Cuspidal curves
on Hirzebruch surfaces

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Summary

The first part of this thesis deals with cuspidal curves on Hirzebruch surfaces. Bounds are given on the number and type of cusps on cuspidal curves, and rational cuspidal curves are constructed using birational transformations.

The second part of this thesis deals with Segre classes of closed subschemes of smooth projective toric varieties. An algorithm for computing Segre classes of closed subschemes of projective spaces is generalized to an algorithm for computing Segre classes of closed subschemes of smooth projective toric varieties.

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Oslo, October 2012 **Torgunn Karoline Moe**

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Notation

$\mathbb{N}$ denotes the natural numbers.
$\mathbb{Z}$ denotes the integers.
$\mathbb{R}$ denotes the real numbers.
$\mathbb{C}$ denotes the complex numbers.
$\mathbb{P}^n$ denotes the complex projective n-space.
$\mathbb{F}_e$ denotes the eth Hirzebruch surface, $e \geq 0$.
$\mathbb{P}^1 \times \mathbb{P}^1 := \mathbb{F}_0$.
$\lfloor a \rfloor$ is the largest integer $\leq a$.
$\lceil a \rceil$ is the smallest integer $\geq a$.
$\mathscr{V}$ denotes the zero set in Part I.
V denotes the zero set in Part II.
F_x denotes the partial derivative of the polynomial F with respect to the variable x.

Introduction

Given an algebraic curve in some ambient space, determining its possible number and types of singularities are classical tasks in geometry. The number and types of singularities on a curve are bounded by fundamental invariants of the curve and structural properties of the ambient space. Nevertheless, there are still open problems concerning the classification of algebraic curves up to its number and type of singularities. Restricting to curves with only unibranched singularities, so-called cuspidal curves, the classification is still not complete, even for rational curves on the complex projective plane. The main aim of this thesis is to give examples of rational cuspidal curves on complex Hirzebruch surfaces, and to provide first results in the classification of such curves.

The classification of cuspidal curves on the complex projective plane begins with the construction of curves of low degree. From the late 19th and early 20th century there are impressive systematical constructions of curves of low degree with prescribed singularities. One early comprehensive collection of such curves can be found in [59] by Salmon, and plane rational curves of degree three and four are thoroughly explained by Telling in [63]. While these results on curves of degree three and four are quite complete, the early attempts to construct and classify cuspidal curves of degree five or higher are more rare and mostly incomplete. Curves of degree five are described by Slobin in [61], but only curves with particularly simple cusps, so-called ordinary cusps, appear in this article. Moreover, there are articles describing examples of curves of degree five with many cusps, see for example [13] by del Pezzo and [19, 20] by Field. There are also articles on curves with very peculiar cusps, see [79] by Yoshihara. The first complete classification of cuspidal curves of degree five that we have found, in particular the first complete list of rational cuspidal curves of degree five, is by Namba in [49] from as late as 1984, see also [22] by Flenner and Zaidenberg and [76] by Wall. The list of plane rational cuspidal curves of degree five is quite short, but it contains several interesting curves, the most exceptional being a curve with four cusps. In higher degrees, we only know the complete classification up to the types of singularities in the case of plane rational cuspidal curves of degree six, provided by Fenske in [16].

From the late 19th and early 20th century there are also results estimating the maximal number of singularities on a curve on the complex projective plane. There is a vast number of articles on this subject, and there are also a few results on the maximal number of cusps on a curve. A first result relevant to our situation is given by Clebsch in [8], where the maximal number of ordinary cusps on a rational, not necessarily cuspidal, curve is determined. This result is independently found by Veronese in [73], using the theory of projections. In [78] Wieleitner restricts this result to a bound on the number of ordinary cusps on a rational cuspidal curve. Subsequently, in [38] Lefschetz gives an exact upper limit for the number of ordinary cusps on a curve of any genus, where again the curve is not necessarily cuspidal. Fundamental to all these estimates are the famous formulas by Plücker [56], and the bounds depend on the degree of the curve.

The study of plane cuspidal curves becomes more interesting in the late 20th century, following the appearance of several articles that connect other topics in algebraic geometry to these curves. In [74] Wakabayashi links the theory of open surfaces to the complements of plane curves, with particularly interesting results for curves with several cusps. In [21] Flenner and Zaidenberg study a special class of open surfaces, surfaces that for example appear as complements of plane rational cuspidal curves. Moreover, in [42] Matsuoka and Sakai compare the degree of a rational cuspidal curve and the maximal multiplicity of its cusps, and the result hints to a specialized version of the unsettled conjecture from the middle of the 20th century bearing the name of Coolidge and Nagata, see [9, 42, 48].

Conjecture (Coolidge–Nagata). *Every plane rational cuspidal curve can be transformed to a line by a birational transformation.*

The turning point in the study of plane rational cuspidal curves comes in 1994, when Sakai poses two unsolved problems on cuspidal curves [30]. The two tasks at hand are first to classify all rational and elliptic cuspidal plane curves, and second to find the maximal number of cusps on a rational cuspidal plane curve. The formulation of the two problems dramatically increases the interest in rational cuspidal curves. The announcement of the problems is followed by a series of articles by many different authors, the most active being Fenske, Flenner, Orevkov, Tono and Zaidenberg, where rational cuspidal plane curves with a certain number or certain kinds of cusps are constructed. In particular, three series of curves with three cusps are constructed using birational transformations by Fenske in [16], and Flenner and Zaidenberg in [22, 23]. Some years later, an upper bound on the number of cusps for a cuspidal plane curve of any genus is found by Tono in [68]. This upper bound is in fact only dependent on the genus of the curve, and for plane rational cuspidal curves the maximal number of cusps is merely eight. The last powerful result on Sakai's problems is a computer based exclusion of curves by Piontkowski in [55]. The result says that up to degree 20, under an additional requirement that is conjectured to hold, there are no other plane rational cuspidal curves with three or more cusps than the ones in the mentioned three series and two curves of degree five with three and four cusps respectively. This observation leads to the following conjecture [55].

Conjecture. *A plane rational cuspidal curve can not have more than four cusps. If it has three or more cusps, then it is either a curve of degree five or in one of the three series of rational cuspidal curves with three cusps.*

This thesis grew out of an attempt to find an answer to the above problems. In the process we realized that it would be interesting to instead study cuspidal curves on surfaces related to the projective plane, and reformulate the problems in this new setting. Although we study cuspidal curves of any genus, our main question is restricted to the rational cuspidal curves.

Question. *How many and what kind of cusps can a rational cuspidal curve on a Hirzebruch surface have?*

The Hirzebruch surfaces are rational ruled surfaces, including the doubly ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$, and these surfaces are linked to the projective plane by birational transformations. Moreover, the Hirzebruch surfaces can be considered as toric varieties with homogeneous coordinate rings, and a curve on one of these surfaces can be given as the zero set of a polynomial in the corresponding homogeneous coordinate ring. With this in mind, we approach the cuspidal curves on the Hirzebruch surfaces from three angles. We first generalize the abstract results given for cuspidal curves on the projective plane. Second, we find bounds and restrictions on the possible number and types of cusps on a curve using properties of the defining polynomial of the curve. Third, we focus on rational cuspidal curves and use plane rational curves to construct examples of rational cuspidal curves on the Hirzebruch surfaces using birational transformations.

To our knowledge there is very little previous work on cuspidal curves on Hirzebruch surfaces. In fact, we have found only two articles, [2] by Ballico and [54] by Piene, that mention curves with cusps on Hirzebruch surfaces. The topic is treated quite briefly in these articles, and the curves are studied from a very different point of view than in this thesis. The results are not considered relevant this time around, but can perhaps inspire further studies.

In the new setting on the Hirzebruch surfaces, our results resemble the known results for the cuspidal curves on the projective plane. First, the general approach gives results parallel to the results on the projective plane. We have found some results concerning the multiplicity sequences of the cusps on the curves, and we have found a bound on the number of cusps of a curve of a given genus. The results on the multiplicity sequences follow from applying general formulas to curves on the Hirzebruch surfaces. The latter result is found mimicking the theorem and the proof by Tono in [68], and the bound is on the same form as the bound for plane cuspidal curves, that is, it depends only on the genus of the curve. Moreover, the bound is independent of the ambient Hirzebruch surface. In particular, the maximal number of cusps on a rational cuspidal curve on a Hirzebruch surface is by this theorem fourteen.

Second, closer inspection of the defining polynomial of a curve gives a few minor results, for example bounds on the multiplicity sequence of a cusp on a curve. In the case of $\mathbb{P}^1 \times \mathbb{P}^1$ there are slightly stronger results following from the symmetry in the homogeneous coordinate ring of this surface.

Third, the constructions of curves using birational transformations give many series of rational cuspidal curves with four and three cusps on the Hirzebruch surfaces, but we do not find any curves with more than four cusps. Furthermore, we observe that we are able to construct more rational cuspidal curves on the first few Hirzebruch surfaces than on a general Hirzebruch surface. We can not stress enough that we do *not* claim to have found all rational cuspidal curves with four or three cusps on the Hirzebruch surfaces. However, finding all such curves is itself an interesting task, and we hope that we have found most of them.

Our main conjecture, based on our results and constructions, is similar to the conjecture on the projective plane.

Conjecture. *A rational cuspidal curve on a Hirzebruch surface can not have more than four cusps. In particular, if it has four cusps, then it is given in Chapter 3.*

Our findings do not prove any of the above conjectures, neither for the plane curves, nor for the curves on the Hirzebruch surfaces. They do, however, support the observation that cuspidal curves on the projective plane and the Hirzebruch surfaces are rare and that they have few cusps. Moreover, our constructions illustrate that the rational cuspidal curves on all these surfaces in some sense are related, and they indicate that plane rational curves of low degree play a crucial role. Since the study of these curves has not given precise answers to the conjectures, we recognize that perhaps a different approach to the curves and surfaces is needed to produce the desired results.

Taking one step back, it is possible to discern an enumerative problem from the above discussion. If we were able to perform some sort of count of the number of curves with a given number and type of cusps, then we could shed light on some of the above conjectures. Even more speculatively, we could wish for a situation where we were able to count curves with a given number of cusps. To prove that a plane rational cuspidal curve can have at most four cusps, we would then only need to check that the number of curves with five to eight cusps is zero.

Counting curves on the projective plane of degree d and genus g going through $3d + g - 1$ points in general position and having only ordinary nodes as singularities is a classical task in enumerative geometry. The first results date back to the second half of the 19th century, but the major breakthroughs on variations of this question appear in the past two decades. The problem is solved on the projective plane by Caporaso and Harris [6] in 1998, and it is solved on rational ruled surfaces by Vakil [72] in 2000. Generally, for an arbitrary surface S with a sufficiently ample line bundle $\mathscr{L}$, similar counts of the number of nodal curves with divisor class $\mathscr{L}$ is calculated by Tzeng [71] in 2010, after a conjecture by Göttsche [28] from 1998. In particular, for $\mathscr{L}$ sufficiently ample, the number of curves is given as a polynomial in the four Chern numbers $c_1(\mathscr{L})^2$, $c_1(\mathscr{L})c_1(S)$, $c_1(S)^2$ and $c_2(S)$. The latter claim is generalized by Tzeng [40] and Rennemo [57] to counts of curves with any prescribed singularities. The methods involved in the works of Tzeng and Rennemo are not applicable to our situation since the approaches only consider counting curves that in some sense are general enough, while we are interested in counting the sometimes very special cases of cuspidal curves.

There is still no way known to us of attacking the problem of counting the number of rational cuspidal curves, neither on the projective plane nor on the Hirzebruch surfaces, with a prescribed number and type of cusps. However, in the enumerative setting, intersection theory and Segre classes are important objects coming into play. Computing Segre classes is therefore the topic of the second part of this thesis.

In [14] from 2011, Eklund, Jost and Peterson give a way of explicitly computing the Segre classes of a subscheme of a projective space given only the ideal of the subscheme. The fundamental prerequisites for their algorithm are the straightforward representation of the Chow ring of projective space, and the fact that a subscheme is given by an ideal in the homogeneous coordinate ring. In the second part of this thesis we show that the algorithm of Eklund–Jost–Peterson generalizes to the case of subschemes of smooth projective toric varieties. This is joint work with Nikolay Qviller. Our main observations are that the smooth projective toric varieties have manageable Chow rings, and that a subscheme again is given by an ideal in the homogeneous coordinate ring of the ambient toric variety. The algorithm is implemented in the computer programs `Sage` and `Macaulay2`.

Note that since both the projective plane and the Hirzebruch surfaces are examples of smooth projective toric varieties, the generalization provides Segre classes in a situation that could prove important for the questions dealt with in the first part of the thesis.

Cuspidal curves on Hirzebruch surfaces is the title of this thesis, and it is the topic of Part I. In Chapter 1 we explain the theoretical foundation for concepts that will appear later on. First, we present general results on properties of cuspidal curves that are known in the literature. Second, we give a brief introduction to the projective plane and the Hirzebruch surfaces. Third, we explain the birational transformations linking these surfaces. This is not original work, and it is included because it provides the necessary prerequisites to perform the constructions of curves in later chapters and to construct and investigate curves using suitable computer programs.

In Chapter 2 we give an overview of the results on cuspidal curves on the projective plane, with particular focus on rational cuspidal curves. We elaborate the conjectures given in the introduction and present lists of cuspidal configurations of rational cuspidal curves. Additionally, we include a construction due to Orevkov [50] of a series of cuspidal curves that links the theory of toric varieties to cuspidal curves. Our contributions in terms of new results in this chapter restrict to a few comments on inflection points and real cuspidal curves.

In Chapter 3 we turn our attention to cuspidal curves on Hirzebruch surfaces. This chapter is the core of the thesis, and where our main results can be found. In this chapter we find general results for cuspidal curves on Hirzebruch surfaces mimicking known results and proofs on the projective plane. We also construct rational cuspidal curves with four and three cusps, and we present conjectures for these curves. Additionally, we provide some minor results for rational cuspidal curves.

In Chapter 4 we take a closer look at the special Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$. Using the structure of $\mathbb{P}^1 \times \mathbb{P}^1$ and a special class of curves we find bounds on the multiplicities of cusps on curves on this surface. Furthermore, we investigate the rational cuspidal curves of low bidegree and describe more rational cuspidal curves on this surface. The images in this chapter are created using `surfex` [32], in cooperation with Georg Muntingh.

The topic of Part II and Chapter 5 is Segre classes of subschemes of smooth projective toric varieties. This article is joint work with Nikolay Qviller. We provide a generalization of the algorithm of Eklund–Jost–Peterson for computing Segre classes of closed subschemes of projective spaces. The algorithm is generalized to computing the Segre classes of closed subschemes of smooth projective toric varieties.

In Appendix A we use `Maple` [77] and the tools from Chapter 1 to find defining polynomials for some of the rational cuspidal curves from Chapter 3.

In Appendix B we give the code in `Macaulay2` [29] and `Sage` [62] for the algorithm to compute Segre Classes of closed subschemes of smooth projective toric varieties presented in Chapter 5.

Part I

Cuspidal curves on Hirzebruch surfaces

Chapter 1

Background

This chapter represents the theoretical foundation of this thesis. In this chapter we make precise what we mean by a cuspidal curve on a surface, define general invariants, and state relevant properties and results. We first present results on curves and cusps that will be applied to specific situations in later chapters. Then we set notation and briefly introduce the projective plane and the Hirzebruch surfaces. Last in this chapter we describe the birational transformations connecting these surfaces.

1.1 General concepts for cusps, curves and surfaces

In this section we explain what we mean by a cuspidal curve on a surface and give essential associated definitions and results. The blowing up of a point on a surface and the minimal embedded resolution of a curve are particularly important tools in this context, and we recall some of the local invariants and results that can be derived from these processes. We also recall a few general concepts and results that will be applied in later chapters.

1.1.1 First definitions and general concepts

We first define the objects of study. By a *variety* we here mean a reduced, irreducible scheme of finite type over $\mathbb{C}$. By a *surface* we mean a 2-dimensional smooth projective variety. By a *curve* we mean any 1-dimensional closed subscheme of a surface. By a *point* we mean an irreducible 0-dimensional closed subscheme of a surface or a curve.

Let X denote a surface, C a curve on X and p a point on X. For any variety Y, let $\mathscr{O}_Y$ denote the structure sheaf of Y, and let $\mathscr{O}_{Y,p}$ denote the stalk of $\mathscr{O}_Y$ at p. Let $\mathfrak{m}_p$ denote the maximal ideal in the local ring $\mathscr{O}_{Y,p}$.

Following [31, Proposition II 3.1, p.82], a curve C is called *reduced and irreducible* if and only if for every open set $U \subset C$, the ring $\mathscr{O}_C(U)$ is an integral domain.

Let p be a point on a curve C. Then p is a *smooth point* on C if the local ring $\mathscr{O}_{C,p}$ is a regular local ring. Otherwise, p is a *singular point* (*singularity*) on C. Note that a reduced and irreducible curve can only have finitely many singularities [31, Theorem I 5.3, p.33].

By a germ of a curve C at p we mean an equivalence class of curves defined in some analytic neighbourhood of p, modulo the equivalence relation of having the same restriction to an open neighbourhood of p [7, p.8]. Let (C, p) denote the germ of a curve C at a point p on a surface X, and let $f \in \mathscr{O}_{X,p}$ be a local equation for C in local coordinates x and y. The *multiplicity* m of p is defined to be the largest integer m such that $f \in \mathfrak{m}_p^m \subset \mathscr{O}_{X,p}$. Note that $m \geq 0$, and that $m \geq 1$ if and only if $p \in C$. Moreover, $m = 1$ if and only if p is a smooth point on C, and $m \geq 2$ if and only if p is a singularity [31, pp.388–389].

Following [7, pp.9–13], a non-empty germ (C, p) is said to be *irreducible* if it cannot be obtained as the sum of two non-empty germs. If p is a smooth point, then the germ (C, p) is irreducible. If p is a singular point and the germ (C, p) is irreducible, then p is called a *cusp*. Thus, a cusp is a unibranched singularity, and an illustration is given in Figure 1.1.

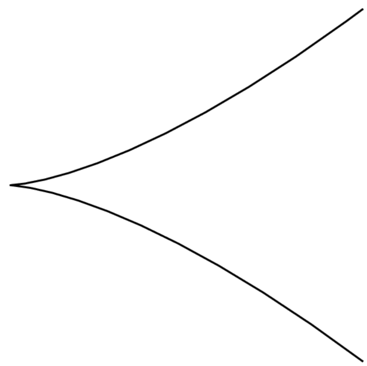


Figure 1.1: A cusp.

Definition 1.1.1. A *cuspidal curve* is a reduced and irreducible curve with only cusps as singularities.

There are several definitions and results associated to cuspidal curves, and next we recall some of them. With (C, p) as above, write $f = \sum_{i+j \geq m} a_{ij} x^i y^j$. Then the tangent cone to C at p is given by the equation $f_m = \sum_{i+j=m} a_{ij} x^i y^j$ [7, p.10]. The reduced linear factor(s) of f_m define the *local tangent line(s)* of C at p. Note that if p is a smooth point or a cusp, then C has a unique tangent line at p. Two curves are said to be *tangent* at a point p if some local tangent lines coincide, and *transversal* otherwise.

Let $\mathrm{Div}(X)$ denote the group of divisors on X, and let $\mathrm{Pic}(X)$ denote the group of divisors on X modulo linear equivalence. A curve C on X corresponds to an effective divisor on X, and following conventions, we will use the same letter for a curve and its corresponding divisor [31, Remark II 6.17.1, p.145]. Moreover, we let $\mathscr{L}(C)$ denote the invertible sheaf associated to C.

Assume that two curves C and C', without common components, meet at a point p. Then p is called an *intersection point* of C and C'. Let f and f' be local equations of C and C' at p. By the *intersection multiplicity* $(C \cdot C')_p$ of C and C' at p we mean

$$(C \cdot C')_p := \dim_{\mathbb{C}} \mathscr{O}_{X,p}/(f, f').$$

For any two divisors C and C' there is a unique, symmetric, bilinear pairing

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z},$$

denoted by $C \,.\, C'$ [31, Theorem V 1.1, pp.357–358]. Moreover, the number $C \,.\, C'$ is called the *intersection number*, and it is calculated using linear equivalence.

The intersection number $C \,.\, C'$ and the intersection multiplicity $(C \cdot C')_p$ are connected through the following proposition [31, Proposition V 1.4, p.360].

Proposition 1.1.2. *If C and C' are two curves on X having no common irreducible component, then*

$$C \,.\, C' = \sum_{p \in C \cap C'} (C \cdot C')_p.$$

Note that the *self intersection number* C^2, defined as

$$C^2 := \deg_{\mathbb{C}}(\mathscr{L}(C) \otimes \mathscr{O}_C),$$

is calculated using linear equivalence [31, Example V 1.4.1, pp.360–361]. Moreover, any curve $Y \cong \mathbb{P}^1$ with $Y^2 = -1$ is called an *exceptional curve of the first kind.*

Recall that a surface is said to be a *relatively minimal model* of its function field if every birational morphism to another surface is necessarily an isomorphism. By [31, p.418], a surface is a relatively minimal model if and only if it contains no exceptional curves of the first kind.

We sometimes view the intersection of C and C' as a 0-cycle, and express this by the notation $C \cdot C'$, where

$$C \cdot C' = \sum_{p \in C \cap C'} (C \cdot C')_p p.$$

For any variety Y of dimension r, we let $\mathrm{H}_i(Y; R)$ denote the ith homology group of Y with coefficients in a ring R. Note that we will sometimes consider Y as a real $2r$-dimensional manifold in this context. In that case, we let $e(Y)$ denote the topological Euler characteristic of Y.

We let $\mathrm{H}^i(Y, \mathscr{F})$ denote the ith cohomology group of Y with respect to a sheaf $\mathscr{F}$. We let $\chi(\mathscr{F})$ denote the Euler characteristic of a coherent sheaf $\mathscr{F}$, so that

$$\chi(\mathscr{F}) := \sum_{i=0}^{r} (-1)^i \dim \mathrm{H}^i(Y, \mathscr{F}).$$

By $p_a(Y)$ we mean the arithmetic genus of Y, where

$$p_a(Y) := (-1)^r(\chi(\mathscr{O}_Y) - 1).$$

In particular, if C is an effective divisor on a surface X, then

$$p_a(C) = 1 - \chi(\mathscr{O}_C).$$

For two divisors C and C', we have by [31, Exercise V 1.3, p.367] that

$$p_a(C + C') = p_a(C) + p_a(C') + C \,.\, C' - 1. \tag{1.1.1}$$

Following [31], we let $\Omega_{X/\mathbb{C}}$ be the sheaf of differentials of $X/\mathbb{C}$, and we let $\omega_X = \bigwedge^2 \Omega_{X/\mathbb{C}}$ be the canonical sheaf of X. By the canonical divisor on X we mean any divisor K linearly equivalent to a divisor corresponding to ω_X. Moreover, $\mathscr{N}_{C/X} = \mathscr{L}(C) \otimes \mathscr{O}_C$ denotes the normal sheaf of C in X.

Connecting some of the above definitions, we now state two important results. First, there is the Adjunction formula [31, Exercise V 1.3, p.366].

Proposition 1.1.3 (Adjunction formula). *For an effective divisor C on a surface X with canonical divisor K,*

$$2p_a(C) - 2 = C \,.\, (C + K).$$

Second, there is the theorem of Riemann–Roch for surfaces [31, Theorem V 1.6, p.362], here in our notation. Note the convention that for a divisor C on a surface X we let $\mathrm{h}^i(C) := \dim_{\mathbb{C}} \mathrm{H}^i(X, \mathscr{L}(C))$.

Theorem 1.1.4 (Riemann–Roch for surfaces). *If C is any divisor and K the canonical divisor on a surface X, then*

$$\mathrm{h}^0(C) - \mathrm{h}^1(C) + \mathrm{h}^0(K - C) = \frac{1}{2} C \,.\, (C - K) + 1 + p_a(X).$$

Closely related to the above concepts is the geometric genus. If Y is a nonsingular variety of dimension r, $\Omega_{Y/\mathbb{C}}$ the sheaf of differentials of $Y/\mathbb{C}$, and $\omega_Y = \bigwedge^r \Omega_{Y/\mathbb{C}}$, then the geometric genus $p_g(Y)$ of Y is defined as

$$p_g(Y) := \dim \mathrm{H}^0(Y, \omega_Y).$$

In particular, if C is a reduced and irreducible nonsingular curve, then $p_a(C) = p_g(C)$ [31, Proposition IV 1.1, p.294]. If C is a reduced and irreducible singular curve, with $\tilde{C}$ its normalization, then we let $g := p_g(\tilde{C})$ denote its geometric genus.

A reduced and irreducible nonsingular curve C is called *rational* if it is birational to $\mathbb{P}^1$. Moreover, a reduced and irreducible nonsingular curve C is rational if and only if $p_a(C) = 0$ [31, Example IV 1.3.5, p.297]. A reduced and irreducible singular curve C is called *rational* if $g = 0$.

Note that we call a divisor D a *rational tree* if all components D_i of D are smooth rational curves and the weighted dual graph $\Gamma(D)$ of D is connected and without cycles.

1.1.2 The minimal embedded resolution

In this section we explain how a minimal embedded resolution of a curve is important in the study of cuspidal curves.

A minimal embedded resultion of a curve consists of a sequence of monoidal transformations. A *monoidal transformation* of a surface X is the blowing up of a single point p on X, where the transformation $\pi : \tilde{X} \to X$ induces an isomorphism of

$\tilde{X} \setminus \pi^{-1}(p)$ onto $X \setminus p$. Note that $\tilde{X}$ is a smooth projective surface. The divisor $E = \pi^{-1}(p)$ on $\tilde{X}$ is referred to as an *exceptional curve*, E is isomorphic to $\mathbb{P}^1$, and $E^2 = -1$ [31, Proposition V 3.1, p.386]. Moreover, the canonical divisor $K_{\tilde{X}}$ of $\tilde{X}$ is given by $K_{\tilde{X}} = \pi^* K_X + E$, and we have $K_{\tilde{X}}^2 = K_X^2 - 1$. Blowing up a point p is referred to as a *blowing up*, and contracting an exceptional curve of the first kind is referred to as a *contraction*.

Let C be an effective divisor on X. With π as above, the divisor $\pi^* C$ is called the the *total transform* of C. The curve $\tilde{C}$, defined as the closure in $\tilde{X}$ of $\pi^{-1}(C \cap (X - P))$, is called the *strict transform* of C. The following proposition [31, Proposition V 3.6, p.389] and corollary [31, Corollary V 3.7, p.389] are essential.

Proposition 1.1.5. *Let C be an effective divisor on X, let p be a point of multiplicity m on C, and let $\pi : \tilde{X} \to X$ be the monoidal transformation with center p. Then*

$$\pi^* C = \tilde{C} + mE.$$

Corollary 1.1.6. *With the same hypotheses as above, we have*

$$\tilde{C} \,.\, E = m,$$

and

$$p_a(\tilde{C}) = p_a(C) - \frac{1}{2} m(m-1).$$

By composing several monoidal transformations, we can ensure that the inverse image of a curve has special properties, and these properties are the main point of performing a minimal embedded resolution of a curve.

We adapt the definition of a simple normal crossing divisor from Iitaka [34, Definition, p.257] to our situation. Let D be a reduced divisor with nonsingular irreducible components on a smooth surface V. Then D is said to have a *simple normal crossing* at a point p if there exists a coordinate neighbourhood U of p, with (x, y) as local coordinates, such that $D \cap U$ is defined by x or $x \cdot y$. If D has only simple normal crossings everywhere, then D is called a *simple normal crossing divisor* (*SNC-divisor*).

By [31, Theorem V 3.9, p.391], there exists for any curve C on a surface X a sequence of t monoidal transformations,

$$V = V_t \xrightarrow{\sigma_t} V_{t-1} \to \cdots \to V_1 \xrightarrow{\sigma_1} V_0 = X,$$

such that the reduced total inverse image of C under the composition $\sigma : V \to X$,

$$D := \sigma^{-1}(C)_{\text{red}},$$

is an SNC-divisor on V. The pair (V, D) and the transformation σ are referred to as an *embedded resolution* of C.

Definition 1.1.7. An embedded resolution of C on X is called a *minimal embedded resolution* of C when t is the smallest integer such that D is an SNC-divisor.

Note that a minimal embedded resolution of a curve C is unique up to the order of the monoidal transformations, and we therefore generally refer to such a resolution as *the* minimal embedded resolution of a curve.

For each monoidal transformation $\sigma_i : V_i \to V_{i-1}$ in the minimal embedded resolution of a curve C there is an exceptional divisor $E_i \subset V_i$ and a center $p_{i-1} \in V_{i-1}$. Moreover, $E_i' \subset V$ denotes the strict transform of E_i by $\sigma_{i+1} \circ \cdots \circ \sigma_t$, and $E_i^{(k)} \subset V_{i+k}$ denotes the strict transform of E_i by $\sigma_{i+1} \circ \cdots \circ \sigma_{i+k}$. On each surface V_i, C_i denotes the strict transform of C under the sequence of monoidal transformations $\sigma_1 \circ \cdots \circ \sigma_i$. Note in particular that the curve $\tilde{C} = C_t$, that is, the strict transform of C under σ, is smooth.

1.1.3 Invariants associated to a cusp

We now take a closer look at a cusp and list invariants associated to the singularity. The invariants listed in this section are given without any index referring to the cusp, and in later sections we add a natural index to the invariants to clarify which cusp they belong to.

In the minimal embedded resolution of a curve, we now focus on the resolution of only one cusp. With (C, p) the germ of C at a cusp p, we perform the monoidal transformations needed to resolve the singularity in such a way that $\sigma^{-1}(p)$ is an SNC-divisor (see [34, Proposition 11.1, p.320]). This process will be referred to as the *minimal resolution* of the cusp p. The sequence of monoidal transformations centers in p and successively in the intersection points p_i of C_i and E_i. Since p is a cusp, the point p_i is unique for each i [17, Proposition 1.3.6., p.22]. The points p_i in the resolution of p are called the *infinitely near points* of p. Abusing notation, we will in this section assume that the cusp p is resolved after t monoidal transformations.

The minimal resolution of a cusp on a curve leads to the notion of the multiplicity sequence of a cusp.

Definition 1.1.8. Let p be a cusp on a curve C, and let m_i denote the multiplicity of $p_i \in C_i$ in the minimal resolution of p, with $m = m_0$. Then the *multiplicity sequence* $\overline{m}$ of the cusp p is defined to be the sequence of integers

$$\overline{m} = [m, m_1, \ldots, m_{t-1}].$$

Note that the multiplicity sequence is a decreasing sequence of integers,

$$m \geq m_1 \geq \ldots \geq m_{t-1}, \tag{1.1.2}$$

and that $m_{t-1} = 1$.

The following lemma by Flenner and Zaidenberg, here adapted to our situation, describes the changes in the intersections of the curve and the exceptional divisors in the course of the minimal resolution of a cusp using the multiplicity sequence [22, Lemma 1.3, p.440].

Lemma 1.1.9. *Let $\overline{m} = [m_0, m_1, \ldots, m_{t-1}]$ be the multiplicity sequence of a cusp p on a curve C. Let $E_i^{(k)} \subset V_{i+k}$ denote the strict transform of the exceptional divisor E_i of σ_i. Then the following hold.*

a) $E_i \,.\, C_i = m_{i-1}$.

b) $E_i^{(k)} \,.\, C_{i+k} = \max\{0, m_{i-1} - m_i - \ldots - m_{i+k-1}\}$, $k > 0$.

c) $E_i^{(1)} \,.\, C_{i+1} = m_{i-1} - m_i$.

Furthermore, there is a proposition by Flenner and Zaidenberg giving restrictions on the multiplicity sequence, and we rewrite the proposition in our notation [22, Proposition 1.2, p.440].

Proposition 1.1.10. *Let* $\overline{m} = [m_0, m_1, \ldots, m_{t-1}]$ *be the multiplicity sequence of a cusp.*

a) *For each* $i = 1, \ldots, t-1$*, there exists an integer* $k \geq 0$ *such that*

$$m_{i-1} = m_i + \ldots + m_{i+k},$$

where

$$m_i = m_{i+1} = \ldots = m_{i+k-1}.$$

b) *The number of ending* 1*'s in the multiplicity sequence equals the smallest* $m_i > 1$*. If* p_{q-1} *is singular on* C_{q-1} *in the minimal resolution and* p_q *is smooth on* C_q*, we have*

$$m_{q-1} = t - q.$$

Conversely, if $\overline{m} = [m_0, m_1, \ldots, m_n]$ *is a non-increasing sequence of positive integers satisfying the above, then* $\overline{m} = \overline{m}_p$ *for some irreducible plane germ* (C, p)*.*

By convention and the above proposition, there is a shorthand notation for the multiplicity sequence. First of all, we may omit the ending 1's in the sequence. Moreover, if some elements in the sequence are equal, they will be listed as one element, and we use a subscript to indicate how many times each value is repeated. For example, we write

$$[6, 6, 2, 2, 2, 1, 1] = [6_2, 2_3].$$

Note that the multiplicity sequence of a cusp determines the topological type of the singularity [4, Theorem 21, p.535]. A cusp with multiplicity sequence [2] will be called an *ordinary* cusp.

The intersection multiplicity of two curves C and C' at a point p can be expressed using the multiplicity sequences, and this formula is referred to as Noether's formula [7, Theorem 3.3.1, p.79]. The theorem holds for points p of any type, but to keep the notation clean, we here restate a version of the theorem for p a smooth point or a cusp. For the purpose of the following theorem, we say that a smooth point has multiplicity sequence [1].

Theorem 1.1.11 (Noether's formula). *Let* p *be a point on two curves* C *and* C'*, and assume that* p *is a smooth point or a cusp. Moreover, let* $\overline{m}$ *and* $\overline{m}'$ *denote the respective multiplicity sequences of the curves at* p*, and append sequences of* 1*'s to the multiplicity sequences as long as* C *and* C' *share points infinitely near* p*. The intersection multiplicity* $(C \cdot C')_p$ *is finite if and only if* C *and* C' *share finitely many points infinitely near* p*, and in that case*

$$(C \cdot C')_p = \sum_{i=0}^{l} m_i m_i',$$

where l *is the largest integer such that* $p_l \in C_l \cap C_l'$*.*

There is an immediate corollary to Theorem 1.1.11 [31, Exercise I 5.4, p.36].

Corollary 1.1.12. *With the same hypothesis as above, let m and m' be the respective multiplicities of C and C' at p. Then*

$$(C \cdot C')_p \geq m \cdot m'.$$

Moreover, there is a lemma by Flenner and Zaidenberg describing the intersection of two curves at a point that is a cusp on one of the curves and smooth on the other curve. We include a modified version of this lemma, here in our notation [23, Lemma 2.6 b), p.101].

Lemma 1.1.13. *Assume that (Γ, p) is a smooth germ, and assume that (C, p) is an irreducible germ of a curve, where the cusp p has multiplicity sequence $\overline{m} = [m, m_1, \ldots, m_{t-1}]$. Then the intersection $(\Gamma \cdot C)_p$ is*

$$(\Gamma \cdot C)_p = m + m_1 + \cdots + m_k$$

for some $k \geq 0$, and moreover, $m = m_1 = \cdots = m_{k-1}$.

Lemma 1.1.13 has an interesting consequence that ultimately gives restrictions on the multiplicity sequence of a cusp, see Lemma 2.1.6 and Theorem 4.1.16.

Lemma 1.1.14. *Assume that T is a smooth curve tangent to a reduced and irreducible curve C at a cusp p. Then*

$$(T \cdot C)_p = k \cdot m + m_k \text{ for some } k \geq 1.$$

In particular,

$$(T \cdot C)_p \geq m + m_1.$$

Proof. Since T is smooth and tangent to C at p, the intersection multiplicity $(T \cdot C)_p > m$. By Lemma 1.1.13, we have $(T \cdot C)_p = m + m_1 + \ldots + m_k$ for some $k \geq 0$, where $m = m_1 = \ldots = m_{k-1}$, and the result follows by compacting the notation. □

Revealing even more subtle properties of cusps, there is a lemma by Fenske that provides insight to the changes of a cusp under a monoidal transformation. This lemma is fundamental in our constructions of curves in later chapters, and we recall it here in an adapted version [17, Lemma 1.4.8, p.31].

Lemma 1.1.15. *Let C and D be curves without common components, and let p be a point on both curves, such that p is a cusp on C with multiplicity sequence $\overline{m}_p = [m, m_1, \ldots, m_{t-1}]$, and p is smooth on D. Let π be the monoidal transformation of X with center p, and let $E = \pi^{-1}(p)$ be the exceptional divisor of π. Let $\tilde{C}$ and $\tilde{D}$ denote the respective strict transforms, and let $\tilde{p}$ denote the point $E \cap \tilde{C}$. Then the following hold.*

a) $\tilde{C}$ has a cusp at p with multiplicity sequence $[m_1, \ldots, m_{t-1}]$, and

$$(\tilde{C} \cdot E)_{\tilde{p}} = m.$$

b) $(C \cdot D)_p = m + m_1 + \cdots m_k$ *with* $m = m_1 = \cdots = m_{k-1}$ *for a* $k \geq 1$, *and moreover,*

$$(\tilde{C} \cdot \tilde{D})_{\tilde{p}} = (C \cdot D)_p - m.$$

There are several invariants linked to a cusp. With f a local equation of a curve C at a singularity p, the *Milnor number* μ is defined to be

$$\mu := \dim_{\mathbb{C}} \mathscr{O}_{C,p}/(f_x, f_y),$$

where x and y are local coordinates, and f_x and f_y are the partial derivatives of f. The *delta invariant* δ of p is defined as [31, Exercise IV 1.8, p.298]

$$\delta := \operatorname{length}(\tilde{\mathscr{O}}_{C,p}/\mathscr{O}_{C,p}),$$

where $\tilde{\mathscr{O}}_{C,p}$ is the integral closure of $\mathscr{O}_{C,p}$. For a cusp, the delta invariant can be calculated using the multiplicity sequence [31, Example V 3.9.3, p.393],

$$\delta = \sum_{i=0}^{t-1} \frac{m_i(m_i - 1)}{2}.$$

For any singularity with b branches, the Milnor number and the delta invariant are linked by the Milnor–Jung-formula, which says that

$$\mu = 2\delta - b + 1.$$

For a cusp, this formula reduces to

$$\mu = 2\delta.$$

Hence, the Milnor number of a cusp can be expressed in terms of the multiplicity sequence,

$$\mu = \sum_{i=0}^{t-1} m_i(m_i - 1).$$

For a cusp p, we additionally use an invariant M, referred to as the *M-number.* It is not clear who introduced this invariant, and the first definitions we have found are by Fenske [17, Definition 1.5.23, p.44] and Orevkov [50, p.659], the latter using a slightly different notation. A blowing up σ_i in the minimal embedded resolution of p is called *inner* if its center is an intersection point of the strict transforms of two exceptional curves of the resolution. Similarly, a blowing up in the resolution is called *outer* if its center is on exactly one exceptional divisor. We let ω denote the number of inner blowing ups, and we let ρ denote the number of outer blowing ups. The first blowing up, σ_1, is by definition neither inner nor outer. Summing up, we have

$$t - 1 = \omega + \rho.$$

Following [17], the M-number of p is defined as

$$M := \eta + \omega - 1,$$

where

$$\eta := \sum_{i=0}^{t-1} (m_i - 1).$$

By [22, Lemma 2.3, p.445] we can express ω in terms of the multiplicity sequence,

$$\omega = \sum_{i=1}^{t-1} \left(\left\lceil \frac{m_{i-1}}{m_i} \right\rceil - 1 \right),$$

where $\lceil a \rceil$ is the smallest integer $\geq a$. Hence, M can be expressed using only the multiplicity sequence,

$$M = \sum_{i=0}^{t-1} (m_i - 1) + \sum_{i=1}^{t-1} \left(\left\lceil \frac{m_{i-1}}{m_i} \right\rceil - 1 \right) - 1.$$

The multiplicity, the Milnor number and the M-number of a cusp can be compared through the following lemma by Orevkov [50, Lemma 4.1 and Corollary 4.2, pp.663–664].

Lemma 1.1.16. *For a cusp p with multiplicity m, M-number M and Milnor number μ, we have*

a) $M - \frac{\mu}{m} > m - 3.$

b) $M \geq \frac{\mu}{m}.$

1.1.4 More definitions and general concepts

We now move back to the situation where we look at the minimal embedded resolution of a cuspidal curve C, where all the cusps are resolved. Considering curves from this perspective, there are global invariants and formulas connecting them. Throughout this section we let C be a cuspidal curve with s cusps. The collection of the multiplicity sequences of the cusps of C will be referred to as the *cuspidal configuration* of the curve. Note that two cuspidal curves C and C' on the same surface X are said to be *equisingular equivalent* if they have the same genus and the same cuspidal configuration. The cusps of the curve C will be denoted by p_j, where $j = 1, \ldots, s$. For each cusp we have the invariants defined in Section 1.1.3, and we will use the index j or p_j as subscripts to link an invariant to its cusp, for example we append an index j to the multiplicity sequence $\overline{m}_j$ of a cusp p_j. Note that we do *not* generally append an index j to elements m_i in the multiplicity sequence. Abusing notation further, we still let p_i denote the centers of the minimal embedded resolution of the curve C. Moreover, we use m_i both to denote the multiplicity of the centers p_i of the minimal embedded resolution, and to denote an element in the multiplicity sequence of a cusp. This slightly ambiguous notation will hopefully not cause any confusion in the frequent transitions from discussions of properties of one cusp to discussions of global properties of a curve.

As before, let $\tilde{C} = C_t$ denote the strict transform of the curve C under the minimal embedded resolution σ. The total transform of the curve C by σ is the divisor

$$\sigma^* C = \tilde{C} + \sum_{i=1}^{t} m_{i-1} E_i.$$

Moreover, the reduced inverse image of C under σ is the divisor D,

$$D := \sigma^{-1}(C)_{\mathrm{red}} = \tilde{C} + \sum_{i=1}^{t} E_i'.$$

The genus formula is the first essential result that can be formulated using the above invariants, here in our notation [31, Example V 3.9.2, p.393].

Proposition 1.1.17 (Genus formula). *Let $\tilde{C}$ denote the normalization of a cuspidal curve C on X, and let K be the canonical divisor on X. Let g denote the geometric genus of C, let t denote the number of monoidal transformations needed in the minimal embedded resolution of C, and let m_i denote the multiplicities of the centers p_i in this transformation. Then*

$$\begin{aligned} g = g(\tilde{C}) &= p_a(C) - \sum_{i=0}^{t-1} \frac{m_i(m_i-1)}{2} \\ &= \frac{C\,.\,(C+K)}{2} + 1 - \sum_{i=0}^{t-1} \frac{m_i(m_i-1)}{2}. \end{aligned}$$

The latter term in the genus formula can be recognized as the sum of the delta invariants δ_j of the cusps p_j of C. If C has s cusps, then we rewrite the above as

$$g = \frac{C\,.\,(C+K)}{2} + 1 - \sum_{j=1}^{s} \delta_j.$$

Moreover, there is an important theorem by Hurwitz, here in a special setting [31, Corollary IV 2.4, p.301].

