

DEPT. OF MATH./CMA  
PURE MATHEMATICS  
ISSN 0806-2439

UNIVERSITY OF OSLO  
NO 4  
JUNE 2012

# A MAXIMUM PRINCIPLE FOR INFINITE HORIZON DELAY EQUATIONS

N.AGRAM\*, S.HAADEM†, B. ØKSENDAL‡§ and F.PROSKE¶

28 June 2012

**Keywords:** Infinite horizon; Optimal control; Stochastic delay equation; Lévy processes; Maximum principle; Hamiltonian; Adjoint process; Partial information.

**2010 Mathematics Subject Classification:**

Primary 93EXX; 93E20; 60J75; 34K50

Secondary 60H10; 60H20; 49J55

## Abstract

We prove a maximum principle of optimal control of stochastic delay equations on infinite horizon. We establish first and second sufficient stochastic maximum principles as well as necessary conditions for that problem. We illustrate our results by an application to the optimal consumption rate from an economic quantity.

## 1 Introduction

To solve the stochastic control problems, there are two approaches: The dynamic programming method (HJB equation) and the maximum principle.

In this paper, our system is governed by the stochastic differential delay equation (SDDE in short):

---

\*Laboratory of Applied Mathematics, University Med Khider, Po. Box 145, Biskra (07000) Algeria.

†Center of Mathematics for Applications (CMA), University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway.

‡Center of Mathematics for Applications (CMA), University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway.

§The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no [228087].

¶Department of Mathematics, University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway.

$$\begin{cases} dX(t) = b(t, X(t), Y(t), A(t), u(t)) dt \\ + \sigma(t, X(t), Y(t), A(t), u(t)) dB(t) \\ + \int_{\mathbb{R}_0} \theta(t, X(t), Y(t), A(t), u(t), z) \tilde{N}(dt, dz); & t \in [0, \infty) \\ X(t) = X_0(t); & t \in [-\delta, 0] \\ Y(t) = X(t - \delta) & t \in [0, \infty) \\ A(t) = \int_{t-\delta}^t e^{-\rho(t-r)} X(r) dr & t \in [0, \infty) \end{cases}$$

which maximise the functional

$$J(u) = E \left[ \int_0^\infty f(t, X(t), Y(t), A(t), u(t)) dt \right] \quad (1.2)$$

where  $u(t)$  is the control process.

The SDDE is not Markovian so we cannot use the dynamic programming method. However, we will prove stochastic maximum principles for this problem.

A sufficient maximum principle in infinite horizon without non-trivial transversality conditions were treated by [4]. The 'natural' transversality condition in the infinite case would be a zero limit condition, meaning in the economic sense that one more unit of good at the limit gives no additional value. But this property is not necessarily verified. In fact [5] provides a counterexample for a 'natural' extension of the finite-horizon transversality conditions. Thus some care is needed in the infinite horizon case. For the case of the 'natural' transversality condition the discounted control problem was studied by [7].

In real life, delay occurs everywhere in our society. For example this is the case in biology where the population growth depends not only on the current population size but also on the size some time ago. The same situation may occur in many economic growth models.

The stochastic maximum principle with delay has been studied by many authors. For example, [3] proved a verification theorem of variational inequality. [10] established the sufficient maximum principle for certain class of stochastic control systems with delay in the state variable. In [4] they studied infinite horizon but without a delay. In [2], they derived a stochastic maximum principle for a system with delay both in the state variable and the control variable. In [12] they studied the finite horizon version of this paper, however, to our knowledge, no one has studied the infinite horizon case for delay equations.

In our paper, we establish two sufficient maximum principles and one necessary for the stochastic delay systems on infinite horizon with jumps.

For backward differential equations see [16], [6] For the infinite horizon BSDE see [14], [13], [17], [1] and [15]. For more details about jump diffusion markets see [11] and for background and details about stochastic fractional delay equations see [8].

Our paper is organised as follows: In the second section, we formulate the problem. The third section is devoted to the first and second sufficient maximum principles with an application to the optimal consumption rate from an

economic quantity described by a stochastic delay equation. In the fourth section, we formulate a necessary maximum principle and we prove an existence and uniqueness of the advanced backward stochastic differential equations on infinite horizon with jumps in the last section.

## 2 Formulation of the problem

Let  $(\Omega, \mathcal{F}, P)$  be a probability space with filtration  $\mathcal{F}_t$  satisfying the usual conditions, on which an  $\mathbb{R}$ -valued standard Brownian motion  $B(\cdot)$  and an independent compensated Poisson random measure  $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz) dt$  are defined.

