Critical Measures and Parameter Space of Jenkins-Strebel Quadratic Differentials

Thesis for the degree of Master of Science in Mathematical Analysis

Anastasia Frolova

Department of Mathematics
University of Bergen

Norway

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Introduction

The thesis is devoted to applications of the theory of quadratic differentials to the problems of construction of equilibrium measures and of description of their support for classical Coulomb potential with a logarithmic weight. It is worth mentioning that classically, quadratic differentials appeared in geometric function theory in late 30-th in the pioneering works by O. Teichmüller [Tei40] who revealed deep relations between extremal problems for conformal maps and quadratic differentials. An heuristic principle named after him states that considering extremum of a continuous functional on the space of conformal normalized embeddings of the unit disk to the complex plane leads us to a certain quadratic differential uniquely defined by the functional. In particular, fixing a value of a function yields the existence of a simple pole of such a differential, and fixing of values of derivatives yields the existence of poles of higher order. M. Schiffer's variational method confirmed this principle in many particular cases. That time a lot of activities in geometric function theory was focused on L. Bieberbach conjecture proved later in 1984 by L. de Branges by using Loewner's method. But that time many partial results were obtained by means of the variational method. However, the original ideas of Teichmüller extended much further to the theory of quasiconformal maps and Teichmüller spaces where quadratic differentials were used for construction of extremal quasiconformal mappings and the Teichmüller metric on the moduli space of Riemann surfaces. Last decade has been marked by a burst of interest to quadratic differentials from specialists in potential theory and approximations where quadratic differentials started to play role in some extremal problems of different nature, in particular, in construction of equilibrium measures, see [MFR11]. We develop this idea and describe quadratic differentials that can be used in this construction.

In the first chapter we introduce the notion of quadratic differential. It is basically a holomorphic (meromorphic) form defined on a Riemann surface. It turns out that this notion gives rise to geometry structure on this Riemann surface. A quadratic differential defines a field of line elements in a natural way. Therefore, one can consider integral curves of this field, that are called trajectories of the quadratic differential. Moreover, a specific conformal invariant metric can be associated to the quadratic differential.

Later on, we deal with a special type of quadratic differentials, so-called Jenkins-Strebel quadratic differentials, with trajectories having finite length with respect to this metric. Such a kind of quadratic differentials was first introduced and studied by J. Jenkins [Jen58] and later by K. Strebel [Str84]. Jenkins-Strebel quadratic differentials were successfully applied to extremal problems for conformal and quasiconformal maps, in particular, by G.V. Kuz'mina [Kuz82], A. Solynin [Sol99], A. Vasiliev [Vas02].

The Chapter 2 describes an application of quadratic differentials to the study of the limit distributions of zeros of polynomial solutions to the generalized Lamé equation.

This ordinary differential equation was introduced 1837 by G. Lamé. It was obtained in a way of separating variables in the Laplace equation with respect to elliptic coordinates. See, for example, [CH89]. The Lamé equation has the following form:

$$q(x)\frac{d^2y}{dx^2} + \frac{q'(x)}{2}\frac{dy}{dx} + (\alpha x + \beta)y = 0, \tag{1}$$

where $q(x) = 4(x-a_1)(x-a_2)(x-a_3)$; a_1, a_2, a_3 are given constants; and α, β are constants that are involved in the separation of variables. So the solution to this equation depends on the choice of α and β. We focus on polynomial solutions to the Lamé equation, whereas solutions of a different form can be found. For instance, for the particular choice $\alpha = 2, \beta = -a_1 - a_2$, the function $y = \sqrt{x - e_1}$ satisfies (1). The question of existence and characterization of polynomial solutions to the Lamé equation was studied by T. Stieltjes, H. Heine, G. Pólya, B. Shapiro. The interest to this problem was caused by Stieltjes' discovery of applications of zeros of the polynomial solutions to an electrostatic extremal problem. Stieltjes [Sti85] considered more general version of the Lamé equation, namely,

$$\prod_{k=1}^{l}(x-a_k)\frac{d^2y}{dx^2} + \sum_{j=1}^{l} b_j \prod_{i\neq j}(x-a_i)\frac{dy}{dx} + V(x)y = 0. \tag{2}$$

Here $a_1 < a_2 < ... < a_l$ are real, b_j, $j = 1, ..., l$, are positive, $V(x)$ is a polynomial of degree at most $l-2$, and the solution depends on the choice of $V(x)$. It was established by Sieltjes and Van-Vleck that in this particular setting there exist

$$\sigma(n) = \left(\begin{array}{c} n+l-2 \\ n \end{array} \right)$$

polynomials $V(x)$ of degree $l-2$, such that (2) admits a polynomial solution of degree n. Let us introduce now a general form of the Lamé equation that plays an important role in the second chapter. Let $A(z)$ be a polynomial with the set of zeros $a_1, ..., a_l$ lying in the complex plane, and let $B(z)$ be a polynomial of degree $l-1$ with a complex leading coefficient α. Finally, let $V(z)$ be a polynomial of degree at most $l-2$, in addition, $V(z)$ is monic in the case $\deg V = l-2$. The generalized Lamé equation has the following form:

$$A(z)\frac{d^2y}{dz^2} + B(z)\frac{dy}{dz} - n(n+\alpha-1)V(z)y = 0. \tag{3}$$

It was proved by Shapiro [Sha11] that for any given A and B there exists a natural number N, such that for any $n \geq N$, there exist $\sigma(n)$ polynomials V of degree $l-2$, for which the generalized Lamé equation has a polynomial solution of degree n.

Stieltjes showed in [Sti85] that the counting measure supported on the set of zeros of y satisfying (2), provides the equilibrium position, in a certain system of charges, which corresponds to this equation. A. Martínez-Finkelstein, E. Rakhmanov [MFR11] studied generalizations of this problem for the Lamé equation of the form (3). They considered so-called critical measures providing critical values of the logarithmic energy of a charge system on the complex plane, such that the system corresponds to (3). It turns out that critical measures are supported on the zero set of solutions to (3) and *-weak limits of these measures have their support lying on trajectories of a quadratic differential represented by a rational function. In Chapter 2 we overview the properties of such measures.

B. Shapiro, K. Takemura, M. Tater in [STT11] considered a particular form of the generalized Lamé equation when the degree of the polynomial A is $l = 3$. They studied sequences of polynomials $V_n(z)$, such that the Lamé equation admits a polynomial solution y_n of degree n. They established convergence of the corresponding to y_n zero-counting measures to measures with certain properties. In the third chapter we consider the problem of characterizing the set of polynomials V_n, such that the corresponding limiting measures are supported on the trajectories of quadratic differential $Q(z)\,dz^2$. For $l = 3$ this problem can be reduced to a problem of describing the space of parameter c of a quadratic differential of the form

$$Q(z)\,dz^2 = -\frac{1}{4\pi^2}\frac{z-c}{(z-a_1)(z-a_2)(z-a_3)}\,dz^2.$$

This question was studied in collaboration with A. Vasiliev and A. Solynin and the results represent our own contribution as well as the description of the parameter space of Jenkins-Strebel quadratic differentials given in the last chapter. Also we completed, revised, and clarified at some points original proofs presented in Chapter 2.

Chapter 1

Quadratic differentials

This chapter is devoted to the properties of a quadratic differential on a Riemann surface. We start with a brief introduction to the notions we frequently use, namely, a Riemann surface and basic structures on it.

Riemann surfaces were introduced in 19th century by B. Riemann and were mainly used in order to represent multivalued functions by single valued ones. Later on, the Riemann theory was developed by K. Weierstrass and others. Nowadays, Riemann surfaces may be understood as one of the basic mathematical structures. Let us now define a Riemann surface.

Let S be a connected Hausdorff topological space, such that whenever we pick a point $p \in S$, there is an open set U containing p, which is homeomorphic to a domain in $\mathbb{C}$. This basically means that S locally has the same topological properties as the complex plane.

We assume that S has an open covering $\{U_\alpha\}$ with corresponding homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{C}$. We call the pairs $(U_\alpha, \varphi_\alpha)$ charts. Two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are compatible if whenever $U_\alpha \cap U_\beta \neq \emptyset$ the maps

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta),$$
$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are analytic. We call the collection $\{U_\alpha, \varphi_\alpha\}$ an atlas on S if its charts are pairwise compatible and their collection $\{U_\alpha\}$ covers S. One can construct so-called maximal atlas by adding all compatible charts to a given atlas. Finally, S together with its maximal atlas is a Riemann surface. In other words, a *Riemann surface* is a complex analytic manifold of dimension 1.

Remark 1.1. The maximal atlas is called a complex structure.

Example 1.2. One of the basic examples of a Riemann surface is the Riemann sphere. We set $S = \hat{\mathbb{C}}$. One can construct an atlas that consists of the following charts: $(\mathbb{C}, \mathrm{id})$, $(\hat{\mathbb{C}} \setminus \{0\}, \frac{1}{z})$ for $z \in \mathbb{C}$, and $\frac{1}{\infty}$ is defined to be zero. Since $\frac{1}{z}$

is analytic on $\mathbb{C} \setminus \{0\}$, the atlas defined above gives a complex structure. Note that the Riemann sphere is homeomorphic to the 2-sphere, and $\frac{1}{z}$ is basically an inversion of the 2-sphere. Both functions defined in the charts are homeomorphisms.

Example 1.3. Consider $\mathbb{C}/\Lambda$, where Λ is a lattice. One can define a quotient topology on $\mathbb{C}/\Lambda$ with respect to the projection map

$$\begin{aligned} p : \mathbb{C} &\to \mathbb{C}/\Lambda, \\ z &\mapsto [z]. \end{aligned}$$

Consider collection of all circles of diameter less than $\frac{1}{2}\inf_{w\in\Lambda\setminus\{0\}}|w|$ on the complex plane. Then $\{(p(V_\alpha), p|_{V_\alpha}{}^{-1})\}$, forms the complex structure for the quotient space, where V_α belongs to the collection defined above.

Let us consider now a chart (U, φ) to be an element of a complex structure. The composition $s \circ \varphi$ is called a local coordinate on U, where s is a coordinate in $\mathbb{C}$. In general, we call (U, φ) a local coordinate system.

Now we introduce briefly some useful objects on a Riemann surface.

A function $f : S \to \mathbb{C}$ is called analytic on S if for any chart (U, φ) on S the function

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$$

is analytic.

Let us pick a point $p \in S$ and a chart (U, φ) about it. Let z be a local coordinate on U. We denote the real and imaginary parts of z by x and y correspondingly. The pair $\left(\frac{\partial}{\partial z}\Big|_p, \frac{\partial}{\partial \bar{z}}\Big|_p\right)$ forms a basis of the tangent space of the Riemann surface S at the point p. Here

$$\begin{aligned} \frac{\partial}{\partial z}\Big|_p &= \frac{1}{2}\left(\frac{\partial}{\partial x}\Big|_p - i\,\frac{\partial}{\partial y}\Big|_p\right), \\ \frac{\partial}{\partial \bar{z}}\Big|_p &= \frac{1}{2}\left(\frac{\partial}{\partial x}\Big|_p + i\,\frac{\partial}{\partial y}\Big|_p\right). \end{aligned}$$

We recall the definition of a partial derivative of the function f at the point p:

$$\frac{\partial}{\partial x}\bigg|_p = \frac{\partial}{\partial r}\bigg|_{\varphi(p)} f \circ \varphi^{-1},$$

where $s = r + it$ is a coordinate on $\mathbb{C}$.

Covectors $dz|_p, d\bar{z}|_p$ form a basis of the cotangent space of S at the point p.

Consider differentials dz, $d\bar{z}$. They are 1-forms that assign the covectors $dz|_p, d\bar{z}|_p$ to the point p.

Consider now a linear function

$$f : V \to \mathbb{C},$$

where V is a vector space. We define a symmetric product of two linear functions $f(x)$, $g(y)$ as follows:

$$f \vee g = \frac{1}{2}\left(f(x)g(y) + f(y)g(x)\right).$$

We call a bilinear function $g(x, y)$ symmetric if $g(x, y) = g(y, x)$.

Remark 1.4. The symmetric product of two linear functions $f : V \to \mathbb{C}$, $g : V \to \mathbb{C}$ is a bilinear symmetric function $f \vee g : V \times V \to \mathbb{C}$.

A symmetric 2-form ω on S assigns a symmetric bilinear function $\omega|_p$ to the point p. By the remark above, the symmetric product of two 1-forms is a symmetric 2-form.

We define a holomorphic (meromorphic) *quadratic differential* as a holomorphic (meromorphic) symmetric 2-form on a Riemann surface S. If z is a local coordinate, quadratic differential is locally represented as $\varphi(z)\,dz^2$, $dz^2 = dz \vee dz$, $\varphi(z)$ is a holomorphic (meromorphic) function on S. Alternatively, we can define holomorphic (meromorphic) quadratic differential as follows.

