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Approximations of Stochastic Partial Differential Equations

Giulia Di Nunno[1] Tusheng Zhang[2]

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Abstract

In this paper we show that solutions of stochastic partial differential equations driven by Brownian motion can be approximated by stochastic partial differential equations forced by pure jump noise/random kicks. Applications to stochastic Burgers equations are discussed.

Key words: Stochastic partial differential equations; Approximations; Jump noise; Tightness; Weak convergence; Stochastic Burgers equations.

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1 Introduction

Stochastic evolution equations and stochastic partial differential equations (SPDEs) are of great interest to many people. There exists a great amount of literature on the subject, see, for example the monographs [PZ], [C].

In this paper, we consider the following stochastic evolution equation:

$$dY_t = -AY_t dt + [b_1(Y_t) + b_2(Y_t)]dt + \sigma(Y_t)dB_t, \tag{1.1}$$

$$Y_0 = h \in H, \tag{1.2}$$

in the framework of a Gelfand triple :

$$V \subset H \cong H^* \subset V^*, \tag{1.3}$$

where H, V are Hilbert spaces, A is the infinitesimal generator of a strongly continuous semigroup, b_1, σ are measurable mappings from H into H, b_2 is a measurable mappings from H into V^*(the dual of V), $B_t, t \geq 0$ is a Brownian motion. The solutions are considered to be weak solutions (in the PDE sense)

[1] Department of Mathematics, University of Oslo, N-0316 Oslo, Norway
[2] School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, England, U.K. Email: tusheng.zhang@manchester.ac.uk

in the space V and not as mild solutions in H as is more common in the literature. The stochastic evolution equations of this type driven by Wiener processes were first studied by in [P] and subsequently in [KR]. For stochastic equations with general Hilbert space valued semimartingales replacing the Brownian motion we refer to [GK1], [GK2], [G] and [RZ].

The aim of this paper is to study the approximations of stochastic evolution equations of the above type by solutions of stochastic evolution equations driven by pure jump processes, namely forced by random kicks. One of the motivations is to shine some light on numerical simulations of SPDEs driven by pure jump noise. To include interesting applications, the drift of the equation (1.1) will consist of a "good" part b_1 and a "bad" part b_2. The crucial step of obtaining the approximation is to establish the tightness of the approximating equations in the space of Hilbert space-valued right continuous paths with left limits. This is tricky because of the nature of the infinite dimensions and weak assumptions on the drift b_2. We first obtain the approximations assuming the diffusion coefficient σ takes values in the smaller space V and then remove the restriction by another layer of approximations. As far as we are aware of, this is the first paper to consider such approximations for SPDEs. The approximations of small jump Lévy processes were considered in [AR]. Robustness of solutions of stochastic differential equations replacing small jump Levy processes by Brownian motion was discussed in [BDK] and [DSE], and for the backward case in [DKV].

The rest of the paper is organized as follows. In Section 2 we lay down the precise framework. The main part is Section 3, where the approximations are established and the applications to stochastic Burgers equations are discussed.

2 Framework

Let V, H be two separable Hilbert spaces such that V is continuously, densely imbedded in H. Identifying H with its dual we have

$$V \subset H \cong H^* \subset V^*, \tag{2.1}$$

where V^* stands for the topological dual of V. We assume that the imbedding $V \subset H$ is compact. Let A be a self-adjoint operator on the Hilbert space H satisfying the following coercivity hypothesis: There exist constants $\alpha_0 > 0$, $\alpha_1 > 0$ and $\lambda_0 \geq 0$ such that

$$\alpha_0 ||u||_V^2 \leq 2 < Au, u > + \lambda_0 |u|_H^2 \leq \alpha_1 ||u||_V^2 \qquad \text{for all } u \in V. \tag{2.2}$$

$< Au, u >= Au(u)$ denotes the action of $Au \in V^*$ on $u \in V$.
We remark that A is generally not bounded as an operator from H into H. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}$ satisfying

the usual conditions. Let $\{B_t, t \geq 0\}$ be a real-valued $\mathcal{F}_t$- Brownian motion, $\nu(dx)$ a σ-finite measure on the measurable space $(R_0, \mathcal{B}(R_0))$, where $R_0 = R \setminus \{0\}$. Let $p = (p(t)), t \in D_p$ be a stationary $\mathcal{F}_t$-Poisson point process on R_0 with characteristic measure ν. Here D_p represents a countable (random) subset of $(0, \infty)$. See [IW] for the details on Poisson point processes. Denote by $N(dt, dx)$ the Poisson counting measure associated with p, i.e., $N(t, A) = \sum_{s \in D_p, s \leq t} I_A(p(s))$. Let $\tilde{N}(dt, dx) := N(dt, dx) - dt\nu(dx)$ be the compensated Poisson random measure. Let b_1, σ be measurable mappings from H into H, and $b_2(\cdot)$ a measurable mapping from V into V^*. Denote by $D([0, T], H)$ the space of all càdlàg paths from $[0, T]$ into H equipped with the Skorohod topology. Consider the stochastic evolution equation:

$$
\begin{aligned}
dX_t &= -AX_t dt + [b_1(X_t) + b_2(X_t)]dt + \sigma(X_t)dB_t, && (2.3)\\
X_0 &= h \in H. && (2.4)
\end{aligned}
$$

Introduce the following conditions:

(H.1) There exists a constant $C < \infty$ such that

$$
\begin{aligned}
& |b_1(y_1) - b_1(y_2)|_H^2 + |\sigma(y_1) - \sigma(y_2)|_H^2 \\
\leq\ & C|y_1 - y_2|_H^2, \qquad \text{for all} \quad y_1, y_2 \in H. \qquad (2.5)
\end{aligned}
$$

(H.2) $b_2(\cdot)$ is a mapping from V into V^* that satisfies

(i) $< b_2(u), u >= 0$ for $u \in V$,

(ii) There exist constants C_1, $\beta < \frac{1}{2}$ such that

$$
\begin{aligned}
& < b_2(y_1) - b_2(y_2), y_1 - y_2 > \\
\leq\ & \beta\alpha_0 ||y_1 - y_2||_V^2 + C_1 |y_1 - y_2|_H^2 (||y_1||_V^2 + ||y_2||_V^2) \\
& \text{for all} \quad y_1, y_2 \in V, \qquad (2.6)
\end{aligned}
$$

(iii) There exists a constant $0 < \gamma < 1$ such that $||b_2(u)||_{V^*} \leq C_2 |u|_H^{2-\gamma} ||u||_V^\gamma$ for $u \in V$.

Under the assumptions (H.1) and (H.2), it is known that equations (2.3) admits a unique solution.

We finish this section with two examples.

Example 2.1 Let D be a bounded domain in $\mathbf{R}^d$. Set $H = L^2(D)$. Let $V = H_0^{1,2}(D)$ denote the Sobolev space of order one with homogenous boundary conditions. Denote by $a(x) = (a_{ij}(x))$ a symmetric matrix-valued function on D satisfying the uniform ellipticity condition:

$$
\frac{1}{c} I_d \leq a(x) \leq c I_d \qquad \text{for some constant} \quad c \in (0, \infty).
$$

Define

$$
Au = -div(a(x)\nabla u(x)).
$$

Then (2.2) is fulfilled for (H, V, A).

