THE AXIOM OF DETERMINATENESS

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I. THE GAMES.

We shall consider certain infinite games with perfect information.

Let X be a set, - we are mostly going to consider the case where $X = N$, the set of natural numbers, or $X = \{0,1\}$. Let A be a subset of X^N . In the <u>game</u> $G_X(A)$ there are two players I and II who successively chooses elements from X , player I starting.

A play, x , is an element $x \in X^N$, and I wins the play x if $x \in A$, otherwise II wins.

A strategy σ is a map from finite sequences of X to elements of X . If player I is using the strategy σ and II is using a strategy τ , let us denote the resulting play by $\sigma * \tau$

A more formal definition of $\sigma * \tau$ is as follows: Let in general $\alpha_0(n) = \alpha(2n)$ and $\alpha_1(n) = \alpha(2n+1)$, then define inductively

$$(\sigma*\tau)_0(n) = \sigma(\ (\sigma*\tau)_1 | n)$$

$$(\sigma*\tau)_1(n) = \tau(\ (\sigma*\tau)_0 | n+1)\ .$$

Thus we see that if $x \in X^N$ is the play produced by I using σ and II τ , then the element x_{2n} of x is obtained by applying σ to the sequence $\langle x_1, x_3, \ldots, x_{2n-1} \rangle$, i.e. to the preceeding choices of II (we have perfect information), and x_{2n+1} is obtained by applying τ to the previous choices of I , i.e. to the sequence $\langle x_0, x_2, \ldots, x_{2n} \rangle$.

A <u>winning strategy</u> for I in the game $G_X(A)$ is a strategy σ such that $\sigma*\tau \in A$ for all counterstrategies τ , and a winning strategy τ for II satisfies $\sigma*\tau \notin A$ for all σ .

A game $G_X(A)$ is determinate if either I or II has a winning strategy, i.e. if

(*) $\exists\sigma\ \forall\tau(\sigma*\tau \in A) \ \vee\ \exists\tau\ \forall\sigma(\sigma*\tau \notin A)$.

A set A is called determinate if the game $G_N(A)$ is determinate.

It is a mathematically interesting problem to decide which games are determinate. But the topic has an interest beyond this purely mathematical question. On a very general level we can make the following remark.

The assumption (*) involves a non-trivial switch of quantifiers. We may rewrite (*) as

$$\forall\tau\exists\sigma(\sigma*\tau \in A) \longrightarrow \exists\sigma\forall\tau(\sigma*\tau \in A) ,$$

i.e. the existence of "local" counterstrategies implies the existence of a "global", i.e. winning, strategy. Usually implications of the type $\forall\exists \longrightarrow \exists\forall$ require some assumptions of finiteness, compactness, uniformly boundedness, or the like.

An unrestricted assumption of the type that games $G_X(A)$ are always determinate, seems like cheating. Instead of proving the existence of a "uniformizing" element, we simply postulate that it exists.

The assumption is non-trivial. As a first example we shall derive the countable axiom of choice from (*).

Let $F = \{X_1, X_2, \ldots\}$ be a countable family of sets $X_i \subseteq N^N$. (We need only assume that card $(\cup\{X \mid X \in F\}) \leq 2^{\aleph_0}$.) We shall show that (*) implies that there is a choice function for F, i.e. a map f such that $f(X) \in X$, for all $X \in F$.

The proof follows by considering a suitable game: Player II wins if and only if whenever n_0 is the first choice of I and $\langle n_1, n_3, \ldots \rangle$ is the sequence of choices of II, then $\langle n_1, n_3, \ldots \rangle \in X_{n_0}$.

More formally let $N^N - A = \{x \mid \langle x_1, x_3, \ldots \rangle \in X_{x_0}\}$. One sees at once that

$$\forall\sigma\ \exists\tau(\sigma*\tau \notin A)\ ,$$

hence by determinateness we obtain

$$\exists\tau\ \forall\sigma(\sigma*\tau \notin A)\ .$$

But this is nothing but an unfamiliar way of asserting the implication

$$\forall n\ \exists f(f \in X_n) \rightarrow \exists f\ \forall n(\lambda y\cdot f(y,n) \in X_n)\ ,$$

the winning strategy for II is the desired choice function.

REMARK. We shall on occasions refer to games $G_X^*(A)$ and $G_X^{**}(A)$. In games of type G^* I may choose arbitrary finite (including empty) sequences from X and II chooses single elements. In games of type G^{**} both I and II may choose arbitrary (non-empty) finite sequences from X .

Intinite games of the above type was apparently first considered in Poland in the 1920's, but not much seems to have been published on the topic.

In 1953 GALE and STEWART worte a paper Infinite games with perfect information [6], proving among other things that open sets (in the games G_N) are determined. Using the axiom of choice they also produced a fairly simple non-determined game. They also gave a series of examples showing that the class of determinate sets has very few desireable algebraic closure properties.

Their work was continued in a game-theoretic context by P. WOLFE and M. DAVIS, the latter author showing in a paper [5] from 1964 that every set belonging to the class $F_{\sigma\delta} \cup G_{\delta\sigma}$ are determined, - this result still being the best obtained, unless large cardinal assumptions are added to set theory.