Theorem 1.1.18 (Hurwitz's theorem). *Let C be a smooth curve of genus g, let $f : C \longrightarrow \mathbb{P}^1$ be a finite morphism, and let $n = \deg f$. Let $e_p = \mathrm{length}(\Omega_{C/\mathbb{P}^1})_p + 1$, with $\Omega_{C/\mathbb{P}^1}$ the sheaf of relative differentials, be the ramification index. Then*

$$2n + 2g - 2 = \sum_{p \in C} (e_p - 1).$$

From the minimal embedded resolution of a cuspidal curve, a relation between the M-numbers of the singularities and the involved divisors can be stated, here using our notation [22, Proposition 2.4, p.445].

Proposition 1.1.19. *Let C be a cuspidal curve with s cusps p_j on a smooth compact complex surface X. Let (V, D) be the minimal embedded resolution of the curve C. Let K_V and K_X denote the canonical divisors on the respective surfaces. Then*

$$K_V\,.\,(K_V + D) = K_X\,.\,(K_X + C) + \sum_{j=1}^{s} M_j,$$

where M_j is the M-number of the cusp p_j.

Another important concept needed in the study of cuspidal curves on surfaces is the *logarithmic Kodaira dimension*. A few preliminary definitions must be stated before the main definition can be recalled.

We first recall the definition of a morphism Ψ_D for any divisor D, and this definition is taken from [17, p.39]. Let V be a nonsingular complete surface and let $\mathscr{L}$ be an invertible sheaf on V. By [31, Theorem II 5.19, p.122], $\mathrm{H}^0(V,\mathscr{L})$ is finite dimensional. Let $\mathrm{H}^0(V,\mathscr{L})$ be generated by sections $s_0,\ldots,s_n$ of $\mathscr{L}$. The base points of $\mathscr{L}$ form a set

$$Z := \{z \in V \mid s_0(z) = \ldots = s_n(z) = 0\}.$$

Let $U \subset V$ be an open set. For each $a \in V \setminus Z$ a trivialization $\phi_{|U} : \mathscr{L}_{|U} \to \mathscr{O}_{V|U}$ can be fixed. The images $\phi_{i,U}$ of s_i under this map form a morphism

$$\begin{aligned} \Psi_U : U \setminus Z &\to \mathbb{P}^n \\ x &\mapsto (\phi_{0,U}(x) : \ldots : \phi_{n,U}(x)). \end{aligned}$$

For an open covering $\{U_i\}_{i\in I}$ of V, the morphisms Ψ_{U_i} define a rational map

$$\Psi_{\mathscr{L}} : V \to \mathbb{P}^n.$$

If D is a divisor corresponding to $\mathscr{L}$, then this map is denoted by Ψ_D.

For the remaining definitions, we follow [34, Chapter 10 and 11]. Let D be an SNC-divisor on V. Note that V is called a *smooth completion* of $V \setminus D$ with *smooth boundary* D. Let $\mathbb{N}(D) := \{n \in \mathbb{N} \mid \mathrm{h}^0(nD) > 0\}$. Then the *$D$-dimension* is

$$\kappa(D,V) = \begin{cases} \max\{\dim \Psi_{nD}(V) \mid n \in \mathbb{N}(D)\}, & \text{if } \mathbb{N}(D) \neq \emptyset, \\ -\infty, & \text{otherwise.} \end{cases}$$

Using this construction, the *Kodaira dimension* $\kappa(V)$ of V is defined to be $\kappa(K_V, V)$ [34, p.309]. A related definition is more useful to us [34, p.326].

Definition 1.1.20. With V and D as above, the *logarithmic Kodaira dimension* $\overline{\kappa}(V \setminus D)$ of $V \setminus D$ is

$$\overline{\kappa}(V \setminus D) := \kappa(K_V + D, V).$$

Note that $\overline{\kappa}(V \setminus D) \in \{-\infty, 0, 1, 2\}$.

Furthermore, we recall a result from Iitaka (see [34, Theorem 10.2, p.301]) that establishes an important property of the logarithmic Kodaira dimension [34, p.326].

Theorem 1.1.21. *With V and D as above, if $\overline{\kappa}(V \setminus D) \geq 0$, then one has $\lambda \in \mathbb{N}$ with $\mathrm{h}^0(\lambda(K_V + D)) > 0$, and $\alpha, \beta > 0$ such that*

$$\alpha n^\kappa \leq \mathrm{h}^0(n\lambda D) \leq \beta n^\kappa, \quad \textit{for}\, n \gg 0.$$

The logarithmic Kodaira dimension is not directly defined for the complement of a reduced and irreducible curve C on a surface X, but we calculate it through the logarithmic Kodaira dimension of its minimal embedded resolution [34, p.332]. This invariant will be frequently referred to in the subsequent Chapters.

Definition 1.1.22. With (V, D) the minimal embedded resolution of a curve C on a surface X, we let

$$\overline{\kappa}(X \setminus C) := \overline{\kappa}(V \setminus D).$$

Another important theorem, often referred to as the *Bogomolev–Miyaoka–Yau*-inequalities (B–M–Y-inequalities), uses the classification of surfaces by the logarithmic Kodaira dimension to link the Euler characteristic and divisors related to the curve and surface in question. The inequalities are given in a very general form in the original literature (see [36, 45]). Here we present the results for a situation that includes the curves and surfaces we investigate, and Theorem 1.1.23 is adapted from [51, Theorem 2.1, p.660].

Let V be a smooth projective surface, D a reduced SNC-divisor, and K_V the canonical divisor on V. If $\overline{\kappa}(V \setminus D) \geq 0$, then there exists a decomposition of $K_V + D$ called the *Zariski decomposition* (see [24, Section 6, pp.527–528]). The decomposition is given by

$$K_V + D = H + N,$$

where H and N are $\mathbb{Q}$-divisors, that is, linear combinations of its prime components with rational coefficients, with the below properties. With $N = \sum n_i N_i$, recall that the intersection matrix $\left[N_i N_j\right]$ is called negative definite if all its eigenvalues are negative.

a) $N = 0$, or N is an effective $\mathbb{Q}$-divisor with negative definite intersection matrix.

b) $H \,.\, C \geq 0$ for any effective divisor $C \in \mathrm{Pic}(X)$.

c) $H \,.\, N_i = 0$ for any prime component N_i of N.

Theorem 1.1.23 (B–M–Y). *With notation as above, the following hold.*

a) *If* $\overline{\kappa}(V \setminus D) \geq 0$*, then*

$$(K_V + D)^2 \leq 3e(V \setminus D).$$

b) *If* $\overline{\kappa}(V \setminus D) = 2$*, then*

$$H^2 \leq 3e(V \setminus D).$$

In the subsequent chapters, there are additionally a few sheaves associated to a curve that we need to recall. First, the tangent sheaf Θ_X of a smooth surface X is defined as $\Theta_X := \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\Omega^1_{X/\mathbb{C}}, \mathscr{O}_X)$. The sheaf Θ_X is locally free of rank 2 [31, p.180].

Second, for D an SNC-divisor on a complete nonsingular surface V, there exists a so-called sheaf of logarithmic 1-forms of V tangent along D. We adapt the definitions found in [34, p.321] to our situation. Assume that V is a smooth completion of $V \setminus D$, and that the SNC-divisor D is its smooth boundary. Any point $q \in D$ is contained in at most two irreducible components of D, say either it is contained in D_x or it is contained in $D_x \cap D_y$. Since D is an SNC-divisor on a surface, we have for every point $q \in D$ a coordinate neighbourhood U_λ with a local coordinate system (x_λ, y_λ). In these coordinates, we have that $D_x \cap U_\lambda = \mathscr{V}(x_\lambda)$ and $D_y \cap U_\lambda = \mathscr{V}(y_\lambda)$.

First, we recall a preliminary definition from [34, p.321].

Definition 1.1.24. The *$\mathscr{O}_V$-module of logarithmic 1-forms of V along D* is defined to be the $\mathscr{O}_V$-submodule $\Omega^1_{V/\mathbb{C}}(\log D)$ of $\Omega^1_{V/\mathbb{C}}(D) = \Omega^1_{V/\mathbb{C}} \bigotimes_{\mathscr{O}_V} \mathscr{O}_V(D)$ such that

a) $\Omega^1_{V/\mathbb{C}}(\log D)|_{V\setminus D} = \Omega^1_{(V\setminus D)/\mathbb{C}}$.

b) At any closed point q of D,

$$\omega_q \in \Omega^1_{V/\mathbb{C}}(\log D)_q \;\Leftrightarrow \omega_q \in \Omega^1_{V/\mathbb{C}}(D)_q$$
$$\text{and}$$
$$\omega_q = \begin{cases} a\frac{dx_\lambda}{x_\lambda} + b\frac{dy_\lambda}{y_\lambda} & \text{if } q \in D_x \cap D_y, \\ a\frac{dx_\lambda}{x_\lambda} + b\, dy_\lambda & \text{if } q \in D_x, \end{cases}$$

where (x_λ, y_λ) is as above and a, $b \in \mathscr{O}_{V,q}$.

Note that $\Omega^1_{V/\mathbb{C}}(\log D)$ is locally free of rank 2 [34, p.321].

We then state the proposed definition, see [34, p.321] for a more general definition.

Definition 1.1.25. The *sheaf of logarithmic 1-forms of V tangent along D* is the sheaf

$$\Theta_V \langle D \rangle := \mathscr{H}\mathrm{om}_{\mathscr{O}_V}(\Omega^1_{V/\mathbb{C}}(\log D), \mathscr{O}_V).$$

Note that, given (V, D), we will sometimes refer to $\Theta_V \langle D \rangle$ as simply the *logarithmic tangent sheaf.*

1.2 Surfaces

This section is a brief introduction to the projective plane and the Hirzebruch surfaces. We describe the surfaces, recall some important observations, and explain what we mean by a change of coordinates.

1.2.1 The projective plane – $\mathbb{P}^2$

Let $\mathbb{P}^2$ denote the projective plane, which is both a rational and a relatively minimal surface. Considered as a toric variety, $\mathbb{P}^2$ can be given as the variety associated to the fan $\Sigma \subset \mathbb{Z}^2$, where the ray generators of Σ are

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

The coordinate ring of $\mathbb{P}^2$ is $\mathbb{C}[x, y, z]$ with standard grading, and the dth graded part is referred to by $\mathbb{C}[x, y, z]_d$. The surface $\mathbb{P}^2$ can be covered by three open affine sets,

$$U_x := \mathbb{P}^2 \setminus \mathscr{V}(x), U_y := \mathbb{P}^2 \setminus \mathscr{V}(y) \text{ and } U_z := \mathbb{P}^2 \setminus \mathscr{V}(z).$$

A subscheme Z of $\mathbb{P}^2$ is given by a graded ideal I in $\mathbb{C}[x, y, z]$. A point p on $\mathbb{P}^2$ will be referred to by its homogeneous coordinates,

$$p = (x : y : z).$$

A reduced and irreducible curve C is given as the zero set $\mathscr{V}(F)$ of a reduced and irreducible homogeneous polynomial $F(x, y, z)$ in $\mathbb{C}[x, y, z]_d$. The curve C is said to have degree d. Note that we by a curve on $\mathbb{P}^2$ mean a reduced and irreducible curve, unless otherwise specified.

The Picard group of $\mathbb{P}^2$ is isomorphic to $\mathbb{Z}$. We let L denote a generator of $\mathrm{Pic}(\mathbb{P}^2)$ with $L^2 = 1$. The divisor corresponding to a curve C of degree d is linearly equivalent to $C \sim dL$, and moreover $C^2 = d^2$. A curve of degree 1 is referred to as a *line*, its divisor linearly equivalent to L. Since $L^2 = 1$, two lines on $\mathbb{P}^2$ intersect in one point. Moreover, the canonical divisor K on $\mathbb{P}^2$ is linearly equivalent to $K \sim -3L$, and $K^2 = 9$ [31].

A point p on $\mathbb{P}^2$ can be moved by a change of coordinates, sometimes referred to as a linear transformation. The linear transformations of $\mathbb{P}^2$ are elements of the group $\mathrm{PGL}_3(\mathbb{C})$, given by (3×3)-matrices. A linear transformation on $\mathbb{P}^2$ is uniquely defined by moving four points, no three of them on the same line.

1.2.2 The Hirzebruch surfaces – $\mathbb{F}_e$

Let $\mathbb{F}_e$ denote the Hirzebruch surface of type e for any $e \geq 0$. Recall that $\mathbb{F}_e$ is a projective ruled surface, with $\mathbb{F}_e = \mathbb{P}(\mathscr{O} \oplus \mathscr{O}(-e))$ and morphism $\pi : \mathbb{F}_e \longrightarrow \mathbb{P}^1$. We have $p_a(\mathbb{F}_e) = 0$ and $p_g(\mathbb{F}_e) = 0$ [31, Corollary V 2.5, p.371]. The Hirzebruch surfaces are rational surfaces, relatively minimal in all cases except $e = 1$. The surface $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blown up in a point, and it contains an exceptional curve $E \cong \mathbb{P}^1$ with $E^2 = -1$.

For any $e \geq 0$, the surface $\mathbb{F}_e$ can be considered as the toric variety associated to a fan $\Sigma_e \subset \mathbb{Z}^2$, where the rays of the fan Σ_e are generated by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ e \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

The coordinate ring of $\mathbb{F}_e$ (see [10]) is denoted by $S_e := \mathbb{C}[x_0, x_1, y_0, y_1]$, where the variables can be given a grading by $\mathbb{Z}^2$,

$$\begin{aligned} \deg x_0 &= (1, 0), \\ \deg x_1 &= (1, 0), \\ \deg y_0 &= (0, 1), \\ \deg y_1 &= (-e, 1). \end{aligned}$$

$\mathbb{F}_e$ can be covered by four affine sets isomorphic to $\mathbb{C}^2$,

$$\mathscr{D}_+(x_i y_j) := \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathscr{V}(x_i y_j),\ i = 0, 1,\ j = 0, 1.$$

A point p on $\mathbb{F}_e$ is, for $e \geq 1$, referred to by its coordinates on the form,

$$p = (x_0 : x_1; y_0, y_1).$$

Observe that the notation reflects that we consider $(x_0 : x_1)$ to be homogeneous coordinates, while the pair (y_0, y_1) have a more complex structure for all $e \geq 1$. This structure will be elaborated later in this section. For $e=0$, we write $p=(x_0:x_1;y_0:y_1)$, meaning homogeneous in both pairs, reflecting the doubly ruled structure of $\mathbb{P}^1\times\mathbb{P}^1$.

A subscheme Z of $\mathbb{F}_e$ is determined by a graded ideal I of S_e. Let $S_e(a,b)$ denote the (a,b) graded part of S_e,

$$S_e(a,b) := \mathrm{H}^0(\mathbb{F}_e, \mathscr{O}_{\mathbb{F}_e}(a,b)) = \bigoplus_{\substack{\alpha_0+\alpha_1-e\beta_1=a \\ \beta_0+\beta_1=b}} \mathbb{C}x_0^{\alpha_0}x_1^{\alpha_1}y_0^{\beta_0}y_1^{\beta_1}.$$

If F is a polynomial in $S_e(a,b)$, then we say that F has bidegree (a,b).

A reduced and irreducible curve C on $\mathbb{F}_e$ is given as the zero set $\mathscr{V}(F)$ of a reduced and irreducible polynomial $F(x_0,x_1,y_0,y_1)\in S_e(a,b)$. The curve C is then said to be of type (a,b). Note that we by a curve on $\mathbb{F}_e$ mean a reduced and irreducible curve, unless otherwise specified.

In the language of divisors, let L be a *fiber* of $\pi:\mathbb{F}_e\longrightarrow\mathbb{P}^1$ and M_0 the *special section* of π. The Picard group of $\mathbb{F}_e$, $\mathrm{Pic}(\mathbb{F}_e)$, is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, and L and M_0 can be chosen as generators of this group. We have [31, Section V 2]

$$L^2=0, \qquad L\,.\,M_0=1, \qquad M_0^2=-e.$$

The canonical divisor K can then be expressed as [31, Corollary V 2.11, p.374]

$$K\sim-(2+e)L-2M_0.$$

To simplify our calculations, we will use another generating set of $\mathrm{Pic}(\mathbb{F}_e)$ [31, Theorem V 2.17, p.379]. Let L and $M\sim eL+M_0$ be the new generating set, and note that this is equivalent to taking as generators the divisors D_1 and D_4 corresponding to the minimal ray generators v_1 and v_4 of the fan Σ_e. Then we have

$$L^2=0, \qquad L\,.\,M=1, \qquad M^2=e.$$

Moreover, in this basis,

$$K\sim(e-2)L-2M \quad\text{and}\quad K^2=8.$$

Note that we from now on, unless otherwise specified, will use L and M as a basis for $\mathrm{Pic}(\mathbb{F}_e)$.

Any irreducible curve $C\neq L, M_0$ corresponds to a divisor on $\mathbb{F}_e$ given by [31, Proposition V 2.20, p.382]

$$C\sim a'L+b'M_0, \quad b'>0,\ a'\geq b'e.$$

Expressing the curve using the preferred generators L and M, we write

$$C\sim aL+bM, \quad b>0,\ a\geq 0.$$

Thus an irreducible curve C of type (a,b) corresponds to a divisor $C\sim aL+bM$. Observe that the ordering of (a,b) is essential, except in the case of $\mathbb{P}^1\times\mathbb{P}^1$.

Note that if $b=0$, then $C\sim aL$, and that any curve of type $(1,0)$, its corresponding divisor linearly equivalent to L, will be referred to as a *fiber*.

The existence of the special section $M_0=\mathscr{V}(y_1)$ on $\mathbb{F}_e$ gives the Hirzebruch surfaces a special structure.

Definition 1.2.1. A point p on $\mathbb{F}_e$ is called *special* if $p \in \mathscr{V}(y_1)$, that is, p has coordinates $(x_0 : x_1; y_0, 0)$. A point p on $\mathbb{F}_e$ is called *general* if $p \notin \mathscr{V}(y_1)$, that is, p has coordinates $(x_0 : x_1; y_0, y_1)$, with $y_1 \neq 0$.

We make the following essential observation of the structure of $\mathbb{F}_e$.

Lemma 1.2.2. *A point p on $\mathbb{F}_e$ determines a unique curve L of type $(1,0)$. Moreover, a general point determines in addition an e-dimensional family of curves of type $(0,1)$. Conversely, a curve L of type $(1,0)$ and an irreducible curve M of type $(0,1)$ determine a general point p on $\mathbb{F}_e$.*

Proof. Let $p = (p_{x_0} : p_{x_1}; p_{y_0}, p_{y_1})$ be a point on $\mathbb{F}_e$. Then $L = \mathscr{V}(p_{x_1}x_0 - p_{x_0}x_1)$ is the unique curve of type $(1,0)$ through p.

Let p be general and let M denote a curve of type $(0,1)$, given as the zero set of a polynomial in $S_e(0,1)$,

$$M = \mathscr{V}(by_0 + \sum_{k=0}^{e} c_k x_0^k x_1^{e-k} y_1).$$

Requiring that M passes through p determines one of the coefficients.

Conversely, given curves L and M of type $(1,0)$ and $(0,1)$, we have that $L \,.\, M = 1$. Hence, the intersection is exactly one point p. Since M is irreducible, the point is general.

□

With the above lemma in mind, we say that two sets of coordinates represent the same general point if they determine the same $(1,0)$-curve and the same collection of $(0,1)$-curves, and the same special point if $y_1 = 0$ and they determine the same $(1,0)$-curve.

Change of coordinates on $\mathbb{F}_e$, $e > 0$

Moving points on $\mathbb{F}_e$ is slightly more difficult than in the case of $\mathbb{P}^2$. For $e > 0$, a change of coordinates must preserve the structure of $\mathbb{F}_e$, hence M_0 must be fixed under such a transformation. Moreover, a change of coordinates must be bidegree preserving. A change of coordinates on $\mathbb{F}_e$ can be given by a map $\nu : \mathbb{F}_e \to \mathbb{F}_e$, explicitly,

$$\begin{array}{c}(x_0 : x_1; y_0, y_1)\\ \big\downarrow\\ (a_{00}x_0 + a_{01}x_1 : a_{10}x_0 + a_{11}x_1; by_0 + y_1(\sum_{k=0}^{e} c_k x_0^k x_1^{e-k}), dy_1),\end{array}$$

where $b, d \in \mathbb{C}^*$, $c_k \in \mathbb{C}$ for $k = 0, \ldots, e$, and $a_{ij} \in \mathbb{C}$ for $i, j \in \{0, 1\}$ with

$$\det \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \neq 0.$$

This map allows us to freely move a general point on $\mathbb{F}_e$ to any other general point, or a special point to any other special point.

1.2.3 The special surface – $\mathbb{P}^1 \times \mathbb{P}^1$

When $e = 0$, the Hirzebruch surface $\mathbb{F}_0$ is $\mathbb{F}_0 = \mathbb{P}(\mathscr{O} \oplus \mathscr{O})$. We refer to this surface as $\mathbb{P}^1 \times \mathbb{P}^1$, and note that it has two morphisms to $\mathbb{P}^1$. The double ruling is reflected in a particular symmetry in the coordinate ring, and the symmetry offers results that are unique to $\mathbb{P}^1 \times \mathbb{P}^1$ and curves on this surface.

Recall that the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$S := \mathbb{C}[x_0, x_1, y_0, y_1],$$

where the four variables are bigraded,

$$\begin{aligned} \deg x_0 = \quad (1,0) \quad &= \deg x_1, \\ \deg y_0 = \quad (0,1) \quad &= \deg y_1. \end{aligned}$$

Moreover, we observe that there is no special section on $\mathbb{P}^1 \times \mathbb{P}^1$, hence points are not classified as special or general in this case. A point on $\mathbb{P}^1 \times \mathbb{P}^1$ will be referred to by its bihomogeneous coordinates $(x_0 : x_1; y_0 : y_1)$.

The (a,b)-graded part of S_0 is

$$S_0(a,b) := \bigoplus_{\substack{a_0+a_1=a \\ b_0+b_1=b}} \mathbb{C}x_0^{a_0}x_1^{a_1}y_0^{b_0}y_1^{b_1}.$$

Recall that a reduced and irreducible curve C on $\mathbb{P}^1 \times \mathbb{P}^1$ is given as the zero set $\mathscr{V}(F)$ of a reduced and irreducible polynomial $F(x_0, x_1, y_0, y_1) \in S_0(a,b)$.

Let L be a curve of type $(1,0)$, and M a curve of type $(0,1)$. With the corresponding divisors L and M a set of generators of $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, we have

$$L^2 = 0, \qquad L \,.\, M = 1, \qquad M^2 = 0.$$

Both curves of type $(1,0)$ and $(0,1)$ will be referred to as fibers of $\mathbb{P}^1 \times \mathbb{P}^1$. A reduced and irreducible curve C of type (a,b) on $\mathbb{P}^1 \times \mathbb{P}^1$ is determined by a reduced and irreducible bihomogeneous polynomial $F \in S_0(a,b)$, and it corresponds to a divisor $C \sim aL + bM$.

The structure of $\mathbb{P}^1 \times \mathbb{P}^1$ is reflected in Lemma 1.2.3, which is a special case of Lemma 1.2.2.

Lemma 1.2.3 (Special case of Lemma 1.2.2). *A point p on $\mathbb{P}^1 \times \mathbb{P}^1$ determines a unique curve L of type $(1,0)$ and a unique curve M of type $(0,1)$. The converse also holds.*

In Figure 1.2, the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is shown embedded in $\mathbb{P}^3$ using the Segre embedding, and we have chosen a suitable affine covering of $\mathbb{P}^3$. The black curves are fibers reflecting the structure of the surface. The image is made in cooperation with Georg Muntingh using `surfex` [32].

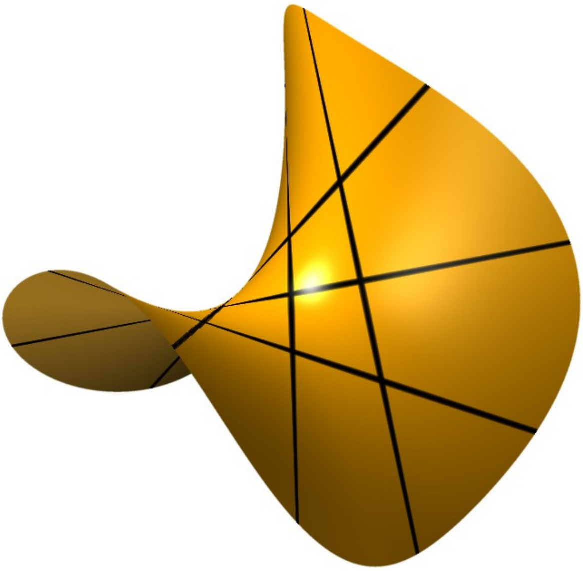


Figure 1.2: The surface $\mathbb{P}^1 \times \mathbb{P}^1$ with fibers.

Change of coordinates

A change of coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ is a bidegree preserving transformation given by linear transformations of the two $\mathbb{P}^1$'s separately. The transformation can be expressed by an element of $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ on the form

$$\begin{bmatrix} a_{00} & a_{01} & | & b_{00} & b_{01} \\ a_{10} & a_{11} & | & b_{10} & b_{11} \end{bmatrix}.$$

Note that we must have

$$\det \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \neq 0 \text{ and } \det \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} \neq 0.$$

Interchanging the two $\mathbb{P}^1$'s changes the order of the bidegree. Since a change of coordinates on $\mathbb{P}^1$ is uniquely determined by moving three distinct points, a change of coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ is uniquely determined by moving three points, no two on the same $(1,0)$- or $(0,1)$-curve.

The action of these transformations reveals an important aspect of the structure of $\mathbb{P}^1 \times \mathbb{P}^1$. A change of coordinates will move two points on the same $(1,0)$- or $(0,1)$-curve to two points with the same property. It is important to notice that two arbitrary points on $\mathbb{P}^1 \times \mathbb{P}^1$ do not generally define a $(1,0)$- or $(0,1)$-curve. Hence, we can not move two points on the same $(1,0)$- or $(0,1)$-curve to two arbitrary points, or conversely, using a change of coordinates.

$(1,1)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$

A $(1,1)$-curve on $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the zero set of a polynomial,

$$c_{00}x_0y_0 + c_{01}x_0y_1 + c_{10}x_1y_0 + c_{11}x_1y_1 = 0, \quad c_{ij} \in \mathbb{C}.$$

These curves have properties that are quite special, and they will eventually provide results for cuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$. The following theorem is fundamental.

Theorem 1.2.4. *Let p, q and r be three distinct points on $\mathbb{P}^1 \times \mathbb{P}^1$ not all on the same $(1,0)$- or $(0,1)$-curve. Then the three points define a unique, possibly reducible, curve Q_{pqr} of type $(1,1)$. Moreover, if no pair of points are on the same $(1,0)$- or $(0,1)$-curve, then the $(1,1)$-curve is irreducible.*

In either case, if the points have coordinates

$$\begin{aligned} p &= (p_{x_0} : p_{x_1}; p_{y_0} : p_{y_1}), \\ q &= (q_{x_0} : q_{x_1}; q_{y_0} : q_{y_1}), \\ r &= (r_{x_0} : r_{x_1}; r_{y_0} : r_{y_1}), \end{aligned}$$

then the defining polynomial of Q_{pqr} is given by

$$F = \det \begin{bmatrix} x_0y_0 & x_0y_1 & x_1y_0 & x_1y_1 \\ p_{x_0}p_{y_0} & p_{x_0}p_{y_1} & p_{x_1}p_{y_0} & p_{x_1}p_{y_1} \\ q_{x_0}q_{y_0} & q_{x_0}q_{y_1} & q_{x_1}q_{y_0} & q_{x_1}q_{y_1} \\ r_{x_0}r_{y_0} & r_{x_0}r_{y_1} & r_{x_1}r_{y_0} & r_{x_1}r_{y_1} \end{bmatrix}.$$

Proof. There are three cases to consider.

(1) The points p and q are on the same $(1,0)$-curve, and q and r are on the same $(0,1)$-curve. Then the product of the defining polynomials of the two curves is the defining polynomial of a unique $(1,1)$-curve passing through the three points.

(2) The points p and q are on the same $(1,0)$-curve, and r is not on the same $(0,1)$-curve as any of the other two. Then r is on a unique $(0,1)$-curve, and the product of the defining polynomials of the two curves gives the defining polynomial of a unique $(1,1)$-curve passing through the three points. By symmetry, the same obviously holds if the points p and q are on the same $(0,1)$-curve, and r is not on the same $(1,0)$-curve as any of the other two.

(3) Let p, q and r be three distinct points on $\mathbb{P}^1 \times \mathbb{P}^1$, not in any of the above configurations. Then the points can be moved by a change of coordinates to

$$\begin{aligned} p &= (1:0;1:0), \\ q &= (0:1;0:1), \\ r &= (1:1;1:1). \end{aligned}$$

Let

$$F = c_{00}x_0y_0 + c_{01}x_0y_1 + c_{10}x_1y_0 + c_{11}x_1y_1$$

be the defining polynomial of a $(1,1)$-curve containing p, q and r. Then $F(p) = 0$ implies $C_{00} = 0$, and $F(q) = 0$ implies $c_{11} = 0$. Moreover, $F(r) = 0$ implies that $c_{01} = -c_{10}$. Thus, the coefficients of the defining polynomial of the $(1,1)$-curve containing p, q and r is uniquely determined up to multiplication with $c \in \mathbb{C}^*$, and the $(1,1)$-curve is unique. Additionally, observe that the curve must be irreducible. If it were reducible, then, contrary to assumption, two of the points must have been on the same $(0,1)$ or $(1,0)$-curve.

The polynomial

$$F = \det \begin{bmatrix} x_0y_0 & x_0y_1 & x_1y_0 & x_1y_1 \\ p_{x_0}p_{y_0} & p_{x_0}p_{y_1} & p_{x_1}p_{y_0} & p_{x_1}p_{y_1} \\ q_{x_0}q_{y_0} & q_{x_0}q_{y_1} & q_{x_1}q_{y_0} & q_{x_1}q_{y_1} \\ r_{x_0}r_{y_0} & r_{x_0}r_{y_1} & r_{x_1}r_{y_0} & r_{x_1}r_{y_1} \end{bmatrix}$$

is the defining polynomial of a $(1,1)$-curve containing p, q and r, and by uniqueness it is the defining polynomial of Q_{pqr}. $\square$

1.3 Birational transformations

In this section we explain the main constructional tool of this thesis. We observe that birational transformations allow us to move from one surface to the other. In this process we blow up points and contract lines, which in turn changes the curves we consider. In combination with changes of coordinates the birational transformations provide the necessary tools to construct curves with prescribed singularities on a given surface.

By a *birational transformation of surfaces* we mean a rational map which admits an inverse. The following theorem ensures that a birational transformation of two surfaces can be composed of a sequence of blowing ups and contractions [31, Theorem V 5.5, p.412].

Theorem 1.3.1. *Let $\phi : X \dashrightarrow Y$ be a birational transformation of surfaces. Then it is possible to factor ϕ into a finite sequence of monoidal transformations and their inverses.*

1.3.1 Birational links

Following [5], we next give detailed descriptions of some fundamental birational transformations between the surfaces $\mathbb{P}^2$ and $\mathbb{F}_e$. These transformations are called *birational links.*

From $\mathbb{P}^2$ to $\mathbb{F}_1$

The map $\epsilon_{I+} : \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ is given in coordinates by

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{F}_1 \\ (x : y : z) &\mapsto (x : y; z, 1). \end{aligned}$$

This map blows up the point $p = (0 : 0 : 1)$. If p is a point of multiplicity m on a curve $C = \mathscr{V}(F)$ of degree d on $\mathbb{P}^2$, then the strict transform of C is a curve $\tilde{C} = \mathscr{V}(\tilde{F})$ of type $(m, d-m)$ on $\mathbb{F}_1$. Substituting the variables using the substitution

$$x = x_0 y_1, \quad y = x_1 y_1, \quad z = y_0,$$

we have

$$F(x, y, z) = y_1^m \tilde{F}(x_0, x_1, y_0, y_1).$$

From $\mathbb{F}_1$ to $\mathbb{P}^2$

The map $\epsilon_{I-} : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$ is given in coordinates by

$$\begin{aligned} \mathbb{F}_1 &\dashrightarrow \mathbb{P}^2 \\ (x_0 : x_1; y_0, y_1) &\mapsto (x_0 y_1 : x_1 y_1 : y_0). \end{aligned}$$

This map contracts the special section of $\mathbb{F}_1$, that is, $\mathscr{V}(y_1)$. If $C = \mathscr{V}(F)$ is a curve of type (a, b) on $\mathbb{F}_1$, then the strict transform of C is a curve $\tilde{C} = \mathscr{V}(\tilde{F})$ of degree $a + b$ on $\mathbb{P}^2$. Substituting the variables using the substitution

$$x_0 = x, \quad x_1 = y, \quad y_0 = z, \quad y_1 = 1,$$

we have

$$F(x_0, x_1, y_0, y_1) = \tilde{F}(x, y, z).$$

Hirzebruch one up

The map $\epsilon_{II+} : \mathbb{F}_i \dashrightarrow \mathbb{F}_{i+1}$ is given in coordinates by

$$\begin{aligned} \mathbb{F}_i &\dashrightarrow \mathbb{F}_{i+1} \\ (x_0 : x_1; y_0, y_1) &\mapsto (x_0 : x_1; x_0 y_0, y_1). \end{aligned}$$

This map blows up the point $p = (0 : 1; 1, 0)$ and contracts the fiber $\mathscr{V}(x_0)$. If p has multiplicity m on a curve $C = \mathscr{V}(F)$ of type (a, b) on $\mathbb{F}_i$, then the strict transform of C is a curve $\tilde{C} = \mathscr{V}(\tilde{F})$ of type $(a - m, b)$ on $\mathbb{F}_{i+1}$. Substituting the variables using the substitution

$$x_0 = x_0', \quad x_1 = x_1', \quad y_0 = y_0', \quad y_1 = x_0' y_1',$$

we have

$$F(x_0, x_1, y_0, y_1) = x_0'^m \tilde{F}(x_0', x_1', y_0', y_1').$$

Hirzebruch one down

The map $\epsilon_{II-} : \mathbb{F}_i \dashrightarrow \mathbb{F}_{i-1}$ is given in coordinates by

$$\begin{aligned} \mathbb{F}_i &\dashrightarrow \mathbb{F}_{i-1} \\ (x_0 : x_1; y_0, y_1) &\mapsto (x_0 : x_1; y_0, x_0 y_1). \end{aligned}$$

This map blows up the point $p = (0 : 1; 0, 1)$ and contracts the fiber $\mathscr{V}(x_0)$. If p has multiplicity m on a curve $C = \mathscr{V}(F)$ of type (a, b) on $\mathbb{F}_i$, then the strict transform of

C is a curve $\tilde{C} = \mathscr{V}(\tilde{F})$ of type $(a+b-m, b)$ on $\mathbb{F}_{i-1}$. Substituting the variables using the substitution

$$x_0 = x_0', \quad x_1 = x_1', \quad y_0 = x_0' y_0', \quad y_1 = y_1',$$

we have

$$F(x_0, x_1, y_0, y_1) = x_0'^m \tilde{F}(x_0', x_1', y_0', y_1').$$

Flip of $\mathbb{P}^1 \times \mathbb{P}^1$

The map $\epsilon_{III} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ that flips the coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ (x_0 : x_1; y_0 : y_1) &\mapsto (y_0 : y_1; x_0 : x_1). \end{aligned}$$

This map only changes the order in the bidegree of the curve. If $C = \mathscr{V}(F)$ is a curve of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1$, then we get a curve $\tilde{C} = \mathscr{V}(\tilde{F})$ of type (b, a).

The next theorem ensures that every birational transformation between two minimal rational surfaces consists of birational links and changes of coordinates [5, Theorem 1.1, p.3].

Theorem 1.3.2 (Noether–Castelnuovo). *Let X and Y be minimal rational surfaces over $\mathbb{C}$ and $\phi : X \dashrightarrow Y$ a birational map between them. Then there are birational links $\epsilon_1, \ldots, \epsilon_r$ and a change of coordinates ν of Y such that $\phi = \nu \circ \epsilon_r \circ \ldots \circ \epsilon_1$.*

The birational transformations transform a curve on one surface into a curve on the other surface. The important point in this construction is to position the curve appropriately with respect to the points that are blown up and the lines that are contracted by the transformation. Our main aim is to construct cuspidal curves, hence we want to resolve existing and avoid constructing new singularities that are not cusps.

In essence, we blow up existing singularities that are not cusps, and contract lines such that we only get cusps as new singularities on the curve. The theory of blowing up ensures that we can get rid of any singularity on the curve by a sequence of blowing ups. Finding lines that when contracted give at most a new cusp on the strict transform of the curve is a bit harder. Ultimately, the line that is contracted can only intersect the curve in one single point, and this point can either be smooth or a cusp, but no other singularity. This is indeed a very limiting restriction. However, since birational transformations between the projective plane and the Hirzebruch surfaces are compositions of blowing ups and contractions that are linked to one another, the situation is even more complicated in practice; we do not necessarily have the desired freedom of choice of points to blow up or lines to contract.

1.3.2 Standard transformations

It is sometimes convenient to have direct expressions for birational transformations between two surfaces. In the following we list maps that are used in the construction of curves on $\mathbb{P}^2$ or in some form appear later on in this thesis.

Cremona transformations

Birational maps from $\mathbb{P}^2$ to $\mathbb{P}^2$, called *Cremona transformations*, are widely used in the construction of cuspidal curves on $\mathbb{P}^2$ (see [17, 22, 23, 67, 46, 64, 58]). There is additionally a conjecture linking Cremona transformations to rational cuspidal curves, the Coolidge–Nagata conjecture (see Chapter 2 and [27, 42]). All Cremona transformations can be generated by linear transformations and three different quadratic maps (see [46]),

$$\begin{aligned}\psi_3 &: (x:y:z) \longmapsto (yz:xz:xy),\\ \psi_2 &: (x:y:z) \longmapsto (z^2:xy:xz),\\ \psi_1 &: (x:y:z) \longmapsto (y^2-xz:yz:z^2).\end{aligned}$$

Notice that these maps are their own inverses.

The main difference between the three maps is the number of basepoints on $\mathbb{P}^2$. The map ψ_3 has three distinct basepoints on $\mathbb{P}^2$, $p=(1:0:0)$, $q=(0:1:0)$ and $r=(0:0:1)$, and it contracts three lines $x=0$, $y=0$ and $z=0$. The strict transform $\tilde{F}$ has degree $2d-m_p-m_q-m_r$, and we have $F(x,y,z)=x^{m_p}y^{m_q}z^{m_r}\tilde{F}(x,y,z)$.

The map ψ_2 has two base points on $\mathbb{P}^2$ and ψ_1 has one base point on $\mathbb{P}^2$. The remaining base points of these maps are infinitely near points of the base points on $\mathbb{P}^2$. Using the maps ψ_2 and ψ_1 directly can be tricky since some of the basepoints are not on $\mathbb{P}^2$. Factoring these Cremona transformations using birational links leads to more control over the situation.

Birational maps of $\mathbb{P}^1\times\mathbb{P}^1$

A birational transformation from $\mathbb{P}^1\times\mathbb{P}^1$ to $\mathbb{P}^1\times\mathbb{P}^1$ can be constructed by blowing up two points on $\mathbb{P}^1\times\mathbb{P}^1$ lying on different $(1,0)$-curves and $(0,1)$-curves, and subsequently contract the two $(1,0)$-curves going through the points. Note that we equivalently could have chosen to contract the $(0,1)$-curves.

Explicitly, we may write $\phi:\mathbb{P}^1\times\mathbb{P}^1 \dashrightarrow \mathbb{P}^1\times\mathbb{P}^1$,

$$\phi:(x_0:x_1;y_0:y_1)\longmapsto(x_0:x_1;x_1y_1:x_0y_0). \tag{1.3.1}$$

Notice that this map is its own inverse. This birational map blows up the two points $\nu_0=(0:1;1:0)$ and $\nu_1=(1:0;0:1)$. It additionally contracts the two fibers $\mathscr{V}(x_0)$ and $\mathscr{V}(x_1)$.

In the construction of cuspidal curves on $\mathbb{P}^1\times\mathbb{P}^1$ starting out with a curve on $\mathbb{P}^1\times\mathbb{P}^1$, a change of coordinates is used to position the curve C appropriately before the birational transformation is applied. If $F(x_0,x_1,y_0,y_1)$ is a bihomogeneous polynomial of bidegree (a,b), then its total transform $F'(x_0,x_1,y_0,y_1)$ under the birational transformation can be factored

$$F'(x_0,x_1,y_0,y_1)=x_0^{m_{\nu_0}}x_1^{m_{\nu_1}}\tilde{F}(x_0,x_1,y_0,y_1),$$

where m_{ν_i} denotes the multiplicity of F at the points ν_i. The bihomogeneous polynomial $\tilde{F}$ has bidegree $(a+b-m_{\nu_0}-m_{\nu_1},b)$, and it is the defining polynomial of the strict transform of the curve $C=\mathscr{V}(F)$. Hence, $\tilde{C}=\mathscr{V}(\tilde{F})$ is a curve of type $(a+b-m_{\nu_0}-m_{\nu_1},b)$.

Maps between $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$

When constructing cuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$, we additionally use birational transformations from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$. Explicit equations for such transformations can be obtained by composing two birational links, for example

$$\begin{aligned} \phi : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ (x : y : z) &\mapsto (x : y; z : x). \end{aligned}$$

This transformation blows up two points on $\mathbb{P}^2$, here $(0 : 1 : 0)$ and $(0 : 0 : 1)$, and contracts the line they span, here $\mathscr{V}(x)$.

The inverse of ϕ can be given by

$$\begin{aligned} \phi^{-1} : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^2 \\ (x_0 : x_1; y_0 : y_1) &\mapsto (x_0 y_1 : x_1 y_1 : x_0 y_0). \end{aligned}$$

This inverse transformation blows up one point on $\mathbb{P}^1 \times \mathbb{P}^1$, here $(0 : 1; 1 : 0)$ and contracts two lines, here $\mathscr{V}(x_0)$ and $\mathscr{V}(y_1)$.

In the construction of cuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ starting out from $\mathbb{P}^2$, we use a change of coordinates to position the curve C on $\mathbb{P}^2$ such that the appropriate points are blown up and the appropriate line is contracted by the birational map. Let $F(x, y, z)$ be the defining polynomial of C on $\mathbb{P}^2$, and let m_y and m_z be the multiplicities of F at the respective base points of ϕ. Applying the transformation then gives

$$F(x_0, x_1, y_0, y_1) = x_0^{m_y} y_1^{m_z} \tilde{F}(x_0, x_1, y_0, y_1),$$

where $\tilde{F}$ is a bihomogeneous polynomial of bidegree $(d - m_y, d - m_z)$ defining the rational curve $\tilde{C}$ of type $(d - m_y, d - m_z)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Conversely, given a curve $C = \mathscr{V}(F(x_0, x_1, y_0, y_1))$ of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1$ with m_p the multiplicity of the point $(0 : 1; 1 : 0)$, we may find a curve $\hat{C}$ on $\mathbb{P}^2$ of degree $a + b - m_p$. The transformation leaves us with the following relation for the defining polynomial,

$$F(x, y, z) = x^{m_p} \hat{F}(x, y, z).$$

Chapter 2

Cuspidal curves on the projective plane

In this chapter we present some of the known results about cuspidal curves on the projective plane. First, we recall general results valid for curves on the projective plane. In particular, we specialize the results in Chapter 1, we consider properties of inflection points on cuspidal curves, we recall a theorem on the logarithmic Kodaira dimension of the complement of a curve by Wakabayashi [74], and we discuss the question of how many cusps a cuspidal curve can have. Then we focus on rational cuspidal curves. We recall a few results for curves of this type, and list the cuspidal configurations of some of the many known rational cuspidal curves. In particular, we investigate three series of curves with three cusps. Moreover, we recall two conjectures on rational cuspidal curves, and consider real rational cuspidal curves. Last in this chapter, we recall a toric construction of a series of cuspidal curves by Orevkov [50]. Note that the results and the curves presented in this chapter are in general not original to this thesis, but they are included to give an overview and because they serve as a starting point for the results and the curves that we find in subsequent chapters.