Example 2.2 Let $Au = -\Delta_\alpha$, where Δ_α denotes the generator of a symmetric α-stable process in R^d, $0 < \alpha \le 2$. Δ_α is called the fractional Laplace operator. It is well known that the Dirichlet form associated with Δ_α is given by

$$\mathcal{E}(u,v) = K(d,\alpha)\int\int_{R^d\times R^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,dxdy,$$

$$D(\mathcal{E}) = \{u \in L^2(R^d): \quad \int\int_{R^d\times R^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}\,dxdy < \infty\},$$

where $K(d,\alpha) = \alpha 2^{\alpha-3}\pi^{-\frac{d+2}{2}} sin(\frac{\alpha\pi}{2})\Gamma(\frac{d+\alpha}{2})\Gamma(\frac{\alpha}{2})$. To study equation (2.3), we choose $H = L^2(\mathbf{R}^d)$, and $V = D(\mathcal{E})$ with the inner product $< u, v >= \mathcal{E}(u,v) + (u,v)_{L^2(R^d)}$. Then (2.2) is fulfilled for (H, V, A). See [FOT] for details about the fractional Laplace operator.

3 Approximations of SPDEs by pure jump type SPDEs

Set, for $\varepsilon \in (0,1)$,

$$\alpha(\epsilon) = \left(\int_{\{|x|\le\epsilon\}} x^2\nu(dx)\right)^{\frac{1}{2}}$$

Consider the following SPDE driven by pure jump noise:

$$\begin{aligned} X_t^\varepsilon &= h - \int_0^t AX_s^\varepsilon ds + \int_0^t [b_1(X_s^\varepsilon) + b_2(X_s^\varepsilon)]ds \\ &\quad + \frac{1}{\alpha(\epsilon)}\int_0^t\int_{|x|\le\varepsilon} \sigma(X_{s-}^\varepsilon)x\tilde{N}(ds,dx). \end{aligned} \quad (3.7)$$

Under the assumptions (H.1) and (H.2), the SPDE above admits a unique solution. See [RZ], [LR] and also [AWZ]. Let X denote the solution to the SPDE (2.3):

$$X_t = h - \int_0^t AX_s ds + \int_0^t [b_1(X_s) + b_2(X_s)]ds + \int_0^t \sigma(X_s)dB_s. \quad (3.8)$$

Denote by μ_ε, μ respectively the laws of X^ε and X on the spaces $D([0,T], H)$ and $C([0,T], H)$. Consider the following conditions:

(H.3) There exists a sequence of mappings $\sigma_n(\cdot): H \to V$ such that
(i) $|\sigma_n(y_1) - \sigma_n(y_2)|_H \le c|y_1 - y_2|_H$, where c is a constant independent of n,
(ii) $|\sigma_n(y) - \sigma(y)|_H \to 0$ uniformly on bounded subsets of H.

Remark 3.1 *In most of the cases, one simply chooses σ_n to be the finite dimensional projection of σ into the space V.*

(H.3)$'$ The mapping $\sigma(\cdot)$ takes the space V into itself and satisfies $||\sigma(y)||_V \leq c(1+||y||_V)$ for some constant c.

(H.4) There exists an orthonormal basis $\{e_k, k \geq 1\}$ of H such that $Ae_k = \lambda_k e_k$ and $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \to \infty$ as $n \to \infty$.

We first prepare some preliminary results needed for the proofs of the main theorems.

The following estimate holds for $\{X^\varepsilon, \varepsilon > 0\}$.

Lemma 3.2 *Let X^ε be the solution of equation (3.7). If $\frac{\varepsilon}{\alpha(\epsilon)} \leq C_0$ for some constant C_0, then we have for $p \geq 2$,*

$$\sup_\varepsilon \{E[\sup_{0\leq t\leq T} |X_t^\varepsilon|_H^p] + E[\left(\int_0^T ||X_s^\varepsilon||_V^2 ds\right)^{\frac{p}{2}}]\} < \infty. \tag{3.9}$$

Proof. We prove the lemma for $p = 4$. Other cases are similar. In view of the assumption (H.2), by Ito's formula, we have

$$\begin{aligned} & |X_t^\varepsilon|_H^2 \\ = \; & |h|_H^2 - 2\int_0^t < X_s^\varepsilon, AX_s^\varepsilon > ds + 2\int_0^t < X_s^\varepsilon, b_1(X_s^\varepsilon) > ds \\ + \; & \int_0^t \int_{|x|\leq\varepsilon} \left(|\frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x|_H^2 + 2 < X_{s-}^\varepsilon, \frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x > \right) \tilde{N}(ds,dx) \\ + \; & \int_0^t \int_{|x|\leq\varepsilon} |\frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x|_H^2 ds\nu(dx). \end{aligned} \tag{3.10}$$

Let

$$M_t = \int_0^t \int_{|x|\leq\varepsilon} \left(|\frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x|_H^2 + 2 < X_{s-}^\varepsilon, \frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x > \right) \tilde{N}(ds,dx).$$

By Burkhölder's inequality, for $t \leq T$, and a positive constant C, we have

$$\begin{aligned} & E[\sup_{0\leq u\leq t} |M_u|_H^2] \leq CE[[M,M]_t] \\ = \; & CE[\int_0^t \int_{|x|\leq\varepsilon} \left(|\frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x|_H^2 + 2 < X_{s-}^\varepsilon, \frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x > \right)^2 N(ds,dx)] \\ = \; & CE[\int_0^t \int_{|x|\leq\varepsilon} \left(|\frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x|_H^2 + 2 < X_{s-}^\varepsilon, \frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x > \right)^2 ds\nu(dx)] \\ \leq \; & CE[\int_0^t (1+|X_s^\varepsilon|_H^4) ds], \end{aligned} \tag{3.11}$$

where the linear growth condition on σ and the fact $\frac{\varepsilon}{\alpha(\epsilon)} \leq C_0$ have been used. Use first (2.2) and then square both sides of the resulting inequality to obtain from (3.10) that

$$
\begin{aligned}
& |X_t^\varepsilon|_H^4 + \left(\int_0^t ||X_s^\varepsilon||_V^2 ds\right)^2 \\
\leq \ & C_T|h|_H^4 + C_T\int_0^t (1+|X_s^\varepsilon|_H^4)ds + C_T M_t^2. \qquad (3.12)
\end{aligned}
$$

Take superemum over the interval $[0,t]$ in (3.12), use (3.11) to get

$$
\begin{aligned}
& E[\sup_{0\leq s\leq t}|X_s^\varepsilon|_H^4] + E[\left(\int_0^t ||X_s^\varepsilon||_V^2 ds\right)^2] \\
\leq \ & C|h|_H^4 + CE[\int_0^t (1+|X_s^\varepsilon|_H^4)ds]. \qquad (3.13)
\end{aligned}
$$

Applying Gronwall's inequality proves the lemma. ■

Proposition 3.3 *Assume (H.1), (H.2), (H.3)′,(H.4) and* $\frac{\varepsilon}{\alpha(\epsilon)} \leq C_0$ *for some constant* C_0. *Then the family* $\{X^\varepsilon, \varepsilon > 0\}$ *is tight on the space* $D([0,T], H)$.

Proof. Write

$$
Y_t^\varepsilon = \frac{1}{\alpha(\epsilon)}\int_0^t\int_{|x|\leq\varepsilon} \sigma(X_{s-}^\varepsilon)x\tilde{N}(ds,dx), \qquad (3.14)
$$

and set $Z_t^\varepsilon = X_t^\varepsilon - Y_t^\varepsilon$. It suffices to prove that both $\{Y^\varepsilon, \varepsilon > 0\}$ and $\{Z^\varepsilon, \varepsilon > 0\}$ are tight. This is done in two steps.

Step 1. Prove that $\{Y^\varepsilon, \varepsilon > 0\}$ is tight.

