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GUARANTEED INVESTMENT CONTRACTS: DISTRIBUTED AND UNDISTRIBUTED EXCESS RETURN

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ABSTRACT. Annual minimum rate of return guarantees are analyzed together with rules for distribution of positive excess return, i.e. investment returns in excess of the guaranteed minimum return. Together with the level of the annual minimum rate of return guarantee both the customer's and the insurer's fractions of the positive excess return are determined so that the market value of the insurer's capital inflow (determined by the fraction of the positive excess return) equals the market value of the insurer's capital outflow (determined by the minimum rate of return guarantee) at the inception of the contract.

The analysis is undertaken both with and without a surplus distribution mechanism. The surplus distribution mechanism works through a bonus account that serves as a buffer in the following sense: in ('bad') years when the investment returns are lower than the minimum rate of return guarantee, funds are transferred from the bonus account to the customer's account. In ('good') years when the investment returns are above the minimum rate of return guarantee, a part of the positive excess return is credited to the bonus account.

In addition to characterizations of fair combinations of the level of the annual minimum rate of return guarantee and the sharing rules of the positive excess return, our analysis indicates that the presence of a surplus distribution mechanism allows the insurer to offer a much wider menu of contracts to the customer than without a surplus distribution mechanism.

1. Introduction

Minimum rate of return guarantees connected to life-insurance products are currently of great practical concern in many countries. More detailed descriptions for the situations in Denmark, Germany, Japan, and the Netherlands may be found in Hansen and Miltersen (2000), Mertens (1999), Matsuyama (1999), and Donselaar (1999), respectively. Historically, the initial level of these guarantees was low and, furthermore, *fixed* throughout the contract period. In the terminology of financial option pricing theory, these

Date: January 1998. This version: January 28, 2000.
Key words and phrases. Annual minimum rate of return guarantee, surplus distribution mechanism, bonus, life insurance.
Earlier versions of this paper have been presented at the *4th Nordic Symposium on Contingent Claims Analysis in Finance and Insurance*, Copenhagen Business School, Copenhagen, Denmark, *Danske Bank Symposium on Securities with Embedded Options*, Stavrby Skov, Denmark, *FIBE conference 1999*, Norwegian School of Economics and Business Administration, Bergen, Norway, *Financial Markets in the Nordic Countries*, the Aarhus School of Business, Århus, Denmark, *9th Annual Derivatives Securities Conference*, Boston, USA, *9th International AFIR Colloquium*, Tokyo, Japan, *The 26th Seminar of the European Group of Risk and Insurance Economists*, Madrid, Spain, and at seminars at Copenhagen University, Copenhagen, Denmark, Tilburg University, Tilburg, the Netherlands, Erasmus University, Rotterdam, the Netherlands, and University of Amsterdam, Amsterdam, the Netherlands. Comments from Knut Aase, Jochen Beißer, Christian Fotland, Anders Grosen, Martin Skovgaard Hansen, Mette Hansen, Ulrich Hege, Bjarne Astrup Jensen, Frank de Jong, Peter Løchte Jørgensen, Monika Mertens, Thomas Møller, Jørgen Aase Nielsen, Ragnar Norberg, Henrik Ramlau-Hansen, Mogens Steffensen, Ton Vorst, and other seminar participants were most appreciated. The first author gratefully acknowledges financial support of the Danish Social Science Research Councils and Danske Bank. Document typeset in LaTeX.

contracts were long maturity products issued far out-of-the-money. Due to the low interest rates experienced by many countries in the 1990s, which also influenced life-insurance companies' realized returns, the probability of minimum rate of return guarantees expiring at-the-money or even in-the-money has increased. As a consequence of this development the value of the annual minimum rate of return guarantee has increased. Naturally, this development triggers the need for a closer examination of minimum rate of return guarantees.

Compared to financial instruments like standard call and put options, life-insurance contracts are more complex products including characteristics like mortality/survival, periodical premiums, and the right to surrender, in addition to annual minimum rate of return guarantees. Moreover, legislation both requires life-insurance companies to set aside funds at the liability side of the balance and restricts the distribution of annual investment surplus. All these factors influence the valuation of life-insurance contracts. Apparently a model including all these factors would be rather complex, and the challenge is to both incorporate the important factors and at the same time keep the model tractable.

Our model consists of an investment/savings plan or contract between two parties called *the insurer* and *the customer*. The contract specifies a benchmark return and a periodic (annual) minimum rate of return guarantee. The valuation of such guarantees has been analyzed by Persson and Aase (1997) and Miltersen and Persson (1999) under various term structure models and investment benchmarks. In this paper these minimum rate of return guarantees are evaluated in connection with a surplus distribution mechanism, i.e. a rule for the distribution of the annual investment return above the minimum rate of return guarantee between the insurer and the customer. We specifically focus on the situation in which the minimum rate of return guarantee and the surplus distribution mechanism are designed in such a way that no additional up-front option premium is required for the minimum rate of return guarantee. This is in contrast to the two papers mentioned which disregard surplus distribution and evaluate an (up-front) option premium (implicitly assuming that the complete surplus is credited to the customer). An alternative surplus distribution mechanism is treated by Grosen and Jørgensen (1999). See also Norberg (1999) for a thorough treatment of surplus distribution and bonus mechanisms in general and Norberg (1997) for a more explicit treatment with some examples of different surplus distribution mechanisms.

At date zero the customer deposits an amount X into an account A, which is invested by the insurer for a period of T years. The insurer promises the customer an annual rate of return on the account A in year i equal to

$$g_i + \alpha(\delta_i - g_i)^+,$$

where the constant g_i is a specified minimum rate of return guarantee in year i, δ_i is the random rate of return of the specified benchmark portfolio in year i, and $\alpha \in [0,1]$ is the fraction of the positive excess rate of return[1] which is credited to the customer's account.

In return for the minimum rate of return guarantee the insurer receives a fraction, β, of the excess rate of return. I.e. the return

$$\beta(\delta_i - g_i)^+$$

is credited to the insurer's account, denoted by C.

In addition, the model includes a surplus distribution mechanism working through the bonus account B, which is managed by the insurer. The part of the overall return neither credited to account A nor C is credited to account B, thus the entire return is distributed between the three accounts. In years when the realized annual rate of return on the benchmark portfolio is greater than the minimum rate of return guarantee, a positive amount will typically be credited to account B. On the other hand, if the realized annual rate of return on the benchmark portfolio is less than the minimum rate of return guarantee, funds are transferred from the bonus account to the account A to cover the minimum rate of return guarantee.

When the contract expires, the customer receives any positive balance on the bonus account, whereas the insurer covers a negative balance. By this mechanism the balance of the bonus account throughout the contract period represents undistributed surplus. In financial terms, the minimum rate of return guarantee is then just a (European) call option issued by the insurer granting the customer the right to the final balance of the bonus account for an exercise price of zero.

The relation between the customer and the insurer can be represented graphically using a standard T-account,

Assets	Liabilities
X	$A + B^+$
	$C - B^-$
X	X

where the first post on the liability side is the customer's claim on the assets, X, and the second post is the insurer's claim on the assets. To further illustrate this contract consider the example in table 1. For the rest of the paper A_t, B_t, and C_t refer to the balance of the accounts A, B, and C by the end of year t. Table 1(a) shows the distribution of returns between the different accounts in the case of high returns, i.e. when the market return is above the minimum guaranteed return. Initially, the deposit $X = 100$ is credited to the account A. The first year's realized rate of return, 30%, is distributed as

[1] As usual the operator z^+ on z denotes the positive part of z, i.e. $z^+ = \max(z, 0)$, similarly the operator z^- on z denotes (the negative of) the negative part of z, i.e. $z^- = \max(-z, 0)$.

Year	Return	X	A	B	C
0		100	100	0	0
1	30%	130	120	5	5
2	30%	169	144	14	11

(a) Scenario One: 'Good'

Year	Return	X	A	B	C
0		100	100	0	0
1	30%	130	120	5	5
2	0%	130	132	-7	5

(b) Scenario Two: 'Bad'

TABLE 1. Example of distributions between the accounts for two given scenarios, 'good' and 'bad', with the following parameter values: $g = 10\%$, $\alpha = 50\%$, $\beta = 25\%$, and $X = 100$.

follows: account A is credited with the amount $A_0\big(g + \alpha(\delta_1 - g)^+\big) = 100(.1 + .5 \times .2) = 20$, account C is credited with the amount $A_0\beta(\delta_1 - g)^+ = 100 \times .25 \times .2 = 5$, the remaining 5 is credited to account B. In our set-up the parameter β determines the share of the positive surplus that is distributed to the insurer. The parameter β thus determines the premium the customer pays for the annual minimum rate of return guarantee (alternatively the maturity guarantee on the bonus account).