Independently some work was also done on infinite games in Poland. In 1962 a short paper by J. MYCIELSKI and H. STEINHAUS, A mathematical axiom contradicting the axiom of choice [13], suggested a new approach. They noted that assumptions of determinateness had remarkable deductive power, and although contradicting the unrestricted axiom of choice, led to a mathematics in some respects more satisfying than the usual one, e.g. determinateness implies that every subset of the real line is Lebesgue measurable. They also noted that analysis when confined to separable spaces, would exist unchanged, i.e. "positive" results such as the Hahn-Banach theorem, the compactness of the Hilbert cube, etc., would still be true (whereas the general Hahn-Banach theorem and the general Tychonoff theorem would fail in the non-separable case).

The 1964 paper of MYCIELSKI, On the axiom of determinateness [11], gives a very complete survey of what was known about determinateness at the time.

Already Mycielski had remarked in [11], using previous work of Specker, that the consistency of ZF (not including AC , the axiom of choice) and AD (= axiom of determinateness) implies the consistency of ZF + AC and the assumption that strongly inaccessible cardinals exists. In 1967 R. SOLOVAY (in a still unpublished paper [19]) showed that the consistency of ZF + AC + MC (where MC = there exists measurable cardinals) follows from the consistency of ZF + AD . This result gives an indication of the strenght of the axiom of determinateness.

In 1967 D. BLACKWELL published a short note, Infinite games and analytic sets [4], where he showed that the reduction principle for co-analytic (i.e. $\underset{\sim}{\Pi}^1_1$) sets follows from the basic result of Gale and Stewart that open sets are determined.

His observation was independently extended by D. MARTIN [7] and by J. ADDISON and Y. MOSCHOVAKIS [3] in 1968. In these papers it was shown that determinateness of certain games had deep consequences for the analytic and projective hierarchies in Bairespace and Cantorspace.

At the same time (but, however, first pubpished in 1970) D. MARTIN [8] showed that adding MC to ZF implies that every analytic set (i.e. $\underset{\sim}{\Sigma}^1_1$ set) is determinate.

This concludes our brief "historical remarks". The present paper is a survey paper. There might be a few refinements to existing results and some new details added, but the aim has been to give a reasonably complete exposition of those parts of the theory which lie closer to "ordinary" mathematics (here: descriptive set theory and real analysis). In this respect we aim at bringing the 1964 survey of J. Mycielski [11] up to date. (However, since there is a large "unpublished literature" on the topic, our survey is probably incomplete.)

We believe that determinateness is an interesting and important topic. As we shall argue in the concluding section of this paper, some form of determinateness assumption might be a reasonable addition to the current set theoretic foundation of mathematics.

REMARK ON NOTATION. Our notation is standard. We assume that the reader has some basic knowledge of set theory and real analysis.

Example: We do not explain the Gödel notion of "constructible set". But since we are giving very few proofs, only a rudimentary knowlegge of constructibility is necessary for following the exposition. From recursion theory we use the standard notation (as e.g. $\boldsymbol{\Sigma}_n^1$, $\boldsymbol{\Pi}_n^1$ and $\boldsymbol{\Delta}_n^1$ for sets on the nth level of the projective hierarchy) and occasionally we refer to some basic fact about "finite path trees", or the like. And in discussing the Lebesgue measurability of various subsets of the real line we assume that the reader know what Lebesgue measure is, but very little beyond that.

II. DETERMINACY AND THE ANALYTIC AND PROJECTIVE HIERARCHIES.

Reduction principles play a central role in the study of hierarchies. Let us recall the basic definitions. A class Q of sets satisfies the reduction principle if for all $X,Y \in Q$ there are sets $X_1, Y_1 \in Q$ such that $X_1 \subseteq X$, $Y_1 \subseteq Y$, $X_1 \cap Y_1 = \emptyset$, and $X_1 \cup Y_1 = X \cup Y$. The main point here is, of course, that X_1 and Y_1 also belong to Q, i.e. is of the same kind of complexity as X and Y.

It is classical that in ZF + AC we have $\mathrm{Red}(\underset{\sim}{\Pi}^1_1)$ and $\mathrm{Red}(\underset{\sim}{\Sigma}^1_2)$, where $\mathrm{Red}(Q)$ means that Q satisfies the reduction principle.

Assuming the axiom of constructibility, $V = L$, ADDISON [1] extended this to $\mathrm{Red}(\underset{\sim}{\Sigma}^1_k)$ for all $k \geq 3$. The idea behind the proof is quite simple. Using $V = L$ one can show that there is a Δ^1_2 well ordering of N^N. Now any set in $\underset{\sim}{\Sigma}^1_k$ is obtained by taking a union over N^N, which by $V = L$ has a "nice" well ordering. So given any element in $X \cap Y$, put it in X_1 if it is generated "earlier" in X than in Y, otherwise put it in Y_1.

Both MARTIN [7] and ADDISON, MOSCHOVAKIS [3] observed that reduction principles obtains in the higher levels of the hierarchies, if one adds suitable assumptions of determinateness. We shall give a brief exposition of the latter author's work (returning to Martin's ideas in the next section).

The basic result is a "prewellordering theorem". A prewellordering on a set is a total, transitive and well-founded binary relation on the set. (Note: If $\leq$ is a pre-wellordering, then we do not assume that $x \leq y$ and $y \leq x$ imply $x = y$. By passing to equivalence classes, $x \sim y$ iff $x \leq y$ and $y \leq x$, we get a true wellordering and hence an associated ordinal.)