In view of the assumptions on σ (H.3)′, we have $Y^\varepsilon \in D([0,T], V)$. Since the imbedding $V \subset H$ is compact, according to Theorem 3.1 in [J], it is sufficient to show that for every $e \in H$, $\{< Y^\varepsilon, e >, \varepsilon > 0\}$ is tight in $D([0,T], R)$. Note that

$$
\begin{aligned}
& \sup_\varepsilon E[\sup_{0\leq t\leq T} < Y_t^\varepsilon, e >^2] \leq \sup_\varepsilon E[\sup_{0\leq t\leq T}|Y_t^\varepsilon|_H^2] \\
\leq \ & C\sup_\varepsilon \frac{1}{\alpha(\epsilon)^2}E[\int_0^T\int_{|x|\leq\varepsilon}|\sigma(X_s^\varepsilon)|_H^2 x^2\nu(dx)ds] \\
= \ & C\sup_\varepsilon E[\int_0^T |\sigma(X_s^\varepsilon)|_H^2 ds] < \infty, \qquad (3.15)
\end{aligned}
$$

and for any stoping times $\tau_\varepsilon \leq T$ and any positive constants $\delta_\varepsilon \to 0$ we have

$$
\begin{aligned}
& E[| < Y_{\tau_\varepsilon}^\varepsilon, e > - < Y_{\tau_\varepsilon+\delta_\varepsilon}^\varepsilon, e > |^2] \leq \frac{1}{\alpha(\epsilon)^2}E[\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\int_{|x|\leq\varepsilon}|\sigma(X_s^\varepsilon)|_H^2 x^2\nu(dx)ds] \\
\leq \ & C\delta_\varepsilon \sup_\varepsilon E[(1+\sup_{0\leq t\leq T}|X_t^\varepsilon|_H^2)] \to 0, \qquad (3.16)
\end{aligned}
$$

as $\varepsilon \to 0$. By Theorem 3.1 in [J], (3.15) and (3.16) yields the tightness of $< Y^\varepsilon, e >, \varepsilon > 0$.

Step 2. Prove that $\{Z^\varepsilon, \varepsilon > 0\}$ is tight.

It is easy to see that Z^ε satisfies the equation:

$$\begin{aligned} Z_t^\varepsilon &= h - \int_0^t AZ_s^\varepsilon ds - \int_0^t AY_s^\varepsilon ds \\ &\quad + \int_0^t b_1(Z_s^\varepsilon + Y_s^\varepsilon) ds + \int_0^t b_2(Z_s^\varepsilon + Y_s^\varepsilon) ds. \end{aligned} \tag{3.17}$$

Recall $\{e_k, k \geq 1\}$ is the othonormal basis of H consisting of eigenvectors of A (see (H.4)). We have

$$\begin{aligned} & < Z_t^\varepsilon, e_k > \\ = & < h, e_k > -\lambda_k \int_0^t < Z_s^\varepsilon, e_k > ds - \lambda_k \int_0^t < Y_s^\varepsilon, e_k > ds \\ & + \int_0^t < b_1(Z_s^\varepsilon + Y_s^\varepsilon), e_k > ds + \int_0^t < b_2(Z_s^\varepsilon + Y_s^\varepsilon), e_k > ds. \end{aligned} \tag{3.18}$$

By Corollary 5.2 in [J], to obtain the tightness of $\{Z^\varepsilon, \varepsilon > 0\}$ we need to show

(i). $\{< Z^\varepsilon, e_k >, \varepsilon > 0\}$ is tight in $D([0,T], R)$ for every k,

(ii). for any $\delta > 0$,

$$\lim_{N\to\infty} \sup_\varepsilon P(\sup_{0\leq t\leq T} R_N^\varepsilon(t) > \delta) = 0, \tag{3.19}$$

where

$$R_N^\varepsilon(t) = \sum_{k=N}^\infty < Z_t^\varepsilon, e_k >^2 .$$

The proof of (i) is similar to that of the tightness of $< Y^\varepsilon, e >, \varepsilon > 0$. It is omitted. Let us prove (ii). By the chain rule, it follows that

$$\begin{aligned} < Z_t^\varepsilon, e_k >^2 &= < h, e_k >^2 -2\lambda_k \int_0^t < Z_s^\varepsilon, e_k >^2 ds - 2\lambda_k \int_0^t < Y_s^\varepsilon, e_k >< Z_s^\varepsilon, e_k > ds \\ &\quad +2\int_0^t < b_1(Z_s^\varepsilon + Y_s^\varepsilon), e_k >< Z_s^\varepsilon, e_k > ds \\ &\quad +2\int_0^t < b_2(Z_s^\varepsilon + Y_s^\varepsilon), e_k >< Z_s^\varepsilon, e_k > ds. \end{aligned} \tag{3.20}$$

By the variation of constants formula, we have

$$\begin{aligned} < Z_t^\varepsilon, e_k >^2 &= e^{-2\lambda_k t} < h, e_k >^2 -2\lambda_k \int_0^t e^{-2\lambda_k(t-s)} < Y_s^\varepsilon, e_k >< Z_s^\varepsilon, e_k > ds \\ &\quad +2\int_0^t e^{-2\lambda_k(t-s)} < b_1(Z_s^\varepsilon + Y_s^\varepsilon), e_k >< Z_s^\varepsilon, e_k > ds \\ &\quad +2\int_0^t e^{-2\lambda_k(t-s)} < b_2(Z_s^\varepsilon + Y_s^\varepsilon), e_k >< Z_s^\varepsilon, e_k > ds. \end{aligned} \tag{3.21}$$

Hence

$$
\begin{aligned}
& R^\varepsilon_N(t) = \sum_{k=N}^\infty < Z^\varepsilon_t, e_k >^2 \\
= \quad & \sum_{k=N}^\infty e^{-2\lambda_k t} < h, e_k >^2 -2\int_0^t \sum_{k=N}^\infty \lambda_k e^{-2\lambda_k(t-s)} < Y^\varepsilon_s, e_k >< Z^\varepsilon_s, e_k > ds \\
& +2\int_0^t \sum_{k=N}^\infty e^{-2\lambda_k(t-s)} < b_1(Z^\varepsilon_s + Y^\varepsilon_s), e_k >< Z^\varepsilon_s, e_k > ds \\
& +2\int_0^t \sum_{k=N}^\infty e^{-2\lambda_k(t-s)} < b_2(Z^\varepsilon_s + Y^\varepsilon_s), e_k >< Z^\varepsilon_s, e_k > ds \\
=: \quad & I^{(1)}_N(t) + I^{(2)}_N(t) + I^{(3)}_N(t) + I^{(4)}_N(t).
\end{aligned} \tag{3.22}
$$

Obviously

$$
I^{(1)}_N(t) \leq \sum_{k=N}^\infty < h, e_k >^2 \to 0, \tag{3.23}
$$

as $N \to \infty$. For the third term on the right side of (3.22), we have

$$
\begin{aligned}
|I^{(3)}_N(t)| \quad \leq \quad & 2\int_0^t e^{-2\lambda_N(t-s)} \sum_{k=N}^\infty | < b_1(Z^\varepsilon_s + Y^\varepsilon_s), e_k >< Z^\varepsilon_s, e_k > |ds \\
\leq \quad & 2\int_0^t e^{-2\lambda_N(t-s)} ds (\sup_{0\leq s\leq T} |Z^\varepsilon_s|_H)(\sup_{0\leq s\leq T} |b_1(Z^\varepsilon_s + Y^\varepsilon_s)|_H) \\
\leq \quad & C\frac{1}{\lambda_N}\left(1 + \sup_{0\leq s\leq T} |Z^\varepsilon_s|^2_H + \sup_{0\leq s\leq T} |Y^\varepsilon_s|^2_H\right).
\end{aligned} \tag{3.24}
$$

Hence,

$$
\begin{aligned}
& \sup_\varepsilon E[\sup_{0\leq t\leq T} |I^{(3)}_N(t)|] \\
\leq \quad & C\frac{1}{\lambda_N}\left(1 + \sup_\varepsilon E[\sup_{0\leq s\leq T} |Z^\varepsilon_s|^2_H] + \sup_\varepsilon E[\sup_{0\leq s\leq T} |Y^\varepsilon_s|^2_H]\right) \\
& \to 0, \quad \text{as} \quad N \to \infty.
\end{aligned} \tag{3.25}
$$