Definition 1.5. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a complex structure on a Riemann surface S. Then Q is a holomorphic (meromorphic) *quadratic differential* on S if for any local coordinate z_α the functional element $Q \circ \varphi_\alpha^{-1}$ is holomorphic (meromorphic) and satisfies the following rule of change of coordinates:

$$Q_\alpha(z_\alpha)\,dz_\alpha^2 = Q_\beta(z_\beta)\,dz_\beta^2,\ dz_\beta = \frac{dz_\beta}{dz_\alpha}dz_\alpha. \tag{1.1}$$

Here z_α and z_β are local coordinates that correspond to the same point on S.

The rule (1.1) establishes the invariance of the quadratic differential under the change of variables. Therefore, we can fix some local coordinate z and denote the quadratic differential by $Q(z)\,dz^2$.

We call zeros and poles of a quadratic differential *critical* points. If $P \in S$ is not critical, we call it *regular*. This definition makes sense because it turns out that the property of having zero or pole at some point $P \in S$ does not depend on the choice of the local coordinate.

Indeed, let us pick a point $P \in S$ and two charts (U_1, φ_1), (U_2, φ_2) about it. Without loss of generality we may assume that $\varphi_1(P) = 0$, $\varphi_2(P) = 0$. We denote by z, $\tilde{z}$ the local coordinates corresponding to the charts chosen above. By the assumption, $z = a_1\tilde{z} + a_2\tilde{z}^2 + a_3\tilde{z}^3 + ...$ Therefore, we have

$$\frac{dz}{d\tilde{z}} = a_1 + 2a_2\tilde{z} + 3a_3\tilde{z}^2 + ...$$

Let Q_1, Q_2 be functional elements of the quadratic differential Q with respect to z and $\tilde{z}$ correspondingly. Let $z = 0$ be the zero of order n for Q_1. Then Q_1 has the following expansion at the origin:

$$Q_1(z) = z^n \left(b_n + b_{n+1}z + b_{n+2}z^2...\right).$$

The rule (1.1) implies that

$$Q_2(\tilde{z}) = \left(a_1\tilde{z} + a_2\tilde{z}^2 + ...\right)^n \left(b_n + b_{n+1}(a_1\tilde{z} + ...) + ...\right) \left(a_1 + 2a_2\tilde{z} + ...\right)^2.$$

Therefore, the expansion of $Q_2(\tilde{z})$ at the origin has the form

$$Q_2(\tilde{z}) = \tilde{z}^n(b_n a_1^{n+2} + ...).$$

Since $a_1 \neq 0$, $b_n \neq 0$, we conclude that Q_2 has a zero of order n at the point $\varphi_2(P)$.

Example 1.6. Consider a quadratic differential Q on the Riemann sphere. Let us set $Q = \frac{(z-1-i)^2}{z^2-1}\,dz^2$ for a fixed local parameter z. This is a meromorphic quadratic differential with zero of order 2 at the point $1+i$, simple poles at 1, -1 and pole of order 4 at ∞.

Consider the complex structure defined in Example 1.2. Let z stand for the chart $(\mathbb{C}, \mathrm{id})$. In this coordinate system Q is locally represented by the function $\frac{(z-1-i)^2}{z^2-1}$, which has simple poles at $z = 1$, $z = -1$ and zero of order 2 at $z = 1+i$. Obviously, their preimages by identity are $-1, 1, 1+i \in \hat{\mathbb{C}}$. We can conclude that these are finite critical points of Q.

Let us change the coordinate $\tilde{z} = \frac{1}{z}$. By the rule of change of coordinates we get

$$Q\left(\tilde{z}\right) = Q(z)\left(\frac{dz}{d\tilde{z}}\right)^2 = \left(\frac{1}{\tilde{z}}\right)^4 \frac{\left(\frac{1}{\tilde{z}} - 1 - i\right)^2}{\left(\frac{1}{\tilde{z}} - 1\right)\left(\frac{1}{\tilde{z}} + 1\right)}$$

in the neighbourhood of ∞. So Q has the pole of order 4 at ∞.

Note that, for instance, this functional element has zero at the point $\frac{1}{1+i} \in \mathbb{C}$. The preimage of this point is $1+i \in \hat{\mathbb{C}}$. This illustrates that the notion of critical point of a quadratic differential is well-defined.

It turns out that once the quadratic differential is defined, one can associate the horizontal and the vertical direction at any regular point on S. Therefore, horizontal and vertical arcs may be considered. In order to investigate the structure of these arcs we need to look at some specific local representations of quadratic differentials. The idea is to introduce a local coordinate such that the local representation of a quadratic differential in terms of it has a simple form. We call this coordinate the natural parameter. It makes sense to distinguish the natural parameters near regular and critical points. This method was described in [Str84].

Let P be a regular point of a quadratic differential $Q(z)\,dz^2$ on a Riemann surface S. So the point P has a neighbourhood small enough to choose a single valued branch of $\sqrt{Q(z)}$. We define the natural parameter near the regular point as:

$$w = \int \sqrt{Q(z)}\,dz. \tag{1.2}$$

By the rule of change of variables we get:

$$Q(z) = Q(w)\left(\frac{dw}{dz}\right)^2.$$

Therefore, $Q(w) \equiv 1$ in this neighbourhood of P. In other words, $Q(z)dz^2 = dw^2$ in the corresponding neighbourhood. Note that the natural parameter near a regular point is defined up to a constant.

Note that the function $\int \sqrt{Q(z)}\,dz$ plays an important role in determining the geometric structure associated to Q.

In the case when $P \in \hat{\mathbb{C}}$ is a pole or a zero of $Q(z)\,dz^2$ we can not always pick a single valued branch of the square root of the functional element in the neighbourhood of P. Therefore, the parameter defined by (1.2) can be not single valued. That is why we have to consider a different form of the natural parameter near the critical points.

Let $P \in S$ be a critical point of order n of the quadratic differential $Q(z)\,dz^2$. In addition, consider n to be odd integer. Without loss of generality we may assume that $z(P) = 0 \in \mathbb{C}$. Then the quadratic differential may be represented in terms of z as:

$$Q(z)dz^2 = z^n\left(a_n + a_{n+1}z + ...\right)\,dz^2 \tag{1.3}$$

where $a_n \neq 0$. We may consider a covering surface $\left(\hat{S}, f\right)$, f is projection map, $f(\hat{P}) = P \in S$. We define f, such that for a local parameter ξ about $\hat{P}$ we get $z = \xi^2$. We define $\hat{Q}(\xi)$ as

$$\hat{Q}(\xi)\,d\xi^2 = Q(z)\,dz^2.$$

The quadratic differential $\hat{Q}$ is called the lift of Q. Then, by the rule of change of variables, the quadratic differential $\hat{Q}$ has the form

$$\xi^{2n+2}\left(4a_n + 4a_{n+1}\xi^2 + 4a_{n+2}\xi^4 + ...\right) d\xi^2.$$

Then, in a small enough neighbourhood of the origin, we can choose a single valued branch of the square root of the functional element. In a punctured neighbourhood of the origin we obtain

$$\sqrt{\hat{Q}(\xi)} = \xi^{n+1}\left(b_0 + b_1\xi^2 + b_2\xi^4 + ...\right). \tag{1.4}$$

Here coefficients b_i correspond to the series representation of the square root of $(4a_n + 4a_{n+1}\xi^2 + 4a_{n+2}\xi^4 + ...)$. Let us integrate the right-hand side of the last expression term by term. We obtain

$$\xi^{n+2}\left(c_0 + c_1\xi^2 + c_2\xi^4 + ...\right).$$

for some c_i. In a small enough neighbourhood of the origin we choose a single valued branch of $(c_0 + c_1\xi^2 + ...)^{\frac{1}{n+2}}$. Suppose it has the series expansion $d_0 + d_1\xi^2 + ...$ at the origin. Then we set a single valued function

$$\zeta = \xi\left(d_0 + d_1\xi^2 + ...\right).$$

By differentiating ζ^{n+2} and squaring we obtain the following representation of the functional element

$$\hat{Q}(\zeta) = (n+2)^2\,\zeta^{2(n+1)}. \tag{1.5}$$

We introduce the parameter $w = \zeta^2$ about P. Then, by the change of variables rule we arrive at

$$\begin{aligned} \hat{Q}(\zeta) &= Q(w)\left(\tfrac{dw}{d\zeta}\right)^2, \\ (n+2)^2\,\zeta^{2(n+1)} &= Q(w)4\zeta^2. \end{aligned} \tag{1.6}$$

Therefore, in a punctured neighbourhood of the origin the quadratic differential in terms of w has the form

$$Q(z)\,dz^2 = \left(\frac{n+2}{2}\right)^2 w^n\,dw^2 \tag{1.7}$$

We call w the natural parameter of Q near the critical point P.

Suppose now P is a zero of even order. For a fixed parameter $z(P) = 0 \in \mathbb{C}$ we obtain representation (1.3), n is even and positive. Then in some neighbourhood of the origin we can choose a single valued branch of the square root of Q and integrate termwise. We obtain the following expression:

$$z^{\frac{n+2}{2}}\left(\tilde{c_0} + \tilde{c_1}z + ...\right).$$

Let $\tilde{d_0} + \tilde{d_1}z + ...$ correspond to a single valued branch of $(\tilde{c_0} + \tilde{c_1}z + ...)^{\frac{2}{n+2}}$. We define a natural parameter of Q near the odd pole P as follows:

$$w = z\left(\tilde{d_0} + \tilde{d_1}z + ...\right).$$

This parameter is well defined. The functional element of Q has representation (1.7).

So we proved the following theorem:

Theorem 1.7. Let Q be a quadratic differential on a Riemann surface S. If $P \in S$ is a critical point of order n of Q, n is either positive or negative and odd, then there exists a local parameter w about P in terms of which the the quadratic differential has the form (1.7).

Remark 1.8. Note that w is defined up to a factor $e^{\frac{2\pi il}{n+2}}, l = 0, ..., n+1$.

In case of an even order pole we get logarithmic terms when integrating the square root of the functional element and we can not use the same algorithm. We deal with the case of a second order pole in the following way.

Let $P \in S$ be a pole of order 2 of the quadratic differential Q. Then locally Q may be represented by (1.3) for $n = -2$. Taking the square root and integrating term by term we get

$$b_0 \log z + b_1 z + ...$$

We put $\log w$ equal to the last expression divided by b_0. Then w has the form

$$w = ze^{\frac{b_1}{b_0}z+...},$$

and the quadratic differential can be represented as

$$Q(z)\,dz^2 = \left(b_0 w^{-1}\,dw\right)^2\,dw^2 = a_{-2}w^{-2}\,dw^2. \tag{1.8}$$

Let us turn now to the case of an even order pole n, $n > 2$. We integrate a single valued branch of the square root of the functional element about a pole P. The local variable z is set such that the pole P is mapped to the origin. After integration we obtain the sum of the logarithmic term $b \log z$ and the powers of z: $z^{\frac{n}{2}+1}(c_0 + ...)$. We put $w = d_0 z + d_1 z^2 + ...$, such that

$$b \log z + z^{\frac{n+2}{2}}(c_0 + ...) = b \log w + w^{\frac{n+2}{2}} + c.$$

Then the quadratic differential Q has the following representation:

$$Q(z)\,dz^2 = \left(bw^{-1} + \frac{n+2}{2}w^{\frac{n}{2}}\right)^2\,dw^2. \tag{1.9}$$

Now we can formulate the theorem:

Theorem 1.9. Let $P \in S$ be a pole of odd order n of the quadratic differential Q. Then there exists a parameter w in a neighbourhood of P, such that

1. In the case n=2 the quadratic differential has the representation (1.8) in terms of w.

2. In the case $n > 2$ the quadratic differential has the representation (1.9) in terms of w.

Now we turn to describing a geometric structure corresponding to a quadratic differential.

A *trajectory* of the quadratic differential $Q(z)\,dz^2$ is a maximal smooth curve $\gamma \in S$, such that

$$\arg Q(z)\,dz^2 = 0 \tag{1.10}$$

along γ.

A maximal curve lying in the Riemann surface S, such that

$$\arg Q(z)\,dz^2 = \pi,$$

is called the *orthogonal trajectory* of the quadratic differential Q.

Remark 1.10. Note that the trajectories of the quadratic differential Q are the orthogonal trajectories of the quadratic differential $-Q$.

The trajectories can be also defined as the maximal curves along which the inequality $Q(z)\,dz^2 > 0$ holds. In other words, the integral curves of the vector field associated to this inequality are the trajectories of the corresponding quadratic differential. We also can consider them as maximal solutions to the equation

$$Q(z)\left(\frac{dz}{du}\right)^2 = 1.$$

Here u stands for the natural parameter of the curve.