2.1 Background and preliminary results

In this section we will state the results for cuspidal curves on the projective plane that follow from the general theorems in Chapter 1 and properties of the projective plane.

Recall from Chapter 1 that a cuspidal curve C on $\mathbb{P}^2$ of degree d is given as the zero set of a reduced and irreducible polynomial $F \in \mathbb{C}[x,y,z]_d$.

First we take a closer look at a point p on a cuspidal curve on $\mathbb{P}^2$. For any point $p \in \mathbb{P}^2$, we have seen that we may make a linear change of coordinates such that $p = (0:0:1)$. In the open neighbourhood U_z of p, the polynomial $F(x,y,1) = f(x,y)$ is the affine defining equation of C. Sorting the polynomial $f = f(x,y)$ by its homogeneous terms in the affine coordinates x and y we get,

$$f = f_0 + f_1 + \cdots + f_i + \cdots + f_d,$$

where $f_i = f_i(x,y)$ denotes the terms of $f(x,y)$ of degree i in x and y. Recall that the multiplicity m of p is the smallest integer m such that $f_m \neq 0$ [31, p.36]. The multiplicity m is restricted by the following lemma.

Lemma 2.1.1. *Let p be a point on a reduced and irreducible curve C of degree $d \geq 2$ on $\mathbb{P}^2$. Then*

$$0 < m < d.$$

Proof. Clearly $d \geq m \geq 0$. Since $p \in C$, we have $0 < m$. Assume for contradiction that $m = d$. Then C would be a union of d lines through p, contrary to the assumption that C was reduced and irreducible, so $m < d$. □

Recall that for a cusp p, the multiplicity is $m \geq 2$. Additionally, we have the multiplicity sequence $\overline{m}$ and the other invariants described in Chapter 1.

Moreover, recall that we can determine that a point p on a curve C is singular using only the defining polynomial.

Definition 2.1.2. Let $C = \mathscr{V}(F)$ be a reduced and irreducible curve on $\mathbb{P}^2$, and p be a point on C. Then p is a *singular* if $p \in \mathscr{V}(F_x, F_y, F_z)$, where F_x, F_y and F_z denote the respective partial derivatives of F.

There are several results concerning cuspidal curves on $\mathbb{P}^2$. The first is a well-known lemma with a straightforward proof.

Lemma 2.1.3. *Let p and p' be two cusps with multiplicities m and m' respectively on a reduced and irreducible plane curve C of degree d. Then we have*

$$m + m' \leq d.$$

Proof. On $\mathbb{P}^2$ there is a unique line L passing through the two cusps. We have $L.C = d$. By Corollary 1.1.12 $(L \cdot C)_p \geq m$ and $(L \cdot C)_{p'} \geq m'$, and the result follows. □

Recall that every smooth point or cusp p on C has a unique local tangent line. On $\mathbb{P}^2$, the local tangent line can be homogenized to a global tangent line $T_p := T_pC$. For a smooth point, the tangent line can be calculated without a change of coordinates,

$$T_p := \mathscr{V}\big(F_x(p)x + F_y(p)y + F_z(p)z\big).$$

Definition 2.1.4. The intersection multiplicity $(C \cdot T_p)_p$ of a curve C and a tangent line T_p to C at a point p is called the *tangential intersection multiplicity.*

The tangent line T_p of a smooth point p on a curve C has the property that $(C \cdot T_p)_p > 1$.

Definition 2.1.5. If p is a smooth point on a reduced and irreducible curve C with tangent T_p at p and we have $(C \cdot T_p)_p \geq 3$, we say that p is an *flex point*, or equivalently an *inflection point*, of type $(C \cdot T_p)_p - 2$.

Let $\mathbb{P}^{2*}$ denote the dual space of $\mathbb{P}^2$, where a line $V(ax + by + cz)$ on $\mathbb{P}^2$ is identified with a point $(a : b : c)$ on $\mathbb{P}^{2*}$. The existence of a unique global tangent line for every smooth point p on a curve C on $\mathbb{P}^2$, allows the definition of a *dual curve* C^* on $\mathbb{P}^{2*}$. The curve C^* is defined as

$$C^* := \overline{\Big(\big\{\big(F_x(p) : F_y(p) : F_z(p)\big) \mid p \text{ smooth on } C\big\}\Big)} \subset \mathbb{P}^{2*}.$$

Another well-known result, resembling Lemma 2.1.3, can be found using properties of the tangent line of a curve at a cusp.

Lemma 2.1.6. *Let p be a cusp on a reduced and irreducible curve C of degree d with multiplicity sequence $\overline{m} = [m_0, m_1, \ldots, m_{t-1}]$. Then*

$$m_0 + m_1 \leq d.$$

Proof. Let T_p denote the tangent line to C at p. By intersection theory, $T_p \,.\, C \leq d$. By Lemma 1.1.14, $m_0 + m_1 \leq (C \cdot T_p)_p$. By Proposition 1.1.2, $(C \cdot T_p)_p \leq T_p \,.\, C$, and the result follows. □

We sometimes need the associated *polar curve* P_pC of a plane curve C with respect to a point $p = (p_x : p_y : p_z) \in \mathbb{P}^2$,

$$P_pC := \mathscr{V}(p_xF_x + p_yF_y + p_zF_z).$$

The polar curve has the property that it intersects the curve C in the singular points of C and in the smooth points of C that have tangent lines that pass through p.

Moreover, the *Hessian curve* $H_C = \mathscr{V}(H_F)$ can be used to find inflection points on a curve $C = \mathscr{V}(F)$, where

$$H_F := \det \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}.$$

The intersection of C and H_C consists of the singular points and the inflection points on C.

2.1.1 Inflection points on plane cuspidal curves

When studying cuspidal curves on the projective plane, the question of how many and what kind of cusps such a curve can have can be answered by classifying them up to genus, degree and cuspidal configuration, that is up to so-called *equisingular equivalence*. This classification is sometimes too coarse to catch interesting features of a curve. Let p and p' be two cusps having the same multiplicity sequences on two curves C and C', and let T and T' denote their respective tangent lines. We saw in Chapter 1 that the multiplicity sequence of a cusp determines its topological type, but two curves with cusps with the same multiplicity sequences may have different intersection multiplicities with the respective tangents. This can be seen in an example with two curves of the same degree and cuspidal configuration.

Example 2.1.7. Let C and C' be two rational cuspidal curves of degree 5 with the same cuspidal configuration $[2_2], [3, 2]$, given by the parametrizations $(s^5 : s^3t^2 : t^5)$ and $(s^5 : s^3t^2 : st^4 + t^5)$ respectively. Let $p \in C$ and $p' \in C'$ denote the cusps with multiplicity sequence $[2_2]$. Let T and T' denote the respective tangent lines. Upon inspection we find that $(T \cdot C)_p = 5$, while $(T' \cdot C')_{p'} = 4$. Moreover, we see that C' has a flex point, while C has no such point.

The seemingly irrelevant difference in the tangential intersections makes the curves very different. For example, the dual curves do not have the same cuspidal configuration.

In the context of this thesis, the above observation is essential. Applying birational transformations to two equisingular equivalent curves on $\mathbb{P}^2$ can possibly lead to very different curves, and the tangential intersection multiplicity plays a fundamental role in the study and construction of plane cuspidal curves. We therefore pay attention to this invariant in our work. Note that the invariant is strongly linked to the number of flex points on the curve.

The significance of the tangential intersection multiplicities of cusps and their effect on the number of inflection points on a cuspidal curve is apparent in the inflection point formula for plane curves, that calculates the number of inflection points, counted with multiplicity. The theorem is a modified version of [4, Theorem 2 (ii), p.586].

Theorem 2.1.8 (Inflection point formula). *Assume that C is a plane cuspidal curve with s cusps p_j, each having tangent T_j. Let m_j denote the multiplicity, δ_j the delta invariant, and $r_j := (T_j \cdot C)_{p_j}$ the tangential intersection multiplicity of p_j. Then the number of inflection points v on C, counted such that an inflection point q_i of type v_i accounts for v_i inflection points, is given by*

$$v = 3d(d-2) - \sum_{j=1}^{s}(6\delta_j + m_j + r_j - 3).$$

Since $v = 6\delta + m + r - 3$ for any inflection point of C, where $\delta = 0$ and r is the tangential intersection multiplicity, we rewrite the inflection point formula in Theorem 2.1.8. This gives

$$\sum_{p \in C \cap H_C} (C \cdot H_C)_p = \sum_{p \in C \cap H_C} (6\delta_p + m_p + r_p - 3).$$

This result is in fact a local result.

Theorem 2.1.9 (Local Hessian intersection formula). *The intersection multiplicity $(C \cdot H_C)_p$ of a cuspidal curve C and its Hessian curve H_C at any point $p \in C$ is*

$$(C \cdot H_C)_p = 6\delta_p + m_p + r_p - 3.$$

Proof. The formula follows from combining a result by Josse and Pène in [35] and a result by Brieskorn and Knörrer in [4]. Since C is cuspidal, all points $p \in C$ are unibranched. For unibranched points, the results in [35, Proposition 25 and 29, pp.18–21] give,

$$(C \cdot H_C)_p = 3(C \cdot P_qC)_p + r_p - 2m_p,$$

where P_qC is the polar curve to C at a point $q \notin T_p$. Since C is cuspidal, by [4, proof of Theorem 2 (i), p.596] we have

$$(C \cdot P_qC)_p = 2\delta_p + m_p - 1,$$

and the result follows. □

2.1.2 Preliminary results for plane cuspidal curves

From the general genus formula given in Proposition 1.1.17 we get the following corollary. This formula gives global restrictions on the multiplicity sequences of the cusps on a cuspidal curve on $\mathbb{P}^2$.

Corollary 2.1.10 (Corollary to Proposition 1.1.17). *Let C be a cuspidal curve of degree d with s cusps p_j, each with multiplicity sequence $\overline{m}_j = [m_0, m_1, \ldots, m_{t_j-1}]$. Then the genus g of C is given by*

$$g = \frac{(d-1)(d-2)}{2} - \sum_{j=1}^{s}\sum_{i=0}^{t_j-1} \frac{m_i(m_i-1)}{2}.$$

Proof. By Proposition 1.1.17 and since $K \sim 3L$,

$$\begin{aligned} g &= \frac{C\,.\,(C+K)}{2} + 1 - \sum_{j=1}^{s} \delta_j \\ &= \frac{dL\,.\,(d-3)L}{2} + 1 - \sum_{j=1}^{s}\sum_{i=0}^{t_j-1} \frac{m_i(m_i-1)}{2} \\ &= \frac{(d-1)(d-2)}{2} - \sum_{j=1}^{s}\sum_{i=0}^{t_j-1} \frac{m_i(m_i-1)}{2}. \end{aligned}$$

□

Moreover, there are the B–M–Y-inequalities for curves on $\mathbb{P}^2$.

Corollary 2.1.11 (Corollary to Theorem 1.1.23). *Let (V, D) be the minimal embedded resolution of a cuspidal curve C of genus g on $\mathbb{P}^2$, and let K_V and H be as in the Zariski decomposition described in Chapter 1.*

a) If $\overline{\kappa}(V \setminus D) \geq 0$, then

$$(K_V + D)^2 \leq 6g + 3.$$

b) If $\overline{\kappa}(V \setminus D) = 2$, then

$$H^2 \leq 6g + 3.$$

Proof. First, $e(V \setminus D) = e(\mathbb{P}^2 \setminus C)$ since $V \setminus D$ is isomorphic to $\mathbb{P}^2 \setminus C$. For a plane cuspidal curve of genus g, the Euler characteristic of the complement is $e(\mathbb{P}^2 \setminus C) = 2g+1$ [68, Proof of Theorem 1.1, p.220]. The conclusion follows from Theorem 1.1.23. □

Another result links the M-numbers to the minimal embedded resolution of a plane curve. This result is a corollary to Theorem 1.1.19.

Corollary 2.1.12 (Corollary of Theorem 1.1.19). *With V, D, K_V and M_j as in Chapter 1, for a cuspidal curve $C \in \mathbb{P}^2$ of degree d with s cusps,*

$$K_V\,.\,(K_V + D) = 9 - 3d + \sum_{j=1}^{s} M_j.$$

Proof. The result follows from Theorem 1.1.19,

$$\begin{aligned} K_V\,.\,(K_V+D) &= K\,.\,(K+C)+\sum_{j=1}^{s} M_j \\ &= -3L\,.\,(d-3)L+\sum_{j=1}^{s} M_j \\ &= 9-3d+\sum_{j=1}^{s} M_j. \end{aligned}$$

□

There is additionally a result on self intersection numbers. This result follows directly from [31, Proposition V 3.2, p.387] and induction.

Lemma 2.1.13. *Let $\tilde{C}$ be the strict transform of the minimal embedded resolution of a plane cuspidal curve C of degree d. Let m_i denote the elements of the multiplicity sequences of the cusps on C. Then*

$$\tilde{C}^2 = d^2 - \sum_i m_i^2.$$

2.2 The logarithmic Kodaira dimension

An important result for curves on $\mathbb{P}^2$ involves the logarithmic Kodaira dimension of the complement of the curve. A theorem by Wakabayashi [74] is fundamental for the development of new results on cuspidal curves on $\mathbb{P}^2$. In particular, the theorem is critical in the production of upper bounds for the number of cusps on a rational cuspidal curve (see [51, 68]).

Theorem 2.2.1 ([74, Theorem, p.157]). *Let C be an irreducible curve of genus g and degree $d \geq 4$ on $\mathbb{P}^2$.*

(*I*) *If $g > 0$, then $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$.*

(*II*) *If $g = 0$ and C has at least three cusps, then $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$.*

(*III*) *If $g = 0$ and C has at least two cusps, then $\overline{\kappa}(\mathbb{P}^2 \setminus C) \geq 0$.*

2.3 On the number of cusps

The question of how many cusps a cuspidal plane curve can have is still unsettled, but there has been a huge effort to solve the problem. Since the cuspidal curves are algebraic, the first conclusion is that the number of cusps on a cuspidal curve on $\mathbb{P}^2$ is finite [31, Theorem I 5.3, p.33].

A more specific upper bound on the number of cusps can be found as a consequence of the genus formula.

Lemma 2.3.1. *The number of cusps s on a cuspidal curve C on $\mathbb{P}^2$ of genus g and degree d is bounded,*

$$s \leq \frac{(d-1)(d-2)}{2} - g.$$

Proof. A curve has the maximal number of cusps whenever the cusps all have multiplicity sequence [2]. The number of ordinary cusps equals the sum of the delta invariants, $s = \sum \delta_j$, and the result follows. □

The question of the number of cusps on a plane curve, not necessarily cuspidal, is intensively studied in articles from the late 19th and early 20th century. In the introduction we mentioned three articles by Clebsch [8], Wieleitner [78] and Lefschetz [38]. We state their results in our notation for completeness.

Proposition 2.3.2 ([8, p.51]). *A rational, not necessarily cuspidal, plane curve C of degree d can have at most $\frac{3}{2}(d-2)$ ordinary cusps.*

Proposition 2.3.3 ([78, p.76]). *A rational cuspidal plane curve C of degree d can have at most $\frac{3}{8}d(d-2)$ ordinary cusps. In particular, for $d > 4$ there does not exist a rational plane curve where all singularities are ordinary cusps.*

Proposition 2.3.4 ([38, p.27–28]). *A plane, not necessarily cuspidal, curve C can have at most $\left\lfloor \frac{d(d-2)}{3} \right\rfloor$ ordinary cusps when $d \leq 13$, and at most $\left\lfloor \frac{d(d+3)}{4} - 4 \right\rfloor$ ordinary cusps when $d \geq 14$.*

The above observations were significantly improved recently. There is a much stronger result by Tono [68], that depends only on the genus g of the curve. This result is so far the best known upper bound for the number of cusps on a plane cuspidal curve. We restate the result using our notation [68, Theorem 1.1, p.216].

Theorem 2.3.5 (On the number of cusps on $\mathbb{P}^2$). *The number of cusps s on a cuspidal curve C of genus g on $\mathbb{P}^2$ has an upper bound,*

$$s \leq \frac{21g+17}{2}.$$

Finding cuspidal curves with many cusps is a quite hard task, and the known examples indicate that the bound in 2.3.5 is not sharp. For example, the largest number of cusps found on a rational cuspidal curve is four. Even more surprisingly, up to equisingular equivalence, only one rational cuspidal curve with four cusps has been found. To our knowledge, this curve was first constructed by Namba in [49].

Example 2.3.6. There exists a plane cuspidal curve C of genus $g = 0$ and degree $d = 5$ with four cusps and cuspidal configuration

$$[2_3], [2], [2], [2].$$

A parametrization of this curve can be found in [49, Theorem 2.3.10, pp.179–182],

$$x = s^4 t,\ y = s^2 t^3 - s^5,\ z = t^5 + 2s^3 t^2.$$

A defining polynomial F can be found by eliminating s and t,

$$F = y^4 z - 2xy^2 z^2 + x^2 z^3 + 2x^2 y^3 - 18x^3 yz - 27x^5.$$

Theorem 2.3.5 indicates that the number of allowed cusps increases with higher genus. With the results for rational cuspidal curves in mind, the following example of a curve of genus $g = 1$ and five cusps fits nicely in this picture. Note that this curve has been thoroughly investigated in the literature, see [39] for details.

Example 2.3.7. There exists a plane cuspidal curve $C = \mathscr{V}(F)$ of genus $g = 1$ and degree $d = 5$ with five ordinary cusps. A defining polynomial F can be found in [19],

$$F = \frac{(\alpha y^2z^2 + \beta x^2z^2 + \gamma x^2y^2 - 6ax^2yz - 6bxy^2z - 6cxyz^2)^2 - (yz + xz + xy)^3(\alpha^2yz + \beta^2xz + \gamma^2xy)}{xyz},$$

where

$$\alpha = -a + b + c,\ \beta = a - b + c,\ \gamma = a + b - c.$$

We can do even better for curves with $g = 1$ and find a cuspidal curve of degree six with nine cusps.

Example 2.3.8. There exists a plane cuspidal curve $C = \mathscr{V}(F)$ of genus $g = 1$ and degree $d = 6$ with nine ordinary cusps. This curve is the dual of a nonsingular plane cubic curve, for example the curve given by $\mathscr{V}(x^3 + y^3 + z^3)$. Computing the dual of the cubic curve, we find a defining polynomial F of C,

$$F = x^6 + y^6 + z^6 - 2x^3y^3 - 2x^3z^3 - 2y^3z^3.$$

The latter two examples show that cuspidal curves that are not rational can have a quite high number of cusps, and the situation is therefore more complicated than the situation for rational cuspidal curves. Theorem 2.3.5 shows that there is an upper bound on the number of cusps on a cuspidal curve of any genus, and again the examples suggest that this bound is not sharp. The main focus of this thesis are rational cuspidal curves, but finding new examples of cuspidal curves and sharp upper bounds for the number of cusps on cuspidal curves of any genus is a task that could be studied in the future.

2.4 Rational cuspidal curves on the projective plane

Although rational cuspidal plane curves have been intensively studied, there are still unanswered questions concerning them. In this section we give an overview of known results and proposed conjectures for the rational cuspidal plane curves. None of the questions from the introduction are completely answered, but they are elaborated, and we present restrictions for the answers.

In recent years the hunt for rational cuspidal curves has been popular, in particular after the announcement of Sakai's open problems on rational cuspidal curves in [30]. There are many approaches to this hunt that have been extensively explored. Indeed, fixing either the degree d of the curve, the largest multiplicity $\hat{m}$ of any cusp, the number of cusps s, or the logarithmic Kodaira dimension of the complement of the curve has lead to quite major results. Combining this with the use of Cremona transformations, the existence of many rational cuspidal curves, even infinite series of rational cuspidal curves, has been established.

In this section we first give a few background results on plane rational cuspidal curves. Then we present some of the known curves and results closing off the hunt for rational cuspidal curves in some directions. For detailed proofs, constructions and defining polynomials, see for example [17, 22, 23, 46, 49, 50, 58, 65, 66, 67, 70].

2.4.1 Background

We now present a few results that are valid only for rational cuspidal curves. By the genus formula we have strong restrictions on the multiplicity sequences for cusps on a rational cuspidal curve. Recall that we do not append an index j to the elements of the multiplicity sequence of a cusp p_j.

Theorem 2.4.1. *Let C be a rational cuspidal curve with s cusps p_j with multiplicity sequences $\overline{m}_j = [m_0, \ldots m_{t_j-1}]$*

$$\frac{(d-1)(d-2)}{2} = \sum_{j=1}^{s} \sum_{i=0}^{t_j-1} \frac{m_i(m_i-1)}{2}.$$

Moreover, the degree d and the maximal multiplicity $\hat{m}$ of all the cusps on C,

$$\hat{m} := \max\{m_j, j = 1, \ldots, s\},$$

must satisfy the following inequalities by Matsuoka and Sakai in [42, Theorem, p.233] and Orevkov in [50, Theorem A, p.657], where the latter estimate is a better estimate for all $d \geq 9$.

Theorem 2.4.2. *Assume that C is a rational cuspidal curve of degree d on $\mathbb{P}^2$ with s cusps. Let $\hat{m}$ be the maximal multiplicity of all the cusps on C, then*

$$\text{(M–S)} \qquad d < 3\hat{m}. \tag{2.4.1}$$

$$\text{(O)} \qquad d < \frac{3+\sqrt{5}}{2}(\hat{m}+1) + \frac{1}{\sqrt{5}}. \tag{2.4.2}$$

Next there is a lemma that has been important in the development of new results for rational cuspidal curves on $\mathbb{P}^2$ [21, Lemma 1.3, p.148]. The lemma is stated in a general setting, namely for $\mathbb{Q}$-acyclic surfaces, and we here recall the lemma using our notation. By [17, Proposition 1.5.16, pp.42–43] the complement of a rational cuspidal curve C on $\mathbb{P}^2$ is $\mathbb{Q}$-acyclic, that is $\mathrm{H}_i(\mathbb{P}^2 \setminus C; \mathbb{Q}) = 0$ for all $i > 0$, so the lemma directly applies to the minimal embedded resolution of such curves. By $c_2(V)$ we mean the second Chern class of V, see [31, pp.431–433].

Lemma 2.4.3 (Lemma for $\mathbb{Q}$-acyclic surfaces). *Assume that V is a smooth complete surface, that D is an SNC-divisor, and that $V \setminus D$ is $\mathbb{Q}$-acyclic. Let $D_1, \ldots, D_r$ be the irreducible components of D. Then the following hold.*

(0) *D is a rational tree.*

(1) $\chi(\Theta_V) = 10 - 2r$.

(2) $K_V^2 = 10 - r$.

(3) $c_2 := c_2(V) = 2 + r$.

(4) $\chi\left(\bigoplus \mathscr{N}_{D_i/V}\right) = r + \sum_{i=1}^{r} D_i^2$.

(5) *We have*

$$\begin{aligned}\chi\Big(\Theta_V \langle D \rangle\Big) &= (K_V + D)^2 + 2 \\ &= -D^2 - r + 8 \\ &= K_V \,.\, (K_V + D).\end{aligned}$$

A conjecture by Fernandéz de Bobadilla, Luengo, Melle-Hernández and Némethi proposes that for a plane rational cuspidal curve C there is an inequality between the dimension of the subgroup of $\mathrm{PGL}_3(\mathbb{C})$ that keeps C fixed, the so-called stabiliser, and the degree d and the M-numbers of the cusps. We include the inequality, alternatively stated using the Euler characteristic of the logarithmic tangent sheaf, for completeness [18, p.420].

Conjecture 2.4.4. *Assume that C is a plane rational cuspidal curve. Let $\mathrm{Stab}_{\mathrm{PGL}_3(\mathbb{C})}(C)$ denote the stabiliser of C. Then*

$$\dim \mathrm{Stab}_{\mathrm{PGL}_3(\mathbb{C})}(C) \geq \chi(\Theta_V \langle D \rangle).$$

Another result with powerful consequences involves the logarithmic Kodaira dimension of the complement of a rational cuspidal plane curve C. The known results can be summed up as follows.

Proposition 2.4.5. *Let C be a plane rational cuspidal curve with s cusps. Then the following hold.*

a) $\overline{\kappa}(\mathbb{P}^2 \setminus C) \neq 0$.

b) If $\overline{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$, *then* $s = 1$.

c) If $s \geq 2$, *then* $\overline{\kappa}(\mathbb{P}^2 \setminus C) \geq 1$.

d) If $s \geq 3$, *then* $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$.

Proof. The first result, $\overline{\kappa}(\mathbb{P}^2 \setminus C) \neq 0$, can be found in [50, Theorem B–c), p.657]. The next results are consequences of Theorem 2.2.1. See [18, p.422]. □

Note that there are examples of unicuspidal and bicuspidal curves with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$ and $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$.

Using the logarithmic Kodaira dimension, Orevkov improves the results in Theorem 2.4.2 with the following result [50, Theorem B–a)–b), p.657].

Theorem 2.4.6. *Assume that C is a plane rational cuspidal curve of degree d and let $\hat{m}$ be the maximal multiplicity of any cusp on C.*

a) If $\overline{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$, then

$$d < \frac{3+\sqrt{5}}{2}\hat{m}.$$

b) If $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$, then

$$d < \frac{3+\sqrt{5}}{2}(\hat{m}+1) - \frac{1}{\sqrt{5}}.$$

2.4.2 Assuming $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$.

In answering the question of how many cusps a rational cuspidal curve on $\mathbb{P}^2$ can have, we are interested in describing curves with three or more cusps. The above discussion tells us that assuming $s \geq 3$ implies that $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$. Conversely, assuming $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$ includes both curves with $s \geq 3$ and curves with $s < 3$. We will now assume that $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$ and find interesting properties of all these curves.

We have already seen one result in Theorem 2.4.6. Moreover, we have further restrictions on the multiplicity sequences of the cusps on C when $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$ (cf. Lemma 2.1.6). A first result is established in [17, 2.3.2 and 2.3.3, p.57].

Theorem 2.4.7. *Let C be a plane rational cuspidal curve of degree d, and assume that $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$. For a cusp with multiplicity sequence $\overline{m} = [m_0, m_1, \ldots, m_{t-1}]$, we have*

$$m_0 + m_1 < d.$$

The next result is a consequence of [68, Lemma 4.1, p.219] and Lemma 2.4.3.

Proposition 2.4.8. *Let C be a rational cuspidal curve on $\mathbb{P}^2$ such that $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$, and let (V, D) be the minimal embedded resolution of C. Then*

$$0 \leq \chi(\Theta_V \langle D \rangle). \tag{2.4.3}$$

Proof. By [68, Remark 4.5], $0 \leq K_V \,.\, (K_V + D)$. The result then follows from Lemma 2.4.3(5). □

The following proposition is a consequence of Proposition 2.4.8 and a result by Orevkov and Zaidenberg. The first inequality is due to Proposition 2.4.8, and was not known to Orevkov and Zaidenberg, who established the second inequality and a weaker version of Corollary 2.4.10 in [51]. We recall the proof to explain the connections.

Proposition 2.4.9. *Let C be a plane rational cuspidal curve with s cusps and $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$. Then*

$$0 \leq \chi(\Theta_V \langle D \rangle) < 5 - \frac{s}{2}.$$

Proof. By Proposition 2.4.8, there is the inequality $0 \leq \chi(\Theta_V \langle D \rangle)$. By Lemma 2.4.3(5), we have $\chi(\Theta_V \langle D \rangle) = (K_V + D)^2 + 2$. By the Zariski-decomposition, $(K_V + D)^2 + 2 = H^2 + N^2 + 2$. Moreover, by the B–M–Y-inequality, Theorem 2.1.11, $H^2 + N^2 + 2 \leq 3 + N^2 + 2 \leq 5 + N^2$. Moreover, it is shown in [51, Lemma 5 and Lemma 6] that in this situation

$$N^2 < -\frac{s}{2}.$$

Hence,

$$0 \leq \chi(\Theta_V \langle D \rangle) < 5 - \frac{s}{2}.$$

□

Corollary 2.4.10. *The number of cusps s on a plane rational cuspidal curve is bounded,*

$$s \leq 9.$$

Note that this estimate is not as good as the one given in Theorem 2.3.5. The bound $s \leq 9$ was established for rational cuspidal curves satisfying $0 \leq \chi(\Theta_V \langle D \rangle)$ by Orevkov and Zaidenberg in [51], but by Proposition 2.4.8 and Theorem 2.2.1, this always holds for plane rational cuspidal curves C with $s \geq 3$.

Moreover, by a result of Iitaka [33, Theorem 6, p.185], for a plane rational cuspidal curve C with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$ it follows that $\mathrm{h}^0(V, \Theta_V \langle D \rangle) = 0$. Hence,

$$\chi(\Theta_V \langle D \rangle) = \mathrm{h}^2(V, \Theta_V \langle D \rangle) - \mathrm{h}^1(V, \Theta_V \langle D \rangle). \tag{2.4.4}$$

By [18, Lemma 5.1, pp.420–422], we have for rational cuspidal curves C with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$ that $\dim \mathrm{Stab}_{\mathrm{PGL}_3(\mathbb{C})}(C) = 0$. Hence, we directly have a modified version of Conjecture 2.4.4.

Conjecture 2.4.11. *If $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$, then*

$$\chi(\Theta_V \langle D \rangle) = 0,$$

and moreover,

$$\sum_{j=1}^{s} M_j = 3(d-3).$$

In the exploration of rational cuspidal curves, there has been a discussion and use of the stronger so-called *rigidity conjecture*, proposed by Flenner and Zaidenberg (cf. [21, 22]), and this conjecture is also still open.

Conjecture 2.4.12 (The rigidity conjecture). *If $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$, then*

$$\mathrm{h}^1(V, \Theta_V \langle D \rangle) = 0 = \mathrm{h}^2(V, \Theta_V \langle D \rangle).$$

In particular,

$$\chi(\Theta_V \langle D \rangle) = 0.$$

2.4.3 Cuspidal curves of low degree

In the literature all rational cuspidal curves of degree $d \leq 6$ have been classified up to equisingular equivalence. In the below tables we recall these curves and their cuspidal configurations. These curves will be extensively used in later chapters, so we include their parametrizations or defining polynomials, as given in [49] and [17].

Curve	**Cuspidal configuration**	**Parametrization**	**Defining polynomial**
C_1	[2]	$(t^2 : t^3 : 1)$	$y^2z - x^3$

Table 2.1: The rational cuspidal cubic [49, Proposition 2.2.1, p.128].

Changing the coordinates, we can make a real affine image of a plane rational cuspidal curve of degree three, where also the inflection point of this curve can be seen. The image is given in Figure 2.1, and it is created with `Maple` [77] and the package `plots`.

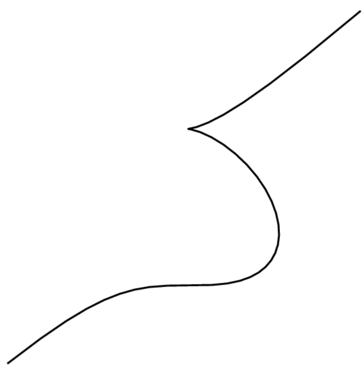


Figure 2.1: A rational cuspidal cubic.

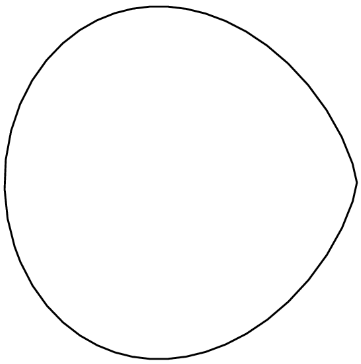
The tables of rational cuspidal curves of degree four and five up to equisingular equivalence counts four and eight curves respectively. It is, however, possible to construct curves with the same cuspidal configuration but different number of inflection points, see [46]. We include these additional curves since the appearance of inflection points allows the construction of other curves on the Hirzebruch surfaces in the subsequent chapters.

Notice in particular that there are few rational cuspidal curves of degree four and five with many cusps, and among them there is only one curve with four cusps.

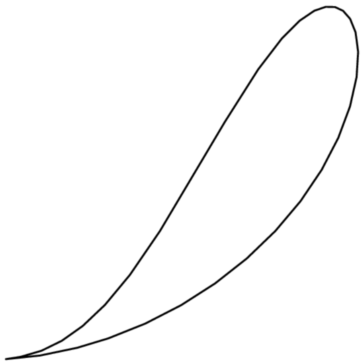
Curve	Cuspidal configuration	Defining polynomial
C_{1A}	$[3]$	$x^4 - y^3z$
C_{1B}	$[3]$	$x^4 - x^3y + y^3z$
C_2	$[2_3]$	$(yz - x^2)^2 - xy^3$
C_3	$[2_2], [2]$	$(yz - x^2)z^2 - x^3y$
C_4	$[2], [2], [2]$	$(2yz + x^2)^2 - 4x^2(x - 2z)(x + y)$

Table 2.2: Rational cuspidal curves of degree four [49, p.135, Theorem 2.2.5, p.146].

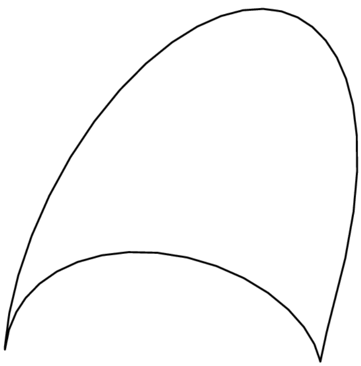
Real affine images of the plane rational cuspidal curves of degree four up to cuspidal configuration is given in Figure 2.2. The images are created with `Maple` [77] and the package `plots`.



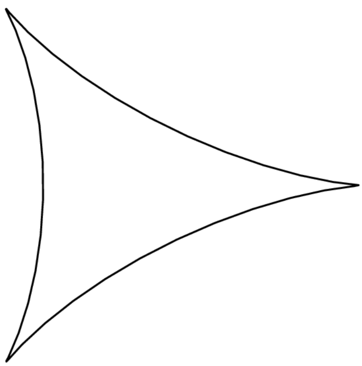
(a) C_1



(b) C_2



(c) C_3



(d) C_4

Figure 2.2: Rational cuspidal curves of degree four up to cuspidal configuration.

Curve	Cuspidal conf.	Parametrization
C_{1A}	$[4]$	$(s^5 : st^4 : t^5)$
C_{1B}	$[4]$	$(s^5 - s^4t : st^4 : t^5)$
C_{1C}	$[4]$	$(s^5 + as^4t - (1+a)s^2t^3 : st^4 : t^5),\ a \neq -1$
C_2	$[2_6]$	$(s^4t : s^2t^3 - s^5 : t^5 - 2s^3t^2)$
C_{3A}	$[3, 2], [2_2]$	$(s^5 : s^3t^2 : t^5)$
C_{3B}	$[3, 2], [2_2]$	$(s^5 : s^3t^2 : st^4 + t^5)$
C_4	$[3], [2_3]$	$(s^4t - \frac{1}{2}s^5 : s^3t^2 : \frac{1}{2}st^4 + t^5)$
C_5	$[2_4], [2_2]$	$(s^4t - s^5 : s^2t^3 - \frac{5}{32}s^5 : -\frac{47}{128}s^5 + \frac{11}{16}s^3t^2 + st^4 + t^5)$
C_6	$[3], [2_2], [2]$	$(s^4t - \frac{1}{2}s^5 : s^3t^2 : -\frac{3}{2}st^4 + t^5)$
C_7	$[2_2], [2_2], [2_2]$	$(s^4t - s^5 : s^2t^3 - \frac{5}{32}s^5 : -\frac{125}{128}s^5 - \frac{25}{16}s^3t^2 - 5st^4 + t^5)$
C_8	$[2_3], [2], [2], [2]$	$(s^4t : s^2t^3 - s^5 : t^5 + 2s^3t^2)$

Table 2.3: Rational cuspidal curves of degree five [49, Theorem 2.3.10, pp.179–182].

The classification of rational cuspidal curves of degree six was completed by Fenske in [17], and from Table 2.4 we see that there are few such curves. Moreover, it is worth noticing that there are no curves with more than three cusps in the list, and only two curves with three cusps.

# Cusps	Curve	Cuspidal configuration
1	C_1	$[5]$
	C_2	$[4, 2_4]$
	C_3	$[3_3, 2]$
2	C_4	$[3_3], [2]$
	C_5	$[3_2, 2], [3]$
	C_6	$[3_2], [3, 2]$
	C_7	$[4, 2_3], [2]$
	C_8	$[4, 2_2], [2_2]$
	C_9	$[4], [2_4]$
3	C_{10}	$[4], [2_3], [2]$
	C_{11}	$[4], [2_2], [2_2]$

Table 2.4: Rational cuspidal curves of degree six [17, Corollary 3.4.6, p.83].

2.4.4 Rational unicuspidal and bicuspidal curves

There are several series of rational cuspidal curves with one or two cusps, and such curves are referred to as uni- and bicuspidal curves, respectively. In this section we present some of these series of curves together with their cuspidal configurations. We additionally recall some related results and results on the classification of such curves.

A rational cuspidal curve with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$ will by Proposition 2.4.5 have only one cusp. By Fernandéz de Bobadilla et al. [18], a classification of these curves can be found in [43], and we list examples of such curves in the following.

A cuspidal curve C is said to be of *Abhyankar–Moh–Suzuki*-type (AMS-type) if there exists a line L such that $C \setminus L$ is isomorphic to $\mathbb{C}$. A rational curve of AMS-type is therefore either a line, an irreducible curve of degree two, or a rational unicuspidal curve that intersects the line L only at the cusp. Note that all AMS-type cuspidal curves have $\overline{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$ [18].

The following series of curves of AMS-type is constructed by Tono using Cremona transformations. Defining equations for these curves are given in [66], and here we present the construction, the degree and the multiplicity sequence of these curves, which is part of [66, Theorem 1.1, pp.47–48].

Theorem 2.4.13 ([66, Theorem 1.1, pp.47–48]). *For a given positive integer k and $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathbb{C}^k \times \mathbb{C}^*$, let $\phi_{\mathbf{a}}$ denote the Cremona transformation*

$$(x : y : z) \mapsto (yz^k : xz^k + \sum_{j=1}^{k+1} a_j y^j z^{k+1-j} : z^{k+1}).$$

For any positive integers $k_1, \ldots, k_{v+1}$, for any $a \in \mathbb{C}$ and any $\mathbf{a}_i \in \mathbb{C}^{k_i} \times \mathbb{C}^*$, there is a cuspidal curve C with one cusp that can be constructed in the following way. Let $F_{v+1} = y + az$, and let $F_{i-1}(x, y, z) = (F_i \circ \phi_{\mathbf{a}_i})(x, y, z)$. Then the defining polynomial of C can be given by F_0. Moreover, the degree of C is

$$\deg C = \prod_{i=1}^{v+1} (k_i + 1).$$

With $d_v := k_{v+1} + 1$ and $d_{i-1} := (k_i + 1)d_i$ for $i = 2, \ldots, v$, the cusp has multiplicity sequence

$$\overline{m} = [k_1 d_1, (d_1)_{2k_1}, k_2 d_2, (d_2)_{2k_2}, \ldots, k_v d_v, (d_v)_{2k_v}, k_{v+1}].$$

If $v = 0$, then $\overline{m} = [k_1]$.

Moreover, there is a curious result by Yoshihara [80] quoted by Tono in [67].

Proposition 2.4.14. *Let $\tilde{C}$ denote the strict transform of a unicuspidal curve C under the minimal embedded resultion of the cusp. For such a curve $\overline{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$ if and only if $\tilde{C}^2 > -2$.*

A cuspidal curve C is said to be of *Lin–Zaidenberg*-type (LZ-type) if there exists a line L such that $C \setminus L$ is homeomorphic to $\mathbb{C}$. It is known that if C is a rational cuspidal curve of LZ-type, then $C \setminus L$ has only one cusp, so C is either uni- or bicuspidal [65].

The rational unicuspidal curves of LZ-type are projectively equivalent to the curves given by [65]

$$y^d + x^{d-1} z = 0.$$

The cuspidal configuration is simply $\overline{m} = [d-1]$, and these curves have $\overline{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$ by Proposition 2.4.14 and Lemma 2.1.13.

For plane rational cuspidal curves with one cusp and $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$, there is a complete classification, and defining polynomials for these curves are constructed by Tono in [67]. Here we recall only the degree and the multiplicity sequences of the unicuspidal curves [67, Theorem 2 and Corollary 2, pp.83–84].

Theorem 2.4.15 ([67, Theorem 2 and Corollary 2, pp.83–84]).

(*I*) *For arbitrary integers $n \geq 2$ and $v \geq 2$, there exists a plane rational unicuspidal curve C with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$ and*

$$\deg C = (n+1)^2(v-1) + 1,$$
$$\overline{m} = \left[n(n+1)(v-1), ((n+1)(v-1))_{2n+1}, (n+1)_{2(v-1)}\right].$$

(*II*) *For an arbitrary integer $n \geq 2$, there exists a plane rational unicuspidal curve C with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$ and*

$$\deg C = \frac{(4n+1)^2 + 1}{2},$$
$$\overline{m} = \left[(n(4n+1))_4, (4n+1)_{2n}, 3n+1, n_3\right].$$

(*III*) *For arbitrary integers $n \geq 2$ and $v > 0$, there exists a plane rational unicuspidal curve C with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$. With $\hat{n} := 4n+1$ and $\hat{v} := 4v-1$, the degree of C and the multiplicity sequence of its cups are*

$$\deg C = \frac{\hat{n}^2\hat{v} + 1}{2},$$

and

$$\overline{m} = \begin{cases} [(3\hat{n}n)_4, (3\hat{n})_{2n}, \hat{n}_3, 3n+1, n_3] & \text{if } v = 1, \\ \left[(\hat{v}\hat{n}n)_4, (\hat{v}\hat{n})_{2n}, (v\hat{n})_3, (v-1)\hat{n}, \hat{n}_{2(v-1)}, 3n+1, n_3\right] & \text{if } v > 1. \end{cases}$$

Moreover, a plane rational unicuspidal curve has $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$ if and only if the multiplicity sequence of the cusp is one of the above.

We additionally recall part of a result on the maximal multiplicity for these curves by Tono [67, Corollary 1, p.84], improving the result in Theorem 2.4.2.

Corollary 2.4.16. *Let C be a plane rational unicuspidal curve of degree d with maximal multiplicity $\hat{m}$.*

a) For the curves of type (I) in Theorem 2.4.15, we have

$$\hat{m} < d \leq \frac{5}{3}\hat{m}.$$

b) For the curves of type (II) and (III) in Theorem 2.4.15, we have

$$2\hat{m} < d \leq \frac{41}{18}\hat{m}.$$

A very special class of rational unicuspidal curves are those found and constructed by Orevkov in [50]. The curves, referred to as Orevkov's curves, are constructed with their defining polynomials by applying compositions of Cremona transformations to simple curves of low degree, and we include this construction here, see [50, Theorem C, p.658].