Let us turn to $I_N^{(2)}(t)$. By Hölder's inequality,

$$
\begin{aligned}
|I_N^{(2)}(t)| &\leq 2\int_0^t (\sum_{k=N}^{\infty} e^{-4\lambda_k(t-s)}\lambda_k < Z_s^\varepsilon, e_k >^2)^{\frac{1}{2}} \\
&\qquad \times(\sum_{k=N}^{\infty} \lambda_k < Y_s^\varepsilon, e_k >^2)^{\frac{1}{2}} ds \\
&\leq 2\int_0^t (\sum_{k=N}^{\infty} e^{-4\lambda_k(t-s)}\lambda_k < Z_s^\varepsilon, e_k >^2)^{\frac{1}{2}} (< AY_s^\varepsilon, Y_s^\varepsilon >^2)^{\frac{1}{2}} ds \\
&\leq C(\sup_{0\leq s\leq T} ||Y_s^\varepsilon||_V)\int_0^t e^{-\lambda_N(t-s)}(\sum_{k=N}^{\infty} e^{-2\lambda_k(t-s)}\lambda_k < Z_s^\varepsilon, e_k >^2)^{\frac{1}{2}} ds \\
&\leq C(\sup_{0\leq s\leq T} ||Y_s^\varepsilon||_V)(\sup_{0\leq s\leq T} |Z_s^\varepsilon|_H)\int_0^t e^{-\lambda_N(t-s)}\frac{1}{\sqrt{t-s}} ds \\
&\leq C(\frac{1}{\sqrt{\lambda_N}}\int_0^\infty e^{-u}\frac{1}{\sqrt{u}} du)(\sup_{0\leq s\leq T} ||Y_s^\varepsilon||_V)(\sup_{0\leq s\leq T} |Z_s^\varepsilon|_H). \qquad (3.26)
\end{aligned}
$$

In view of the assumption (H.2), the last term on the right side of (3.22) can be estimated as follows:

$$
\begin{aligned}
|I_N^{(4)}(t)| &= |\int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k(t-s)} < b_2(X_s^\varepsilon), e_k >< Z_s^\varepsilon, e_k > ds| \\
&= |\int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k(t-s)}\sqrt{\lambda_0+\lambda_k} < (A+\lambda_0 I)^{-\frac{1}{2}} b_2(X_s^\varepsilon), e_k >< Z_s^\varepsilon, e_k > ds| \\
&\leq C\int_0^t \left(\sum_{k=N}^{\infty} e^{-4\lambda_k(t-s)} < (A+\lambda_0 I)^{-\frac{1}{2}} b_2(X_s^\varepsilon), e_k >^2\right)^{\frac{1}{2}} \\
&\qquad \times\left(\sum_{k=N}^{\infty}(\lambda_0+\lambda_k) < Z_s^\varepsilon, e_k >^2\right)^{\frac{1}{2}} ds \\
&\leq C\int_0^t ||Z_s^\varepsilon||_V e^{-2\lambda_N(t-s)}\left(\sum_{k=N}^{\infty} < (A+\lambda_0 I)^{-\frac{1}{2}} b_2(X_s^\varepsilon), e_k >^2\right)^{\frac{1}{2}} ds \\
&\leq C\int_0^t ||Z_s^\varepsilon||_V e^{-2\lambda_N(t-s)}||b_2(X_s^\varepsilon)||_{V^*} ds \\
&\leq C\int_0^t ||Z_s^\varepsilon||_V e^{-2\lambda_N(t-s)}|X_s^\varepsilon|_H^{2-\gamma}||X_s^\varepsilon||_V^{\gamma} ds. \qquad (3.27)
\end{aligned}
$$

This yields that

$$
\begin{aligned}
|I_N^{(4)}(t)| &\le C \sup_{0\le s\le T} |X_s^\varepsilon|_H^{2-\gamma} \int_0^t ||Z_s^\varepsilon||_V e^{-2\lambda_N(t-s)} ||X_s^\varepsilon||_V^\gamma ds \\
&\le C \sup_{0\le s\le T} |X_s^\varepsilon|_H^{2-\gamma} \int_0^t e^{-2\lambda_N(t-s)} (||X_s^\varepsilon||_V^{1+\gamma} + ||X_s^\varepsilon||_V^\gamma ||Y_s^\varepsilon||_V) ds \\
&\le C \sup_{0\le s\le T} |X_s^\varepsilon|_H^{2-\gamma} \left(\int_0^t e^{-\frac{4}{1-\gamma}\lambda_N(t-s)} ds \right)^{\frac{1-\gamma}{2}} \\
&\qquad \times \left(\int_0^T \left(||X_s^\varepsilon||_V^{1+\gamma} + ||X_s^\varepsilon||_V^\gamma ||Y_s^\varepsilon||_V \right)^{\frac{2}{1+\gamma}} ds \right)^{\frac{1+\gamma}{2}} \\
&\le C(\frac{1}{\lambda_N})^{\frac{1-\gamma}{2}} \sup_{0\le s\le T} |X_s^\varepsilon|_H^{2-\gamma} \left(\int_0^T \left(||X_s^\varepsilon||_V^{1+\gamma} + ||X_s^\varepsilon||_V^\gamma ||Y_s^\varepsilon||_V \right)^{\frac{2}{1+\gamma}} ds \right)^{\frac{1+\gamma}{2}}
\end{aligned} \tag{3.28}
$$

Hence,

$$
\begin{aligned}
&\sup_\varepsilon E[\sup_{0\le t\le T} |I_N^{(4)}(t)|] \\
\le\ & C(\frac{1}{\lambda_N})^{\frac{1-\gamma}{2}} \sup_\varepsilon E\left[\sup_{0\le s\le T} |X_s^\varepsilon|_H^{2-\gamma} \right. \\
&\left. \times \left(\int_0^T \left(||X_s^\varepsilon||_V^{1+\gamma} + ||X_s^\varepsilon||_V^\gamma ||Y_s^\varepsilon||_V \right)^{\frac{2}{1+\gamma}} ds \right)^{\frac{1+\gamma}{2}} \right] \\
\le\ & C(\frac{1}{\lambda_N})^{\frac{1-\gamma}{2}} \sup_\varepsilon E\left[\sup_{0\le s\le T} |X_s^\varepsilon|_H^{2-\gamma} \right. \\
&\left. \times \left(\int_0^T \left(C||X_s^\varepsilon||_V^2 + c||Y_s^\varepsilon||_V^2 \right) ds \right)^{\frac{1+\gamma}{2}} \right] \\
&\to 0, \quad \text{as} \quad N\to\infty,
\end{aligned} \tag{3.29}
$$

where we used the fact that

$$
|ab| \le C(|a|^p + |b|^q), \quad \frac{1}{p} + \frac{1}{q} = 1.
$$

Putting together (3.22)—(3.29) and applying the Chebychev inequality we obtain (3.19). ■

Let $\mathcal{D}$ denote the class of functions $f \in C_b^3(H)$ that satisfy (i) $f'(z) \in D(A)$ and $|Af'(z)|_H \le C(1+|z|_H)$ for some constant C, where $f'(z)$ stands for the Frechet derivative of f, (ii) f'', f''' are bounded.

