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A SPDE Maximum Principle for Stochastic Differential Games under Partial Information with Application to Optimal Portfolios on Fixed Income Markets

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Abstract. In this paper we aim at establishing a necessary and sufficient maximum principle for partial information control of general stochastic games, where the controlled process is given by a stochastic reaction-diffusion equation with jumps. As an application of this result we study a zero-sum stochastic differential game on a fixed income market, that is we solve the problem of finding an optimal strategy for portfolios of constant maturity interest rate derivatives managed by a trader who plays against various "market scenarios". Here we permit the restriction that the trader has limited access to market information.

Keywords. Stochastic differential games, optimal portfolios, SPDE control.

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1 Introduction

The field of game theory initiated by the path breaking works of von Neumann and Morgenstern [14] has been an indispensable tool in economics to analyze complex strategic interactions between agents. Game theory as a branch of mathematics has also received much attention in other areas of applied sciences. For example it has been proven useful in social sciences as an approach to model decision making of interacting individuals in certain social situations. Other applications of this theory pertain e.g. to the description of evolutionary processes in biology, the modeling of interactive computation or the design of fair division in political science.

In this paper we study a zero-sum stochastic differential game under partial information: The total benefit of the players in this game following a strategy based on partial information always adds to zero. In other words, we consider the antagonistic interventions

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of two players A and B: There is a payoff function depending on the partial information strategies of A and B which stands for a reward for A but a cost for B. More specifically, the player A in our game is represented by a trader who tries to optimize his portfolio of constant maturity interest rate derivatives against various "market scenarios" symbolized by B. On the one hand the trader aims at maximizing his payoff, that is maximizing the expected terminal (cumulative) utility of his portfolio under the constraint of limited market information. On the other hand the market endeavors to create "reasonable" market prices by minimizing the payoff function. The portfolio managed by the trader is composed of fixed income instruments with constant time-to-maturity. Thus the portfolio value evolves in time and space (i.e. time-to-maturity) and necessaries an infinite dimensional modeling approach. Here in this paper we use stochastic partial differential equations (SPDE's) to describe the portfolio dynamics. In order to solve the min-max problem we want to employ a stochastic maximum principle for SPDE's.

We remark that there is a rich literature on the stochastic maximum principle. See e.g. [3], [2], [9], [18], [19] and the references therein. The authors in [1] derive a stochastic maximum principle for stochastic differential games, where the controlled process is given by a stochastic differential equation (SDE) and the control processes are assumed to be adapted to a sub-filtration of a filtration generated by a Lévy process. Our paper is an extension of [1] to the setting of SPDE's. We shall finally mention [12], where the authors invoke stochastic dynamic programming to study stochastic differential games.

In Section 2 we prove a sufficient (and necessary) maximum principle for zero-sum games (Theorem 2.1, 2.2). Then in Section 3 we apply the results of the previous section to construct an optimal strategy for the above mentioned stochastic differential game on fixed income markets.

2 The stochastic maximum principle for zero-sum games

In this section we want to study the stochastic maximum principle for stochastic differential games in the framework of SPDE control.

2.1 A sufficient maximum principle

Let $\Gamma(t,x)$ be our *controlled process* described by stochastic reaction-diffusion equation:

$$\begin{aligned} \Gamma(t,x) &= \xi(x) + \int_0^t \left[L\Gamma(s,x) + b(s,x,\Gamma(s,x),u_0(s,x)) \right] ds \\ &+ \int_0^t \sigma(s,x,\Gamma(s,x),u_0(s,x)) dB_s \qquad (1) \\ &+ \int_0^t \int_{\mathbb{R}} \psi(s,x,\Gamma(s,x),u_1(s,x,z)) \widetilde{N}(ds,dz), \qquad (t,x) \in [0,T] \times G, \end{aligned}$$

with boundary condition

$$\begin{aligned}\Gamma(0,x) &= \xi(x),\ x\in\overline{G},\\ \Gamma(t,x) &= \eta(t,x),\ \ (t,x)\in(0,T)\times\partial G,\end{aligned}$$

where $\{B_s\}_{0\le s\le T}$ is a $1-$dimensional Brownian motion and $\widetilde{N}(ds,dz) = N(ds,dz) - ds\nu(dz)$ a compensated Poisson random measure associated with a Lévy process defined on the filtered probability space

$$(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\le t\le T},P).$$

Here L is a partial differential operator of order m acting on the space variable $x\in\mathbb{R}^d$ and $G\subset\mathbb{R}^d$ is an open set. Further $U\subset\mathbb{R}^n$ is a closed set and the functions

$$\begin{aligned}b &: [0,T]\times G\times\mathbb{R}\times U\longrightarrow\mathbb{R},\\ \sigma &: [0,T]\times G\times\mathbb{R}\times U\longrightarrow\mathbb{R},\\ \psi &: [0,T]\times G\times\mathbb{R}\times U\times\mathbb{R}_0\longrightarrow\mathbb{R},\\ \xi &: \overline{G}\longrightarrow\mathbb{R},\\ \eta &: (0,T)\times\partial G\longrightarrow\mathbb{R}\end{aligned}$$

are Borel measurable. The processes

$$u_0:[0,T]\times G\times\Omega\longrightarrow U \text{ and } u_1:[0,T]\times G\times\mathbb{R}_0\times\Omega\longrightarrow U$$

are the *control processes* which are required to be càdlàg and adapted to a given subfiltration

$$\mathcal{E}_t\subseteq\mathcal{F}_t,\ \ t\ge 0.$$

We shall define *performance criterion* by

$$J(u)=\mathbb{E}\left[\int_0^T\int_G f(t,x,\Gamma(t,x),u_0(t,x))dxdt+\int_G g(x,\Gamma(T,x))dx\right], \tag{2}$$

provided that for $u=(u_0,u_1)$

$$\Gamma=\Gamma^{(u)} \text{ admits a unique strong solution of (1)} \tag{3}$$

and that

$$\mathbb{E}\left[\int_0^T\int_G |f(t,x,X(t,x),u_0(t,x))|\,dxdt+\int_G |g(x,X(T,x))|\,dx\right]<\infty, \tag{4}$$

for some given continuous functions

$$\begin{aligned}f &: [0,T]\times G\times\mathbb{R}\times U\longrightarrow\mathbb{R},\\ g &: G\times\mathbb{R}\longrightarrow\mathbb{R}.\end{aligned}$$

We call $u = (u_0, u_1)$ an *admissible control* if conditions (3) and (4) are satisfied. As for general conditions which guarantee the existence and uniqueness of strong solutions of SPDE's of the type (1) the reader is referred to [6]. From now on we assume that our controls $u = (u_0, u_1)$ have components of the form

$$u_0(t,x) = (\theta_0(t,x), \pi_0(t,x)),\ (t,x) \in [0,T] \times G, \tag{5}$$

$$u_1(t,x,z) = (\theta_1(t,x,z), \pi_1(t,x,z)),\ (t,x,z) \in [0,T] \times G \times \mathbb{R}_0. \tag{6}$$

Further we shall denote by Θ (resp. Π) the class of $\theta = (\theta_0, \theta_1)$ (resp. $\pi = (\pi_0, \pi_1)$) such that controls u of the form (5) and (6) are admissible.

The *partial information control problem for zero-sum stochastic differential games* amounts to determining a $(\theta^*, \pi^*) \in \Theta \times \Pi$ such that

$$\Phi_{\mathcal{E}} = J(\theta^*, \pi^*) = \sup_{\pi \in \Pi}(\inf_{\theta \in \Theta} J(\theta, \pi)). \tag{7}$$

A control $(\theta^*, \pi^*) \in \Theta \times \Pi$ solving the min-max problem (7) is called *optimal control.* The min-max problem (7) is inspired by game theory and arise for e.g. from antagonistic actions of two players, I and II, where player I pursues to minimize and player II to maximize the cost functional J.