The distribution of the second year's return is similar. However, observe that whereas the value of the benchmark portfolio by the end of the first year, denoted X_1, is used as the base when the percentage return of the benchmark portfolio is determined (39 represents 30% of 130), the value of the insurer's account by the end of the first year A_1 is used as the base when investment returns are credited to the accounts A and C. E.g. in year two the balance of account A is increased by 20% of A_1 and the balance of C is increased by 5% of A_1. The amount that is credited to account B is residually determined and can be divided into two components. First, the remaining 5% of the investment surplus using A_1 as the base (for year two this amount is 6), then a correction term for using A_1 as the base instead of X_1. The correction term takes the value 3 in year two (the difference between 30% of 130 and 30% of 120). In general, this correction term will be positive when the last year's balance of X is larger than last year's balance of A and negative if last year's A is larger than last year's X.

If the contract matures at date two, the customer receives the balances of account A and B, in total $144 + 14 = 158$. Under this scenario the cashflow credited to the insurer is 11.

Table 1(b) illustrates the case when the minimum rate of return guarantee is binding in the second year whereas the return distribution of year one is the same as in scenario one. In year two the realized rate of return is 0% and the minimum rate of return guarantee is triggered. Account A is credited with the amount $A_1(g + \alpha(\delta_2 - g)^+ = 120(.1) = 12$, account C is credited with the amount $A_1\beta(\delta_2 - g)^+ = 0$, therefore the amount 12 is subtracted from account B. If the contract matures at date two, the customer now receives only the balance of account A, in total 132, whereas the insurer has to cover the negative balance of -7 of the bonus account. Under this scenario the accumulated cashflow credited to the insurer is $5 - 7 = -2$.

These tables illustrate that the insurer may either lose or win money. The parameter β determines the fraction of the return credited to the insurer and hence determines the (option) premium the customer pays for the minimum rate of return guarantee. Our main objective of the paper is to determine the parameter β together with the parameters α and g_i in order to make the contract, in a specific sense, fair.

A fair contract is a contract where the initial market value of capital inflow (premiums) equals the initial market value of the capital outflow (benefits). This pricing principle is therefore in the spirit of the classical principle of equivalence known from the actuarial sciences. For our set-up the consequence of this principle is that the date zero market value of the sum of the final balance of account A and the call option on the final balance of account B is equal to the initial investment in the benchmark portfolio X. An implication of this principle is that the market value at date zero of the final balance of account C equals the market value at date zero of the promise to cover a potential negative final balance of account B, i.e. the market price of the insurer's income stream, the final balance of account C, equals the market value of the insurer's net expenses, the potential negative balance of account B.

We use standard theory from financial economics based on no-arbitrage arguments to calculate initial market prices. Any other price would lead to one of the following situations: if the date zero market value of the sum of the final balance of account A and the call option on the final balance of account B was greater than the initial investment in the benchmark portfolio, the customer could make arbitrary high profits by increasing the number of such contracts. In the opposite case, the insurer could make arbitrary high profits. Neither of these situations are consistent with any sensible economic model with a frictionless market where both the underlying benchmark portfolio and these insurance contracts are traded simultaneously. The same argument is used by Briys and de Varenne (1997) also in the context of life insurance though analyzing different issues. The same valuation principle can be used in a competitive market (i.e. with free entry for new insurers offering this type of contracts) for this type of contracts, cf. e.g. Hansen and Miltersen (2000).

The central role of the insurer in our set-up is to serve as a financial intermediator. Instead of investing directly in the underlying benchmark portfolio, the insurer offers alternative investment possibilities based on the same underlying benchmark portfolio which may include an annual minimum rate of return guarantee and a surplus distribution mechanism. In good years the insurer keeps part of the surplus, in bad years the insurer provides additional yield. Note carefully that the annual minimum rate of return guarantee and the call option on the bonus account by construction have market value equal to X at the time of inception of the contract. Thus the customer does not have to pay any extra up-front premium, i.e. the contract is a fair zero-sum game between the insurer and the customer.

An additional point concerns the parameter α. Although we focus on annual returns instead of benefits, a high α means that the return of the benchmark portfolio has a high impact on the annual return of the contract. This situation resembles unit-linked life-insurance policies.[2] In contrast, $\alpha = 0$ corresponds to a deterministic rate of return of the contract which is more in the spirit of traditional life-insurance contracts. The traditional life-insurance contract is normally associated with a surplus distribution mechanism, which we have also included in our modeling framework.

In the fall of 1998 a major Norwegian insurance company introduced a new savings product. It turns out, as we demonstrate below, that this product fits exactly into our model and thus may be analyzed within our framework. Also both the Dutch 'click' funds (Klikfondsen), where the gain is locked in and the exercise price is adjusted accordingly when the price process of the underlying security hits certain prespecified levels, and the Dutch investment contracts, where savings are accumulated in order to mimic an annuity profile for bullet mortgage loans, are closely related and can be priced with similar methods.

The paper is organized as follows. Section 2 outlines the model and explains the cash flows between the three accounts. Section 3 treats the case of Gaussian return on the benchmark portfolio and deterministic short term interest rates. A closed form solution for the value of the customer's account is derived. The value of the bonus account is solved by Monte Carlo simulations in section 4. Corresponding values of annual minimum rate of return guarantees and the fractions of the excess return distributed to the customer's account and the bonus account are plotted for fair contracts. Finally, section 5 concludes.

2. The Model

We analyze two different situations, one including the bonus account, the other excluding the bonus account.

From now on we work with logarithmic (or continuously compounded) returns in contrast to the arithmetic returns used in our initial example from table 1.

2.1. **No Bonus Account.**

At first we will ignore the existence of the bonus account and only work with the customer's account A and the insurer's account C. At the end of year t the total amount on account A can be written recursively using the amount on the account at the end of the preceding year as

$$A_t = A_{t-1} e^{g_t + \alpha(\delta_t - g_t)^+}.$$

That is, the balance at the end of year t is simply the balance at the end of year $t-1$ with interest accrued according to the guaranteed minimum rate of return, g_t, and a fraction of a positive excess rate

[2]Such contracts are also called *equity-linked* contracts, among other names, cf. e.g. Brennan and Schwartz (1976), Brennan and Schwartz (1979), and Aase and Persson (1994).

of return. The initial amount, A_0, on this account equals the invested sum at date zero, X. Hence, A_t can be written as

$$A_t = Xe^{\sum_{i=1}^{t}\left(g_i+\alpha(\delta_i-g_i)^+\right)}. \tag{1}$$

The remaining amount is credited to the insurer's account C. Hence, the amount on the account C at the end of year t can residually be determined as

$$C_t = Xe^{\sum_{i=1}^{t}\delta_i} - A_t. \tag{2}$$

The insurer is not actually required to invest the amount X in the benchmark portfolio at date zero. The actual investment strategy followed by the insurer is not of any concern to the customer as long as the correct amount is credited to the account A. The correct amount is calculated on the basis of X and the rate of return on the benchmark portfolio. However, for the purpose of finding the value, at date zero, of the different accounts we can assume (without loss of generality) that the insurer actually does invest the amount X in the benchmark portfolio, as the following simple no-arbitrage argument shows: suppose that there was an alternative investment strategy that would give a higher date zero value than the investment of the amount X in the benchmark portfolio. Then any investor could create an arbitrage opportunity by shorting the amount X in the benchmark portfolio and investing the money by following the alternative strategy.

As explained from our fair pricing principle with no bonus account, the market price at date zero of C_T equals zero. If the date t market value operator is denoted by $V_t(\cdot)$,[3] we obtain the following restriction

$$V_0(C_T) = 0. \tag{3}$$

That is, the insurer gets a fair share of the excess rate of return on the benchmark portfolio for issuing the annual minimum rate of return guarantee for the customer's account A. Combining equations (2) and (3), we have

$$X = V_0\left(Xe^{\sum_{i=1}^{T}\delta_i}\right) = V_0(A_T),$$

which implies, using equation (1), that

$$V_0\left(\frac{A_T}{X}\right) = V_0\left(e^{\sum_{i=1}^{T}(g_i+\alpha(\delta_i-g_i)^+)}\right) = 1. \tag{4}$$

[3]Formally, in this paper we will work in a dynamically complete market so that the date t market value can be calculated as

$$V_t(Z_T) = e^{-r(T-t)}E_t^Q[Z_T],$$

where $E_t^Q[\cdot]$ denotes the conditional expectation under an equivalent martingale measure, Q, given the information at date t, Z_T is a (stochastic) payoff at date T, and r is the instantaneous short term interest rate.

The final condition determines possible specifications of the annual minimum rate of return guarantees from date zero to the end of year T, $\{g_i\}_{i=1}^T$, and the fraction α of the excess rate of return credited to the customer for contracts fulfilling the assumption of the fair pricing principle.