For $f \in \mathcal{D}$, define

$$
\begin{aligned}
&L^{\varepsilon} f(z) = - < Af'(z), z > + < b_1(z), f'(z) > + < b_2(z), f'(z) > \\
&+ \int_{|x| \le \varepsilon} \left[f(z + \frac{1}{\alpha(\epsilon)} \sigma(z) x) - f(z) - < f'(z), \frac{1}{\alpha(\epsilon)} \sigma(z) x > \right] \nu(dx),
\end{aligned} \tag{3.30}
$$

and

$$
\begin{aligned}
Lf(z) &= - < Af'(z), z > + < b_1(z), f'(z) > + < b_2(z), f'(z) > \\
&\quad + \frac{1}{2} < f''(z)\sigma(z), \sigma(z) > .
\end{aligned} \tag{3.31}
$$

Lemma 3.4 *Assume* $\lim_{\varepsilon \to 0} \frac{\varepsilon}{\alpha(\varepsilon)} = 0$. *For* $f \in \mathcal{D}$, *it holds that*

$$
L^{\varepsilon} f(z) \to Lf(z) \quad \text{uniformly on bounded subsets of} \quad H \tag{3.32}
$$

as $\varepsilon \to 0$.

Proof. Note that

$$
f(y + w) - f(y) - < f'(y), w >= \int_0^1 d\alpha \int_0^{\alpha} < f''(y + \beta w) w, w > d\beta.
$$

Thus

$$
\begin{aligned}
&L^{\varepsilon} f(z) - Lf(z) \\
= &\int_{|x| \le \varepsilon} \int_0^1 d\alpha \int_0^{\alpha} d\beta < f''(z + \beta \frac{1}{\alpha(\epsilon)} \sigma(z) x) \frac{1}{\alpha(\epsilon)} \sigma(z) x, \frac{1}{\alpha(\epsilon)} \sigma(z) x > \nu(dx) \\
&\quad - \int_0^1 d\alpha \int_0^{\alpha} d\beta < f''(z) \sigma(z), \sigma(z) > \\
= &\frac{1}{\alpha(\epsilon)^2} \int_{|x| \le \varepsilon} x^2 \nu(dx) \int_0^1 d\alpha \int_0^{\alpha} d\beta \Big[< f''(z + \beta \frac{1}{\alpha(\epsilon)} \sigma(z) x) \sigma(z), \sigma(z) > \\
&\quad - < f''(z) \sigma(z), \sigma(z) >]
\end{aligned} \tag{3.33}
$$

Hence, for $z \in B_N = \{z \in H; |z|_H \le N\}$ we have

$$
\begin{aligned}
&|L^{\varepsilon} f(z) - Lf(z)| \\
\le\; &C \frac{1}{\alpha(\epsilon)^2} \int_{|x| \le \varepsilon} x^2 \nu(dx) \int_0^1 d\alpha \int_0^{\alpha} d\beta \beta \frac{1}{\alpha(\epsilon)} |\sigma(z)|_H |x| |\sigma(z)|_H^2 \\
\le\; &C_N \frac{\varepsilon}{\alpha(\epsilon)} \to 0,
\end{aligned} \tag{3.34}
$$

uniformly on B_N as $\varepsilon \to 0$, where we have used the local Lipschtiz continuity of $f''(z)$. ■

Theorem 3.5 *Suppose (H.1), (H.2),(H.3)′, (H.4) hold and* $\lim_{\varepsilon\to 0}\frac{\varepsilon}{\alpha(\varepsilon)}=0$. *Then, for any* $T>0$, μ_ε *converges weakly to* μ, *for* $\varepsilon\to 0$, *on the space* $D([0,T],H)$ *equipped with the Skorohod topology.*

Proof. Since the mapping σ takes values in the space V, by Proposition 3.3, the family $\{\mu_\varepsilon,\varepsilon>0\}$ is tight. Let μ_0 be the weak limit of any convergent sequence $\{\mu_{\varepsilon_n}\}$ on the canonical space $(\Omega=D([0,T],H),\mathcal{F})$ as $\varepsilon_n\to 0$. We will show that $\mu_0=\mu$. Denote by $X_t(\omega)=w(t),\omega\in\Omega$ the coordinate process. Set $J(X)=\sup_{0\le s\le T}|X_s-X_{s-}|_H$. Since

$$\begin{aligned}
& E^{\mu_\varepsilon}[J(X)]=E[J(X^\varepsilon)]\\
\le\ & \frac{\varepsilon}{\alpha(\epsilon)}E[\sup_{0\le s\le T}|\sigma(X_s^\varepsilon)|_H]\\
\le\ & C\frac{\varepsilon}{\alpha(\epsilon)}(1+E[\sup_{0\le s\le T}|X_s^\varepsilon|_H])\to 0, \qquad (3.35)
\end{aligned}$$

as $\varepsilon\to 0$, it follows from Theorem 13.4 in [B] that μ_0 is supported on the $C([0,T],H)$, the space of H-valued continuous functions on $[0,T]$. As a consequence, the finite dimensional distributions of μ_{ε_n} converge to that of μ_0.

Let $f\in\mathcal{D}$. By Ito's formula,

$$\begin{aligned}
& f(X_t^\varepsilon)-f(h)-\int_0^t L^\varepsilon f(X_s^\varepsilon)ds\\
=\ & \int_0^t\int_{|x|\le\varepsilon}\{f(X_{s-}^\varepsilon+\frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x)-f(X_{s-}^\varepsilon)\}\tilde{N}(ds,dx) \qquad (3.36)
\end{aligned}$$

is a martingale. Hence, for any $s_0<s_1<...<s_n\le s<t$ and $f_0,f_1,...f_n\in C_b(H)$ it holds that

$$\begin{aligned}
& E^{\mu_\varepsilon}[\left(f(X_t)-f(X_s)-\int_s^t L^\varepsilon f(X_u)du\right)f(X_{s_0})...f(X_{s_n})]\\
=\ & 0. \qquad (3.37)
\end{aligned}$$

For any positive constant $M>0$, by Lemma 3.4 we have

$$\lim_{n\to\infty}E^{\mu_{\varepsilon_n}}[\int_s^t|L^{\varepsilon_n}f(X_u)-Lf(X_u)|du,\ \sup_{0\le u\le T}|X_u|_H\le M]=0. \qquad (3.38)$$

On the other hand, in view of the assumptions on f we have

$$\begin{aligned}
& \sup_n E^{\mu_{\varepsilon_n}}[\int_s^t|L^{\varepsilon_n}f(X_u)-Lf(X_u)|du,\ \sup_{0\le u\le T}|X_u|_H>M]\\
\le\ & C\frac{1}{M}\sup_n E^{\mu_{\varepsilon_n}}[\sup_{0\le u\le T}|X_u|_H^3]\le C'\frac{1}{M} \qquad (3.39)
\end{aligned}$$

Combining (3.38) with (3.39) we arrive at

$$\lim_{n\to\infty} E^{\mu_{\varepsilon_n}}[\int_s^t |L^{\varepsilon_n} f(X_u) - Lf(X_u)|du] = 0. \tag{3.40}$$

By the weak convergence of μ_{ε_n} and the convergence of finite distributions, it follows from (3.37) and (3.40) that

$$\begin{aligned} & E^{\mu_0}[(f(X_t) - f(X_s) - \int_s^t Lf(X_u)du)f(X_{s_0})...f(X_{s_n})] \\ = & \lim_{n\to\infty} E^{\mu_{\varepsilon_n}}[(f(X_t) - f(X_s) - \int_s^t Lf(X_u)du)f(X_{s_0})...f(X_{s_n})] \\ = & \lim_{n\to\infty} E^{\mu_{\varepsilon_n}}[(f(X_t) - f(X_s) - \int_s^t L^{\varepsilon_n} f(X_u)du)f(X_{s_0})...f(X_{s_n})] \\ = & \ 0. \end{aligned} \tag{3.41}$$