Let ψ denote the Cremona transformation given by the composition of the formulas in Table 2.5. Applying the substitutions to a polynomial $F(x',y',z')$ leads to a polynomial $F(x,y,z)$. With $C = \mathscr{V}(F)$, $\psi(C)$ here denotes the strict transform of C under the Cremona transformation ψ.

$x_1 = xy$	$x_2 = x_1z_1 - y_1^2$	$x_2' = y_2z_2$	$x_1' = x_2'z_2'$	$x' = {x_1'}^2$
$y_1 = y^2$	$y_2 = y_1z_1$	$y_2' = x_2z_2$	$y_1' = y_2'z_2' + {x_2'}^2$	$y' = x_1'y_1'$
$z_1 = yz - x^2$	$z_2 = z_1^2$	$z_2' = x_2y_2$	$z_1' = {z_2'}^2$	$z' = x_1'z_1' + {y_1'}^2$

Table 2.5: The Cremona transformations used to construct Orevkov's curves.

Given four curves,

$$\begin{aligned} C_{-3} &= \mathscr{V}(F_{-3}), \text{ where } F_{-3} = x,\\ C_{-1} &= \mathscr{V}(F_{-1}), \text{ where } F_{-1} = y,\\ C_0 &= \mathscr{V}(F_0), \text{ where } F_0 = 3x + 3y + z,\\ \text{and } C_0^\star &= \mathscr{V}(F_0^\star), \text{ where } F_0^\star = 21x^2 - 22xy + 21y^2 - 6xz - 6yz + z^2, \end{aligned}$$

define recursively

$$\begin{aligned} C_k &= \psi(C_{k-4}), \qquad k \geq 3, \qquad k \not\equiv 2 \mod 4,\\ C_k^\star &= \psi(C_{k-4}^\star), \qquad k > 0, \qquad k \equiv 0 \mod 4. \end{aligned}$$

Then the curves C_k and $C_k^\star$ have particular properties, given in the following theorem.

Theorem 2.4.17 (Orevkov's cuves [50]). *Let φ_k denote the kth Fibonacci number, with $\varphi_0 = \varphi_1 = 1$. Then there exist unicuspidal curves*

(I) C_k *for any* $k > 1$, $k \equiv 1 \mod 4$, $\kappa(\mathbb{P}^2 \setminus C_k) = -\infty$,

(II) C_k *for any* $k > 0$, $k \equiv 3 \mod 4$, $\kappa(\mathbb{P}^2 \setminus C_k) = -\infty$,

(III) C_k *for any* $k > 0$, $k \equiv 0 \mod 4$, $\kappa(\mathbb{P}^2 \setminus C_k) = 2$, *and*

(IV) $C_k^\star$ *for any* $k > 0$, $k \equiv 0 \mod 4$, $\kappa(\mathbb{P}^2 \setminus C_k^\star) = 2$.

The curve C_k has degree $d_k = \varphi_{k+2}$ and a single cusp of multiplicity $m_k = \varphi_k$. The multiplicity sequence is

$$\overline{m}_k = [\varphi_k, S_k, S_{k-4}, \ldots, S_\nu],$$

where $k = 4j + \nu$, $\nu = 3, 4, 5$, $j \in \mathbb{N} \cup \{0\}$ and

$$\begin{aligned} S_i &= [(\varphi_i)_5, (\varphi_i - \varphi_{i-4})], \qquad i > 5,\\ S_3 &= [2_5],\\ S_4 &= [3_5],\\ S_5 &= [5_5]. \end{aligned}$$

The curve $C_k^\star$ has degree $d_k^\star = 2\varphi_{k+2}$ and a single cusp of multiplicity $m_k^\star = 2\varphi_k$. The multiplicity sequence of the cusp on these curves is $2\overline{m}_k$.

Let p denote the cusp on C_k (respectively $C_k^\star$). Observe that the tangent T to C_k at p has $(T \cdot C_k)_p = 2\varphi_k$ (respectively $(T \cdot C_k^\star)_p = 4\varphi_k$).

Observe that in Theorem 2.4.17 there are examples of unicuspidal curves with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$. The unicuspidal curves with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$ are not classified, but there is the following result by Tono [69, Theorem 1, p.1].

Theorem 2.4.18. *Let C be a rational unicuspidal plane curve with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$. Then C is projectively equivalent to one of Orevkov's curves C_k or $C_k^\star$ with $k \equiv 0 \mod 4$ if and only if $\tilde{C}^2 = -2$.*

More generally, there is the following theorem by Tono for unicuspidal curves with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$ [67, Corollary 3, p.84].

Proposition 2.4.19. *Let C be a rational unicuspidal plane curve with $\tilde{C}$ the strict transform of C under the minimal embedded resolution of its cusp. Then $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$ if and only if $\tilde{C}^2 \leq -2$ and the multiplicity sequence of the cusp is not given in Theorem 2.4.15.*

The search for more cuspidal curves leads Fenske to the discovery of essentially eight different series of rational unicuspidal and bicuspidal curves [15, Theorem 1.1, p.310]. The curves are found using suitable Cremona transformations to transform cuspidal curves of degree d,

$$\mathscr{V}(xy^{d-1} - z^d) \qquad \text{and} \qquad \mathscr{V}(xy^{d-1} - z^d - yz^{d-1}).$$

An overview is given in Table 2.6. Note that since the curves are strict transforms of the above curves, their degree is given as a function of d.

Curve	**Degree**	**Cuspidal configuration**
C_1	$da + d$	$[da, d_{a+b}, d-1)], [d_{a-b}]$
$C_{1'}$	$da + d$	$[(da, d_{2a}, d-1]$
C_2	$da + d$	$[da, d_{a+b}], [d_{a-b}, d-1]$
$C_{2'}$	$da + d$	$[da, d_{2a}], [d-1]$
C_3	$da + d + 1$	$[da+1, d_a], [d_{a+1}]$
C_4	$da + d + 1$	$[da, d_{a+1}], [(d+1)_a]$
C_5	$da + d + 1$	$[da, d_a], [(d+1)_a, d]$
C_6	$da + d + 2$	$[da+1, d_a], [(d+a)_{a+1}]$
C_7	$da + 2d - 1$	$[da+d-1, d_a, d-1], [d_{a+1}, d-1]$
C_8	$a + 2$	$[a], [2_a]$

$d \geq 2$ and $0 \leq b < a$, with a, b integers.

Table 2.6: Fenske's plane rational unicuspidal and bicuspidal curves [15, Theorem 1.1, p.310].

Fenske furthermore finds and constructs all rational uni- and bicuspidal curves C of degree d with maximal multiplicity $\hat{m} = d-2$, and with $\hat{m} = d-3$, up to equisingular equivalence [17]. His results are summarized in Table 2.7 and Table 2.8. The curves exist for integers a and b, with $a \geq 1$ and $0 \leq b < a$. Conversely, there are no other uni- or bicuspidal rational curves of degree d with $\hat{m} = d-2$ or $\hat{m} = d-3$. Note that the curves in Table 2.7 were independently found by Sakai and Tono in [58].

# Cusps	**Curve**	**Degree**	**Cuspidal configuration**
1	C_1	$2a+2$	$[2a, 2_{2a}]$
2	C_2	$a+2$	$[a], [2_a]$
	C_3	$2a+3$	$[2a+1, 2_a], [2_{a+1}]$
	C_4	$2a+2$	$[2a, 2_{a+b}], [2_{a-b}]$

Table 2.7: Series of plane rational unicuspidal and bicuspidal curves with $\hat{m} = d-2$ [17, Theorem 3.1.4, Theorem 3.1.5, pp.62–63].

# Cusps	**Curve**	**Degree**	**Cuspidal configuration**
1	C_1	5	$[2_6]$
	C_2	$3a+3$	$[3_a, 3_{2a}, 2]$
2	C_3	7	$[4], [3_3]$
	C_4	6	$[3], [3_2, 2]$
	C_5	5	$[2_4], [2_2]$
	C_6	$2a+3$	$[2a, 2_a], [3_a, 2]$
	C_7	$2a+4$	$[2a+1, 2_a], [3_{a+1}]$
	C_8	$2a+3$	$[2a, 2_{a+1}], [3_a]$
	C_9	$3a+3$	$[3a, 3_{2a}], [2]$
	C_{10}	$3a+4$	$[3a+1, 3_a], [3_{a+1}]$
	C_{11}	$3a+3$	$[3a, 3_{a+b}, 2], [3_{a-b}]$
	C_{12}	$3a+3$	$[3a, 3_{a+b}], [3_{a-b}, 2]$
	C_{13}	$3a+5$	$[3a+2, 3_a, 2], [3_{a+1}, 2]$

Table 2.8: Series of plane rational unicuspidal and bicuspidal curves with $\hat{m} = d-3$ [17, Theorem 3.4.1, Theorem 3.4.2, pp.73–74].

In [65], Tono succeeds in classifying all rational bicuspidal curves with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$. The rational bicuspidal curves with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$ are given in [65] by defining polynomials and Cremona transformations. We will not list the curves and the multiplicity sequences here because of the complex notation, but we quote a few results.

Theorem 2.4.20 ([65, Theorem 1, p.1]). *Let C be a plane rational bicuspidal curve. Then C is of LZ-type if and only if $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$.*

Theorem 2.4.21 ([65, Corollary 4.5, p.17]). *Let C be a rational bicuspidal plane curve and $\tilde{C}$ its strict transform under the minimal embedded resolution. Then $\tilde{C}^2 \leq 0$, and the equality holds if and only if $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$.*

There is additionally a result on the maximal multiplicity of both unicuspidal and bicuspidal curves of LZ-type.

Theorem 2.4.22 ([65, Corollary 1.2, p.3]). *If C is a rational cuspidal curve of LZ-type of degree d and maximal multiplicity $\hat{m}$, then $d \leq 2\hat{m}$.*

Assuming $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$, Tono finds a new series of plane rational bicuspidal curves in [70, Theorem 2, p.2]. Table 2.9 presents the cuspidal configuration of all rational bicuspidal curves C with $\tilde{C}^2 = -1$, and they exist for the integers a and b specified in the table. Note that this does not violate Proposition 2.4.19, since C here is assumed to be bicuspidal. Observe also that some of these curves are described by Fenske, see Table 2.6.

# Cusps	Curve	Degree	Cuspidal configuration
	C_1	$2ab+b-1$	$[ab+b-1, ab-1, b_{a-1}, b-1], [(ab)_2, b_a]$
2	C_2	$2ab+b+1$	$[ab+b, ab, b_a], [(ab+1)_2, b_a]$
	C_3	$2ab+1$	$[ab+1, ab-b+1, b_{a-1}], [(ab)_2, b_a]$
	C_4	$2ab+2b-1$	$[ab+b, ab, b_a], [(ab+b-1)_2, b_a, b-1]$

Integers $a > 0$ and $b \geq 2$ for C_1 and C_2.

Integers $a > 0$ and $b \geq 3$ for C_3 and C_4.

Table 2.9: Series of plane rational bicuspidal curves with $\tilde{C}^2 = -1$ [70, Theorem 2, p.2].

2.4.5 Three or more cusps

Finding curves with three or more cusps has proved significantly harder than finding cuspidal curves with fewer cusps. The below list represents the only curves that have been found up to equisingular equivalence.

(I) For $d = 5$, the only rational cuspidal curves with three or more cusps have one of these cuspidal configurations [49, Theorem 2.3.10, pp.179–182]:

$$[3], [2_2], [2],$$
$$[2_2], [2_2], [2_2],$$
$$[2_3], [2], [2], [2].$$

(II) For any $a \geq b \geq 1$ there exists a rational cuspidal plane curve C of degree $d = a + b + 2$ with three cusps [22, Theorem 3.5, p.448],

$$[d-2], [2_a], [2_b].$$

(III) For any $a \geq 1$, there exists a rational cuspidal plane curve C of degree $d = 2a+3$ with three cusps [23, Theorem 1.1, p.94],

$$[d-3, 2_a], [3_a], [2].$$

(IV) For any $a \geq 1$, there exists a rational cuspidal plane curve C of degree $d = 3a+4$ with three cusps [16, Theorem 1.1, p.512],

$$[d-4, 3_a], [4_a, 2_2], [2].$$

All these curves are constructed explicitly by successive Cremona transformations of plane curves of low degree. In cases (II) and (III) it is proved by Flenner and Zaidenberg in [22, 23] that these are the only tricuspidal curves with maximal multiplicity of this kind. The same is proved by Fenske in [16] for case (IV) under the assumption that $\chi(\Theta_V \langle D \rangle) = 0$. Note that this result is originally proved with $\chi(\Theta_V \langle D \rangle) \leq 0$, but by Proposition 2.4.8 we only need $\chi(\Theta_V \langle D \rangle) = 0$. The curves in ($II$), ($III$) and ($IV$) constitute three series of cuspidal curves with three cusps, with infinitely many curves in each series.

Further research by Fenske in [17, Section 5] implies that there are no other tricuspidal curves. In [17], Fenske in particular shows the following theorem, here in our notation [17, Theorem 5.1.2, Corollary 5.1.3, pp.111–112].

Theorem 2.4.23. *Up to equisingular equivalence there are only finitely many rational curves of degree d with at least three cusps and $\hat{m} = d - k$ that do* not *possess the following cuspidal configuration,*

$$\overline{m}_1 = [d-k, (k-1)_a, \overline{n}_1], \tag{2.4.5}$$
$$\overline{m}_2 = [k_b, \overline{n}_2], \tag{2.4.6}$$
$$\overline{m}_3 = [2_c]. \tag{2.4.7}$$

where a, b, c are suitable positive integers, and $\overline{n}_1, \overline{n}_2$ are suitable multiplicity sequences.

In particular, there exists no infinite series of plane rational cuspidal curves with at least four cusps.

Under the extra conditions that $d-k = (k-1)a$ and $a = b$, Fenske additionally proves that for $k > 4$ there exists no series of tricuspidal curves with the cuspidal configuration from Theorem 2.4.23. These conditions hold for the known series (III) and IV, with $k = \{3, 4\}$ respectively [17, Theorem 5.2.2, p.127].

Piontkowski proves in [55] that, assuming $\chi(\Theta_V \langle D \rangle) = 0$, there are no other curves with three or more cusps of degree $d \leq 20$ than the curves in the above list. This observation leads to the following conjecture [55, Conjecture 1.4, p.252].

Conjecture 2.4.24 (Piontkowski). *Any rational cuspidal plane curve with at least three cusps is contained in the above list.*

As a curiosity we mention the following theorem by Tono.

Theorem 2.4.25 ([64, Theorem 5, p.4]). *If C is a rational cuspidal plane curve having exactly three cusps, then $\tilde{C}^2 \leq -2$. Moreover, $\tilde{C}^2 = -2$ if and only if C is the quartic curve with three cusps.*

2.4.6 Inflection points on tricuspidal curves

Closer investigation of the defining polynomials of rational cuspidal curves with three cusps indicates that most of them have inflection points. Contrary to the intuitive feeling that these curves have a lot of ramification in their cusps, inflection points seem to exist on these curves from a certain degree on. The existence of inflection points is not immediately apparent, and we have not been able to prove the claim in general.

The problem of inflection points is linked to the tangential intersection multiplicities for the cusps, and we list these for the cusps of the tricuspidal curves in series (II), (III), and (IV) in Table 2.10, Table 2.11, and Table 2.12.

Case	$[\mathbf{d-2}]$	$[\mathbf{2_a}]$	$[\mathbf{2_b}]$
$\mathbf{a=b=1}$	3	3	3
$\mathbf{a>1, b=1}$	$d-1$	$r_{IIa}(a)$	3
$\mathbf{a, b>1}$	$d-1$	$r_{IIa}(a)$	$r_{IIb}(b)$

Table 2.10: Tangential intersection multiplicities for the cusps on the rational cuspidal curves in series (II).

Case	$[\mathbf{d-3, 2_a}]$	$[\mathbf{3_a}]$	$[\mathbf{2}]$
$\mathbf{a=1}$	4	4	3
$\mathbf{a>1}$	$d-1$	$r_{III}(a)$	3

Table 2.11: Tangential intersection multiplicities for the cusps on the rational cuspidal curves in series (III).

Case	$[\mathbf{d-4, 3_a}]$	$[\mathbf{4_a, 2_2}]$	$[\mathbf{2}]$
$\mathbf{a=1}$	6	6	3
$\mathbf{a>1}$	$d-1$	$r_{IV}(a)$	3

Table 2.12: Tangential intersection multiplicities for the cusps on the rational cuspidal curves in series (IV).

Note that the value of the unknown tangential intersection appears to be constant in all the examples that we have checked, with $r_{IIa}(a) = r_{IIb}(b) = 4$, $r_{III}(a) = 6$ and $r_{IV}(a) = 8$, but we have not been able to prove this. If we could prove these equalities, the existence of inflection points would follow from Theorem 2.1.8. Note that in all series, the tangential intersection at the cusp with highest multiplicity is $d-1$ by Lemma 1.1.14, and an ordinary cusp with multiplicity sequence [2] has tangential intersection 3. It follows that in series (II), the inflection point formula in Theorem 2.1.8 gives that the number of inflection points counted with multiplicity, v, is $v = d+2-r_{IIa}(a)-r_{IIb}(b)$. In series (III) the formula reduces to $v = d-1-r_{III}(a)$, and in series (IV) it reduces to $v = d-1-r_{IV}(a)$.

However, it is still possible to prove the existence of inflection points on some of the curves using the formula in Theorem 2.1.8. As an example, we show that most curves in series (III) have inflection points.

Example 2.4.26. For $a = 1$, all the tangential intersections can be found by inspection. For $a > 1$, the only unknown tangential intersection multiplicity is that of the cusp with multiplicity sequence $[3_a]$. We have checked that up to degree $d = 13$, the value of $r_{III}(a)$ is independent of a, and in fact $r_{III}(a) = 6$. By Theorem 2.1.8, the number of inflection points counted with multiplicity is $v = d - 1 - r_{III}(a)$. Assume for contradiction that there are no inflection points, that is $r_{III}(a) = d - 1$. By Lemma 1.1.14, we have either $r_{III}(a) = 3k$ for some integer $k \leq a$, or $r_{III}(a) = 3a + 1$. In the first case, $2a + 3 - 1 = 3k$, which is only possible if $a \equiv 2 \mod 3$. In the second case, $2a + 3 - 1 = 3a + 1$, which is true only for $a = 1$. By contradiction, if $a \equiv 0$ or 1 mod 3, then the curve must have inflection points.

Remark 2.4.27. It is important to notice that most rational cuspidal curves with three cusps have inflection points. Through the inflection point formula, the occurrence of inflection points gives useful information about lines intersecting the curve, that is, tangential intersection multiplicities of both smooth points and cusps. Control over the tangential intersection multiplicities of both smooth points and cusps of a cuspidal curve is essential in the construction of new curves on Hirzebruch surfaces.

2.4.7 An important conjecture

With Piontkowski's conjecture in mind, there is an obvious upper bound for the number of cusps on a rational cuspidal curve.

Conjecture 2.4.28. *A plane rational cuspidal curve can not have more than four cusps.*

This seemingly elementary observation has not been proved. The best upper bound so far is the consequence of Theorem 2.3.5 [68, Corollary 1.2, p.216].

Corollary 2.4.29 (Corollary to Theorem 2.3.5). *A rational cuspidal plane curve has no more than eight cusps.*

Hence, the answer to the question of how many cusps s a rational cuspidal plane curve can have is narrowed down to $s \in \{4, 5, 6, 7, 8\}$.

The question of the number of cusps on a rational curve has also been studied for other curves than projective rational curves. Borodzik and Zoladek study this question in [3] for plane algebraic annuli, that is, reduced algebraic curves $C \subset \mathbb{C}^2$ that are homeomorphic to $\mathbb{C}^*$. They believe that their methods can be extended to all rational curves on $\mathbb{C}^2$, but claim that the computations in that situation are highly complex. We recall the theorem of Borodzik and Zoladek and restate it as a corollary applicable to our situation.

Theorem 2.4.30 ([3, Main Theorem, p.1]). *Any algebraic curve in $\mathbb{C}^2$ homeomorphic to $\mathbb{C}^*$ has at most three singular points.*

Corollary 2.4.31. *Let C be a plane rational cuspidal curve that admits a parametrization of the following form in a suitable affine covering,*

$$\begin{aligned} x &= \phi(t) = t^p + a_1 t^{p-1} + \ldots + a_{p+r} t^{-r} \\ y &= \psi(t) = t^q + b_1 t^{q-1} + \ldots + b_{q+s} t^{-s}, \qquad a_{p+r} b_{q+s} \neq 0. \end{aligned}$$

Then C can have at most five cusps.

Proof. This corollary follows from Theorem 2.4.30. Curves on the above form are homeomorphic to $\mathbb{C}^*$ and may have at most three cusps for finite values of t, in addition to at most two singularities at $t = \{0, \infty\}$ (see [3, p.5]). Note that Theorem 2.4.30 is proved in [3] using the above invariants, p, q, r and s, and the B–M–Y-inequality. □

Example 2.4.32. We observe that the cuspidal quintic with four cusps admits a parametrization as above. One reason for this is that there exists a line that intersects the curve in exactly two points. This line is here the tangent line of the cusp with multiplicity sequence $[2_3]$, which additionally intersects the curve transversally in a smooth point. Explicitly, we have the parametrization

$$\begin{aligned} \phi(t) &= t^4 + 2t, \\ \psi(t) &= t^2 - t^{-1}. \end{aligned}$$

2.4.8 Real cuspidal curves

Can all cusps on a cuspidal curve on the projective plane have real coordinates? By elementary properties of changes of coordinates of $\mathbb{P}^2$, the answer to this question is yes for all rational cuspidal curves with four or fewer cusps.

If we ask the same question for curves with defining polynomials that have only real coefficients, the answer is less obvious. We say that a plane curve $C = \mathscr{V}(F)$ is real if $F \in \mathbb{R}[x, y, z]$.

For real curves, there is the following formula, originally due to Klein, and generalized by Schuh [60], here presented as an adapted version of [75, Theorem, p.361] by Wall.

Theorem 2.4.33 (Generalized Klein formula). *Let C be a real plane cuspidal curve of degree d with real cusps p_j, $j = 1, \ldots, s$, and let m_j denote the multiplicity of p_j. Let C^* be the dual curve of C with degree d^* and real cusps $p'_{j'}$, $j' = 1, \ldots, s'$, and let $m'_{j'}$ denote the multiplicity of $p'_{j'}$. For every real cuspidal curve C on $\mathbb{P}^2$,*

$$d - \sum_{p_j \in \operatorname{Sing} C(\mathbb{R})} (m_j - 1) = d^* - \sum_{p'_{j'} \in \operatorname{Sing} C^*(\mathbb{R})} (m'_{j'} - 1).$$

For the known rational cuspidal curve with four cusps, using the generalized Klein formula, we show in [46] that if the curve is real, then the cusps can not all have real coordinates. We recall the result.

Proposition 2.4.34. *Let C be the plane rational cuspidal quintic with four cusps, and assume that C is a real curve. Then the cusps on C can not all have real coordinates.*

Proof. Let C be the quintic with cuspidal configuration $[2_3],[2],[2],[2]$, and assume that C is a real curve. The dual curve C^* is a quartic with cuspidal configuration $[2_3]$. Assume for contradiction that all the cusps on C have real coordinates. Since C is a real curve, the cusp on C^* must also have real coordinates. Then C contradicts Theorem 2.4.33,

$$5 - 4 \cdot (2-1) \neq 4 - (2-1).$$

Hence, the cusps on C can not all have real coordinates. □

Moreover, it is shown by construction in [16, 22, 23] that the known series of rational cuspidal curves with three cusps can be defined over $\mathbb{R}$, in fact over $\mathbb{Q}$. Investigation of the rational quintic with cuspidal configuration $[2_2],[2_2],[2_2]$, using the real parametrization given in Table 2.3 and calculating in `Maple`, shows that the cusps of this real curve can have real coordinates too. From these observations we conclude that all known rational cuspidal curves with three cusps can be defined over $\mathbb{R}$.

2.4.9 The Coolidge–Nagata conjecture

The question of which rational curves can be transformed into a line via Cremona transformations has been thoroughly investigated. In this section we will clarify some definitions and give a conjecture linked to the mentioned question and plane rational cuspidal curves.

We begin in a general setting. Let C be a plane rational curve of degree d and with singular points p_j of multiplicity m_j. In [9, pp.396-399], Coolidge claims to prove that such a curve can be transformed into a line via Cremona transformations if and only if C has no *special adjoint of any index*. A curve C_s of degree $d-3s$ is called a *special adjoint* to C of index $s \geq 1$ if C_s has multiplicity $\geq m_j - s$ at every point p_j. The proof of Coolidge's claim given in [9] is noticed to be incomplete by Garcia in [27].

Using new techniques, an equivalent result is properly proved by Mohan Kumar and Murthy in [47]. Let $\sigma : V \to \mathbb{P}^2$ be the minimal embedded resolution of $C \subset \mathbb{P}^2$, and let $\tilde{C}$ be the strict transform of C. Then with $\kappa(\tilde{C}, V)$ as defined in Section 1, let $\tilde{\kappa}(C, \mathbb{P}^2) := \kappa(\tilde{C}, V)$. The claim by Coolidge is then restated and proved with the following theorem, here in our notation.

Theorem 2.4.35 ([47, Theorem 2.6, p.772]). *Let $C \subset \mathbb{P}^2$ be an irreducible rational curve. Then there exists a Cremona transformation ψ of $\mathbb{P}^2$ such that $\psi(C)$ is a line if and only if $\tilde{\kappa}(C, \mathbb{P}^2) = -\infty$.*

Note that Mohan Kumar and Murthy in [47] additionally give a direct proof of Coolidge's claim, but this proof is also incomplete.

The specialization from general plane curves to cuspidal curves is first discussed by Matsuoka and Sakai in [42]. Let C be a plane curve of degree d. Let p_i be the centers of the blowing ups in the minimal embedded resolution of C. Renumber the points p_i such that the multiplicities are ordered $m_1 \geq m_2 \geq \ldots \geq m_{t-1} \geq 1$. If $t-1 \geq 2$, define k to be the maximal positive integer such that $m_1 \geq m_2 + \ldots + m_k$.

Theorem 2.4.36 ([42, Theorem A, p.245]). *Let C be a plane curve of degree d. With m_i and k as above, if C is transformable into a line by a Cremona transformation, then either* *a)* $d < m_1 + 2m_{k+1}$ $(2 \leq k < t-1)$,

b) $d = m_1 + 1$ $(t - 1 = 1$ *or* $k = t - 1 \geq 2, m_2 = \cdots = m_k = 1)$, *or*

c) $d = 1$ *or* 2 $(t - 1 = 0)$.

A straightforward observation reveals that the condition of not having any special adjoint of any index implies that $d < m_1 + m_2 + m_3$, with the p_i ordered as in Theorem 2.4.36 (see [9]). This obviously implies that $d < 3m_1$. Because of the result for rational cuspidal curves given in Theorem 2.4.2, Matsuoka and Sakai conjecture the following.

Conjecture 2.4.37 (Coolidge–Nagata [42, Corollary, p.234]). *Every plane rational cuspidal curve can be transformed into a line by a Cremona transformation.*

The classification of plane rational cuspidal curves by the logarithmic Kodaira dimension of their complements gives a partial answer to the conjecture. In this Chapter, we have seen that there is a complete classification of all plane rational cuspidal curves with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = \{-\infty, 1\}$, and these curves can all be transformed into lines by Cremona transformations. For all known curves with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$, it is shown that they can be transformed into lines by Cremona transformations. Since the classification of rational cuspidal curves with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 2$ is not complete, the conjecture is not proved in general.

Recently, it is shown by Palka in [52] that the Coolidge–Nagata conjecture holds for rational cuspidal curves with more than four cusps. The proof relies on the B–M–Y-inequality, Zariski decomposition and properties of the minimal embedded resolution of a rational cuspidal curve. An estimation of the number of maximal twigs (see proof of Theorem 3.3.2) of such a resolution forms the essential step of the proof. With Conjecture 2.4.24 in mind, this result is not surprising.

2.5 A toric construction

In this section we will revisit the special class of rational unicuspidal curves found and constructed by Orevkov in [50]. As mentioned, these curves are originally constructed using Cremona transformations. Orevkov does, however, additionally explain a way to construct the curves using theory from toric geometry, and we will repeat his construction in the following. This section does not introduce new results, but it is included as an illustration of how toric geometry can be used to shed light on the rational cuspidal curves.

Let φ_k denote the kth Fibonacci number, with $\varphi_1 = \varphi_2 = 1$. Recalling Theorem 2.4.17, we know that there exist unicuspidal curves C_k of degree φ_{k+2}, where the cusp has multiplicity $m_k = \varphi_k$ and multiplicity sequence

$$\overline{m}_k = [\varphi_k, S_k, S_{k-4}, \ldots, S_\nu],$$

where $k = 4j + \nu,\ \nu = 3, 4, 5,\ j \in \mathbb{N} \cup \{0\}$ and

$$\begin{aligned} S_i &= [(\varphi_i)_5, (\varphi_i - \varphi_{i-4})], \qquad i > 5, \\ S_3 &= [2_5], \\ S_4 &= [3_5], \\ S_5 &= [5_5]. \end{aligned}$$

In the following we make a summary of Orevkov's toric construction of the curves C_k for odd values of k, where $k \geq 3$.

Given a smooth surface X and a linear chain of rational curves $D = D_1 + \ldots + D_n$, we have a sequence of integers $(-D_1^2, \ldots, -D_n^2)$. By [50] we may then uniquely, up to the action of $\mathrm{SL}_2(\mathbb{Z})$, encode these self intersections by a sequence of vectors

$$c(D) = (v_0, \ldots, v_{n+1}), \text{ where } v_i \in \mathbb{Z}^2,$$

with the following properties. With $(a, b) \wedge (c, d) = ad - bc$, the v_i's in the sequence have the properties that

$$v_i \wedge v_{i+1} = 1 \text{ and } v_{i-1} \wedge v_{i+1} = -D_i^2.$$

For any vector $v \in \mathbb{Z}^2$, let A_v denote the automorphism defined by $A_v u = u + (v \wedge u)v$. Observe that $A_v^{-1} u = u - (v \wedge u)v$ and that $A_v v = v$. Define the discriminant $d(D)$ of D to be the determinant of the intersection matrix of D. Moreover, let b_i denote the positively oriented angle from v_i to v_{i+1} and define the rotation number $r(D)$ of D, $r(D) := \sum_{i=0}^{n} b_i$. Then there is the following proposition by Orevkov [50].

Proposition 2.5.1 ([50, Proposition 7.1, p.670]).

a) If $\tilde{X}$ is obtained by blowing up $D_i \cap D_{i+1}$, $i = 0, \ldots, n$, and $\tilde{D}$ is the total transform of D, then

$$c(\tilde{D}) = (v_0, \ldots v_i, v_i + v_{i+1}, v_{i+1}, \ldots, v_{n+1}).$$

b) If X' is obtained by blowing up a smooth point of D_i, $i = 1, \ldots, n$, and D' is the strict transform of D, then

$$c(D') = (v_0, \ldots, v_i, A_{v_i} v_{i+1}, \ldots, A_{v_i} v_{n+1}).$$

c) $d(D) = v_0 \wedge v_{n+1}$.

d) D can be blown down to a smooth point if and only if $v_0 \wedge v_{n+1} = 1$ and $r(D) < \pi$.

Fixing an odd k, we may consider three vectors in $\mathbb{Z}^2$,

$$v_0 = -(\varphi_k^2, \varphi_{k+2}^2), \quad v_1 = (\varphi_{k-2}, \varphi_{k+2}), \quad v_2 = (\varphi_k, \varphi_{k+4}).$$

By easily proved relations for Fibonacci numbers (see [50]), we have

$$v_0 \wedge v_1 = \varphi_{k+2}, \quad v_1 \wedge v_2 = 3, \quad v_2 \wedge v_0 = \varphi_k. \tag{2.5.1}$$

The key to the construction of the curves C_k is to manipulate the above vectors in such a way that Proposition 2.5.1 can be applied. Roughly speaking, we find a divisor that can be contracted, and in addition a curve that transforms into the appropriate plane curve in the course of the contraction.

First take the complete fan $\Sigma = \mathrm{span}\{v_0, v_1, v_2\}$. By the relations in (2.5.1), the cones $\sigma_{ij} = \mathrm{Cone}(v_i, v_j)$, $i, j = 0, 1, 2$, are singular cones, that is, the minimal generators of the cones do not form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$. Let Σ' denote the refinement of Σ, such that the associated toric surface $X_{\Sigma'}$ is smooth. The vectors spanning Σ' can be written as a sequence of vectors,

$$(v_0, u_1, \ldots, v_1, \ldots, u_k, \ldots, v_2, \ldots, u_n, v_0).$$

Second, let D_i denote the closure of the 1-dimensional orbit corresponding to v_i, $i = 0, 1, 2$, and let E_k correspond to u_k, $k = 1, \ldots, n$. Denote by D the closure of $X_{\Sigma'} \setminus (X_0 \cup D_0)$, where X_0 is the open orbit of $X_{\Sigma'}$. By properties of resolutions of singularities on toric varieties, we may consider D as a finite linear chain of rational curves on the smooth surface $X_{\Sigma'}$, and write

$$D = E_1 + \ldots + D_1 + \ldots + D_2 + \ldots + E_n.$$

By [50], there is the convenient correspondence

$$c(D) = (v_0, u_1, \ldots, v_1, \ldots, u_k, \ldots, v_2, \ldots, u_n, v_0).$$

We then blow up generic points p_1 and p_2 on D_1 and D_2 respectively and let D' denote the strict transform of D. We let F_1 and F_2 denote the exceptional divisors of the blowing ups. Let e_1 and e_2 denote the unit vectors of $\mathbb{Z}^2$. By Proposition 2.5.1 b), we get

$$\begin{aligned}
c(D') &= (v_0, u_1, \ldots, v_1, \ldots, A_{v_1}u_k, \ldots, A_{v_1}v_2, \ldots, A_{v_1}A_{v_2}u_n, A_{v_1}A_{v_2}v_0) \\
&= (A_{v_1}^{-1}v_0, A_{v_1}^{-1}u_1, \ldots, A_{v_1}^{-1}v_1, \ldots, u_k, \ldots, v_2, \ldots, A_{v_2}u_n, A_{v_2}v_0) \\
&= (v_0 - (v_1 \wedge v_0)v_1, u_1 - (v_1 \wedge u_1)v_1, \ldots, v_1, \ldots, u_k, \ldots \\
&\qquad \ldots, v_2, \ldots, u_n + (v_2 \wedge u_n)v_2, v_0 + (v_2 \wedge v_0)v_2) \\
&= (e_1, \ldots, v_1, \ldots, v_2, \ldots, e_2).
\end{aligned}$$

Considering this new sequence of vectors, we see that $d(D') = e_1 \wedge e_2 = 1$ and $r(D') = \frac{\pi}{2} < \pi$. By Proposition 2.5.1 d), the divisor D' can be blown down to a smooth point. In total, we have performed the same number of blowing ups and contractions, hence we are back in $\mathbb{P}^2$. Moreover, by the shape and weights of the dual graph of the divisor $D' + F_2$, we see that F_2 is mapped onto a rational unicuspidal curve C_k. The characteristic pair of the cusp p can be calculated using [50, Proposition 3.2, p.662] and Proposition 2.5.1 c),

$$\begin{aligned}
\mathrm{Ch}(p) &= (e_1 \wedge v_1, v_2 \wedge e_2) \\
&= (\varphi_k, \varphi_{k+1}).
\end{aligned}$$

Hence, the strict transform of F_2 has the desired properties, it is indeed the plane rational unicuspidal curve C_k.

Chapter 3

Cuspidal curves on Hirzebruch surfaces

In this Chapter we will investigate cuspidal curves on Hirzebruch surfaces. Motivated by the results on cuspidal curves on the projective plane, our main focus will be finding bounds on the number of cusps and constructing series of rational curves with many cusps.

3.1 Background and preliminary results

We begin our investigation of cuspidal curves on $\mathbb{F}_e$ with a few general facts. Recall from Chapter 1 that a reduced and irreducible curve C of type (a, b) on $\mathbb{F}_e$ is given by $\mathscr{V}(F)$, where $F \in S_e(a, b)$ is a reduced and irreducible polynomial, and

$$S_e(a,b) = \bigoplus_{\substack{\alpha_0+\alpha_1-e\beta_1=a \\ \beta_0+\beta_1=b}} \mathbb{C}x_0^{\alpha_0}x_1^{\alpha_1}y_0^{\beta_0}y_1^{\beta_1}.$$

In the language of divisors we have that $C \sim aL + bM$. The notions of a singular point, its multiplicity and multiplicity sequence are local, and the results from Chapter 1 apply.

We start with a closer look on an affine part of a curve around a point p. Assuming that the point p on the curve $C = \mathscr{V}(F)$ is a general point on $\mathbb{F}_e$, we can move it by a change of coordinates to the point $(0 : 1; 0, 1)$. In the affine neighbourhood $\mathscr{D}_+(x_1y_1)$ we consider the polynomial $f(x, y) = F(x, 1, y, 1)$. Splitting $f = f(x, y)$ into its homogeneous terms, we write

$$f = f_0 + \cdots + f_i + \cdots + f_{a+be},$$

where $f_i = f_i(x, y)$ denotes the terms of $f(x, y)$ of degree i in x and y.

Assuming, on the other hand, that the point p on the curve $C = \mathscr{V}(F)$ is a special point on $\mathbb{F}_e$, so that $p \in \mathscr{V}(y_1)$, we can move it by a change of coordinates to the point $(0 : 1; 1, 0)$. In the affine neighbourhood $\mathscr{D}_+(x_1y_0)$ we consider the polynomial $f(x, y) = F(x, 1, 1, y)$. Splitting $f = f(x, y)$ into its homogeneous terms, we write

$$f = f_0 + \cdots + f_i + \cdots + f_{a+b(e+1)},$$

where $f_i = f_i(x, y)$ denotes the terms of $f(x, y)$ of degree i in x and y. Note that this is the situation for all points when $e = 0$.

In both the above cases, when p is a point on C we have

$$f = f_m + \cdots ,$$

with m the the multiplicity of C at p. Recall that if p is smooth or a cusp, then the term f_m defines a unique *local tangent line* of C at p. If p is smooth, we have by the Taylor expansion of f that $f_1(x, y) = f_x(p)x + f_y(p)y$. Note that the local tangent line in general is *not* the restriction of a curve on $\mathbb{F}_e$ since its defining polynomial normally is not an element of S_e. There are exceptions to this general observation, for example when the local tangent line is a fiber, that is, a $(1, 0)$-curve, or the special section $\mathscr{V}(y_1)$.

With the above in mind, we classify points on $\mathbb{F}_e$ in the following way.

Definition 3.1.1. Let p be a point on a curve $C = \mathscr{V}(F) \subset \mathbb{F}_e$.

a) The point p is called *fiber tangential* if a fiber is tangent to the curve C at the point p.

b) The point p is called *special tangential* if the special section is tangent to the curve C at the point p.

Note that both fiber tangential and special tangential points can be either smooth or singular.

Note that the tangential property of fiber and special tangential points can be used when we construct curves. Setting up the right situation, we may choose to contract the tangent line, and its intersection multiplicity with the curve will affect the singularities of the curve.

In general, we can make additional small observations connecting local tangent lines to $(0, 1)$-curves. When $e = 0$, a $(0, 1)$-curve can in fact be a tangent line, since in that case $(0, 1)$-curves are fibers. For $e \geq 1$, the situation is more subtle. A special point p can not be on any irreducible $(0, 1)$-curve. This can be seen by considering the possible defining polynomials of $(0, 1)$-curves through p, all of which must have y_1 as a factor. For a general point, however, a local tangent line can be uniquely homogenized to a $(0, 1)$-curve, but this is not well defined as a tangent line in the global sense, except in the case of $e = 1$. Lemma 3.1.2 clarifies the situation.

Lemma 3.1.2. *Let p be a smooth general point on a curve $C = \mathscr{V}(F)$ on the surface $\mathbb{F}_e$, $e \geq 1$. Then there exists an $(e-1)$-dimensional family of $(0, 1)$-curves, where the curves are possibly reducible, tangent to C at p.*

Proof. By Lemma 1.2.2 there exists an e-dimensional family of $(0, 1)$-curves through any general point p on $\mathbb{F}_e$. Moving the point p to coordinates $(0 : 1; 0, 1)$, the defining polynomials of curves in this family are on the form

$$by_0 + \sum_{k=1}^{e} c_k x_0^k x_1^{e-k} y_1,$$

with b and c_k complex coefficients. As above, let $f(x,y) = F(x,1,y,1)$. Then the Taylor expansion of f gives

$$f(x,y) = f_x(p)x + f_y(p)y + H(2),$$

where $H(2)$ denotes higher order terms. Note that in these coordinates, we have $f_x(p) = F_{x_0}(p)$ and $f_y(p) = F_{y_0}(p)$. Requiring the $(0,1)$-curves through p to be tangent to C at p determines the two coefficients $b = F_{y_0}(p)$ and $c_1 = F_{x_0}(p)$. Then we are left with an affine $(e-1)$-dimensional family of curves tangent to C at p. □

We now prove a simple observation, similar to the Euler relation on projective spaces.

Lemma 3.1.3. *For a polynomial $F \in S_e(a,b)$ the following hold,*

$$\begin{aligned} x_0F_{x_0} + x_1F_{x_1} - ey_1F_{y_1} &= aF, \\ y_0F_{y_0} + y_1F_{y_1} &= bF. \end{aligned}$$

Proof. We show this for the first expression, and the second equation can be proved in the same way. Notice that if the expression holds for a monomial $F = x_0^{\alpha_0}x_1^{\alpha_1}y_0^{\beta_0}y_1^{\beta_1} \in S_e(a,b)$, then it also holds for a general polynomial in $S_e(a,b)$. So it suffices to show that the formula holds for a monomial, say F.

$$\begin{aligned} x_0F_{x_0} + x_1F_{x_1} - ey_1F_{y_1} &= \alpha_0F + \alpha_1F - e\beta_1F \\ &= (\alpha_0 + \alpha_1 - e\beta_1)F \\ &= aF. \end{aligned}$$

□

Note that we can *not* evaluate F and the partial derivatives at a point to get an element of any graded part of the coordinate ring of $\mathbb{F}_e$ from Lemma 3.1.3, except in the case when $e = 0$.

Moreover, for a point $p \in \mathscr{V}(F)$, where $p \notin \mathscr{V}(x_0x_1y_0y_1)$, the relations imply that if $F_{y_0}(p) = 0$, then $F_{y_1}(p) = 0$, and conversely. If either $F_{y_0}(p) = 0$ or $F_{y_1}(p) = 0$, the same holds for $F_{x_i}(p)$. If follows that the definition of a singular point given in Chapter 1 is equivalent to saying that p is *singular* if $F_{x_0}(p) = F_{x_1}(p) = F_{y_0}(p) = F_{y_1}(p) = 0$.

A first result concerning curves on $\mathbb{F}_e$ regards the genus g of the curve.

