0 & 0 & -\bar{a} \end{pmatrix} \middle| \, \alpha \in \mathbb{R}; a,b \in \mathbb{C} \right\};$$

[3,4]
$$\mathfrak{g} = \mathfrak{sc}(4;\mathbb{R})$$
$$= \left\{ \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \sigma & \tau & \nu & -\gamma \\ \mu & \varepsilon & -\tau & \beta \\ \rho & -\mu & \sigma & -\alpha \end{pmatrix} \middle| \, \alpha,\beta,\gamma,\delta,\sigma,\tau,\nu,\mu,\varepsilon,\rho \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} \alpha E + A & B \\ C & -\alpha E + A \end{pmatrix} \middle| \, \alpha \in \mathbb{R};\ A,B,C \in \mathfrak{sl}(2;\mathbb{R}) \right\},$$
$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \tau & \nu & -\gamma \\ 0 & \varepsilon & -\tau & \beta \\ 0 & 0 & 0 & -\alpha \end{pmatrix} \middle| \, \alpha,\beta,\gamma,\delta,\tau,\nu,\varepsilon \in \mathbb{R} \right\};$$

[3,5]
$$\mathfrak{g} = \mathfrak{sc}(4,\mathbb{R}),$$
$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \sigma & \tau & \nu & -\gamma \\ 0 & 0 & -\tau & \beta \\ 0 & 0 & \sigma & -\alpha \end{pmatrix} \middle| \, \alpha,\beta,\gamma,\delta,\sigma,\tau,\nu \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} \alpha E + A & B \\ 0 & -\alpha E + A \end{pmatrix} \middle| \, \alpha \in \mathbb{R};\ A,B \in \mathfrak{sl}(2;\mathbb{R}) \right\};$$

[3,6] $\mathfrak{g} = \mathfrak{sc}(2,\mathbb{H})$

$$= \left\{ \begin{pmatrix} a & b & \sqrt{-1}\alpha & c \\ -\bar{b} & \bar{a} & -\bar{c} & -\sqrt{-1}\alpha \\ \sqrt{-1}\beta & d & -\bar{a} & b \\ -\bar{d} & -\sqrt{-1}\beta & -\bar{b} & -a \end{pmatrix} \middle| \begin{array}{c} \alpha,\beta \in \mathbb{R}; \\ a,b,c,d \in \mathbb{C} \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} \alpha E + A & B \\ C & -\alpha E + A \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A,B,C \in \mathfrak{su}(2) \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha E + A & B \\ 0 & -\alpha E + A \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A,B \in \mathfrak{su}(2) \right\};$$

[3,7] $\mathfrak{g} = \mathfrak{sl}(4,\mathbb{R})$,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \operatorname{tr} A & B \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{gl}(3,\mathbb{R}),\ B \in \mathbb{R}^3 \right\};$$

[4,1] $\mathfrak{g} = \mathfrak{su}(3)$,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \sqrt{-1}\alpha & a & 0 \\ -\bar{a} & \sqrt{-1}\beta & 0 \\ 0 & 0 & -\sqrt{-1}(\alpha+\beta) \end{pmatrix} \middle| \alpha,\beta \in \mathbb{R};\ a \in \mathbb{C} \right\};$$

[4,2] $\mathfrak{g} = \mathfrak{su}(2,1)$

$$= \left\{ \begin{pmatrix} \sqrt{-1}\alpha & a & b \\ -\bar{a} & \sqrt{-1}\beta & c \\ \bar{b} & \bar{c} & -\sqrt{-1}(\alpha+\beta) \end{pmatrix} \middle| \alpha,\beta \in \mathbb{R};\ a,b,c \in \mathbb{C} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \sqrt{-1}\alpha & a & 0 \\ -\bar{a} & \sqrt{-1}\beta & 0 \\ 0 & 0 & -\sqrt{-1}(\alpha+\beta) \end{pmatrix} \middle| \alpha,\beta \in \mathbb{R};\ a \in \mathbb{C} \right\};$$

[4,3] $\mathfrak{g} = \mathfrak{su}(2,1)$,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\sqrt{-1}(\beta+\gamma) & 0 & 0 \\ 0 & \sqrt{-1}\beta & c \\ 0 & \bar{c} & \sqrt{-1}\gamma \end{pmatrix} \middle| \beta,\gamma \in \mathbb{R};\ c \in \mathbb{C} \right\};$$

[4,4]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & \alpha E + B \\ -\alpha E + B & C \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A,B,C \in \mathfrak{sl}(2,\mathbb{R}) \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \middle| A,C \in \mathfrak{sl}(2,\mathbb{R}) \right\};$$

[4,5]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & \sqrt{-1}\alpha E + B \\ -\sqrt{-1}\alpha E + B & \bar{A} \end{pmatrix} \middle| \begin{array}{c} \alpha \in \mathbb{R},\ A \in \mathfrak{sl}(2,\mathbb{C}), \\ B \in \mathfrak{sl}(2,\mathbb{R}) \end{array} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \middle| A \in \mathfrak{sl}(2,\mathbb{C}) \right\};$$

[4,6]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & \alpha E + \sqrt{-1}B \\ -\alpha E + \sqrt{-1}B & \bar{A} \end{pmatrix} \middle| \begin{array}{c} \alpha \in \mathbb{R},\ A \in \mathfrak{sl}(2,\mathbb{C}), \\ B \in \mathfrak{sl}(2,\mathbb{R}) \end{array} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \middle| A \in \mathfrak{sl}(2,\mathbb{C}) \right\};$$

[4,7]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & \alpha E + B \\ -\alpha E + B & C \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A, B, C \in \mathfrak{su}(2) \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \middle| A, C \in \mathfrak{su}(2) \right\};$$

[4,8]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & \sqrt{-1}\alpha E + \sqrt{-1}B \\ -\sqrt{-1}\alpha E + \sqrt{-1}B & C \end{pmatrix} \middle| \begin{array}{c} \alpha \in \mathbb{R}, \\ A, B, C \in \mathfrak{su}(2) \end{array} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \middle| A, C \in \mathfrak{su}(2) \right\};$$

[4,9] $\mathfrak{g} = \mathfrak{sl}(4,\mathbb{R})$,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \middle| A, B, C \in \mathfrak{sl}(2,\mathbb{R});\ \operatorname{tr} A + \operatorname{tr} C = 0 \right\};$$

[4,10] $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{H})$

$$= \left\{ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \middle| A_{ij} \in \mathbb{H};\ \operatorname{tr} A_{11} + \operatorname{tr} A_{22} = 0 \right\},$$

where $\mathbb{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \right\}$,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \middle| A_{ij} \in \mathbb{H};\ \operatorname{tr} A_{11} + \operatorname{tr} A_{22} = 0 \right\};$$

[4,11]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & \sqrt{-1}\alpha E + B \\ \sqrt{-1}\beta E + C & -{}^t\bar{A} \end{pmatrix} \middle| \begin{array}{c} \alpha, \beta \in \mathbb{R},\ A \in \mathfrak{gl}(2,\mathbb{C}), \\ B, C \in \mathfrak{su}(2),\ \operatorname{tr} A = \operatorname{tr} \bar{A} \end{array} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & \sqrt{-1}\alpha E + B \\ 0 & -{}^t\bar{A} \end{pmatrix} \middle| \begin{array}{c} \alpha \in \mathbb{R},\ A \in \mathfrak{gl}(2,\mathbb{C}), \\ B \in \mathfrak{su}(2),\ \operatorname{tr} A = \operatorname{tr} \bar{A} \end{array} \right\};$$

[4,12] $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{gl}(2,\mathbb{C}),\ B \in \mathbb{C}^2 \right\};$$

[4,13] $\mathfrak{g} = \mathfrak{sl}(5,\mathbb{R})$,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{gl}(4,\mathbb{R}),\ B \in \mathbb{R}^4 \right\}.$$

Remark. Here each pair $(\mathfrak{g}, \mathfrak{g}_0)$ is equipped with two numbers $[n, m]$, where $n = \operatorname{codim} \mathfrak{g}_0$ and m is the ordinal number of the pair.