Since $s_0 < s_1 < ... < s_n \leq s < t$ are arbitrary, (3.41) implies that for any $f \in \mathcal{D}$,

$$M_t^f = f(X_t) - f(h) - \int_0^t Lf(X_s)ds, \quad t \geq 0,$$

is a martingale under μ_0. In particular, let $f(z) =< e_k, z >$ and $f(z) =< e_k, z >< e_j, z >$ respectively to obtain that under μ_0

$$\begin{aligned} M_t^k \ := \ & < e_k, X_t > - < e_k, h > + \int_0^t < Ae_k, X_s > ds - \int_0^t < b_1(X_s), e_k > ds \\ & - \int_0^t < b_2(X_s), e_k > ds \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} & M_t^{k,j} :=< e_k, X_t >< e_j, X_t > - < e_k, h >< e_j, h > \\ & + \int_0^t \{< Ae_k, X_s >< e_j, X_s > + < Ae_j, X_s >< e_k, X_s >\}ds \\ & - \int_0^t < b_1(X_s), e_k < e_j, X_s > +e_j < e_k, X_s >> ds \\ & - \int_0^t < b_2(X_s), e_k < e_j, X_s > +e_j < e_k, X_s >> ds \\ & - \int_0^t < \sigma(X_s), e_k >< \sigma(X_s), e_j > ds \end{aligned} \tag{3.43}$$

are martingales. This together with Ito's formula yields that

$$< M^k, M^j >_t = \int_0^t < \sigma(X_s), e_k >< \sigma(X_s), e_j > ds, \tag{3.44}$$

where $< M^k, M^j >$ stands for the sharp bracket of the two martingales. Now by Theorem 18.12 in [K] (or Theorem 7.1′ in [IW]), there exists a probability space $(\Omega', \mathcal{F}', P')$ with a filtration $\mathcal{F}'_t$ such that on the standard extension

$$(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathcal{F}_t \times \mathcal{F}'_t, \mu_0 \times P')$$

of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ there exists a Brownian motion $B_t, t \geq 0$ such that

$$M_t^k = \int_0^t < \sigma(X_s), e_k > dB_s, \tag{3.45}$$

namely,

$$\begin{aligned} & < e_k, X_t > - < e_k, h > \\ = & - \int_0^t < Ae_k, X_s > ds + \int_0^t < b_1(X_s), e_k > ds + \int_0^t < b_2(X_s), e_k > ds \\ & + \int_0^t < \sigma(X_s), e_k > dB_s \end{aligned} \tag{3.46}$$

for any $k \geq 1$. Thus, under μ_0, $X_t, t \geq 0$ is a weak solution (both in the probabilistic and in PDE sense) of the SPDE:

$$X_t = h - \int_0^t AX_s ds + \int_0^t b_1(X_s) ds + \int_0^t b_2(X_s) ds + \int_0^t \sigma(X_s) dB_s$$

By the uniqueness of the above equation, we conclude that $\mu_0 = \mu$ completing the proof of the theorem. ■

Theorem 3.6 *Suppose (H.1), (H.2), (H.3) and (H.4) hold and* $\lim_{\varepsilon \to 0} \frac{\varepsilon}{\alpha(\varepsilon)} = 0$. *Then, for any* $T > 0$, μ_ε *converges weakly to* μ, *for* $\varepsilon \to 0$, *on the space* $D([0,T], H)$ *equipped with the Skorohod topology.*

Proof. Let $\sigma_n(\cdot)$ be the mapping specified in (H.3). Let $X^{n,\varepsilon}, X^n$ be the solutions of the SPDEs:

$$\begin{aligned} X_t^{n,\varepsilon} &= h - \int_0^t AX_s^{n,\varepsilon} ds + \int_0^t b_1(X_s^{n,\varepsilon}) ds + \int_0^t b_2(X_s^{n,\varepsilon}) ds \\ & + \frac{1}{\alpha(\epsilon)} \int_0^t \int_{|x| \leq \varepsilon} \sigma_n(X_{s-}^{n,\varepsilon}) x \tilde{N}(ds, dx). \end{aligned} \tag{3.47}$$

$$\begin{aligned} X_t^n &= h - \int_0^t AX_s^n ds + \int_0^t b_1(X_s^n) ds + \int_0^t b_2(X_s^n) ds \\ & + \int_0^t \sigma_n(X_s^n) dB_s. \end{aligned} \tag{3.48}$$

We claim that for any $\delta > 0$,

$$\lim_{n\to\infty} \sup_{\varepsilon} P(\sup_{0\le t\le T} |X_t^{n,\varepsilon} - X_t^{\varepsilon}| > \delta) = 0. \tag{3.49}$$

$$\lim_{n\to\infty} P(\sup_{0\le t\le T} |X_t^{n} - X_t|^2 > \delta) = 0. \tag{3.50}$$

Let us only prove (3.49). The proof of (3.50) is simpler. As the proof of (3.9), we can show that

$$\sup_n \sup_{\varepsilon} \{E[\sup_{0\le t\le T} |X_t^{n,\varepsilon}|_H^2] + E[\int_0^T ||X_s^{n,\varepsilon}||_V^2 ds]\} < \infty. \tag{3.51}$$

$$\sup_n \{E[\sup_{0\le t\le T} |X_t^{n}|_H^2] + E[\int_0^T ||X_s^{n}||_V^2 ds]\} < \infty. \tag{3.52}$$

By Ito's formula, we have

$$\begin{aligned}
& e^{-\gamma\int_0^t (||X_s^{n,\varepsilon}||_V^2 + ||X_s^{\varepsilon}||_V^2) ds} |X_t^{n,\varepsilon} - X_t^{\varepsilon}|_H^2 \\
= & -\gamma \int_0^t e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^{\varepsilon}||_V^2) du} |X_s^{n,\varepsilon} - X_s^{\varepsilon}|_H^2 (||X_s^{n,\varepsilon}||_V^2 + ||X_s^{\varepsilon}||_V^2) ds \\
& -2 \int_0^t e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^{\varepsilon}||_V^2) du} < X_s^{n,\varepsilon} - X_s^{\varepsilon}, A(X_s^{n,\varepsilon} - X_s^{\varepsilon}) > ds \\
& +2 \int_0^t e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^{\varepsilon}||_V^2) du} < X_s^{n,\varepsilon} - X_s^{\varepsilon}, b_1(X_s^{n,\varepsilon}) - b_1(X_s^{\varepsilon}) > ds \\
& +2 \int_0^t e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^{\varepsilon}||_V^2) du} < X_s^{n,\varepsilon} - X_s^{\varepsilon}, b_2(X_s^{n,\varepsilon}) - b_2(X_s^{\varepsilon}) > ds \\
+ & \int_0^t \int_{|x|\le\varepsilon} e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^{\varepsilon}||_V^2) du} \Big(|\frac{1}{\alpha(\epsilon)} (\sigma_n(X_{s-}^{n,\varepsilon})x - \sigma(X_{s-}^{\varepsilon})x)|_H^2 \\
& \quad +2 < (X_s^{n,\varepsilon} - X_s^{\varepsilon}), \frac{1}{\alpha(\epsilon)} (\sigma_n(X_{s-}^{n,\varepsilon})x - \sigma(X_{s-}^{\varepsilon})x) > \Big) \tilde{N}(ds, dx) \\
+ & \int_0^t \int_{|x|\le\varepsilon} e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^{\varepsilon}||_V^2) du} |\frac{1}{\alpha(\epsilon)} (\sigma_n(X_{s-}^{n,\varepsilon})x - \sigma(X_{s-}^{\varepsilon})x)|_H^2 ds\nu(dx) \\
:= & \sum_{k=1}^{6} I_k^{n,\varepsilon}(t).
\end{aligned} \tag{3.53}$$

In view of the assumption (2.6), we see that

$$\begin{aligned}
& I_1^{n,\varepsilon}(t) + I_2^{n,\varepsilon}(t) + I_4^{n,\varepsilon}(t) \\
\le & -(1-2\beta)\alpha_0 \int_0^t e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^{\varepsilon}||_V^2) du} ||X_s^{n,\varepsilon} - X_s^{\varepsilon}||_V^2 ds,
\end{aligned} \tag{3.54}$$

if $\gamma \ge 2C_1$, where C_1 is the constant appeared in (2.6).