We consider the following stochastic control system with delay :

$$\begin{cases} dX(t) = b(t, X(t), Y(t), A(t), u(t)) dt + \sigma(t, X(t), Y(t), A(t), u(t)) dB(t) \\ \quad + \int_{\mathbb{R}_0} \theta(t, X(t), Y(t), A(t), u(t), z) \tilde{N}(dt, dz); t \in [0, \infty) \\ X(t) = X_0(t); \quad t \in [-\delta, 0] \\ Y(t) = X(t - \delta) \quad A(t) = \int_{t-\delta}^t e^{-\rho(t-r)} X(r) dr \end{cases} \quad (2.1)$$

$\delta > 0, \rho > 0$  are given constants.

$b : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \rightarrow \mathbb{R}$ ,

$\sigma : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \rightarrow \mathbb{R}$ ,

$\theta : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \times \Omega \rightarrow \mathbb{R}$ ,

are given functions such that for all  $t, b(t, x, y, a, u, \cdot), \sigma(t, x, y, a, u, \cdot)$  and  $\theta(t, x, y, a, u, z, \cdot)$  are  $\mathcal{F}_t$ -mesurable for all  $x \in \mathbb{R}, y \in \mathbb{R}, a \in \mathbb{R}, u \in \mathcal{U}$  and  $z \in \mathbb{R}_0$ .

Let  $\mathcal{E}_t \subset \mathcal{F}_t$  be a given subfiltration, representing the information available to the controller at time  $t$ .

Let  $\mathcal{U}$  be a non-empty subset of  $\mathbb{R}$ . We let  $\mathcal{A}_{\mathcal{E}}$  denote the family of admissible  $\mathcal{E}_t$ -adapted control processes.

An element of  $\mathcal{A}_{\mathcal{E}}$  is called an admissible control.

The corresponding performance functional is

$$J(u) = E \left[ \int_0^\infty f(t, X(t), Y(t), A(t), u(t)) dt \right]; u \in \mathcal{A}_{\mathcal{E}}, \quad (2.2)$$

where we assume that

$$E \int_0^\infty \left\{ |f(t, X(t), Y(t), A(t), u(t))| + \left| \frac{\partial f}{\partial x_i}(t, X(t), Y(t), A(t), u(t)) \right|^2 \right\} dt < \infty \quad (2.3)$$

The value function  $\Phi$  is defined as

$$\Phi(X_0) = \sup_{u \in \mathcal{A}_\varepsilon} J(u) \quad (2.4)$$

An admissible control  $u^*(.)$  is called an optimal control for (2.1) if it attains the maximum of  $J(u(.))$  over  $\mathcal{A}_\varepsilon$ . (2.1) is called the state equation, the solution  $X^*$  corresponding to  $u^*(.)$  is called an optimal trajectory.

### 3 A sufficient maximum principle

Our objective is to establish a sufficient maximum principle.

#### 3.1 Hamiltonian and time-advanced BSDEs for adjoint equations

We now introduce the adjoint equations and the Hamiltonian function for our problem.

The Hamiltonian is

$$\begin{aligned} H(t, x, y, a, u, p, q, r(.)) &= f(t, x, y, a, u) + b(t, x, y, a, u)p + \sigma(t, x, y, a, u)q \\ &\quad + \int_{\mathbb{R}_0} \theta(t, x, y, a, u, z)r(z)\nu(dz), \end{aligned} \quad (3.1)$$

where

$$H : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathfrak{H} \times \Omega \rightarrow \mathbb{R}$$

and  $\mathfrak{H}$  is the set of functions  $r: \mathbb{R}_0 \rightarrow \mathbb{R}$  such that the terms in (3.1) converges and  $U$  is the set of possible control values.

We suppose that  $b$ ,  $\sigma$  and  $\theta$  are  $C^1$  functions with respect to  $(x, y, a, u)$  and that

$$\begin{aligned} E \left[ \int_0^\infty \left\{ \left| \frac{\partial b}{\partial x_i}(t, X(t), Y(t), A(t), u(t)) \right|^2 + \left| \frac{\partial \sigma}{\partial x_i}(t, X(t), Y(t), A(t), u(t)) \right|^2 \right. \right. \\ \left. \left. + \int_{\mathbb{R}_0} \left| \frac{\partial \theta}{\partial x_i}(t, X(t), Y(t), A(t), u(t)) \right|^2 \nu(dz) \right\} dt \right] < \infty \end{aligned} \quad (3.2)$$

for  $x_i = x, y, a$  and  $u$ .