Proof. The statement of the lemma follows immediately from the previous lemmas.

1.8. Suppose $k = \mathbb{R}$, $\mathfrak{g}$ is semisimple but not simple, $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$, $\mu(\mathfrak{g}_0) = \{0\}$, and $\operatorname{codim} \mathfrak{g}_0 \leqslant 4$. Then $(\mathfrak{g}, \mathfrak{g}_0)$ can be represented as one and only one of the following pairs:

[3,8] $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$,

$\mathfrak{g}_0 = \{(A, A) \mid A \in \mathfrak{sl}(2,\mathbb{R})\}$;

[3,9] $\mathfrak{g} = \mathfrak{su}(2) \times \mathfrak{su}(2)$,

$\mathfrak{g}_0 = \{(A, A) \mid A \in \mathfrak{sl}(2)\}$.

1.9. A Lie algebra $\mathfrak{g}$ is called *quasi-reductive* if its nilpotent radical $\mathfrak{s}(\mathfrak{g})$ is a semisimple $\mathfrak{g}$-module and contains no trivial simple submodules. It is obvious that the nilpotent radical of a quasi-reductive Lie algebra is commutative, and the direct product of quasi-reductive Lie algebras is again quasi-reductive.

A Levi–Cartan subalgebra $\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{h}$ of a quasi-reductive Lie algebra $\mathfrak{g}$ is reductive in $\mathfrak{g}$ and

$$\mathfrak{s}(\mathfrak{g}) = \big(Z(\mathfrak{p})\big)^{+}(\mathfrak{h}) \oplus [\mathfrak{p}, \mathfrak{s}(\mathfrak{g})],$$

where the ideals $\big(Z(\mathfrak{p})\big)^{+}(\mathfrak{h})$ and $[\mathfrak{p}, \mathfrak{s}(\mathfrak{g})]$ do not depend on the choice of a Levi–Cartan subalgebra.

The following conditions are equivalent:

(*i*) a Lie algebra $\mathfrak{g}$ is quasi-reductive;

(*ii*) the Frattini ideal $\varphi(\mathfrak{g})$ of $\mathfrak{g}$ is equal to zero.

1.10. Lemma. *If $\mathfrak{g}$ is a quasi-reductive Lie algebra, then every ideal $\mathfrak{g}_0$ in $\mathfrak{g}$ is also quasi-reductive, and moreover, there exists a quasi-reductive subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ complementary to $\mathfrak{g}_0$.*

Proof. Let $\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{h}$ be a Levi–Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$ is a Levi subalgebra of $\mathfrak{g}_0$ and $\mathfrak{r}(\mathfrak{g}_0) = \mathfrak{r}(\mathfrak{g}) \cap \mathfrak{g}_0$ (Chapter I, §6, no. 9, Corollary 4 to Theorem 5). In addition, $\mathfrak{s}(\mathfrak{g}_0) = [\mathfrak{g}_0, \mathfrak{r}(\mathfrak{g}_0)] \subset \mathfrak{s}(\mathfrak{g})$ and

$$\mathfrak{r}(\mathfrak{g}_0) = \big(\mathfrak{h} \cap \mathfrak{r}(\mathfrak{g}_0)\big) \oplus \big(\big(Z(\mathfrak{p})\big)^{+}(\mathfrak{h}) \cap \mathfrak{s}(\mathfrak{g}_0)\big) \oplus \big([\mathfrak{p}, \mathfrak{s}(\mathfrak{g})] \cap \mathfrak{s}(\mathfrak{g}_0)\big).$$

Let us note that the $\mathfrak{g}_0$-module $\mathfrak{s}(\mathfrak{g}_0)$ is semisimple (Chapter I, §6, no. 5, Corollary 4 to Theorem 4) and that in the Lie algebra $Z(\mathfrak{p}_0) \cap \mathfrak{g}_0 \subset \mathfrak{r}(\mathfrak{g}_0)$ there is a Cartan subalgebra $\mathfrak{h}_0$ that lies in $Z\big(\mathfrak{h} \cap \mathfrak{r}(\mathfrak{g}_0)\big)$ (Chapter VII, §2, no. 3, Proposition 10). It is clear that

$$\mathfrak{h}_0 = \big(\mathfrak{h} \cap \mathfrak{r}(\mathfrak{g}_0)\big) \oplus \big(\mathfrak{s}(\mathfrak{g}) \cap \mathfrak{h}_0\big),$$

and consequently, $\mathfrak{h}_0$ is commutative and $\mathfrak{q}_0 = \mathfrak{p}_0 \oplus \mathfrak{h}_0$ is a Levi–Cartan subalgebra of the quasi-reductive Lie algebra $\mathfrak{g}_0$.

Now put $\mathfrak{p}_1 = Z(\mathfrak{p}_0) \cap \mathfrak{p}$ and let V be a subspace of $\mathfrak{h}$ complementary to $\mathfrak{h} \cap \mathfrak{r}(\mathfrak{g}_0)$, and $\mathfrak{b}$ a submodule of the $\mathfrak{g}$-module $\mathfrak{s}(\mathfrak{g})$ complementary to the submodule $\mathfrak{s}(\mathfrak{g}_0)$. Then the subspace $\mathfrak{g}_1 = \mathfrak{p}_1 \oplus V \oplus \mathfrak{b}$ is a subalgebra complementary to $\mathfrak{g}_0$. Note that $\mathfrak{p}_1$ is a Levi subalgebra of $\mathfrak{g}_1$, $\mathfrak{r}(\mathfrak{g}_1) = V \oplus \mathfrak{b}$, and the $\mathfrak{g}_1$-module $\mathfrak{b}$ is semisimple (Chapter I, §6, no. 5, Theorem 4); in addition, in the Lie algebra $Z(\mathfrak{p}_1) \cap \mathfrak{g}_1 \subset \mathfrak{r}(\mathfrak{g}_1)$ there is a Cartan subalgebra $\mathfrak{h}_1$ that belongs to $Z(V)$. Obviously, $\mathfrak{q}_1 = \mathfrak{p}_1 \oplus \mathfrak{h}_1$ is a Levi–Cartan subalgebra of the quasi-reductive Lie algebra $\mathfrak{g}_1$.

1.11. Assume that a Lie algebra $\mathfrak{g}$ is quasi-reductive but not reductive and that $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$ and $\mu(\mathfrak{g}_0) = \{0\}$. Then $\mathfrak{g}_0$ is a Levi–Cartan subalgebra of $\mathfrak{g}$, the $\mathfrak{g}_0$-module $\mathfrak{s}(\mathfrak{g})$ is faithful and simple, and

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{s}(\mathfrak{g}).$$

If now $\bar{k}$ is the algebraic closure of the field k, then the Lie algebra $\bar{\mathfrak{g}} = \bar{k} \otimes_k \mathfrak{g}$ is also quasi-reductive (Chapter I, §6, no. 10 and §3, no. 8).