2.2. **The Bonus Account.** We now introduce the bonus account B. It is natural to determine the bonus residually for given values of α and β, hence account C is not, as in the previous case, residually determined. Instead the balance of account C is given by[4]

$$C_t = C_{t-1} + A_{t-1}(e^{\beta(\delta_t - g_t)^+} - 1).$$

The last term represents the share of year t's excess return that is credited to the insurer. Observe that the balance of the customer's account A is used as the base. The initial balance of the account C is zero. We can thus write C_t, for $t = 1, \ldots, T$, as

$$C_t = \sum_{i=1}^{t} (e^{\beta(\delta_i - g_i)^+} - 1) A_{i-1}. \tag{5}$$

The bonus account B is residually determined as

$$B_t = X e^{\sum_{i=1}^t \delta_i} - A_t - C_t.$$

The insurer's obligation is to cover a potential deficit on the account B at date T.

The fair pricing principle employed earlier again sets the condition $V_0(C_T) - V_0(B_T^-) = 0$ for the contract, leading to the condition

$$X = V_0(A_T) + V_0(B_T^+)$$

to be satisfied by different combinations of α, β, and $\{g_i\}_{i=1}^T$. Different combinations of αs and βs give us different contracts. The special case $\alpha = 0$ resembles a stylized standard life-insurance investment contract with a surplus distribution mechanism. The case of no bonus account, cf. the previous subsection, is equivalent to the case where the insurer keeps the balance of the bonus account, whether positive or negative, at date T. This contract resembles a unit-linked policy including an annual minimum rate of return guarantee, but without a surplus distribution mechanism. Our set-up is thus fairly general[5]

[4]Note that there are no interest payments on the balance of the account C. It is a modeling issue whether one prefers the amount of the insurer's account to earn interest or not. The customer does not benefit from these missing interest payments at the expense of the insurer since we use the fair pricing principle to argue that the value of the insurer's claim is zero. We would have used exactly the same principle if there had been interest payments on the insurer's account. Hence, the result would simply have been a lower value of the parameter β, *ceteris paribus*.

[5]The previous case with no bonus account can be seen as an (exotic) special case of the general set-up including a bonus account where the parameter $\beta = \infty$. This condition ensures that no positive amounts will be credited to the bonus account, hence the balance of the bonus account will be non-positive with probability one and will therefore be covered by the insurer.

and, particularly, includes the link property of unit-linked insurance, annual minimum rate of return guarantees, and a surplus distribution mechanism.

3. Closed-form Solutions in the Gaussian Case with Deterministic Interest Rates

Assume that the interest rate r is constant and that the annual continuously compounded rate of return from the benchmark portfolio, δ_i, is normally distributed and independent over different years. Hence δ can be modeled (under an equivalent martingale measure Q) as

$$\delta_t = r - \frac{1}{2}\sigma^2 + \sigma(W_t - W_{t-1}), \tag{6}$$

where σ is the volatility of the rate of return on the benchmark portfolio and $W = \{W_t, t \geq 0\}$ is a standard Wiener process under the probability measure Q. Note that we have implicitly assumed that there are no dividend payments[6] on the assets included in the benchmark portfolio since the drift term of δ in equation (6)) is $r - \frac{1}{2}\sigma^2$. The return on the benchmark portfolio follows the process in equation (6) if e.g. we assume that the price process of the benchmark portfolio follows a standard geometric Brownian motion as e.g. in the Black-Scholes model, cf. Black and Scholes (1973) or Merton (1973).

3.1. **The Value of the Account A.** In order to find the date zero value of the customer's account A from equation (1) for a given annual minimum rate of return guarantee, $\{g_i\}_{i=1}^T$, and the fraction of the excess rate of return that the customer gets, α, we will evaluate

$$\begin{aligned}
V_0\big(\frac{A_T}{X}\big) &= V_0\big(e^{\sum_{i=1}^T (g_i + \alpha(\delta_i - g_i)^+)}\big) \\
&= E^Q\big[e^{-rT} e^{\sum_{i=1}^T (g_i + \alpha(\delta_i - g_i)^+)}\big] \\
&= E^Q\big[e^{-rT} e^{\sum_{i=1}^T (g_i \vee (\alpha\delta_i + (1-\alpha)g_i))}\big] \\
&= E^Q\big[e^{-rT} e^{\sum_{i=1}^T ((\alpha g_i \vee \alpha\delta_i) + (1-\alpha)g_i))}\big] \\
&= E^Q\big[e^{-rT} e^{(1-\alpha)\sum_{i=1}^T g_i} e^{\sum_{i=1}^T (\alpha g_i \vee \alpha\delta_i)}\big] \\
&= e^{(1-\alpha)\sum_{i=1}^T g_i} \prod_{i=1}^T E^Q\big[e^{-r} e^{(\alpha g_i \vee \alpha\delta_i)}\big] \\
&= e^{(1-\alpha)\sum_{i=1}^T g_i} \prod_{i=1}^T E^Q\big[e^{-r}(e^{\alpha g_i} \vee e^{\alpha\delta_i})\big].
\end{aligned} \tag{7}$$

[6] An equivalent interpretation is that potential dividends on the assets included in the benchmark portfolio are immediately reinvested into the benchmark portfolio.

In order to evaluate $E^Q\big[e^{-r}(e^{\alpha g_i} \vee e^{\alpha\delta_i})\big]$ from equation (7), we make the following observation

$$E^Q\big[e^{-r}(e^{\alpha g_i} \vee e^{\alpha\delta_i})\big] = e^{-r}E^Q\big[(e^{\alpha\delta_i} - e^{\alpha g_i})^+ + e^{\alpha g_i}\big] \tag{8}$$
$$= e^{-r}E^Q\big[(e^{\alpha\delta_i} - e^{\alpha g_i})^+\big] + e^{\alpha g_i - r},$$

where $\vee$ denotes the max operator. I.e. $X \vee Y = \max(X, Y)$.

Hence, we have the value of a European call option on a modified underlying security with payoff $e^{\alpha\delta_i}$ at the maturity of the option. The value of this modified underlying security is

$$e^{-r}E^Q\big[e^{\alpha\delta_i}\big] = e^{-r}e^{\alpha(r-\frac{1}{2}\sigma^2)+\frac{1}{2}\alpha^2\sigma^2} = e^{(\alpha-1)(r+\frac{1}{2}\alpha\sigma^2)}$$

and its volatility is $\alpha\sigma$. Therefore, we can evaluate

$$e^{-r}E^Q\big[(e^{\alpha\delta_i} - e^{\alpha g_i})^+\big] = e^{(\alpha-1)(r+\frac{1}{2}\alpha\sigma^2)}\Phi\Big(\frac{r - g_i - \frac{1}{2}\sigma^2 + \alpha\sigma^2}{\sigma}\Big) - e^{\alpha g_i - r}\Phi\Big(\frac{r - g_i - \frac{1}{2}\sigma^2}{\sigma}\Big)$$

using the Black-Scholes formula. Here Φ denotes the cumulative distribution function of the standard normal distribution. If the reader is uncomfortable with our economic argument, the result can also be derived by brute force evaluation of the expectation. Thus, from equation (8),

$$E^Q\big[e^{-r}(e^{\alpha g_i} \vee e^{\alpha\delta_i})\big] = e^{(\alpha-1)(r+\frac{1}{2}\alpha\sigma^2)}\Phi\Big(\frac{r - g_i - \frac{1}{2}\sigma^2 + \alpha\sigma^2}{\sigma}\Big) + e^{\alpha g_i - r}\Phi\Big(\frac{g_i - r + \frac{1}{2}\sigma^2}{\sigma}\Big).$$

Finally, from equation (7),

$$V_0\big(\frac{A_T}{X}\big) = \prod_{i=1}^{T}\Big(e^{(1-\alpha)(g_i - r - \frac{1}{2}\alpha\sigma^2)}\Phi\Big(\frac{r - g_i - \frac{1}{2}\sigma^2 + \alpha\sigma^2}{\sigma}\Big) + e^{g_i - r}\Phi\Big(\frac{g_i - r + \frac{1}{2}\sigma^2}{\sigma}\Big)\Big), \tag{9}$$

which gives a closed form solution for the date zero value of the account A. For the special case of $\alpha = 1$ and constant minimum rate of return guarantee, g, this result has earlier been derived by Hipp (1996) and Miltersen and Persson (1999).

The case without a bonus account can now be analyzed from equations (4) and (9). Assuming that the annual minimum rate of return guarantee is the same each year, i.e. $g_i = g$, for all i, we have depicted corresponding values of αs and annual minimum rate of return guarantees, g, that provide a date zero value of the account A equal to *one* in figure 1. Note that when the annual minimum rate of return guarantee is the same each year the solutions, i.e. the αs and corresponding gs, to

$$V_0\big(\frac{A_T}{X}\big) = 1$$

are independent of T as it can be seen from equation (9). Hence, figure 1 is valid for any maturity.