The adjoint processes  $(p(t), q(t), r(t, z))$ ,  $t \in [0, \infty)$ ,  $z \in \mathbb{R}$  are assumed to satisfy the equation :

$$dp(t) = E [\mu(t) \mid \mathcal{F}_t] dt + q(t)dB_t + \int_{\mathbb{R}_0} r(t, z) \tilde{N}(dt, dz); t \in [0, \infty), \quad (3.3)$$

where

$$\begin{aligned} \mu(t) = & -\frac{\partial H}{\partial x}(t, X_t, Y_t, A_t, u_t, p(t), q(t), r(t, \cdot)) \\ & -\frac{\partial H}{\partial y}(t + \delta, X_{t+\delta}, Y_{t+\delta}, A_{t+\delta}, u_{t+\delta}, p(t + \delta), q(t + \delta), r(t + \delta, \cdot)) \\ & -e^{\rho t} \left( \int_t^{t+\delta} \frac{\partial H}{\partial a}(s, X_s, Y_s, A_s, u_s, p(s), q(s), r(s, \cdot)) e^{-\rho s} ds \right) \end{aligned} \quad (3.4)$$

### 3.2 A first sufficient maximum principle

**Theorem 3.1** *Let  $\hat{u} \in \mathcal{A}_{\mathcal{E}}$  with corresponding state processes  $\hat{X}(t)$ ,  $\hat{Y}(t)$  and  $\hat{A}(t)$  and adjoint processes  $\hat{p}(t)$ ,  $\hat{q}(t)$  and  $\hat{r}(t, z)$  assumed to satisfy the ABSDE (3.3)- (3.4). Suppose that the following assertions hold:*

- (i)  $E \left[ \overline{\lim}_{t \rightarrow \infty} \hat{p}(t)(X(t) - \hat{X}(t)) \right] \geq 0$ .
- (ii) *The function*

$$(x, y, a, u) \rightarrow H(t, x, y, a, u, \hat{p}, \hat{q}, \hat{r}(t, \cdot)),$$

*is concave for each  $t \in [0, \infty)$  a.s.*

(iii)

$$\begin{aligned} E \left[ \int_0^\infty \left\{ \hat{p}^2(t) \left( \sigma^2(t) + \int_{\mathbb{R}_0} \theta^2(t, z) \nu(dz) \right) \right. \right. \\ \left. \left. + X^2(t) \left( \hat{q}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t, z) \nu(dz) \right) \right\} dt \right] < \infty, \end{aligned} \quad (3.5)$$

*for all  $u \in \mathcal{A}_{\mathcal{E}}$ .*

(iiii)

$$\begin{aligned} & \max_{v \in \mathcal{A}_{\mathcal{E}}} E \left[ H(t, \hat{X}(t), \hat{X}(t - \delta), \hat{A}(t), v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \mid \mathcal{E}_t \right] \\ & = E \left[ H(t, \hat{X}(t), \hat{X}(t - \delta), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \mid \mathcal{E}_t \right], \end{aligned}$$

*for all  $t \in [0, \infty)$  a.s.*

*Then  $\hat{u}(t)$  is an optimal control for the problem (2.4).*

**Proof.** Choose an arbitrary  $u \in \mathcal{A}_{\mathcal{E}}$ , and consider

$$J(u) - J(\hat{u}) = I_1 \quad (3.6)$$

where

$$I_1 = E \left[ \int_0^\infty \left\{ f(t, X(t), Y(t), A(t), u(t)) - f\left(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t)\right) \right\} dt \right]. \quad (3.7)$$

By the definition (3.1) of  $H$  and the concavity, we have

$$\begin{aligned} I_1 \leq E \left[ \int_0^\infty \left\{ \frac{\partial \hat{H}}{\partial x}(t)(X(t) - \hat{X}(t)) + \frac{\partial \hat{H}}{\partial y}(t)(Y(t) - \hat{Y}(t)) + \frac{\partial \hat{H}}{\partial a}(t)(A(t) - \hat{A}(t)) \right. \right. \\ \left. \left. + \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t)) - (b(t) - \hat{b}(t))\hat{p}(t) - (\sigma(t) - \hat{\sigma}(t))\hat{q}(t) \right. \right. \\ \left. \left. - \int_{\mathbb{R}_0} (\theta(t, z) - \hat{\theta}(t, z))\hat{r}(t, z)\nu(dz) \right\} dt \right], \quad (3.8) \end{aligned}$$

where we have used the simplified notation

$$\frac{\partial \hat{H}}{\partial x}(t) = \frac{\partial \hat{H}}{\partial x} \left( t, \hat{X}_t, \hat{Y}_t, \hat{A}_t, \hat{u}_t, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot) \right).$$

Applying the Itô formula to  $\hat{p}(t)(X(t) - \hat{X}(t))$  we get

$$\begin{aligned} 0 &\leq E \left[ \overline{\lim}_{T \rightarrow \infty} \hat{p}(T)(X(T) - \hat{X}(T)) \right] \\ &= E \left[ \overline{\lim}_{T \rightarrow \infty} \left( \int_0^T (b(t) - \hat{b}(t))\hat{p}(t)dt + \int_0^T (X(t) - \hat{X}(t))E[\hat{\mu}(t) | \mathcal{F}_t]dt \right. \right. \\ &\quad \left. \left. + \int_0^T (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)dt + \int_0^T \int_{\mathbb{R}_0} (\theta(t, z) - \hat{\theta}(t, z))\hat{r}(t, z)\nu(dz)dt \right) \right] \\ &= E \left[ \overline{\lim}_{T \rightarrow \infty} \left( \int_0^T (b(t) - \hat{b}(t))\hat{p}(t)dt + \int_0^T (X(t) - \hat{X}(t))\hat{\mu}(t)dt \right. \right. \\ &\quad \left. \left. + \int_0^T (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)dt \right. \right. \\ &\quad \left. \left. + \int_0^T \int_{\mathbb{R}_0} (\theta(t, z) - \hat{\theta}(t, z))\hat{r}(t, z)\nu(dz)dt \right) \right]. \quad (3.9) \end{aligned}$$