Let $\mathfrak{p}_0$ be a Levi subalgebra of $\mathfrak{g}_0$ and $\mathfrak{z}_0$ the center of $\mathfrak{g}_0$; then $\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{z}_0$. Now let $\mathfrak{h}_0$ be a Cartan subalgebra in $\mathfrak{p}_0$. It is clear that $\mathfrak{h}_0 \oplus \mathfrak{z}_0$ is a Cartan subalgebra in $\mathfrak{g}_0$. For a simple submodule V of the $\bar{\mathfrak{g}}_0$-module $\mathfrak{s}(\bar{\mathfrak{g}})$, a linear form $\mathbf{\Lambda}$ on $\bar{\mathfrak{h}}_0$, and a linear form ν on $\bar{\mathfrak{z}}_0$, put

$$V^{\mathbf{\Lambda}+\nu}(\bar{\mathfrak{h}}_0 \oplus \bar{\mathfrak{z}}_0) = \big\{ v \in V \bigm| [h+z, v] = \big(\mathbf{\Lambda}(h) + \nu(z)\big)(v) \quad \forall h \in \bar{\mathfrak{h}}_0,\ z \in \bar{\mathfrak{z}}_0 \big\}.$$

If $V^{\mathbf{\Lambda}+\nu}(\bar{\mathfrak{h}}_0 \oplus \bar{\mathfrak{z}}_0) \neq \{0\}$, then the form $\mathbf{\Lambda}+\nu$ is called a *weight of the $\bar{\mathfrak{g}}_0$-module V with respect to the Cartan subalgebra $\bar{\mathfrak{h}}_0 \oplus \bar{\mathfrak{z}}_0$*, and the set $\varkappa = \varkappa(V, \bar{\mathfrak{h}}_0 \oplus \bar{\mathfrak{z}}_0)$ of all such weights is called the *weight system of the $\bar{\mathfrak{g}}_0$-module V with respect to $\bar{\mathfrak{h}}_0 \oplus \bar{\mathfrak{z}}_0$*. Besides, $\mu(H_\alpha) \in \mathbb{Z}$ for all $\mu \in \varkappa$ and $\alpha \in R = R(\bar{\mathfrak{p}}_0, \bar{\mathfrak{h}}_0)$, and also

$$V = \bigoplus_{\mu \in \varkappa} V^{\mu}(\bar{\mathfrak{h}}_0 \oplus \bar{\mathfrak{z}}_0)$$

(Chapter VIII, §7, no. 1, Proposition 1). In addition

$$[\bar{\mathfrak{p}}_0^{\alpha}(\bar{\mathfrak{h}}_0), V^{\mu}(\bar{\mathfrak{h}}_0 \oplus \bar{\mathfrak{z}}_0)] \subset V^{\alpha+\mu}(\bar{\mathfrak{h}}_0 \oplus \bar{\mathfrak{z}}_0)$$

for all $\alpha \in R$ and $\mu \in \varkappa$ (Chapter VII, §1, no. 3, Proposition 10). If B is a base of the root system R, then there exists a unique $\omega \in \varkappa$ such that $\omega + \alpha \notin \varkappa$ for all $\alpha \in B$. It is called the *highest weight of the $\bar{\mathfrak{g}}_0$-module V with respect to B*. We have $\omega(H_\alpha) \in \mathbb{Z}_+$ for all $\alpha \in B$ and

$$\dim V^{\omega}(\bar{\mathfrak{h}}_0 \oplus \bar{\mathfrak{z}}_0) = 1$$

(Chapter VIII, §6, no. 2, Proposition 3 and §7, no. 2, Theorem 1). The highest weight $\omega = \mathbf{\Lambda} + \nu$, where $\mathbf{\Lambda} \in \bar{\mathfrak{h}}_0^*$ and $\nu \in \bar{\mathfrak{z}}_0^*$, is normally given by a diagram. The vertex of the Dynkin diagram of B corresponding to a root $\alpha \in B$ is marked by the number $\omega(H_\alpha) = \mathbf{\Lambda}(H_\alpha)$. The form ν is specified next to the diagram. (If $\omega(H_\alpha) = 0$, the mark can be omitted.)

1.12. Suppose $k = \mathbb{C}$, $\mathfrak{g}$ is quasi-reductive but not reductive, $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$, $\mu(\mathfrak{g}_0) = \{0\}$, and $\operatorname{codim} \mathfrak{g}_0 = 4$. Then the $\mathfrak{g}_0$-module $\mathfrak{s}(\mathfrak{g})$ can be given by one of the following diagrams:

(ε)	$\operatorname{codim} \mathfrak{g}_0 = 1;$
$\overset{1}{\circ}$ (ε)	$\operatorname{codim} \mathfrak{g}_0 = 2;$
$\overset{1}{\circ}$	$\operatorname{codim} \mathfrak{g}_0 = 2;$
$\overset{1}{\circ}\!-\!\circ$ (ε)	$\operatorname{codim} \mathfrak{g}_0 = 3;$
$\overset{1}{\circ}\!-\!\circ$	$\operatorname{codim} \mathfrak{g}_0 = 3;$
$\overset{2}{\circ}$ (ε)	$\operatorname{codim} \mathfrak{g}_0 = 3;$
$\overset{2}{\circ}$	$\operatorname{codim} \mathfrak{g}_0 = 3;$
$\overset{1}{\circ}\!-\!\circ\!-\!\circ$ (ε)	$\operatorname{codim} \mathfrak{g}_0 = 4;$
$\overset{1}{\circ}\!-\!\circ\!-\!\circ$	$\operatorname{codim} \mathfrak{g}_0 = 4;$
$\overset{1}{\circ}\Leftarrow\circ$ (ε)	$\operatorname{codim} \mathfrak{g}_0 = 4;$
$\overset{1}{\circ}\Leftarrow\circ$	$\operatorname{codim} \mathfrak{g}_0 = 4;$
$\overset{1}{\circ}\ \overset{1}{\circ}$ (ε)	$\operatorname{codim} \mathfrak{g}_0 = 4;$
$\overset{1}{\circ}\ \overset{1}{\circ}$	$\operatorname{codim} \mathfrak{g}_0 = 4;$
$\overset{3}{\circ}$ (ε)	$\operatorname{codim} \mathfrak{g}_0 = 4;$
$\overset{3}{\circ}$	$\operatorname{codim} \mathfrak{g}_0 = 4.$

The sign (ε) attached to a diagram shows that the subalgebra $\mathfrak{g}_0$ has one-dimensional center $\mathfrak{z}_0 = \mathbb{C}e$, $\varepsilon \in \mathfrak{z}_0^*$, and $\varepsilon(e) = 1$.

1.13. Suppose $k = \mathbb{R}$, $\mathfrak{g}$ is a quasi-reductive but not reductive Lie algebra, $\mathfrak{g}_0 \in \mathcal{M}(\mathfrak{g})$, $\mu(\mathfrak{g}_0) = \{0\}$, and $\operatorname{codim} \mathfrak{g}_0 \leqslant 4$. Then the pair $(\mathfrak{g}, \mathfrak{g}_0)$ can be written in one and only one of the forms:

[1,2]

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\};$$

[2,8]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{gl}(2, \mathbb{R}),\ B \in \mathbb{R}^2 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{gl}(2, \mathbb{R}) \right\};$$

[2,9]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sl}(2, \mathbb{R}),\ B \in \mathbb{R}^2 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sl}(2, \mathbb{R}) \right\};$$

[2,10]

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in \mathbb{C} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a \in \mathbb{C} \right\};$$

[2,11]

$$\mathfrak{g} = \left\{ \begin{pmatrix} e^{\sqrt{-1}\varphi}\alpha & b \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R},\ b \in \mathbb{C} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} e^{\sqrt{-1}\varphi}\alpha & 0 \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\},$$

where $\varphi \in [0, \pi/2]$;

[3,23]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{gl}(3, \mathbb{R}),\ B \in \mathbb{R}^3 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{gl}(3, \mathbb{R}) \right\};$$

[3,24]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sl}(3, \mathbb{R}),\ B \in \mathbb{R}^3 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sl}(3, \mathbb{R}) \right\};$$

[3,25]

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha E + A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A \in \mathfrak{so}(p, q; \mathbb{R}),\ B \in \mathbb{R}^3 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha E + A & 0 \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A \in \mathfrak{so}(p, q; \mathbb{R}) \right\},$$

where $p > q,\ p + q = 3$;

[3,26]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{so}(p, q; \mathbb{R}),\ B \in \mathbb{R}^3 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{so}(p, q; \mathbb{R}) \right\},$$

where $p > q,\ p + q = 3$;

[4,50]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{gl}(4, \mathbb{R}),\ B \in \mathbb{R}^4 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{gl}(4,\mathbb{R}) \right\};$$

[4,51]

$$\mathfrak{g} \;= \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sl}(4,\mathbb{R}),\ B \in \mathbb{R}^4 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sl}(4,\mathbb{R}) \right\};$$