Corollary 3.1.4 (Proposition 1.1.17). *A cuspidal curve C on $\mathbb{F}_e$ of type (a,b) with cusps p_j, for $j = 1,\dots,s$, and multiplicity sequences $\overline{m}_j = [m_0, m_1, \dots, m_{t_j-1}]$ has genus g, where*

$$g = \frac{(b-1)(2a-2+be)}{2} - \sum_{j=1}^{s}\sum_{i=0}^{t_j-1}\frac{m_i(m_i-1)}{2}.$$

Proof. Recall that $C \sim aL + bM$, $K \sim (e-2)L - 2M$, $L^2 = 0$, $L\,.\,M = 1$ and $M^2 = e$. By Proposition 1.1.17, we have

$$g = \frac{(aL+bM)\,.\,(aL+bM+(e-2)L-2M)}{2} + 1 - \sum_{j=1}^{s}\delta_j.$$

This gives

$$\begin{aligned} g &= \frac{b^2e - 2be + ab + be - 2a + ab - 2b}{2} + 1 - \sum_{j=1}^{s}\sum_{i=0}^{t_j-1}\frac{m_i(m_i-1)}{2} \\ &= \frac{(b-1)(2a-2+be)}{2} - \sum_{j=1}^{s}\sum_{i=0}^{t_j-1}\frac{m_i(m_i-1)}{2}. \end{aligned}$$

□

The structure of $\mathbb{F}_e$ gives restrictions on the multiplicity sequence of a cusp on a curve.

Theorem 3.1.5. *Let p be a cusp on a reduced and irreducible curve $C = \mathscr{V}(F)$ on $\mathbb{F}_e$ of type (a, b) with multiplicity sequence $\overline{m} = [m, m_1, \ldots, m_{t-1}]$. Then $m \leq b$.*

Proof. The point p determines a unique $(1, 0)$-curve L by Lemma 1.2.2. By Corollary 1.1.12, $m \leq (L \cdot C)_p$. By Proposition 1.1.2, $(L \cdot C)_p \leq L\,.\,C$. By intersection theory, $L\,.\,C = b$. Hence, $m \leq (L \cdot C)_p \leq L\,.\,C = b$. □

Further restrictions on the type of points on a curve on $\mathbb{F}_e$ can be found using Hurwitz's theorem [31, Corollary IV 2.4, p.301]. First, the general result in this situation.

Theorem 3.1.6 (Hurwitz's theorem for $\mathbb{F}_e$). *Let C be a curve on $\mathbb{F}_e$, $e > 0$, of genus g and type (a, b), where $b > 0$. Let $\tilde{C}$ denote the normalization of C, and let ν be the composition of the normalization map $\tilde{C} \to C$ and the projection map $C \to \mathbb{P}^1$ of degree b. Let e_p denote the ramification index of a point $p \in \tilde{C}$ with respect to ν. Then the following equality holds,*

$$2b + 2g - 2 = \sum_{p \in \tilde{C}} (e_p - 1).$$

When $e = 0$, for curves C of genus g and type (a, b), with $a, b > 0$, we have

$$2\min\{a, b\} + 2g - 2 = \sum_{p \in \tilde{C}} (e_p - 1).$$

Proof. The result follows from [31, Corollary IV 2.4, p.301]. With ν as above, we get

$$2b + 2g - 2 = \sum_{p \in \tilde{C}} (e_p - 1).$$

When $e = 0$, use the projection map of lower degree, $\min\{a, b\}$. □

The first corollary gives restrictions on the multiplicities of cusps on a curve.

Corollary 3.1.7. *Let C be a cuspidal curve on $\mathbb{F}_e$ of type (a, b) and genus g with $s > 0$ cusps p_j with multiplicities m_j. Then the following inequality holds,*

$$2b + 2g - 2 \geq \sum_{j=1}^{s} (m_j - 1).$$

Proof. The cusps p_j of C gives branching points with ramification index bigger than or equal to the multiplicity m_j, so the result follows from Theorem 3.1.6. $\square$

The second corollary gives restrictions on the number of smooth fiber tangential points.

Corollary 3.1.8. *Let C be a cuspidal curve on $\mathbb{F}_e$ of type (a, b) and genus g with $s > 0$ cusps p_j with multiplicities m_j. Then the number θ of smooth fiber tangential points is finite, and bounded above,*

$$\theta \leq 2b + 2g - 2 - \sum_{j=1}^{s}(m_j - 1).$$

Proof. Since a smooth fiber tangential point has ramification index $e_p \geq 2$, the result follows from Theorem 3.1.6 and Corollary 3.1.7. $\square$

We now establish a result on the Euler characteristic of the complement of a curve C on $\mathbb{F}_e$. In this case we view C and $\mathbb{F}_e$ as real manifolds.

Lemma 3.1.9. *Let C be a cuspidal curve of genus g and type (a, b) on $\mathbb{F}_e$. Then*

$$e(\mathbb{F}_e \setminus C) = 2g + 2.$$

Proof. For the pair $(\mathbb{F}_e, C)$ we have the long exact sequence of cohomology groups

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^0(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & \mathrm{H}^0(\mathbb{F}_e; \mathbb{Z}) & \longrightarrow & \mathrm{H}^0(C; \mathbb{Z}) \\
 & \longrightarrow & \mathrm{H}^1(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & \mathrm{H}^1(\mathbb{F}_e; \mathbb{Z}) & \longrightarrow & \mathrm{H}^1(C; \mathbb{Z}) \\
 & \longrightarrow & \mathrm{H}^2(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & \mathrm{H}^2(\mathbb{F}_e; \mathbb{Z}) & \longrightarrow & \mathrm{H}^2(C; \mathbb{Z}) \\
 & \longrightarrow & \mathrm{H}^3(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & \mathrm{H}^3(\mathbb{F}_e; \mathbb{Z}) & \longrightarrow & 0 \\
 & \longrightarrow & \mathrm{H}^4(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & \mathrm{H}^4(\mathbb{F}_e; \mathbb{Z}) & \longrightarrow & 0.
\end{array}$$

It is well known that the Hirzebruch surfaces have cohomology groups of the following form,

$$\begin{aligned}
&\mathrm{H}^0(\mathbb{F}_e; \mathbb{Z}) \cong \mathbb{Z}, \\
&\mathrm{H}^1(\mathbb{F}_e; \mathbb{Z}) \cong 0, \\
&\mathrm{H}^2(\mathbb{F}_e; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}, \\
&\mathrm{H}^3(\mathbb{F}_e; \mathbb{Z}) \cong 0, \\
&\mathrm{H}^4(\mathbb{F}_e; \mathbb{Z}) \cong \mathbb{Z}.
\end{aligned}$$

Since a cuspidal curve is homeomorphic to its normalization, we have the following cohomology groups for a cuspidal curve C of genus g (see [17, Proof of Proposition 1.5.16 pp.42–43]),

$$\begin{aligned}
&\mathrm{H}^0(C; \mathbb{Z}) \cong \mathbb{Z}, \\
&\mathrm{H}^1(C; \mathbb{Z}) \cong \mathbb{Z}^{2g}, \\
&\mathrm{H}^2(C; \mathbb{Z}) \cong \mathbb{Z}.
\end{aligned}$$

We get the long exact sequence

$$\begin{array}{cccccc} 0 & \longrightarrow & \mathrm{H}^0(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & \longrightarrow & \mathrm{H}^1(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}^{2g} \\ & \longrightarrow & \mathrm{H}^2(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & \longrightarrow & \mathrm{H}^3(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & 0 \\ & \longrightarrow & \mathrm{H}^4(\mathbb{F}_e, C; \mathbb{Z}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

Using Poincaré-Lefschetz duality, we have $\mathrm{H}_i(\mathbb{F}_e \setminus C; \mathbb{Z}) \cong \mathrm{H}^{4-i}(\mathbb{F}_e, C; \mathbb{Z})$ for $i = 0, \ldots, 4$. Taking dimensions in the long exact sequence, we find that $e(\mathbb{F}_e \setminus C) = 2g + 2$. $\square$

We may now establish the B–M–Y-inequalities for curves on $\mathbb{F}_e$.

Corollary 3.1.10 (Theorem 1.1.23). *Let (V, D) be the minimal embedded resolution of a cuspidal curve C of genus g on $\mathbb{F}_e$, and let K_V and H be as in the Zariski decomposition described in Chapter 1.*

a) If $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$, then

$$(K_V + D)^2 \leq 3e(\mathbb{F}_e \setminus C) = 6g + 6.$$

b) If $\overline{\kappa}(\mathbb{F}_e \setminus C) = 2$, then

$$H^2 \leq 3e(\mathbb{F}_e \setminus C) = 6g + 6.$$

3.2 The logarithmic Kodaira dimension

In this section we establish a result parallel to Theorem 2.2.1 by Wakabayashi [74] concerning the logarithmic Kodaira dimension of complements of curves on the Hirzebruch surfaces.

Theorem 3.2.1 (On the logarithmic Kodaira dimension on $\mathbb{F}_e$, $e \geq 0$). *Let C be an irreducible curve on $\mathbb{F}_e$ of genus g and type (a, b), with $b > 2$ and $a > 2 - \frac{1}{2}be$, $a > 0$.*

(I) If $g > 0$, then $\overline{\kappa}(\mathbb{F}_e \setminus C) = 2$.

(II) If $g = 0$ and C has at least three cusps, then $\overline{\kappa}(\mathbb{F}_e \setminus C) = 2$.

(III) If $g = 0$ and C has at least two cusps, then $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$.

We prove this theorem closely following the proof given by Wakabayashi in [74] for the parallel theorem for curves on the projective plane, replacing only the details for $\mathbb{P}^2$ with the corresponding details for $\mathbb{F}_e$ where necessary. Note that the proof goes through without essential changes, but that we have different indices in some parts of the proof.

We start by recalling the essential definitions from Chapter 1. Let L and M denote the generators of $\mathrm{Pic}(\mathbb{F}_e)$. Let $\sigma : V \longrightarrow \mathbb{F}_e$ be a finite sequence of monoidal transformations,

$$V = V_t \xrightarrow{\sigma_t} V_{t-1} \to \cdots \to V_1 \xrightarrow{\sigma_1} V_0 = \mathbb{F}_e.$$

In short,

$$\sigma = \sigma_1 \circ \cdots \circ \sigma_t : V \to \mathbb{F}_e.$$

Each transformation σ_i has exceptional divisor $E_i \subset V_i$ and is centered in $p_{i-1} \in V_{i-1}$. Let E_i' denote the strict transform of E_i by $\sigma_{i+1} \circ \cdots \circ \sigma_t$. By abuse of notation, we also use the symbol E_i for $(\sigma_{i+1} \circ \cdots \circ \sigma_t)^* E_i$, M for $\sigma^* M$ and L for $\sigma^* L$.

Before giving the proof of the theorem, we need a lemma and a proposition. The formulation and proofs of these are simply adjustments to the ones found in [74].

Lemma 3.2.2. *Let $\sigma : V \to \mathbb{F}_e$, M, L and E_i be as above. For any $\hat{a}, \hat{b} \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$ we have*

$$\dim \mathrm{H}^0\Big(V, \mathscr{O}\big(\hat{a}L + \hat{b}M - \sum_{i=1}^{t} n_i E_i\big)\Big) \geq \frac{(\hat{b}+1)(2\hat{a}+2+\hat{b}e)}{2} - \sum_{i=1}^{t} \frac{n_i(n_i+1)}{2}.$$

Proof. Most of the proof of [74, Lemma, p.157] goes unchanged, since it only concerns local properties of points. A calculation of the dimension of the vector space of polynomials of bigrading $(\hat{a}, \hat{b})$ can be found in [37, Proposition 2.3, p.129]. □

Let C be an irreducible curve on $\mathbb{F}_e$ of type (a, b). Let $\sigma : V \to \mathbb{F}_e$ be the minimal embedded resolution of its singularities, such that its reduced inverse image D is an SNC-divisor. Let C_i denote the strict transform of C by $\sigma_1 \circ \cdots \circ \sigma_i$, and let m_i be the multiplicity of p_i on C_i. Let $\tilde{C}$ denote the strict transform of C by σ. Finally, let K_V denote the canonical divisor on V. Then with the sloppy notation introduced above, we have

$$\begin{aligned} D &= \tilde{C} + \sum_{i=1}^{t} E_i', \\ K_V &\sim (e-2)L - 2M + \sum_{i=1}^{t} E_i, \\ aL + bM &\sim C = \tilde{C} + \sum_{i=1}^{t} m_{i-1} E_i. \end{aligned}$$

Hence,

$$D + K_V \sim (a+e-2)L + (b-2)M + \sum_{i=1}^{t} E_i' - \sum_{i=1}^{t} (m_{i-1} - 1) E_i. \tag{3.2.1}$$

Proposition 3.2.3. *With C, a, b, D, K_V, L and M as above, suppose that for sufficiently large $k \in \mathbb{N}$*

$$\lambda k(D + K_V) \sim \lambda\Big((a+e-2)L + (b-2)M\Big) + G_k, \tag{3.2.2}$$

where λ is a suitable positive number independent of k, and G_k is a suitable non-negative divisor on V dependent on k. Then $\overline{\kappa}(\mathbb{F}_e \setminus C) = 2$.

Proof. Choose a k such that (3.2.2) holds. Since G_k is non-negative, then for any $n \in \mathbb{N}$ we have

$$\dim \mathrm{H}^0\Big(V, \mathscr{O}\big(n\lambda k(D+K_V)\big)\Big) \geq \dim \mathrm{H}^0\Big(V, \mathscr{O}\big(n\lambda((a+e-2)L+(b-2)M)\big)\Big).$$

For readability, we will use an even more sloppy notation and write $C+K$ instead of $(a+e-2)L+(b-2)M$. By Riemann–Roch [31, Theorem V 1.6, p.362], we have that

$$\begin{aligned}
&\mathrm{h}^0\Big(n\lambda(C+K)\Big) - \mathrm{h}^1\Big(n\lambda(C+K)\Big) + \mathrm{h}^0\Big(K_V - n\lambda(C+K)\Big) \\
&\quad = \frac{1}{2}n\lambda\big((a+e-2)L+(b-2)M\big) \,.\, \Big(n\lambda\big((a+e-2)L+(b-2)M\big) - K_V\Big) + 1 + p_a(V).
\end{aligned}$$

Rewriting this equation, using that $p_a(V) = p_a(\mathbb{F}_e) = 0$ and $\mathrm{h}^1\Big(n\lambda(C+K)\Big) \geq 0$, we find that

$$\begin{aligned}
\mathrm{h}^0\Big(n\lambda(C+K)\Big) &= \frac{1}{2}n\lambda\big((a+e-2)L+(b-2)M\big) \,.\, \Big(n\lambda\big((a+e-2)L+(b-2)M\big) - K_V\Big) \\
&\quad + 1 + \mathrm{h}^1\Big(n\lambda(C+K)\Big) - \mathrm{h}^0\Big(K_V - n\lambda(C+K)\Big) \\
&\geq \frac{1}{2}n\lambda\big((a+e-2)L+(b-2)M\big) \,.\, \Big(n\lambda\big((a+e-2)L+(b-2)M\big) - K_V\Big) \\
&\quad - \mathrm{h}^0\Big(K_V - n\lambda(C+K)\Big) \\
&\geq \frac{1}{2}n^2\lambda^2(b-2)(2a+be-4) + \frac{1}{2}n\lambda(2a+2b+be-8) \\
&\quad - \mathrm{h}^0\Big(K_V - n\lambda(C+K)\Big) \\
&\geq \frac{1}{2}n^2\lambda^2(b-2)(2a+be-4) - \mathrm{h}^0\Big(K_V - n\lambda(C+K)\Big).
\end{aligned}$$

Next we show that

$$\mathrm{h}^0\Big(K_V - n\lambda((a+e-2)L+(b-2)M)\Big) \leq 0.$$

Assume to the contrary that on V there exists a positive divisor

$$P \sim K_V - n\lambda\big((a+e-2)L+(b-2)M\big). \tag{3.2.3}$$

Then $\sigma(P)$ must be effective. Considering $\mathbb{F}_e$ as a toric variety, the divisor $L+M$ is in the interior of the nef cone, hence it is ample (see [10, Example 6.1.16, p.273]). Therefore, $\sigma(P) \,.\, (L+M) \geq 0$. Because of (3.2.3) and the conditions on a and b, we have that

$$\begin{aligned}
P \,.\, \sigma^{-1}(L+M) &= \sigma(P) \,.\, (L+M) \\
&= \big((e-2)L - 2M - n\lambda(a+e-2)L - n\lambda(b-2)M\big) \,.\, (L+M) \\
&= -e-4-n\lambda\big(a+b-4+e(b-1)\big) \\
&< 0.
\end{aligned}$$

This is a contradiction to the above assumption, hence

$$\mathrm{h}^0\Big(n\lambda\big((a+e-2)L+(b-2)M\big)\Big) \geq n^2\lambda^2\frac{(b-2)(2a+be-4)}{2}.$$

In other words

$$\dim \mathrm{H}^0\Big(V, \mathscr{O}\big(n\lambda\big((a+e-2)L+(b-2)M\big)\big)\Big) \geq c\cdot n^2$$

for a suitable constant $c > 0$ independent of n. Note that $c > 0$ because of the conditions on a and b. By definition of the logarithmic Kodaira dimension (see Theorem 1.1.21), we then have $\overline{\kappa}(\mathbb{F}_e \setminus C) = 2$. □

We now prove Theorem 3.2.1 in the same way that Wakabayashi proves the result for curves on $\mathbb{P}^2$ in [74].

Proof of Theorem 3.2.1 (cf. Wakabayashi [74]).
Case (I). Let C be an irreducible curve with $g(C) \geq 1$, $b > 2$ and $a > 2 - \frac{1}{2}be$, $a > 0$. The genus formula ensures that

$$g(C) = \frac{(b-1)(2a-2+be)}{2} - \sum_{i=0}^{t-1}\frac{m_i(m_i-1)}{2} \geq 1.$$

With $\hat{a} = a+e-2$, $\hat{b} = b-2$, and $n_i = m_{i-1}-1$ in Lemma 3.2.2, we get

$$\dim \mathrm{H}^0\Big(V, \mathscr{O}\big((a+e-2)L+(b-2)M - \sum_{i=1}^{t}(m_{i-1}-1)E_i\big)\Big) \geq 1.$$

Hence, the below vector space is non-zero,

$$\mathrm{H}^0\Big(V, \mathscr{O}\big((a+e-2)L+(b-2)M - \sum_{i=1}^{t}(m_{i-1}-1)E_i\big)\Big) \neq 0.$$

Therefore,

$$(a+e-2)L+(b-2)M \sim \sum_{i=1}^{t}(m_{i-1}-1)E_i + G,$$

where G is a positive divisor on V. This implies that

$$\begin{aligned} k(D+K_V) &\sim (a+e-2)L+(b-2)M+(k-1)\Big((a+e-2)L+(b-2)M\Big) \\ &\quad + k\sum_{i=1}^{t}E_i' - k\sum_{i=1}^{t}(m_{i-1}-1)E_i \\ &\sim (a+e-2)L+(b-2)M+(k-1)G+k\sum_{i=1}^{t}E_i' - \sum_{i=1}^{t}(m_{i-1}-1)E_i. \end{aligned}$$

Each E_i, that is each $(\sigma_{i+1}\circ\cdots\circ\sigma_t)^*E_i$, is a linear combination of the strict transforms E_j', $j \geq i$, so for large k the latter three terms in the above sum constitute a non-negative divisor. Hence, we can use Proposition 3.2.3, with $\lambda = 1$, to conclude that

$\overline{\kappa}(\mathbb{F}_e \setminus C) = 2.$

Case (II). Let C be a rational cuspidal curve on $\mathbb{F}_e$. We first assume that C has only one cusp. Let q denote the index with the property that p_{q-1} is singular on C_{q-1} and p_q is non-singular on C_q. As before, we let t be the number of monoidal transformations such that D is the minimal embedded resolution of p on C. We write

$$\begin{aligned} E_q &= E'_q + E_{q+1} + \cdots + E_t, \\ E_{t-1} &= E'_{t-1} + E'_t, \\ t - q &= m_{q-1}. \end{aligned} \tag{3.2.4}$$

Using the strategy from [74], we first look at the following vector space,

$$\mathrm{H}^0\Big(V, \mathscr{O}\big((a+e-2)L + (b-2)M - \sum_{i \neq q}(m_{i-1}-1)E_i - (m_{q-1}-2)E_q - E_{q+1} - \cdots - E_{t-2}\big)\Big), \tag{3.2.5}$$

and show that this vector space is non-zero. Changing the index in the genus formula gives

$$g(C) = \frac{(b-1)(2a-2+be)}{2} - \frac{1}{2}\sum_{i=1}^{t} m_{i-1}(m_{i-1}-1) = 0.$$

Rewriting this expression, we have

$$\frac{(b-1)(2a-2+be)}{2} - \frac{1}{2}\sum_{i \neq q} m_{i-1}(m_{i-1}-1) - \frac{1}{2}(m_{q-1}-1)(m_{q-1}-2) - (m_{q-1}-2) = 1.$$

Using Lemma 3.2.2, we conclude that the vector space in (3.2.5) above is non-zero.

This implies that we may write

$$(a+e-2)L + (b-2)M \sim \sum_{i \neq q}(m_{i-1}-1)E_i + (m_{q-1}-2)E_q + E_{q+1} + \cdots + E_{t-2} + G_p,$$

where G_p is a positive divisor.

The latter observation can be used together with (3.2.4) to get an expression for $k(D + K_V)$.

$$\begin{aligned} k(D+K_V) \sim\ & (a+e-2)L + (b-2)M + (k-1)\big((a+e-2)L + (b-2)M\big) \\ & + k\sum_{i=1}^{t} E'_i - k\sum_{i=1}^{t}(m_{i-1}-1)E_i, \end{aligned}$$

$$\begin{aligned} k(D+K_V) \sim\ & (a+e-2)L + (b-2)M \\ & + (k-1)\Big(\sum_{i \neq q}(m_{i-1}-1)E_i + (m_{q-1}-2)E_q + E_{q+1} + \cdots + E_{t-2} + G_p\Big) \\ & + k\sum_{i=1}^{t} E'_i - k\sum_{i=1}^{t}(m_{i-1}-1)E_i, \end{aligned}$$

$$
\begin{aligned}
k(D+K_V) \sim\ & (a+e-2)L+(b-2)M+(k-1)G_p \\
& -\sum_{i\neq q}(m_{i-1}-1)E_i-k(m_{q-1}-1)E_q+(k-1)(m_{q-1}-2)E_q \\
& +(k-1)\left(E_{q+1}+\cdots+E_{t-2}\right)+k\sum_{i=1}^{t}E_i',
\end{aligned}
$$

$$
\begin{aligned}
k(D+K_V) \sim\ & (a+e-2)L+(b-2)M+(k-1)G_p-\sum_{i=1}^{t}(m_{i-1}-1)E_i \\
& -(k-1)E_q+(k-1)\left(E_{q+1}+\cdots+E_{t-2}\right)+k\sum_{i=1}^{t}E_i',
\end{aligned}
$$

$$
\begin{aligned}
k(D+K_V) \sim\ & (a+e-2)L+(b-2)M+(k-1)G_p-\sum_{i=1}^{t}(m_{i-1}-1)E_i \\
& +(k-1)\left(-E_q+E_{q+1}+\cdots+E_{t-2}\right)+k\sum_{i=1}^{t}E_i',
\end{aligned}
$$

$$
\begin{aligned}
k(D+K_V) \sim\ & (a+e-2)L+(b-2)M+(k-1)G_p-\sum_{i=1}^{t}(m_{i-1}-1)E_i \\
& +(k-1)\left(-E_q'-E_{t-1}-E_t\right)+k\sum_{i=1}^{t}E_i',
\end{aligned}
$$

$$
\begin{aligned}
k(D+K_V) \sim\ & (a+e-2)L+(b-2)M+(k-1)G_p \\
& -\sum_{i=1}^{t}(m_{i-1}-1)E_i+k\sum_{i=1}^{t}E_i'-(k-1)\left(E_q'+E_{t-1}'+2E_t'\right)
\end{aligned}
$$

Then we make the assumption that C has three cusps, p_1, p_2 and p_3. Note that the following procedure also works if we assume that C has more than three cusps. We perform successive minimal embedded resolutions of the cusps, and take one cusp at the time until we reach V. Let $\hat{t}_j$ denote the number of monoidal transformations needed to resolve the cusps $p_1,\ldots,p_j$, but not $p_{j+1},\ldots$, $j=1,2,3$. To resolve the three singularities in such a way that D is an SNC-divisor, we must apply in total $t:=\hat{t}_3$ successive monoidal transformations to the curve. We let $\hat{q}_j$ denote the smallest index such that the cusps $p_1,\ldots,p_{j-1}$ are resolved and that in the process of resolving p_j, the curve $C_{\hat{q}_j-1}$ is singular at $p_{\hat{q}_j-1}$, but $C_{\hat{q}_j}$ is non-singular at $p_{\hat{q}_j}$. For each cusp p_j we have that $\hat{t}_j=\hat{q}_j+m_{\hat{q}_j-1}$.

The minimal embedded resolution of the curve can be viewed in three different ways, and we use this to find three positive divisors $G_{p,j}$ and similar expressions to the above for $k(D+K_V)$ on the surface V. Note that we now sum up to t. For each j we may write

$$\begin{aligned}
k(D+K_V) \sim{} & (a+e-2)L+(b-2)M \\
& + (k-1)G_{p,j} - \sum_{i=1}^{t}(m_{i-1}-1)E_i \\
& + k\sum_{i=1}^{t} E'_i - (k-1)\left(E'_{\hat{q}_j} + E'_{\hat{t}_j-1} + 2E'_{\hat{t}_j}\right).
\end{aligned}$$

We then add the three expressions and get

$$\begin{aligned}
3k(D+K_V) \sim{} & 3\big((a+e-2)L+(b-2)M\big) \\
& + (k-1)\sum_{j=1}^{3} G_{p,j} - 3\sum_{i=1}^{t}(m_{i-1}-1)E_i \\
& + 3k\sum_{i=1}^{t} E'_i - (k-1)\sum_{j=1}^{3}\left(E'_{\hat{q}_j} + E'_{\hat{t}_j-1} + 2E'_{\hat{t}_j}\right).
\end{aligned}$$

The latter two lines of the sum constitutes a non-negative divisor for large k. The conclusion then follows by Proposition 3.2.3, and we have $\overline{\kappa}(\mathbb{F}_e \setminus C) = 2$.

Case (III). If C has two cusps p_1 and p_2, then as in Case (II) we can look at each cusp separately and find two expressions on the form

$$(a+e-2)L+(b-2)M \sim \sum_{i\neq q}(m_{i-1}-1)E_i + (m_{q-1}-2)E_q + E_{q+1} + \cdots + E_{t-2} + G_{p,j},$$

where $G_{p,j}$ is a positive divisor for each $j = 1, 2$.

By performing the blowing-ups of the cusps successively, with the same indices as in Case (II), we can use (3.2.1),

$$D + K_V \sim \sum_{i=1}^{t} E'_i + G_{p,j} - E_{\hat{q}_j} + E_{\hat{q}_j+1} + \cdots + E_{\hat{t}_j-2}.$$

Summing these expressions, we get

$$2(D+K_V) \sim 2\sum_{i=1}^{t} E'_i + \sum_{j=1}^{2} G_{p,j} + \sum_{j=1}^{2}(-E_{\hat{q}_j} + E_{\hat{q}_j+1} + \cdots + E_{\hat{t}_j-2}).$$

Using (3.2.4), we then get

$$2(D+K_V) \sim 2\sum_{i=1}^{t} E'_i + \sum_{j=1}^{2} G_{p,j} - \sum_{j=1}^{2}(E'_{\hat{q}_j} + E'_{\hat{t}_j-1} + 2E'_{\hat{t}_j}).$$

The right hand side is a positive divisor, hence

$$\mathrm{H}^0\Big(V, \mathscr{O}\big(2(D+K_V)\big)\Big) \neq 0.$$

It follows that $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$. $\square$

3.3 On the number of cusps

In this section we find an upper bound for the number of cusps on a rational cuspidal curve on a Hirzebruch surface. This result is a modification of Theorem 2.3.5 by Tono [68] and its proof, and essentially everything in the proof goes unchanged.

We first introduce a few preliminary definitions and results needed in the proof. Let D be a reduced effective SNC-divisor on a nonsingular projective surface V. We write D as the sum of its irreducible components D_i, that is, $D = D_1 + \ldots + D_r$.

We have from [24, 44, 68] a number of important notions related to D. We define the *branching number* of D_i, $\beta(D_i) = (D - D_i) \,.\, D_i$. The component D_i is called an *isolated component* of D if $\beta(D_i) = 0$. If $\beta(D_i) = 1$, then D_i is called a *tip*. If $\beta(D_i) \geq 3$, then D_i is called a *branching component* of D. A partial sum of components of D, say $L = D_1 + \ldots + D_m$, is called a *linear chain* of D if $\beta(D_1) = 1$, $\beta(D_i) = 2$ for $2 \leq i \leq m-1$, and $D_i \,.\, D_{i+1} = 1$ for $1 \leq i \leq m-1$. If $\beta(D_m) = 1$, then L is called a *rod*. If $\beta(D_m) = 2$, then L is called a *twig*. In the latter case, L is connected to D by a component $D_{m+1} \notin L$. If $\beta(D_{m+1}) \geq 3$, that is D_{m+1} is a branching component, then L is called a *maximal twig*. A linear chain is called *rational* if D_i is a rational curve for every i. It is called *admissible* if $D_i^2 \leq -2$ for every i.

A divisor on V is called *contractible* if the intersection matrix of its irreducible components is negative definite. If a linear chain L of D is rational and admissible, then it is contractible [68]. Moreover, there exists a unique $\mathbb{Q}$-divisor $\mathrm{Bk}(L)$, called the *bark* of L, with the property that $(K + D).D_i = \mathrm{Bk}(L).D_i$ for every i.

A component F of D consisting of three rational admissible maximal twigs and a rational curve F_1 is called a *fork* if $(K + F + B).F_1 < 0$, where B is the sum of the barks of the three maximal twigs. A fork is called admissible if $F_1^2 \leq -2$, and a fork is admissible if and only if it is contractible [68].

The *bark* of D, $\mathrm{Bk}(D)$ is defined to be the sum of the barks of all rational admissible rods, rational admissible forks and the remaining rational admissible twigs.

We call the the pair (V, D) *almost minimal* if for every irreducible curve M in V, either $(K + D - \mathrm{Bk}(D)) \,.\, M \geq 0$ or $(K + D - \mathrm{Bk}(D)) \,.\, E < 0$ and $\mathrm{Bk}(D) + M$ is *not* contractible.

We also need the following proposition before we state and prove the main theorem. This proposition holds for nonsingular projective surfaces defined over $\mathbb{C}$, and it is proved by Tono in [68, Corollary 4.4, p.219], here stated for our situation.

Proposition 3.3.1. *Let l denote the number of rational maximal twigs of D. If $\overline{\kappa}(V \setminus D) = 2$, if the pair (V, D) is almost minimal, and if D contains neither a rod consisting of (-2)-curves nor a fork consisting of (-2)-curves, then*

$$l \leq 12e(V \setminus D) + 5 - 3p_a(D). \tag{3.3.1}$$

We are now ready to give an upper bound on the number of cusps on a rational cuspidal curve on $\mathbb{F}_e$. The result is similar to the one given by Tono in [68] for $\mathbb{P}^2$ (see Theorem 2.3.5).

Theorem 3.3.2 (On the number of cusps on $\mathbb{F}_e$, $e \geq 0$). *The number of cusps s on a cuspidal curve C of genus g on $\mathbb{F}_e$ has an upper bound,*

$$s \leq \frac{21g + 29}{2}.$$

Proof (cf. Tono [68]). The proof given by Tono in [68] for $\mathbb{P}^2$ is directly applicable in this situation, and the following is essentially the same proof. At some places we have chosen to write out some details more carefully than in the original proof.

The aim of the proof is to set up a situation where we can apply Proposition 3.3.1, then the theorem follows.

We now construct the surface to which we can apply Proposition 3.3.1, and first show that two of the prerequisites in the proposition hold for this surface. Let $C = \mathscr{V}(F)$ be a cuspidal curve of genus g on $\mathbb{F}_e$. Let s denote the number of cusps on C. We are looking for an upper bound of s, hence we may assume that $s \geq 3$.

Let $\sigma : V \longrightarrow \mathbb{F}_e$ denote the minimal embedded resolution of C, and let $D = \tilde{C} + \sum_{i=1}^{t} E_i$ be the reduced total inverse image of C on the surface V. For a cusp p, the dual graph of $\sigma^{-1}(p) + \tilde{C}$ has the shape given in Figure 3.1.

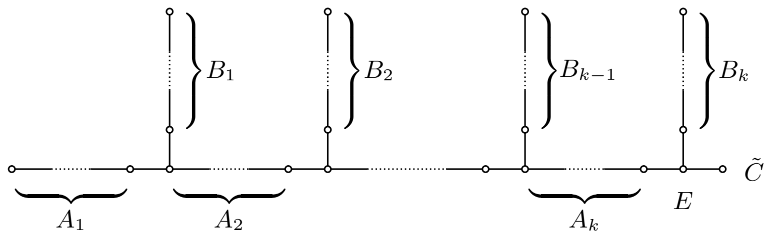


Figure 3.1: The dual graph of $\sigma^{-1}(p) + \tilde{C}$.

In Figure 3.1, E denotes the last blowing up in the resolution of p, and we have $E^2 = -1$. All other curves in D have self intersection ≤ -2. Notice that the morphism σ can be viewed as successive contractions in a way that can be handled with a quite clean notation. For a cusp p, σ first contracts $E + A_k + B_k$ in Figure 3.1 to a (-1)-curve E'. The process then continues in the same manner, with the contraction of $E' + A_{k-1} + B_{k-1}$ to another (-1)-curve and so on, until we reach $\mathbb{F}_e$.

Considering the graph of the minimal embedded resolution of all cusps on C, we see that D is connected. Notice that D contains s curves E_j with self intersection -1, all of which are branching components, and that the strict transform $\tilde{C}$ of C is also a branching component when $s \geq 3$.

We do not know if the pair (V, D) is almost minimal, so we cannot use Proposition 3.3.1 on this surface directly. We solve this problem by applying a theorem by Tsunoda in [44] (see [68, Lemma 3.2, p.218]). By [44, Theorem 1.11, p.226], there exists a birational morphism $\mu : V \longrightarrow V'$, consisting of successive contractions of (-1)-curves such that, with $D' = \mu_* D$, the pair (V', D') is almost minimal and $\overline{\kappa}(V \backslash D) = \overline{\kappa}(V' \backslash D')$. Since we assume that $s \geq 3$, by Theorem 3.2.1, Definition 1.1.22 and [44, Theorem 1.11, p.226], we have

$$2 = \overline{\kappa}(\mathbb{F}_e \setminus C) = \overline{\kappa}(V \setminus D) = \overline{\kappa}(V' \setminus D').$$

Before we show that the third prerequisite in Proposition 3.3.1 holds for (V', D'), we must estimate some of the invariants involved in the formula in Proposition 3.3.1. We begin with the Euler characteristic. By Lemma 3.1.9 we have $e(V \setminus D) = e(\mathbb{F}_e \setminus C) = 2g + 2$. To determine $e(v' \setminus D')$ we investigate the morphism μ more closely. The morphism μ is a composition of contractions, and we let $M_1, \dots, M_n \subset V$ denote the strict transforms of the (-1)-curves that are contracted by μ and *not* contained in D. Observe that μ possibly contracts some (-1)-curves contained in D in addition to the M_j's, but these contractions would not affect the Euler characteristic of the complement. Since D is connected, we must by Tono [68, Lemma 3.4, p.218] have $D \,.\, M_j \leq 1$. Moreover, since $V \setminus D \cong \mathbb{F}_e \setminus C \cong \mathscr{D}_+(F)$, it is affine. Hence, $M_j \cong \mathbb{P}^1$ cannot be contained in $V \setminus D$, and therefore $D \,.\, M_j = 1$. Using this information, we calculate the Euler characteristic,

$$e(V' \setminus D') = e(V \setminus D) - n = 2g + 2 - n.$$

Note that since $V' \setminus D' \cong V \setminus D$ is affine, it follows that $e(V' \setminus D') > 0$.

Our next aim is to ensure that the number l of rational maximal twigs of D' can be estimated by the number of cusps on C. This estimate relies on the fact that some of the components of D cannot be contracted by μ.

For each cusp p_j, $j = 1, \dots, s$, let E_j denote the (-1)-curve that intersects $\tilde{C}$ in the minimal embedded resolution. We will now show by contradiction that μ does not contract any E_j. Assume for contradiction that one E_j, say E, is contracted by μ. Now E is a branching component of D, hence it cannot be directly contracted by μ [68, Lemma 3.4, p.218], and μ must contract a (-1)-curve that intersects E. Contracting the (-1)-curve intersecting E turns E into a curve with nonnegative self intersection. Then E cannot be contracted, contrary to the assumption. We conclude that E_j cannot be contracted for any cusp.

We additionally have to ensure that $\tilde{C}$ is not contracted by μ. This can be shown by induction on the number of blowing downs in the morphism μ. Note that this part of the proof is also by Tono (personal communication). Let $\mu = \mu_\nu \circ \cdots \circ \mu_1$, $\nu \geq n$, be a decomposition of μ. Then μ_1 cannot contract $\tilde{C}$ since $s \geq 3$ makes $\tilde{C}$ a branching component of D, that is, $\beta_D(\tilde{C}) \geq 3$. That would contradict [68, Lemma 3.4, p.218]. So suppose that $\mu_k \circ \cdots \circ \mu_1$, $k < n$, does not contract the strict transform of $\tilde{C}$ by $\mu_{k-1} \circ \cdots \circ \mu_1$. Let D^k and $\tilde{C}^k$ denote the strict transforms of D and $\tilde{C}$ under $\mu_k \circ \cdots \circ \mu_1$. Now since μ does not contract any of the last exceptional curves E_j for any cusp p_j, $\tilde{C}^k$ will still be a branching component of D^k. Then μ_{k+1} cannot contract $\tilde{C}^k$, because that would contradict [68, Lemma 3.4, p.218]. Hence, $\mu_{k+1} \circ \cdots \circ \mu_1$ does not contract $\tilde{C}$. So by induction, $\tilde{C}$ cannot be contracted by μ.

The number l of rational maximal twigs of D' can now be estimated by the number of cusps on C. For each cusp p, let $A = \sigma^{-1}(p) - E - B_k$. The morphism μ affects the tree of rational curves $A + E + B_k$, and contracts it at most to another tree of rational curves. Since E is not contracted by μ, $\mu(E)$ must be a curve with self intersection ≥ -1. Then by [68, Lemma 3.5, p.218], $\mu(E)$ cannot be part of any rational linear chain or fork, hence not part of any rational maximal twig of D'. This implies that A cannot be contracted to a point by μ. Furthermore, $\mu(B_k)$ can be contracted to a point, but then $\mu(A)$ has to contain at least two rational maximal twigs in order to avoid that $\mu(E)$ is part of a rational maximal twig. Summing up, we observe that D' must have at least two rational maximal twigs per cusp, so we have $2s \leq l$.

Now we note that the third prerequisite in Proposition 3.3.1 holds for (V', D'). The morphism μ does not disconnect D, so D' is connected. Since D' is connected and additionally has at least $6 \leq 2s$ maximal twigs, it is impossible that it contains a rod consisting of (-2)-curves or a fork consisting of (-2)-curves.

Proposition 3.3.1 additionally involves the invariant $p_a(D')$, which is equal to g in this case. Indeed, since $\tilde{C}$ is nonsingular, $p_a(\tilde{C}) = g$. Since $D' = \tilde{C} + \sum E_i$, not necessarily for all i, and since D' is an SNC-divisor, we can successively apply formula (1.1.1) on p.6, and find that $p_a(D') = p_a(\tilde{C}) = g$.

By Proposition 3.3.1 applied to (V', D') and the above estimates, we then find the desired upper bound on the number of cusps,

$$\begin{aligned} 2s &\leq 12(2g + 2 - n) + 5 - 3g \\ &\leq 21g + 29 - 12n \\ &\leq 21g + 29. \end{aligned}$$

□

We immediately get a corollary for rational cuspidal curves on $\mathbb{F}_e$.

Corollary 3.3.3. *A rational cuspidal curve on $\mathbb{F}_e$ can not have more than 14 cusps.*

3.4 Rational cuspidal curves on Hirzebruch surfaces

In this section we give examples of rational cuspidal curves on Hirzebruch surfaces, and our aim is to shed light on the question of how many and what kind of cusps a rational cuspidal curve on a Hirzebruch surface can have. Constructing rational cuspidal curves with many cusps is a difficult task. On the projective plane we have seen that there are very few such curves, and they are indeed constructed with care. A natural place to look for curves on the Hirzebruch surfaces is to start out with the rational cuspidal curves on the projective plane, and this is our main point of attack. From these curves we construct rational cuspidal curves on the Hirzebruch surfaces with four and three cusps, using birational transformations. Moreover, the connections between the projective plane and the Hirzebruch surfaces allow us to look for rational cuspidal curves starting out with curves with other singularities in addition to cusps. We do not explore the latter strategy in depth, but show an example and discuss some of the issues we face using this approach.

3.4.1 Rational cuspidal curves with four cusps

We begin with the construction of rational cuspidal curves with four cusps on the Hirzebruch surfaces. Perhaps not surprisingly, we are not able to construct many such curves. Indeed, on each $\mathbb{F}_e$, with $e \geq 0$, we construct one infinite series of rational cuspidal curves with four cusps. On $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ we construct another three infinite series of rational cuspidal curves with four cusps, and on $\mathbb{F}_2$ we construct a single additional rational cuspidal curve with four cusps.

The following theorem presents the series of rational fourcuspidal curves that consists of curves on all the Hirzebruch surfaces. In Appendix A we construct some of the curves from a plane fourcuspidal curve using the birational maps from Chapter 1.

Theorem 3.4.1. *For all $e \geq 0$ and $k \geq 0$, except for the pair $(e,k) = (0,0)$, there exists a rational cuspidal curve $C_{e,k}$ on $\mathbb{F}_e$ of type $(2k+1,4)$ with four cusps and cuspidal configuration*

$$[4_{k-1+e}, 2_3], [2], [2], [2].$$

Proof. We will show that for each $e \geq 0$ there is an infinite series of curves on $\mathbb{F}_e$, and we show this by induction on k. The proof is split in two, and we treat the case of k odd and even separately. We construct the series of curves $C_{e,0}$ for $e \geq 1$, and then we construct the initial series $C_{e,1}$ and $C_{e,2}$, with $e \geq 0$. We only treat the induction to prove the existence of $C_{e,k}$ for odd values of k, as the proof for even values of k is completely parallel.

Let C be the rational cuspidal curve of degree 5 on $\mathbb{P}^2$ with cuspidal configuration $[2_3], [2], [2], [2]$. Let p be the cusp with multiplicity sequence $[2_3]$, and let T be the tangent line to C at p. Then $T \cdot C = 4p + r$, with r a smooth point on C. Blowing up at r, the strict transform of C is a curve $C_{1,0}$ of type $(1,4)$ on $\mathbb{F}_1$ with cuspidal configuration $[2_3], [2], [2], [2]$. Letting $T_{1,0}$ denote the strict transform of T and $p_{1,0}$ the strict transform of p, we have $T_{1,0} \cdot C_{1,0} = 4p_{1,0}$. We observe that $p_{1,0}$ is fiber tangential. Let E_1 denote the special section on $\mathbb{F}_1$, and let $s_{0,1} = E_1 \cap T_{1,0}$.

From $C_{1,0}$ we can proceed with the construction of curves on Hirzebruch surfaces in three ways.