Let $P \in S$ be a regular point of the quadratic differential $Q(z)\,dz^2$. Let w be the natural parameter of Q at P. Then the quadratic differential has the form dw^2 in terms of w. The curve γ_w in the $w-$plane along which the equality $\arg w = 0$ is satisfied is simply a horizontal line. So the preimage $w^{-1}(\gamma_w)$ is an arc of the trajectory of the quadratic differential in the neighbourhood of the regular point P.

The trajectory structure near critical points of the quadratic differential is more complicated. But we also can use the representation of the quadratic differential in terms of the natural parameter w near critical points in order to describe the trajectory structure.

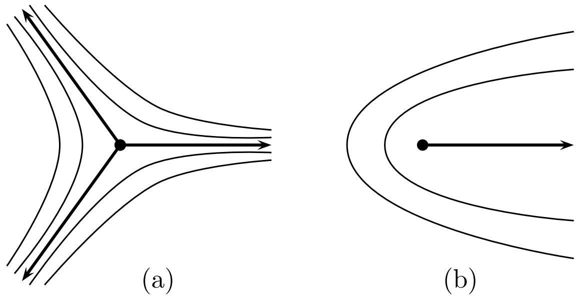

Figure 1.1: The local trajectory structure near (a) simple zero, (b) simple pole

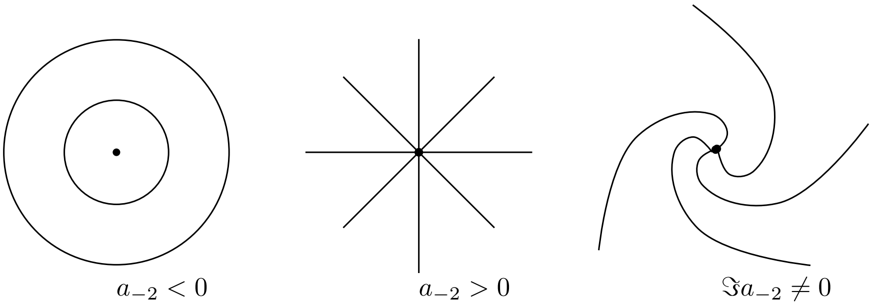

Figure 1.2: The local trajectory structure near a double pole

Remark 1.11. Analytic homeomorphisms $\varphi_k : U_k \to \mathbb{C}$ belonging to the complex structure on S are automatically conformal. Therefore, both topological and geometrical structures are preserved under these mappings and their inverses. That is why, the trajectory structure and the structure of the homeomorphic images of the trajectories are identical locally.

Let P be a zero of order n. Then, according to Theorem 1.7, the quadratic differential Q can be represented as in (1.7). Integrating the square root of Q we get $w^{\frac{n+2}{2}}$. Then the $w-$plane is divided into $n+2$ sectors and the function $v = w^{\frac{n+2}{2}}$ maps each of them to a half-plane. The trajectories on the $w-$plane are mapped by $w^{\frac{n+2}{2}}$ to the horizontal lines in the $v-$plane.

Analogously we reveal the trajectory structure near a simple pole. So the trajectory structure about a zero and a simple pole is visualized in Figure 1.1.

In the case of a pole of order 2 we use Theorem 1.9 and set $v = \sqrt{a_{-2}} \log w$. Depending on whether a_{-2} is negative, positive or non-real, we get three different pictures corresponding to the trajectory structure on the $w-$plane.

Now let P be a pole of higher odd order. We use Theorem 1.7 and put

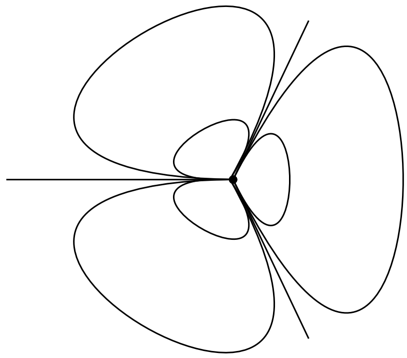

Figure 1.3: The local trajectory structure near a pole of order 5

$v = w^{\frac{n+2}{2}}$, where n stands for the power of leading term of the functional element expansion. The trajectories of Q in the $w-$plane are mapped by v to horizontal straight lines. When n is even, it is a bit tricky to deal with the logarithmic term. But introducing new parameters and getting the preimages as above we obtain a similar picture: the $w-$plane is divided into $|n| - 2$ sectors, in each sector the trajectories tend to $w(P)$ in two directions as on the Figure 1.3.

One can associate a conformal metric with the quadratic differential $Q(z)\,dz^2$ by setting the length element to be $|dw| = |Q(z)|^{\frac{1}{2}}|dz|$. The length of some curve $\gamma \in S$ can be defined as:

$$|\gamma|_Q = \int_{\gamma_w} |dw|,$$

where γ_w is the image of the curve γ in the $w-$plane. Note that we can continue analytically the branch of square root along γ in order to get the whole image of the curve.

The corresponding area element is $|Q(z)|^{\frac{1}{2}}\,dx\,dy$.

We define an L_1-norm of Q as

$$\|Q\| = \iint_S |Q(z)|\,dx\,dy.$$

If the closure of a trajectory of the quadratic differential Q contains a critical point of Q, it is called a critical trajectory. Let us denote by Φ the union of all critical trajectories and their closures. Then $S \setminus \Phi$ consists of a certain number of domains, we call it a domain decomposition. The comprehensive description of this global geometric structure for certain types of surfaces and quadratic differentials was given by J. Jenkins in Basic structure theorem, [Jen58] and K. Strebel [Str84]. Let us consider several types of domains, which may be associated to a quadratic differential.

Definition 1.12. Let $D \subset S$ be a maximal doubly connected domain, such that it does not contain critical points and whenever a trajectory passes through a point in D, it lies entirely in D. Moreover, there exists a map

$$v = \exp\{c \int \sqrt{Q(z)}\, dz\},\ c \neq 0,$$

which maps D onto a ring. Then we call D a *ring domain.*

Definition 1.13. Let $D \subset S$ be a maximal simply connected domain, such that it contains a double pole p and is swept out by trajectories separating the pole from δD, and whenever a trajectory passes through a point in D, it lies entirely in D. Moreover, there can be found a map $v = \exp\{c \int \sqrt{Q(z)}\, dz\}$, $c \neq 0$ that maps $D \setminus p$ to a disc $|v| < R$. The pole is mapped to a origin. Such D is called a *circular domain.*

Definition 1.14. Let $D \subset S$ be a maximal simply connected domain, such that it is swept out by trajectories connecting two double poles lying on δD, and whenever a trajectory passes through a point in D, it lies entirely in D. Moreover, there is a map $v = \int \sqrt{Q(z)}\, dz$, which maps D onto a strip $a < \Im v < b$. Such D is called a *strip domain.*

Definition 1.15. Let $D \subset S$ be a maximal simply connected domain, such that it is swept out by trajectories having both their limiting end points at a pole of order $n > 2$, and whenever a trajectory passes through a point in D, it lies entirely in D. Moreover, there is a map $v = \int \sqrt{Q(z)}\, dz$, which maps D onto an upper or a lower halfplane. Then D is called an *ending domain.*

Let S be a compact Riemann surface and Q be a meromorphic quadratic differential on S.

Note that a quadratic differential on a compact Riemann surface has a finite L_1-norm if and only if it does not have poles of order higher than two.

Let P be a regular point of the quadratic differential $Q(z)\, dz^2$. Consider natural parameter $w = \int \sqrt{Q(z)}$ of $Q(z)\, dz^2$ near P. We can obtain representation of a trajectory containing the point P in terms of inverse of the map $w = \int \sqrt{Q(z)}$.

There exists a neighbourhood U of P which is mapped homeomorphically and conformally onto a disk V centered in the origin in the $w-$plane. Let us construct the analytic continuation of w^{-1} along the chain of discs centered on the real axis. We denote the real axes by $\mathbb{R}$. Let us pick a point $u \in V \cap \mathbb{R}$.

There is a neighbourhood $\tilde{U}$ around $w^{-1}(u_1)$, which is mapped homeomorphically and conformally onto a disk $\tilde{V}$. We choose the branches of w such that they coincide on $U \cap \tilde{U}$. Then we may choose a point $\tilde{u} \in \tilde{V} \cap \mathbb{R}$ and proceed the procedure. We denote the obtained chain of discs by C. Then w^{-1} is uniquely defined on C. In addition, it is a conformal homeomorfism of C onto its image. Let u belong to $C \cap \mathbb{R}$, then $w^{-1}(u)$ defines a trajectory γ passing through P. By this representation, $\gamma = w^{-1}(I)$, where I is an interval (u_1, u_2) on the real axis. Note that the $Q-$length of γ is equal to $u_2 - u_1$.

The point P divides γ into two rays. More precisely, we define a trajectory ray γ^+ to be $w^{-1}([0, u_2))$, and a trajectory ray γ^- to be $w^{-1}((u_1, 0])$. Moreover, they are oriented such that they start at the point P. We define $A^+ = \lim_{u \to u_2} \overline{w^{-1}([u, u_2))}$ to be a limit set of the trajectory ray γ^+. Analogously, $A^- = \lim_{u \to u_1} \overline{w^{-1}((u_1, u])}$ is a limit set of γ^-. We call a trajectory ray γ^+ *recurrent*, if $P \in A^+$. It turns out that if γ^+ is recurrent, the interior of the corresponding limit set is a domain. Moreover, it is bounded by critical trajectories with finite $Q-$length. Note that critical trajectory γ of Q has a finite $Q-$length if and only if both γ^+ and γ^- tend to either a zero or a simple pole of Q.

Now we turn to describing the global trajectory structure of the quadratic differential Q on S.

Remark 1.16. All situations conformally equivalent to the special cases, when $S = \hat{C}$, $Q(z)\, dz^2 = dz^2$ or $Q(z)\, dz^2 = \frac{re^{i\alpha}}{z^2}\, dz^2$, where $r > 0$ and α is real, are out of following consideration.

Let γ be the trajectory of Q. Suppose, one of the corresponding rays tends to a zero or a simple pole. Then the other ray either tends to a critical point or is recurrent.

If both γ^+ and γ^- tend to a pole of order $n \geq 2$, then a strip domain or an ending domain appears in the domain decomposition.

If both γ^+ and γ^- are recurrent, a spiral domain $D = int\overline{\gamma}$ can be associated to the quadratic differential.

If non-critical closed trajectories appear in the global trajectory structure, they sweep out the circular domains and the ring domains.

The recurrent rays of the trajectories may induce so-called *dense structure*, when the trajectory is dense in a certain domain.

We conclude the following

- We associate ring domains, circular domains, ending domains, strip domains, spiral domains and dense structures with the meromorphic quadratic differential Q on the compact Riemann surface S.

- If Q has a finite L_1-norm, it possesses the domain decomposition consisting of ring domains and dense structures.
- If the trajectories of Q have finite $Q-$length, the domain decomposition consists of ring domains, circular domains, ending domains and strip domains.

Note that in the most of cases dense structures appear in the domain decomposition.

See more details on the global trajectory structure in [Vas02, Str84].

The trajectory structure of a quadratic differential has not only its own interest. This theory is applicable to various problems related to conformal and quasiconformal maps. For instance, quadratic differential gives a solution to the problem of maximizing reduced moduli of punctured discs, which was posed and solved by O. Teichmüller in late 30-s [Tei40]. Another example is the problem of maximizing weighted moduli sum. The solution was given by J. Jenkins in a form of Jenkins-Strebel differentials.

Chapter 2

Lamé equation and critical measures

In this chapter we will be mainly concerned with the generalized Lamé equation

$$A(z)\frac{d^2y}{dz^2} + B(z)\frac{dy}{dz} - n(n+\alpha-1)V(z)y = 0, \tag{2.1}$$

where A and B are polynomials of degree l and $l-1$ correspondingly; V is a polynomial as well, $\deg V \le l-2$. The polynomials A and B are fixed, while V can vary. This is a natural generalization of the classical Lamé equation given in the introduction.

In [MFR11] the authors described the limit distribution of the zeros of the polynomial solutions to the generalized Lamé equation. However, the motivation for studying the polynomial solutions to the Lamé equation was given by Stieltjes in 1880-s [Sti85]. His electrostatic model involves a system of charges, which can be associated with a particular kind of the Lamé equation. Stieltjes showed that the zeros of the polynomial solution to this equation correspond to the equilibrium position of the charge system.