[4,52]

$$\mathfrak{g} \;= \left\{ \begin{pmatrix} \alpha E + A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A \in \mathfrak{so}(p,q;\mathbb{R}),\ B \in \mathbb{R}^4 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha E + A & 0 \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A \in \mathfrak{so}(p,q;\mathbb{R}) \right\},$$

where $\quad p \geqslant q,\ p+q=4;$

[4,53]

$$\mathfrak{g} \;= \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{so}(p,q;\mathbb{R}),\ B \in \mathbb{R}^4 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{so}(p,q;\mathbb{R}) \right\},$$

where $\quad p \geqslant q,\ p+q=4;$

[4,54]

$$\mathfrak{g} \;= \left\{ \begin{pmatrix} \alpha E + A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A \in \mathfrak{sp}(4,\mathbb{R}),\ B \in \mathbb{R}^4 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha E + A & 0 \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A \in \mathfrak{sp}(4,\mathbb{R}) \right\};$$

[4,55]

$$\mathfrak{g} \;= \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sp}(4,\mathbb{R}),\ B \in \mathbb{R}^4 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sp}(4,\mathbb{R}) \right\};$$

[4,56]

$$\mathfrak{g} \;= \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{gl}(2,\mathbb{C}),\ B \in \mathbb{C}^2 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{gl}(2,\mathbb{C}) \right\};$$

[4,57]

$$\mathfrak{g} = \left\{ \begin{pmatrix} e^{\sqrt{-1}\varphi}\alpha E + A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A \in \mathfrak{sl}(2,\mathbb{C}),\ B \in \mathbb{C}^2 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} e^{\sqrt{-1}\varphi}\alpha E + A & 0 \\ 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R};\ A \in \mathfrak{sl}(2,\mathbb{C}) \right\},$$

where $\varphi \in [0, \pi/2]$;

[4,58]

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & {}^tB \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sl}(2,\mathbb{C});\ B \in \mathbb{C}^2 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{sl}(2,\mathbb{C}) \right\};$$

[4,59]

$$\mathfrak{g} = \left\{ \begin{pmatrix} 3\alpha+\delta & \beta & 0 & 0 & \sigma \\ 3\gamma & \alpha+\delta & 2\beta & 0 & \tau \\ 0 & 2\gamma & -\alpha+\delta & 3\beta & \nu \\ 0 & 0 & \gamma & -3\alpha+\delta & \mu \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \alpha,\beta,\gamma,\delta,\sigma,\tau,\nu,\mu \in \mathbb{R} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} 3\alpha+\delta & \beta & 0 & 0 & 0 \\ 3\gamma & \alpha+\delta & 2\beta & 0 & 0 \\ 0 & 2\gamma & -\alpha+\delta & 3\beta & 0 \\ 0 & 0 & \gamma & -3\alpha+\delta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \alpha,\beta,\gamma,\delta \in \mathbb{R} \right\};$$

[4,60]

$$\mathfrak{g} = \left\{ \begin{pmatrix} 3\alpha & \beta & 0 & 0 & \sigma \\ 3\gamma & \alpha & 2\beta & 0 & \tau \\ 0 & 2\gamma & -\alpha & 3\beta & \nu \\ 0 & 0 & \gamma & -3\alpha & \mu \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \alpha,\beta,\gamma,\sigma,\tau,\nu,\mu \in \mathbb{R} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} 3\alpha & \beta & 0 & 0 & 0 \\ 3\gamma & \alpha & 2\beta & 0 & 0 \\ 0 & 2\gamma & -\alpha & 3\beta & 0 \\ 0 & 0 & \gamma & -3\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \alpha,\beta,\gamma \in \mathbb{R} \right\}.$$

TABLE 1

Effective subalgebras of codimension $\leqslant 4$ in reductive complex Lie algebras.

[1] $\mathfrak{g} = \mathbb{C}$:

$\mathfrak{g}_0 = \{0\}$,

$N(\mathfrak{g}_0) = \mathfrak{g}$, $\operatorname{codim} \mathfrak{g}_0 = 1$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$;

[2] $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$:

a) $\mathfrak{g}_0 = \mathfrak{st}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \middle| \, a, b \in \mathbb{C} \right\}$,

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 1$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$;

b) $\mathfrak{g}_0 = \mathfrak{sa}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \middle| \, a \in \mathbb{C} \right\}$,

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 2$;

c) $\mathfrak{g}_0 = \mathfrak{sb}(2, \mathbb{C}) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| \, b \in \mathbb{C} \right\}$,

$N(\mathfrak{g}_0) = \mathfrak{st}(2, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 2$;

d) $\mathfrak{g}_0 = \{0\}$,

$N(\mathfrak{g}_0) = \mathfrak{g}$, $\operatorname{codim} \mathfrak{g}_0 = 3$.

[3] $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$:

a) $\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \middle| \, A \in \mathfrak{gl}(2, \mathbb{C}), \ B \in \mathbb{C}^2 \right\}$,

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 2$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$;

b) $\mathfrak{g}_0 = \mathfrak{st}(3, \mathbb{C}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & -a_{11} - a_{22} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{C} \right\}$,

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

c) $\mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & B \\ 0 & A \end{pmatrix} \middle| \, A \in \mathfrak{sl}(2, \mathbb{C}), \ B \in \mathbb{C}^2 \right\}$,

$N(\mathfrak{g}_0) = [3a]$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

d) $\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{tr} A & 0 \\ 0 & A \end{pmatrix} \middle| \, A \in \mathfrak{gl}(2, \mathbb{C}) \right\}$,

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

e) $\mathfrak{g}_0 = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & -a_{11} - a_{22} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{C} \right\}$,

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

$$f)\ \mathfrak{g}_0 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & (A-1)a_{11} & a_{23} \\ 0 & 0 & -Aa_{11} \end{pmatrix} \middle| a_{ij} \in \mathbb{C} \right\},$$

where $A \in \mathbb{C}$; $|A| < 1$ or $|A| = 1$, $\arg A \in [0, \pi]$,

$N(\mathfrak{g}_0) = \mathfrak{st}(3, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

[4] $\mathfrak{g} = \mathfrak{sc}(4, \mathbb{C})$

$$= \left\{ \begin{pmatrix} a & b & c & d \\ e & f & g & -c \\ h & i & -f & b \\ j & -h & e & -a \end{pmatrix} \middle| a, b, c, d, e, f, g, h, i, j \in \mathbb{C} \right\}$$

$$= \left\{ \begin{pmatrix} aE + A & B \\ C & -aE + A \end{pmatrix} \middle| a \in \mathbb{C};\ A, B, C \in \mathfrak{sl}(2, \mathbb{C}) \right\} :$$

$$a)\ \mathfrak{g}_0 = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & f & g & -c \\ 0 & i & -f & b \\ 0 & 0 & 0 & -a \end{pmatrix} \middle| a, b, c, d, f, g, i \in \mathbb{C} \right\},$$

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 3$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$;

$$b)\ \mathfrak{g}_0 = \left\{ \begin{pmatrix} aE + A & B \\ 0 & -aE + A \end{pmatrix} \middle| a \in \mathbb{C};\ A, B \in \mathfrak{sl}(2, \mathbb{C}) \right\},$$

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 3$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$;

$$c)\ \mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 & d \\ 0 & f & g & 0 \\ 0 & i & -f & 0 \\ j & 0 & 0 & -a \end{pmatrix} \middle| a, d, f, g, i, j \in \mathbb{C} \right\},$$

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 4$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$;

$$d)\ \mathfrak{g}_0 = \left\{ \begin{pmatrix} aE + A & B \\ 0 & -aE + A \end{pmatrix} \middle| \begin{matrix} a \in \mathbb{C},\ A \in \mathfrak{st}(2, \mathbb{C}), \\ B \in \mathfrak{sl}(2, \mathbb{C}) \end{matrix} \right\},$$

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

$$e)\ \mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & b & c & d \\ 0 & f & g & -c \\ 0 & i & -f & b \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| b, c, d, f, g, i \in \mathbb{C} \right\},$$

$N(\mathfrak{g}_0) = [4a]$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

$$f)\ \mathfrak{g}_0 = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \middle| A, B \in \mathfrak{sl}(2, \mathbb{C}) \right\},$$

$N(\mathfrak{g}_0) = [4b]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

[5] $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$:

$$a)\ \mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{gl}(3, \mathbb{C}),\ B \in \mathbb{C}^3 \right\},$$

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 3$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$;

$b)$ $\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \middle| A, B, C \in \mathfrak{gl}(2, \mathbb{C});\ \operatorname{tr} A + \operatorname{tr} B = 0 \right\}$,
$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 4$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$;

$c)$ $\mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & B \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{sl}(3, \mathbb{C}),\ B \in \mathbb{C}^3 \right\}$,
$N(\mathfrak{g}_0) = [5a]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

[6] $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{C})$:

$\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{gl}(4, \mathbb{C}),\ B \in \mathbb{C}^4 \right\}$,
$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 4$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$.