Figure 1 depicts (for all maturities simultaneously) combinations of αs and annual minimum rate of return guarantees for contracts fulfilling the fair pricing principle. Not surprisingly, an increase of the

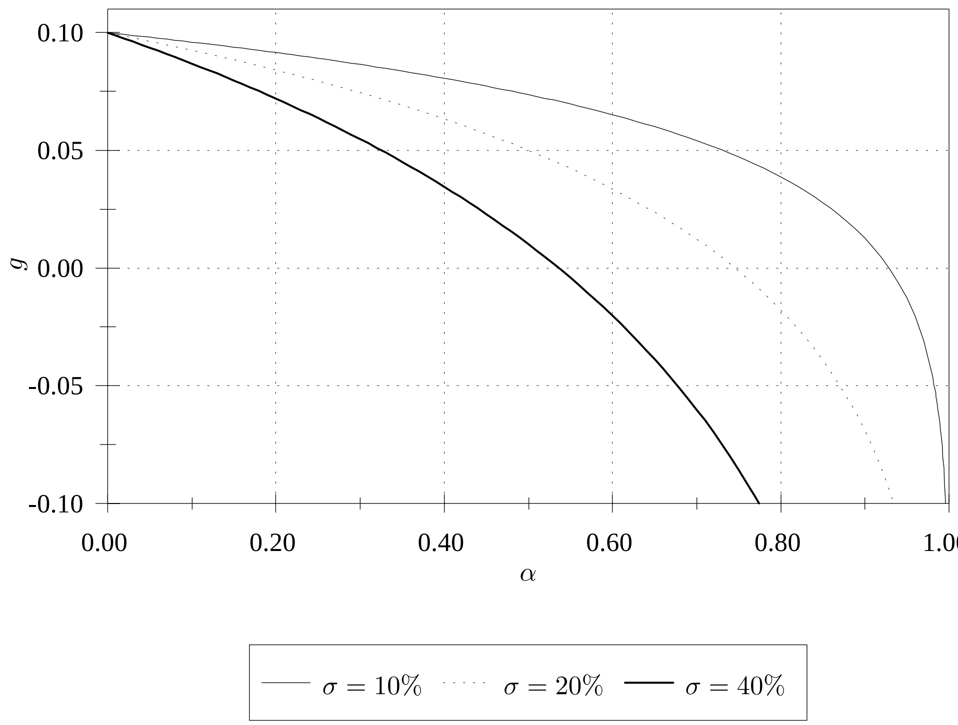

FIGURE 1. Corresponding values of α and g that implies fair contracts for three different volatilities ($\sigma = 10\%$, $\sigma = 20\%$, $\sigma = 40\%$) and $r = 10\%$.

volatility of the benchmark return lowers the annual minimum rate of return guarantee to the customer, *ceteris paribus*. The three graphs in figure 1 can also be considered as a kind of efficient frontier of fair contract combinations without surplus distribution mechanisms. Contracts with combinations of αs and gs below the efficient frontier must have a surplus distribution mechanism in order to fulfill the fair pricing principle.

Example 3.1. *The bank division (Gjensidige Bank AS) of a Norwegian insurance company recently (September 1998) offered a product very similar to the one described above. The investment period is 8 years ($T = 8$). The annual minimum rate of return guarantee is $g_i = 0\%$. The underlying benchmark portfolio is an arithmetic average of a selected series of national stock indices from various European stock exchanges. The sales brochure indicate an α in the range 50–60%. The initial investment amount, X, is at least NOK 10,000.*[7] *For this product equation* (9) *reduces to*

$$V_0\Big(\frac{A_T}{X}\Big) = \Big(e^{(\alpha-1)(r+\frac{1}{2}\alpha\sigma^2)}\Phi\Big(\frac{r-\frac{1}{2}\sigma^2+\alpha\sigma^2}{\sigma}\Big) + e^{-r}\Phi\Big(\frac{-r+\frac{1}{2}\sigma^2}{\sigma}\Big)\Big)^T.$$

[7] *From this amount administrative expenses in the range .5–2.5% is subtracted. In this treatment we disregard administrative expenses and for this particular product we interpret them as such and not as an additional up-front payment charged for the financial risk.*

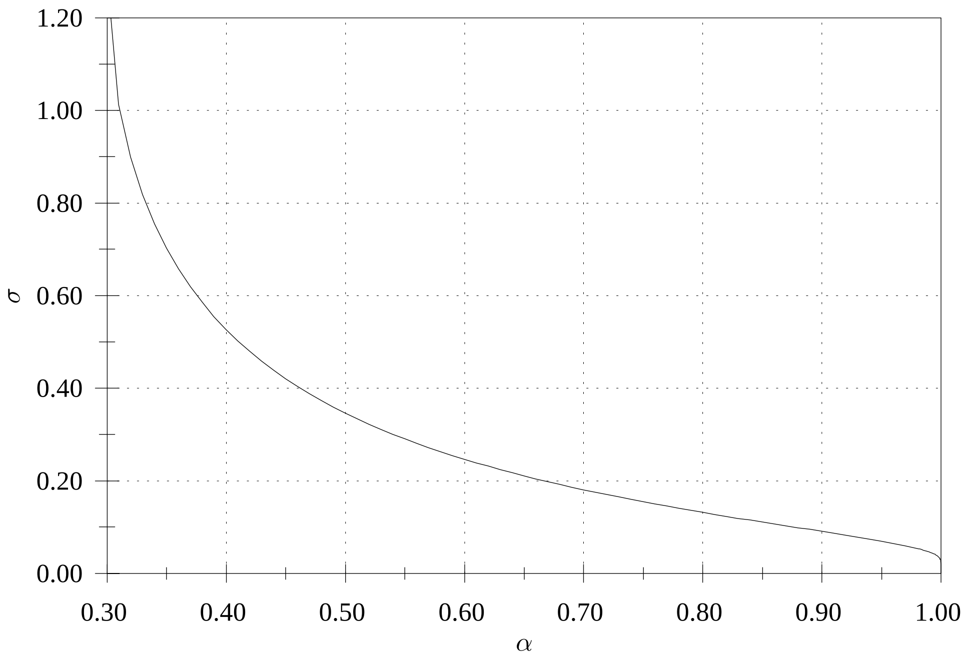

FIGURE 2. Corresponding values of α and σ that imply fair contracts for $g = 0\%$ and $r = 8\%$.

The requirement that $V_0\left(\frac{A_T}{X}\right) = 1$ *is thus equivalent to the condition*

$$e^{(\alpha-1)(r+\frac{1}{2}\alpha\sigma^2)}\Phi\left(\frac{r-\frac{1}{2}\sigma^2+\alpha\sigma^2}{\sigma}\right)+e^{-r}\Phi\left(\frac{-r+\frac{1}{2}\sigma^2}{\sigma}\right)=1.$$

In figure 2 we have plotted combinations of α *and* σ *satisfying this condition under the assumption that* $r = 8\%$ *(roughly the interest rate level in Norway in September 1998).*

From the graph we see that the implied volatility of this product is between 25% and 35% depending on the exact value of α*. Taking into account that the underlying benchmark portfolio in this case is an average of six indices even 25% volatility seems high. Moreover, such stock indices are not usually adjusted for dividend payments. Finally, there are no adjustments for depreciations or appreciations of the exchange rates, e.g. as one would expect since the interest level was relatively high in September 1998 in Norway relative to the six countries from where the indices were taken.*

As seen from the graph, our model as well as each of these arguments which are not formally included in our analysis all indicate that this product is over-priced. This may be one reason for that this product never became a big success.

3.2. **The Value of the Account C.** We now assume that the annual minimum rate of return guarantee is the same each year, i.e. $g_i = g$, for all i. Denote

$$\pi_A(t) = V_0\big(\frac{A_t}{X}\big) = V_0\big(e^{\sum_{i=1}^{t}(g+\alpha(\delta_i-g)^+)}\big),$$

which has the closed form expression (9). The similar market value, at date zero, for the account C at date $t \leq T$ can be derived according to equation (5) as

$$\begin{aligned} V_0\big(\frac{C_t}{X}\big) &= \sum_{i=1}^{t} V_0\big((e^{\beta(\delta_i-g_i)^+}-1)\frac{A_{i-1}}{X}\big) \\ &= \sum_{i=1}^{t} V_{i-1}\big((e^{\beta(\delta_i-g_i)^+}-1)\big)e^{-r(t-i+1)}V_0\big(\frac{A_{i-1}}{X}\big) \\ &= \pi_H(1)e^{-rt}\sum_{i=1}^{t}\pi_A(i-1)e^{r(i-1)}, \end{aligned}$$

where $\pi_A(0) = 1$ and

$$\begin{aligned} \pi_H(1) &= V_{t-1}\big(e^{\beta(\delta_t-g)^+}-1\big) \\ &= V_0\big(e^{\beta(\delta_1-g)^+}-1\big) \\ &= e^{(\beta-1)(r+\frac{1}{2}\beta\sigma^2)-\beta g}\Phi\Big(\frac{r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\Big) - e^{-r}\Phi\Big(\frac{r-g-\frac{1}{2}\sigma^2}{\sigma}\Big), \end{aligned}$$

which can be derived by a minor modification of equation (9).