In the following denote by $\mathcal{R}$ be the collection of functions

$$r : [0,T] \times G \times \mathbb{R}_0 \longrightarrow \mathbb{R}.$$

In order to solve problem (7) we shall proceed as in [1] and apply a SPDE maximum principle for stochastic differential games. In our setting the *Hamitonian function* H : $[0,T] \times G \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \longrightarrow \mathbb{R}$ gets the following form:

$$\begin{aligned} H(t,x,\gamma,u,p,q,r(t,x,\cdot)) &= f(t,x,\gamma,u) + b(t,x,\gamma,u)p \\ &\quad + \sigma(t,x,\gamma,u)q + \int_{\mathbb{R}} \psi(t,x,\gamma,u,z) r(t,x,z)\nu(dz), \end{aligned} \tag{8}$$

and the *adjoint equation* which fits into our framework is given by the following backward stochastic partial differential equation (BSPDE) in the unknown predictable processes $p = p(t,x), q = q(t,x)$ and $r = r(t,x,z)$:

$$\begin{aligned} dp(t,x) = &-\left[\frac{\partial H}{\partial \gamma}(t,x,\Gamma^{(u)}(t,x), u(t,x), p(t,x), q(t,x), r(t,x,\cdot)) + L^* p(t,x)\right] dt \\ &+ q(t,x)dB_t + \int_{\mathbb{R}_0} r(t,x,z)\widetilde{N}(dt,dz), \qquad (t,x) \in [0,T) \times G \end{aligned} \tag{9}$$

with

$$\begin{aligned} p(T,x) &= \frac{\partial g}{\partial \gamma}(x, \Gamma^{(u)}(T,x)),\ x \in \overline{G} \\ p(t,x) &= 0,\ (t,x) \in (0,T) \times \partial G. \end{aligned}$$

Here L^* is the adjoint of the operator L, that is

$$(L^* f, g)_{L^2(G)} = (f, Lg)_{L^2(G)}$$

for all $f, g \in C_0^\infty(G)$. Let us mention that BSPDE's of the form (9) have been studied e.g. in [15].

We are now coming to a verification theorem for the optimization problem (7):

Theorem 1. *Let $(\hat{\theta}, \hat{\pi}) \in \Theta \times \Pi$ and denote by $\widehat{\Gamma}(t,x) = \Gamma^{(\hat{\theta},\hat{\pi})}(t,x)$ the corresponding solution of (1). Further set $\Gamma^\theta(t,x) = \Gamma^{(\theta,\hat{\pi})}(t,x)$ and $\Gamma^\pi(t,x) = \Gamma^{(\hat{\theta},\pi)}(t,x)$. Require that $\hat{p}(t,x), \hat{q}(t,x)$ and $\hat{r}(t,x,z)$ solve the adjoint equation (9) in the strong sense and assume that the following conditions are fulfilled: For all $u \in \mathcal{A}$,*

$$E\Big[\int_G \int_0^T \big(\Gamma^\theta(t,x) - \widehat{\Gamma}(t,x)\big)^2 \Big\{\hat{q}^2(t,x) + \int_{\mathbb{R}_0} \hat{r}^2(t,x,z)\nu(dz)\Big\} dt dx\Big] < \infty, \tag{10}$$

$$E\Big[\int_G \int_0^T \big(\Gamma^\pi(t,x) - \widehat{\Gamma}(t,x)\big)^2 \Big\{\hat{q}^2(t,x) + \int_{\mathbb{R}_0} \hat{r}^2(t,x,z)\nu(dz)\Big\} dt dx\Big] < \infty, \tag{11}$$

and

$$\begin{aligned} &E\Big[\int_G \int_0^T \hat{p}^2(t,x)\Big\{\sigma(t,x,\Gamma^\theta(t,x),\theta_0(t,x),\hat{\pi}_0^2(t,x)) \\ &+ \int_{\mathbb{R}_0} \psi^2(t,x,\Gamma^\theta(t,x),\theta_1(t,x,z),\hat{\pi}_1(t,x,z),z)\Big\}\nu(dz) dt dx\Big] < \infty, \end{aligned} \tag{12}$$

$$\begin{aligned} &E\Big[\int_G \int_0^T \hat{p}(t,x)^2\Big\{\sigma(t,x,\Gamma^\pi(t,x),\hat{\theta}_0(t,x),\pi_0^2(t,x)) \\ &+ \int_{\mathbb{R}_0} \psi^2(t,x,\Gamma^\pi(t,x),\hat{\theta}_1(t,x,z),\pi_1(t,x,z),z)\Big\}\nu(dz) dt dx\Big] < \infty. \end{aligned} \tag{13}$$

Furthermore, assume that for all $(t,x) \in [0,T] \times G$ the following partial information maximum principle *holds:*

$$\begin{aligned} &\inf_{\theta\in\Theta} E[H(t,x,\Gamma^\theta(t,x),\theta(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,\cdot))\,|\mathcal{E}_t] \\ &= E[H(t,x,\hat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,\cdot))\,|\mathcal{E}_t] \\ &= \sup_{\pi\in\Pi} E[H(t,x,\Gamma^\pi(t,x),\hat{\theta}(t,x),\pi(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,\cdot))\,|\mathcal{E}_t]\,. \end{aligned} \tag{14}$$

Then for all $(t,x) \in [0,T] \times G$, we have:

(i) If $g(x,\gamma)$ is concave and $H(t,x,\gamma,\theta,\pi,\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,\cdot))$ is concave for all $\theta = \hat{\theta}$ then

$$J(\hat{\theta},\hat{\pi}) \geq J(\hat{\theta},\pi) \text{ for all } \pi \in \Pi,$$

and

$$J(\hat{\theta},\hat{\pi}) = \sup_{\pi\in\Pi} J(\hat{\theta},\pi).$$

(ii) If $g(x,\gamma)$ is convex and $H(t,x,\gamma,\theta,\pi,\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,\cdot))$ is convex for all $\pi = \hat{\pi}$ then

$$J(\hat{\theta},\hat{\pi}) \le J(\theta,\hat{\pi}) \textit{ for all } \theta \in \Theta,$$

and

$$J(\hat{\theta},\hat{\pi}) = \inf_{\theta\in\Theta} J(\theta,\hat{\pi}).$$

(iii) If the conditions in (i) and (ii) are satisfied (i.e. g is linear) then $(\theta^*,\pi^*) := (\hat{\theta},\hat{\pi})$ is an optimal control and

$$\Phi_{\mathcal{E}} = \sup_{\pi\in\Pi}\left(\inf_{\theta\in\Theta} J(\theta,\pi)\right) = \inf_{\theta\in\Theta}\left(\sup_{\pi\in\Pi} J(\theta,\pi)\right). \tag{15}$$

Proof. i) Fix $\hat{\theta} \in \Theta$. Let $\pi \in \Pi$ be an arbitrary admissible control with corresponding solution $\Gamma^{\pi}(t,x) = \Gamma^{(\hat{\theta},\pi)}(t,x)$. Then we have

$$\begin{aligned} & J(\hat{\theta},\hat{\pi}) - J(\hat{\theta},\pi) = \\ & E\Big[\int_0^T \int_G \{f(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)) - f(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x))\}dxdt \\ & \quad + \int_G \{g(x,\widehat{\Gamma}(T,x)) - g(x,\Gamma^{\pi}(T,x))\}dx\Big]. \end{aligned} \tag{16}$$