Consider a set $\{a_1, ..., a_l\}$, $a_k \in \mathbb{R}$, such that $a_k < a_{k+1}$ for $k = 1, ..., l-1$. These points correspond to the positions of l external charges. Let b_j, $j = 1, ..., l$, be positive numbers corresponding to the values of the external charges. For each $k = 1, ..., l-1$ we put n_k unit charges in the interval (a_k, a_{k+1}). Let us set $n = n_1 + ... + n_{l-1}$. We define an external field as

$$\phi(x) = \Re\left(\sum_{k=1}^{l} \frac{b_k}{2}\log\frac{1}{x-a_k}\right).$$

Let us consider the weighted potential energy

$$E^d = \sum_{i\neq k}\log\frac{1}{|\xi_i - \xi_k|} + 2\sum_{k=1}^{n}\phi(\xi_k), \tag{2.2}$$

where the points ξ_k correspond to the positions of the unit charges, and let us denote the Dirak delta distribution at ξ_k by δ_{ξ_k}. Then the discrete measure $\mu = \sum_{k=1}^{n} \delta_{\xi_k}$ can be associated with a system of unit charges.

We denote by M_n^d the class of measures of the form $\mu = \sum_{k=1}^{n} \delta_{z_k}$, $z_k \in \mathbb{C}$. Let us define a class M^d of measures on the complex plane as the union over n of the classes M_n^d. Finally, we define a class of measures M', consisting of measures $\mu = \sum_{k=1}^{n} \delta_{x_k} \in M_n^d$ supported on the interval (a_1, a_l), such that $\mu((a_k, a_{k+1})) = n_k$ for $k = 1, ..., l-1$.

Consider a problem of minimizing the weighted energy (2.2) in the class M'. In other words, we are looking for a measure corresponding to the equilibrium position of the system of charges.

Denote the minimizer by $\hat{\mu}$. Stieltjes proved in [Sti85] that the global minimizer corresponds to a unique equilibrium position. Moreover, the support of the minimizer $\hat{\mu}$ is formed by the zeros of the polynomial solution y of degree n to the corresponding Lamé equation.

$$\prod_{k=1}^{l}(x-a_k)\frac{d^2y}{dx^2} + \sum_{j=1}^{l} b_j \prod_{i\neq j}(x-a_i)\frac{dy}{dx} + V(x)y = 0. \tag{2.3}$$

It was proved by Shapiro in [Sha11] that for any given A and B there exists a natural number N, such that for any $n \geq N$, there exist $\sigma(n)$ polynomials V of degree $l-2$, for which the generalized Lamé equation has a polynomial solution of degree n.

The problem of minimizing the discrete energy of the charge system can be extended to analogous problem for a continuous energy.

We define M^c to be a class of probability Borel measures m with a compact support on the complex plane. We choose an external field $\phi = \Re\Phi(z)$, where $\Phi(z)$ is an analytic function. Assume in addition, that the field is an integrable function with respect to measures in M^c. Then we construct a continuous weighted energy

$$E^c = \iint_{\mathbb{C}} \log\frac{1}{|x-y|}\, dm(x)dm(y) + 2\int_{\mathbb{C}} \phi\, dm. \tag{2.4}$$

The extremal problem can be reformulated as the problem of finding the minimum of the energy with respect to the class M^c.

We call a minimizer $\hat{m}$ an equilibrium measure. Such minimizer exists and is unique under additional conditions [ST97].

Let us turn to a more general situation, when the charges are placed on the complex plane. This leads us to a study of measures corresponding to the critical points of the weighted logarithmic energy on the complex plane. This is the main subject of the paper by A. Martínez-Finkelstein and E. Rakhmanov [MFR11].

Definition 2.1. Let us choose a set $A \subset \mathbb{C}$, such that cap$A = 0$. Consider function $\phi = \Re\Phi(z)$, where $\Phi(z)$ is analytic and $\dfrac{d\Phi}{dz}$ is single valued. Let μ be a discrete measure in the class M^d, which is supported on $\mathbb{C} \setminus A$. So $\mu = \sum_{k=1}^{n} \delta_{z_k}$. Assume in addition, that the points z_k are pairwise distinct. We associate to μ and ϕ the weighted discrete energy E^d of the form (2.2) in the complex plane. We call μ $(A, \phi)-$*critical* in the complex plane if

$$\frac{\partial}{\partial z} E^d\Big|_{z=z_k} = 0 \tag{2.5}$$

for any $k = 1, ..., n$.

Consider now the generalized form (2.1) of the Lamé equation. Suppose,

$$\frac{B(z)}{A(z)} = \sum_{k=1}^{l} \frac{b_k}{z - a_k}.$$

for real b_k. The following theorem shows the relation between the extremal problem and the solution to the corresponding Lamé equation.

Theorem 2.2. Let A be the set of points $a_1, .., a_l$, $a_j \neq a_k$ when $j \neq k$. Choose ϕ, such that $\phi(x) = \Re\left(\sum_{k=1}^{l} \dfrac{b_k}{2} \log \dfrac{1}{z - a_k}\right)$, b_k are real. Then μ is $(A, \phi)-$critical if and only if there is a polynomial $V(z)$ such that zeros of the corresponding polynomial solution $y(z)$ form a support of μ.

Proof. Consider the derivative of the discrete logarithmic energy

$$\frac{\partial}{\partial z_k}\left(\sum_{i\neq k} \log \frac{1}{|z_i - z_k|}\right) = -\sum_{i\neq k} \frac{\partial}{\partial z_k} \log|z_i - z_k| = -\frac{1}{2}\sum_{i\neq k} \frac{1}{z_i - z_k}.$$

Here we use that

$$\begin{aligned}
\frac{\partial}{\partial z} \log|z - w| &= \frac{1}{2}\frac{\partial}{\partial z} \log|z - w|^2 \\
&= \frac{1}{2}\frac{1}{|z - w|^2}\frac{\partial}{\partial z}\left((z - w)\overline{(z - w)}\right) \\
&= \frac{1}{2}\frac{\overline{z - w}}{|z - w|^2} \\
&= \frac{1}{2}\frac{1}{z - w}.
\end{aligned} \tag{2.6}$$

By definition of critical measure, $\phi(z)$ has a single valued derivative

$$\begin{aligned}\frac{\partial}{\partial z}\phi(z) &= -\frac{\partial}{\partial z}\Re\left(\sum_{k=1}^{l}\frac{b_k}{2}\log(z-a_k)\right)\\ &= -\sum_{k=1}^{l}\frac{b_k}{2}\frac{\partial}{\partial z}\log|z-a_k|\\ &= -\sum_{k=1}^{l}\frac{b_k}{4}\frac{1}{z-a_k}.\end{aligned}$$

So in our case the condition (2.5) is equivalent to

$$-\frac{1}{2}\sum_{i\neq k}\frac{1}{z_i-z_k}-\sum_{k=1}^{l}\frac{b_k}{4}\frac{1}{z_k-a_k}=\sum_{i\neq k}\frac{1}{z_i-z_k}+\frac{1}{2}\frac{B(z_k)}{A(z_k)}=0.$$

We put $y(z)=(z-z_1)...(z-z_l)$. Let us rewrite the last expression in terms of y.

$$\left(\frac{y''(z_k)}{y'(z_k)}+\frac{B(z_k)}{A(z_k)}\right)=0.$$

This holds for any $k=1,...,n$. Then a polynomial $A(z)y''(z)+B(z)y'(z)$ has the degree $l+n-2$ and is divisible by $y(z)$. This implies that

$$A(z)y''(z)+B(z)y'(z)=V(z)y(z),$$

where $V(z)$ is a polynomial of degree $l-2$. This proves Theorem (2.2) and shows that if n unit charges are placed in a field of l external charges with values b_k, $k=1,...,l$, then the critical point of the resulting potential energy is provided by the zeros of solutions to the Lamé equation. □

B. Shapiro in [Sha11] obtained that the zeros of polynomial solutions to the Lamé equation corresponding to the external field of the form

$$\phi=\Re\left(\sum_{k=1}^{l}\frac{b_k}{2}\log\frac{1}{z-a_k}\right),$$

where b_k are complex, are included into the convex hull of the set $A=\{a_1,...,a_l\}$.

Let us turn to the continuous case.

We use the variational derivative of the energy functional with respect to a measure in order to define the continuous critical measure. Let D be a domain in the complex plane. We consider a continuous function $h : \hat{D} \to \mathbb{C}$. This function induces a variation F^t of a set $F \subset \mathbb{C}$ as

$$F^t = \{z + th(z) | z \in F\},$$

where t is a complex number. We define a variation m^t of a measure m as $m^t(F^t) = m(F)$. In the differential form we define $dm^t(x^t) = dm(x)$. In addition, we consider a variation of energy while the external charges are fixed, i.e. $h(z)$ vanishes at the points corresponding to the external charges.

Definition 2.3. Consider the function $\phi = \Re\Phi(z)$, where $\Phi(z)$ is analytic and $\dfrac{d\Phi}{dz}$ is single valued. Let A be a subset of the domain D, such that $\mathrm{cap} A = 0$.

Let $m \in M^c$ be a signed measure supported in the domain D. We call m continuous $(A, \phi)-$critical if for any h continuous in $D \setminus A$, such that $h|_A = 0$, we have

$$\frac{d}{dt} E^c\left(m^t\right)\Big|_{t=0} = \lim_{t\to 0} \frac{E^c(m^t) - E^c(m)}{t} = 0. \tag{2.7}$$

If an external field $\phi = 0$, we call such m continuous $A-$critical measure.

Remark 2.4. Suppose a set A consists of finitely many points in the plane. Then for the discrete weighted logarithmic energy with the external field of the form $\phi = \Re\left(\sum_{k=1}^{l} \dfrac{b_k}{2} \log \dfrac{1}{z - a_k}\right)$, $b_k \in \mathbb{C}$, two definitions of critical measures agree. Note that definition 2.1 involves the Frechét derivative, while definition 2.3 involves the Gâteaux derivative.

The following equivalent condition is convenient to use.

Theorem 2.5. Let D be a simply connected domain. Consider $\phi = \Re\Phi$, where is Φ is analytic in D. Then condition (2.7) is equivalent to the equation

$$\iint_D \frac{h(x) - h(y)}{x - y}\, dm(x)dm(y) - 2\int_D \Phi'(x)h(x)dm(x) = 0. \tag{2.8}$$

Proof. In order to prove this statement it is enough to show that

$$E^c\left(m^t\right) - E^c(m) = -\Re\big(tf + O\left(t^2\right)\big), \tag{2.9}$$

where f denotes

$$\iint_D \frac{h(x) - h(y)}{x - y}\, dm(x)dm(y) - 2\int_D \Phi'(x)h(x)dm(x).$$

The variation of the logarithmic energy has the form

$$\begin{aligned} &\textstyle\iint_D \log \dfrac{1}{|x^t - y^t|} dm^t(x^t)\, dm^t(y^t) \\ &= \textstyle\iint_D \log \dfrac{1}{|x + th(x) - y - th(y)|}\, dm(x)dm(y). \end{aligned} \tag{2.10}$$

Then

$$\begin{aligned}
&\iint_D \log \frac{1}{|x^t - y^t|} dm^t\left(x^t\right) dm^t\left(y^t\right) - \iint_D \log \frac{1}{|x-y|} dm\left(x\right) dm\left(y\right) \\
&= - \iint_D \log \left| \frac{(x-y) + t(h(x) - h(y))}{x-y} \right| dm(x) dm(y) \\
&= - \iint_D \log \left| 1 + t\frac{h(x) - h(y)}{x-y} \right| dm(x) dm(y) \\
&= -\Re \iint_D \log \left(1 + t\frac{h(x) - h(y)}{x-y} \right) dm(x) dm(y).
\end{aligned} \tag{2.11}$$

Expanding logarithm in the Taylor series we obtain

$$\begin{aligned}
\iint_D \log \frac{1}{|x^t - y^t|} dm^t\left(x^t\right) dm^t\left(y^t\right) &- \iint_D \log \frac{1}{|x-y|} dm\left(x\right) dm\left(y\right) \\
&= -\Re \iint_D \left(t\frac{h(x) - h(y)}{x-y} + O\left(t^2\right) \right) dm(x) dm(y)
\end{aligned} \tag{2.12}$$

for small t.

For the external field we get

$$\begin{aligned}
&\int_{D^t} \phi\left(x^t\right)\, dm^t(x^t) - \int_D \phi(x)\, dm(x) \\
&= \int_D \phi(x + th(x))\, dm(x) - \int_D \phi(x)\, dm(x) \\
&= \Re \left(\int_D \Phi\left(x + th(x)\right)\, dm(x) - \int_D \Phi(x)\, dm(x) \right).
\end{aligned} \tag{2.13}$$

Note that

$$\Phi^{'}(x) = \frac{1}{2} \lim_{t \to 0} \frac{\Phi(x + th(x)) - \Phi(x)}{th(x)}.$$

This leads us to

$$\int_{D^t} \phi\left(x^t\right)\, dm^t(x^t) - \int_D \phi(x)\, dm(x) = 2\Re \int_D \left(t\Phi^{'}(x)h(x) + O(t^2) \right)\, dm(x). \tag{2.14}$$

for small t. Hence, (2.9) follows from (2.12) and (2.14) letting t tend to zero. □

The authors discovered in [MFR11] that for a certain choice of the set A and the external field ϕ the $(A, \phi)-$critical measure is supported on the critical trajectories of a quadratic differential represented by a rational function.