Similar to the proofs of (3.11), (3.13), using Burkhölder's inequality, we obtain from (3.53), (3.54) that for $t \le T$,

$$\begin{aligned}
& E[\sup_{0\le s\le t} e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}|X_s^{n,\varepsilon}-X_s^{\varepsilon}|_H^2] \\
& +E[\int_0^t e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}||X_s^{n,\varepsilon}-X_s^{\varepsilon}||_V^2 ds] \\
\le\ & \frac{1}{4}E[\sup_{0\le s\le t} e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}|X_s^{n,\varepsilon}-X_s^{\varepsilon}|_H^2] \\
+\ & CE[\int_0^t e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}|X_s^{n,\varepsilon}-X_s^{\varepsilon}|_H^2 ds] \\
+\ & CE[\int_0^t\int_{|x|\le\varepsilon} e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}|\frac{1}{\alpha(\epsilon)}(\sigma_n(X_{s-}^{n,\varepsilon})x-\sigma_n(X_{s-}^{\varepsilon})x)|_H^2 ds\nu(dx)] \\
+\ & CE[\int_0^t\int_{|x|\le\varepsilon} e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}|\frac{1}{\alpha(\epsilon)}(\sigma_n(X_{s-}^{\varepsilon})x-\sigma(X_{s-}^{\varepsilon})x)|_H^2 ds\nu(dx)] \\
\le\ & \frac{1}{4}E[\sup_{0\le s\le t} e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}|X_s^{n,\varepsilon}-X_s^{\varepsilon}|_H^2] \\
+\ & CE[\int_0^t e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}|X_s^{n,\varepsilon}-X_s^{\varepsilon}|_H^2) ds] \\
+\ & CE[\int_0^t\int_{|x|\le\varepsilon} e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}|\frac{1}{\alpha(\epsilon)}(\sigma_n(X_{s-}^{\varepsilon})x-\sigma(X_{s-}^{\varepsilon})x)|_H^2 ds\nu(dx)],
\end{aligned} \tag{3.55}$$

where uniform Lipschitz constant of σ_n (H.3)(i) has been used. Applying the Gronwall's inequality we obtain

$$\begin{aligned}
& E[\sup_{0\le s\le t} e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}|X_s^{n,\varepsilon}-X_s^{\varepsilon}|_H^2] \\
& +E[\int_0^t e^{-\gamma\int_0^s(||X_u^{n,\varepsilon}||_V^2+||X_u^{\varepsilon}||_V^2)du}||X_s^{n,\varepsilon}-X_s^{\varepsilon}||_V^2 ds] \\
\le\ & CE[\int_0^T |\sigma_n(X_s^{\varepsilon})-\sigma(X_s^{\varepsilon})|_H^2 ds].
\end{aligned} \tag{3.56}$$

For any $M > 0$, we have

$$\begin{aligned}
& E[\int_0^T |\sigma_n(X_s^\varepsilon) - \sigma(X_s^\varepsilon)|_H^2 ds] \\
= \ & E[\int_0^T |\sigma_n(X_s^\varepsilon) - \sigma(X_s^\varepsilon)|_H^2 ds, \sup_{0\le s\le T} |X_s^\varepsilon|_H \le M] \\
+ \ & E[\int_0^T |\sigma_n(X_s^\varepsilon) - \sigma(X_s^\varepsilon)|_H^2 ds, \sup_{0\le s\le T} |X_s^\varepsilon|_H > M] \\
\le \ & T \sup_{|z|\le M} |\sigma_n(z) - \sigma(z)|_H^2 + CT\frac{1}{M}(1 + E[\sup_{0\le s\le T} |X_s^\varepsilon|_H^3]) \\
\le \ & \sup_{|z|\le M} |\sigma_n(z) - \sigma(z)|_H^2 + CT\frac{1}{M}, \qquad (3.57)
\end{aligned}$$

where (3.9) has been used. Since M can be chosen as large as we wish, together with (3.56) and (H.3)(ii) we deduce that

$$\begin{aligned}
& \lim_{n\to\infty} \sup_\varepsilon E[\sup_{0\le s\le T} e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^\varepsilon||_V^2)du} |X_s^{n,\varepsilon} - X_s^\varepsilon|_H^2] \\
= \ & 0. \qquad (3.58)
\end{aligned}$$

For any given $\delta_1 > 0$, in view of (3.51), (3.52), we can choose a positive constant M_1 such that

$$\begin{aligned}
& \sup_{n,\varepsilon} P(\sup_{0\le t\le T} |X_t^{n,\varepsilon} - X_t^\varepsilon|_H > \delta, \int_0^T (||X_s^{n,\varepsilon}||_V^2 + ||X_s^\varepsilon||_V^2)ds > M_1) \\
\le \ & \sup_{n,\varepsilon} P(\int_0^T (||X_s^{n,\varepsilon}||_V^2 + ||X_s^\varepsilon||_V^2)ds > M_1) \le \frac{\delta_1}{2}. \qquad (3.59)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \sup_\varepsilon P(\sup_{0\le t\le T} |X_t^{n,\varepsilon} - X_t^\varepsilon|_H^2 > \delta, \int_0^T (||X_s^{n,\varepsilon}||_V^2 + ||X_s^\varepsilon||_V^2)ds \le M_1) \\
\le \ & \sup_\varepsilon P(\sup_{0\le s\le T} e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^\varepsilon||_V^2)du} |X_s^{n,\varepsilon} - X_s^\varepsilon|_H^2 \ge e^{-\gamma M_1}\delta^2) \\
\le \ & e^{\gamma M_1}\frac{1}{\delta^2} \sup_\varepsilon E[\sup_{0\le s\le T} e^{-\gamma\int_0^s (||X_u^{n,\varepsilon}||_V^2 + ||X_u^\varepsilon||_V^2)du} |X_s^{n,\varepsilon} - X_s^\varepsilon|_H^2]. \qquad (3.60)
\end{aligned}$$

It follows from (3.58) and (3.60) that there exists $N > 0$ such that for $n \ge N$,

$$\begin{aligned}
& \sup_{n,\varepsilon} P(\sup_{0\le t\le T} |X_t^{n,\varepsilon} - X_t^\varepsilon|_H > \delta, \int_0^T (||X_s^{n,\varepsilon}||_V^2 + ||X_s^\varepsilon||_V^2)ds \le M_1) \\
\le \ & \frac{\delta_1}{2}. \qquad (3.61)
\end{aligned}$$

Combining (3.59) and (3.61) together yields (3.49)

Finally we prove that μ^ε converges to μ. Let μ_n^ε, μ_n denote respectively the laws of $X^{n,\varepsilon}$ and X^n . Let G be a bounded, uniformly continuous function on $E := D([0,T],H)$. For any $n \geq 1$, we write

$$\begin{aligned}
& \int_E G(w)\mu^\varepsilon(dw) - \int_E G(w)\mu(dw) \\
= & \int_E G(w)\mu^\varepsilon(dw) - \int_E G(w)\mu_n^\varepsilon(dw) + \int_E G(w)\mu_n^\varepsilon(dw) - \int_E G(w)\mu_n(dw) \\
& \qquad + \int_E G(w)\mu_n(dw) - \int_E G(w)\mu(dw) \\
= & E[G(X^\varepsilon) - G(X^{n,\varepsilon})] + (\int_E G(w)\mu_n^\varepsilon(dw) - \int_E G(w)\mu_n(dw)) \\
& \quad + E[G(X^n) - G(X)]. \qquad (3.62)
\end{aligned}$$