[7] $\mathfrak{g} = \mathbb{C} \times \mathfrak{sl}(2, \mathbb{C})$:

$a)$ $\mathfrak{g}_0 = \left\{ \left(a, \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \right) \middle| a, b \in \mathbb{C} \right\}$,
$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{st}(2, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 2$;

$b)$ $\mathfrak{g}_0 = \left\{ \left(a, \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right) \middle| a \in \mathbb{C} \right\}$,
$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{sa}(2, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

$c)$ $\mathfrak{g}_0 = \left\{ \left(b, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right) \middle| b \in \mathbb{C} \right\}$,
$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{sb}(2, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 3$.

[8] $\mathfrak{g} = \mathbb{C} \times \mathfrak{sl}(3, \mathbb{C})$:

$a)$ $\mathfrak{g}_0 = \left\{ \left(\operatorname{tr} A, \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \right) \middle| A \in \mathfrak{gl}(2, \mathbb{C}),\ B \in \mathbb{C}^2 \right\}$,
$N(\mathfrak{g}_0) = \mathbb{C} \times [3a]$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

$b)$ $\mathfrak{g}_0 = \left\{ \left((A-1)a_{11} - a_{22}, \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & -a_{11} - a_{22} \end{pmatrix} \right) \middle| a_{ij} \in \mathbb{C} \right\}$,
where $A \in \mathbb{C}$; $|A| < 1$ or $|A| = 1$, $\arg A \in [0, \pi]$,
$N(\mathfrak{g}_0) = \mathbb{C} \times [3b]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

[9] $\mathfrak{g} = \mathbb{C} \times \mathfrak{sc}(4, \mathbb{C})$:

$a)$ $\mathfrak{g}_0 = \left\{ \left(a, \begin{pmatrix} a & b & c & d \\ 0 & f & g & -c \\ 0 & i & -f & b \\ 0 & 0 & 0 & -a \end{pmatrix} \right) \middle| a, b, c, d, f, g, i \in \mathbb{C} \right\}$,
$N(\mathfrak{g}_0) = \mathbb{C} \times [4a]$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

$b)$ $\mathfrak{g}_0 = \left\{ \left(a, \begin{pmatrix} aE + A & B \\ 0 & -aE + A \end{pmatrix} \right) \middle| a \in \mathbb{C}; A, B \in \mathfrak{sl}(2, \mathbb{C}) \right\}$,
$N(\mathfrak{g}_0) = \mathbb{C} \times [4b]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

[10] $\mathfrak{g} = \mathbb{C} \times \mathfrak{sl}(4, \mathbb{C})$:

$$\mathfrak{g}_0 = \left\{ \left(\operatorname{tr} A, \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \right) \middle| A \in \mathfrak{gl}(3, \mathbb{C}),\ B \in \mathbb{C}^3 \right\},$$

$N(\mathfrak{g}_0) = \mathbb{C} \times [5a]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

[11] $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$:

a) $\mathfrak{g}_0 = \{ (A, A) \mid A \in \mathfrak{sl}(2, \mathbb{C}) \}$,

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 3$, $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$;

b) $$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \begin{pmatrix} Aa & c \\ 0 & -Aa \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\},$$

where $A \in \mathbb{C}^*$; $|A| < 1$ or $|A| = 1$, $\arg A \in [0, \pi]$,

$N(\mathfrak{g}_0) = \mathfrak{st}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

c) $$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} Aa & b \\ 0 & -Aa \end{pmatrix} \middle| a, b \in \mathbb{C} \right\},$$

where $A \in \mathbb{C}^*$ and $\arg A \in [0, \pi)$,

$N(\mathfrak{g}_0) = \mathfrak{sa}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

d) $$\mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & c \\ 0 & -b \end{pmatrix} \middle| b, c \in \mathbb{C} \right\},$$

$N(\mathfrak{g}_0) = \mathfrak{sb}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

e) $$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \middle| a, b \in \mathbb{C} \right\},$$

$N(\mathfrak{g}_0) = \mathfrak{g}_0$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

[12] $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(3, \mathbb{C})$:

$$\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} C \operatorname{tr} A & a \\ 0 & -C \operatorname{tr} A \end{pmatrix}, \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \right) \middle| \begin{matrix} a \in \mathbb{C},\ B \in \mathbb{C}^2, \\ A \in \mathfrak{gl}(2, \mathbb{C}) \end{matrix} \right\},$$

where $C \in \mathbb{C}^*$,

$N(\mathfrak{g}_0) = \mathfrak{st}(2, \mathbb{C}) \times [3a]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

[13] $\mathfrak{g} = \mathbb{C} \times \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$:

a) $$\mathfrak{g}_0 = \left\{ \left(Aa - c, \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & -c \end{pmatrix} \right) \middle| a, b, c, d \in \mathbb{C} \right\},$$

where $A \in \mathbb{C}^*$; $|A| < 1$ or $|A| = 1$, $\arg A \in [0, \pi]$,

$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{st}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

b) $$\mathfrak{g}_0 = \left\{ \left(a, \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \begin{pmatrix} Aa & c \\ 0 & -Aa \end{pmatrix} \right) \middle| a, b, c, \in \mathbb{C} \right\},$$

where $A \in \mathbb{C}^*$; $|A| < 1$ or $|A| = 1$, $\arg A \in [0, \pi]$,

$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{st}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

$c)\ \mathfrak{g}_0 = \left\{ \left(Aa - b, \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} b & c \\ 0 & -b \end{pmatrix} \right) \middle| a, b, c, \in \mathbb{C} \right\},$

where $A \in \mathbb{C}^*$ and $\arg A \in [0, \pi)$,

$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{sa}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C}),\ \operatorname{codim} \mathfrak{g}_0 = 4;$

$d)\ \mathfrak{g}_0 = \left\{ \left(a - b, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & -a \end{pmatrix} \right) \middle| a, b, c, \in \mathbb{C} \right\},$

$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{sb}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C}),\ \operatorname{codim} \mathfrak{g}_0 = 4.$

[14] $\mathfrak{g} = \mathbb{C} \times \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(3, \mathbb{C})$:

$$\mathfrak{g}_0 = \left\{ \left(a - C \operatorname{tr} A, \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \right) \middle| \begin{array}{c} a, b \in \mathbb{C},\ B \in \mathbb{C}^2, \\ A \in \mathfrak{gl}(2, \mathbb{C}) \end{array} \right\},$$

where $C \in \mathbb{C}^*$,

$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{st}(2, \mathbb{C}) \times [3a],\ \operatorname{codim} \mathfrak{g}_0 = 4.$

[15] $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$:

$$\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} Aa + Bb & c \\ 0 & -Aa - Bb \end{pmatrix}, \begin{pmatrix} a & d \\ 0 & -a \end{pmatrix}, \begin{pmatrix} b & e \\ 0 & -b \end{pmatrix} \right) \middle| a, b, c, d, e \in \mathbb{C} \right\},$$

where $A, B \in \mathbb{C}^*$ and one of the following conditions holds:

(i) $|A| < |B| \leqslant 1$;

(ii) $|A| = |B| < 1$ and $\arg A \leqslant \arg B$;

(iii) $|A| = |B| = 1$ and $2 \arg A \leqslant \arg B \leqslant 2\pi - \arg A$,

$N(\mathfrak{g}_0) = \mathfrak{st}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C}),\ \operatorname{codim} \mathfrak{g}_0 = 4.$