Using the definition (3.4) of  $\mu$  we see that

$$\begin{aligned}
E \left[ \overline{\lim}_{T \rightarrow \infty} \left( \int_0^T (X(t) - \hat{X}(t)) \hat{\mu}(t) dt \right) \right] &= E \left[ \overline{\lim}_{T \rightarrow \infty} \left( \int_{\delta}^{T+\delta} (X(t-\delta) - \hat{X}(t-\delta)) \hat{\mu}(t-\delta) dt \right) \right] \\
&= E \left[ \overline{\lim}_{T \rightarrow \infty} \left( - \int_{\delta}^{T+\delta} \frac{\partial \hat{H}}{\partial x}(t-\delta) (X(t-\delta) - \hat{X}(t-\delta)) dt \right. \right. \\
&\quad \left. \left. - \int_{\delta}^{T+\delta} \frac{\partial \hat{H}}{\partial y}(t) (Y(t) - \hat{Y}(t)) dt - \int_{\delta}^{T+\delta} \left( \int_{t-\delta}^t \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} ds \right) \right. \right. \\
&\quad \left. \left. e^{\rho(t-\delta)} (X(t-\delta) - \hat{X}(t-\delta)) dt \right) \right] \tag{3.10}
\end{aligned}$$

Using integration by parts and substituting  $r = t - \delta$ , we obtain

$$\begin{aligned}
&\int_0^T \frac{\partial \hat{H}}{\partial a}(s) (A(s) - \hat{A}(s)) ds \\
&= \int_0^T \frac{\partial \hat{H}}{\partial a}(s) \int_{s-\delta}^s e^{-\rho(s-r)} (X(r) - \hat{X}(r)) dr ds \\
&= \int_0^T \left( \int_r^{r+\delta} \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} ds \right) e^{\rho r} (X(r) - \hat{X}(r)) dr \\
&= \int_{\delta}^{T+\delta} \left( \int_{t-\delta}^t \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} ds \right) e^{\rho(t-\delta)} (X(t-\delta) - \hat{X}(t-\delta)) dt \tag{3.11}
\end{aligned}$$

Combining (3.9), (3.10) and (3.11); we get

$$\begin{aligned}
0 \leq E \left[ \overline{\lim}_{T \rightarrow \infty} \hat{p}(T) (X(T) - \hat{X}(T)) \right] &= E \left[ \left( \int_0^{\infty} (b(t) - \hat{b}(t)) \hat{p}(t) dt \right. \right. \\
&\quad \left. \left. - \int_0^{\infty} \frac{\partial \hat{H}}{\partial x}(t) (X(t) - \hat{X}(t)) dt - \int_0^{\infty} \frac{\partial \hat{H}}{\partial y}(t) (Y(t) - \hat{Y}(t)) dt \right. \right. \\
&\quad \left. \left. - \int_0^{\infty} \frac{\partial \hat{H}}{\partial a}(t) (A(t) - \hat{A}(t)) dt + \int_0^{\infty} (\sigma(t) - \hat{\sigma}(t)) \hat{q}(t) dt \right. \right. \\
&\quad \left. \left. + \int_0^{\infty} \int_{\mathbb{R}_0} (\theta(t, z) - \hat{\theta}(t, z)) \hat{r}(t, z) \nu(dz) dt \right) \right] \tag{3.12}
\end{aligned}$$

Subtracting and adding  $\int_0^{\infty} \frac{\partial \hat{H}}{\partial u}(t) (u(t) - \hat{u}(t)) dt$  in (3.12) we conclude