Give any $\delta > 0$. Since G is uniformly continuous, there exists $\delta_1 > 0$ such that

$$|E[(G(X^\varepsilon) - G(X^{n,\varepsilon})), \sup_{0\leq s\leq T} |X_s^{n,\varepsilon} - X_s^\varepsilon|_H \leq \delta_1]| \leq \frac{\delta}{4} \qquad (3.63)$$

for all $n \geq 1, \varepsilon > 0$. In view of (3.49) and (3.50), there exists N_1,

$$\begin{aligned}
& \sup_\varepsilon |E[(G(X^\varepsilon) - G(X^{N_1,\varepsilon})), \sup_{0\leq s\leq T} |X_s^{N_1,\varepsilon} - X_s^\varepsilon|_H > \delta_1]| \\
\leq & \; C \sup_\varepsilon P(\sup_{0\leq s\leq T} |X_s^{N_1,\varepsilon} - X_s^\varepsilon|_H > \delta_1) \leq \frac{\delta}{4}, \qquad (3.64)
\end{aligned}$$

and

$$|E[(G(X^{N_1}) - G(X))]| \leq \frac{\delta}{4}. \qquad (3.65)$$

On the other hand, by Theorem 3.4, there exists $\varepsilon_1 > 0$ such that for $\varepsilon \leq \varepsilon_1$,

$$|\int_E G(w)\mu_{N_1}^\varepsilon(dw) - \int_E G(w)\mu_{N_1}(dw))| \leq \frac{\delta}{4}. \qquad (3.66)$$

Putting (3.62)—(3.66) together we obtain that for $\varepsilon \leq \varepsilon_1$,

$$|\int_E G(w)\mu^\varepsilon(dw) - \int_E G(w)\mu(dw)| \leq \delta.$$

Since $\delta > 0$ is arbitrarily small, we deduce that

$$\lim_{\varepsilon\to 0} \int_E G(w)\mu^\varepsilon(dw) = \int_E G(w)\mu(dw)$$

finishing the proof of the Theorem. ■

Example 3.7 *Approximations of stochastic Burgers equations*

Consider the stochastic Burgers equations on $[0,1]$:

$$du(t,\xi) = \frac{\partial^2}{\partial \xi^2} u(t,\xi)dt + \frac{1}{2}\frac{\partial}{\partial \xi}[u^2(t,\xi)]dt + \sigma(u(t,\xi))dB_t, \qquad (3.67)$$

$$u(t,0) = u(t,1) = 0, t > 0, \qquad (3.68)$$

$$du^\varepsilon(t,\xi) = \frac{\partial^2}{\partial \xi^2} u^\varepsilon(t,\xi)dt + \frac{1}{2}\frac{\partial}{\partial \xi}[(u^\varepsilon)^2(t,\xi)]dt \qquad (3.69)$$

$$+\frac{1}{\alpha(\epsilon)}\int_{|x|\le\varepsilon} \sigma(u^\varepsilon(t-,\xi))x\tilde{N}(dt,dx), \qquad (3.70)$$

$$u^\varepsilon(t,0) = u^\varepsilon(t,1) = 0, t > 0, \qquad (3.71)$$

where $\sigma(\cdot)$ is a Lipschitz continuous function with $\sigma(0) = 0$.

Let $V = H_0^1(0,1)$ with the norm

$$||v||_V := \Big(\int_0^1 (\frac{\partial u(\xi)}{\partial \xi})^2 \, d\xi\Big)^{\frac{1}{2}} = ||v||.$$

Let $H := L^2(0,1)$ be the L^2-space with inner product $(\cdot)$.
Set

$$Au = -\frac{\partial^2}{\partial \xi^2} u(\xi), \forall u \in H^2(0,1) \cap V.$$

Define for $k \ge 1$,

$$e_k(\xi) = \sqrt{2} sin(k\pi\xi), \xi \in [0,1].$$

Then $e_k, k \ge 1$ are eigenvectors of the operator A with eigenvalues $\lambda_k = \pi^2 k^2$, which forms an orthonormal basis of the Hilbert space H. For $u \in V$, define

$$B(u) := u(\xi)\frac{\partial}{\partial \xi} u(\xi), \quad \sigma(u) := \sigma(u(\xi)).$$

By the Lipschitz continuity of σ, it is easily seen that

$$||\sigma(u)||_V \le C(1 + ||u||_V), \qquad (3.72)$$

hence (H.3)$'$ holds. Now let us show that $B(u)$ satisfies the condition (H.2). First (H.2)(i) holds, in fact

$$< B(u), u > = \int_0^1 u^2(\xi)\frac{\partial}{\partial \xi} u(\xi) d\xi = \frac{1}{3}[u^3(1) - u^3(0)] = 0.$$

Note that $\bar{e}_k = \frac{1}{\sqrt{\lambda_k}} e_k, k \ge 1$ forms an orthonomal basis of V. Thus, for

$u \in V$, we have

$$
\begin{aligned}
||B(u)||^2_{V^*} &= \sum_{k=1}^{\infty} < B(u), \bar{e}_k >^2 \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{2} \frac{1}{\sqrt{\lambda_k}} \int_0^1 \frac{\partial}{\partial \xi}[u^2(\xi)] e_k(\xi) d\xi \right)^2 \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{2} \frac{1}{\sqrt{\lambda_k}} \int_0^1 u^2(\xi) \frac{\partial}{\partial \xi} e_k(\xi) d\xi \right)^2 \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{2} \int_0^1 u(\xi)^2 \sqrt{2} cos(k\pi\xi) d\xi \right)^2 \\
&\leq C \int_0^1 u^\varepsilon(s,\xi)^4 d\xi = C|u|^4_{L^4}, \qquad (3.73)
\end{aligned}
$$

where we have used the fact that $\{\sqrt{2}cos(k\pi\xi); k \geq 1\}$ also forms an orthonormal system of $L^2(0,1)$. Using the following well known interpolation inequality

$$
|u|^4_{L^4} \leq C|u|^3_H ||u||_V, \qquad (3.74)
$$

we obtain from (3.73) that

$$
||B(u)||_{V^*} \leq C|u|^{\frac{3}{2}}_H ||u||^{\frac{1}{2}}_V
$$

proving (H.2)(iii) with $\gamma = \frac{1}{2}$. Finally we will check (H.2)(ii). Let $u, v \in V$. We have

$$
\begin{aligned}
& < B(u) - B(v), u - v > = \frac{1}{2} \int_0^1 \frac{\partial}{\partial \xi}[u^2(\xi) - v^2(\xi)](u(\xi) - v(\xi)) d\xi \\
= & -\frac{1}{2} \int_0^1 (u^2(\xi) - v^2(\xi)) \frac{\partial}{\partial \xi}(u(\xi) - v(\xi)) d\xi \\
\leq & \frac{1}{2} \int_0^1 (\frac{\partial}{\partial \xi}(u(\xi) - v(\xi)))^2 d\xi \\
& + C \int_0^1 (u(\xi) - v(\xi))^2 (u(\xi) + v(\xi))^2 d\xi \\
\leq & \frac{1}{2}||u - v||^2_V + C|u - v|^2_H (||u||^2_\infty + ||v||^2_\infty) \\
\leq & \frac{1}{2}||u - v||^2_V + C|u - v|^2_H (||u||^2_V + ||V||^2_V), \qquad (3.75)
\end{aligned}
$$

which is (H.2)(ii).

Now we can apply Theorem 3.5 to obtain the following convergence of the solutions of stochastic Burgers equations.

Theorem 3.8 *Let u^ε, u be solutions to the stochastic Burgers equations (3.69) and (3.67). Then u^ε converges weakly to u in the space $D([0,T]; H)$.*

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