Let A be a set of points a_k, $k = 1, ..., l$, in the complex plane. We assume $a_k \neq a_j$ whenever $k \neq j$. Suppose,

$$\frac{B(z)}{A(z)} = \sum_{k=1}^{l} \frac{b_k}{z - a_k}, \tag{2.15}$$

where $b_k \in \mathbb{C}$. Let domain D be the complex plane punctured at the points a_k. We put $\phi = \Re\Phi(z)$, where

$$\Phi(z) = \sum_{k=1}^{l} \frac{b_k}{2} \log \frac{1}{z - a_k}, \Phi^{'}(z) = -\frac{1}{2}\frac{B(z)}{A(z)}.$$

Theorem 2.6. Consider the a set A and an external field described above. Then for any corresponding $(A, \phi)-$critical measure there exists a quadratic differential $-Q(z)\,dz^2$, where $Q(z)$ is rational, such that

- $Q(z)$ has the second order poles at the points $a_k \in A$, unless $b_k = 0$, $k = 1, ..., l$. In case $b_k = 0$, a_k is either a simple pole or regular point of $Q(z)$.
- The support of the measure is included into the union of trajectories of $-Q\,dz^2$.

Proof. Let us split the proof into two parts:

- Suppose m is a $(A, \phi)-$critical measure, where A, ϕ are defined as above. Then there exists a rational function $Q(z)$, which satisfies the first property of Theorem 2.6. Moreover, the principal value of the Cauchy transform of m
$$C(z) = \lim_{\varepsilon \to 0} \int_{|x-z|>\varepsilon} \frac{dm(x)}{x - z}$$
and the function Q satisfy the equality
$$Q(z) = \Big(C(z) + \Phi^{'}(z)\Big)^2 \tag{2.16}$$
almost everywhere with respect to the Lebesgue measure.
- Let m belong to the class M^c. Suppose there are rational functions Q, R, such that
$$Q(z) = (C(z) + R(z))^2 \tag{2.17}$$
almost everywhere with respect to the Lebesgue measure in the plane. Then $\operatorname{supp} m$ in included into the union of trajectories of the quadratic differential $-Q(z)\,dz^2$.

We need to prove that (2.16) holds for any point z, such that $C(z)$ converges absolutely. We use the fact that for such z the equality

$$\lim_{\varepsilon \to 0} \int_{|x-z|<\varepsilon} \frac{dm(x)}{|x - z|} = 0 \tag{2.18}$$

holds almost everywhere with respect to the Lebesgue measure in the plane.

Consider a disk $D(z,r)$ centered at z of radius $r \in (0,1)$. Then $m(D(z,r))$ is a function of r. Suppose $r \in (0,1)$. If m is a positive measure, the function $m(r)$ increases monotonically and is continuous from the left. As a consequence, it is almost everywhere differentiable with respect to the Lebesgue measure. Let m be a real measure. By Hahn decomposition theorem, it can be represented as $m = m^+ - m^-$, where m^+, m^- are positive and negative variations of the measure m. Being positive measures, they are monotonically increasing and continuous from the left functions of r. Therefore, m is almost everywhere differentiable with respect to the Lebesgue measure.

Let us fix r, such that $m^{'}(r)$ exists. For $\varepsilon \in (0,1)$ we define an auxiliary function F as

$$F(x) = \begin{cases} 0, & 0 \leq x < 1-\varepsilon, \\ \dfrac{(x-1+\varepsilon)^2(2\varepsilon+1-x)}{4\varepsilon^3}, & 1-\varepsilon \leq x < 1+\varepsilon, \\ 1, & x \geq 1+\varepsilon. \end{cases} \tag{2.19}$$

The function F is continuous with respect to $x \geq 0$. Moreover, we can estimate the absolute value of its derivative with respect to x. For $-\varepsilon < x-1 < \varepsilon$ the derivative

$$\frac{dF}{dx} = \frac{2((x-1)+\varepsilon)(2\varepsilon-(x-1)) - ((x-1)+\varepsilon)^2}{4\varepsilon^3} < \frac{c}{\varepsilon},$$

where c is an integer, which does not depend on ε. Then for $x \geq 0$ the following estimate holds

$$\left|\frac{dF}{dx}\right| < \frac{c}{\varepsilon}. \tag{2.20}$$

Further on, we use Theorem 2.5. Since m is $(A,\phi)-$critical, the condition (2.8) holds for any variation $h \in C(\mathbb{C} \setminus A)$ vanishing at the zeros of $A(z)$. So we pick

$$h(w) = F\left(\frac{|w-z|}{r}\right)\frac{A(w)}{w-z}.$$

Observe that

$$F\left(\frac{|w-z|}{r}\right) = \begin{cases} 0, & 0 \leq |w-z| \leq r(1-\varepsilon), \\ 1, & |w-z| \geq r(1+\varepsilon). \end{cases} \tag{2.21}$$

Let us denote by D the disk centered at z with radius $r(1-\varepsilon)$, by E the ring $r(1-\varepsilon) \leq |w-z| \leq r(1+\varepsilon)$, and by G the set $|w-z| \geq r(1+\varepsilon)$. Then the variation h has the form

$$h(w) = \begin{cases} 0, & w \in D, \\ \dfrac{A(w)}{w-z}, & w \in G. \end{cases} \tag{2.22}$$

Consider the integral

$$\begin{aligned}\iint_{\mathbb{C}} \frac{h(x)-h(y)}{x-y}\,dm(x)dm(y) &= I(D\times D)+I(E\times E)+I(G\times G)\\ &+2I(D\times E)+2I(D\times G)+2I(G\times E),\end{aligned} \tag{2.23}$$

where $I(S)$ denotes the corresponding integral over the set S.

Observe that, by construction, $I(D\times D)=0$.

The aim is to analyze the behaviour of the left-hand side of (2.23) as $\varepsilon\to 0$.

Let w belong to the ring E. Then h has the form

$$h(w)=\frac{A(w)}{w-z}F\left(\frac{|w-z|}{r}\right).$$

Let us estimate the gradient of F for $w\in E$.

$$\frac{\partial}{\partial\overline{w}}F\left(\frac{|w-z|}{r}\right)=\frac{1}{r}F'\left(\frac{|w-z|}{r}\right)\frac{\partial}{\partial\overline{w}}|w-z|=\frac{1}{r}F'\left(\frac{|w-z|}{r}\right)\frac{w-z}{|w-z|}. \tag{2.24}$$

Then the last expression together with (2.20) yields

$$\frac{1}{2}\left\|\operatorname{grad}F\left(\frac{|w-z|}{r}\right)\right\|=\left|\frac{\partial}{\partial\overline{w}}F\left(\frac{|w-z|}{r}\right)\right|\le\frac{c}{r\varepsilon}.$$

Therefore, by the mean value theorem, we obtain

$$\left|\frac{h(x)-h(y)}{x-y}\right|\leqq\frac{\tilde c}{r\varepsilon},$$

where x, y lie in the ring E and $\tilde c$ does not depend on ε. Therefore,

$$|I(E\times E)|\le\iint_{E\times E}\left|\frac{h(x)-h(y)}{x-y}\right|dm(x)dm(y)\le\frac{\tilde c}{r\varepsilon}(m(E))^2. \tag{2.25}$$

By assumption, the measure is almost everywhere differentiable as a function of r. Denote by $D_{r+\varepsilon r}$ the set of all w, such that $|w-z|\le r+\varepsilon r$, then

$$m(E)=m(D_{r+\varepsilon r}\setminus D)=m(r+\varepsilon r)-m(r-\varepsilon r).$$

Therefore,

$$\lim_{\varepsilon\to 0}\frac{m(E)}{\varepsilon}=2rm'(r).$$

Taking into account (2.25) we conclude that $I(E\times E)=o(1)$ for $\varepsilon\to 0$.

Now we come to estimating the integral $I(E \times D)$. By construction,

$$h(x) = \frac{A(x)}{x-z} F\left(\frac{|x-z|}{r}\right)$$

for $x \in E$, and $h(y) = 0$ for $y \in D$. We obtain

$$\frac{h(x)-h(y)}{x-y} = \frac{A(x)}{(x-z)(x-y)} F\left(\frac{|x-z|}{r}\right).$$

If $r - 2r\varepsilon < |y-z| < r - r\varepsilon$ and $x \in E$ we can use analogous arguments and obtain that the integral is $o(1)$ as $\varepsilon \to 0$ for this choice of y. If $|y-z| < r-2r\varepsilon$ and $x \in E$, the situation is different, since domain of integration contains a singularity. There is a constant c, such that

$$\left|\frac{h(x)-h(y)}{x-y}\right| \le \frac{c}{r(1-\varepsilon)|x-y|} \le \frac{c}{r(1-\varepsilon)(|x|-|y|)}.$$

Then

$$\begin{gathered} \left|\int_{|y-z|<r-2r\varepsilon}\int_{x\in E} \frac{h(x)-h(y)}{x-y}\,dx\,dy\right| \le \\ \frac{\tilde{c}}{r(1-\varepsilon)} \int_{|y-z|<r-2r\varepsilon}\int_{x\in E} \frac{1}{|x|-|y|}\,dx\,dy. \end{gathered} \tag{2.26}$$

After change of variables the last integral has the form

$$\frac{\text{const}}{r(1-\varepsilon)} \int_0^{r-2r\varepsilon}\int_{r-r\varepsilon}^{r+r\varepsilon} \frac{ts}{t-s}\,dt\,ds$$

and is $o(1)$ as $\varepsilon \to 0$. Similarly, $I(E \times G) = o(1)$ as $\varepsilon \to 0$.

Binding the estimates of the integrals together we arrive at

$$\begin{aligned} &\lim_{\varepsilon\to 0} \iint_{\mathbb{C}} \frac{h(x)-h(y)}{x-y} dm(x)dm(y) \\ &= \iint_{|x-z|\ge r, |y-z|\ge r} \frac{\tilde{h}(x)-\tilde{h}(y)}{x-y} dm(x)dm(y) \\ &+2\iint_{|x-z|\ge r, |y-z|< r} \frac{\tilde{h}(x)-\tilde{h}(y)}{x-y} dm(x)dm(y), \end{aligned} \tag{2.27}$$

where

$$\tilde{h}(w) = \begin{cases} 0, & |w-z| < r, \\ \frac{A(w)}{w-z}, & |w-z| > r. \end{cases} \tag{2.28}$$

Now let us look at the behaviour of the left-hand side of (2.27) as $r \to 0$.

By (2.18) the Fubini theorem is applicable to the integral

$$\begin{aligned}&\iint_{|x-z|\geq r,|y-z|<r}\frac{\tilde{h}(x)-\tilde{h}(y)}{x-y}dm(x)dm(y)\\&=\iint_{|x-z|\geq r,|y-z|<r}\frac{A(x)}{(x-y)(x-z)}dm(x)dm(y).\end{aligned}\tag{2.29}$$

We obtain that

$$\begin{aligned}&\iint_{|x-z|\geq r,|y-z|<r}\tfrac{A(x)}{(x-y)(x-z)}dm(x)dm(y)\\&=\int_{|x-z|\geq r}\tfrac{A(x)}{x-z}\left(\int_{|y-z|<r}\tfrac{1}{x-y}\,dm(y)\right)\,dm(x).\end{aligned}\tag{2.30}$$

Condition (2.18) implies that the last integral tends to zero as $r\to 0$.