Observe the following simple example: If $A \in \underset{\sim}{\Sigma}^0_1$, i.e. $A = \{x \mid \exists n\, R(x,n)\}$, where x may be any (finite) sequence of function and number variables, then the relation

$$x \leq_0 y \text{ iff } x,y \in A \text{ and } \mu n R(x,n) \leq \mu m R(y,m)$$

is a prewellordering of A .

To explain the general result announced by Addison and Moschovakis we need some further terminology. Let E^1_k be Σ^1_k if k is even and Π^1_k if k is odd. A subset C of $N \times X$, where X may be any finite product of N and N^N , is called E^1_k-universal for X if C is E^1_k and for every E^1_k-subset A of X there is some $n \in N$ such that $x \in A$ iff $\langle n,x\rangle \in C$.

PREWELLORDERING THEOREM. Let X be as above, let l be an even number, and let k be l or $l+1$. If every $\underset{\sim}{\Delta}^1_l$ subset of N_N is determinate, then there is a subset $W_k \subseteq N \times X$ which is E^1_k-universal for X , and a prewellordering $\leq_k$ of W_k whose initial segments are uniformly $\underset{\sim}{\Delta}^1_k$.

We shall briefly indicate the basic ideas of the proof. The proof goes by induction. In the basis the simple example $\leq_0$ above will serve.

In the induction step one must distinguish between the case where $k = l$ is even or $k = l+1$ is odd. In the former case we pass from a Π to a Σ class, hence we get by by taking infima of wellorderings, just as in the "ablolute" case, i.e. from Π^1_1 to Σ^1_2 . In detail: let $z \in W_k$ iff $\exists \alpha (z,\alpha) \in W_{k-1}$ and define $z <_k w$ iff $z,w \in W_{k-1}$ and the $\leq_{k-1}$ - least (z,α) is $\leq_{k-1}$ - less than the $\leq_{k-1}$ - least (w,β) .

In the odd case one has to go from a Σ to a Π class. This means that one has to take suprema of wellorderings in order to get $\underset{\sim}{\Delta}_k^1$ definitions of the segments of $\leq_k$. But this cannot be done using ordinary function quantifiers, a fact which has obstructed the passage from Σ_2^1 to Π_3^1 in "absolute" recursion theory. Assuming $V = L$ Addison gave one "solution" to the problem. Assuming determinacy of projective sets gives another, and perhaps is some respects, more satisfactory "solution" to the problem.

A required E_k^1-universal set has the form $W_k = \{x \mid \forall\alpha(x,\alpha) \in W_{k-1}\}$. Given $x,y \in W_k$ define a set $B_{x,y}$ by setting

$$B_{x,y} = \{\alpha \in N^N \mid (x,\alpha_0) \leq_{k-1} (y,\alpha_1)\} ,$$

where α_0 and α_1 are defined from α as in the first section of this paper. One sees that $B_{x,y} \in \underset{\sim}{\Delta}_1^1$, hence is determinate by assumption. The ordering $\leq_k$ on W_k is now defined by

$$x \leq_k y \equiv x,y \in W_k \text{ and } \exists\tau\, \forall\sigma(\sigma*\tau \in B_{x,y}) .$$

The definition is "natural" which is seen by considering the case $k = 1$ and $l = 0$. In this case W_1 has the form: $W_1 = \{x \mid \forall\alpha\, \exists n\, R(x,\bar{\alpha}(n))\}$, and $\alpha \in B_{x,y}$ iff $\mu n\, R(x,\bar{\alpha}_0(n)) \leq \mu m\, R(y,\bar{\alpha}_1(n))$. (Note that in this case $B_{x,y}$ is provably determinate by the main result of Gale, Stewart [6].)

Consider now the usual sequence trees T_x and T_y associated with R, x and y. If $x,y \in W_1$, the trees are well-founded, so let $O(T_x)$ and $O(T_y)$ denote the corresponding ordinals. The so-called "basic tree lemma" (see e.g. Rogers [16]) asserts that $O(T_x) < O(T_y)$ iff (except for degenerate cases) there exists a branch tree T_y' of T_y such that for all branch trees T_x' of T_x, $O(T_x') < O(T_y')$. The existence of a winning strategy τ, as required in the definition of $\leq_1$ thus follows from the basic tree lemma, and a proof of the prewellordering theorem is easily obtained in this case. But in order to generalize one needs a proof within a game-theoretic context.

We shall not prove in detail that $\leq_k$ has the required properties. As quite typical for the proof we shall verify that the relation is total: Assume that $x,y \in W_k$ and $\neg(x \leq_k y)$, i.e.

$$\neg \exists\tau \ \forall\sigma(\sigma*\tau \in B_{x,y}) .$$

By the suitable assumption of determinateness this implies

$$\exists\sigma \ \forall\tau(\sigma*\tau \notin B_{x,y}) ,$$

which means that

$$\exists\sigma \ \forall\tau((y,(\sigma*\tau)_1) \leq_{k-1} (x,(\sigma*\tau)_0)) .$$

But since player II always can imitate player I , one easily infers that

$$\exists\tau \ \forall\sigma((y,(\sigma*\tau)_0) \leq_{k-1} (x,(\sigma*\tau)_1)) ,$$

i.e. $y \leq_1 x$.