First, we show by induction on e that the curves $C_{e,0}$ exist on the Hirzebruch surfaces $\mathbb{F}_e$, for all $e \geq 1$. We have already seen that $C_{1,0}$ exists on $\mathbb{F}_1$, and that there exists a fiber $T_{1,0}$ with the property that $T_{1,0} \cdot C_{1,0} = 4p_{1,0}$ for the first cusp $p_{1,0}$. Now assume $e \geq 2$ and that the curve $C_{e-1,0}$ of type $(1,4)$ exists on $\mathbb{F}_{e-1}$ with cuspidal configuration $[4_{e-2}, 2_3], [2], [2], [2]$, where $p_{e-1,0}$ denotes the first cusp and $T_{e-1,0}$ has the property that $T_{e-1,0} \cdot C_{e-1,0} = 4p_{e-1,0}$. Then, with E_{e-1} the special section of $\mathbb{F}_{e-1}$, blowing up at the intersection $s_{e-1,0} \in E_{e-1} \cap T_{e-1,0}$ and contracting $T_{e-1,0}$, we get $C_{e,0}$ on $\mathbb{F}_e$ of type $(1,4)$ with cuspidal configuration $[4_{e-1}, 2_3], [2], [2], [2]$. Moreover, we note that there exists a fiber $T_{e,0}$ with $T_{e,0} \cdot C_{e,0} = 4p_{e,0}$. So the series exists on all $e \geq 1$ for $k = 0$.

Second, note that from the curve $C_{1,0}$ on $\mathbb{F}_1$ it is possible to construct the curve $C_{0,1}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up at $p_{1,0}$ before contracting $T_{1,0}$. The curve $C_{0,1}$ is a curve of type $(3,4)$ with cuspidal configuration $[2_3], [2], [2], [2]$, and there is a fiber $T_{0,1}$ such that $T_{0,1} \,.\, C_{0,1} = 4p_{0,1}$. Blowing up at a point $s_{0,1} \in T_{0,1} \setminus \{p_{0,1}\}$ and contracting $T_{0,1}$ result in the curve $C_{1,1}$ on $\mathbb{F}_1$ of type $(3,4)$ with cuspidal configuration $[4, 2_3], [2], [2], [2]$. Moreover, there exists a fiber $T_{1,1}$ with $T_{1,1} \cdot C_{1,1} = 4p_{1,1}$ and $p_{1,1} \notin E_1$. The same induction on e as above proves that the series exists for $k = 1$.

Third, note that from the curve $C_{1,0}$ on $\mathbb{F}_1$ it is possible to construct the curve $C_{0,2}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up at a point $t_{1,0} \in T_{1,0} \setminus \{p_{1,0}, s_{1,0}\}$ before contracting $T_{1,0}$. The curve $C_{0,2}$ is a curve of type $(5,4)$ with cuspidal configuration $[4, 2_3], [2], [2], [2]$, and there is a fiber $T_{0,2}$ such that $T_{0,2} \cdot C_{0,2} = 4p_{0,2}$. Blowing up at a point $s_{0,2} \in T_{0,2} \setminus \{p_{0,2}\}$ and contracting $T_{0,2}$ give the curve $C_{1,2}$ on $\mathbb{F}_1$ of type $(5,4)$ with cuspidal configuration $[4_2, 2_3], [2], [2], [2]$. Moreover, there exists a fiber $T_{1,2}$ with $T_{1,2} \cdot C_{1,2} = 4p_{1,2}$ and $p_{1,2} \notin E_1$. The same induction on e as above proves that the series exists for $k = 2$.

Next assume $k \geq 3$, with k odd, and that there exists a series of curves $C_{e,k-2}$ of type $(2k-3,4)$ on $\mathbb{F}_e$ for all $e \geq 0$ with cuspidal configuration $[4_{e+k-3}, 2_3], [2], [2], [2]$. Then, in particular, the curve $C_{1,k-2}$ on $\mathbb{F}_1$ with cuspidal configuration $[4_{k-2}, 2_3], [2], [2], [2]$

exists. Moreover, there exists a fiber $T_{1,k-2}$ on $\mathbb{F}_1$ such that $T_{1,k-2} \cdot C_{1,k-2} = 4p_{1,k-2}$, where $p_{1,k-2}$ denotes the cusp with multiplicity sequence $[4_{k-2}, 2_3]$. With E_1 the special section on $\mathbb{F}_1$, let $s_{1,k-2} \in E_1 \cap T_{1,k-2}$. We now blow up at a point $t_{1,k-2} \in T_{1,k-2} \setminus \{p_{1,k-2}, s_{1,k-2}\}$ and subsequently contract $T_{1,k-2}$. This gives the curve $C_{0,k}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(2k+1, 4)$ with cuspidal configuration $[4_{k-1}, 2_3], [2], [2], [2]$. With $T_{0,k}$ the strict transform of the exceptional line of the latter blowing up, we have $T_{0,k} \cdot C_{0,k} = 4p_{0,k}$. Blowing up at a point $s_{0,k} \in \Theta_V \langle D \rangle_{0,k} \setminus \{p_{0,k}\}$ and contracting $T_{0,k}$ gives the curve $C_{1,k}$ on $\mathbb{F}_1$ of type $(2k+1, 4)$ with cuspidal configuration $[4_k, 2_3], [2], [2], [2]$. Moreover, there is a fiber $T_{1,k}$ with the property that $T_{1,k} \cdot C_{1,k} = 4p_{1,k}$. With the same induction on e as above, we get the series of curves $C_{e,k}$. □

There are three infinite series of rational fourcuspidal curves that can be found on the Hirzebruch surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$. We will shortly list these three series.

First we consider the rational cuspidal curves with four cusps on $\mathbb{F}_1$ that we can get by blowing up a single point on $\mathbb{P}^2$. These curves represent examples from the series.

Theorem 3.4.2. *Let C be the rational cuspidal curve with four cusps of degree 5 on $\mathbb{P}^2$. The following rational cuspidal curves on $\mathbb{F}_1$ with four cusps can be constructed from C by blowing up a single point on $\mathbb{P}^2$.*

# Cusps	Curve	Type	Cuspidal configuration
	C_1	$(0,5)$	$[2_3], [2], [2], [2]$
4	C_2	$(1,4)$	$[2_3], [2], [2], [2]$
	C_3	$(2,3)$	$[2_2], [2], [2], [2]$

Table 3.1: Rational cuspidal curves on $\mathbb{F}_1$ with four cusps.

Proof. The curve C_1 is constructed by blowing up any point r on $\mathbb{P}^2 \setminus C$. Note that if r is on the tangent line to a cusp on C, then C_1 has cusps that are fiber tangential. If r is only on tangent lines of smooth points on C, then C_1 has smooth fiber tangential points.

The curve C_2 is constructed by blowing up any smooth point r on C. Again, if r is on a tangent line of C, C_2 will have points that are fiber tangential.

The curve C_3 is constructed by blowing up the cusp with multiplicity sequence $[2_3]$. □

The fact that we can construct curves with fiber tangential points is crucial in the later constructions. Although we do not get new cuspidal configurations in this first step, the fiber tangential points can sometimes account for enough intersection with the fiber such that a cusp can be constructed later on.

We now give the three series of rational cuspidal curves with four cusps on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$. For notational purposes we denote these surfaces by $\mathbb{F}_h$, with $h \in \{0, 1\}$, in the theorems.

Theorem 3.4.3. *For $h \in \{0, 1\}$ and all integers $k \geq 0$, except the pair $(h, k) = (0, 0)$, there exists a rational cuspidal curve $C_{h,k}$ on $\mathbb{F}_h$ of type $(3k + 1 - h, 5)$ with four cusps and cuspidal configuration*

$$[4_{2k-1+h}, 2_3], [2], [2], [2].$$

Proof. The proof is by construction and induction on k. Let C be a rational cuspidal curve of degree 5 on $\mathbb{P}^2$ with cuspidal configuration $[2_3], [2], [2], [2]$. Let p be the cusp with multiplicity sequence $[2_3]$, and let T be the tangent line to C at p. There is a smooth point $r \in C$, such that $T \cdot C = 4p + r$. Blowing up at any point $t \in T \setminus \{p, r\}$, we get the curve $C_{1,0}$ of type $(0, 5)$ and cuspidal configuration $[2_3], [2], [2], [2]$ on $\mathbb{F}_1$. Moreover, with $T_{1,0}$ the strict transform of T and $p_{1,0}$, $r_{1,0}$ the strict transforms of the points p and r, we have $T_{1,0} \cdot C_{1,0} = 4p_{1,0} + r_{1,0}$.

Now assume that the curve $C_{1,k-1}$ of type $(3(k-1), 5)$ exists on $\mathbb{F}_1$ with cuspidal configuration $[4_{2(k-1)}, 2_3], [2], [2], [2]$, and the intersection $T_{1,k-1} \cdot C_{1,k-1} = 4p_{1,k-1} + r_{1,k-1}$ for a fiber $T_{1,k-1}$ and points as above. Then blowing up at $r_{1,k-1}$ and contracting $T_{1,k-1}$, we get a curve $C_{0,k}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3k + 1, 5)$ and cuspidal configuration $[4_{2k-1}, 2_3], [2], [2], [2]$. Moreover, there is a fiber $T_{0,k}$ with the property that $T_{0,k} \cdot C_{0,k} = 4p_{0,k} + r_{0,k}$. Blowing up at $r_{0,k}$ and contracting $T_{0,k}$, we get a rational cuspidal curve $C_{1,k}$ of type $(3k, 5)$ on $\mathbb{F}_1$ with cuspidal configuration $[4_{2k}, 2_3], [2], [2], [2]$. □

Theorem 3.4.4. *For $h \in \{0, 1\}$ and all integers $k \geq 0$, except the pair $(h, k) = (0, 0)$, there exists a rational cuspidal curve on $\mathbb{F}_h$ of type $(2k + 2 - h, 4)$ with four cusps and cuspidal configuration*

$$[3_{2k-1+h}, 2], [2_3], [2], [2].$$

Proof. The proof is by construction and induction on k. Let C be the rational cuspidal curve of degree 5 on $\mathbb{P}^2$ with cuspidal configuration $[2_3], [2], [2], [2]$. Let q be one of the cusps with multiplicity sequence $[2]$, and let T be the tangent line to C at q. Then there are smooth points $r, s \in C$, such that $T \cdot C = 3q + r + s$. Blowing up at s, we get the curve $C_{1,0}$ of type $(1, 4)$ and cuspidal configuration $[2_3], [2], [2], [2]$ on $\mathbb{F}_1$. Moreover, with $T_{1,0}$ the strict transform of T and $p_{1,0}$, $r_{1,0}$ the strict transforms of the points p and r, we have $T_{1,0} \cdot C_{1,0} = 3p_{1,0} + r_{1,0}$.

Now assume that the curve $C_{1,k-1}$ of type $(2k-1, 4)$ exists on $\mathbb{F}_1$ with cuspidal configuration $[3_{2k-2}, 2], [2_3], [2], [2]$, and the intersection $T_{1,k-1} \cdot C_{1,k-1} = 3p_{1,k-1} + r_{1,k-1}$ for a fiber $T_{1,k-1}$ and points as above. Then blowing up at $r_{1,k-1}$ and contracting $T_{1,k-1}$, we get a curve $C_{0,k}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(2k + 2, 4)$ and cuspidal configuration $[3_{2k-1}, 2], [2_3], [2], [2]$. Moreover, there is a fiber $T_{0,k}$ with the property that $T_{0,k} \cdot C_{0,k} = 3p_{0,k} + r_{0,k}$. Blowing up at $r_{0,k}$ and contracting $T_{0,k}$, we get a rational cuspidal curve $C_{1,k}$ of type $(2k+1, 4)$ on $\mathbb{F}_1$ with cuspidal configuration $[3_{2k}, 2], [2_3], [2], [2]$. □

Theorem 3.4.5. *For $h \in \{0, 1\}$, all integers $k \geq 2$, and every choice of $n_j \in \mathbb{N}$, with $j = 1, \ldots, 4$, such that $\sum_{j=1}^{4} n_j = 2k + h$, there exists a rational cuspidal curve on $\mathbb{F}_h$ of type $(k + 1 - h, 3)$ with four cusps and cuspidal configuration*

$$[2_{n_1}], [2_{n_2}], [2_{n_3}], [2_{n_4}].$$

Proof. We prove the existence of the curves on $\mathbb{P}^1 \times \mathbb{P}^1$ by induction on k. In the proof we show that any curve on $\mathbb{F}_1$ can be reached from a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ by an elementary transformation, hence we prove the theorem for $h \in \{0, 1\}$.

First we observe that a choice of n_j such that the condition $\sum_{j=1}^{4} n_j = 2k$ means that either all four n_j are odd, two are odd and two are even, or all four are even. We split the proof into these three cases, and prove only the first case completely. The other two can be dealt with in the same way once we have proved the existence of a first curve.

We now prove the theorem when all n_j are odd. Let C be a rational cuspidal curve on $\mathbb{P}^2$ of degree 4 with three cusps and cuspidal configuration $[2],[2],[2]$ for cusps p_j, $j=1,2,3$. Let p_4 be a general smooth point on C and let T be the tangent line to C at p_4. Then $T \cdot C = 2p_4 + t_1 + t_2$, where t_1, t_2 are two smooth points on C. Blowing up at t_1 and t_2 and contracting T, we get a rational cuspidal curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3,3)$ with four ordinary cusps.

Fixing notation, we say that we have a curve C_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3,3)$ and four cusps p_j^2, $j=1,\ldots,4$, all with multiplicity sequence $[2]$. Since the choice of $p_4 \in \mathbb{P}^2$ was general, there are four $(1,0)$-curves L_j^2 such that

$$L_j^2 \cdot C_2 = 2p_j^2 + r_j^2,$$

for smooth points $r_j^2 \in C_2$. Now assume that we have a curve C_{k-1} on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $((k-1)+1,3)$, with cuspidal configuration $[2_{n_1-2}],[2_{n_2}],[2_{n_3}],[2_{n_4}]$ such that all n_j are odd, and such that there exist fibers L_j^{k-1} with

$$L_j^{k-1} \cdot C_{k-1} = 2p_j^{k-1} + r_j^{k-1},$$

for smooth points r_j^{k-1} on C_{k-1}.

We blow up at r_1^{k-1}, contract the corresponding L_1^{k-1} and get a curve $C_{1,k-1}$ on $\mathbb{F}_1$ of type $(k-1,3)$ with cuspidal configuration $[2_{n_1-1}],[2_{n_2}],[2_{n_3}],[2_{n_4}]$. Moreover, since r_1^{k-1} was not fiber tangential, we have that $r_1^{1,k-1} \notin E_1$, and the strict transform of the exceptional fiber of the blowing up, $L_1^{1,k-1}$, has intersection with $C_{1,k-1}$,

$$L_1^{1,k-1} \cdot C_{1,k-1} = 2p_1^{1,k-1} + r_1^{1,k-1}.$$

Blowing up at $r_1^{1,k-1}$ and contracting $L_1^{1,k-1}$ bring us back to $\mathbb{P}^1 \times \mathbb{P}^1$ and a curve C_k of type $(k+1,3)$ and cuspidal configuration $[2_{n_1}],[2_{n_2}],[2_{n_3}],[2_{n_4}]$. This takes care of the case when all n_j are odd.

To prove the theorem when two n_j are even or all n_j are even, we only show that there exist curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of the right type and cuspidal configurations $[2_2],[2_2],[2],[2]$ and $[2_2],[2_2],[2_2],[2_2]$. The rest of the argument is then similar to the above. To get the first curve, we blow up C_2 in r_1^2 and r_2^2 and contract L_1^2 and L_2^2. This is a curve C_3 of type $(4,3)$ with cuspidal configuration $[2_2],[2_2],[2],[2]$. The curve is on $\mathbb{P}^1 \times \mathbb{P}^1$ since it can be shown with direct calculations in `Maple` that r_1^2 and r_2^2 are not on the same $(0,1)$-curve. To get the second curve, we blow up at the analogous r_3^3 and r_4^3 on the curve C_3, before contracting L_3^3 and L_4^3. We are again on $\mathbb{P}^1 \times \mathbb{P}^1$ by a similar argument to the above, and the curve C_4 is of type $(5,3)$ and has cuspidal configuration $[2_2],[2_2],[2_2],[2_2]$. □

Note that the construction of the curves in Theorem 3.4.5 can also be done from the rational cuspidal cubic on $\mathbb{P}^2$.

Alternative proof of Theorem 3.4.5. Let C be the rational cuspidal cubic on $\mathbb{P}^2$. Let s be a general point on $\mathbb{P}^2$, where general here means that s is neither on C, nor the tangent line to the cusp, nor the tangent line to the inflection point on C. For example we can choose $y^2z - x^3$ as the defining polynomial of C, and take $s = (0:1:1)$. Then the polar curve of C with respect to the point s, given by the defining polynomial $2yz + y^2$, intersects C in three smooth points, $(2^{\frac{2}{3}} : -2 : 1)$, $(2^{-\frac{1}{3}}(-1+\sqrt{3}i) : -2 : 1)$ and $(2^{-\frac{1}{3}}(-1-\sqrt{3}i) : -2 : 1)$. Blowing up at s brings us to $\mathbb{F}_1$ and a curve of type $(0,3)$ with one ordinary cusp, say p_4. We additionally have three fibers L_j, $j = 1, \ldots, 3$, with the property that $L_j \cdot C = 2p_j + r_j$ for smooth points p_j and r_j on C. Blowing up at the r_j's and contracting the L_j's, we get the desired series of curves. □

The series in Theorem 3.4.5 can be extended to a series of rational cuspidal curves with less than four cusps in an obvious way. We state this as a corollary.

Corollary 3.4.6. *For $h \in \{0,1\}$, all integers $k \geq 0$, and every choice of $n_j \in \mathbb{N} \cup \{0\}$, with $j = 1, \ldots, 4$, such that $\sum_{j=1}^{4} n_j = 2k + h$, there exists a rational cuspidal curve on $\mathbb{F}_h$ of type $(k+1-h, 3)$ with $s \in \{0,1,2,3,4\}$ cusps and cuspidal configuration*

$$[2_{n_1}], [2_{n_2}], [2_{n_3}], [2_{n_4}].$$

Proof. These curves can be constructed from the curves in Theorem 3.4.5 by a similar construction. In order to construct the curves with less than four cusps we have to blow up cusps on the curves in the series from Theorem 3.4.5. □

Last in this section we provide an example of a curve not represented in any of the above series. This is the only example we have found of such a curve, and in particular the only such curve on $\mathbb{F}_2$.

Theorem 3.4.7. *There exists a rational cuspidal curve on $\mathbb{F}_2$ of type $(0,3)$ with four cusps and cuspidal configuration*

$$[2], [2], [2], [2].$$

Proof. Let C be the plane rational fourcuspidal curve of degree 5. Let p be the cusp $[2_3]$ and p_i, $i = 1,2,3$, the cusps with multiplicity sequence $[2]$. Let T be the tangent line to C at p. Let L_i denote the line through p and p_i, with $i = 1,2,3$. There are smooth points s and r_i, $i = 1,2,3$, on C, such that

$$T \cdot C = 4p + s, \qquad L_i \cdot C = 2p + 2p_i + r_i.$$

Blowing up at p gives a $(2,3)$-curve on $\mathbb{F}_1$ with cuspidal configuration $[2_2], [2], [2], [2]$. Let C' denote the strict transform of C, T' and L_i' the strict transforms of T and L_i, and let E' be the special section on $\mathbb{F}_1$. Let p' be the cusp $[2_2]$, p_i' the other cusps, and s' and r_i' the strict transforms of the points s and r_i. Then we have the following intersections,

$$E' \cdot C' = 2p', \qquad T' \cdot C' = 2p' + s', \qquad L_i' \cdot C' = 2p_i' + r_i'.$$

Since $p' \in E'$, blowing up at p' and contracting T', we get a cuspidal curve on $\mathbb{F}_2$ of type $(0,3)$ and cuspidal configuration $[2], [2], [2], [2]$.

□

3.4.2 Rational cuspidal curves with three cusps

In this section we present lists of rational cuspidal curves on Hirzebruch surfaces with three cusps. We do not give all the details in the proofs of the existence of these curves, but in most cases we describe the arrangement of lines on $\mathbb{P}^2$ needed in the construction of the curves on $\mathbb{F}_e$. An arrangement of lines on $\mathbb{P}^2$ determines a fibration of $\mathbb{F}_1$, and properties of this fibration are crucial in the construction of the cuspidal curves using birational transformations. Note that by the curve C' on $\mathbb{F}_1$ in this section we mean the strict transform of a plane curve C under a transformation given by blowing up a point on $\mathbb{P}^2$.

In order to construct curves with three cusps on $\mathbb{F}_e$ from curves on $\mathbb{P}^2$, we need appropriate line arrangements or a high number of cusps on a curve C on $\mathbb{P}^2$. We therefore proceed with the construction starting out with plane curves of low degree and the three series of plane rational tricuspidal curves.

The following three theorems list rational tricuspidal curves that exist on $\mathbb{F}_e$ and are constructed from curves on $\mathbb{P}^2$ of degree $d \leq 5$, see Table 2.1, Table 2.2 and Table 2.3. In some cases, the same series of curves can be constructed from two different curves, and we then only include the construction from the curve of lower degree. For notational purposes, we give the type of the curve as (k, b), where b is determined and k is a suitable integer. Given a surface $\mathbb{F}_e$, $e \geq 0$, and a rational curve C with its cuspidal configuration, the integer k can be computed from the genus formula in Corollary 3.1.4. Note that k depends on the conditions on the cuspidal configurations.

Theorem 3.4.8. *For all integers $e \geq 0$ and $l \geq 0$ such that the conditions on m, n_1 and n_2 are met, the following rational cuspidal curves exist on $\mathbb{F}_e$ for suitable k.*

Curve	Type	Cuspidal configuration	Conditions
C_1	$(k,3)$	$[3_m], [2_{n_1}], [2_{n_2}]$	$n_1, n_2 \geq 1$ $m \geq e-1$, $m \geq 1$ $m + n_1 + n_2 = 2l + e$
C_2	$(k,3)$	$[3_m, 2], [2_{n_1}], [2_{n_2}]$	$n_1, n_2 \geq 1$ $m \geq e-1$, $m \geq 0$ $m + n_1 + n_2 = 2l + e - 1$

Table 3.2: Rational tricuspidal curves on $\mathbb{F}_e$ from a plane rational cuspidal cubic.

Proof. Let C be a plane rational cuspidal cubic curve. It has one ordinary cusp and one inflection point.

For the construction of C_1, let p be the cusp of C. Let q be the inflection point of C, and let T_q be the inflection tangent. We have $T_q \cdot C = 3q$. Let r be a smooth point on C, and let T_r be the tangent line of C at r. Then $T_r \cdot C = 2r + r_1$ for another smooth point r_1 on C. Let s denote the intersection point of T_q and T_r. Notice that the line L between p and s has the following intersection with C, $L \cdot C = 2p + s_1$, where s_1 is a smooth point on C. Then blowing up at s brings us to $\mathbb{F}_1$. The strict transform of C is a curve C' of type $(0, 3)$. The strict transforms of T_q, T_r and L are fibers T_q', T_r' and

L', and nothing has changed in the intersections of C' and these fibers. Appropriate elementary transformations on T'_r, T'_q and L' give the series.

For the construction of C_2, let p denote the cusp of C. Let T_p be the tangent line of C at p, with $T_p \cdot C = 3p$. Let t be a point on $T_p \setminus \{p\}$, $t \notin T_q$, and let P_tC be the polar curve to C at t. Using $F = y^2z - x^3$ as the defining polynomial of C, the polar curve to C at $t = (l : 0 : 1)$ for a parameter $l \in \mathbb{C}^*$ is reducible, given by $\mathscr{V}((y - \sqrt{3l}x)(y + \sqrt{3l}x))$. By intersection theory, we have $P_tC \cdot C = 4p + r + s$ for two smooth points r and s on C. Then the tangent lines T_r, T_s and T_p intersect at t. Moreover, $T_r \cdot C = 2r + r_1$ and $T_s \cdot C = 2s + s_1$ for two smooth points r_1 and s_1 on C. Blowing up at t brings us to $\mathbb{F}_1$. The strict transform of C is a curve C' of type $(0, 3)$, which the strict transforms T'_p, T'_q and T'_r of T_p, T_q and T_r intersect as before. Appropriate elementary transformations on these three fibers give the series.

□

Theorem 3.4.9. *For all integers $e \geq 0$ and $l \geq 0$ such that the conditions on m and n are met, the following rational cuspidal curves exist on $\mathbb{F}_e$ for suitable k.*

Curve	Type	Cuspidal configuration	Conditions
C_1	$(k, 4)$	$[4_m, 2_2], [3_n], [2]$	$n \geq 1$ $m \geq e - 1$, $m \geq 0$ $m + n = 2l + e - 1$

Table 3.3: Rational tricuspidal curves on $\mathbb{F}_e$ from rational cuspidal quartics.

Proof. Let C be a rational cuspidal quartic with two cusps, cuspidal configuration $[2_2], [2]$, and one inflection point. Let p be the cusp with multiplicity sequence $[2_2]$, and let q be the inflection point of C. The tangent lines T_p and T_q intersect C as follows, $T_p \cdot C = 4p$ and $T_q \cdot C = 3q + r$ for a smooth point $r \in C$. Let s denote the intersection of T_p and T_q, and note that $s \notin C$. Blowing up at s gives a curve C' on $\mathbb{F}_1$, and the strict transforms T'_p and T'_q of T_p and T_q intersect C' as before. Appropriate elementary transformations on the two fibers T'_p and T'_q give the series. □

Theorem 3.4.10. *For all integers $e \geq 0$ and $l \geq 0$ such that the conditions on m and n are met, the following rational cuspidal curves exist on $\mathbb{F}_e$ for suitable k.*

Curve	Type	Cuspidal configuration	Conditions
C_1	$(k,4)$	$[4_m,3],[2_2],[2]$	$m \geq e-1,\ m \geq 0$ $m = 2l+e-1$
C_2	$(k,4)$	$[4_m,2_2],[3_n],[2]$	$n \geq 1$ $m \geq e-1,\ m \geq 0$ $m+n = 2l+e$
C_3	$(k,4)$	$[4_m,2_2],[2_2],[2_2]$	$m \geq e-1,\ m \geq 0$ $m = 2l+e-1$

Table 3.4: Rational tricuspidal curves on $\mathbb{F}_e$ from rational cuspidal quintics.

Proof. We construct C_1 from a rational cuspidal curve C of degree 5 with cuspidal configuration $[3],[2_2],[2]$. Let p denote the cusp with multiplicity sequence $[3]$. Let T_p be the tangent line to C at p. Then $T_p \cdot C = 4p + r$ for a smooth point $r \in C$. Blowing up at r, we get a curve C' of type $(1,4)$ on $\mathbb{F}_1$. Moreover, the strict transform T_p' of T_p intersects C' only at p', with $T_p' \cdot C' = 4p'$. Appropriate elementary transformations on the fiber T_p' give the series.

We construct C_2 from a rational cuspidal quintic C with cuspidal configuration $[3],[2_2],[2]$. Let p and q denote the first two cusps. Let T_q be the tangent line to C at q. Then $T_q \cdot C = 4q + r$ for a smooth point $r \in C$. Let L be the line between r and p, then $L \cdot C = 3p + r + s$ for a smooth point $s \in C$. Blowing up at r, we get a curve C' of type $(1,4)$ on $\mathbb{F}_1$. Moreover, the strict transforms T_q' and L' of T_q and L, intersect C' as follows, $T_q' \cdot C' = 4q'$ and $L' \cdot C' = 3p' + s'$. Appropriate elementary transformations on the two fibers T_q' and L' give the series. Note that this series is not the same as the series constructed in Theorem 3.4.9, but that the two series complement each other. Indeed, given e, it is possible to construct curves C_2 for pairs m,n that satisfy the conditions on m and n, and $m+n = 2l+e$ for some integer l, while it is possible to construct curves C_1 in Theorem 3.4.9 for pairs m,n that satisfy the conditions on m and n, and $m+n = 2l+e-1$ for some integer l.

We construct C_3 from a rational cuspidal quintic C with cuspidal configuration $[2_2],[2_2],[2_2]$. Let p be one of the cusps. Let T_p be the tangent line to C at p. Then $T_p \cdot C = 4p + r$ for a smooth point $r \in C$. Blowing up at r, we get a curve C' of type $(1,4)$ on $\mathbb{F}_1$. Moreover, the strict transform T_p' of T_p intersects C' only at p'. Appropriate elementary transformations on the fiber T_p' give the series.

□

From the curves of low degree we get even more rational cuspidal curves with three cusps on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ in addition to the above.

Theorem 3.4.11. *For $h \in \{0,1\}$ and all integers $l \geq 0$ such that the conditions on m, n, n_1, n_2 and n_3 are met, the following rational cuspidal curves exist on $\mathbb{F}_h$ for suitable k.*

Curve	**Type**	**Cuspidal configuration**	**Conditions**
C_1	$(k,4)$	$[3_m,2],[2],[2]$	$m \geq 0$ $m = 2l+h-1$
C_2	$(k,4)$	$[3_m,2],[3_n,2],[2]$	$m,n \geq 0$ $m+n = 2l+h-1$
C_3	$(k,4)$	$[3_m,2],[3_n],[2_2]$	$n \geq 1$ $m \geq 0$ $m+n = 2l+h-1$
C_4	$(k,4)$	$[3_m],[3_n],[2_3]$	$m,n \geq 1$ $m+n = 2l+h-1$
C_5	$(k,4)$	$[3_{n_1}],[3_{n_2}],[3_{n_3}]$	$n_1,n_2,n_3 \geq 1$ $n_1+n_2+n_3 = 2l+h$

Table 3.5: Rational tricuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ from rational cuspidal quartics.

Proof. We construct C_1 from a rational cuspidal quartic C with cuspidal configuration $[2],[2],[2]$. Let p be one of the cusps, and let T_p be the tangent line to C at p. We have $T_p \cdot C = 3p + r$ for a smooth point $r \in C$. Blowing up at a point $s \in T_p \setminus \{p, r\}$ gives a curve C' on $\mathbb{F}_1$ of type $(0,4)$. The strict transform T'_p of T_p intersects C' as before. Appropriate elementary transformations on the fiber T'_p give the series.

We construct C_2 from a rational cuspidal quartic C with cuspidal configuration $[2],[2],[2]$. Let p, q denote two of the cusps, and let T_p and T_q be the respective tangent lines to C. We have $T_p \cdot C = 3p + r_1$ and $T_q \cdot C = 3q + r_2$ for smooth points $r_1, r_2 \in C$. Let s be the intersection point of T_p and T_q. Notice that $s \notin C$, hence blowing up at s gives a curve C' on $\mathbb{F}_1$ of type $(0,4)$. The strict transforms T'_p and T'_q of T_p and T_q intersect C' as before. Appropriate elementary transformations on the fibers T'_p and T'_q give the series. Notice the similarities in the constructions of C_1 and C_2.

We construct C_3 from a rational cuspidal quartic C with cuspidal configuration $[2_2],[2]$ and one inflection point. Let p be the cusp with multiplicity sequence $[2]$, and let q be the unique inflection point on C. Then the respective tangents T_p and T_q intersect C in the following way, $T_p \cdot C = 3p + r_1$ and $T_q \cdot C = 3q + r_2$, where $r_1, r_2 \in C$ are smooth points. Let s denote the intersection point of T_p and T_q. Blowing up at s gives a curve C' on $\mathbb{F}_1$ of type $(0,4)$. The strict transforms T'_p and T'_q of T_p and T_q intersect C' as before. Appropriate elementary transformations on the fibers T'_p and T'_q give the series.

We construct C_4 from a rational cuspidal quartic C with cuspidal configuration $[2_3]$ and three inflection points. Let p and q be two inflection points on C. Then the tangents T_p and T_q intersect C in the following way, $T_p \cdot C = 3p + r_1$ and $T_q \cdot C = 3q + r_2$, where $r_1, r_2 \in C$ are smooth points. Let s be the intersection point of T_p and T_q. Since $s \notin C$, blowing up at s gives a curve C' on $\mathbb{F}_1$ of type $(0,4)$. The strict transforms T'_p and T'_q of T_p and T_q intersect C' as before. Appropriate elementary transformations on the fibers T'_p and T'_q give the series.

We construct C_5 from a rational cuspidal quartic C with cuspidal configuration $[3]$ and two inflection points. Let p denote the cusp, and let q and r be the two inflection points on C. Then the tangents T_q and T_r intersect C in the following way, $T_q \cdot C = 3q + s_1$ and $T_r \cdot C = 3r + s_2$, where $s_1, s_2 \in C$ are smooth points. Let t be the intersection point of T_q and T_r. Let L be the line between p and t. Since $t \notin C, T_p$, we have $L \cdot C = 3p + s_3$, for a smooth point $s_3 \in C$. Blowing up at t gives a curve C' on $\mathbb{F}_1$ of type $(0,4)$. The strict transforms T'_q, T'_r and L' of T_q, T_r and L intersect C' as before. Appropriate elementary transformations on the fibers L', T'_q and T'_r give the series. □

Theorem 3.4.12. *For $h \in \{0,1\}$ and all integers $l \geq 0$ such that the conditions on m and n are met, the following rational cuspidal curves exist on $\mathbb{F}_h$ for suitable k.*

Curve	**Type**	**Cuspidal configuration**	**Conditions**
C_1	$(k,4)$	$[3_m,2],[3_n],[2_2]$	$n \geq 1$ $m \geq 0$ $m+n=2l+h$
C_2	$(k,4)$	$[3_m],[3_n],[2_3]$	$m,n \geq 1$ $m+n=2l+h$
C_3	$(k,4)$	$[3_m],[2_4],[2_2]$	$m \geq 1$ $m=2l+h$
C_4	$(k,4)$	$[3_m],[3_n,2],[2_2]$	$n \geq 0$ $m \geq 1$ $m+n=2l+h$
C_5	$(k,5)$	$[4_m,2_2],[4_n,3],[2]$	$m,n \geq 0$ $m+n=2l+h-1$
C_6	$(k,5)$	$[4_m,2_2],[2_2],[2_2]$	$m \geq 0$ $m=2l+h-1$
C_7	$(k,5)$	$[4_m,2_2],[4_n,2_2],[2_2]$	$m,n \geq 0$ $m+n=2l+h-1$

Table 3.6: Rational tricuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ from rational cuspidal quintics.

Proof. We construct C_1 from a rational cuspidal quintic C with cuspidal configuration $[3,2],[2_2]$ and one inflection point. Let p be the cusp with multiplicity sequence $[3,2]$, and let q be the inflection point of C. Let T_q be the tangent line to C at q. We have $T_q \cdot C = 3q + r_1 + s$ for two smooth points $r_1, s \in C$. Let L be the line between s and the cusp p. We have $L \cdot C = 3p + r_2 + s$, where $r_2 \in C$ is a smooth point. Blowing up at s gives a curve C' on $\mathbb{F}_1$ of type $(1, 4)$. Moreover, for the strict transforms T'_q and L' of T_q and L we have the intersections $T'_q \cdot C' = 3q' + r'_1$ and $L' \cdot C' = 3p + r_2$. Appropriate elementary transformations on the fibers T'_q and L' give the series. Note that this series is not the same as the series C_3 constructed in Theorem 3.4.11, but that the two series complement each other. Indeed, given h, it is possible to construct curves C_1 for pairs m, n that satisfy the conditions on m and n, and $m+n = 2l+h$ for some integer l, while it is possible to construct curves C_3 in Theorem 3.4.11 for pairs m, n that satisfy the conditions on m and n, and $m + n = 2l + h - 1$ for some integer l.

We construct C_2 from a rational cuspidal quintic C with cuspidal configuration $[3], [2_3]$ and one inflection point. The construction is parallel to the construction of C_1.

We construct C_3 from a rational cuspidal quintic C with cuspidal configuration $[2_4], [2_2]$ and one inflection point. Let q denote the inflection point, with tangent line T_q. We have $T_q \cdot C = 3q + r + s$ for smooth points $r, s \in C$. Blowing up at s we get a curve C' on $\mathbb{F}_1$ of type $(1, 4)$. Moreover, for the strict transform T'_q of T_q we have the intersection $T'_q \cdot C' = 3q' + r'$. Appropriate elementary transformations on the fiber T'_q give the series.

We construct C_4 from a rational cuspidal quintic C with cuspidal configuration $[3], [2_2], [2]$. Let p and q be the cusps with multiplicity sequence $[3]$ and $[2]$ respectively. Let T_q be the tangent line at q, with $T_q \cdot C = 3q + r_1 + s$ for smooth points $r_1, s \in C$. Let L be the line between s and p, with $L \cdot C = 3p + r_2 + s$ for a smooth point $r_2 \in C$. Blowing up at s we get a curve C' on $\mathbb{F}_1$ of type $(1, 4)$. For the strict transforms T'_q and L' of T_q and L, we have the intersections $T'_q \cdot C' = 3q' + r'_1$ and $L' \cdot C' = 3p' + r'_2$. Appropriate elementary transformations on the fibers T'_q and L' give the series.

We construct C_5 from a rational cuspidal quintic C with cuspidal configuration $[3], [2_2], [2]$. Let p and q be the cusps with multiplicity sequence $[3]$ and $[2_2]$ respectively. Let T_p and T_q be the respective tangent lines, with $T_p \cdot C = 4p + r_1$ and $T_q \cdot C = 4q + r_2$ for smooth points $r_1, r_2 \in C$. Let s be the intersection point of T_p and T_q. Since $s \notin C$, blowing up at s we get a curve C' on $\mathbb{F}_1$ of type $(0, 5)$. For the strict transforms T'_p and T'_q of T_p and T_q we have intersections with C' as before. Appropriate elementary transformations on the fibers T'_p and T'_q give the series.

We construct C_6 from a rational cuspidal quintic C with cuspidal configuration $[2_2], [2_2], [2_2]$. Let p be one of the cusps. We have $T_p \cdot C = 4p + r$ for a smooth point r on C. Let $s \notin C$ be a point on the tangent line T_p. Blowing up at s gives a curve C' on $\mathbb{F}_1$ of type $(0, 5)$. For the strict transforms T'_p and T'_q of T_p and T_q we have intersections with C' as before. Appropriate elementary transformations on the fiber T'_p give the series. We construct C_7 from a rational cuspidal quintic C with cuspidal configuration $[2_2], [2_2], [2_2]$. Let p, q be two of the cusps, and T_p, T_q their respective tangent lines. We have $T_p \cdot C = 4p + r_1$ and $T_q \cdot C = 4q + r_2$, with $r_1, r_2 \in C$ smooth points. Let $s \notin C$ denote the intersection point of T_p and T_q. Blowing up at s we get a curve C' on $\mathbb{F}_1$ of type $(0, 5)$. Appropriate elementary transformations on the fibers T'_p and T'_q give the series. Notice that the cuspidal configuration of C_6 is included in C_7, but that C_7 has fiber tangential properties. □

From the three series of tricuspidal rational curves on $\mathbb{P}^2$, there are more tricuspidal rational curves on $\mathbb{F}_e$. The curves that only exist on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ are marked with a $\star$.

Theorem 3.4.13. *For all integers $e \geq 0$, $h \in \{0, 1\}$ and $l \geq 0$ such that the conditions on m are met, the following rational cuspidal curves exist on $\mathbb{F}_e$ and $\mathbb{F}_h$ for integers $a, b \geq 1$, where $a + b = d - 2$, and suitable k.*

Curve	Type	Cuspidal configuration	Conditions
C_1	$(k, d-2)$	$[(d-2)_m], [2_{a-1}], [2_b]$	$m \geq e-1$, $m \geq 1$ $m = 2l + e$
C_2	$(k, d-1)$	$[(d-1)_m, d-2], [2_a], [2_b]$	$m \geq e-1$, $m \geq 0$ $m = 2l + e - 1$
$\star C_3$	(k, d)	$[(d-1)_m, d-2], [2_a], [2_b]$	$m \geq 1$ $m = 2l + h - 1$
$\star C_4$	$(k, d-1)$	$[(d-2)_m], [2_a], [2_b]$	$m \geq 1$ $m = 2l + h$

Table 3.7: Rational tricuspidal curves on $\mathbb{F}_e$ from the plane tricuspidal curves with $\hat{m} = d - 2$.

Proof. Let C be a plane rational cuspidal curve of degree d with three cusps p_1, p_2 and p_3 with multiplicity sequences $[d-2]$, $[2_a]$ and $[2_b]$ respectively, where $a, b \geq 1$ and $a + b = d - 2$.

We construct the series C_1 from C by blowing up at p_2. We then get a curve C' of type $(2, d-2)$ on $\mathbb{F}_1$. Let L be the line on $\mathbb{P}^2$ between p_1 and p_2, and notice that $L \cdot C = (d-2)p_1 + 2p_2$. The strict transform L' of L intersects C' only at the cusp p_1', and we have $L' \cdot C' = (d-2)p_1'$. Appropriate elementary transformations on the fiber L' give the series.

We construct the series C_2 from C. Let T be the tangent line to C at p_1. Then we have $T \cdot C = (d-1)p_1 + r$ for a smooth point $r \in C$. Blowing up at r, we get a curve C' of type $(1, d-1)$ on $\mathbb{F}_1$. The strict transform T' of T intersects C' only at the cusp p_1', and we have $T' \cdot C' = (d-1)p_1'$. Appropriate elementary transformations on the fiber T' give the series.

We construct the series C_3 from C by blowing up a point $s \in T \setminus \{p_1, r\}$. Then we have a curve C' on $\mathbb{F}_1$ of type $(0, d)$. Notice that the strict transform T' of T intersects C' in two points, $T' \cdot C' = (d-1)p_1' + r'$. Appropriate elementary transformations on the fiber T' give the series on $\mathbb{F}_h$.

We construct the series C_4 from C by blowing up a smooth point $t \in C$. Let L_t be the line between t and p_1. We have $L_t \cdot C = (d-2)p_1 + t + u$, for a smooth point $u \in C$. Blowing up at t, we get a curve C' on $\mathbb{F}_1$ of type $(1, d-1)$, and for the strict transform L_t' of L_t we have $L_t' \cdot C' = (d-2)p_1' + u'$. Appropriate elementary transformations on the fiber L_t' give the series on $\mathbb{F}_h$. $\square$

Theorem 3.4.14. *For all integers $e \geq 0$, $h \in \{0,1\}$ and $l \geq 0$ such that the conditions on m, n_1 and n_2 are met, the following rational cuspidal curves exist on $\mathbb{F}_e$ and $\mathbb{F}_h$ for integers $a \geq 1$, where $d = 2a+3$, and suitable k.*

Curve	**Type**	**Cuspidal configuration**	**Conditions**
C_1	$(k, d-3)$	$[(d-3)_m, 2_a], [3_{a-1}], [2]$	$m \geq 0$ $m = 2l + e$
C_2	$(k, d-1)$	$[(d-1)_m, d-3, 2_a], [3_a], [2]$	$m \geq 0$ $m = 2l + e - 1$
C_3	$(k, 3)$	$[3_{a+m}], [2_{a+n_1}], [2_{1+n_2}]$	$n_1, n_2 \geq 0$ $m \geq e-1$, $m \geq 0$ $m + n_1 + n_2 = 2l + e - 1$
$\star C_4$	(k, d)	$[(d-1)_m, d-3, 2_a], [3_a], [2]$	$m \geq 0$ $m = 2l + h - 1$

Table 3.8: Rational tricuspidal curves on $\mathbb{F}_e$ from the plane tricuspidal curves with $\hat{m} = d-3$.