However,

$$\begin{aligned}&\iint_{|x-z|\geq r,|y-z|\geq r}\frac{\tilde{h}(x)-\tilde{h}(y)}{x-y}dm(x)dm(y)\\&=\iint_{|x-z|\geq r,|y-z|\geq r}\left(\frac{A(x)}{(x-y)(x-z)}-\frac{A(y)}{(x-y)(y-z)}\right).\end{aligned}\tag{2.31}$$

There exists a polynomial $P(z,x,y)$ of degree $l-2$, such that

$$(y-z)A(x)-(x-z)A(y)+(x-y)A(z)=(y-z)(x-z)(x-y)P(z,x,y).$$

Therefore,

$$\frac{A(x)}{(x-y)(x-z)}-\frac{A(y)}{(x-y)(y-z)}=-\frac{A(z)}{(x-z)(y-z)}+P(z,x,y).$$

Since

$$\begin{aligned}&\iint_{|x-z|\geq r,|y-z|\geq r}\frac{A(z)}{(x-z)(y-z)}\,dm(x)dm(y)\\&=A(z)\int_{|x-z|\geq r}\frac{1}{x-z}\,dm(x)\int_{|y-z|\geq r}\frac{1}{y-z}\,dm(y),\end{aligned}\tag{2.32}$$

we obtain

$$\begin{aligned}&\lim_{r\to 0}\iint_{|x-z|\geq r,|y-z|\geq r}\left(\frac{A(x)}{(x-y)(x-z)}-\frac{A(y)}{(x-y)(y-z)}\right)\\&=P_1(z)-A(z)\left(C(z)\right)^2,\end{aligned}\tag{2.33}$$

where polynomial function $P_1(z)$ stands for

$$P_1(z)=\iint_{\mathbb{C}}P(z,x,y)\,dm(x)dm(y).$$

Therefore,

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\iint_{\mathbb{C}} \frac{h(x)-h(y)}{x-y} dm(x)dm(y) = P_1(z) - A(z)\left(C(z)\right)^2. \tag{2.34}$$

Analyzing similarly, we come to

$$\begin{aligned} &\int \Phi'(x)h(x)\,dm(x) = \\ &\int_E \Phi'(x)h(x)\,dm(x) + \int_G \Phi'(x)h(x)\,dm(x). \end{aligned} \tag{2.35}$$

Analogous estimates for $x \in E$ imply that

$$\lim_{\varepsilon\to 0}\int \Phi'(x)h(x)\,dm(x) = \int_{|x-z|\le r} \Phi'(x)\frac{A(x)}{x-z}\,dm(x).$$

By the choice of the external field,

$$\Phi'(x)\frac{A(x)}{x-z} = -\frac{1}{2}\frac{B(x)}{x-z}. \tag{2.36}$$

Therefore, we have

$$\int_{|x-z|\le r}\frac{B(x)}{x-z} = \int_{|x-z|\le r}\frac{B(x)-B(z)}{x-z}\,dm(x) + B(z)\int_{|x-z|\le r}\frac{1}{x-z}\,dm(x). \tag{2.37}$$

We denote the integral

$$\int_{|x-z|\le r}\frac{B(x)-B(z)}{x-z}\,dm(x)$$

by $P_2(z)$. Note that P_2 is rational. We obtain

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\int \Phi'(x)h(x)\,dm(x) = -\frac{1}{2}\left(P_2(z) + B(z)C(z)\right).$$

We put the last equality together with (2.34) and use Theorem 2.5. Then

$$P_1(z) - A(z)\left(C(z)\right)^2 + P_2(z) + B(z)C(z) = 0.$$

We rewrite it as

$$C^2(z) - \frac{B(z)}{A(z)}C(z) + \left(\frac{1}{2}\frac{B(z)}{A(z)}\right)^2 = \frac{P_1(z)+P_2(z)}{A(z)} + \left(\frac{1}{2}\frac{B(z)}{A(z)}\right)^2.$$

We put

$$Q(z) = \frac{P_1(z)+P_2(z)}{A(z)} + \left(\Phi'(z)\right)^2.$$

So we found a rational function $Q(z)$ with possible poles at the zeros of $A(z)$, such that (2.16) holds. Moreover, whenever $b_k \neq 0$, the corresponding a_k is a double pole of $Q(z)$. This concludes the first part of the proof.

Consider the natural parameter w of Q near a regular point

$$w(z) = \int \sqrt{Q(z)}\, dz.$$

We pick a simply connected domain D around the chosen point, such that the domain does not contain critical points of the quadratic differential and is bounded by two horizontal and two vertical arcs. Then the natural parameter maps D conformally onto a rectangle in the $w-$plane. We define a sign function

$$s(w) = \operatorname{sgn}\left(\frac{C+R}{\sqrt{Q}}(z(w))\right).$$

For $z \in D$ we rewrite (2.17) as

$$s(w(z))\sqrt{Q(z)} = C(z) + R(z). \tag{2.38}$$

Note that

$$\frac{\partial}{\partial \overline{z}} C(z) = \pi m(z),$$

in a sense of generalized functions. Here the conjugate derivative is defined as

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Therefore,

$$\frac{\partial}{\partial \overline{z}}\left(C(z) + R(z)\right) = \pi m(z).$$

Differentiating the left-hand side of (2.38) we get

$$\frac{\partial}{\partial \overline{z}}\left(s(w(z))\sqrt{Q(z)}\right) = \frac{\partial}{\partial \overline{w}} s(w(z)) \overline{\left(\frac{\partial w(z)}{\partial z}\right)} \sqrt{Q(z)}. \tag{2.39}$$

Taking into account that

$$\overline{\left(\frac{\partial w(z)}{\partial z}\right)} = \overline{\sqrt{Q(z)}},$$

we obtain

$$\frac{\partial}{\partial \overline{w}} s(w(z)) = \frac{\pi m(z)}{|Q(z)|}. \tag{2.40}$$

From the last expression we conclude that the generalized partial derivative of the sign function s vanishes along the axis $\{\Re w = 0\}$. Therefore, $s(w)$ is a function of $\Re w$. Whenever $s(\Re w_0) \neq 0$ for some w_0, it takes the same value along the whole vertical line passing through the point w_0. Then the set

$$\sigma = \left\{ w \,\middle|\, \frac{\partial}{\partial \overline{w}} s(w) \neq 0 \right\}$$

consists of certain vertical lines.

Therefore, $\operatorname{supp} m \cap D$ is generated by a union of trajectories of the quadratic differential $-Q(z)\,dz^2$.

Consider now positive measure m. Then, by (2.40), we get

$$\frac{\partial}{\partial \overline{w}} s(w) \geq 0,$$

where $w \in w(D)$. The function $s(w)$ changes sign only once in $w(D)$ and, therefore, at most one vertical arc is contained in $\operatorname{supp} m \cap D$.

□

Remark 2.7. Suppose $b_k \in \mathbb{R}$, $k = 1, ..., l$ in the representation (2.15). Then the support of the $(A, \varphi)-$critical measure is provided by the Jenkins-Strebel quadratic differential. See the details in [MFR11].

We conclude that any continuous critical measure is supported on the trajectories of a certain quadratic differential. But in general, there is no one-to-one correspondence between continuous critical measures and the quadratic differentials providing their support.

Example 2.8. Consider the quadratic differential Q with four simple poles and two simple zeros with critical trajectories as on the Figure 2.1. Let us pick the external energy $\varphi = 0$ and the set A consisting of poles of Q. Then three different $(A, \varphi)-$critical measures can be associated with the quadratic differential Q. On the figure 2.1 three copies of the critical graph of Q are illustrated. The highlighted with bold lines correspond to the support of the corresponding $(A, \varphi)-$critical measures.

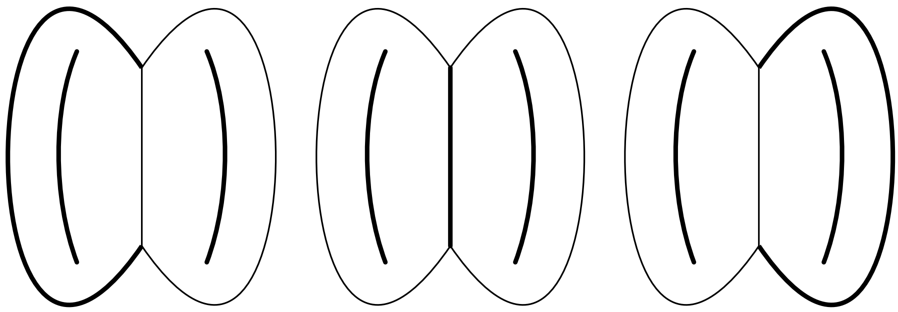

Figure 2.1: Three measures associated with a Jenkins-Strebel differential.

Chapter 3

Parameter space of Jenkins-Strebel quadratic differential

Recall the Lamé equation

$$\prod_{k=1}^{l}(z-a_k)\frac{d^2y}{dz^2}+\sum_{j=1}^{l}b_j\prod_{i\neq j}(z-a_i)\frac{dy}{dz}+V(z)y=0. \tag{3.1}$$

Here a_k, b_k are fixed points on the complex plane. Polynomial $V(z)$ of degree at most $l-2$ can vary. It was proved by B. Shapiro in [Sha11] that there is a number $N \in \mathbb{N}$, such that for any $n \geq N$, there exist

$$\begin{pmatrix} n+l-2 \\ n \end{pmatrix}$$

polynomials $V(z)$ of degree $l-2$, for which a polynomial $y(z)$ of degree n satisfies (3.1).

Sequences of $V(z) = V_n(z)$ and corresponding $y(z) = y_n(z)$ for such n were considered in [MFR11].

When the values of b_k are real, the measures μ_n counting zeros of $y_n(z)$ provide the minimum of discrete weighted potential energy of a charge system, which corresponds to (3.1). So we can associate a sequence of measures μ_n to the sequences $V_n(z)$ and $y_n(z)$.

Remark 3.1. B. Shapiro showed in [Sha11] that the zeros of $V_n(z)$ lie in an $\epsilon-$neighbourhood of the convex hull of the set $A = \{a_1, ..., a_l\}$, for n greater than some number N.

Authors proved in [MFR11] a theorem about limit of sequence of zero counting measures associated to the Lamé equation. We observe this theorem for the particular case when $l = 3$. Then the sequence $V_n(z)$ is a sequence of monomials. By Remark 3.1, the sequence of roots of $V_n(z)$ has a limiting point. This is equivalent to that the sequence $V_n(z)$ has a limiting polynomial. We suppose that for the sequence $V_n(z)$ there is a limiting polynomial $\tilde{V}(z)$ with leading coefficient 1.

Theorem 3.2. *Suppose there exists a polynomial $\tilde{V}$ with the leading coefficient 1, $deg\tilde{V} = 1$, such that*

$$\lim_{n\to\infty} V_n = \tilde{V}.$$

Then the normalized zero counting measure μ_n/n^2 converges to a continuous $A-$critical measure $m \in M^c$ in a weak- sense.*

In this case the Cauchy transform of m satisfies the equality

$$C^2(z) = \frac{\tilde{V}(z)}{\prod_{k=1}^{3}(z-a_k)} \tag{3.2}$$

almost everywhere with respect to the Lebesgue measure. Recall that A-critical measure provides a saddle point of continuous logarithmic energy associated with (3.1).

B. Shapiro, K. Takemura and M. Tater [STT11] analysed an analogous problem for the Heun equation, and therefore, for the Lamé equation of degree $l = 3$. They considered the sequences of polynomials V_n of degree 1 and corresponding sequences of so-called normalized polynomials $\tilde{V_n} = V_n/v$, where v is the leading coefficient of V_n. By Remark 3.1, there exists a monic polynomial $\tilde{V}$, which is the limit of the sequence of normalized polynomials $\tilde{V_n}$. Moreover, it was proved in [STT11], that the sequence of normalized zero counting measures μ_n/n^2 corresponding to the polynomial solutions $y_n(z)$ weakly converge to a measure $m \in M^c$ with the Cauchy transform satisfying (3.2) almost everywhere with respect to the Lebesgue measure.

Note that the limiting measures are supported on the trajectories of a Jenkins Strebel quadratic differential

$$Q(z)\,dz^2 = -\frac{\tilde{V}(z)}{\prod_{k=1}^{3}(z-a_k)}\,dz^2.$$

This description of the weak-* limits of the zero counting measures corresponding to (3.1) depends on the limit polynomials $\tilde{V}$. One can consider the problem of characterizing the set of such $\tilde{V}$. We choose the degree $l = 3$ because this case is the simplest non-trivial case of this problem. We reformulate it as follows.

Consider the quadratic differential

$$Q(z)\,dz^2 = -\frac{1}{4\pi^2}\frac{z-c}{(z-a_1)(z-a_2)(z-a_3)}\,dz^2. \tag{3.3}$$

The points a_1, a_2, a_3 are given, the parameter c can vary. The goal is to characterize the set of all possible $\tilde{V} = z - c$ by describing the parametric space of c, for which the quadratic differential $Q(z)\,dz^2$ is a Jenkins-Strebel quadratic differential.

This quadratic differential has a simple zero at the point c, simple poles at a_1, a_2 and a_3, and a second order pole at ∞. The trajectory structure in large of $Q(z)\,dz^2$ consists of at most two domains. Denote by D_∞ the domain containing ∞. It is a circular domain. The other domain D can vary. There are three cases

i. Domain D degenerates, i.e. $D = \emptyset$.

ii. Domain D forms a density domain.

iii. Domain D is a ring domain. It separates the simple poles from the zero c and ∞.