Transitivity and well-foundedness is obtained by simultaneously playing several games. In the verification of well foundedness one also needs the axiom of dependent choices (DC) .

Using the prewellordering theorem one may now lift all theorems of "abstract" or "generalized" recursion theory to every level of the hierarchy. Many of these results are listed in the announcement of Addison and Moschovakis [3].

REMARK. Our understanding of the prewellordering theorem has greatly benefitted from many discussions with T. OTTESEN, who in a seminar at Oslo has worked out the various consequences of the pwo theorem, and also added several refinements.

In conclusion let us return to the reduction principles. Assuming $V = L$ one has the series

$$\mathrm{Red}(\underset{\sim}{\Pi}^1_1),\ \mathrm{Red}(\underset{\sim}{\Sigma}^1_2),\ \mathrm{Red}(\underset{\sim}{\Sigma}^1_3),\ \mathrm{Red}(\underset{\sim}{\Sigma}^1_4),\ \ldots$$

The prewellordering theorem gives the series

$$\mathrm{Red}(\underset{\sim}{\Pi}^1_1),\ \mathrm{Red}(\underset{\sim}{\Sigma}^1_2),\ \mathrm{Red}(\underset{\sim}{\Pi}^1_3),\ \mathrm{Red}(\underset{\sim}{\Sigma}^1_4),\ \ldots$$

More precisely one proves that DC (= axiom of dependent choices) and determinateness of $\underset{\sim}{\Delta}^1_{2k}$ sets imply $\mathrm{Red}(\underset{\sim}{\Pi}^1_{2k+1})$ and $\mathrm{Red}(\underset{\sim}{\Sigma}^1_{2k+2})$. The proof is straight forward since we can use the prewellordering theorem to separate elements in the intersection.

One sees that AD and $V = L$ both give answers, but conflicting ones. Note that it is AD which continues the pattern of the classical, or "absolute" case.

The next topic would be the extension of uniformization principles using AD . There is one paper by MARTIN and SOLOVAY, A Basis theorem for $\underset{\sim}{\Sigma}^1_3$ sets of reals [9], which makes a one-step extension (going by way of measurable cardinals), but the field seems to be wide open. Reduction principles and "generalized" recursion theory seems to need very little beyond the assignment of ordinals (existence of prewellorderings). Uniformization principles seems to lie much deeper.

III. DETERMINACY AND THE REAL LINE.

It is a standard fact of measure theory that there are subsets of the real line which are not Lebesgue measurable. One remarkable consequence of AD is the following result of MYCIELSKI and SWIERCZKOWSKI [14].

THEOREM. AD implies that every subset of the real line is Lebesgue measurable.

We shall give a brief sketch of the proof and indicate some refinements. Rather than working on the real line we shall consider the space 2^N and the usual product measure μ on 2^N (i.e. $\mu(\{\alpha \in 2^N \mid \alpha(n) = 0\}) = \mu(\{\alpha \in 2^N \mid \alpha(n) = 1\}) = \frac{1}{2}$ for all $n \in N$).

Let $X \subseteq 2^N$ and let $r = \langle r_n \rangle$ be a sequence of numbers of the form $r_n = 2^{-t_n}$ where each t_n is a natural number and $2 < t_1 < t_2 < \ldots$. Associated with X and r there will be a game $G(X,r)$, and the assumption that these games are determinate will imply that every subset of 2^N is μ-measurable.

Let J_k be the class of subsets $S \subseteq 2^N$ which satisfy the following requirements:

(i) Each S is a union of basic neighborhoods.

(ii) Each S is contained in a basic neighborhood of diameter $\leq 2^{-k}$.

(iii) The μ-measure of S is

$$\mu(S) = r_1 \, r_2 \cdot \ldots \cdot r_k = 2^{-(t_1+\ldots+t_k)}$$

The game $G(X,r)$ is played as follows:

Move 1: Player I chooses a set $S_1 \in J_1$.

Move 2: Player II chooses a set $S_2 \in J_2$ such that $S_2 \subseteq S_1$.

Move 3: Player I chooses a set $S_3 \in J_3$ such that $S_3 \subseteq S_2$.

etc.

Note that because of (i) and (ii) $\bigcap_{n=1}^{\infty} S_n$ reduces to one point. If the point determined by the play $S_o = 2^N, S_1, S_2, \ldots$ belongs to X , then I wins. If the point belongs to $2^N - X$, then II wins.

Let μ_* denote the inner measure on 2^N associated with μ . Mycielski and Swierczkowski obtained the following estimates for $\mu_*(X)$ and $\mu_*(2^N - X)$:

(a) If I has a winning strategy in the game $G(X,r)$, then

$$\mu_*(X) \geq r_1 \cdot \prod_{n=1}^{\infty} (1-2 \cdot r_{2n}) .$$

(b) If II has a winning strategy in the game $G(X,r)$, then

$$\mu_*(2^N - X) \geq \prod_{n=1}^{\infty} (1-2 \cdot r_{2n-1}) .$$

(To be accurate: Mycielski and Swierczkowski worked with the interval $[0,1]$. An analysis of their proof shows that it works as well for the space 2^N , some extra care being needed to obtain the sets S_n^j , see the sketch below.)