Putting

$$I_1 = E\Big[\int_0^T \int_G \{\hat{f} - f^{\pi}\}dxdt\Big], \tag{17}$$

and

$$I_2 = E\Big[\int_G \{\hat{g} - g^{\pi}\}dx\Big] \tag{18}$$

where

$$\begin{aligned} \hat{f} &= f(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)), \\ f^{\pi} &= f(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x)), \\ \hat{g} &= g(x,\widehat{\Gamma}(T,x)) \text{ and } g^{\pi} = g(x,\Gamma^{\pi}(T,x)). \end{aligned}$$

Similarly we put

$$\begin{aligned} \hat{b} &= b(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)), b^{\pi} = b(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x)), \\ \hat{\sigma} &= \sigma(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)), \sigma^{\pi} = \sigma(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x)), \\ \hat{\psi} &= \psi(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),z), \psi^{\pi} = \psi(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x),z). \end{aligned}$$

Moreover, we set

$$\begin{aligned} \widehat{H} &= H(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.)), \\ H^{\pi} &= H(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.)). \end{aligned}$$

Since $g(x,\gamma)$ is concave in γ, we have

$$\hat{g} - g^{\pi} \geq \frac{\partial g}{\partial \gamma}(x, \widehat{\Gamma}(T,x)).(\widehat{\Gamma}(T,x) - \Gamma^{\pi}(T,x)). \tag{19}$$

Putting $\widetilde{\Gamma}(t,x) = \widehat{\Gamma}(t,x) - \Gamma^{\pi}(t,x)$ and using integration by part formula for jump diffusions we get,

$$\begin{aligned}
I_2 &\geq E\Big[\int_G \frac{\partial g}{\partial \gamma}(x, \widehat{\Gamma}(T,x)).\widetilde{\Gamma}(T,x)dx\Big] = E\Big[\int_G \hat{p}(T,x).\widetilde{\Gamma}(T,x)dx\Big] \\
&= E\Big[\int_G \Big(\hat{p}(0,x).\widetilde{\Gamma}(0,x) \\
&\quad + \int_0^T \big\{\widetilde{\Gamma}(t,x)d\hat{p}(t,x) + \hat{p}(t,x)d\widetilde{\Gamma}(t,x) + (\hat{\sigma} - \sigma^{\pi})\hat{q}(t,x)\big\}dt \\
&\quad + \int_0^T \int_{\mathbb{R}} (\widehat{\psi} - \psi^{\pi})\hat{r}(t,x,z)\nu(dz)dt\Big)dx\Big] \\
&= E\Big[\int_G \Big(\int_0^T \widetilde{\Gamma}(t,x)\Big\{-\Big(\frac{\partial H}{\partial \gamma}\Big)^{\wedge} - L^*\hat{p}(t,x)\Big\}dt \\
&\quad + \int_0^T \Big\{\hat{p}(t,x)[L\widetilde{\Gamma}(t,x) + (\hat{b} - b^{\pi})] + (\hat{\sigma} - \sigma)\hat{q}(t,x)\Big\}dt \\
&\quad + \int_0^T \int_{\mathbb{R}} (\hat{\psi} - \psi^{\pi})\hat{r}(t,x,z)\nu(dz)dt\Big)dx\Big],
\end{aligned} \tag{20}$$

where

$$\Big(\frac{\partial H}{\partial \gamma}\Big)^{\wedge} = \frac{\partial H}{\partial \gamma}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.)). \tag{21}$$

By definition of H we have

$$\begin{aligned}
I_1 = E\Big[\int_0^T \int_G \Big\{&\hat{H} - H^{\pi} - (\hat{b} - b^{\pi})\hat{p}(t,x) - (\hat{\sigma} - \sigma)\hat{q}(t,x) \\
&- \int_{\mathbb{R}} (\hat{\psi} - \psi)\hat{r}(t,x,z)\nu(dz)\Big\}dxdt\Big].
\end{aligned} \tag{22}$$

On the other hand, we have for all $(t,x) \in (0,T) \times \partial G$

$$\widetilde{\Gamma}(t,x) = \hat{p}(t,x) = 0,$$

and

$$\int_G \{\widetilde{\Gamma}(t,x)L^*\hat{p}(t,x) - \hat{p}(t,x)L\widetilde{\Gamma}(t,x)\}dx = 0 \text{ for all } t \in (0,T). \tag{23}$$

Combining this with (20) and (22) we get

$$J(\hat{\theta},\hat{\pi}) - J(\hat{\theta},\pi) \geq E\Big[\int_G \Big(\int_0^T \Big\{\hat{H} - H^{\pi} + \Big(\frac{\partial H}{\partial \gamma}\Big)^{\wedge} \cdot \widetilde{\Gamma}(t,x)\Big\}dt\Big)dx\Big]. \tag{24}$$

From the concavity of H we get

$$\hat{H} - H^{\pi} \geq \Big(\frac{\partial H}{\partial \gamma}\Big)^{\wedge} \cdot \widetilde{\Gamma}(t,x) + \Big(\frac{\partial H}{\partial \pi}\Big)^{\wedge} \cdot (\hat{\pi} - \pi) \tag{25}$$

where

$$\Big(\frac{\partial H}{\partial \pi}\Big)^{\wedge} = \frac{\partial H}{\partial \pi}(t, x, \hat{\Gamma}(t,x), \hat{\theta}(t,x), \pi(t,x), \hat{p}(t,x), \hat{q}(t,x), \hat{r}(t,x,.))$$

Since

$$\pi \to E[H^{\pi}(t, x, \Gamma^{\pi}(t,x), \hat{\theta}(t,x), \pi(t,x), \hat{p}(t,x), \hat{q}(t,x), \hat{r}(t,x,.))|\mathcal{E}_t]$$

is maximum at $\pi(t,x) = \hat{\pi}(t,x)$ and $\pi(t,x), \hat{\pi}(t,x)$ are $\mathcal{E}_t$-measurable, we get

$$E\Big[\Big(\frac{\partial H}{\partial \pi}\Big)^{\wedge}(\hat{\pi} - \pi)\Big|\mathcal{E}_t\Big] = (\hat{\pi} - \pi)E\Big[\Big(\frac{\partial H}{\partial \pi}\Big)^{\wedge}\Big|\mathcal{E}_t\Big]_{\pi=\hat{\pi}} \geq 0. \tag{26}$$

This gives

$$\hat{H} - H^{\pi} \geq \frac{\partial H}{\partial \gamma} \cdot \widetilde{\Gamma}(t,x). \tag{27}$$

Hence

$$J(\hat{\theta}, \hat{\pi}) - J(\hat{\theta}, \pi) \geq 0. \tag{28}$$

Since $\pi \in \Pi$ is arbitrary this prove (i).

ii) Fix $\hat{\pi} \in \Pi$. Let $\theta \in \Theta$ be an arbitrary admissible control. Prove in the same way as done in (i) we can show that

$$J(\hat{\theta}, \hat{\pi}) - J(\theta, \hat{\pi}) \leq 0. \tag{29}$$

ii) If both (i) and (ii) hold then

$$J(\hat{\theta}, \pi) \leq J(\hat{\theta}, \hat{\pi}) \leq J(\theta, \hat{\pi})$$

for any $(\theta, \pi) \in \Theta \times \Pi$. Thereby

$$J(\hat{\theta}, \hat{\pi}) \leq \inf_{\theta \in \Theta} J(\theta, \hat{\pi}) \leq \sup_{\pi \in \Pi}\big(\inf_{\theta \in \Theta} J(\theta, \pi)\big).$$

On the other hand

$$J(\hat{\theta}, \hat{\pi}) \geq \sup_{\pi \in \Pi} J(\hat{\theta}, \pi) \geq \inf_{\theta \in \Theta}\big(\sup_{\pi \in \Pi} J(\theta, \pi)\big)$$

Now due to the inequality

$$\inf_{\theta \in \Theta}\big(\sup_{\pi \in \Pi} J(\theta, \pi)\big) \geq \sup_{\pi \in \Pi}\big(\inf_{\theta \in \Theta} J(\theta, \pi)\big)$$

we have

$$\Phi_{\mathcal{E}}(x) = \sup_{\pi \in \Pi}\big(\inf_{\theta \in \Theta} J(\theta, \pi)\big) = \inf_{\theta \in \Theta}\big(\sup_{\pi \in \Pi} J(\theta, \pi)\big).$$

$\square$

2.2 A necessary maximum principle for zero-sum games

As in [1], we give a necessary maximum principle for zero-sum game. In addition to the assumptions in Section 2.1 we shall now assume the following:

(A1) For all $t \in (0,T)$ and all $\mathcal{E}_t$-measurable random variables α, ρ the controls

$$\beta_\alpha(s,x) := \alpha(\omega)\chi_{[t,T]}(s)\chi_G(x),$$

and

$$\eta_\rho(s,x) := \rho(\omega)\chi_{[t,T]}(s)\chi_G(x)$$

belong to Θ and Π, respectively.