Proof. Let C be a plane rational cuspidal curve of degree d with three cusps p_1, p_2 and p_3 with multiplicity sequences $[d-3, 2_a]$, $[3_a]$ and $[2]$ respectively, with $a \geq 1$ and $d = 2a+3$.

We construct the series C_1 from C by blowing up at p_2. We then get a curve C' of type $(3, d-3)$ on $\mathbb{F}_1$. Let L_2 be the line on $\mathbb{P}^2$ between p_1 and p_2, and notice that $L_2 \cdot C = (d-3)p_1 + 3p_2$. The strict transform L_2' of L_2 intersects C' only at the cusp p_1', and we have $L_2' \cdot C' = (d-3)p_1'$. Appropriate elementary transformations on the fiber L_2' give the series.

We construct the series C_2 from C. Let T be the tangent line to C at p_1. Then we have $T \cdot C = (d-1)p_1 + r$ for a smooth point $r \in C$. Blowing up at r, we get a curve C' of type $(1, d-1)$ on $\mathbb{F}_1$. The strict transform T' of T intersects C' only at the cusp p_1', and we have $T' \cdot C' = (d-1)p_1'$. Appropriate elementary transformations on the fiber T' give the series.

We construct the series C_3 from C by blowing up at p_1. Let L_3 be the line on $\mathbb{P}^2$ between p_1 and p_3, with $L_3 \cdot C = (d-3)p_1 + 2p_3 + t$ for a smooth point $t \in C$. Blowing up at p_1 we get a curve C' on $\mathbb{F}_1$ of type $(d-3, 3)$. The strict transforms L_2', L_3' and T' of L_2, L_3 and T intersect C' such that appropriate elementary transformations on these three fibers give the series.

We construct the series C_4 from C by blowing up a point $s \in T \setminus \{p_1, r\}$. Then we have a curve C' on $\mathbb{F}_1$ of type $(0, d)$. Notice that the strict transform T' of T intersects C' in two points, and we have $T' \cdot C' = (d-1)p_1' + r'$. Appropriate elementary transformations on the fiber T' give the series on $\mathbb{F}_h$.

□

Theorem 3.4.15. *For all integers $e \geq 0$, $h \in \{0,1\}$ and $l \geq 0$ such that the conditions on m and n are met, the following rational cuspidal curves exist on $\mathbb{F}_e$ and $\mathbb{F}_h$ for integers $a \geq 1$, where $d = 3a+4$, and suitable k.*

Curve	**Type**	**Cuspidal configuration**	**Conditions**
C_1	$(k, d-4)$	$[(d-4)_m, 3_a], [4_{a-1}, 2_a], [2]$	$m \geq 0$ $m = 2l + e$
C_2	$(k, d-1)$	$[(d-1)_m, d-4, 3_a], [4_a, 2_2], [2]$	$m \geq 0$ $m = 2l + e - 1$
C_3	$(k, 4)$	$[3_{a+m}], [4_{a+n}, 2_2], [2]$	$m, n \geq 0$ $m + n = 2l + e - 1$
$\star C_4$	(k, d)	$[(d-1)_m, d-4, 3_a], [4_a, 2_2], [2]$	$m \geq 0$ $m = 2l + h - 1$

Table 3.9: Rational tricuspidal curves on $\mathbb{F}_e$ from the plane tricuspidal curves with $\hat{m} = d - 4$.

Proof. Let C be a plane rational cuspidal curve of degree d with three cusps p_1, p_2 and p_3 with multiplicity sequences $[d-4, 3_a]$, $[4_a, 2_2]$ and $[2]$ respectively, with $a \geq 1$ and $d = 3a + 4$.

We construct the series C_1 from C by blowing up at p_2. We then get a curve C' of type $(4, d-4)$ on $\mathbb{F}_1$. Let L_2 be the line on $\mathbb{P}^2$ between p_1 and p_2, and notice that $L_2 \cdot C = (d-4)p_1 + 4p_2$. The strict transform L_2' of L_2 intersects C' only at the cusp p_1', and we have $L_2' \cdot C' = (d-4)p_1'$. Appropriate elementary transformations on the fiber L_2' give the series.

We construct the series C_2 from C. Let T be the tangent line to C at p_1. Then we have $T \cdot C = (d-1)p_1 + r$ for a smooth point $r \in C$. Blowing up at r, we get a curve C' of type $(1, d-1)$ on $\mathbb{F}_1$. The strict transform T' of T intersects C' only at the cusp p_1', and we have $T' \cdot C' = (d-1)p_1'$. Appropriate elementary transformations on the fiber T' give the series.

We construct the series C_3 from C by blowing up at p_1. We get a curve C' on $\mathbb{F}_1$ of type $(d-4, 4)$. The strict transforms L_2' and T' of L_2 and T intersect C' such that appropriate elementary transformations on these two fibers give the series.

We construct the series C_4 from C by blowing up a point $s \in T \setminus \{p_1, r\}$. Then we have a curve C' on $\mathbb{F}_1$ of type $(0, d)$. Notice that the strict transform T' of T intersects C' in two points, and we have $T' \cdot C' = (d-1)p_1' + r'$. Appropriate elementary transformations on the fiber T' give the series on $\mathbb{F}_h$.

□

There are additionally a few other series of curves arising from tricuspidal curves of low degree $6 \leq d \leq 10$.

Theorem 3.4.16. *For all integers $e \geq 0$, $h \in \{0,1\}$ and $l \geq 0$ such that the conditions on m and n are met, the following rational cuspidal curves exist on $\mathbb{F}_e$ and $\mathbb{F}_h$ for suitable k.*

Curve	Type	Cuspidal configuration	Conditions
$\star C_1$	$(k,5)$	$[4_m, 2_2], [4_n], [2_2]$	$m \geq 0$ $n \geq 1$ $m = 2l + h$
$\star C_2$	$(k,5)$	$[4_m, 2_3], [4_n], [2]$	$m \geq 0$ $n \geq 1$ $m = 2l + h$
C_3	$(k,6)$	$[6_m, 3_2], [4, 2_2], [2]$	$m \geq 0$ $m = 2l + e - 1$
$\star C_4$	$(k,7)$	$[6_m, 4, 2_2], [6_n, 3_2], [2]$	$m, n \geq 0$ $m = 2l + h - 1$
$\star C_5$	$(k,9)$	$[8_m, 4_2, 2_2], [6, 3_2], [2]$	$m \geq 0$ $m = 2l + h - 1$

Table 3.10: Rational tricuspidal curves on $\mathbb{F}_e$ from special plane tricuspidal curves.

Proof. We construct the series C_1 from a plane rational cuspidal curve C of degree 6 with three cusps p_1, p_2 and p_3 with multiplicity sequences $[4]$, $[2_2]$ and $[2_2]$ respectively. Let T be the tangent line to C at p_2. Then $T \cdot C = 4p_2 + r + s$ for smooth points $r, s \in C$. Note that the line L between r and p_1 has the following intersection with C, $L \cdot C = 4p_1 + r + t$ for a smooth point $t \in C$. Blowing up at r, we get a curve C' on $\mathbb{F}_1$ of type $(1,5)$. The strict transforms T' and L' of T and L intersect the curve C' in ≤ 2 such that appropriate elementary transformations on the two fibers give the series on $\mathbb{F}_h$.

We construct the series C_2 from a plane rational cuspidal curve C of degree 6 with three cusps p_1, p_2 and p_3 with multiplicity sequences $[4]$, $[2_3]$ and $[2]$ respectively. Let T be the tangent line to C at p_2. Then $T \cdot C = 4p_2 + r + s$ for smooth points $r, s \in C$. Note that the line L between r and p_1 has the following intersection with C, $L \cdot C = 4p_1 + r + t$ for a smooth point $t \in C$. Blowing up at r, we get a curve C' on $\mathbb{F}_1$ of type $(1,5)$. The strict transforms T' and L' of T and L intersect the curve C' such that appropriate elementary transformations on the two fibers give the series on $\mathbb{F}_h$.

We construct the series C_3 from a plane rational cuspidal curve C of degree 7 with three cusps p_1, p_2 and p_3 with multiplicity sequences $[4, 2_2]$, $[3_2]$ and $[2]$ respectively. Let T be the tangent line to C at p_2. Then $T \cdot C = 6p_2 + r$ for a smooth point $r \in C$. Blowing up at r, we get a curve C' on $\mathbb{F}_1$ of type $(1,6)$. The strict transform T' of T intersects the curve C' only at the cusp p_2'. Appropriate elementary transformations on the fiber T' give the series.

We construct the series C_4 from a plane rational cuspidal curve C of degree 7 with three cusps p_1, p_2 and p_3 with multiplicity sequences $[4, 2_2]$, $[3_2]$ and $[2]$ respectively. Let T_1 and T_2 be the tangent lines to C at p_1 and p_2. We have $T_1 \cdot C = 6p_1 + r_1$ and $T_2 \cdot C = 6p_2 + r_2$ for two smooth points $r_1, r_2 \in C$. Let s denote the intersection point of T_1 and T_2. Blowing up at s we get a curve C' on $\mathbb{F}_1$ of type $(0, 7)$. The fibers T_1' and T_2' intersect C' such that appropriate elementary transformations on these two fibers give the series on $\mathbb{F}_h$.

We construct the series C_5 from a plane rational cuspidal curve C of degree 10 with three cusps p_1, p_2 and p_3 with multiplicity sequences $[6, 3_2]$, $[4_2, 2_2]$ and $[2]$ respectively. Let T be the tangent line to C at p_2. Then $T \cdot C = 8p_2 + r + s$ for smooth points $r, s \in C$. Blowing up at r, we get a curve C' on $\mathbb{F}_1$ of type $(1, 9)$. The strict transform T' of T intersects C' in the following way, $T' \cdot C' = 8p_2' + s'$. Appropriate elementary transformations on the fiber T' give the series on $\mathbb{F}_h$. $\square$

3.4.3 Rational cuspidal curves from noncuspidal curves

In this section we show by an example that we can use a noncuspidal curve on the projective plane to construct a cuspidal curve on a Hirzebruch surface. This example supports the claim that we have more flexibility and more rational cuspidal curves on a Hirzebruch surface than on the projective plane. This approach could be interesting to explore in depth and detail, but we argue that it has some major downsides.

We begin with the announced example.

Example 3.4.17. Let C be a rational curve on $\mathbb{P}^2$ of degree 5 with four real cusps and two real nodes. Recall that a node is a singularity with two transversal branches. A parametrization of such a curve can be found in [20], where λ_1, λ_2 are arbitrary constants.

$$\begin{aligned}
x &= (t - \lambda_1)^2\big((\lambda_1 - 2\lambda_2 + 2\lambda_1\lambda_2)t - \lambda_1\lambda_2\big),\\
y &= \lambda_1^3(t-1)^2(t-\lambda_2),\\
z &= (t-1)^2(t-\lambda_1)^2\big((\lambda_1 - 2\lambda_2)t - \lambda_1\lambda_2\big).
\end{aligned}$$

Blowing up at one of the nodes on the curve on $\mathbb{P}^2$, we get a curve on $\mathbb{F}_1$ of type $(2, 3)$ with four cusps and one node. There is a fiber L on $\mathbb{F}_1$ intersecting this curve in the remaining node and one further point. In particular, the node is not on the special section of $\mathbb{F}_1$. Blowing up at the node and contracting L, we get a rational cuspidal curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3, 3)$ with four ordinary cusps.

The above example is the beginning of a third construction of the rational cuspidal curves with four cusps with multiplicity sequences $[2_{n_j}]$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$. Notice in particular that the four cusps have real coordinates because the transformation from $\mathbb{P}^2$ is real at all stages.

Moreover, Example 3.4.17 shows that we can get cuspidal curves on a Hirzebruch surface from a curve that is not cuspidal on the projective plane. The example even shows that we can start out with curves with *more than one* singularity that is not a cusp.

A natural place to start looking for cuspidal curves on Hirzebruch surfaces would be the curves on $\mathbb{P}^2$ with only one multiple point in addition to cusps. In particular, it should be possible to resolve the singularity in one blow-up. More precisely, we can handle multiple points with b branches, all of which are smooth or cusps, where the b tangent directions are distinct. We call such a singularity an *easily resolvable multiple point.*

Our main task would be to look for curves with many cusps. Among the plane rational curves of degree $d \leq 5$, there seems to be no curves with $s \geq 5$ cusps and only one easily resolvable multiple point. The first possible curve would be a quintic with five ordinary cusps and one node, but there is no such curve. Indeed, using Hurwitz's theorem 1.1.18 on the composition of the normalization map and a projection to $\mathbb{P}^1$ from the node, leads to a contradiction. Moving to degree $d = 6$, the situation becomes more complicated, since the singularities now can be more complex, and we have yet to find a good candidate in this degree or higher.

We could also start our search with rational curves on $\mathbb{P}^2$ with more multiple points, as in Example 3.4.17. Then there are even more restrictions that come into play, and the situation again runs out of hand.

The major downside of approaching rational cuspidal curves on $\mathbb{F}_e$ from noncuspidal curves on $\mathbb{P}^2$ is first of all the fact that we know very little about the rational curves on $\mathbb{P}^2$ of degrees $d \geq 6$. It seems that finding curves that fit even the simplest restrictions is hard, and more research is needed.

3.4.4 Reflexions and conjectures

So far in this chapter we have seen that we can construct many rational cuspidal curves on Hirzebruch surfaces from plane rational curves. We have in particular considered curves with four and three cusps. Constructing curves with fewer cusps is of course also possible. Reflecting on the structure of the birational maps between the surfaces, we argue that it is natural that there are more cuspidal curves on the Hirzebruch surfaces than on the projective plane. However, the observation that there seems to be no rational cuspidal curve with more than four cusps on any Hirzebruch surface either, still suggests that the rational cuspidal curves are very special.

It is very easy to construct examples of curves with two cusps or less. Indeed, inspired by Fenske in [17] we can start out with simple plane rational cuspidal curves and construct several series of curves on Hirzebruch surfaces. We will not go into details here, only briefly describe the curves.

The first plane curves that easily give a lot of curves are the binomial cuspidal curves. For every pair m, n, where $n > m$ and with $\gcd(m, n) = 1$, let C be the plane cuspidal curve $C = \mathscr{V}(y^m z^{n-m} - x^n)$. By symmetry, we may assume that $n - m > m$. If $m = 1$, then C is a unicuspidal curve with an inflection point. The multiplicity sequence of the cusp is $[n-1]$. The intersection multiplicity of the tangent line and the curve equals n at both the cusp and the inflection point. If $m > 1$, then the curve is bicuspidal. The two cusps have multiplicities $n - m$ and m, and their multiplicity sequences can fairly easily be calculated using the Enriques–Chisini algorithm, see [4, Theorem 12, p.516]. The intersection multiplicity of the tangent line and the curve equals n at both cusps. The delta invariants are $\frac{(n-m-1)(n-1)}{2}$ and $\frac{(m-1)(n-1)}{2}$.

The second plane curves that easily give a lot of rational cuspidal curves on the Hirzebruch surfaces are the curves given by $C = \mathscr{V}(yz^{n-1} - x^n - x^{n-1}z)$. These curves are unicuspidal, with one cusp with multiplicity sequence $[n-1]$ and one inflection point. The intersection multiplicity of the tangent line and the curve equals n at the cusp and $n-1$ at the inflection point. Blowing up appropriate points on these curves gives curves on $\mathbb{F}_1$ with fibers intersecting the curves in such a way that cuspidal curves can be constructed. As above, the key to the constructions is finding the lines that intersect the curve in few points, and general lines through the cusps and tangent lines at cusps and inflection points are examples here.

The constructions of curves in this chapter additionally lead to the claim that the surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ allow more rational cuspidal curves than $\mathbb{P}^2$ and the other Hirzebruch surfaces. This claim is actually plausible just from looking at the birational transformations from one surface to the other, without the vast amount of curves provided in this chapter.

Starting out with a curve on $\mathbb{P}^2$, moving to $\mathbb{F}_1$ only requires blowing up a single point. This is extremely flexible since we do not contract any lines.

Moving from $\mathbb{F}_1$ to $\mathbb{P}^1 \times \mathbb{P}^1$ requires one blowing up and one contraction, where the point we blow up is general on $\mathbb{F}_1$. This is still quite flexible, but in order to avoid constructing multiple points we need a fiber with at most two intersections with our curve. If there are two intersections between the curve and the fiber, then at least one intersection must be transversal for a general point.

Moving from $\mathbb{F}_1$ to $\mathbb{F}_2$ also requires one blowing up and one contraction, but this time the point we blow up is special on $\mathbb{F}_1$. This is less flexible than the elementary transformation from $\mathbb{F}_1$ to $\mathbb{P}^1 \times \mathbb{P}^1$. To avoid constructing multiple points we again need a fiber with at most two intersections with our curve. If there are two intersections between the curve and the fiber, then one of them must be transversal and on the special section.

To get a series of cuspidal curves on $\mathbb{F}_e$ we see that we must have a fiber that intersects the curve in only one point.

If we want to get back to $\mathbb{P}^2$, we must perform several elementary transformations and one contraction of the special section on $\mathbb{F}_1$, and in each step we have to avoid creating multiple points. This turns out to be very difficult, and accounts perhaps for the rigidity we see in the curves on $\mathbb{P}^2$.

The fact that cuspidal curves can be constructed on Hirzebruch surfaces from non-cuspidal plane curves complicates the picture further. We have already noted that more research is needed, and although we do not go into this topic, as a first observation we say that a plane curve that can be transformed into a cuspidal curve on a Hirzebruch surface is a *semi-cuspidal curve.*

Regardless of the theorems, the curves and the above discussion, it is important to notice that we have *not* found any rational cuspidal curves with more than four cusps. This observation leads to a natural conjecture.

Conjecture 3.4.18. *Let C be a rational cuspidal curve on the projective plane or on a Hirzebruch surface. Then C has at most four cusps.*

Another observation is the fact that the curves of type (a, b) with four cusps all have $b \leq 5$, and by Theorem 3.1.5 the cusps then have multiplicity $m \leq 5$. In fact, the cusps in the series we have found all have multiplicity $m \leq 4$. These observations might be interesting points of attack for a future proof of the above conjecture.

3.5 More results for rational cuspidal curves

First in this section we state and prove two lemmas for rational cuspidal curves on Hirzebruch surfaces, the first analogous to Lemma 2.4.3 by Flenner and Zaidenberg, and the other a lemma bounding the sum of the M-numbers. Second, we use these lemmas to give an explicit formula for $\chi(\Theta_V \langle D \rangle)$ in this case. Third, we use the lemmas and some other results to find a lower bound on the highest multiplicity of a cusp on a rational cuspidal curve on a Hirzebruch surface. Last in this section, we investigate real cuspidal curves on Hirzebruch surfaces.

3.5.1 Two lemmas

We now state and prove two lemmas for rational cuspidal curves on Hirzebruch surfaces. First, we prove a lemma that is a variant of Lemma 2.4.3.

Lemma 3.5.1 (Variant of 2.4.3). *Let C be a rational cuspidal curve on $\mathbb{F}_e$. Let (V, D) be the minimal embedded resolution of C. Let $D_1, \ldots, D_r$ be the irreducible components of D. Then the following hold.*

(0) *D is a rational tree.*

(1) $\chi(\Theta_V) = 8 - 2r$.

(2) $K_V^2 = 9 - r$.

(3) $c_2 := c_2(V) = 3 + r$.

(4) $\chi\left(\bigoplus \mathscr{N}_{D_i/V}\right) = r + \sum_{i=1}^{r} D_i^2$.

(5) $\chi(\Theta_V \langle D \rangle) = K_V \,.\, (K_V + D) - 1$.

Proof. Note that the proof is very similar to the proof of Lemma 2.4.3 given in [21], only small details are changed.

(0) Since (V, D) is the minimal embedded resolution of C, D is an SNC-divisor. Since C is a rational curve, $\tilde{C}$ is smooth, and all exceptional divisors are smooth and rational, then all components of D are smooth and rational. The dual graph of D, say Γ, is necessarily a connected graph, and since C is cuspidal, Γ will not contain cycles. Thus, D is a rational tree by the definition given in Chapter 1.

(3) Since V is obtained by $r-1$ blowing ups, we have that the Chern class $c_2 := c_2(V)$ is

$$c_2(V) = c_2(\mathbb{F}_e) + r - 1.$$

Moreover, by [31, Corollary V 2.5, p.371] we have that $\chi(\mathscr{O}_{\mathbb{F}_e}) = 1$. With K the canonical divisor on $\mathbb{F}_e$, we apply the formula in [31, Remark V 1.6.1, p.363],

$$\begin{aligned} 12\chi(\mathscr{O}_{\mathbb{F}_e}) &= K^2 + c_2(\mathbb{F}_e) \\ 12 &= 8 + c_2(\mathbb{F}_e). \end{aligned}$$

We get $c_2(\mathbb{F}_e) = 4$, hence,

$$\begin{aligned} c_2(V) &= 4 + r - 1 \\ &= 3 + r. \end{aligned}$$

(2) We have by [31, Proposition V 3.4, p.387] that $\chi(\mathscr{O}_V) = \chi(\mathscr{O}_{\mathbb{F}_e}) = 1$. By the formula in [31, Remark V 1.6.1, p.363] again, we get

$$\begin{aligned} K_V^2 &= 12\chi(\mathscr{O}_V) - c_2 \\ &= 12 - (3 + r) \\ &= 9 - r. \end{aligned}$$

(4) Since D_i is a rational curve on the surface V for all i, we have that $g(D_i) = 0$. By [31, Proof of Proposition II 8.20, p.182] we have that

$$\mathscr{N}_{D_i/V} \cong \mathscr{L}(D_i) \otimes \mathscr{O}_{D_i}.$$

Hence, by the Riemann–Roch theorem for curves [31, p.362],

$$\begin{aligned} \chi\left(\bigoplus \mathscr{N}_{D_i/V}\right) &= \chi\left(\bigoplus \mathscr{L}(D_i) \otimes \mathscr{O}_{D_i}\right) \\ &= \sum_{i=1}^{r} \left(D_i^2 + 1\right) \\ &= r + \sum_{i=1}^{r} D_i^2. \end{aligned}$$

(1) By the Hirzebruch–Riemann–Roch theorem for surfaces [31, Theorem A 4.1, p.432], we have that for any locally free sheaf $\mathscr{E}$ on V of rank s with Chern classes $c_i(\mathscr{E})$,

$$\chi(\mathscr{E}) = \frac{1}{2} c_1(\mathscr{E}) \,.\, \big(c_1(\mathscr{E}) + c_1(\Theta_V)\big) - c_2(\mathscr{E}) + s \cdot \chi\left(\mathscr{O}_V\right).$$

Moreover, by [31, Example A 4.1.2, p.433], Θ_V has rank $s = 2$ and $c_1(\Theta_V) = -K_V$. We have by the previous results,

$$\begin{aligned} \chi(\Theta_V) &= \frac{1}{2}(-K_V) \,.\, (-2K_V) - c_2(\Theta_V) + 2\chi\left(\mathscr{O}_V\right) \\ &= K_V^2 - c_2 + 2 \\ &= 9 - r - (3 + r) + 2 \\ &= 8 - 2r. \end{aligned}$$

(5) Observe first that since D is an SNC-divisor, we have by direct calculation

$$\begin{aligned} D^2 &= \sum_{i=1}^{r} D_i^2 + \sum_{i \neq j} D_i D_j \\ &= \sum_{i=1}^{r} D_i^2 + (1 + 2(r-2) + 1) \\ &= \sum_{i=1}^{r} D_i^2 + 2r - 2. \end{aligned}$$

Since D is an effective divisor, we have by definition that $p_a(D) = 1 - \chi(\mathscr{O}_D)$. Since D is a rational tree, by [21, Lemma 1.2, p.148], $p_a(D) = 0$. So by Theorem 1.1.3,

$$K_V \,.\, D = -D^2 - 2.$$

Using the additivity of χ on the short exact sequence (see [21, pp.147,162]),

$$0 \longrightarrow \Theta_V \langle D \rangle \longrightarrow \Theta_V \longrightarrow \bigoplus \mathscr{N}_{D_i/V} \longrightarrow 0,$$

and the above results and remarks, we get

$$\begin{aligned}
\chi\Big(\Theta_V \langle D \rangle\Big) &= \chi(\Theta_V) - \chi\left(\bigoplus \mathscr{N}_{D_i/V}\right) \\
&= (8 - 2r) - \left(r + \sum_{i=1}^{r} D_i^2\right) \\
&= 8 - 2r - (r + D^2 - 2r + 2) \\
&= 6 - r - D^2 \\
&= K_V^2 - D^2 - 3 \\
&= K_V^2 + 2K_V \,.\, D - 2(-D^2 - 2) - D^2 - 3 \\
&= (K_V + D)^2 + 1 \\
&= K_V \,.\, (K_V + D) - 1.
\end{aligned}$$

□

The second lemma bounds the sum of the M-numbers by the type of the curve, and this work is inspired by Orevkov (see [50]).

Lemma 3.5.2. *For a rational cuspidal curve C of type (a, b) on $\mathbb{F}_e$ with s cusps and $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$, we have*

$$\sum_{j=1}^{s} M_j \leq 2(a + b) + be.$$

Proof. Let (V, D) and $\sigma = \sigma_1 \circ \ldots \circ \sigma_t$ be the minimal embedded resolution of C. Write $\sigma^*(C) = \tilde{C} + \sum_{i=1}^{t} m_{i-1}E_i$, with $\tilde{C}$, m_i and E_i as before. Then by induction and [31, Proposition V 3.2, p.387] we find that

$$\begin{aligned}
\tilde{C}^2 &= \left(\sigma^*(C) - \sum_{i=1}^{t} m_{i-1}E_i\right)^2 \\
&= C^2 - \sum_{i=0}^{t-1} m_i^2 \\
&= b^2 e + 2ab - \sum_{i=0}^{t-1} m_i^2.
\end{aligned}$$

By the genus formula, we may rewrite this,

$$\begin{aligned}\tilde{C}^2 &= b^2e + 2ab - \sum_{i=0}^{t-1} m_i^2 \\ &= be + 2(a+b) - 2 - \sum_{i=0}^{t-1} m_i.\end{aligned}$$

Moreover, we have that for $D = \tilde{C} + \sum_{i=1}^{t} E_i'$,

$$\begin{aligned}D^2 &= \tilde{C}^2 + 2\tilde{C}\,.\Big(\sum_{i=1}^{t} E_i'\Big) + \Big(\sum_{i=1}^{t} E_i'\Big)^2 \\ &= \tilde{C}^2 + 2s + \Big(\sum_{i=1}^{t} E_i'\Big)^2.\end{aligned}$$

Now we split the latter term in this sum into the sum of the strict transforms of the exceptional divisors for each cusp,

$$\sum_{i=1}^{t} E_i' = \sum_{j=1}^{s} E_{p_j},$$

where s denotes the number of cusps. By [42, Lemma 2, p.235], we still have

$$\omega_j = -E_{p_j}^2 - 1.$$

Combining the above results, we get

$$\begin{aligned}D^2 &= be + 2(a+b) - 2 - \sum_{i=0}^{t-1} m_i + 2s - \sum_{j=1}^{s}(\omega_j + 1) \\ &= be + 2(a+b) - 2 - \sum_{i=0}^{t-1} m_i - \sum_{j=1}^{s}(\omega_j - 1).\end{aligned}$$

By the proof of Lemma 3.5.1, we have

$$6 - r - D^2 = (K_V + D)^2 + 1.$$

Note that r denotes the number of components of the divisor D. This number is equal to the total number of blowing ups needed to resolve the singularity, plus one component from the strict transform of the curve itself. Following the notation established, we have $r = t + 1$. Moreover, by assumption, $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$. By Corollary 3.1.10, we then have that

$$(K_V + D)^2 \leq 6.$$

So we get

$$\begin{aligned}
0 &\leq 1 + r + D^2 \\
&\leq 1 + r + be + 2(a+b) - 2 - \sum_{i=0}^{t-1} m_i - \sum_{j=1}^{s}(\omega_j - 1) \\
&\leq -1 + 1 + be + 2(a+b) - \sum_{i=0}^{t-1}(m_i - 1) - \sum_{j=1}^{s}(\omega_j - 1) \\
&\leq be + 2(a+b) - \sum_{j=1}^{s} M_j.
\end{aligned}$$

Hence,

$$\sum_{j=1}^{s} M_j \leq 2(a+b) + be.$$

□

3.5.2 An expression for $\chi\left(\Theta_V \langle D \rangle\right)$

In this section we show that the value of $\chi(\Theta_V \langle D \rangle)$ for curves C on $\mathbb{F}_e$ is more subtle than in the case of curves on $\mathbb{P}^2$, even if we require $\overline{\kappa}(\mathbb{F}_e \setminus C) = 2$. We first provide an explicit formula for $\chi\left(\Theta_V \langle D \rangle\right)$.

Theorem 3.5.3. *For an irreducible rational cuspidal curve C of type (a,b) on $\mathbb{F}_e$ with s cusps p_j with respective M-numbers M_j, we have*

$$\chi\Big(\Theta_V \langle D \rangle\Big) = 7 - 2a - 2b - be + \sum_{j=1}^{s} M_j.$$

Proof. By Proposition 1.1.19,

$$K_V \,.\, (K_V + D) = K_{\mathbb{F}_e} \,.\, (K_{\mathbb{F}_e} + C) + \sum_{j=1}^{s} M_j.$$

By Lemma 3.5.1, we then get

$$\begin{aligned}
\chi\Big(\Theta_V \langle D \rangle\Big) &= K_V \,.\, (K_V + D) - 1 \\
&= K_{\mathbb{F}_e} \,.\, (K_{\mathbb{F}_e} + C) + \sum_{j=1}^{s} M_j - 1 \\
&= ((e-2)L - 2M) \,.\, \Big((a+e-2)L + (b-2)M\Big) - 1 + \sum_{j=1}^{s} M_j \\
&= 7 - 2a - 2b - be + \sum_{j=1}^{s} M_j.
\end{aligned}$$

□

With the above result in mind, we investigate $\chi(\Theta_V \langle D \rangle)$ further. Let C be a rational cuspidal curve on $\mathbb{F}_e$ of type (a, b), and let (V, D) be as before. By the above, we have that

$$\begin{aligned}\chi(\Theta_V \langle D \rangle) &:= \mathrm{h}^0(V, \Theta_V \langle D \rangle) - \mathrm{h}^1(V, \Theta_V \langle D \rangle) + h^2(V, \Theta_V \langle D \rangle),\\ &= K_V \,.\, (K_V + D) - 1,\\ &= 7 - 2(a+b) - be + \sum_{j=1}^{s} M_j.\end{aligned}$$

Moreover, when $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$, we see from Lemma 3.5.2 that

$$\chi(\Theta_V \langle D \rangle) \leq 7.$$

If $\overline{\kappa}(\mathbb{F}_e \setminus C) = 2$, then it follows from a result by Iitaka [33, Theorem 6] that $\mathrm{h}^0(V, \Theta_V \langle D \rangle) = 0$. Then

$$\chi(\Theta_V \langle D \rangle) = h^2(V, \Theta_V \langle D \rangle) - \mathrm{h}^1(V, \Theta_V \langle D \rangle).$$

If $\overline{\kappa}(\mathbb{F}_e \setminus C) = 2$ *and* (V, D) is almost minimal, we can apply a lemma by Tono [68, Lemma 4.1, p.219]. In this case $K_V \,.\, (K_V + D) \geq 0$, hence

$$-1 \leq \chi(\Theta_V \langle D \rangle) \leq 7.$$

Going back to the proof of Theorem 3.3.2, we see that for curves of genus g on $\mathbb{F}_e$, we have that $n < 2g + 2$. As before, n is the number of exceptional curves not in D that will be contracted by the minimalization morphism.

For rational cuspidal curves, we see that we have $n = \{0, 1\}$. This means that we, in contrast to the situation on $\mathbb{P}^2$, are not directly in the situation that the resolution of a rational curve gives an almost minimal pair (V, D). Therefore, $\chi(\Theta_V \langle D \rangle)$ is not necessarily bounded below in this case. For rational cuspidal curves with four cusps on Hirzebruch surfaces $\mathbb{F}_e$ and $\mathbb{F}_h$, where $e \geq 0$, $h \in \{0, 1\}$, we have the following table.

Type	**Cuspidal configuration**	$\chi(\mathbf{\Theta_V \langle D \rangle})$	**Surface**
$(2k+1, 4)$	$[4_{k-1+e}, 2_3], [2], [2], [2]$	$1-k-e$	$\mathbb{F}_e$
$(3k+1-h, 5)$	$[4_{2k-1+h}, 2_3], [2], [2], [2]$	-1	$\mathbb{F}_h$
$(2k+2-h, 4)$	$[3_{2k-1+h}, 2], [2_3], [2], [2]$	0	$\mathbb{F}_h$
$(k+1-h, 3)$	$[2_{n_1}], [2_{n_2}], [2_{n_3}], [2_{n_4}]$	-1	$\mathbb{F}_h$
$(0, 3)$	$[2], [2], [2], [2]$	-1	$\mathbb{F}_2$

Table 3.11: $\chi(\Theta_V \langle D \rangle)$ for rational cuspidal curves with four cusps on $\mathbb{F}_e$ and $\mathbb{F}_h$. For the three first series, $k \geq 0$. For the fourth series, $k \geq 2$ and $\sum_{j=1}^{4} n_j = 2k + h$.

An important observation from this list is the fact that $\chi(\Theta_V \langle D \rangle) \leq 0$ for all these curves. We reformulate this observation in a conjecture.

Conjecture 3.5.4. *Let C be a rational cuspidal curve on $\mathbb{F}_e$ with four or more cusps. Then $\chi(\Theta_V \langle D \rangle) \leq 0$.*

3.5.3 On the multiplicity

In the following we establish a result on the multiplicities of the cusps on a rational cuspidal curve on a Hirzebruch surface. Note that this work is inspired by Orevkov (see [50]).

Assume that C is a rational cuspidal curve on a Hirzebruch surface $\mathbb{F}_e$. Let $p_1, \ldots, p_s$ denote the cusps of C, and $m_{p_1}, \ldots, m_{p_s}$ their multiplicities. Renumber the cusps such that $m_{p_1} \geq m_{p_2} \geq \ldots \geq m_{p_s}$. Then for curves with $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$ we are able to establish a lower bound on m_{p_1}.

Theorem 3.5.5. *A rational cuspidal curve C on $\mathbb{F}_e$ of type (a,b) with $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$ and s cusps must have at least one cusp p_1 with multiplicity $m := m_{p_1}$ that satisfies the below inequality,*

$$m > \frac{3}{2} + a + b - \frac{1}{2}\sqrt{1 + 20(a+b) + 4(a^2+b^2) + 4be(1-b)}.$$

Proof. Using Lemma 3.5.2 and Lemma 1.1.16, we get

$$\begin{aligned}
2(a+b) + be &\geq \sum_{j=1}^{s} M_j \\
&\geq M_1 + \sum_{j=2}^{s} M_j \\
&> \frac{\mu_1}{m} + m - 3 + \sum_{j=2}^{s} M_j \\
&= \frac{(b-1)(2a-2+be)}{m} + m - 3 + \sum_{j=2}^{s}\Big(M_j - \frac{\mu_j}{m}\Big) \\
&\geq \frac{2ab - 2(a+b) + 2 + b^2e - be}{m} + m - 3 + \sum_{j=2}^{s}\Big(M_j - \frac{\mu_j}{m_j}\Big) \\
&\geq \frac{2ab - 2(a+b) + 2 + b^2e - be}{m} + m - 3.
\end{aligned}$$

This means that

$$0 > \frac{2ab - 2(a+b) + 2 + b^2e - be}{m} + m - 3 - 2(a+b) - be.$$

Let

$$g(a,b,m) = \frac{2ab - 2(a+b) + 2 + b^2e - be}{m} + m - 3 - 2(a+b) - be.$$

Factoring g, we have that $g < 0$ for

$$m > \frac{3}{2} + a + b - \frac{1}{2}\sqrt{1 + 20(a+b) + 4(a^2+b^2) + 4be(1-b)}.$$

□

Corollary 3.5.6. *A rational cuspidal curve C on $\mathbb{F}_e$ of type (a,b) with two or more cusps must have at least one cusp p_1 with multiplicity m that satisfies the below inequality,*

$$m > \frac{3}{2} + a + b - \frac{1}{2}\sqrt{1 + 20(a+b) + 4(a^2+b^2) + 4be(1-b)}.$$

Note that this theorem will exclude some potential curves. For example, a rational cuspidal curve of type $(a,4)$ on $\mathbb{F}_1$ with two or more cusps must have at least one cusp with multiplicity $m = 3$ for any $a \geq 6$. We also have that any rational cuspidal curve of type $(a,5)$ on $\mathbb{F}_1$ with two or more cusps must have at least one cusp with multiplicity $m = 3$ for any $a \geq 3$. Similarly, any rational cuspidal curve of type (a,b) on $\mathbb{F}_1$ with two or more cusps and $b \geq 6$ must have at least one cusp with multiplicity $m = 3$.

3.5.4 Real cuspidal curves

We observed in Chapter 2 that the known plane rational cuspidal curves with three cusps could be defined over $\mathbb{R}$. That was not the case for the plane rational cuspidal quintic curve with cuspidal configuration $[2_3], [2], [2], [2]$.

On the Hirzebruch surfaces, the question whether all cusps on real cuspidal curves can have real coordinates is still hard to answer. Recall that we call $C = \mathscr{V}(F)$ a real curve if the polynomial F has real coefficients. However, all known curves on $\mathbb{F}_e$ can be constructed from curves on $\mathbb{P}^2$. Since the birational links are real transformations, if it is possible to arrange the curve on $\mathbb{P}^2$ such that the preimages of the cusps have real coordinates, then the cusps will have real coordinates on the curve on the Hirzebruch surface as well. Note the possibility that this arrangement is not always attainable.

Considering the rational cuspidal curves on $\mathbb{F}_e$ with four cusps, we see that most of them are constructed from the plane rational cuspidal quintic with cuspidal configuration $[2_3], [2], [2], [2]$. Hence, we expect that the cusps on these curves can not all have real coordinates when the curve is real. Contrary to this intuition, however there are examples of fourcuspidal curves with this property.

Proposition 3.5.7. *The series of rational cuspidal curves on $\mathbb{F}_h$ of type $(k+1-h, 3)$, $k \geq 2$, with four cusps and cuspidal configuration $[2_{n_1}], [2_{n_2}], [2_{n_3}], [2_{n_4}]$, where the indices satisfy $\sum_{j=1}^{4} n_j = 2k + h$, has the property that all cusps can be given real coordinates on a real curve.*

Proof. We have seen that the series of curves can be constructed using the plane rational cuspidal quartic C with three cusps. Let

$$y^2z^2 + x^2z^2 + x^2y^2 - 2xyz(x+y+z)$$

be a real defining polynomial of C. Then it is possible to find a tangent line to C that intersects C in three real points. For example, choose the line T defined by

$$\frac{2048}{125}x + \frac{2048}{27}y - \frac{1048576}{3375}z = 0.$$

This line is tangent to C at the point $(\frac{64}{9} : \frac{64}{25} : 1)$, and it intersects C transversally at the points $(16 : \frac{16}{25} : 1)$ and $(\frac{4}{9} : 4 : 1)$. With this configuration, there exists a birational transformation from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ that preserves the real coordinates of the cusps on C and constructs a fourth cusp with real coordinates on the strict transform of C on $\mathbb{P}^1 \times \mathbb{P}^1$. We blow up the two real points at the transversal intersections and contract the tangent line T, using the birational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ described in Chapter 1. The strict transform is a real curve C' on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3,3)$ and cuspidal configuration $[2],[2],[2],[2]$, and all the cusps have real coordinates.

On $\mathbb{P}^1 \times \mathbb{P}^1$, since the cusps p_j have real coordinates, a fiber, say L_j, intersecting C' at a cusp is real. Using the defining polynomial of L_j to substitute one of the variables x_0 or x_1 in the defining polynomial of C and removing the factor of x_i^3, we are left with a polynomial with real coefficients in y_0 and y_1 of degree 3. This polynomial has a double real root, and one simple, hence real, root. The double root corresponds to the y-coordinates of the cusp p_j, and the simple root to the y-coordinates of a smooth intersection point r_j of C and L_j. Successively blowing up at any r_j and contracting the corresponding L_j lead to the desired series of curves. Since the points we blow up have real coordinates, the transformations preserve the real coordinates of the cusps. Hence, all the curves in the series can have four cusps with real coordinates. □

An image of a real rational cuspidal curve of type $(3,3)$ with four ordinary cusps on $\mathbb{P}^1 \times \mathbb{P}^1$ is given in Figure 3.2. In the figure, the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^3$ using the Segre embedding, and we have chosen a suitable affine covering of $\mathbb{P}^3$. The image is created by Georg Muntingh with `surfex` [32].

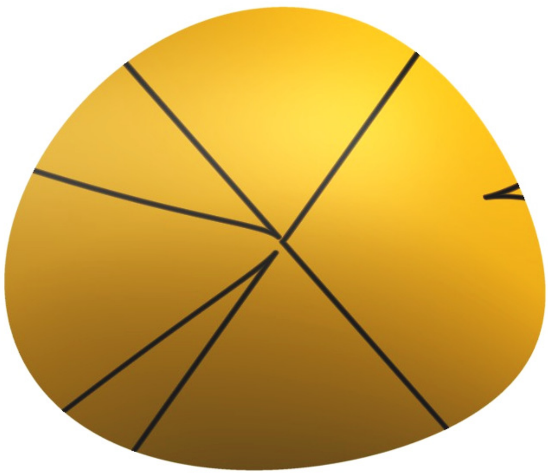


Figure 3.2: A real rational cuspidal curve of type $(3,3)$ with four ordinary cusps on $\mathbb{P}^1 \times \mathbb{P}^1$.

Chapter 4

The special case of $\mathbb{P}^1 \times \mathbb{P}^1$

In this Chapter we explore the nature of cuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ in more detail. We use properties of the defining polynomial of a curve and the structure of $\mathbb{P}^1 \times \mathbb{P}^1$ to give bounds on the multiplicity and multiplicity sequences of a cusp. We additionally give more examples of rational cuspidal curves.

4.1 Bounds on the multiplicity

The defining polynomial of a curve and the structure of $\mathbb{P}^1 \times \mathbb{P}^1$ give results about the possible cuspidal curves on this surface. In this section we first consider the defining polynomial of a curve and show that this ultimately gives bounds on the multiplicity and multiplicity sequence of a cusp. Then we look at the $(1,1)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and get general results from these curves reflecting the structure of $\mathbb{P}^1 \times \mathbb{P}^1$. Some of these results also give direct bounds on the multiplicity and the multiplicity sequence of a cusp on a curve.

4.1.1 From the defining polynomial

Since $\mathbb{P}^1 \times \mathbb{P}^1$ has a double ruling and no special section, the defining polynomial of a curve $C = \mathscr{V}(F)$ may be described in more detail. We recall the construction from Chapter 3 and derive special results in the case of curves on $\mathbb{P}^1 \times \mathbb{P}^1$.