We indicate how (a) and (b) are proved: Let I be using a strategy σ . Let

$$S_o = 2^N, S_1, \ldots, S_{2n-1}$$

be a position in a play of the game. One observes that there is a finite number of sets $S_{2n}^1, \ldots, S_{2n}^m \in J_{2n}$ such that the sets

$$\hat{S}_{2n}^i = \sigma(\langle S_o, \ldots, S_{2n-1}, S_{2n}^i \rangle)$$

are disjoint for $n = 1,\dots,m$, and such that

$$\mu(\bigcup_{i=1}^{m} \hat{S}^i_{2n}) \geq \mu(S_{2n-1})\cdot(1-2r_{2n}) .$$

Let A_m be the union of all possible such sets at move $2m+1$ in plays where I has been using the strategy σ and II at each of his moves has picked one of the sets S^i_{2n} . One proves that $A_{m+1} \subseteq A_m$ and, using the above inequality, that

$$\mu(A_m) \geq r_1 \cdot \prod_{n=1}^{m}(1-2r_{2n}) .$$

If $x \in \bigcap_{m=1}^{\infty} A_m$, there exists a strategy τ_x for II such that the play $\sigma * \tau_x$ produces the point x , viz. II always chooses the unique set S^i_{2n} such that $x \in \sigma(<S_o,\dots,S^i_{2n}>)$, the latter sets being disjoint.

If σ is a winning strategy for I , then the argument shows that $\cap A_m \subseteq X$, hence the estimate above on the μ-measure of A_m gives the inequality of (a) . In a similar way one proves (b) .

The theorem is an immediate consequence: If there are non-measurable sets, then by standard measure theory there will be a set $X \subseteq 2^N$ such that $\mu_*(X) = 0$ and $\mu_*(2^N - X) = 0$. But by (a) and (b) this is impossible, - if every game $G(X,r)$ is determined.

To obtain a refinement we choose a particular type of sequence $r = <r_n>$, viz. we set

$$r_{2n-1} = 2^{-(K\cdot n+1)} \qquad \text{for } n \geq 1 ,$$

and

$$r_{2n} = 2^{-K\cdot(n+1)} \qquad \text{for } n \geq 0 ,$$

where K is some natural number > 1 . We then see that

$$\prod_{n=1}^{\infty}(1-2r_{2n-1}) = \prod_{n=1}^{\infty}(1-2^{-K\cdot n})$$

$$\geq e^{-\frac{2}{2K-1}}$$

and we observe that $e^{-\frac{2}{2K-1}} \longmapsto 1$ from below when $K \to \infty$.

Assume now that every game $G(X,r)$ is determined, where X is a $\underset{\sim}{\Pi}_n^1$ subset of 2^N and r is a sequence of the type considered above.

To derive a contradiction let X_1 be a non-measurable $\underset{\sim}{\Pi}_n^1$ subset of 2^N. Standard measure theory (needing nothing more than a countable version of the axiom of choice) shows that there is a Borel set $F \subseteq X_1$ such that $\mu_*(X) = \mu(F)$. Hence if we set $X = X_1 - F$, then X is $\underset{\sim}{\Pi}_n^1$ and $\mu_*(X) = 0$.

Since X also must be non-measurable, we see that $\mu_*(2^N-X)<1$. Choose K such that

$$e^{-\frac{2}{2K-1}} > \mu_*(2^N - X) .$$

By assumption the game $G(X,r)$ is determinate, where r is the particular sequence determined by K. So either (a) or (b) should obtain. But both are impossible, hence the given $\underset{\sim}{\Pi}_n^1$ subset X_1 of 2^N must be μ-measurable.

The game $G(X,r)$ is not in standard form. But we see from the "effectiveness" of the clauses (i) - (iii) in the definition of the class J_k, and from the fact that the sequence r depends upon a single number parameter K, that it can - without too much effort - be recast as a game of the type G_N, and further that determinateness of the associated $\underset{\sim}{\Pi}_n^1$ game G_N gives the determinateness of the considered $G(X,r)$ game. Thus our analysis yields the following corollary to the theorem of Mycielski and Swierczkowski:

THEOREM. If the game $G_N(A)$ is determined for every $\underset{\sim}{\Pi}_n^1$ set A, then every $\underset{\sim}{\Pi}_n^1$ subset of 2^N is μ-measurable.

REMARK. AD also implies further "nice" properties of the real line or the space 2^N, e.g. every non-denumerable subset contains a perfect subset and every subset has the property of Baire (i.e. there is some open set such that the symmetric difference of the given set and the open set is a set of first category). We return to these properties in the next section.

Not every consequence of AD with respect to the real line is "nice". From results we have mentioned it follows that the uncountable axiom of choice fails. Further we get incomparability of many cardinal numbers, e.g. $\aleph_1$ and $2^{\aleph_0}$ will be incomparable. And we may conclude that there is no well-ordering of the real line.