4. Numerical Results in the Case of a Bonus Account

In the previous section we derived closed form solutions for the initial market values of the final balances of the accounts A and C. It is then straightforward to determine the initial market value of the account B since the sum of these three market values equal the initial investment amount X. The interesting problem is to calculate the initial market value of the total cashflow to the customer $A_T + B_T^+$ and since no similar closed form expression is available for B_T^+, we have to resort to numerical methods.

We have implemented a numerical simulation algorithm in order to calculate the expectation under the equivalent martingale measure, Q,

$$\frac{V_0(A_T+B_T^+)}{X} = e^{-rT}E^Q\Big[\frac{A_T+B_T^+}{X}\Big].$$

For $T = 5$ (years), $r = 10\%$, and $\sigma = 10\%$ figure 3 depicts combinations of αs, βs, and annual minimum rate of return guarantees for contracts fulfilling the fair pricing principle. That is, on top of the simulation algorithm we have a numerical search algorithm searching for combinations of g, α, and β such that the sum of the value of account A and the positive part of account B is equal to X.

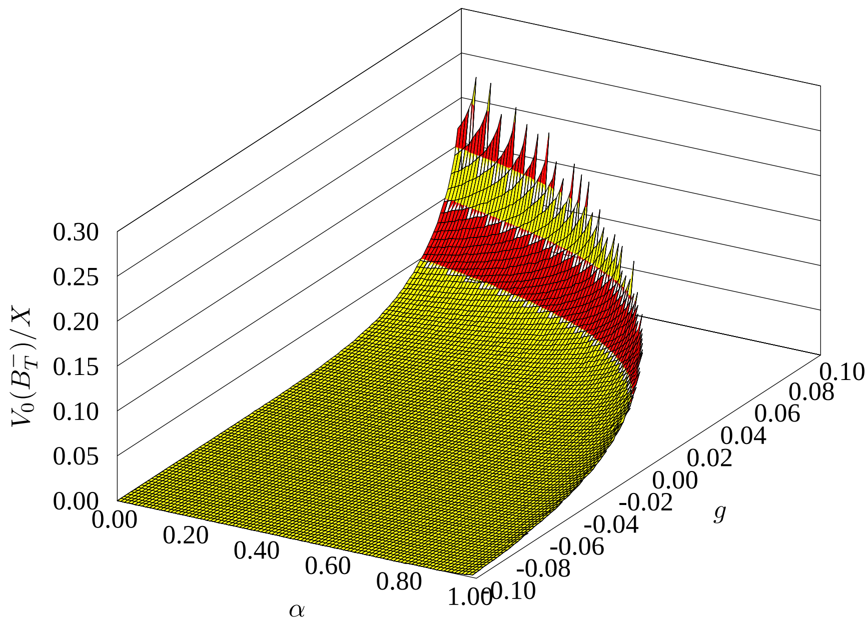

FIGURE 3. Corresponding values of α, g, and $\frac{V_0(B_T^-)}{X}$ for $\sigma = 10\%$, $r = 10\%$, and $T = 5$.

In figure 3 β is not presented directly. Although the parameter β determines the cost of the annual minimum rate of return guarantee for the customer, it has no direct interpretation as the cost of the annual minimum rate of return guarantee, i.e. in a more absolute sense. In order to quantify this cost at the inception of the contract we calculate[8]

$$\frac{V_0(C_T)}{X} = \frac{V_0(B_T^-)}{X}.$$

$\frac{V_0(B_T^-)}{X}$ (or $\frac{V_0(C_T)}{X}$) can be interpreted as the fair percentage up-front premium the customer will have to pay (instead of sharing the excess return with the insurer) for the minimum rate of return guarantee. By comparing figure 1 and 3 we see that we are only able to characterize fair contracts at or below the efficient frontier of fair contract combinations without surplus distribution mechanisms. We also see that the up-front premium that the customer will have to pay (instead of sharing the excess return with the insurer) for the minimum rate of return guarantee is higher the closer the contracts are to the efficient frontier.

We have further illustrated this point in two two-dimensional cuts of figure 3. Figures 4 and 5 show $\frac{V_0(B_T^-)}{X}$ as a function of α. We present graphs for three different levels of the minimum rate of return

[8]For contracts fulfilling the fair pricing principle, $V_0(A_T + B_T^+) = X$, hence, $V_0(C_T - B_T^-) = 0$, since $V_0(A_T + B_T^+ - B_T^- + C_T) = X$.

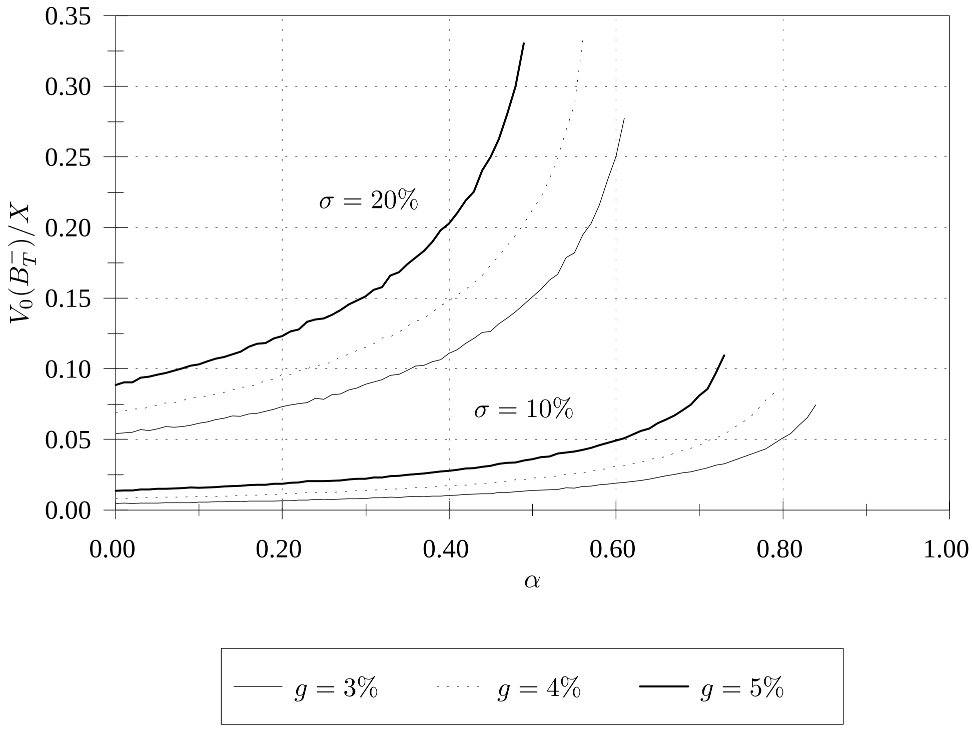

FIGURE 4. Corresponding values of α and $\frac{V_0(B_T^-)}{X}$ for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$), two different volatilities ($\sigma = 10\%$, $\sigma = 20\%$), $r = 10\%$, and $T = 5$.

guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$) and two different volatilities ($\sigma = 10\%$, $\sigma = 20\%$). Figure 4 is for a contract period of five years ($T = 5$), and figure 5 is for a contract period of thirty years ($T = 30$). Note that the graphs stop when the contract reaches the terms which are fair even without a surplus distribution mechanism, i.e. when they hit the efficient frontier. E.g. with a volatility of 20% and a minimum rate of return guarantee of 3% we can see from the efficient frontier in figure 1 that the contract is fair even without a surplus distribution mechanism for α just above 60% independent on the maturity of the contract. Hence, with a surplus distribution mechanism we are only able to find fair contracts for α up to 60%. This is why the graphs for $g = 3\%$ and $\sigma = 20\%$ stop at α just above 60% in both figures 4 and 5. The reason for the jagged graphs in figures 4 and 5 (contrary to figure 1) is that we are using a combined numerical simulation method and search procedure to find the fair contract combinations.