[16] $\mathfrak{g} = \mathbb{C} \times \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$:

$$\mathfrak{g}_0 = \left\{ \left(a - Bb - Cc, \begin{pmatrix} a & d \\ 0 & -a \end{pmatrix}, \begin{pmatrix} b & e \\ 0 & -b \end{pmatrix}, \begin{pmatrix} c & f \\ 0 & -c \end{pmatrix} \right) \middle| \begin{array}{c} a, b, c \in \mathbb{C} \\ d, e, f \in \mathbb{C} \end{array} \right\},$$

where $B, C \in \mathbb{C}^*$ and one of the following conditions holds:

(i) $|B| < |C| \leqslant 1$;

(ii) $|B| = |C| < 1$ and $\arg B \leqslant \arg C$;

(iii) $|B| = |C| = 1$ and $2 \arg B \leqslant \arg C \leqslant 2\pi - \arg B$,

$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{st}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C}) \times \mathfrak{st}(2, \mathbb{C}),\ \operatorname{codim} \mathfrak{g}_0 = 4.$

[17] All other pairs of this class have the form

$$\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}''$$
$$\mathfrak{g}_0 = \mathfrak{g}'_0 \times \mathfrak{g}''_0,$$

where the pairs $(\mathfrak{g}', \mathfrak{g}'_0)$ and $(\mathfrak{g}'', \mathfrak{g}''_0)$ are also of this class.

TABLE 2

Nonmaximal effective subalgebras of codimension $\leqslant 4$
in reductive real Lie algebras.

(1) $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$:

[2,5] $\mathfrak{g}_0 = \mathfrak{sa}(2,\mathbb{R}) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\},$
$N(\mathfrak{g}_0) = \mathfrak{g}_0, \ \operatorname{codim} \mathfrak{g}_0 = 2;$

[2,6] $\mathfrak{g}_0 = \mathfrak{sb}(2,\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \middle| \beta \in \mathbb{R} \right\},$
$N(\mathfrak{g}_0) = \mathfrak{st}(2,\mathbb{R}), \ \operatorname{codim} \mathfrak{g}_0 = 2;$

[3,10] $\mathfrak{g}_0 = \{0\},$
$N(\mathfrak{g}_0) = \mathfrak{g}, \ \operatorname{codim} \mathfrak{g}_0 = 3.$

(2) $\mathfrak{g} = \mathfrak{su}(2)$:

[3,11] $\mathfrak{g}_0 = \{0\},$
$N(\mathfrak{g}_0) = \mathfrak{g}, \ \operatorname{codim} \mathfrak{g}_0 = 3.$

(3) $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$:

[3,12] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} e^{\sqrt{-1}\varphi}\,\alpha & a \\ 0 & -e^{\sqrt{-1}\varphi}\,\alpha \end{pmatrix} \middle| \alpha \in \mathbb{R}, \ a \in \mathbb{C} \right\},$
where $\varphi \in [0, \pi/2]$,
$N(\mathfrak{g}_0) = \mathfrak{st}(2,\mathbb{C}), \ \operatorname{codim} \mathfrak{g}_0 = 3;$

[4,14] $\mathfrak{g}_0 = \mathfrak{sb}(2,\mathbb{C}) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| b \in \mathbb{C} \right\},$
$N(\mathfrak{g}_0) = \mathfrak{st}(2,\mathbb{C}), \ \operatorname{codim} \mathfrak{g}_0 = 4;$

[4,15] $\mathfrak{g}_0 = \mathfrak{sa}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \middle| a \in \mathbb{C} \right\},$
$N(\mathfrak{g}_0) = \mathfrak{g}_0, \ \operatorname{codim} \mathfrak{g}_0 = 4;$

[4,16] $\mathfrak{g}_0 = \mathfrak{st}(2,\mathbb{R}),$
$N(\mathfrak{g}_0) = \mathfrak{g}_0, \ \operatorname{codim} \mathfrak{g}_0 = 4.$

(4) $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$:

[3,13] $\mathfrak{g}_0 = \mathfrak{st}(3,\mathbb{R}) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & -\alpha_{11}-\alpha_{22} \end{pmatrix} \middle| \alpha_{ij} \in \mathbb{R} \right\},$
$N(\mathfrak{g}_0) = \mathfrak{g}_0, \ \operatorname{codim} \mathfrak{g}_0 = 3;$

[3,14] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & B \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{sl}(2,\mathbb{R}), \ B \in \mathbb{R}^2 \right\},$
$N(\mathfrak{g}_0) = [2,4], \ \operatorname{codim} \mathfrak{g}_0 = 3;$

[4,17] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{tr} A & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{gl}(2,\mathbb{R}) \right\},$

$N(\mathfrak{g}_0) = \mathfrak{g}_0, \ \operatorname{codim} \mathfrak{g}_0 = 4;$

[4,18] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & -\alpha_{11} - \alpha_{22} \end{pmatrix} \middle| \alpha_{ij} \in \mathbb{R} \right\},$

$N(\mathfrak{g}_0) = \mathfrak{g}_0, \ \operatorname{codim} \mathfrak{g}_0 = 4;$

[4,19] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & (A-1)\alpha_{11} & \alpha_{23} \\ 0 & 0 & -A\alpha_{11} \end{pmatrix} \middle| \alpha_{ij} \in \mathbb{R} \right\},$

where $A \in [-1, 1]$,

$N(\mathfrak{g}_0) = \mathfrak{st}(3,\mathbb{R}), \ \operatorname{codim} \mathfrak{g}_0 = 4.$

(5)

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b & \sqrt{-1}\alpha \\ c & \bar{a} - a & -\bar{b} \\ \sqrt{-1}\beta & -\bar{c} & -\bar{a} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R}; \ a, b, c \in \mathbb{C} \right\} :$$

[4,20] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} \mathrm{e}^{\sqrt{-1}\varphi}\alpha & a & \sqrt{-1}\beta \\ 0 & (\mathrm{e}^{-\sqrt{-1}\varphi} - \mathrm{e}^{\sqrt{-1}\varphi})\alpha & -\bar{a} \\ 0 & 0 & -\mathrm{e}^{-\sqrt{-1}\varphi}\alpha \end{pmatrix} \middle| \begin{array}{c} \alpha, \beta \in \mathbb{R}, \\ a \in \mathbb{C} \end{array} \right\},$

where $\varphi \in [0, \pi/2]$,

$N(\mathfrak{g}_0) = [3,3], \ \operatorname{codim} \mathfrak{g}_0 = 4.$

(6) $\mathfrak{g} = \mathfrak{sc}(4,\mathbb{R})$ (see [3,4]):

[4,21] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \tau & \nu & -\gamma \\ 0 & 0 & -\tau & \beta \\ 0 & 0 & 0 & -\alpha \end{pmatrix} \middle| \alpha, \beta, \gamma, \delta, \tau, \nu \in \mathbb{R} \right\},$

$N(\mathfrak{g}_0) = \mathfrak{g}_0, \ \operatorname{codim} \mathfrak{g}_0 = 4;$

[4,22] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & \beta & \gamma & \delta \\ 0 & \tau & \nu & -\gamma \\ 0 & \varepsilon & -\tau & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \beta, \gamma, \delta, \tau, \nu, \varepsilon \in \mathbb{R} \right\},$

$N(\mathfrak{g}_0) = [3,4], \ \operatorname{codim} \mathfrak{g}_0 = 4;$

[4,23] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \middle| A, B \in \mathfrak{sl}(2,\mathbb{R}) \right\},$

$N(\mathfrak{g}_0) = [3,5], \ \operatorname{codim} \mathfrak{g}_0 = 4.$

(7) $\mathfrak{g} = \mathfrak{sc}(2,\mathbb{H})$ (see [3,6]):

[4,24] $\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \middle| A, B \in \mathfrak{su}(2) \right\},$

$N(\mathfrak{g}_0) = [3,6], \ \operatorname{codim} \mathfrak{g}_0 = 4.$

(8) $\mathfrak{g} = \mathfrak{sl}(4,\mathbb{R})$:

$$[4,25] \quad \mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & B \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{sl}(3,\mathbb{R}),\ B \in \mathbb{R}^3 \right\},$$
$N(\mathfrak{g}_0) = [3,7]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

(9) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sl}(2,\mathbb{R})$:

$$[2,7] \quad \mathfrak{g}_0 = \left\{ \left(\alpha, \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} \right) \middle| \alpha, \beta \in \mathbb{R} \right\},$$
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{st}(2,\mathbb{R})$, $\operatorname{codim} \mathfrak{g}_0 = 2$;

$$[3,15] \quad \mathfrak{g}_0 = \left\{ \left(\alpha, \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} \right) \middle| \alpha \in \mathbb{R} \right\},$$
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{so}(2,\mathbb{R})$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

$$[3,16] \quad \mathfrak{g}_0 = \left\{ \left(\alpha, \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \right) \middle| \alpha \in \mathbb{R} \right\},$$
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{sa}(2,\mathbb{R})$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

$$[3,17] \quad \mathfrak{g}_0 = \left\{ \left(\beta, \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \right) \middle| \beta \in \mathbb{R} \right\},$$
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{sb}(2,\mathbb{R})$, $\operatorname{codim} \mathfrak{g}_0 = 3$.