(A2) For given $\theta, \beta \in \Theta$ and $\pi, \eta \in \Pi$ with β, η are bounded, there exists a $\delta > 0$ such that

$$\theta + y\beta \in \Theta \text{ and } \pi + v\eta \in \Pi$$

for all $y, v \in (-\delta, \delta)$.

Set $\Gamma^{\theta+y\beta}(t,x) = \Gamma^{(\theta+y\beta,\pi)}(t,x)$ and $\Gamma^{\pi+v\eta}(t,x) = \Gamma^{(\theta,\pi+v\eta)}(t,x)$. For a given $\theta, \beta \in \Theta$ and $\pi, \eta \in \Pi$ with β, η bounded, we define the processes $Y^\theta(t,x)$ and $Y^\pi(t)$ (if existing) by

$$Y^\theta(t,x) = \frac{d}{dy}\Gamma^{\theta+y\beta}(t,x)\Big|_{y=0}, \tag{30}$$

$$Y^\pi(t,x) = \frac{d}{dv}\Gamma^{\pi+v\eta}(t,x)\Big|_{v=0}. \tag{31}$$

Further let us assume that $Y^\theta(t,x)$ and $Y^\pi(t)$ satisfy the equations:

$$dY^\theta(t,x) = (LY^\theta(t,x) + \lambda^\theta(t,x))dt + \xi^\theta(t,x)dB(t) + \int_{\mathbb{R}} \zeta^\theta(t,x,z)\widetilde{N}(dt,dz), \tag{32}$$

and

$$dY^\pi(t) = (LY^\pi(t,x) + \lambda^\pi(t,x))dt + \xi^\pi(t,x)dB(t) + \int_{\mathbb{R}} \zeta^\pi(t,x,z)\widetilde{N}(dt,dz), \tag{33}$$

where

$$\begin{cases} \lambda^\theta(t,x) = & \frac{\partial b}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^\theta(t,x) \\ & +\frac{\partial b}{\partial \theta}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x), \\ \xi^\theta(t,x) = & \frac{\partial \sigma}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^\theta(t,x) \\ & +\frac{\partial \sigma}{\partial \theta}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x), \\ \zeta^\theta(t,x) = & \frac{\partial \psi}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^\theta(t,x) \\ & +\frac{\partial \psi}{\partial \theta}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x), \end{cases} \tag{34}$$

and

$$
\begin{cases}
\lambda^{\pi}(t,x) = & \frac{\partial b}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^{\theta}(t,x) \\
& +\frac{\partial b}{\partial \pi}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x), \\
\xi^{\pi}(t,x) = & \frac{\partial \sigma}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^{\theta}(t,x) \\
& +\frac{\partial \sigma}{\partial \pi}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x), \\
\zeta^{\pi}(t,x) = & \frac{\partial \psi}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^{\theta}(t,x) \\
& +\frac{\partial \psi}{\partial \pi}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x).
\end{cases}
\tag{35}
$$

Theorem 2. *Suppose $\hat{\theta} \in \Theta$ and $\hat{\pi} \in \Pi$ are respectively a local minimum and a maximum for $J(\theta, \pi)$, in the sense that for all bounded $\beta \in \Theta$ and $\eta \in \Pi$ there exist a $\delta > 0$ such that $\hat{\theta} + y\beta \in \Theta$ and $\hat{\pi} + v\eta \in \Pi$ for all $y \in (-\delta, \delta)$ and $v \in (-\delta, \delta)$ and*

$$
h(y,v) := J(\hat{\theta} + y\beta, \hat{\pi} + v\eta), \qquad y, v \in (-\delta, \delta)
$$

attains a minimum at $y = 0$ and a maximum at $v = 0$.

Suppose there exists a solution $\hat{p}(t,x), \hat{q}(t,x), \hat{r}(t,x,.)$ of the associated adjoint equation

$$
\begin{cases}
d\hat{p}(t,x) &= -\Big(\frac{\partial H}{\partial \gamma}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.)) \\
& \quad +L^{*}\hat{p}(t,x)\Big)\,dt + \hat{q}(t,x)\,dB(t) + \int_{\mathbb{R}^n} \hat{r}(t^-,x,z)\widetilde{N}(\,dt,\,dz), \\
\hat{p}(T,x) &= \frac{\partial g}{\partial \gamma}(x,\widehat{\Gamma}(T,x)),\ x \in \bar{G};\ p(t,x) = 0,\ (t,x) \in (0,T) \times \partial G.
\end{cases}
\tag{36}
$$

Moreover, adopting the notation in (32)-(35), assume that

$$
E\Big[\int_G \int_0^T Y^{\hat{\theta}}(t,x)^2 \Big\{\hat{q}^2(t,x) + \int_{\mathbb{R}} \hat{r}^2(t,x,z)\nu(dz)\Big\} dxdt\Big] < \infty, \tag{37}
$$

$$
E\Big[\int_G \int_0^T Y^{\hat{\pi}}(t,x)^2 \Big\{\hat{q}^2(t,x) + \int_{\mathbb{R}} \hat{r}^2(t,x,z)\nu(dz)\Big\} dxdt\Big] < \infty, \tag{38}
$$

and

$$
\begin{aligned}
E\Big[\int_G \int_0^T \hat{p}^2(t,x) \Big\{ & \xi^{\hat{\theta}}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)) \\
& + \int_{\mathbb{R}} \psi^2(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x))\nu(dz)\Big\} dxdt\Big] < \infty,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
E\Big[\int_G \int_0^T \hat{p}^2(t,x) \Big\{ & \xi^{\hat{\pi}}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)) \\
& + \int_{\mathbb{R}} \psi^2(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x))\nu(dz)\Big\} dxdt\Big] < \infty.
\end{aligned}
\tag{40}
$$

Then for a.a. $t \in [0,T]$, we have

$$
\begin{aligned}
& E\Big[\frac{\partial H}{\partial \theta}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.))\Big|\mathcal{E}_t\Big] \\
& = E\Big[\frac{\partial H}{\partial \pi}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.))\Big|\mathcal{E}_t\Big] = 0.
\end{aligned}
\tag{41}
$$