$$\begin{aligned}
0 &\leq E \left[ \overline{\lim_{T \rightarrow \infty}} \hat{p}(T)(X(T) - \hat{X}(T)) \right] = E \left[ \left( \int_0^\infty (b(t) - \hat{b}(t)) \hat{p}(t) dt \right. \right. \\
&\quad - \int_0^\infty \frac{\partial \hat{H}}{\partial x} (t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t, \cdot)) (X(t) - \hat{X}(t)) dt \\
&\quad - \int_0^\infty \frac{\partial \hat{H}}{\partial y} (t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t, \cdot)) (Y(t) - \hat{Y}(t)) dt \\
&\quad - \int_0^\infty \frac{\partial \hat{H}}{\partial a} (t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t, \cdot)) (A(t) - \hat{A}(t)) dt \\
&\quad + \int_0^\infty (\sigma(t) - \hat{\sigma}(t)) \hat{q}(t) dt + \int_0^\infty \int_{\mathbb{R}_0} (\theta(t, z) - \hat{\theta}(t, z)) \hat{r}(t, z) \nu(dz) dt \\
&\quad \left. \left. - \int_0^\infty \frac{\partial \hat{H}}{\partial u} (t)(u(t) - \hat{u}(t)) dt + \int_0^\infty \frac{\partial \hat{H}}{\partial u} (t)(u(t) - \hat{u}(t)) dt \right) \right] \\
&\leq -I_1 + E \left[ \int_0^\infty E \left[ \frac{\partial \hat{H}}{\partial u} (t)(u(t) - \hat{u}(t)) \mid \mathcal{E}_t \right] dt \right].
\end{aligned}$$

Hence

$$I_1 \leq E \left[ \int_0^\infty E \left[ \frac{\partial \hat{H}}{\partial u} (t) \mid \mathcal{E}_t \right] (u(t) - \hat{u}(t)) dt \right] \leq 0.$$

Since  $u \in \mathcal{A}_{\mathcal{E}}$  was arbitrary, this proves *Theorem 1*. ■

### 3.3 A second sufficient maximum principle

We extend the result in [10] to infinite horizon with jump diffusions.

Consider again the system

$$\begin{cases}
dX(t) = b(t, X(t), Y(t), A(t), u(t)) dt \\
\quad + \sigma(t, X(t), Y(t), A(t), u(t)) dB(t) \\
\quad + \int_{\mathbb{R}_0} \theta(t, X(t), Y(t), A(t), u(t), z) \tilde{N}(dt, dz); & t \in [0, \infty) \\
X(t) = X_0(t); & t \in [-\delta, 0] \\
Y(t) = X(t - \delta) & t \in [0, \infty) \\
A(t) = \int_{t-\delta}^t e^{-\rho(t-r)} X(r) dr & t \in [0, \infty)
\end{cases}$$

Let  $X_t \in C[-\delta, 0]$  be the segment of the path of  $X$  from  $t - \delta$  to  $t$ , i.e

$$X_t(s) = X(t + s),$$



for  $s \in [-\delta, 0]$ . We now give an Itô formula which is proved in [3] without jumps. Adding the jump parts are just an easy observation.

**Lemma 3.2** *The Itô formula for delay*  
*Consider a function*

$$G(t) = F(t, X(t), A(t)), \quad (3.13)$$

where  $F$  is a function in  $C^{1,2,1}(\mathbb{R}^3)$  and

$$Y(t) = \int_{-\delta}^0 e^{\lambda s} X(t+s) ds.$$

then

Then

$$\begin{aligned} dG(t) = & LF dt + \sigma(t, x, y, a, u) \frac{\partial F}{\partial x} dB(t) \\ & + \int_{\mathbb{R}_0} \left\{ F(t, X(t^-), A(t^-)) + \theta(t, X(t), Y(t), A(t), u, z) \right. \\ & \quad \left. - F(t, X(t^-), A(t^-)) \right. \\ & \quad \left. - \frac{\partial F}{\partial x}(t, X(t^-), A(t^-)) + \theta(t, X(t), Y(t), A(t), u, z) \right\} \nu(dz) dt \\ & + \int_{\mathbb{R}_0} \left\{ F(t, X(t^-), A(t^-)) + \theta(t, X(t), Y(t), A(t), u, z) \right. \\ & \quad \left. - F(t, X(t^-), A(t^-)) \right\} \tilde{N}(dt, dz) \\ & + [x - \lambda y - e^{-\lambda \delta} a] \frac{\partial F}{\partial a} dt \end{aligned}$$

where

$$LF = LF(t, x, y, a, u) = \frac{\partial F}{\partial t} + b \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2}$$

Now, define the Hamiltonian,  $H : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathcal{R} \rightarrow \mathbb{R}$  as

$$\begin{aligned} H(t, x, y, a, u, p, q, r(\cdot)) & \quad (3.14) \\ = & f(t, x, y, a, u) + b(t, x, y, a, u)p_1 + (x - \lambda y - e^{-\lambda \delta} a)p_2 \\ & + \sigma(t, x, y, a, u)q_1 + \int_{\mathbb{R}_0} \theta(t, x, y, a, u, z)r(z)\nu(dz) \end{aligned}$$

where  $p = (p_1, p_2, p_3)^T \in \mathbb{R}^3$  and  $q = (q_1, q_2) \in \mathbb{R}^2$  For each  $u \in \mathcal{A}$  the associated adjoint equations are the following backward stochastic differential equations in

the unknown  $\mathcal{F}_t$ -adapted precesses  $(p(t), q(t), r(t, \cdot))$  given by;

$$\begin{aligned} dp_1(t) &= -\frac{\partial H}{\partial x}(t, X(t), Y(t), A(t), u(t), p(t), q(t))dt + q_1(t)dB(t) \\ &\quad + \int_{\mathbb{R}_0} r(t, z)\bar{N}(dt, dz), \\ dp_2(t) &= -\frac{\partial H}{\partial y}(t, X(t), Y(t), A(t), u(t), p(t), q(t))dt + q_2(t)dB(t), \end{aligned} \quad (3.15)$$

$$dp_3(t) = -\frac{\partial H}{\partial a}(t, X(t), Y(t), A(t), u(t), p(t), q(t))dt, \quad (3.16)$$