The first case can be associated with the well-known Chebotarev problem of finding a continuum of minimal logarithmic capacity, such that it contains the points a_1, a_2, a_3. The extremal continuum exists, is unique and is well-known as the Chebotarev continuum (see [Kuz82]). We denote it by $E = E(a_1, a_2, a_3)$. The Chebotarev continuum can be given by the closure of the critical graph Ψ of the quadratic differential

$$Q(z)\,dz^2 = -\frac{1}{4\pi^2}\frac{z-c_0}{(z-a_1)(z-a_2)(z-a_3)}\,dz^2.$$

The closures of three trajectories $\gamma_{0,k} \in \Psi$, $k = 1, 2, 3$ have a common point $c_0 = c_0(a_1, a_2, a_3)$. Let us set an orientation of the curves $\gamma_{0,k}$ such that the closure of $\gamma_{0,k}$ goes from the point a_k to c_0. We denote by $-\gamma_{0,k}$ a curve with reverse orientation.

If the points a_1, a_2, a_3 are collinear, one of the critical trajectories $\gamma_{0,k}$ degenerates. The properties of the point c_0 corresponding to the Chebotarev continuum are studied by G. I. Kuz'mina in [Kuz82].

In the non-degenerate case (iii) the domain decomposition of Q consists of a circular domain D_∞ and a ring domain D.

We are interested in the first and the last cases, since we are concerned with the parameter c, for which the quadratic differential is Jenkins-Strebel.

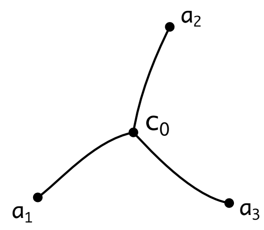

Figure 3.1: The Chebotarev continuum.

It was proved in [Str84] that the set of Jenkins-Strebel quadratic differentials in the space of quadratic differentials represented by a meromorphic function with fixed poles. Therefore, the set C of c, such that Q is Jenkins-Strebel, is dense in a parametric space $\mathbb{C}$. The geometric structure of C has a particular interest. In this chapter we show that the set C consists of countably many disjoint Jordan curves, starting at c_0 and ending at ∞ in the parametric space.

There is one-to-one correspondence between curves in the parametric space and homotopy classes of closed Jordan curves on $\mathbb{C} \setminus \{a_1, a_2, a_3\}$. In order to see the connection, we define first these classes in few steps.

Suppose that the points a_1, a_2, a_3 do not lie on one line. We define the winding number of a curve γ with respect to the point a_3 as

$$\mathrm{ind}_{a_3}\gamma = \left[\frac{1}{2\pi}\Delta \arg \frac{z-a_3}{a_1-a_3}\right], \tag{3.4}$$

where $\Delta \arg f(z)$ stands for the change of argument of $f(z)$ while $z \in \gamma$. We define $G_k^{'} = G_k^{'}[a_1, a_2; a_3]$ to be a homotopy class of Jordan curves $\gamma^{'}$ on the punctured plane $\mathbb{C}\setminus\{a_3\}$ connecting the points a_1 and a_2, such that the winding number $\mathrm{ind}_{a_3}\gamma^{'} = k$. In order to consider non-Jordan curves as well, we extend the defined class as follows. Let $G_k = G_k[a_1, a_2; a_3]$ be a homotopy class of continuous curves connecting the points a_1 and a_2 on $\mathbb{C}$, such that for any curve $\gamma \in G_k$ there is a curve $\gamma^{'} \in G_k^{'}$ homotopic γ on $\mathbb{C} \setminus \{a_1, a_2, a_3\}$.

Later on, we use a similar extension process. Let us call it extension by homotopy.

Note that for any k the class $G_k[a_1, a_2; a_3]$ has one representative of the form:

$$\gamma_k = \begin{cases} \underbrace{\gamma_1^0 \cup (-\gamma_3^0) \cup \gamma_3^0 \cup (-\gamma_1^0)}_{k \text{ times}} \cup \gamma_1^0 \cup (-\gamma_2^0) & k \geq 0, \\ \underbrace{\gamma_1^0 \cup (-\gamma_3^0) \cup \gamma_3^0 \cup (-\gamma_1^0)}_{|k|-1 \text{ times}} \cup \gamma_1^0 \cup (-\gamma_3^0) \cup \gamma_3^0 \cup (-\gamma_2^0) & k < 0. \end{cases} \tag{3.5}$$

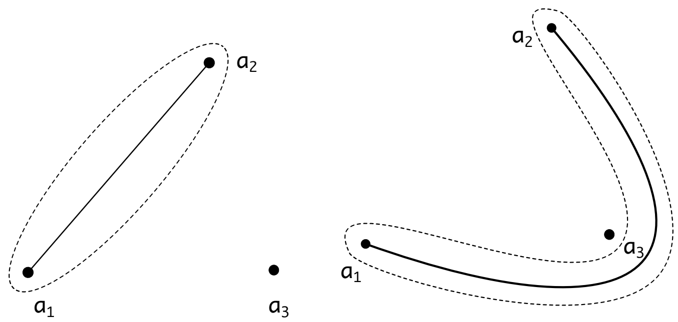

Figure 3.2: Representatives of the homotopy classes $G_0[a_1, a_2; a_3]$, $\Gamma_0[a_1, a_2; a_3]$, $G_{-1}[a_1, a_2; a_3]$, $\Gamma_{-1}[a_2, a_1; a_3]$.

In the next step we associate to $G'_k[a_1, a_2; a_3]$ a class $\Gamma'_k[a_1, a_2; a_3]$ of closed Jordan curves $\tilde{\gamma}'$, which separate a_1, a_2 from a_3 and ∞. More precisely, the class Γ'_k consists of curves $\tilde{\gamma}'$, such that there is a curve $\gamma' \in G'$ separated by $\tilde{\gamma}'$ from the points a_3, ∞. Let $\Gamma_k[a_1, a_2; a_3]$ be extension by homotopy of the class $\Gamma'_k[a_1, a_2; a_3]$.

Finally we define $\mathbf{\Gamma}[a_1, a_2; a_3]$ as the union $\{\Gamma_k[a_1, a_2; a_3]\}_{k\in\mathbb{Z}}$.

Remark 3.3. Note that in the families $\mathbf{\Gamma}[a_1, a_2; a_3]$ and $\mathbf{\Gamma}[a_2, a_1; a_3]$ there are two pairs of homotopy classes, which have the same curves up to orientation. So we identify the pairs of classes $\Gamma_{-1}[a_1, a_2; a_3]$, $\Gamma_0[a_2, a_1; a_3]$, and $\Gamma_{-1}[a_2, a_1; a_3]$, $\Gamma_0[a_1, a_2; a_3]$. From now on, we consider the families $\mathbf{\Gamma}[a_{i_1}, a_{i_2}; a_{i_3}]$, where $(a_{i_1}, a_{i_2}, a_{i_3})$ denotes the permutations of the triple (a_1, a_2, a_3), to be different up to this identification.

On the left part of the Figure 3.2 the representatives of homotopy class $G_0[a_1, a_2; a_3] = G_{-1}[a_2, a_1; a_3]$ and homotopy class $\Gamma_0[a_1, a_2; a_3] = \Gamma_{-1}[a_2, a_1; a_3]$ are illustrated with regular and dotted line correspondingly. On the right part of the figure we can see the representatives of $G_{-1}[a_1, a_2; a_3] = G_0[a_2, a_1; a_3]$ (regular line) and $\Gamma_{-1}[a_2, a_1; a_3] = \Gamma_0[a_1, a_2; a_3]$ (dotted line).

Let Γ^0 be the homotopy class of the closed curves on $\mathbb{C}$ separating ∞ from the points a_1, a_2 and a_3.

Later on, we consider moduli of domains with respect to the classes Γ^0 and Γ_k.

We call domains D_k, $D_{k,\infty}$ are *admissible* for the homotopy classes $\Gamma_k[a_1, a_2; a_3]$ and Γ^0 if

1. D_k, $D_{k,\infty}$ do not intersect,

2. $D_{k,\infty} \ni \infty$ is simply connected,

3. D is a doubly connected domain on $\mathbb{C}\backslash\{a_1, a_2, a_3\}$, such that its boundary components are separated by the curves from the class $\Gamma_k[a_1, a_2; a_3]$.

Consider the Jenkins modulus problem for the homotopy classes $\Gamma_k[a_1, a_2; a_3]$ and Γ^0 and a positive weight α. Denote by $M(D_k)$ the modulus of D_k with respect to the family of curves separating its boundary components, and by $D_{k,\infty}$ the reduced modulus of $D_{k,\infty}$ at the point ∞. Then the problem is to find the maximum

$$\max\{\alpha\, M(D_k) + M(D_{k,\infty})\} \tag{3.6}$$

for a fixed $\alpha > 0$.

It follows from [Jen58, Jen57] that for any classes Γ_k, Γ^0 and any $\alpha > 0$ there exists a unique pair of extremal domains $D_k^* = D_k^*(\alpha; a_1, a_2; a_3)$ and $D_{k,\infty}^* = D_{k,\infty}^*(\alpha; a_1, a_2; a_3)$. Moreover, these domains belong to the domain configuration of the quadratic differential (3.3).

The zero

$$c = c_k(\alpha; a_1, a_2; a_3) \tag{3.7}$$

of this quadratic differential defines the extremal domains. So the solution of the problem of finding the maximum (3.6) depends only on this parameter.

By remark 3.3, the homotopy class $\Gamma_{-1}[a_{i_1}, a_{i_2}; a_{i_3}]$ is identical to $\Gamma_0[a_{i_2}, a_{i_1}; a_{i_3}]$, and $\Gamma_{-1}[a_{i_2}, a_{i_1}; a_{i_3}]$ is identical to $\Gamma_0[a_{i_1}, a_{i_2}; a_{i_3}]$. Thus the corresponding zeros c_k coincide. More precisely,

$$\begin{aligned} c_{-1}(\alpha; a_{i_1}, a_{i_2}; a_{i_3}) = c_0(\alpha; a_{i_2}, a_{i_1}; a_{i_3}), \\ c_0(\alpha; a_{i_1}, a_{i_2}; a_{i_3}) = c_{-1}(\alpha; a_{i_2}, a_{i_1}; a_{i_3}). \end{aligned} \tag{3.8}$$

It is known that for any $\Gamma_k[a_1, a_2; a_3]$ there exists a *transition value* $\alpha_k^* = \alpha_k^*(a_1, a_2, a_3)$ such that

i. If the weight $\alpha \le \alpha_k^*$ then the doubly connected extremal domain degenerates.

ii. If $\alpha > \alpha_k^*$ then D_k^* is non-empty.

In the case (i) the domain decomposition of the quadratic differential providing maximum (3.6) consists of one simply connected domain $D_{k,\infty}^* = \hat{\mathbb{C}} \setminus E(a_1, a_2, a_3)$, where $E(a_1, a_2, a_3)$ is the Chebotarev continuum.

Therefore, the transition value α_k^* can be found from the equation

$$\alpha_k^* = 2 \int_{\gamma_k} |Q_0(z)|^{\frac{1}{2}}\, |dz|, \tag{3.9}$$

where $Q_0(z)\, dz^2$ is the quadratic differential of the form (3.3) for $c = c_0(a_1, a_2, a_3)$.

The residue of $\sqrt{Q_0(z)}$ at $z = \infty$ is $\frac{1}{2\pi i}$ and $\sqrt{Q_0(z)}\,dz$ is positive along trajectories. Therefore, it follows that

$$2\int_{E(a_1,a_2,a_3)} \sqrt{Q_0(z)}\,dz = 2\int_{E(a_1,a_2,a_3)} \sqrt{|Q_0(z)|}\,|dz| = 1. \tag{3.10}$$

We conclude from (3.9) and (3.10) that $0 < \alpha_0^* \leq 1$. For $k \neq 0$ we obtain $\alpha_k^* > 1$.

Using linear transformation, we rewrite (3.3) in the form:

$$Q(z)\,dz^2 = -\frac{1}{4\pi^2}\frac{z-c}{(z^2-1)(z-a)}\,dz^2. \tag{3.11}$$

Since the trajectory structure of a quadratic differential is invariant under conformal mapping, we may assume that $\Im a \geq 0, \Re a \geq 0$.

Consider now the limit cases of the problem (3.6).

The first limit case corresponds to the transition value $\alpha = \alpha_0^*$. G.V. Kuz'mina gave transcendental equations describing the zero parameter c_0, see theorem 1.6 [Kuz82]. She showed as well that c_0 lies in the interior of the triangle with vertices at $-1, 1, a$ and $\Re c_0 > 0$.

The second limit case corresponds to the Teichmüller problem of finding the maximal modulus of a doubly connected domain D with respect to the family of curves separating the points -1 and 1 from a and ∞. The curves γ separating the boundary components of D belong to the class $\Gamma_m[-1, 1; a]$, $m \in \mathbb{Z}$.