Proof. See [1]. □

3 Application to constant maturity interest rate derivatives

In the following denote by $F(t,T)$ the (market) price of an interest rate derivative at time $t \geq 0$ which expires at maturity $T < \infty$. In this Section we want to study optimal portfolio strategies for constant maturity interest rate derivatives, that is we aim at constructing optimal hedging strategies with respect to fixed income market contracts with constant time-to-maturity x. In our framework the price of such a contract at time t is assumed to be $F(t, t+x)$. Examples of such financial instruments are bonds on 6 month LIBOR rates or more general contracts on forward rates with constant time-to-maturity. In a wider sense such instruments also comprise constant maturity swaps. See e.g. Hull [10]. We shall mention that these derivatives steadily gain importance in asset liability management and are e.g. used by life insurance companies to match their liabilities. Suppose that for each $x \geq 0$ our portfolio S^x is a portfolio made up of a risk-free asset and a constant maturity contract with constant time-to-maturity x . We are interested to find an optimal portfolio strategy for the entirety of portfolios $\{S^x\}_{x\in J}$ (J subset of $[0,\infty)$) managed by a trader who only has limited access to market information. In the sequel let us consider a market model consisting of a risk-free asset and an interest rate derivative with maturity T specified by

$$\text{(risk-free asset)} \qquad dP_0(t) = \rho(t)P_0(t)dt, \; P_0(0) = 1 \tag{42}$$

$$\begin{aligned}\text{(interest rate derivative)} \qquad dF(t,T) &= F(t^-,T)\Big[\alpha(t,T)dt + \sigma(t,T)dW_t \\ &+ \int_{\mathbb{R}_0} \gamma(t,T,z)\widetilde{N}(dt,dz)\Big], \; F(0,T) \quad > 0\end{aligned} \tag{43}$$

for all $T > 0$, where $(\rho(t))_{t\geq 0}$, $(\alpha(t,T))_{0\leq t\leq T<\infty}$, $(\sigma(t,T))_{0\leq t\leq T<\infty}$ and $(\gamma(t,T,z))_{0\leq t\leq T<\infty}$ are $\mathcal{F}_t-$ predictable processes such that

$$\begin{aligned}E\Big[\int_0^\infty \Big\{ \mid \rho(s) \mid + |\alpha(s,T)| + \frac{1}{2}\sigma^2(s,T) \\ + \int_{\mathbb{R}_0} |\log(1+\gamma(s,T,z)) - \gamma(s,T,z)|\nu(dz)\Big\}ds\Big] < \infty,\end{aligned} \tag{44}$$

for all $T \geq 0$. We require that

$$\gamma(t,T,z) > -1 \qquad \text{for } (\omega,t,z) \in \Omega \times [0,T] \times \mathbb{R}_0 \text{ a.e. for all } T \geq 0.$$

We assume that the dynamics of the short rate $\rho(t)$ is stochastic and governed by

$$\begin{cases} d\rho(t) &= a(t)dt + b(t)dW_t + \int_{\mathbb{R}_0} c(t,z)\widetilde{N}(dt,dz) \\ \rho(0) &= 0. \end{cases} \tag{45}$$

where $a(t), b(t)$ and $c(t,z)$ are predictable and sufficiently integrable.

Let $\mathcal{E}_t \subseteq \mathcal{F}_t$ be a given sub-filtration. Denote by $\phi(t,T), t \geq 0$ the fraction of wealth invested in $F(t,T)$ based on the partial market information $\mathcal{E}_t \subseteq \mathcal{F}_t$ being available at time t. Thus we impose on $\{\phi(t,T)\}_{t\geq 0, T\geq 0}$ to be $\mathcal{E}_t-$ predictable. Then for each T the total wealth $V^{(\phi)}(t,T)$ of the portfolio S^T is given by the SDE

$$\begin{cases} dV^{(\phi)}(t,T) &= V^{(\phi)}(t^-,T)\Big[\{\rho(t)+(\alpha(t,T)-\rho(t))\phi(t,T)\}dt \\ &+ \phi(t,T)\sigma(t,T)dW_t + \phi(t,T)\int_{\mathbb{R}_0}\gamma(t,T,z)\widetilde{N}(dt,dz)\Big], \\ V^{(\phi)}(0,T) &= w(T). \end{cases} \tag{46}$$

Let us rewrite the dynamics of the total wealth as an integral evolution equation in infinite dimensions by viewing terms of (46) as functions of maturity T. So we see that

$$\begin{aligned} V^{(\phi)}(t,\cdot) &= w(\cdot) + \int_0^t V^{(\phi)}(s,\cdot)\{\rho(s)+(\alpha(s,\cdot)-\rho(s))\phi(s,\cdot)\}ds \\ &\quad + \int_0^t V^{(\phi)}(s,\cdot)\phi(s,\cdot)\sigma(s,\cdot)dW_s \\ &\quad + \int_0^t \int_{\mathbb{R}_0} V^{(\phi)}(s^-,\cdot)\phi(s,\cdot)\gamma(s,\cdot,z)\widetilde{N}(ds,dz). \end{aligned} \tag{47}$$

Define

$$\begin{aligned} V_t^{(\phi)}(x) &= V^{(\phi)}(t,t+x),\ \phi_t(x)=\phi(t,t+x), \alpha_t(x)=\alpha(t,t+x), \\ \sigma_t(x) &= \sigma(t,t+x),\ \gamma_t(x,z)=\gamma(t,t+x,z), t,x\geq 0, z\in\mathbb{R}_0. \end{aligned}$$

Set $T=t+x$ in (46). Then differentiation of both sides of (46) w.r.t. time t (formally) yields

$$\begin{aligned} dV_t^{(\phi)}(x) &= \Big(AV_t^{(\phi)}(x) + V_{t^-}^{(\phi)}(x)\left\{\rho(t)+(\alpha_t(x)-\rho(t))\phi_t(x)\right\}\Big)dt \\ &+ V_{t^-}^{(\phi)}(x)\phi_t(x)\Big\{\sigma_t(x)dW_t + \int_{\mathbb{R}_0}\gamma_t(x,z)\widetilde{N}(dt,dz)\Big\}, \end{aligned} \tag{48}$$

where A is the densely defined operator given by

$$A = \frac{d}{dx}.$$

We may think of A as the generator of a strongly continuous left shift operator on an appropriate Hilbert space H. In the case of a constant maturity bond portfolio one could e.g. choose H to be the weighted Sobolev space $H_\gamma, \gamma>0$, consisting of functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$\parallel f \parallel_\gamma^2 := \int_0^\infty f^2(x)e^{-\lambda x}dx + \int_0^\infty \Big(\frac{d}{dx}f(x)\Big)^2 e^{-\lambda x}dx < \infty,$$

where the derivative $\frac{d}{dx}$ is in the distributional sense (See [7]). Criteria ensuring the existence and uniqueness of (strong) solutions of first order (quasi-) linear SPDE's of the type (48) can be e.g. in [11].

Let us also mention that the type of SPDE obtained in (48) is often referred to as "Musiela equation" in the theory of interest rate modeling [5]. Usually a no-arbitrage condition in terms of a volatility process and a risk premium is imposed on the Musiela equation to enforce a risk-free evolution of forward curves (see e.g. [5]). In this paper we won't necessarily require such a condition on the dynamics of the portfolio value $V_t^{(\phi)}(x)$ (or on (43)), since we are interested in a general portfolio optimization problem.

Definition 3.1. The set $\mathcal{A}$ of *admissible portfolios* of all processes $\phi = \phi(t,x), t \in [0,T]$, such that

(i) $0 \leq \phi_t(x) \leq 1$;

(ii) ϕ permits a strong solution of the SPDE (48);

(iii) $\int_0^\infty \{|\rho(s) + (\alpha_s(x) - \rho(s))\phi_s(x)| + \phi_s^2(x)(\sigma_s^2(x) + \int_{\mathbb{R}_0} \gamma_s^2(x,z)\nu(dz))\}ds < \infty$;

(iv) $\phi_t(x)\gamma_t(x,z) > -1 \qquad (\omega, t, z) -$ a.e..