**Theorem 3.3 (An infinite horizon maximum principle for delay equations)**

Suppose  $\hat{u} \in \mathcal{A}$  and let  $(\hat{X}, \hat{Y}, \hat{A})$  and  $(p(t), q(t), r(t, \cdot))$  be the corresponding solutions of (3.15)-(3.16), respectively. Suppose that

$$H(t, \cdot, \cdot, \cdot, \cdot, p(t), q(t), r(t, \cdot))$$

are concave for all  $t \geq 0$ ,

$$\begin{aligned} &E \left[ H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) | \mathcal{E}_t \right] \\ &= \max_{u \in U} E \left[ H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) | \mathcal{E}_t \right]. \end{aligned} \quad (3.17)$$

Further, assume that

$$E[\lim p_1(t)(X(t) - \hat{X}(t))] \geq 0, \quad (3.18)$$

and

$$E[\lim p_2(t)(Y(t) - \hat{Y}(t))] \geq 0. \quad (3.19)$$

In addition assume that

$$p_3(t) = 0, \quad (3.20)$$

for all  $t$ . Then  $\hat{u}$  is an optimal control.

**Proof.** To simplify notation we put

$$\zeta(t) = (X(t), Y(t), A(t)),$$

and

$$\hat{\zeta}(t) = (\hat{X}(t), \hat{Y}(t), \hat{A}(t)).$$

Let

$$I := E \left[ \int_0^\infty (f(t, \hat{\zeta}(t), \hat{u}(t)) - f(t, \zeta(t), u(t)))dt \right]$$

Then we have that

$$\begin{aligned}
I &= E\left[\int_0^\infty (H(t, \hat{\zeta}(t), \hat{u}(t), p(t), q(t), r(t, \cdot)) - H(t, \zeta(t), u(t)), p(t), q(t), r(t, \cdot))dt\right] \\
&\quad - E\left[\int_0^\infty (b(t, \hat{\zeta}(t), \hat{u}(t)) - b(t, \zeta(t), u(t)))p_1(t)dt\right] \\
&\quad - E\left[\int_0^\infty \{(\hat{X}(t) - \lambda\hat{Y}(t) - e^{-\lambda\delta}\hat{A}(t)) - (X(t) - \lambda Y(t) - e^{-\lambda\delta}A(t))\}p_2(t)dt\right] \\
&\quad - E\left[\int_0^\infty \{\sigma(t, \hat{\zeta}(t), \hat{u}(t)) - \sigma(t, \zeta(t), u(t))\}q_1(t)dt\right] \\
&\quad - E\left[\int_0^\infty \int_{\mathbb{R}_0} (\theta(t, \hat{\zeta}(t), \hat{u}(t), z) - \theta(t, \zeta(t), u(t), z)) \times r(t, z)\nu(dz)dt\right] \\
&=: I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.21}
\end{aligned}$$

Since  $(\zeta, u) \rightarrow H(\zeta, u)$  is concave and (3.12), we have that

$$\begin{aligned}
H(\zeta, u) - H(\hat{\zeta}, \hat{u}) &\leq H_\zeta(\hat{\zeta}, \hat{u}) \cdot (\zeta - \hat{\zeta}) + H_u(\hat{\zeta}, \hat{u}) \cdot (u - \hat{u}) \\
&\leq H_\zeta(\hat{\zeta}, \hat{u}) \cdot (\zeta - \hat{\zeta})
\end{aligned}$$

where  $H_\zeta = (\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial a})$ . From this we get that

$$\begin{aligned}
I_1 &\geq E\left[\int_0^\infty -H_\zeta(t, \hat{\zeta}(t), \hat{u}(t), p(t), q(t)) \cdot (\zeta(t) - \hat{\zeta}(t))dt\right] \\
&= E\left[\int_0^\infty (\zeta(t) - \hat{\zeta}(t))dp(t) - \int_0^\infty (X(t) - \hat{X}(t))q_1(t)dB(t) \right. \\
&\quad \left. - \int_0^\infty (Y(t) - \hat{Y}(t))q_2(t)dB(t)\right] \\
&= E\left[\int_0^\infty (X(t) - \hat{X}(t))dp_1(t) + \int_0^\infty (Y(t) - \hat{Y}(t))dp_2(t)\right]. \tag{3.22}
\end{aligned}$$