The extremal domain is given by a ring domain of the domain decomposition of the quadratic differential

$$Q_m^T(z; -1, 1, a)\,dz^2 = e^{2i\varphi_m}\frac{dz^2}{(z^2-1)(z-a)}, \tag{3.12}$$

where $\varphi_m = \varphi_m(-1, 1; a)$ is uniquely determined by the points -1, 1, a.

In order to describe extremal domain D^*, we will use an auxiliary complex plane of a variable u, following G.V. Kuz'mina in [Kuz82], such that

$$z = 2k^2\mathrm{sn}^2(u, k) - 1, \tag{3.13}$$

with

$$k^2 = k^2(a) = \frac{a+1}{2}, \tag{3.14}$$

Let P be fundamental parallelogram of the function (3.13). Then the vertices of P can be chosen to be at the points

$$u^{-1} = 0, u^1 = \mathbf{K}(k) + i\mathbf{K}'(k), u^a = \mathbf{K}(k), u^\infty = i\mathbf{K}'(k). \tag{3.15}$$

Here u^z denotes $u(z)$. Let Λ be the lattice with the initial point at $u = 0$. Then

$$\Lambda = \{u^w + 2l\mathbf{K}(k) + 2in\mathbf{K}^{'}(k)\},\ n, l = \pm 1, \pm 2, \dots;\ w = -1, 1, a, \infty.$$

The function

$$\tau(k^2) = \frac{i\mathbf{K}^{'}(k)}{\mathbf{K}(k)}$$

takes values in the region

$$T = \left\{ \tau \left| \left| \tau \pm \frac{1}{2} \right| \geq \frac{1}{2}, -1 \leq \Re\tau \leq 1, \Re\tau \geq 0, \Im\tau \geq 0 \right. \right\}.$$

The function $\tau(k^2)$ is called the modular function, see [Ahl78]. Functions $\mathbf{K}^{'}(k)$, $\mathbf{K}(k)$ are positive for $k^2 \in (0, 1)$. If $\Im k^2 > 0$, we extend these function analytically along any curve γ, such that intγ lies in the half-plane $\{\Im k^2 > 0\}$.

We denote by $\gamma_{-1,1}$ and $\gamma_{a,\infty}$ the bounded and unbounded boundary components the domain D correspondingly.

Quadratic differential (3.12) can be written in terms of variable u as

$$Q(z)\, dz^2 = 2e^{i\varphi_m}\, du^2. \tag{3.16}$$

Consider now the images of the boundary components $\gamma_{-1,1}$ and $\gamma_{a,\infty}$ by the mapping $u(z)$.

First we suppose that the curves separating the boundary components of D belong to the class $\Gamma_0[-1, 1; a]$. Then the image of the curve $\gamma_{-1,1}$ corresponds to a family of parallel lines, which is obtained from a line L_1 connecting the points $u = 0$ and $u = \mathbf{K}(k) - i\mathbf{K}^{'}(k)$, by a transformation of the form

$$f(u) = u + n(\mathbf{K}(k) + i\mathbf{K}^{'}(k)),\ n = \pm 1, \pm 2, ... \tag{3.17}$$

The corresponding image of $\gamma_{a,\infty}$ is generated by all transformations (3.17) of a line L_2 connecting the points $\mathbf{K}(k)$, $i\mathbf{K}^{'}(k)$ in the $u-$plane.

Then the angle φ_0 can be expressed as

$$\varphi_0 = -2\arg(\mathbf{K}(k) - i\mathbf{K}^{'}(k)).$$

Suppose

$$u(-1) = 0.$$

For the ring domain D, there exists a conformal mapping $\xi = g(z)$ of domain D onto the ring $\{1 < |\xi| < R\}$, $R > 0$. Then

$$\frac{1}{2\pi}\log R = M(D).$$

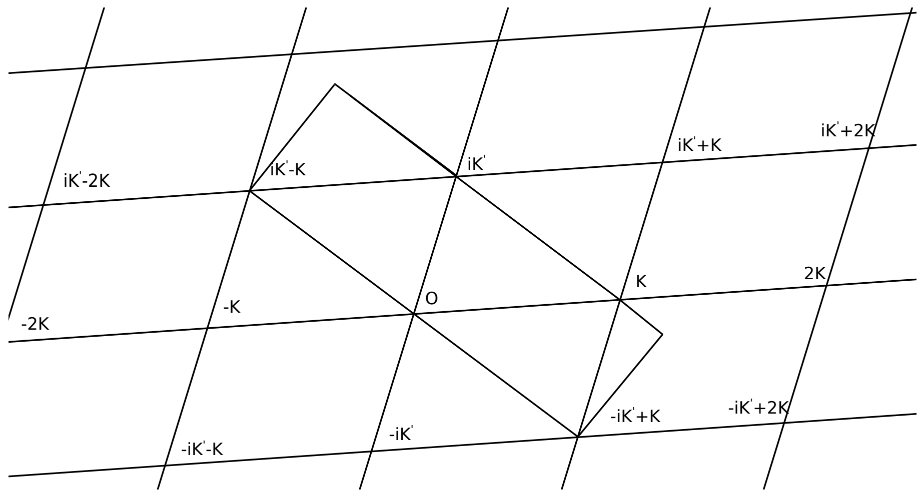

Figure 3.3: The rectangle Π.

Let γ be an orthogonal critical trajectory of the quadratic differential $Q(z)$, such that the point $z = 1$ belongs to its closure. We set $D^{'} = D \setminus \gamma$. Then $D^{'}$ is mapped by $g(z)$ onto a domain $g(D^{'})$, which is equal to a ring $\{1 < |\xi| < R\}$ with a slit along the interval $[1, R]$. On the other hand, the image $u(D^{'})$ of $D^{'}$ by the function $u(z)$ is the interior of one of rectangles $\pm\Pi$, which are described as follows. One side of Π connects the points $-\mathbf{K}(k) + i\mathbf{K}^{'}(k)$, $\mathbf{K}(k) - i\mathbf{K}^{'}(k)$, and the opposite side lies on the line L_2. Denote by $2a$ the length of the longer side, and by b the length of the shorter side. Then the modulus of Π with respect to family of trajectories is equal to $b/(2a)$. The Figure 3.3 shows the rectangle Π.

In order to obtain a formula for the modulus of the extremal domain, we use calculations different to that suggested Kuz'mina.

By the property of conformal invariance, the moduli of $D^{'}$, $u(D^{'})$ and $g(D^{'})$ with respect to corresponding families of curves coincide. On the other hand, the moduli of the ring $\{1 < |\xi| < R\}$ and the ring with the slit coincide (see, for example, Chapter 2.2 in [Vas02]). More precisely,

$$\frac{1}{2\pi} \log R = M(\Pi) = \frac{b}{2a}. \tag{3.18}$$

Therefore, it is sufficient to calculate a/b in order to find the modulus of the extremal domain D. To do so, we use some basic geometry.

Let us prove that

$$\frac{a}{b} = \pi \Im \frac{i\mathbf{K}^{'}(k)}{\mathbf{K}(k) - i\mathbf{K}^{'}(k)}. \tag{3.19}$$

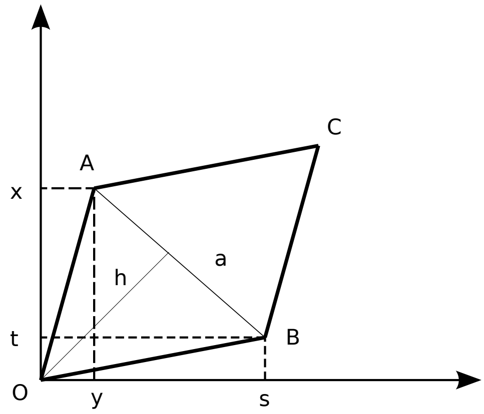

Figure 3.4:

Let $i\mathbf{K}'(k) = y + ix$, $\mathbf{K}(k) = s + it$, where $x, y, s, t > 0$. Then we get

$$\Im \frac{i\mathbf{K}'(k)}{\mathbf{K}(k) - i\mathbf{K}'(k)} = \frac{xs - yt}{(s-y)^2 + (t-x)^2}. \tag{3.20}$$

We denote by O, A, B, C the vertices of parallelogram P of the lattice Λ corresponding to the points 0, $i\mathbf{K}'(k)$, $\mathbf{K}(k)$, $\mathbf{K}(k) + i\mathbf{K}'(k)$. Let h be an altitude of the triangle OAB through the vertex O. Then $b/a = h/a$. The denotations are visualized on the Figure 3.4.

Obviously,

$$a = AB = \sqrt{(s-y)^2 + (x-t)^2}. \tag{3.21}$$

Denote by θ an angle between the side OA and the imaginary axis. Let us rotate the triangle OAB by θ. The resulting triangle is denoted by $OA'B'$. Then, by the triangle area formula, we obtain

$$h = \frac{|OA| \cdot B_1'}{a}, \tag{3.22}$$

where B_1' stands for the real part of the point B' and $|OA|$ denotes the length of the side OA. In order to get the value of B_1', we use a rotation matrix. Obviously,

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \ \cos\theta = \frac{x}{\sqrt{x^2+y^2}}. \tag{3.23}$$

Then the B_1' has form

$$B_1' = \frac{xs}{\sqrt{x^2+y^2}} - \frac{yt}{\sqrt{x^2+y^2}}. \tag{3.24}$$

Then we obtain for h

$$h = \frac{xs - yt}{\sqrt{(s-y)^2 + (x-t)^2}},$$

which together with (3.21) concludes the proof. We can rewrite (3.19) and (3.18) as

$$\log R = \pi \Im \frac{\tau(k^2)}{1 - \tau(k^2)}. \tag{3.25}$$

The angle parameter has the form

$$\varphi_0 = -2\arg(\mathbf{K}(k) - i\mathbf{K}^{'}(k)).$$

Suppose that D is a doubly connected domain, such that curves separating the boundary components $\gamma_{a,\infty}$ and $\gamma_{-1,1}$ belong to the class $\Gamma_1[-1,1;a]$. Then the corresponding domain $D^{'}$ is mapped by $u(z)$ onto a rectangle Π_1 with one side connecting points $i\mathbf{K}^{'}(k) - 3\mathbf{K}(k)$, $-i\mathbf{K}^{'}(k) + 3\mathbf{K}(k)$ and the opposite side lying on the line going through the points $\mathbf{K}(k)$, $i\mathbf{K}^{'}(k) - 2\mathbf{K}(k)$. The corresponding modulus of Π_1 is evidently smaller than the modulus of a domain with respect to curves from the class $\Gamma_0[-1,1;a]$.

So the modulus of a doubly connected domain with respect to curves of class $\Gamma_m[-1,1;a]$ separating its boundary components decreases as $|m|$ increases. It is defined by equalities

$$\log R = \pi \Im \frac{\tau(k^2)}{1 - (2|m|+1)\tau(k^2)},\ m = 0, 1, ... \tag{3.26}$$

The angle parameter φ_m has the form

$$\varphi_m = 2\,\mathrm{sign}(m)\arg\{-i\mathbf{K}^{'}(k) + (2|m|+1)\mathbf{K}(k)\}. \tag{3.27}$$

Therefore, the maximal modulus is provided by curves belonging to $\Gamma_0[-1,1;a]$.

Let us change the parameter k to $1/k$. Then

$$\mathbf{K}\left(\frac{1}{k}\right) = k\{\mathbf{K}(k) + i\mathbf{K}^{'}(k)\},$$

$$\mathbf{K}^{'}\left(\frac{1}{k}\right) = k\mathbf{K}^{'}(k).$$

We put $k := 1/k$ and rewrite (3.25) as

$$\log R = \pi \Im \tau(k^2),\ k^2 = \frac{2}{a+1}. \tag{3.28}$$

Therefore, for the new $k^2 = 2/(a+1)$ we get

$$\varphi_0 = -\arg(k^2\mathbf{K^2}(k)),\ k^2 = \frac{2}{a+1}. \tag{3.29}$$

We proved a theorem

Theorem 3.4. *Consider the points*

$$a_1 = -1, a_2 = 1, a_3 = a; \quad \Im a \geq 0, \Re a \geq 0.$$

Let D_m be a family of doubly connected domains, such that the curves separating its boundary components belong to the homotopy class Γ_m. Then the maximum of a modulus of D_m is attained for $m = 0$ and is given by equality (3.28). The extremal domain is a ring domain of the quadratic differential (3.12) with the angle parameter defined by (3.29).

When $m \neq 0$, the modulus of the domain D_m is described by equality (3.26), where $k = (a+1)/2$. D_m is a ring domain of the quadratic differential (3.12), where φ_k is given by (3.27).

For k with $\Im k > 0$ we understand $\mathbf{K}(k)$, $\mathbf{K}'(k)$ as analytic continuation of $\mathbf{K}(k)$, $\mathbf{K}'(k)$ for $0 < k < 1$ along any path, which does not intersect the imaginary axis.

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