We now introduce a family $\mathcal{Q}$ of measures Q_θ parametrized by process $\theta = (\theta^0(t,x), \theta^1(t,x,z))$ such that

$$dQ(\omega) = Z^{(\theta)}(T,x)dP(\omega) \qquad \text{on } \mathcal{F}_t \tag{49}$$

where

$$\begin{cases} dZ^{(\theta)}(t,x) &= Z^{(\theta)}(t^-,x)[-\theta^0(t,x)dW_t - \int_{\mathbb{R}} \theta^1(t,x,z)\widetilde{N}(dt,dz)], \\ Z^\theta(0,x) &= 1. \end{cases} \tag{50}$$

We assume that

$$\theta^1(t,x,z) \leq 1, \qquad \text{for } (\omega,t,z) \text{ a.s} \tag{51}$$

and

$$\int_0^t \Big\{\theta^0(s,x)^2 + \int_{\mathbb{R}} \theta^1(s,x,z)^2\Big\}ds < \infty \qquad \text{a.s.} \tag{52}$$

Setting

$$Z_t^{(\theta)}(x) = Z^{(\theta)}(t,x);\ \theta_t^0(x) = \theta^0(t,x);\ \theta_t^1(x,z) = \theta^1(t,x,z) \tag{53}$$

Differentiating both sides of (50), we get

$$dZ_t^{(\theta)}(x) = -Z_t^{(\theta)}(x)\theta_t^0(x)dW_t - \int_{\mathbb{R}} Z_t^{(\theta)}(x)\theta_t^1(x,z)\widetilde{N}(dt,dz)\Big). \tag{54}$$

The set of all $\theta = (\theta^0, \theta^1)$ such that (51)-(52) hold is denoted by Θ. These are the *admissible controls of the market.* Fix a utility function $U : G \times [0,\infty) \to [-\infty,\infty)$, assumed to be increasing, concave and twice continuously differentiable on $(0,\infty)$.

The problem is to find $\theta^* \in \Theta$ and $\phi^* \in \mathcal{A}$ such that

$$\Phi(y_1,y_2) = \inf_{\theta\in\Theta}\Big(\sup_{\phi\in\mathcal{A}} E_{Q_\theta}\Big[\int_G U(x, V_T^{(\phi)}(x))dx\Big]\Big), \tag{55}$$

where G is the set of all time-to-maturity.

This is a problem of the type as described in the previous section. Here player I is the trader and player II is market. The trader wants to find a optimal strategy for portfolios which maximizes the utility of the terminal wealth of constant maturity interest rate derivatives and the market "wants" to choose a scenario (represented by a probability measure) which minimizes this maximal utility. Thus to solve (55) by stochastic control methods, we have to look at the following three-dimensional state process $Y(t,x)$ as following:

$$\begin{aligned} dY(t,x) &= \begin{bmatrix} dY_1(t,x) \\ dY_2(t,x) \\ dY_3(t,x) \end{bmatrix} = \begin{bmatrix} d\rho(t) \\ dZ_t^{\theta}(x) \\ dV_t^{(\phi)}(x) \end{bmatrix} \\ &= \begin{bmatrix} a(t) \\ 0 \\ AV_t^{(\phi)}(x) + V_{t^-}^{(\phi)}\{\rho(t) + (\alpha_t(x) - \rho(t))\phi_t(x)\} \end{bmatrix} dt \\ &+ \begin{bmatrix} b(t) \\ -Z_{t^-}^{\theta}(x)\theta_t^0(x) \\ V_{t^-}^{(\phi)}(x)\sigma_t(x)\phi_t(x) \end{bmatrix} dW_t + \int_{\mathbb{R}} \begin{bmatrix} c(t,z) \\ -Z_{t^-}^{(\theta)}(x)\theta_t^1(x,z) \\ V_{t^-}^{(\phi)}(x)\phi_t(x)\gamma_t(x,z) \end{bmatrix} \widetilde{N}(dt,dz). \end{aligned} \tag{56}$$

The Hamiltonian is defined as following

$$\begin{aligned} &H(t,x,y_1,y_2,y_3,\theta,\phi,p,q,r(t,x,\cdot)) \\ &= a(t)p_1(t,x) + y_3\{y_1 + (\alpha_t(x) - y_1)\phi_t(x)\}p_3 \\ &+ b(t)q_1(t,x) - y_2\theta_t^0(x)q_2 + y_3\sigma_t(x)\phi_t(x)q_3 \\ &+ \int_{\mathbb{R}} \{c(t)r_1(t,x,z) - y_2\theta_t^1(x,z)r_2(t,x,z) \\ &\qquad + y_3\phi_t(x)\gamma_t(x,z)r_3(t,x,z)\}\nu(dz). \end{aligned} \tag{57}$$

And the adjoint equations are defined by

$$\begin{cases} dp_1(t,x) &= -y_3(1-\phi_t(x))p_3(t,x)dt + q_1(t,x)dW_t + \int_{\mathbb{R}} r_1(t,x,z)\widetilde{N}(dt,dz) \\ p_1(T,x) &= U_{y_1}(x,y_3), x \in \bar{G}; \qquad p_1(t,x) = 0, (t,x) \in (0,T)\times\partial G, \end{cases} \tag{58}$$

$$\begin{cases} dp_2(t,x) &= \Big[\theta_t^0(x)q_2(t,x) + \int_{\mathbb{R}} \theta_t^1(x,z)r_2(t,x,z)\nu(dz)\Big]dt \\ &\quad +q_2(t,x)dW_t + \int_{\mathbb{R}} r_2(t,x,z)\widetilde{N}(dt,dz) \\ p_2(T,x) &= U_{y_2}(x,y_3), x \in \bar{G}; \qquad p_2(t,x) = 0, (t,x) \in (0,T)\times\partial G, \end{cases} \tag{59}$$

and

$$\begin{cases} dp_3(t,x) &= \Big[-\{y_1 + (\alpha_t(x) - y_1)\phi_t(x)\}p_3(t,x) \\ &\quad -\sigma_t(x)\phi_t(x)q_3(t,x) - \int_{\mathbb{R}} \phi_t(x)\gamma_t(x,z)r_3(t,x,z)\nu(dz) \\ &\quad -A^*p_3(t,x)\Big]dt + q_3(t,x)dW_t + \int_{\mathbb{R}} r_3(t,x,z)\widetilde{N}(dt,dz) \\ p_3(T,x) &= U_{y_3}(x,y_3), x \in \bar{G}; \qquad p_3(t,x) = 0, (t,x) \in (0,T)\times\partial G. \end{cases} \tag{60}$$

Suppose $(\hat{\theta}, \hat{\phi})$ is an optimal control and $\widehat{Y}(t) = (\widehat{Y}_1(t,x), \widehat{Y}_2(t,x), \widehat{Y}_3(t,x))$ is the corresponding optimal process associated with the solution $\hat{p}(t,x) = (\hat{p}_1(t,x), \hat{p}_2(t,x))$, $\hat{q}(t,x) = (\hat{q}_1(t,x), \hat{q}_2(t,x)), \hat{r}(t,x,\cdot) = (\hat{r}_1(t,x,\cdot), \hat{r}_2(t,x,\cdot))$ of the adjoint equations. Maximizing the Hamiltonian $E[H(t,x,y_1,y_2,\theta,\phi,p,q,r) \mid \mathcal{E}_t]$ over all $\phi \in \mathcal{A}$ lead to the following first order condition for the maximum point $\hat{\phi}$:

$$\begin{aligned} &E[(\alpha_t(x) - y_1)\hat{p}_3(t,x) \mid \mathcal{E}_t] + E[\sigma_t(x)\hat{q}_3(t,x) \mid \mathcal{E}_t] \\ &\qquad + \int_{\mathbb{R}} E[\gamma_t(x,z)\hat{r}_3(t,z) \mid \mathcal{E}_t]\nu(dz) = 0 \end{aligned} \tag{61}$$