From figure 4 (not surprisingly) we see that the percentage up-front premium is increasing in α, g, and σ. The same is true for the longer contract period in figure 5. Comparing figures 4 and 5, we investigate the percentage up-front premium with respect to the contract period. We find that short term contracts are more risky for the insurer in the sense that the option premium is a higher fraction of the contract value

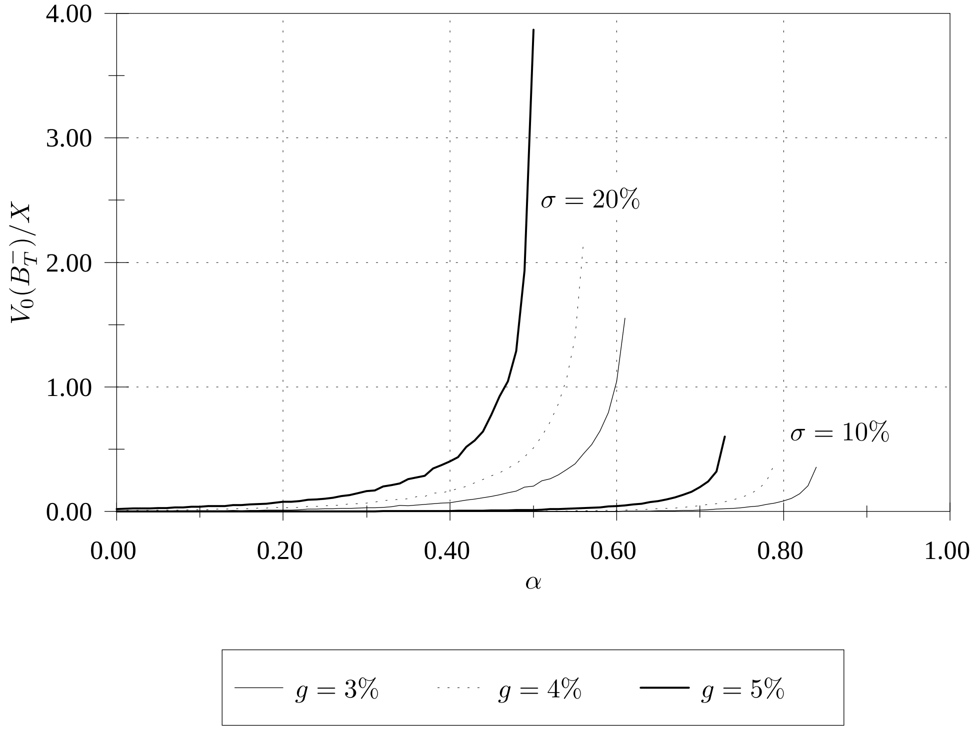

FIGURE 5. Corresponding values of α and $\frac{V_0(B_T^-)}{X}$ for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$), two different volatilities ($\sigma = 10\%$, $\sigma = 20\%$), $r = 10\%$, and $T = 30$.

compared to long term contracts for low values of αs. For high values of αs the situation is the opposite. This point is related to the smoothing effect of the bonus account and is perhaps easier illustrated in figures 6 and 7 where we have collected the graphs by volatility instead of contract period. Figures 6 and 7 show again $\frac{V_0(B_T^-)}{X}$ as a function of α. Here we also present graphs for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$) and two different contract periods ($T = 5$, $T = 30$). In figure 6 the volatility is 10% ($\sigma = 10\%$) and in figure 7 the volatility is 20% ($\sigma = 20\%$). The up-front premium of a minimum rate of return guarantee is increasing in maturity of the contract, cf. e.g. Miltersen and Persson (1999). For contracts close to the efficient frontier the inflow of funds to the bonus account is very limited (this can be seen from figure 8 below). Therefore the smoothing effect of the bonus account is very limited—frankly speaking smoothing requires both inflows and outflows. This explains why the percentage up-front premium is increasing in the maturity of the contracts for contracts close to the efficient frontier. However, for contracts further below the efficient frontier the inflow of funds to the bonus account is higher, cf. figure 8 below. Hence, the smoothing effect of the bonus account works much better. Naturally, a smoothing mechanism works better for long maturity contracts where there are more years to smooth over than for short maturity contracts, *ceteris paribus*. It can be seen from

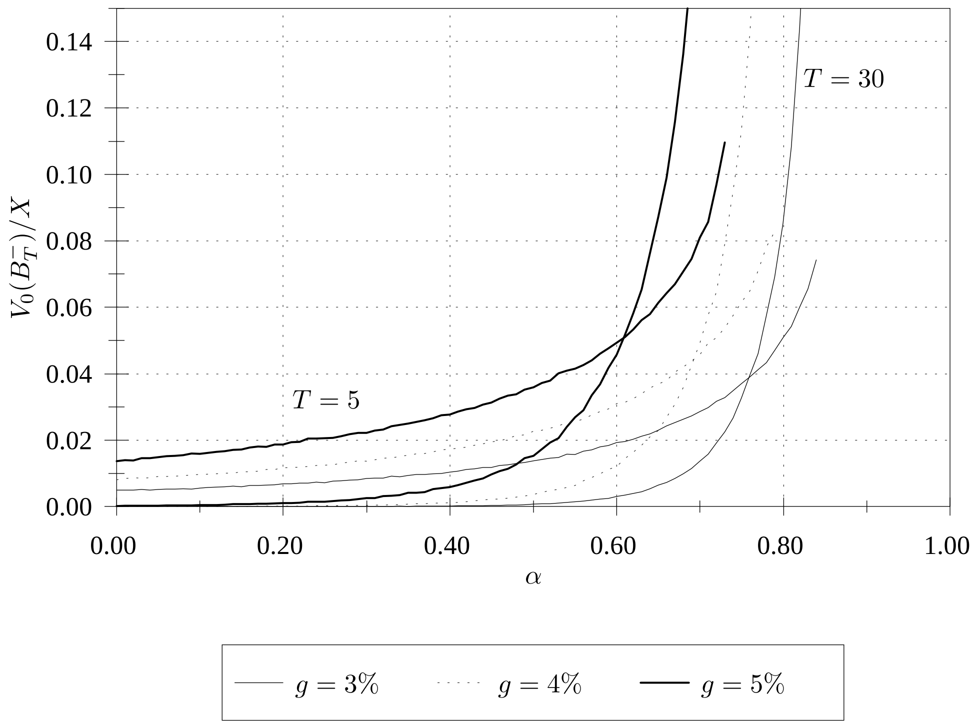

FIGURE 6. Corresponding values of α and $\frac{V_0(B_T^-)}{X}$ for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$), two different times to maturity ($T = 5$, $T = 30$), $r = 10\%$, and $\sigma = 10\%$.

figures 6 and 7 that the advantage of better smoothing for the longer maturity contracts outperforms the effect that the up-front value of a minimum rate of return guarantee without a smoothing mechanism is increasing in the maturity of the contract for contracts sufficiently below the efficient frontier. That is, for contracts sufficiently below the efficient frontier the percentage up-front premium is actually lower for the long maturity contracts than the short maturity contracts, *ceteris paribus*.

Instead of focusing on the corresponding up-front premium the customer would alternatively have had to pay (if she did not want to share the excess return with the insurer) for the minimum rate of return guarantee, we can depict the up-front value of the right to the potential positive balance on the bonus account, $V_0(B_T^+)$. In the same way as $\frac{V_0(B_T^-)}{X}$ is interpreted as the percentage of the value of the contract attributed to the alternative up-front option premium for the minimum rate of return guarantee, $\frac{V_0(B_T^+)}{X}$ can be interpreted as the fraction of the total value of the contract attributed to the right to receive a potential positive balance of the bonus account. We have illustrated this point in figure 8. In this figure we see how the fraction of the contract value attributed to the right to receive a potential positive balance of the bonus account decreases both in g and α down to zero which is hit exactly when the contracts are

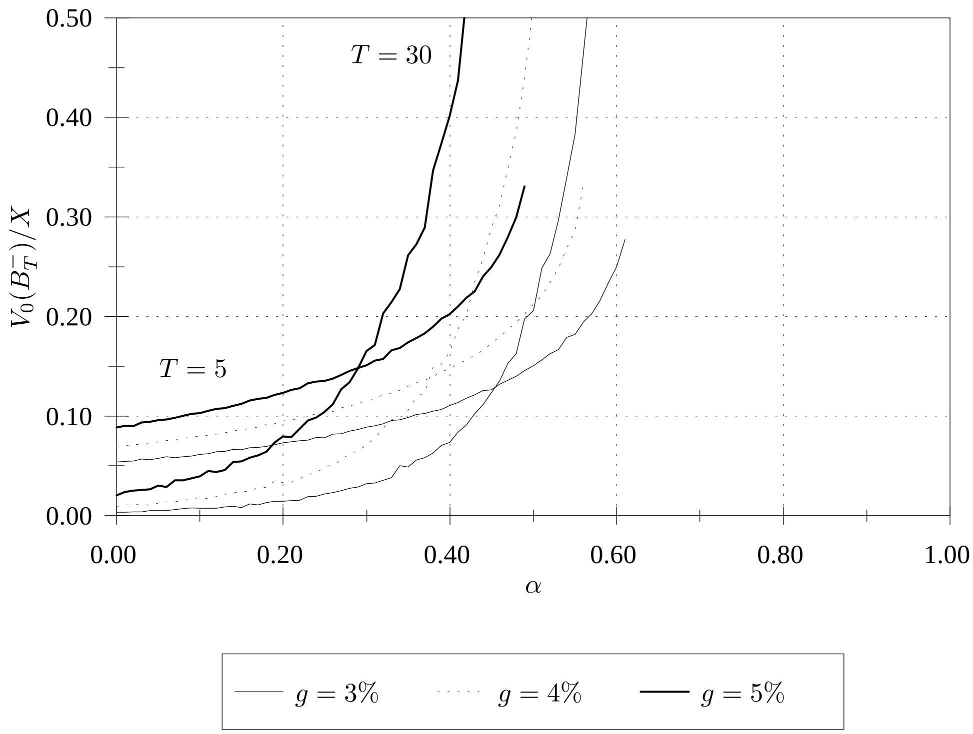

FIGURE 7. Corresponding values of α and $\frac{V_0(B_T^-)}{X}$ for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$), two different times to maturity ($T = 5$, $T = 30$), $r = 10\%$, and $\sigma = 20\%$.