(10) $\mathfrak{g} = \mathbb{R} \times \mathfrak{su}(2)$:

$$[3,18] \quad \mathfrak{g}_0 = \left\{ \left(\alpha, \begin{pmatrix} \sqrt{-1}\alpha & 0 \\ 0 & -\sqrt{-1}\alpha \end{pmatrix} \right) \middle| \alpha \in \mathbb{R} \right\},$$
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{su}_0(2)$ (see $[2,2]$), $\operatorname{codim} \mathfrak{g}_0 = 3$.

(11) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sl}(2,\mathbb{C})$:

$$[3,19] \quad \mathfrak{g}_0 = \left\{ \left(\sqrt{-1}\left(\mathrm{e}^{-\sqrt{-1}\varphi}\, a - \mathrm{e}^{\sqrt{-1}\varphi}\, \bar{a} \right), \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \right) \middle| a, b \in \mathbb{C} \right\},$$
where $\varphi \in [0, \pi/2]$,
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{st}(2,\mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

$$[4,26] \quad \mathfrak{g}_0 = \left\{ \left(\alpha, \begin{pmatrix} \mathrm{e}^{\sqrt{-1}\varphi}\, \alpha & a \\ 0 & -\mathrm{e}^{\sqrt{-1}\varphi}\, \alpha \end{pmatrix} \right) \middle| \alpha \in \mathbb{R},\ a \in \mathbb{C} \right\},$$
where $\varphi \in [0, \pi/2]$,
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{st}(2,\mathbb{C})$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

(12) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sl}(3,\mathbb{R})$:

$$[3,20] \quad \mathfrak{g}_0 = \left\{ \left(\operatorname{tr} A, \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \right) \middle| A \in \mathfrak{gl}(2,\mathbb{R}),\ B \in \mathbb{R}^2 \right\},$$
$N(\mathfrak{g}_0) = \mathbb{R} \times [2,4]$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

$$[4,27] \quad \mathfrak{g}_0 = \left\{ \left((A-1)\alpha_{11} - \alpha_{22}, \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & -\alpha_{11} - \alpha_{22} \end{pmatrix} \right) \middle| \alpha_{ij} \in \mathbb{R} \right\},$$
where $A \in [-1, 1]$,
$N(\mathfrak{g}_0) = \mathbb{R} \times [3,13]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

(13)

$$\mathfrak{g} = \mathbb{R} \times \left\{ \begin{pmatrix} a & b & \sqrt{-1}\alpha \\ c & \bar{a} - a & -\bar{b} \\ \sqrt{-1}\beta & -\bar{c} & -\bar{a} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R}; a, b, c \in \mathbb{C} \right\} :$$

[4,28] $$\mathfrak{g}_0 = \left\{ \left(\sqrt{-1}\left(\mathrm{e}^{-\sqrt{-1}\varphi}\, a - \mathrm{e}^{\sqrt{-1}\varphi}\, \bar{a}\right), \begin{pmatrix} a & b & \sqrt{-1}\alpha \\ 0 & \bar{a} - a & -\bar{b} \\ 0 & 0 & -\bar{a} \end{pmatrix} \right) \middle| \begin{matrix} \alpha \in \mathbb{R}, \\ a, b \in \mathbb{C} \end{matrix} \right\},$$

where $\varphi \in [0, \pi/2]$,

$N(\mathfrak{g}_0) = \mathbb{R} \times [3,3]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

(14) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sc}(4, \mathbb{R})$:

[4,29] $$\mathfrak{g}_0 = \left\{ \left(\alpha, \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \tau & \nu & -\gamma \\ 0 & \varepsilon & -\tau & \beta \\ 0 & 0 & 0 & -\alpha \end{pmatrix} \right) \middle| \alpha, \beta, \gamma, \delta, \tau, \nu, \varepsilon \in \mathbb{R} \right\},$$

$N(\mathfrak{g}_0) = \mathbb{R} \times [3,4]$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

[4,30] $$\mathfrak{g}_0 = \left\{ \left(\alpha, \begin{pmatrix} \alpha E + A & B \\ 0 & -\alpha E + A \end{pmatrix} \right) \middle| \alpha \in \mathbb{R}; A, B \in \mathfrak{sl}(2, \mathbb{R}) \right\},$$

$N(\mathfrak{g}_0) = \mathbb{R} \times [3,5]$, $\operatorname{codim} \mathfrak{g}_0 = 4$.

(15) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sc}(2, \mathbb{H})$:

[4,31] $$\mathfrak{g}_0 = \left\{ \left(\alpha, \begin{pmatrix} \alpha E + A & B \\ 0 & -\alpha E + A \end{pmatrix} \right) \middle| \alpha \in \mathbb{R};\ A, B \in \mathfrak{su}(2) \right\},$$

$N(\mathfrak{g}_0) = \mathbb{R} \times [3,6]$.

(16) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sl}(4, \mathbb{R})$:

[4,32] $$\mathfrak{g}_0 = \left\{ \left(\operatorname{tr} A, \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \right) \middle| A \in \mathfrak{gl}(3, \mathbb{R}),\ B \in \mathbb{R}^3 \right\},$$

$N(\mathfrak{g}_0) = \mathbb{R} \times [3,7]$.

(17) $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$:

[3,21] $$\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} A\alpha & \gamma \\ 0 & -A\alpha \end{pmatrix}, \right) \middle| \alpha, \beta, \gamma \in \mathbb{R} \right\},$$

where $A \in [-1, 1]$ and $A \neq 0$,

$N(\mathfrak{g}_0) = \mathfrak{st}(2, \mathbb{R}) \times \mathfrak{st}(2, \mathbb{R})$, $\operatorname{codim} \mathfrak{g}_0 = 3$;

[4,33] $$\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}, \begin{pmatrix} A\alpha & \beta \\ 0 & -A\alpha \end{pmatrix}, \right) \middle| \alpha, \beta \in \mathbb{R} \right\},$$

where $A \in \mathbb{R}^*_+$,

$N(\mathfrak{g}_0) = \mathfrak{so}(2, \mathbb{R}) \times \mathfrak{st}(2, \mathbb{R})$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

[4,34] $$\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} A\alpha & \beta \\ 0 & -A\alpha \end{pmatrix} \right) \middle| \alpha, \beta \in \mathbb{R} \right\},$$

where $A \in \mathbb{R}^*_+$,

$N(\mathfrak{g}_0) = \mathfrak{sa}(2, \mathbb{R}) \times \mathfrak{st}(2, \mathbb{R})$, $\operatorname{codim} \mathfrak{g}_0 = 4$;

[4,35] $\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \beta & \gamma \\ 0 & -\beta \end{pmatrix} \right) \middle| \beta, \gamma \in \mathbb{R} \right\},$
$N(\mathfrak{g}_0) = \mathfrak{sb}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}),$ codim $\mathfrak{g}_0 = 4;$

[4,36] $\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} \right) \middle| \alpha, \beta \in \mathbb{R} \right\},$
$N(\mathfrak{g}_0) = \mathfrak{g}_0,$ codim $\mathfrak{g}_0 = 4.$