We then minimize $E[H(t,x,y_1,y_2,\theta,\phi,p,q,r) \mid \mathcal{E}_t]$ over all $\theta = (\theta^0, \theta^1)$ and get the following first order conditions for a minimum point $\hat{\theta} = (\hat{\theta}^0, \hat{\theta}^1)$:

$$E[-\widehat{Y}_2(t,x)\hat{q}_2(t,x) \mid \mathcal{E}_t] = 0 \tag{62}$$

and

$$\int_{\mathbb{R}} E[-\widehat{Y}_2(t,x)\widehat{r}_2(t,x,z) \mid \mathcal{E}_t]\nu(dz) = 0 \tag{63}$$

We try a process $\hat{p}_2(t,x)$ of the form

$$\hat{p}_2(t,x) = f(t, \widehat{Y}_1(t,x))U(x, \widehat{Y}_3(t,x)) \qquad \text{with } f(T, y_1) = 0 \text{ for all } y_1 \tag{64}$$

Differentiating (64) we get

$$\begin{aligned} d\hat{p}_2(t,x) = &\Big\{ f_t + \widetilde{A}(t,x)f + \widetilde{B}(t,x)f_{y_1} + \frac{1}{2}b^2(t)f_{y_1y_1} \\ &+ \int_{\mathbb{R}} \{f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1) - c(t,z)f_{y_1}\}\nu(dz)\Big\}dt \\ &+ \Big(b(t)f_{y_1} + \widehat{Y}_3\sigma_t\phi_t\frac{U'}{U}f\Big)dW_t \\ &+ \int_{\mathbb{R}} \Big\{\frac{f}{U}[U(\widehat{Y}_3(1+\gamma_t\phi_t)) - U(\widehat{Y}_3)] + [f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1)]\Big\}\widetilde{N}(dt,dz) \end{aligned}$$

where

$$\begin{aligned} \widetilde{A}(t,x) &= \Big(\widehat{Y}_3\big(\widehat{Y}_1 + (\alpha_t - \widehat{Y}_1)\phi_t\big)\Big)\frac{U'}{U} + \frac{1}{2}\widehat{Y}_3^2\sigma_t^2\phi_t^2\frac{U''}{U} \\ &+ \frac{1}{U}\int_{\mathbb{R}} \{U(\widehat{Y}_3(1+\gamma_t\phi_t)) - U(\widehat{Y}_3) - \widehat{Y}_3\gamma_t\phi_t U'\}\nu(dz) \end{aligned} \tag{65}$$

$$\widetilde{B}(t,x) = a(t) + \widehat{Y}_3 b(t)\sigma_t\phi_t\frac{U'}{U} \tag{66}$$

and

$$\begin{aligned} 0 = &f_t + \widetilde{A}(t,x)f + \widetilde{B}(t,x)f_{y_1} + \frac{1}{2}b^2(t)f_{y_1y_1} \\ &+ \int_{\mathbb{R}} \{f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1) - c(t,z)f_{y_1}\}\nu(dz) \end{aligned} \tag{67}$$

Comparing this with equation (59) by equating the dt, dW_t and $\widetilde{N}(dt,dz)$ coefficients respectively, we get

$$\hat{q}_2(t,x) = b(t)f_{y_1} + \widehat{Y}_3\sigma_t\phi_t\frac{U^{'}}{U}f \tag{68}$$

and

$$\hat{r}_2(t,x) = \frac{f}{U}[U(\widehat{Y}_3(1+\gamma_t\phi_t)) - U(\widehat{Y}_3)] + [f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1)] \tag{69}$$

Combining (68) and (62) we get

$$\phi_t(x) = -E\Big[\frac{b(t)}{\sigma_t(x)}\frac{U}{\widehat{Y}_3U^{'}}\frac{f_{y_1}}{f}\Big|\mathcal{E}_t\Big] \tag{70}$$

Try the process $\hat{p}_3(t,x)$ of the form

$$\hat{p}_3(t,x) = f(t,\widehat{Y}_1(t,x))\widehat{Y}_2(t,x)U^{'}(x,\widehat{Y}_3(t,x)) \tag{71}$$

Differentiating both side of equation (71) we get

$$\begin{aligned} d\hat{p}_3(t,x) = &\Big\{U^{'}f_t + A\hat{p}_3(t,x) + \widetilde{C}(t,x)f + \widetilde{D}(t,x)f_{y_1} + \frac{1}{2}b^2(t)f_{y_1y_1} \\ &+ \int_{\mathbb{R}}\{f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1) - c(t,z)f_{y_1}\}\nu(dz)\Big\}dt \\ &+ \Big(\widehat{Y}_3\sigma_t\phi_tU^{''}f - \theta_t^0U^{'}f + b(t)U^{'}f_{y_1}\Big)dW_t \\ &+ \int_{\mathbb{R}}\Big\{f[U^{'}(\widehat{Y}_3(1+\gamma_t\phi_t)) - U^{'}(\widehat{Y}_3)] \\ &+ U^{'}[f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1)] - \theta_t^1U^{'}f\Big\}\widetilde{N}(dt,dz) \end{aligned} \tag{72}$$

where

$$\begin{aligned} \widetilde{C}(t,x) &= \widehat{Y}_3(\widehat{Y}_1 + (\alpha_t - \widehat{Y}_1)\phi_t)U^{''} \\ &+ \frac{1}{2}\widehat{Y}_3^2\sigma_t^2\phi_t^2U^{'''} + \widehat{Y}_3\sigma_t\phi_t\theta_t^1U^{''} \\ &+ \int_{\mathbb{R}}\{U^{'}(\widehat{Y}_3(1+\gamma_t\phi_t)) - U^{'}(\widehat{Y}_3) - \widehat{Y}_3\gamma_t\phi_tU^{''}\}\nu(dz) \end{aligned} \tag{73}$$

and

$$\widetilde{D}(t,x) = a(t)U^{'} + \widehat{Y}_3b(t)\sigma_t\phi_tU^{''} - b(t)\theta_t^0U^{'} \tag{74}$$

Comparing this with equation (60) by equating the dt, dW_t and $\widetilde{N}(dt,dz)$ coefficients respectively, we get

$$\hat{q}_3(t,x) = \widehat{Y}_3\sigma_t\phi_tU^{''}f - \theta_t^0U^{'}f + b(t)U^{'}f_{y_1} \tag{75}$$

$$\begin{aligned} \hat{r}_3(t,x) = &f[U^{'}(\widehat{Y}_3(1+\gamma_t\phi_t)) - U^{'}(\widehat{Y}_3)] \\ &+ U^{'}[f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1)] - \theta_t^1U^{'}f \end{aligned} \tag{76}$$

and

$$
\begin{aligned}
&U^{'} f_t + A\hat{p}_3(t,x) + \widetilde{C}(t,x) f + \widetilde{D}(t,x) f_{y_1} + \frac{1}{2} b^2(t) f_{y_1 y_1} \\
&+ \int_{\mathbb{R}} \{ f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1) - c(t,z) f_{y_1} \} \nu(dz) \\
= &-\{\widehat{Y}_1 + (\alpha_t - \widehat{Y}_1)\phi_t\} \hat{p}_3(t,x) - \sigma_t \phi_t \hat{q}_3(t,x) \\
&- \int_{\mathbb{R}} \phi_t \gamma_t \hat{r}_3(t,x,z) \nu(dz) - A^* \hat{p}_3(t,x)
\end{aligned} \tag{77}
$$