The freedom of the change of coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ ensures that we may move any point p to $(0:1;0:1)$. In the affine neighbourhood $\mathscr{D}_+(x_1y_1)$, we now keep (x_0, y_0) as coordinates, and consider the defining polynomial $f(x_0, y_0) = F(x_0, 1, y_0, 1)$. Splitting $f = f(x_0, y_0)$ into its homogeneous terms, we have

$$f = f_0 + \ldots + f_i + \ldots + f_{a+b},$$

where $f_i = f_i(x_0, y_0)$ denotes the terms of $f(x_0, y_0)$ of degree i in x_0 and y_0. Note that there may exist a k such that $f_i = 0$ for all $k < i \leq a+b$. Recall that this is not the case for curves on $\mathbb{P}^2$. In this situation we may state and prove a small lemma.

Lemma 4.1.1. *Let $C = \mathscr{V}(F)$ be a reduced and irreducible curve of type (a,b) on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $p = (0:1,0:1) \in C$, and let $f(x_0, y_0) = F(x_0, 1, y_0, 1)$. Then the following hold.*

a) *If $a + b \geq 2$, then $f_i \neq 0$ for at least two i.*

b) *The polynomial f has to contain at least one monomial with factor x_0^a and at least one monomial with factor y_0^b.*

c) *The integer k above must have the property that $k \geq \max\{a, b\}$.*

Proof.

a) Assume for contradiction that $f = f_i$ for some i. Then obviously the polynomial F would be reducible.

b) Assume for contradiction that f does not contain a monomial with factor x_0^a. Bi-homogenizing f to F with x_1, y_1, we observe that all monomials of F must contain at least one factor of x_1, hence it would be reducible.

c) This follows from *b*).

□

Recall that the smallest integer m such that $f_m \neq 0$ in the above decomposition of f gives the multiplicity of p, and that f_m gives the local tangent line(s) of C at p. The notion of a global tangent line on $\mathbb{P}^1 \times \mathbb{P}^1$ is only defined in the particular case that the local tangent line is a fiber.

We have the following theorem about the multiplicity m of a cusp p.

Theorem 4.1.2. *For the multiplicity m of a cusp p on an irreducible cuspidal curve of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1$ we have*

$$m \leq \min\{a, b\}.$$

We can prove this result using the defining polynomial directly.

Proof. Choose an open affine neighbourhood of a cusp $p = (0, 0)$ with multiplicity m, and let (x_0, y_0) denote local coordinates. As above, we have $f = f_m + \ldots + f_{a+b}$. Without loss of generality we may assume that $b \geq a$. Assume for contradiction that $m > a$. Since x_0 is a factor of all terms of f_m of power $\leq a$, y_0 must also be a factor of f_m. The same is true for all the other homogeneous terms of f since they have degrees $\geq m > a$. Hence, y_0 must be a factor of f, which is a contradiction to the irreducibility of C. □

Alternatively, we may prove the result using the notion of divisors.

Alternative proof of Theorem 4.1.2. Without loss of generality we let $a \leq b$. The point p determines uniquely a curve M of type $(0, 1)$. By intersection theory $M \,.\, C = a$. Hence, $m \leq (M \cdot C)_p \leq a$. □

We have two immediate corollaries.

Corollary 4.1.3. *For the multiplicities of two cusps p and q on an irreducible cuspidal curve of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1$ we have*

$$m_p + m_q \leq 2 \cdot \min\{a, b\} \leq a + b.$$

Corollary 4.1.4. *For the multiplicity sequence $\overline{m} = [m, m_1, \dots m_{t-1}]$ of a cusp on a curve of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1$, we have that*

$$m + m_1 \leq 2 \cdot \min\{a, b\} \leq a + b.$$

Proof. We always have $m \geq m_1$, so the result follows from Theorem 4.1.2. $\square$

4.1.2 From properties of $(1, 1)$-curves

In this section we take a closer look at the $(1, 1)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$ introduced in Chapter 1. In the investigation of these curves, we observe that we get some restrictions on the multiplicity sequence of a cusp on a curve on $\mathbb{P}^1 \times \mathbb{P}^1$.

We begin with a consequence of Theorem 1.2.4.

Theorem 4.1.5. *Let p, q and r be three distinct points on an irreducible cuspidal curve C of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1$. Then the multiplicities of the points satisfy the inequality,*

$$m_p + m_q + m_r \leq a + b.$$

Proof. The theorem follows from intersection theory and Theorem 1.2.4. We split the proof into four cases to give the nuance that we sometimes get even better estimates.

(1) The points p, q and r are on the same $(1, 0)$-curve or $(0, 1)$-curve. In this case, we have by intersection theory

$$m_p + m_q + m_r \leq b \text{ or } m_p + m_q + m_r \leq a.$$

In any case, we have

$$m_p + m_q + m_r \leq a + b.$$

(2) The points p and q are on the same $(1, 0)$-curve, and q and r are on the same $(0, 1)$-curve. In this case, we have

$$\begin{aligned} m_p + m_q &\leq b, \\ m_q + m_r &\leq a. \end{aligned}$$

Hence,

$$m_p + m_q + m_r \leq m_p + 2m_q + m_r \leq a + b.$$

(3) The points p and q are on the same $(1, 0)$-curve, and r is not on the same $(0, 1)$-curve as p or q. In this case, we have

$$m_p + m_q \leq b.$$

The point r is on a unique $(0, 1)$-curve, so $m_r \leq a$. Hence,

$$m_p + m_q + m_r \leq a + b.$$

(4) The points p, q and r are general. These three points define by Theorem 1.2.4 a unique curve Q_{pqr} of type $(1,1)$. Intersecting C and Q_{pqr}, we get

$$m_p + m_q + m_r \leq a + b.$$

□

The theorem has a fairly obsolete consequence, and we state and prove it for completion.

Corollary 4.1.6. *Let C be a cuspidal curve of type (a,b), where $a, b \geq 2$, on $\mathbb{P}^1 \times \mathbb{P}^1$ with at least two cusps p and q. Then*

$$m_p + m_q < a + b.$$

Recall that by Corollary 4.1.3 we always have $m_p + m_q \leq 2\min\{a, b\}$. This corollary provides a better bound only when $a = b$.

Proof. If p and q are on the same $(0,1)$- or $(1,0)$-curve, then by intersection theory we have either $m_p + m_q \leq b < a + b$ or $m_p + m_q \leq a < a + b$. So assume that p and q are not on the same $(0,1)$- or $(1,0)$-curve. Pick a general point r on C and let Q_{pqr} denote the unique $(1,1)$-curve passing through p, q and r. Then $m_r = 1$, and by Theorem 4.1.5 we have

$$m_p + m_q \leq a + b - 1 < a + b.$$

□

We now show that we can find $(1,1)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with certain tangential properties with respect to a curve. Since a point can be freely moved on $\mathbb{P}^1 \times \mathbb{P}^1$, we first show the theorems for the point $p = (0:1;0:1)$.

Theorem 4.1.7. *Let $p = (0:1;0:1)$ be a smooth point on a curve $C = \mathscr{V}(F)$ of type (a,b), $a, b > 0$, such that no fiber is tangent to C at p. Let k be as on p.103. Then the following hold.*

a) There exists a net of $(1,1)$-curves intersecting C at p. These curves are given by

$$c_{00}x_0y_0 + c_{01}x_0y_1 + c_{10}x_1y_0 = 0, \quad c_{00}, c_{01}, c_{10} \in \mathbb{C}.$$

b) There exists a pencil of $(1,1)$-curves T_p intersecting C at p non-transversally, that is $(T_p \cdot C)_p \geq 2$. These curves are given by

$$c_{00}x_0y_0 + F_{x_0}(p)x_0y_1 + F_{y_0}(p)x_1y_0 = 0, \quad c_{00} \in \mathbb{C}.$$

c) If $k \geq 3$, there exists a unique $(1,1)$-curve O_p such that $(O_p \cdot C)_p > 2$. This curve is given by

$$\left(F_{x_0y_0}(p) - \frac{F_{x_0x_0}(p)F_{y_0}(p)}{2F_{x_0}(p)} - \frac{F_{y_0y_0}(p)F_{x_0}(p)}{2F_{y_0}(p)}\right)x_0y_0 + F_{x_0}(p)x_0y_1 + F_{y_0}(p)x_1y_0 = 0.$$

Remark 4.1.8. Note that *a*) holds for any point $p \in C$.

Proof. Let

$$c_{00}x_0y_0 + c_{01}x_0y_1 + c_{10}x_1y_0 + c_{11}x_1y_1 = 0, \quad c_{ij} \in \mathbb{C}$$

denote a general $(1,1)$-curve.

a) Requiring that a general $(1,1)$-curve passes through p determines one coefficient, say $c_{11} = 0$, which is what we need to show.

b) If two curves have non-transversal intersection at a point, then locally the tangent lines must coincide. In an affine neighbourhood of p, F is given by

$$f(x_0, y_0) = H(2) + f_{x_0}(p)x_0 + f_{y_0}(p)y_0,$$

where $H(2)$ denotes the higher order terms of f. Locally, a general $(1,1)$-curve passing through p is given by the equation

$$c_{00}x_0y_0 + c_{01}x_0 + c_{10}y_0 = 0.$$

Demanding locally coinciding tangent lines determines two of the coefficients up to multiplication by an element of $\mathbb{C}^*$. Hence, we set

$$(c_{01} : c_{10}) = (f_{x_0}(p) : f_{y_0}(p)).$$

By choice of p, we have $f_{x_0}(p) = F_{x_0}(p)$ and $f_{y_0}(p) = F_{y_0}(p)$. Observe that since no fiber is tangent to C at p, we have $F_{x_0}(p), F_{y_0}(p) \neq 0$, and the curves are irreducible for all values of c_{00}. The conclusion follows.

c) Let $T = c_{00}x_0y_0 + F_{x_0}(p)x_0y_1 + F_{y_0}(p)x_1y_0$ be the polynomial defining the pencil of $(1,1)$-curves from (II), and let $t(x_0, y_0) = T(x_0, 1, y_0, 1)$. We will now perform a series of operations on the polynomials f and t to show that there exists a unique value of c_{00} such that $(O_p \cdot C)_p > 2$ for the curve O_p given by the vanishing of the corresponding polynomial in the pencil.

For two constants k_1 and k_2, we consider the polynomial $P = f - t - k_1y_0t - k_2x_0t$. This gives

$$\begin{aligned} P = H(3) &+ \big(\tfrac{1}{2}f_{x_0x_0}(p) - k_2f_{x_0}(p)\big)x_0^2 \\ &+ \big(f_{x_0y_0}(p) - k_1f_{x_0}(p) - k_2f_{y_0}(p) - c_{00}\big)x_0y_0 \\ &+ \big(\tfrac{1}{2}f_{y_0y_0}(p) - k_1f_{y_0}(p)\big)y_0^2. \end{aligned}$$

If the three coefficients in the above expression vanish, then $(T \cdot C)_p > 2$. This is the case whenever

$$\begin{aligned} k_1 &= \frac{f_{y_0y_0}(p)}{2f_{y_0}(p)}, \\ k_2 &= \frac{f_{x_0x_0}(p)}{2f_{x_0}(p)}, \\ c_{00} &= f_{x_0y_0}(p) - \frac{f_{y_0y_0}(p)f_{x_0}(p)}{2f_{y_0}(p)} - \frac{f_{x_0x_0}(p)f_{y_0}(p)}{2f_{x_0}(p)}. \end{aligned}$$

By choice of p, we have that $f_{x_0x_0}(p) = F_{x_0x_0}(p)$, $f_{x_0y_0}(p) = F_{x_0y_0}(p)$ and $f_{y_0y_0}(p) = F_{y_0y_0}(p)$. Hence, the unique curve O_p with the property that $(O_p \cdot C)_p > 2$ is given by

$$\left(F_{x_0y_0}(p) - \frac{F_{x_0x_0}(p)F_{y_0}(p)}{2F_{x_0}(p)} - \frac{F_{y_0y_0}(p)F_{x_0}(p)}{2F_{y_0}(p)}\right)x_0y_0 + F_{x_0}(p)x_0y_1 + F_{y_0}(p)x_1y_0 = 0.$$

□

Example 4.1.9. Take C to be the $(2,1)$-curve $\mathscr{V}(x_0x_1y_1 + x_1^2y_0 + x_0^2y_0)$. The point $p = (0:1;0:1)$ is smooth on C, and there is a pencil of curves

$$T_p = \mathscr{V}(c_{00}x_0y_0 + x_0y_1 + x_1y_0), \quad c_{00} \in \mathbb{C},$$

intersecting C at p with $(T_p \cdot C)_p \geq 2$. The unique curve O_p in this pencil intersecting C at p with $(O_p \cdot C)_p = 3 > 2$, is given by

$$O_p = \mathscr{V}(x_0y_1 + x_1y_0).$$

In Figure 4.1, the curve C is visualized in black on $\mathbb{P}^1 \times \mathbb{P}^1$ in a neighbourhood of p. Two curves in the pencil T_p, for $c_{00} \in \{-1, 1\}$, having the intersection $(T_p \cdot C)_p = 2$, are displayed in red, and the curve O_p is displayed in white. In the figure, the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^3$ using the Segre embedding, and we have chosen a suitable affine covering of $\mathbb{P}^3$. The image is made with `surfex` [32].

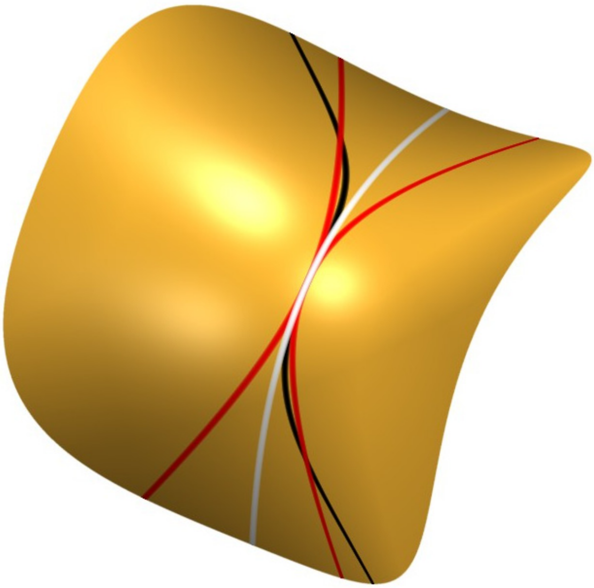


Figure 4.1: $(1,1)$-curves intersecting a curve at a smooth point non-transversally.

Corollary 4.1.10. *Let $p = (A : 1; B : 1)$ be a smooth point on a curve $C = \mathscr{V}(F)$ of type (a, b), $a, b > 0$, such that no fiber is tangent to C at p. Let k be as on p.103. Then the following hold.*

a) *There exists a net of $(1, 1)$-curves intersecting C at p.*

b) *There exists a pencil of $(1, 1)$-curves T_p intersecting C at p non-transversally, that is $(T_p \cdot C)_p \geq 2$. These curves are given by*

$$cx_0y_0 + (F_{x_0}(p) - Bc)x_0y_1 + (F_{y_0}(p) - Ac)x_1y_0 + (ABc + F_{x_1}(p) + F_{y_1}(p))x_1y_1 = 0, \quad c \in \mathbb{C}.$$

c) *If $k \geq 3$, there exists a unique $(1, 1)$-curve O_p such that $(O_p \cdot C)_p > 2$. This curve is given by*

$$\sum_{i,j=0}^{1} \Big(F_{x_iy_j}(p) - \frac{F_{x_ix_i}(p)F_{y_j}(p)}{2F_{x_i}(p)} - \frac{F_{y_jy_j}(p)F_{x_i}(p)}{2F_{y_j}(p)} \Big) x_iy_j = 0.$$

Proof.

a) We can determine one of the coefficients of the general $(1, 1)$-curve by requiring it to pass through p.

b) Abusing notation we move $p = (0 : 1; 0 : 1)$ to $p = (A : 1; B : 1)$ by substituting x_0 by $x_0 - Ax_1$ and y_0 by $y_0 - By_1$. Moreover, we observe that the partial derivatives with respect to x_0 and y_0 evaluated at the respective points do not change under this translation. In the new coordinates, we have by the Euler relations,

$$\begin{aligned} AF_{x_0}(p) + F_{x_1}(p) &= aF(p) = 0 \\ BF_{y_0}(p) + F_{y_1}(p) &= bF(p) = 0. \end{aligned}$$

Moving the curves T_p by the mentioned translation and using the Euler relations, we get the desired expression.

c) This follows using the same strategy as in case (II). Let the constant $c = c_{00}$ from case (II) be as in Theorem 4.1.7. Note that by differentiating the Euler relations, we have the following relations on the second order partial derivatives evaluated at $p = (A : 1; B : 1)$.

$$\begin{aligned} AF_{x_0x_0}(p) + F_{x_0x_1}(p) &= (a - 1)F_{x_0}(p) \\ AF_{x_0x_1}(p) + F_{x_1x_1}(p) &= (a - 1)F_{x_1}(p) \\ AF_{x_0y_0}(p) + F_{x_1y_0}(p) &= aF_{y_0}(p) \\ AF_{x_0y_1}(p) + F_{x_1y_1}(p) &= aF_{y_1}(p) \\ BF_{y_0y_0}(p) + F_{y_0y_1}(p) &= (b - 1)F_{y_0}(p) \\ BF_{y_0y_1}(p) + F_{y_1y_1}(p) &= (b - 1)F_{y_1}(p) \\ BF_{x_0y_0}(p) + F_{x_0y_1}(p) &= bF_{x_0}(p) \\ BF_{x_1y_0}(p) + F_{x_1y_1}(p) &= bF_{x_1}(p) \end{aligned}$$

Straightforward calculations give the desired symmetry in the defining equation.

□

The notion of $(1,1)$-curves also triggers the definition of the equivalent on $\mathbb{P}^1 \times \mathbb{P}^1$ of an inflection point on $\mathbb{P}^2$. We mention this only as a curiosity.

Definition 4.1.11. Let p be a smooth point on a curve $C = \mathscr{V}(F)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

a) The point p is called a *flex point of normal type* if it is a smooth point that is not fiber tangential and $(O_p \cdot C)_p > 3$.

b) The point p is called a *fiber tangential flex point* if it is a fiber tangential smooth point and $(T \cdot C)_p \geq 3$ for T the tangent $(1,0)$- or $(0,1)$-curve of C at p.

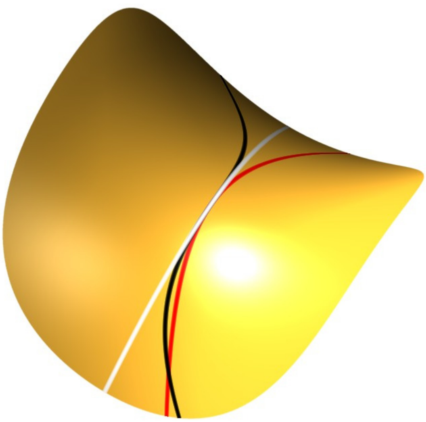
Example 4.1.12. In Figure 4.2 we show two images of curves on $\mathbb{P}^1 \times \mathbb{P}^1$ in a neighbourhood of the point $p = (0:1;0:1)$. The first curve is the same as in Example 4.1.9, that is,

$$C_a = \mathscr{V}(x_0x_1y_1 + x_1^2y_0 + x_0^2y_0).$$

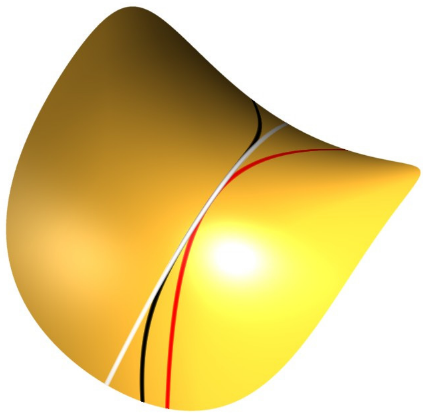
The second curve is given by

$$C_b = \mathscr{V}(x_0x_1y_1^3 + x_1^2y_0y_1^2 + x_0^2y_0^3).$$

As in Figure 4.1, the curves are visualized in black. The two curves have the same pencil of tangential $(1,1)$-curves at p, $T_p = \mathscr{V}(c_{00}x_0y_0 + x_0y_1 + x_1y_0)$, where $c_{00} \in \mathbb{C}$, and the same unique curve $O_p = \mathscr{V}(x_0y_1 + x_1y_0)$, such that $(O_p \cdot C)_p > 2$. For the first curve, we have $(O_p \cdot C_a)_p = 3$. For the second curve, we have $(O_p \cdot C_b)_p = 5$, hence p is a flex point of normal type on C_b. In the images in Figure 4.2, we display a curve in the pencil in red, here $T_p = \mathscr{V}(x_0y_0 + x_0y_1 + x_1y_0)$, and we display the curve $O_p = \mathscr{V}(x_0y_1 + x_1y_0)$ in white. In the images, the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^3$ using the Segre embedding, and we have chosen a suitable affine covering of $\mathbb{P}^3$. The images are made with `surfex` [32].



(a) Curve with a smooth point.



(b) Curve with a flex point of normal type.

Figure 4.2: $(1,1)$-curves intersecting a curve at a smooth point (a) and a flex point of normal type (b).

Theorem 4.1.13. *A smooth point $p = (0:1;0:1)$ on a curve $C = \mathscr{V}(F)$ is a flex point of normal type if $F_{x_0}(p), F_{y_0}(p) \neq 0$ and*

$$\frac{F_{x_0y_0^2}(p) - 2k_2c_{00} - 2k_4F_{y_0}(p)}{2F_{x_0}(p)} = \frac{F_{x_0^2y_0}(p) - 2k_1c_{00} - 2k_3F_{x_0}(p)}{2F_{y_0}(p)},$$

where

$$k_1 = \tfrac{F_{y_0y_0}(p)}{2F_{y_0}(p)}, \qquad k_2 = \tfrac{F_{x_0x_0}(p)}{2F_{x_0}(p)},$$
$$k_3 = \tfrac{F_{y_0^3}(p)}{6F_{y_0}(p)}, \qquad k_4 = \tfrac{F_{x_0^3}(p)}{6F_{x_0}(p)},$$
$$c_{00} = F_{x_0y_0}(p) - \frac{F_{y_0y_0}(p)F_{x_0}(p)}{2F_{y_0}(p)} - \frac{F_{x_0x_0}(p)F_{y_0}(p)}{2F_{x_0}(p)}.$$

Proof. Consider the local polynomials f and t of C and O_p. Then we investigate the polynomial $P = f - t - k_1ty_0 - k_2tx_0 - k_3ty_0^2 - k_4tx_0^2 - k_5tx_0y_0$. With k_1, k_2 and c_{00} as in the proof of Theorem 4.1.7, we get

$$\begin{aligned} P = H(4) &+ \big(\frac{1}{6}f_{x_0^3}(p) - k_4f_{x_0}(p)\big)x_0^3 \\ &+ \big(\frac{1}{2}f_{x_0^2y_0}(p) - k_2c_{00} - k_4f_{y_0}(p) - k_5f_{x_0}(p)\big)x_0^2y_0 \\ &+ \big(\frac{1}{2}f_{x_0y_0^2}(p) - k_1c_{00} - k_3f_{x_0}(p) - k_5f_{y_0}(p)\big)x_0y_0^2 \\ &+ \big(\frac{1}{6}f_{y_0^3}(p) - k_3f_{y_0}(p)\big)y_0^3. \end{aligned}$$

If the four coefficients in the above expression vanish, then $(T \cdot C)_p > 3$. This is the case whenever

$$c_{00} = f_{x_0y_0}(p) - \frac{f_{y_0y_0}(p)f_{x_0}(p)}{2f_{y_0}(p)} - \frac{f_{x_0x_0}(p)f_{y_0}(p)}{2f_{x_0}(p)},$$

$$k_1 = \tfrac{F_{y_0y_0}(p)}{2F_{y_0}(p)}, \qquad k_2 = \tfrac{F_{x_0x_0}(p)}{2F_{x_0}(p)},$$
$$k_3 = \tfrac{F_{y_0^3}(p)}{6F_{y_0}(p)}, \qquad k_4 = \tfrac{F_{x_0^3}(p)}{6F_{x_0}(p)},$$
$$k_5 = \frac{F_{x_0^2y_0}(p) - 2k_2c_{00} - 2k_4F_{y_0}(p)}{2F_{x_0}(p)} = \frac{F_{x_0y_0^2}(p) - 2k_1c_{00} - 2k_3F_{x_0}(p)}{2F_{y_0}(p)}.$$

Hence, the point p is a flex point whenever such a k_5 exists. □

Theorem 4.1.14. *Let p be a cusp of multiplicity m on a reduced and irreducible curve $C = \mathscr{V}(F)$ of type (a,b) on $\mathbb{P}^1 \times \mathbb{P}^1$ such that the fibers are not tangent to C at p. Then there exists a pencil of $(1,1)$-curves T_p such that $(T_p \cdot C)_p > m$.*

Proof. We move p to $(0:1;0:1)$ and observe that F has affine defining polynomial that can be written

$$f(x_0, y_0) = (d_{01}x_0 + d_{10}y_0)^m + H(m+1)$$

for some $d_{01}, d_{10} \in \mathbb{C}^*$. The $(1,1)$-curves intersecting C at p non-transversally, so that $(T_p \cdot C)_p > m$, must be on the form

$$dx_0y_0 + d_{01}x_0y_1 + d_{10}x_1y_0 = 0, \quad d \in \mathbb{C}.$$

□

Example 4.1.15. As an example, consider the $(3,2)$-curve

$$C = \mathscr{V}(x_0^2x_1y_1^2 + 2x_0x_1^2y_0y_1 + x_1^3y_0^2 + x_0^3y_1^2).$$

This curve has an ordinary cusp at the point $p = (0:1;0:1)$, and it can be visualized as the black curve in Figure 4.3. A pencil of $(1,1)$-curves intersecting C non-transversally at p is given by

$$T_p = \mathscr{V}(dx_0y_0 + x_0y_1 + x_1y_0), \quad d \in \mathbb{C}.$$

In Figure 4.3, elements of this pencil are displayed in white for $d \in \{-0.5, 0, 0.5\}$. In the figure, the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^3$ using the Segre embedding, and we have chosen a suitable affine covering of $\mathbb{P}^3$. The image is made with `surfex` [32].

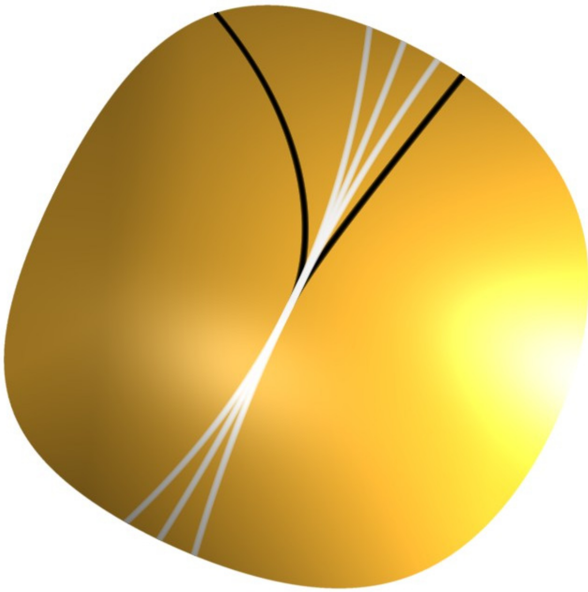


Figure 4.3: A pencil of $(1,1)$-curves intersecting a curve at a cusp non-transversally.

Theorem 4.1.16. *Let p and q be two cusps on a reduced and irreducible curve $C = \mathscr{V}(F)$ of type (a,b) on $\mathbb{P}^1 \times \mathbb{P}^1$, and let p and q have multiplicity sequences $\overline{m}_p = [m_0^p, m_1^p, \ldots]$ and $\overline{m}_q = [m_0^q, m_1^q, \ldots]$ respectively. Then*

$$m_0^p + m_1^p + m_0^q \leq a + b.$$

Proof. By Theorem 4.1.14 there is a pencil of $(1,1)$-curves T_p through p that each intersects C with intersection multiplicity $(T_p \cdot C)_p > m_0^p$. By Lemma 1.1.14, we have $(T_p \cdot C)_p \geq m_0^p + m_1^p$. If p is moved to $(0:1;0:1)$, this pencil is given by

$$dx_0y_0 + d_{01}x_0y_1 + d_{10}x_1y_0 = 0, \quad d \in \mathbb{C},$$

with d_{01} and d_{10} as in the proof of Theorem 4.1.14. Requiring that the $(1,1)$-curve should pass through q, determines the coefficient d, hence a curve $O_{p,q}$ uniquely. We have $(O_{p,q} \cdot C)_q \geq m_0^q$. The result then follows from Proposition 1.1.2. □

4.2 More on rational cuspidal curves

In this section we present some rational cuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ in addition to the ones presented in Chapter 3. First we consider curves of low bidegree, and then we look at curves that can be constructed from a local parametrization.

4.2.1 Low bidegree

Let C be a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of type (a,b), and by symmetry we may let $a \leq b$. With a fairly low, the search for cuspidal curves is quite easy since the multiplicity is restricted by $\min\{a,b\}$ and because of Hurwitz's theorem and the genus formula. For $a=1$, the quest stops immediately, since a rational curve of type $(1,b)$ must be nonsingular for all $b \geq 1$.

Moving on to curves C of type $(2,b)$ we get that the multiplicity of any cusp can not exceed 2. From Hurwitz's theorem we get that C can have at most two cusps. This leaves us with the following list of possible cuspidal configurations for curves of type $(2,b)$ for $b \leq 6$.

Type	**Cuspidal configuration**	**# Cusps**
$(2,2)$	$[2]$	1
$(2,3)$	$[2_2]$	1
	$[2],[2]$	2
$(2,4)$	$[2_3]$	1
	$[2_2],[2]$	2
$(2,5)$	$[2_4]$	1
	$[2_3],[2]$	2
	$[2_2],[2_2]$	2
$(2,6)$	$[2_5]$	1
	$[2_4],[2]$	2
	$[2_3],[2_2]$	2

Table 4.1: Rational cuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(2,b)$ for $b \leq 6$.

Table 4.1 can be extended in a natural way with curves of type $(2,b)$ with either just one cusp with multiplicity sequence $[2_{b-1}]$ or exactly two cusps with multiplicity sequences $[2_k],[2_l]$, where $k+l=b-1$.

Theorem 4.2.1. *For every $b \geq 2$ there exists a rational cuspidal curve C on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(2,b)$ with one cusp and cuspidal configuration $[2_{b-1}]$. For every $b \geq 3$ there additionally exist $\lfloor \frac{b-1}{2} \rfloor$ rational cuspidal curves with two cusps and cuspidal configuration $[2_k],[2_l]$, where $k+l=b-1$. Moreover, these curves can be constructed from the series of plane rational cuspidal curves of degree d with three cusps and largest multiplicity $d-2$ using a birational transformation from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. The existence follows from the construction. From Chapter 2 we know that for every $d \geq 4$ there exists a series of plane rational cuspidal curves of degree d with three cusps and cuspidal configuration $[d-2],[2_m],[2_n]$, where $m+n=d-2$.

We first construct the unicuspidal curves. Let $\hat{C}$ be the plane tricuspidal curve of degree d as described above, with $n=1$ and $m=d-3$. Then blowing up the two cusps with multiplicity sequences $[d-2]$ and $[2]$ and contracting the line between them gives a curve C on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(2,d-2)$ with one cusp with multiplicity sequence $[2_{d-3}]$. Letting $b=d-2$, we have the result.

We now construct the bicuspidal curves. Let $\hat{C}$ be the plane tricuspidal curve of degree d with cuspidal configuration $[d-2],[2_m],[2_n]$, where $m,n \geq 1$ and $n+m=d-2$. The tangent line to $\hat{C}$ at the cusp with multiplicity sequence $[d-2]$ intersects the curve in the cusp, say p, and in one smooth point, say r. Applying the birational transformation from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ that blows up p and r and contracts the line between them, we get a curve C on $\mathbb{P}^1 \times \mathbb{P}^1$. C is a rational cuspidal curve of type $(2,d-1)$ and has two cusps with multiplicity sequences $[2_m]$ and $[2_n]$, $m,n \geq 1$ and $n+m=d-2$. Letting $b=d-1$, $k=m$, and $l=n$, we have the result. The number of different cuspidal configurations can easily be counted for each b. □

Letting $a=3$, we get a lot more curves. However, the multiplicity of a cusp can not exceed 3, and by Hurwitz's theorem, we can maximally have four cusps with multiplicity 2 on such a curve. The below tables lists the curves of type $(3,3)$ and $(3,4)$.

Type	**Cuspidal configuration**	**# Cusps**
	$[3,2]$	1
	$[2_4]$	1
	$[2_3],[2]$	2
$(3,3)$	$[2_2],[2_2]$	2
	$[3],[2]$	2
	$[2_2],[2],[2]$	3
	$[2],[2],[2],[2]$	4

Table 4.2: Rational cuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3,3)$.

Type	Cuspidal configuration	# Cusps
	$[3_2]$	1
	$[3], [3]$	2
	$[3, 2], [2_2]$	2
	$[3_2], [2], [2]$	3
	$[3], [2_2], [2]$	3
	$[2_6]$	1
$(3, 4)$	$[2_5], [2]$	2
	$[2_4], [2_2]$	2
	$[2_3], [2_3]$	2
	$[2_4], [2], [2]$	3
	$[2_3], [2_2], [2]$	3
	$[2_2], [2_2], [2_2]$	3
	$[2_3], [2], [2], [2]$	4
	$[2_2], [2_2], [2], [2]$	4

Table 4.3: Rational cuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3, 4)$.

Note that the curves in Table 4.2 and Table 4.3 all exist. The curves can be constructed using birational transformations from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ of for example plane rational cuspidal curves of degree 4 or 5, see Chapter 3.

Letting $a = 4$, we quickly loose control over the list of possible curves of type $(4, b)$ and their cuspidal configurations. Already for $b = 4$, finding the list of cuspidal configurations of existing rational cuspidal curves is hard. We are not even able to construct all the curves in the list of curves with four or more cusps. In fact, using the bounds on the multiplicity sequences, the genus formula and Hurwitz's theorem, the list of possible rational cuspidal curves of type $(4, 4)$ with four or more cusps counts 26 curves. We have found only one of them, its cuspidal configuration being $[3, 2], [2_3], [2], [2]$. As discussed in Chapter 3, we do not expect that any of the other 25 curves exist.

It is possible to exclude some of the curves from the list of rational fourcuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(4, 4)$ using a birational transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ and showing that the curve reached on $\mathbb{P}^2$ does not exist. A problem with this approach is that, in general, the curves reached on $\mathbb{P}^2$ will be hard to exclude because the singularities on them can be quite complex, and the degrees of the curves in question can be quite high. Moreover, there can be a certain ambiguity concerning fiber tangential properties of the cusps, further complicating the picture. As an example we can, however, exclude one of the cuspidal configurations appearing in the mentioned list.

Example 4.2.2. Assume that C is a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(4, 4)$ with cuspidal configuration $[4], [2], [2], [2]$. Blowing up at the cusp with multiplicity sequence $[4]$, we get a curve C_1 of type $(0, 4)$ on $\mathbb{F}_1$ that does not intersect the special section. Moreover, there is a smooth point p_1 on C_1, and a fiber L_1 on $\mathbb{F}_1$, such that L_1 intersects C_1 in the following way, $L_1 \cdot C_1 = 4p_1$. Contracting the special section on $\mathbb{F}_1$, the strict transform of C_1 is a curve $\tilde{C}$ on $\mathbb{P}^2$ of degree 4 with three ordinary cusps and an inflection point. This curve would contradict the inflection point formula in Theorem 2.1.8, thus C does not exist.

4.2.2 Bicuspidal curves from local parametrization

As a final curiosity we observe that we have bicuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with local parametrization on the following form,

$$(x(t) : 1; y(t) : 1),$$

$$\begin{aligned} x(t) &= t^m, \\ y(t) &= t^r + \sum_{i=0}^{n-1} a_i t^{\alpha_i m} + a_n t^{\alpha_n m}, \end{aligned}$$

where $r, m \in \mathbb{N}$, with $r > m$ and $\gcd(m, r) = 1$, the coefficients $a_i \in \mathbb{C}$, and the exponents $\alpha_i \in \mathbb{N}$.

Bihomogenizing, we find that these curves of type $(m, \alpha_n m)$ have two cusps with Puiseux pairs (m, r) and $(m, 2\alpha_n m - r)$ respectively. The multiplicity sequences may be calculated from the Puiseux pairs using the Enriques–Chisini algorithm, see [4, Theorem 12, p.516].

Part II

Segre classes on toric varieties

Appendix B

The toricSegreClass algorithm

We include the implementation in `Macaulay2` and `Sage` of the `toricSegreClass` algorithm used in Chapter 5 to compute Segre classes of closed subschemes of toric varieties.

B.1 M2 part of the algorithm

```
-- TORIC SEGRE CLASS VERSION 1.0

needsPackage "FourierMotzkin";
needsPackage "NormalToricVarieties";

-- Correction of nef method, so that returned nef cone depends on specified basis

nef NormalToricVariety := List => X ->
(
	if not isComplete X then return false
	else
	(
		n := #rays X;
		A := transpose matrix degrees ring X;
		outer := 0 * A_{0};
		for s in max X do
		(
			sc := select(n, i -> not member(i,s));
			outer = outer | (fourierMotzkin A_sc)#0;
		);
		return ((fourierMotzkin outer)#0)^{0..(n-dim X-1)};
	);
);

-- INPUT: Two positive integers, m and l
-- OUTPUT: A list of lists, corresponding to all possible ways
-- to write m as a a length l sum of non-negative integers

listings = (m,l) -> if l == 1 then return {{m}} else
(
	poss := {};
	for i from 0 to m do
	(
```

```
                for p in listings(m-i,l-1) do
                (
                        poss = append(poss,append(p,i));
                );
        );
        return poss;
);

-- INPUT: A toric variety X and an ideal I in its Cox ring
-- OUTPUT: An admissible multidegree, two help lists

toricSegreClass = (X, I) ->
(

        -- Setup
        S = ring X;
        B = ideal X;
        r = length rays X;
        k = dim X;
        nefCone = nef X;

        gensI = flatten entries sort gens I;
        degs = degrees I;
        len = length degs;

        -- Find admissible multidegree, i.e., apex of the
        -- admissible cone
        P = convexHull(transpose matrix {degs_0}, nefCone);

        for i from 1 to (len-1) do
        (
                P = intersection(P, convexHull(transpose matrix {degs_i}, nefCone));
        );
        temp = flatten entries transpose vertices P;

        degAlpha = {};
        for i from 0 to (length temp -1) do
        (
                degAlpha = append(degAlpha, floor temp_i);
        );

        -- Create random variables in I of admissible multidegree
        f = for i from 1 to k list sum(gensI, g -> g * random(degAlpha - degree g, S));

        -- Create residual schemes
        for d from 1 to k do
        (
                J_d = saturate(ideal(take(f,d)),B);
                R_d = saturate(J_d, I);
        );

        for i from 0 to r-1 do
        (
                deg_i = degree x_i;
        );
```

```
        gammaLists = {};
        integerLists = {};

        -- Find degrees of intersections of the R_d with
        -- monomials in torus-invariant divisors...
        for d from 1 to k do
        (
                currentGammaList = {};

                currentIntegerList = listings(k-d,r);

                for p in currentIntegerList do
                (
                        -- Intersect residual scheme with monomial in torus-
                        -- invariant divisors determined by p
                        K = R_d;
                        for i from 0 to r-1 do
                        (
                                for j from 0 to (p#i)-1 do
                                (
                                        K = K + ideal random(deg_i, S);
                                );
                        );

                        -- Add degree of intersection to list of degrees
                        currentGammaList = append(currentGammaList,
                                multidegree saturate(K,B));
                );

                gammaLists = append(gammaLists, currentGammaList);
                integerLists = append(integerLists, currentIntegerList);

        );

        -- Return everything (user should manually input this in Sage)
        return (degAlpha, integerLists, gammaLists);
);
```

B.2 Sage part of the algorithm

```
# TORIC SEGRE CLASS VERSION 1.0

# INPUT: A toric divisor div, an integer m
# OUTPUT: A Chow cycle corresponding to intersecting
# the divisor m-1 times with itself

def intDiv(div, m):
        if m == 1:
                return div.Chow_cycle()
        else:
                return intDiv(div,m-1).intersection_with_divisor(div)

# INPUT: A Chow cycle, a divisor div, an integer m
# OUTPUT: A Chow cycle, the original cycle intersected with div m times
```

```
def intCycleDiv(cycle, div, m):
        if m == 1:
                return cycle.intersection_with_divisor(div)
        else:
                return intCycleDiv(cycle, div, m-1).intersection_with_divisor(div)

# INPUT: The fan of a toric variety X, an ideal in its Cox ring,
# an admissible multidegree, two lists returned from the M2 part
# OUTPUT: The Segre class of the subscheme defined by I,
# expressed in the Chow ring of X

def toricSegreClass(fan, I, degAlpha, integerLists, gammaLists):
# Setup

        X = ToricVariety(fan)
        k = X.dimension()

        S = X.coordinate_ring()
        X.inject_variables()
        Z = X.subscheme(I)
        n = Z.dimension()

        A = X.Chow_group()
        r = S.ngens()
        D = X.toric_divisor_group().gens()

        residualClasses = []

        # Compute residual classes R_d by solving equations
        for d in range(k-n, k+1):
                # Collect Chow ring generators of correct degree
                generators = A.gens(degree = k-d)
                l = len(generators)

                # Initialize equation and variable lists
                eqns = []
                v = list(var('v_%d' % i) for i in range(0,l))

                currI = len(integerLists[d-1])

                # Consider all possible intersections of Chow generators
                # with monomials in the torus-invariant divisors
                for counter in range(0, currI):
                        p = integerLists[d-1][counter]
                        betaList = []
                        for b in generators:
                                cycle = b
                                for i in range(0,r):
                                        for j in range(1,p[i]+1):
                                                cycle = cycle.intersection_with_divisor(D[i])
                                # Add degree of resulting cycle to list of betas
                                betaList.append(cycle.count_points())

                        currentGamma = gammaLists[d-1][counter]
```

```
                # Create equations
                eq = betaList[0]*v[0]

                if (l > 1):
                        for i in range(1, l):
                                eq = eq + betaList[i]*v[i]

                # Add new equation to list of equations
                eqns.append(eq == currentGamma)

        # Solve equations (test cases required for proper formatting)
        if (len(eqns) == 1):
                if (len(v) == 1):
                        c = [solve(eqns, v[0])]
                else:
                        c = [solve(eqns, v)]
        else:
                if (len(v) == 1):
                        c = solve(eqns, v[0])
                else:
                        c = solve(eqns, v)

        a = []
        for i in range(0,len(c[0])):
                 # Convert symbolic solution to integer and add to list a
                a.append(ZZ(c[0][i].rhs()))

        # Create and append the new residual class
        Rd = sum (a[i]*generators[i] for i in range(0,l))
        residualClasses.append(Rd)

# Find (components of) Segre class using
# the residual intersection formula
s = []
alpha = degAlpha[0]*D[0]

for i in range(1,len(degAlpha)):
        alpha = alpha + degAlpha[i]*D[i]

s.append(intDiv(alpha,k-n) + (-1)*residualClasses[0])

for i in range(1,n+1):
        temp = 0
        for j in range(0,i):
                temp = temp + binomial(i+k-n,i-j)*intCycleDiv(s[j],alpha,i-j)

        s.append(intDiv(alpha,i+k-n) + (-1)*residualClasses[i] + (-1)*temp)

return sum (s[i] for i in range(0,n+1))
```

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