From (3.18), (3.19) and (3.20) we get that

$$\begin{aligned}
0 &\geq -E[\lim p_1(t)(X(t) - \hat{X}(t)) + \lim p_2(t)(Y(t) - \hat{Y}(t))] \\
&= -E\left[\int_0^\infty (X(t) - \hat{X}(t))dp_1(t) + \int_0^\infty p_1(t)d(X(t) - \hat{X}(t)) \right. \\
&\quad + \int_0^\infty [\sigma(t, \zeta(t), u(t)) - \sigma(t, \hat{\zeta}(t), \hat{u}(t))]q_1(t)dt \\
&\quad + \int_0^\infty \int_{\mathbb{R}_0} (\theta(t, \hat{\zeta}(t), \hat{u}(t), z) - \theta(t, \zeta(t), u(t), z)) \times r(t, z)\nu(dz)dt \\
&\quad \left. + \int_0^\infty (Y(t) - \hat{Y}(t))dp_2(t) + \int_0^\infty p_2(t)d(Y(t) - \hat{Y}(t))\right].
\end{aligned}$$

Combining this with (3.21) and (3.22) we have that so that

$$-I = I_1 + I_2 + I_3 + I_4 + I_5 \leq 0.$$

Hence  $J(\hat{u}) - J(u) = I \geq 0$ , and  $\hat{u}$  is an optimal control for our problem.

■

**Example 3.4 (A non-delay infinite horizon example)** *Let us first consider a non-delay example. Assume we are given*

$$J(u) = E \left[ \int_0^\infty e^{-\rho t} \frac{1}{\gamma} (u(t)X(t))^\gamma dt \right],$$

where

$$\begin{cases} dX(t) &= [X(t)\mu - u(t)X(t)] dt \\ &\quad + \sigma(t, X(t), u(t)) dB(t); t \geq 0, \\ X(t) &= X_0 \end{cases}$$

$\gamma \in (0, 1)$  and  $\rho, \delta > 0$ . In this case the Hamiltonian (3.14) takes the form

$$\begin{aligned} H(t, u, x, p, q) &= e^{-\rho t} \frac{1}{\gamma} (ux)^\gamma + [x\mu - ux]p_1 \\ &\quad + [x - e^{-\rho\delta}a]p_2 + \sigma(t, x, y, a, u)q, \end{aligned}$$

so that we get the partial derivative

$$\nabla_u H(t, u, x, p, q) = e^{-\rho t} u^{\gamma-1} x^\gamma - xp_1 - \frac{\partial \sigma}{\partial u} q_1.$$

This gives us that

$$p_1(t) = e^{-\rho t} x^{\gamma-1} u(t)^{\gamma-1} - \frac{\partial \sigma}{\partial u} \frac{1}{x} q_1.$$

We now see that the adjoint equations are given by:

$$\begin{aligned} dp_1(t) &= - \left[ e^{-\rho t} (u(t))^\gamma X(t)^{\gamma-1} \right. \\ &\quad \left. + (\mu - u(t))p_1(t) + p_2(t) + \frac{\partial \sigma}{\partial x} q_1(t) \right] dt + q_1(t) dB(t), \\ dp_2(t) &= -q_2(t) dB(t), \\ dp_3(t) &= - \left[ -e^{-\rho\delta} p_2(t) + \frac{\partial \sigma}{\partial a} q_1(t) \right] dt. \end{aligned}$$

Since  $p_3(t)$  must be 0, we then get  $q_1 = q_2 = 0$ . and

$$p_2(t) = 0,$$

which gives us that

$$\begin{aligned} dp_1(t) &= - [e^{-\rho t} (u(t))^\gamma X(t)^{\gamma-1} dt + (\mu - u(t)) p_1(t)] dt, \\ dp_2(t) &= 0, \end{aligned}$$

and

$$p_1(t) = e^{-\rho t} X(t)^{\gamma-1} u(t)^{\gamma-1}.$$

So

$$\begin{aligned} dp_1(t) &= - [e^{-\rho t} (u(t))^\gamma X(t)^{\gamma-1} dt + (\mu - u(t)) p_1(t)] dt, \\ &= -\mu p_1(t) dt \end{aligned}$$

which gives

$$p_1(t) = p_1(0) e^{-\mu t},$$

for some constant  $p_1(0)$ , so that

$$u(t) = \frac{p_1(0)^{\frac{1}{\gamma-1}}}{X(t)} e^{\frac{1}{\gamma-1}(\rho t - \mu t)},$$

for all  $t > 0$ . Inserting  $u$  into the dynamics of  $X$ , we get that

$$dX(t) = \left[ \mu X(t) - p_1(0)^{\frac{1}{\gamma-1}} e^{\frac{1}{\gamma-1}(\rho t - \mu t)} \right] dt.$$