But we can still consider prewellorderings of the continuum. Let

$$\underset{\sim}{\delta}_n^1 = \text{the least ordinal not the type of a } \underset{\sim}{\Delta}_n^1 \text{ prewellordering of the real line.}$$

It is a classical result that $\underset{\sim}{\delta}_1^1 = \omega_1$. We have the recent results of MOSCHOVAKIS [10]:

THEOREM. Assume AD and DC. Then each $\underset{\sim}{\delta}_n^1$ is a regular cardinal and $\underset{\sim}{\delta}_n^1 \geq \omega_n$.

It is tempting to formulate the conjecture: $\underset{\sim}{\delta}_n^1 = \omega_n$, which according to Moschovakis [10], ought to be true on notational grounds alone. D. Martin (unpublished) proved that $\underset{\sim}{\delta}_2^1 \leq \omega_2$ is a theorem of ZF + AC , so that the conjecture is verified for $n = 1,2$. Recently he is reported to have settled the conjecture in the negative by showing that AD implies that ω_3 is a singular cardinal. Thus AD gives a rather complicated and unfamiliar theory of cardinals. The reader must judge for himself whether this is an argument for or against AD . In this connection he may also contemplate the following noteworthy result.

THEOREM. AD implies that ω_1 is a measurable cardinal.

This means that there is a countable additive, two-valued measure defined on the powerset of ω_1 such that ω_1 has measure 1 and each point has measure 0 .

The theorem is due to SOLOVAY [19], but it is now possible to give a very different and much simpler proof based on a result of MARTIN [7].

THEOREM. Let D be the set of all degrees of undecidability and let E be an arbitrary subset of D . Then there exists a degree d_0 such that either $d \in E$ for all $d \geq d_0$, or $d \in D - E$ for all $d \geq d_0$.

The proof is simple and worth repeating: Let E^* be the set of all sequences in 2^N whose degree belongs to E . Assume that I has a winning strategy in the game $G_2(E^*)$, and let d_0 be its degree. Let $d \geq d_0$ and let α be a sequence of degree d . If II plays according to α and I plays according to his winning strategy, the sequence produced will have degree d . Hence $d \in E$.

From this one gets a countably additive 0-1 measure on the set of degrees D . Let $E \subseteq D$, we define

$$\lambda(E) = \begin{cases} 1 & \text{if } (\exists d_0)(\forall d \geq d_0)[d \in E] \\ 0 & \text{ow.} \end{cases}$$

And using the map $f: D \longrightarrow \omega_1$ defined by $f(d) = \omega_1^d$ where ω_1^d is the least (countable) ordinal not recursive in d, we get a measure μ on ω_1 by the formula

$$\mu(A) = \lambda(f^{-1}(A)) ,$$

for A an arbitrary subset of ω_1.

REMARK. Martin in [7] used the measure λ to lift many of the results of general recursion theory (such as reduction principles) to all levels of the analytic and projective hierarchies. The use of λ to get a measure μ on ω_1 was noticed later.

We have now an immediate proof of the following theorem of SOLOVAY [19]:

THEOREM. Con(ZF + AD) implies Con(ZF + AC + MC) .

Let μ be the measure constructed above. The inner model L_μ will then be the required model for ZF + MC .

Stronger results are known. We have e.g. the following result of Solovay:

THEOREM. Con(ZF + AD + DC) implies Con(ZF + AC + $\forall\alpha\, \exists M$ (M is an inner transitive model with α measurable cardinals)).

We shall conclude this part of our survey by relating AD to Souslin hypothesis.

Let L be a totally ordered set without a first or last element which is connected in the usual interval topology. Souslin conjectured that if each family of disjoint open intervals in L is countable, then L is topologically the real line.

It is well known that if we instead require L to be separable, then it is equivalent to the real line. We can therefore rephrase Souolin's conjecture in the following way. We call L a Souslin line if we in addition to the properties considered by Souslin also require L to be not separable. The conjecture is now: There is no Souslin line.

The conjecture has an equivalent form in terms of trees. A Souslin tree is a tree of cardinality ω_1 such that each chain and each antichain of the tree is countable. The basic fact is: Souslin lines exist iff Souslin trees exist. (For a proof and an elementary survey of the topic, see [17].)

THEOREM. AD implies that there are no Souslin trees.

The proof uses the fact that ω_1 is measurable. Assume that $\langle T, \leq \rangle$ is a Souslin tree. Since the cardinality of T is ω_1, we may assume that T also is measurable, hence carries a two-valued measure μ. For $\rho < \omega_1$ let T_ρ denote the set of elements in the tree of level ρ. Since T is a Souslin tree and each T_ρ is an antichain in T, T_ρ is countable. And since the cardinality of T is ω_1, each T_ρ is non-empty.

Let $\rho < \omega_1$. Define $D_\rho = \bigcup_{\gamma<\rho} T_\gamma$, we see that $\mu(D_\rho) = 0$. Let for $t \in T$:

$$f(t) = \{t' \in T \mid t \leq t'\} .$$

Then $\{D_\rho\} \cup \{f(t) \mid t \in T_\rho\}$ is a countable partition of T, hence there is exactly one $t \in T_\rho$ such that $\mu(f(t)) = 1$. Thus we may define for each $\rho < \omega_1$

$$t_\rho = \text{the unique } t \in T_\rho \text{ such that } \mu(f(t)) = 1 .$$

It is not difficult to verify that $\{t_\rho \mid \rho < \omega_1\}$ is an uncountable chain in T , which contradicts the assumption that T is a Souslin tree.