Substituting $\hat{p}_3(t,x), \hat{q}_3(t,x)$ and $\hat{r}_3(t,x,z)$ into equation (61) we have the following

$$
\begin{aligned}
&\theta_t^0(x) E[\sigma_t(x) \mid \mathcal{E}_t] - \int_{\mathbb{R}} \theta_t^1(x,z) E[\gamma_t(x,z) \mid \mathcal{E}_t] \nu(dz) \\
= &E[(\alpha_t(x) - \rho(t)) | \mathcal{E}_t] + E\Big[b(t)\sigma_t(x) \frac{f_{y_1}}{f} \Big| \mathcal{E}_t \Big] - E\Big[b(t)\sigma_t(x) \frac{UU^{''}}{U^{'}U^{'}} \frac{f_{y_1}}{f} \Big| \mathcal{E}_t \Big] \\
+ &\int_{\mathbb{R}} E\Big[\gamma_t(x,z) \Big(\frac{1}{U^{'}} [U^{'}(\hat{Y}_3(1 + \gamma_t(x)\phi_t(x))) - U^{'}(\hat{Y}_3)] \\
&\quad + \frac{1}{f}[f(\hat{Y}_1 + c(t,z)) - f(\hat{Y}_1)] \Big) \Big| \mathcal{E}_t \Big] \nu(dz)
\end{aligned} \tag{78}
$$

We have proved the following result:

Theorem 3. *A portfolio $\phi(t,x) \in \mathcal{A}$ is a maximum point for the problem* (55) *if it satisfies the equation* (70) *and if the optimal measure $Q_{\hat{\theta}}$ has an optimizer $\hat{\theta}(t,x) = (\hat{\theta}_t^0(x), \hat{\theta}_t^1(x))$ which satisfies the equation* (78).

Remark. When the short rate $\rho(t)$ is deterministic, we can easily see from (70) and (78) that

$$\phi(t,x) = 0$$

and

$$\theta_t^0(x) E[\sigma_t(x) | \mathcal{E}_t] + \int_{\mathbb{R}} \theta_t^1((x,z) E[\gamma_t(x,z) \mid \mathcal{E}_t] \nu(dz) = E[(\alpha_t(x) | \mathcal{E}_t] - \rho(t)$$

This case is analogous to the result obtained in [1], where the authors deal with SDE control.

Example 3.1. Let us consider in the continuous case, i.e. $c(t,z) = 0, \gamma_t(x) = 0, \theta_t^1(x) = 0$, and the *power utility*, i.e.

$$U(x,u) = \frac{1}{\eta} u^{\eta}, \qquad u > 0, \tag{79}$$

where $\eta \in (-\infty, 1) \backslash \{0\}$ is a constant. Using the separation

$$f(t, y_1) = g(t) e^{\beta(t) y_1} \tag{80}$$

with terminal conditions $\beta(T) = 0$ and $g(t) = 1$ we get the optimal for portfolio is

$$\phi_t(x) = -\frac{1}{\eta} \frac{E[b(t)\beta(t) | \mathcal{E}_t]}{E[\sigma_t(x) | \mathcal{E}_t]} \tag{81}$$

provided that

$$0 \leq -\frac{1}{\eta} \frac{E\left[b(t)\beta(t) \mid \mathcal{E}_t\right]}{E\left[\sigma_t(x) \mid \mathcal{E}_t\right]} \leq 1.$$

In this case the equation (67) becomes

$$\begin{aligned} 0 &= g^{'} + \Big(\beta^{'} + \frac{b(t)}{\sigma_t(x)}\beta + \eta\Big) y_1 g \\ &+ \Big\{\frac{1}{2} b(t)\Big(\frac{\eta-1}{\eta} - b(t)\Big)\beta^2 + \Big(a(t) - \frac{\alpha_t(x)b(t)}{\sigma_t(x)}\Big)\beta\Big\} g \end{aligned} \tag{82}$$

The function f will be meaningful if we get an ODE for g which does not include the short rate y_1. Hence β should be calculated so that the term of y_1 in (82) becomes zero, i.e.,

$$\beta^{'} = -\frac{b(t)}{\sigma_t(x)}\beta - \eta \qquad \text{with } \beta(T) = 0 \tag{83}$$

This leads to

$$\beta(t) = -\frac{\eta \sigma_t(x)}{b(t)} \Big(e^{-\frac{b(t)}{\sigma_t(x)}(T-t)} - 1\Big) \tag{84}$$

Then the optimal the market is to choose the scenario $Q_{\hat{\theta}}$ satisfies the equation

$$\begin{aligned} \theta^0_t(x) E[\sigma_t(x) \mid \mathcal{E}_t] &= E[(\alpha_t(x) - \rho(t))|\mathcal{E}_t] + E[b(t)\sigma_t(x)\beta|\mathcal{E}_t] \\ &- \frac{\eta-1}{\eta} E[b(t)\sigma_t(x)\beta|\mathcal{E}_t]. \end{aligned} \tag{85}$$

Example 3.2. Keep the utility function as above example and consider to the case when the dynamic of short rate ρ is described by a Vasicek model:

$$d\rho(t) = (\zeta - \mu\rho(t))dt + b dW_t \tag{86}$$

where ζ, μ, b are constants. The Vasicek model is an affine rate model and now $\beta(t) = \frac{1}{\mu}(1 - e^{-\mu(T-t)})$. In this case the optimal controls for portfolio and for the market simplify:

$$\phi_t(x) = -\frac{bE[(1 - e^{\mu(T-t)}) \mid \mathcal{E}_t]}{\mu\eta E[\sigma_t(x) \mid \mathcal{E}_t]} \tag{87}$$

and

$$\begin{aligned} &\theta^0_t(x) E[\sigma_t(x)|\mathcal{E}_t] + \int_{\mathbb{R}} \theta^1_t(x,z) E[\gamma_t(x,z) \mid \mathcal{E}_t]\nu(dz) \\ &= E[(\alpha_t(x) - \rho(t))|\mathcal{E}_t] + \frac{b}{\mu\eta} E[\sigma_t(x)(1 - e^{-\mu(T-t)}) \mid \mathcal{E}_t] \\ &+ \int_{\mathbb{R}} E[\gamma_t(x,z)\{(1 + \gamma_t(x,z)\phi_t(x))^{\eta-1} + (e^{\frac{c(t,z)}{\mu}(1-e^{-\mu(T-t)})} - 1)\} \mid \mathcal{E}_t]\nu(dz). \end{aligned} \tag{88}$$

Remark. a) Let us consider the case, when $Z^{(\theta)}_t(x) \equiv 1$ in (55). So our stochastic differential game reduces to an ordinary optimization problem for the SPDE (48) w.r.t. the portfolio strategy $\phi_t(x)$. In this case one can compare the optimal strategy $\phi_t(x)$ for constant maturity contracts with the corresponding one in the classical portfolio optimization problem of Merton in [16]: As a result one finds that optimal hedging based on constant maturity instruments presumes knowledge of the *whole* "term structure of volatility" $x \mapsto \sigma_t(x)$, whereas derivatives expiring at a fixed maturity only require information of *single* points (i.e. $\sigma(t,T)$ for T fixed) on volatility curves.

b) In practice one may be interested in hedging a constant maturity portfolio for a certain time-to-maturity $x_0 > 0$. By inspecting (70) and (78) we observe that the optimal hedging strategies are independent of the domain G in (55). By choosing $G = (x_0 - \varepsilon, x_0 + \varepsilon)$ ($\varepsilon > 0$ sufficiently small) one can argue that we may replace the performance functional in (55) by

$$J(\phi,\theta) = E_{Q_\theta}\left[U(x, V_T^{(\phi)}(x_0)\right],$$

if e.g. $\left(x \longmapsto E_{Q_\theta}\left[U(x, V_T^{(\phi)}(x)\right]\right)$ is continuous.

c) Our optimization problem can be easily generalized to the case of an investor who is allowed to consume portfolio wealth.

d) In the framework of Malliavin calculus a SPDE optimization problem related to (48) is studied in [13].

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