(18) $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{su}(2):$

[4,37] $\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} A\alpha & \beta \\ 0 & -A\alpha \end{pmatrix}, \begin{pmatrix} \sqrt{-1}\alpha & 0 \\ 0 & -\sqrt{-1}\alpha \end{pmatrix} \right) \middle| \alpha, \beta \in \mathbb{R} \right\},$
where $A \in \mathbb{R}^*_+,$
$N(\mathfrak{g}_0) = \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{su}_0(2),$ codim $\mathfrak{g}_0 = 4.$

(19) $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{C}):$

[4,38] $\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} \sqrt{-1}(\bar{A}a - A\bar{a}) & \alpha \\ 0 & -\sqrt{-1}(\bar{A}a - A\bar{a}) \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \right) \middle| \begin{matrix} \alpha \in \mathbb{R}, \\ a, b \in \mathbb{C} \end{matrix} \right\},$
where $A \in \mathbb{C}^*$ and $\arg A \in [0, \pi/2],$
$N(\mathfrak{g}_0) = \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{C}),$ codim $\mathfrak{g}_0 = 4.$

(20) $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R}):$

[4,39] $\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} C \operatorname{tr} A & \alpha \\ 0 & -C \operatorname{tr} A \end{pmatrix}, \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \right) \middle| \begin{matrix} \alpha \in \mathbb{R}, \ B \in \mathbb{R}^2, \\ A \in \mathfrak{gl}(2,\mathbb{R}) \end{matrix} \right\},$
where $C \in \mathbb{R}^*,$
$N(\mathfrak{g}_0) = \mathfrak{st}(2,\mathbb{R}) \times [2,4],$ codim $\mathfrak{g}_0 = 4.$

(21) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}):$

[3,22] $\mathfrak{g}_0 = \left\{ \left(A\alpha - \gamma, \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} \gamma & \delta \\ 0 & -\gamma \end{pmatrix} \right) \middle| \alpha, \beta, \gamma, \delta \in \mathbb{R} \right\},$
where $A \in [-1, 1]$ and $A \neq 0,$
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}),$ codim $\mathfrak{g}_0 = 3;$

[4,40] $\mathfrak{g}_0 = \left\{ \left(\alpha, \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} A\alpha & \gamma \\ 0 & -A\alpha \end{pmatrix} \right) \middle| \alpha, \beta, \gamma \in \mathbb{R} \right\},$
where $A \in [-1, 1]$ and $A \neq 0,$
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}),$ codim $\mathfrak{g}_0 = 4;$

[4,41] $\mathfrak{g}_0 = \left\{ \left(A\alpha - \beta, \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}, \begin{pmatrix} \beta & \gamma \\ 0 & -\beta \end{pmatrix} \right) \middle| \alpha, \beta, \gamma \in \mathbb{R} \right\},$
where $A \in \mathbb{R}^*_+,$
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{so}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}),$ codim $\mathfrak{g}_0 = 4;$

[4,42] $\mathfrak{g}_0 = \left\{ \left(A\alpha - \beta, \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} \beta & \gamma \\ 0 & -\beta \end{pmatrix} \right) \middle| \alpha, \beta, \gamma \in \mathbb{R} \right\},$
where $A \in \mathbb{R}^*_+,$
$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{sa}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}),$ codim $\mathfrak{g}_0 = 4;$

[4,43] $\mathfrak{g}_0 = \left\{ \left(\alpha - \beta, \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & \gamma \\ 0 & -\alpha \end{pmatrix} \right) \middle| \alpha, \beta, \gamma \in \mathbb{R} \right\},$

$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{sb}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}),\ \operatorname{codim} \mathfrak{g}_0 = 4.$

(22) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{su}(2):$

[4,44] $\mathfrak{g}_0 = \left\{ \left(\alpha - A\gamma, \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} \sqrt{-1}\gamma & 0 \\ 0 & -\sqrt{-1}\gamma \end{pmatrix} \right) \middle| \alpha, \beta, \gamma \in \mathbb{R} \right\},$

where $A \in \mathbb{R}^*_+$,

$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{su}_0(2),\ \operatorname{codim} \mathfrak{g}_0 = 4.$

(23) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{C}):$

[4,45] $\mathfrak{g}_0 = \left\{ \left(\alpha - \sqrt{-1}(\bar{A}a - A\bar{a}), \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \right) \middle| \begin{matrix} \alpha, \beta \in \mathbb{R} \\ a, b \in \mathbb{C} \end{matrix} \right\},$

where $A \in \mathbb{C}^*$ and $\arg A \in [0, \pi/2]$,

$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{C}),\ \operatorname{codim} \mathfrak{g}_0 = 4.$

(24) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R}):$

[4,46] $\mathfrak{g}_0 = \left\{ \left(\alpha - C \operatorname{tr} A, \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \right) \middle| \begin{matrix} \alpha, \beta \in \mathbb{R},\ B \in \mathbb{R}^2 \\ A \in \mathfrak{gl}(2,\mathbb{R}) \end{matrix} \right\},$

where $C \in \mathbb{R}^*$,

$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{st}(2,\mathbb{R}) \times [2,4],\ \operatorname{codim} \mathfrak{g}_0 = 4.$

(25) $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}):$

[4,47] $\mathfrak{g}_0 = \left\{ \left(\begin{pmatrix} A\alpha + B\beta & \gamma \\ 0 & -A\alpha - B\beta \end{pmatrix}, \begin{pmatrix} \alpha & \delta \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} \beta & \sigma \\ 0 & -\beta \end{pmatrix} \right) \middle| \begin{matrix} \alpha, \beta, \gamma \in \mathbb{R} \\ \delta, \sigma \in \mathbb{R} \end{matrix} \right\},$

where $A, B \in \mathbb{R}^*$ and one of the following conditions holds:

(*i*) $|A| < |B| \leqslant 1$;

(*ii*) $A = |B| \leqslant 1$,

$N(\mathfrak{g}_0) = \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}),\ \operatorname{codim} \mathfrak{g}_0 = 4.$

(26) $\mathfrak{g} = \mathbb{C} \times \mathfrak{sl}(2,\mathbb{C}):$

[4,48] $\mathfrak{g}_0 = \left\{ \left(a, \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \right) \middle| a, b \in \mathbb{C} \right\},$

$N(\mathfrak{g}_0) = \mathbb{C} \times \mathfrak{st}(2,\mathbb{C}),\ \operatorname{codim} \mathfrak{g}_0 = 4.$

(27) $\mathfrak{g} = \mathbb{R} \times \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}):$

[4,49] $\mathfrak{g}_0 = \left\{ \left(\alpha - B\beta - C\gamma, \begin{pmatrix} \alpha & \delta \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} \beta & \sigma \\ 0 & -\beta \end{pmatrix}, \begin{pmatrix} \gamma & \tau \\ 0 & -\gamma \end{pmatrix} \right) \middle| \begin{matrix} \alpha, \beta, \gamma \in \mathbb{R} \\ \delta, \sigma, \tau \in \mathbb{R} \end{matrix} \right\},$

where $B, C \in \mathbb{R}^*$,

$N(\mathfrak{g}_0) = \mathbb{R} \times \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}),\ \operatorname{codim} \mathfrak{g}_0 = 4.$

(28) All other pairs of this class have the form

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}' \times \mathfrak{g}'' \\ \mathfrak{g}_0 &= \mathfrak{g}_0' \times \mathfrak{g}_0'', \end{aligned}$$

where the pairs $(\mathfrak{g}', \mathfrak{g}_0')$ and $(\mathfrak{g}'', \mathfrak{g}_0'')$ are either maximal or also belong to this class.

To make the picture complete, we write out one more maximal pair:

$$[1,0] \quad \begin{aligned} &\mathfrak{g} = \mathbb{R} \\ &\mathfrak{g}_0 = \{0\} \\ &N(\mathfrak{g}_0) = \mathfrak{g}, \ \operatorname{codim} \mathfrak{g}_0 = 1. \end{aligned}$$

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