at the efficient frontier, (i.e. when the contracts are fair even without a surplus distribution mechanism) cf. figure 1.

Again, we have further illustrated this point in two two-dimensional cuts of figure 8. Figures 9 and 10 show $\frac{V_0(B_T^+)}{X}$ as a function of α. Again, we present graphs for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$) and two different volatilities ($\sigma = 10\%$, $\sigma = 20\%$). In figure 9 the contract period is five years ($T = 5$) and in figure 10 the contract period is thirty years ($T = 30$).

At first thought it might seem surprising that the fraction of the contract value attributed to the right to receive a potential positive balance of the bonus account, $\frac{V_0(B_T^+)}{X}$, for $\alpha = 0$ is independent of the volatility, σ, cf. figures 9 and 10. However, there is a natural explanation. For $\alpha = 0$ the development of the customer's account A is purely deterministic: it earns the minimum rate of return guarantee, g, each period. Hence, $V_0(A_T)$ is independent of the volatility, σ. But since the contracts are fair

$$\frac{V_0(B_T^+)}{X} = 1 - \frac{V_0(A_T)}{X}.$$

Hence, $\frac{V_0(B_T^+)}{X}$ will also be independent of the volatility, σ.

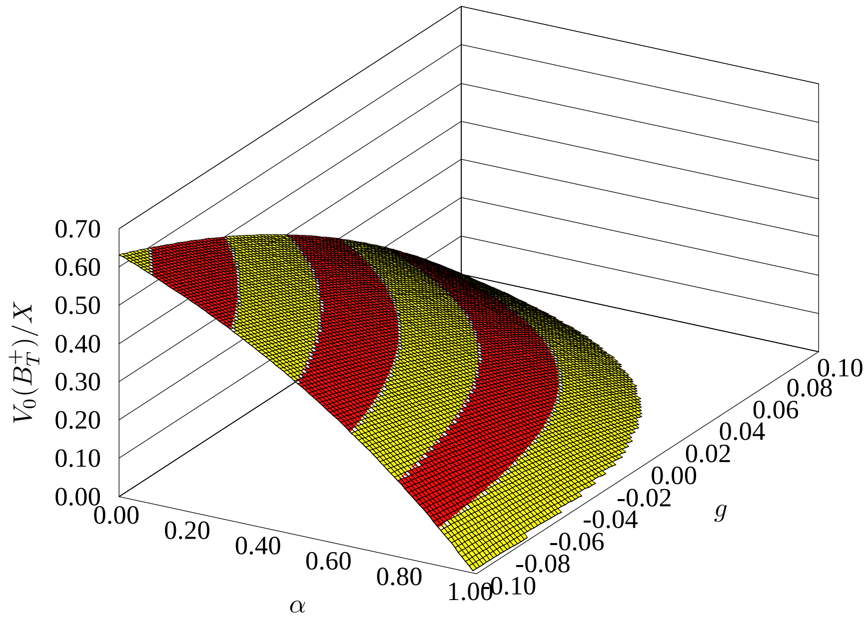

FIGURE 8. Corresponding values of α, g, and $\frac{V_0(B_T^+)}{X}$ for $\sigma = 10\%$, $r = 10\%$, and $T = 5$.

From figure 9 (not surprisingly) we see that the fraction of the contract value attributed to the right to receive a potential positive balance of the bonus account, $\frac{V_0(B_T^+)}{X}$, is decreasing in α, g, and σ. The same is true for the longer contract period in figure 10. By comparing figures 9 and 10 we see that the fraction of the contract value attributed to the right to receive a potential positive balance of the bonus account, $\frac{V_0(B_T^+)}{X}$, is increasing with the contract period. This point is perhaps easier illustrated in figures 11 and 12 where we have collected the graphs by volatility instead of contract period. Figures 11 and 12 show again $\frac{V_0(B_T^+)}{X}$ as a function of α. Also here we present graphs for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$) and two different contract periods ($T = 5$, $T = 30$). In figure 11 the volatility is 10% ($\sigma = 10\%$) and in figure 12 the volatility is 20% ($\sigma = 20\%$).

5. Concluding remarks

We have presented a framework which we believe constitutes a suitable starting point for analyzing the connection between annual minimum rate of return guarantees and the distribution of surplus. The contract we study is closely related to many real-life contracts and further properties of such contracts can easily be included in our set-up and analyzed using our simulation method. E.g. we can easily

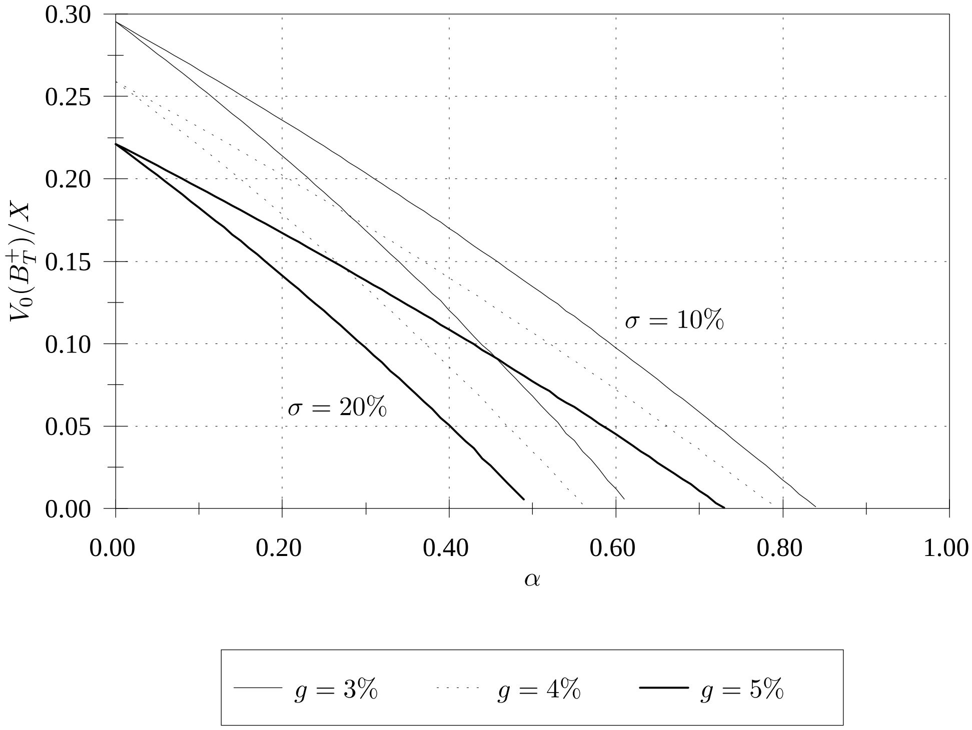

FIGURE 9. Corresponding values of α and $\frac{V_0(B_T^+)}{X}$ for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$), two different volatilities ($\sigma = 10\%$, $\sigma = 20\%$), $r = 10\%$, and $T = 5$.

include mortality risk[9] and annual premium payments instead of a single lump sum payment. It is also straightforward to include e.g. stochastic interest rates in our framework.

In our model the total initial amount X is credited to account A. Other possibilities are to deposit parts of the initial deposit X into account B and/or C when the contract is initiated. An immediate effect of this would be higher α and g, *ceteris paribus*. At first sight such an arrangement may seem more beneficial for the customer. On the other hand the initial balance of the account A will decrease and thereby the base for the annual minimum rate of return guarantees decreases. A strictly positive initial deposit to account C may be interpreted as (a sort of) up-front premium for the minimum rate of return guarantee.