We would like to conclude from this that AD implies that Souslin hypothesis is true. However, the construction of a Souslin tree from a Souslin line requires the uncountable axiom of choice, which we know is inconsistent with AD . Hence there remains a puzzling problem: Is it consistent with AD to have Souslin lines, but not Souslin trees ?

On the other hand (as we shall point out in the concluding section of this paper) a decent mathematics on the basis of ZF + AD requires that we impose conditions of separability. And, as seen above, adding separability makes Souslin's conjecture trivial.

REMARK. The reader will have noticed that what we really proved above, is that if $\varkappa$ is a measurable cardinal, then there are no Aronszajn $\varkappa$-tree (or fake Souslin tree of cardinality $\varkappa$). This has been known to the experts on trees for a long time.

IV. WHICH GAMES ARE DETERMINATE ?

We hope to have convinced the reader that assumptions of determinateness leads to many interesting results, which have great potential importance for mathematics, - if we can answer the basic questions: Which games are provably determinate? Which games can consistently be assumed to be determinate?

The basic positive result is due to GALE, STEWART [6]. Their main result is:

THEOREM. Let X be either N or $\{0,1\}$. Then the game $G_X(A)$ is determinate if A is open or A is closed.

This result was extended by DAVIS [5], whose main result is:

THEOREM. The game $G_N(A)$ is determinate if $A \in F_{\sigma\delta} \cup G_{\delta\sigma}$.

This is as far as we at present can go within ZF + AC . One of the difficulties in extending the result is that the class of determinate sets has no nice closure properties, e.g. it is not closed under unions, intersections, and complementation.

Perhaps Davis' result is optimal? The main open problem is, of course, whether Borelgames are determinate. Dropping the replacement axiom form ZF H. Friedman (unpublished) has produced counterexamples. It is also known that $\underset{\sim}{\Delta}_1^1$ represents the limit of what is obtainable in ZF . As we shall indicate below there are $\underset{\sim}{\Pi}_1^1$ non-determinate sets in ZF + V = L .

The situation is much better understood with respect to games of the type $G_X^*(A)$ and $G_X^{**}(A)$. We shall first state the known relationships between games G, G^*, and G^{**}. To do this we need some notation. Let:

AD	=	Every game $G_N(A)$ is determinate
AD_2	=	Every game $G_2(A)$ (i.e. $A \subseteq 2^N$) is determinate
AD^*	=	Every game $G_N^*(A)$ is determinate
AD_2^*	=	Every game $G_2^*(A)$ is determinate
AD^{**}	=	Every game $G_N^{**}(A)$ is determinate
AD_2^{**}	=	Every game $G_2^{**}(A)$ is determinate

The following relationships are known (see [11]):

$$\begin{array}{ccccccccc} AD_2 & \longleftrightarrow & AD & \rightarrow & AD^* & \rightarrow & AD^{**} & \longleftrightarrow & AD_2^{**} \\ & & & & \downarrow & & & & \\ & & & & AD_2^* & & & & \end{array}$$

M. DAVIS [5] has shown that player I has a winning strategy in games $G_2^*(A)$ iff A contains a perfect subset, and that II has a winning strategy iff A is countable.

In the Banach-Mazur games $G_N^{**}(A)$ J. OXTOBY [15] (building on previous work of Banach and Mazur) has proved that II wins iff A is of the first category, and I wins iff A is of the first category in some open subset of N^N.

Restating these results one gets (see [11]):

I. AD_2^* is equivalent to the assertion that every uncountable subset of 2^N has a perfect subset.

II. AD^{**} is equivalent to the asserting that every subset of N^N has the property of Baire (i.e. is congruent to an open set modulo sets of first category).

These are the "classical" results, Recently we have the following remarkable result of D. MARTIN [8] :

THEOREM. If measurable cardinals exists, then every $\underset{\sim}{\Sigma}_1^1$ set is determined.

Nothing seems to be known about the determinateness of $\underset{\sim}{\Delta}_2^1$ sets. (Note that $\underset{\sim}{\Delta}_2^1$ determinateness is the first non-trivial assumption of determinateness needed for the prewellordering theorem.)

Counterexamples are easier to come by. E.g. a non Lebesgue measurable subset of the real line must be non-determinate.

If we work within $ZF + V = L$ we immediately get a $\underset{\sim}{\Delta}_2^1$ counterexample: $ZF + V = L$ implies $\mathrm{Red}(\Pi_3^1)$. If every $\underset{\sim}{\Delta}_2^1$ set were determined, the prewellordering theorem would give $\mathrm{Red}(\underset{\sim}{\Pi}_3^1)$. And as is well known we cannot at the same time have $\mathrm{Red}(\underset{\sim}{\Sigma}_n^1)$ and $\mathrm{Red}(\underset{\sim}{\Pi}_n^1)$.