In this paper we have only considered individual undistributed surplus modeled by the customer's own bonus account. In many real-life life-insurance contracts the undistributed surplus mechanism pools the bonus accounts for a large group of customers, cf. Hansen and Miltersen (2000). The introduction of

[9]Under the assumption that the insurer has a large pool of customers with independent mortality risk, which is also uncorrelated with the return on the benchmark portfolio, a simple argument based on the law of large numbers can be applied to eliminate the mortality risk as seen from the insurer's point of view. In financial terms mortality risk is diversifiable. Therefore, the insurer would be willing to provide life-insurance contracts based on standard mortality tables, i.e. simply using average mortality data for the different age groups of the population.

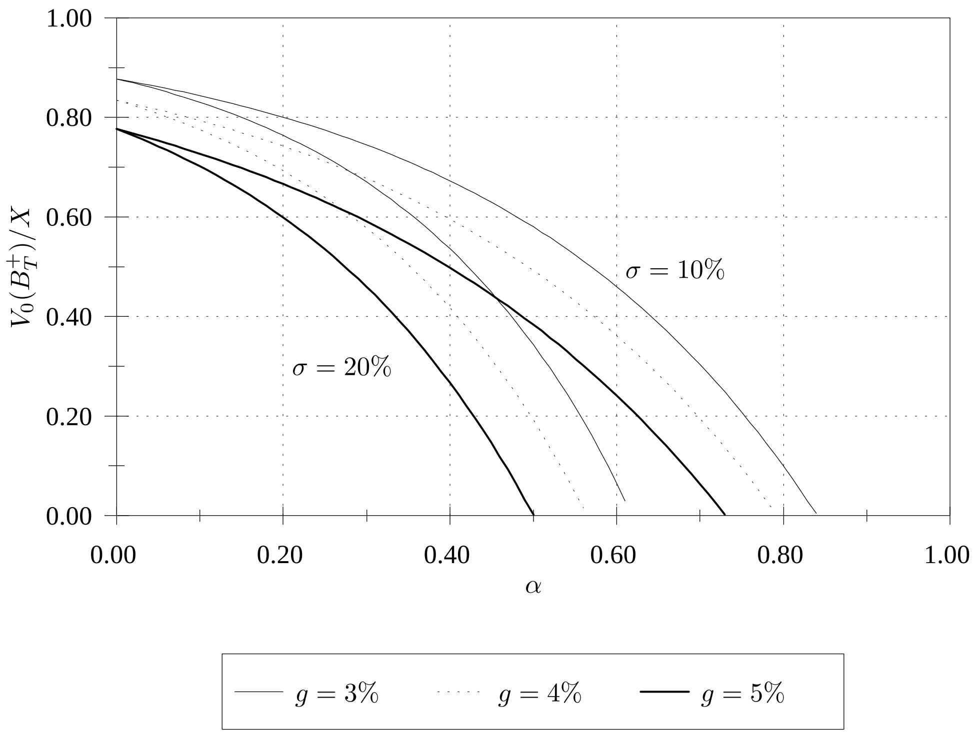

FIGURE 10. Corresponding values of α and $\frac{V_0(B_T^+)}{X}$ for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$), two different volatilities ($\sigma = 10\%$, $\sigma = 20\%$), $r = 10\%$, and $T = 30$.

a pooled undistributed surplus mechanism opens up a number of new and interesting issues including game theoretical considerations for the individual customers of when to enter into these life-insurance contracts and when to surrender if the contract includes a surrender option.[10] We consider these new issues outside the scope of the present paper.

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[10]For more on surrender options cf. e.g. Grosen and Jørgensen (1997).

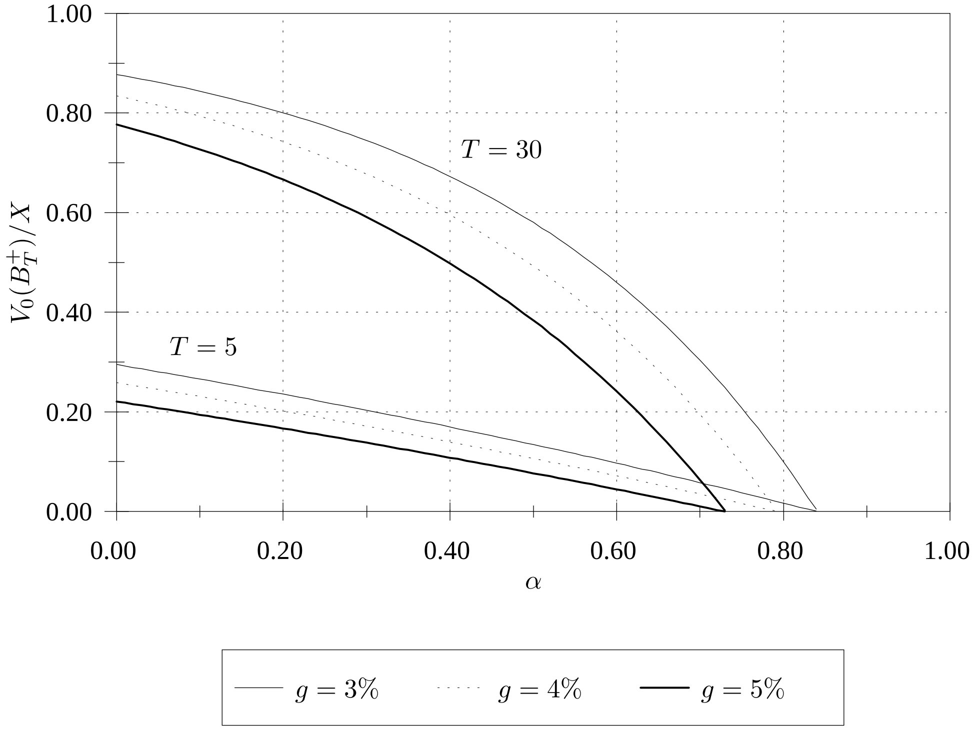

FIGURE 11. Corresponding values of α and $\frac{V_0(B_T^+)}{X}$ for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$), two different times to maturity ($T = 5$, $T = 30$), $r = 10\%$, and $\sigma = 10\%$.

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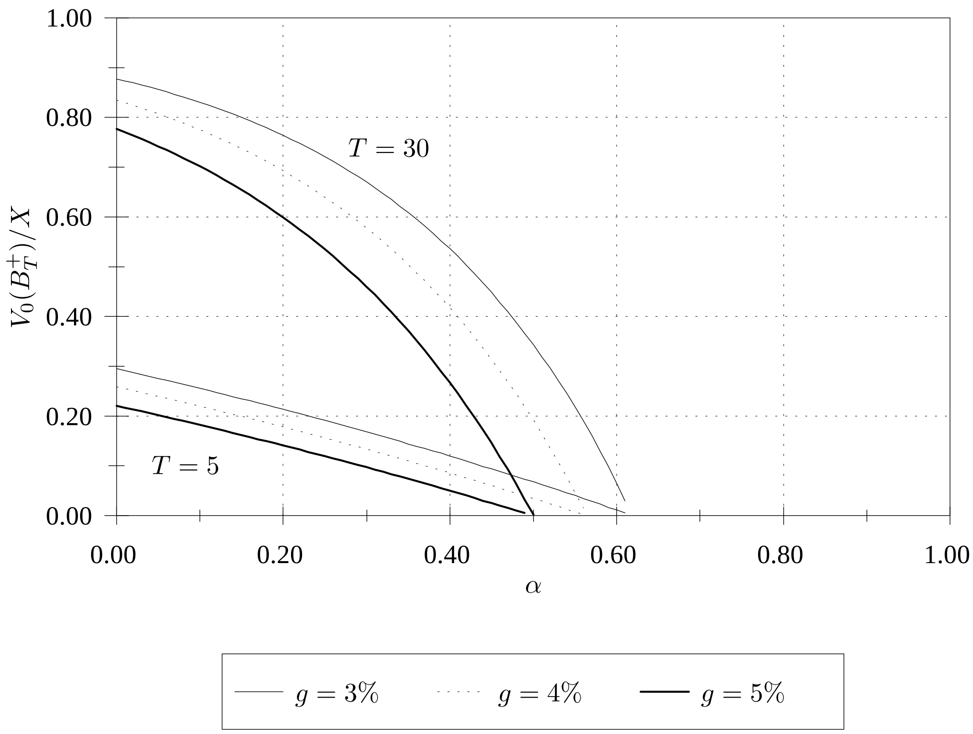

FIGURE 12. Corresponding values of α and $\frac{V_0(B_T^+)}{X}$ for three different levels of the minimum rate of return guarantee ($g = 3\%$, $g = 4\%$, $g = 5\%$), two different times to maturity ($T = 5$, $T = 30$), $r = 10\%$, and $\sigma = 20\%$.

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