The following is a $\underset{\sim}{\Pi}_1^1$ counterexample: Using the Δ_2^1 wellordering of N^N which follows from $V = L$, we may define a $\underset{\sim}{\Sigma}_2^1$ subset of N^N which contains one code α for each ordinal less than ω_1 . This is an uncountable set, and we claim that it cannot contain a perfect subset. For if so, this subset would also be uncountable and hence cofinal in the given set, and therefore could be used to define a $\underset{\sim}{\Sigma}_1^1$ set $WO = \{\alpha \in N^N \mid \alpha$ is a wellordering of $N\}$, viz. $\alpha \in WO$ iff α is a linear ordering of N and there is a function in the perfect set (which is a closed set) such that α is orderisomorphic to a segment of this function. But this is impossible since $WO \in \underset{\sim}{\Pi}_1^1 - \underset{\sim}{\Sigma}_1^1$.

By the Kondo-Addison uniformization theorem every $\underset{\sim}{\Sigma}_2^1$ set is a 1-1 projection of a $\underset{\sim}{\Pi}_1^1$ - set. And since a 1-1 continuous image of a perfect set is a perfect set, we get a $\underset{\sim}{\Pi}_1^1$ - counterexample.

We shall make some further comments on the relationship between AD and MC . In section III we saw that AD implies that ω_1 is measurable, hence that the consistency of ZF + AC + MC follows from the consistency of ZF + AD . Further results of Solovay were quoted showing that AD is a much stronger assumption than MC . Martin's theorem quoted above is a step in the converse direction, MC implies that every $\underset{\sim}{\Sigma}_1^1$ (or $\underset{\sim}{\Pi}_1^1$) set is determined. This result can be used to obtain different proofs of some results which follows from the existence of measurable cardinals.

Classically one knows that an uncountable $\underset{\sim}{\Sigma}_1^1$ set contains a perfect subset. Unrestricted AD extends this to every uncountable subset of N^N . By Martin's theorem we can conclude that every $\underset{\sim}{\Pi}_1^1$ set is determined, and looking closer at the characterization theorem of M. Davis, it is not difficult to conclude that every uncountable $\underset{\sim}{\Pi}_1^1$ set contains a perfect subset. Using the Kondo-Addison uniformization theorem, the result is lifted to $\underset{\sim}{\Sigma}_2^1$. Thus we have obtained a quite different proof a result of Solovay and Mansfield (see SOLOVAY [18]). (This proof was also noted by Martin in [8].)

The existence of a $\underset{\sim}{\Pi}_1^1$ counterexample, as presented above, depended essensially on the fact that $\omega_1 = \omega_1^L$. Assuming MC , hence determinacy of every $\underset{\sim}{\Pi}_1^1$ set, we are led to conclude that $\omega_1^L < \omega_1$, or more generally that $\omega_1^{L[a]} < \omega_1$ for all $a \subseteq \omega$. (For details see Solovay [18].) Thus we get a rather different proof of a theorem of Rowbottom and Gaifman.

Consistency of AD still remains unsettled. Concerning AD^* and AD^{**} the following result is an immediate corollary of SOLOVAY [20]:

THEOREM. Con(ZF + AC + $\exists\alpha$ (α inaccessible cardinal)) implies Con(ZF + DC + AD_2^* + AD^{**}) .

It has been suggested that perhaps L[R] , the minimal model of ZF which contains all reals and all ordinals, is a model for AD . It is known that DC is true in the model.

V. CONCLUDING REMARKS.

How seriously is AD to be regarded as a new axiom of set theory? In their note [13] Mycielski and Steinhaus are rather cautious. They note that AD has a remarkable deductive power, that it implies many desirable properties such as Lebesgue measurability of every subset of the real line. (And we may add: it implies the prewellordering theorem.) It does contradict the general axiom of choice, but implies a form of the countable axiom of choice which is sufficient for many of the most important applications to analysis. This is particulararily so if one is careful to add suitable assumptions of separability and hence will have no need for the most general forms of theorems such as Hahn - Banach, Tychonoff etc.

So if one is willing to judge the credibility of an axiom by its consequences, there are many arguments in favor of AD . However, one may argue that one would like to have a more direct insight into the "true" universe of set theory before one committs oneself to AD . And above all one would like to have consistency results, which are notably lacking in this field.

Perhaps one has not to go all the way to AD in order to obtain a significant strenghtening of the current set theoretic foundation. Let PD stand for the asseetion that every projective set is determinate, i.e. that the game $G_N(A)$ is determinate for any A belonging to some class $\underset{\sim}{\Delta}_n^1$, $n \in N$. The theory ZF + AC + PD could be the sought for strenghtening.

Many constructions of real analysis and probability theory are initially carried out within the class of Borelsets, $\underset{\sim}{\Delta}_1^1$, and one would be perfectly happy to remain there. But the Borelsets are not sufficiently closed with respect to operations analysts like to perform. The full projective hierarchy seems to be the next class beyond the Borelsets which has reasonable closure properties. But in passing from the Borelsets to all of the projective hierarchy one is not able at the same time to extend the nice properties which hold at the lower levels of the hierarchy. The strenght of the theory ZF + AD + PD is that many of the desirable properties of the first few levels provably extends to the whole hierarchy (e.g. Lebesgue measurability, reduction and separation principles etc.) (And we may add that from this point of view it is a defect of ZF + V = L that in this theory we can find counterexamples to many of these properties within the projective hierarchy, even at low levels.)

Is ZF + AC + PD consistent? It has been suggested that large cardinal assumptions will do, but not much seems to be known. (See Martin [8] for some further remarks.)

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