So

$$X(t) = e^{\mu t} \left[ X(0) - p_1(0)^{\frac{1}{\gamma-1}} \int_0^t \exp\left(-\mu - \frac{1}{1-\gamma}(\lambda - \mu)s\right) ds \right].$$

To ensure that  $X(t)$  is always non-negative, we get the optimal  $p(0)$  as

$$p_1(0) = \left[ \frac{X(0)}{\int_0^\infty \exp\left(-\mu - \frac{1}{1-\gamma}(\lambda - \mu)s\right) ds} \right]^{\gamma-1}.$$

We now see that  $\lim p_1(t) = 0$ , so that we have

$$E[\lim \hat{p}_1(t)(X(t) - \hat{X}(t))] \geq 0.$$

This tells us that  $\hat{u}$  is an optimal control.

**Example 3.5 (An infinite horizon example with delay)** Now let us consider a case where we have delay. This is an infinite horizon version of Example 1 in [10]. Let

$$J(u) = E \left[ \int_0^\infty e^{-\rho t} \frac{1}{\gamma} (u(t)(X(t) + Y(t)e^{\rho\delta}\beta))^\gamma dt \right],$$

where

$$\begin{cases} dX(t) &= [X(t)\mu + Y(t)\alpha + \beta A(t) - u(t)(X(t) + Y(t)e^{\rho\delta}\beta)]dt \\ &\quad + \sigma(t, X(t), Y(t), A(t), u(t))dB(t); t \geq 0, \\ X(t) &= X_0(t); t \in [-\delta, 0], \end{cases}$$

$\gamma \in (0, 1)$  and  $\rho, \delta > 0$ . In this case the Hamiltonian (3.20) takes the form

$$\begin{aligned} H(t, u, x, y, a, p, q) &= e^{-\rho t} \frac{1}{\gamma} (u(x + ye^{\rho\delta}\beta))^\gamma + [x\mu + \alpha y + \beta a - u(x + ye^{\rho\delta}\beta)]p_1 \\ &\quad + [x - \lambda y - e^{-\rho\delta}a]p_2 + \sigma(t, x, y, a, u)q, \end{aligned}$$

so that we get the partial derivative

$$\nabla_u H(t, u, x, y, a, p, q) = e^{-\rho t} u^{\gamma-1} (x + ye^{\rho\delta}\beta)^\gamma - (x + ye^{\rho\delta}\beta)p_1 - \frac{\partial \sigma}{\partial u} q_1.$$

This gives us that

$$p_1(t) = e^{-\rho t} (x + ye^{\rho\delta}\beta)^{\gamma-1} u(t)^{\gamma-1} - \frac{\partial \sigma}{\partial u} \frac{1}{(x + ye^{\rho\delta}\beta)} q_1.$$

We now see that the adjoint equations are given by:

$$\begin{aligned} dp_1(t) &= -[e^{-\rho t} (u(t))^\gamma (X(t) + Y(t)e^{\rho\delta}\beta)^{\gamma-1} dt \\ &\quad + (\mu - u(t))p_1(t) + p_2(t) + \frac{\partial \sigma}{\partial x} q_1(t)]dt + q_1(t)dB(t), \\ dp_2(t) &= -[e^{-\rho t} (u(t))^\gamma (X(t) + Y(t)e^{\rho\delta}\beta)^{\gamma-1} e^{\rho\delta}\beta dt \\ &\quad + (\alpha - u(t)e^{\rho\delta}\beta)p_1(t) - \lambda p_2(t) + \frac{\partial \sigma}{\partial y} q_1(t)]dt + q_2(t)dB(t), \\ dp_3(t) &= -[\beta p_1(t) - e^{-\rho\delta}p_2(t) + \frac{\partial \sigma}{\partial a} q_1(t)]dt. \end{aligned}$$

Let us try to choose  $q_1 = q_2 = 0$ . Since  $p_3(t) = 0$ , we then get

$$p_1(t) = \frac{e^{-\rho\delta}}{\beta} p_2(t),$$

which gives us that

$$\begin{aligned} dp_1(t) &= -[e^{-\rho t} (u(t))^\gamma (X(t) + Y(t)e^{\rho\delta}\beta)^{\gamma-1} dt + (\mu - u(t))p_1(t) + e^{\rho\delta}\beta p_1(t)]dt, \\ dp_2(t) &= -[e^{-\rho t} (u(t))^\gamma (X(t) + Y(t)e^{\rho\delta}\beta)^{\gamma-1} e^{\rho\delta}\beta dt + (\alpha \frac{e^{-\rho\delta}}{\beta} - u(t))p_2(t) - \lambda p_2(t)]dt, \end{aligned}$$

and

$$p_1(t) = e^{-\rho t} (X(t) + Y(t)e^{\rho\delta}\beta)^{\gamma-1} u(t)^{\gamma-1}$$

or

$$u(t) = \frac{e^{\frac{\rho t}{\gamma-1}} p_1^{\frac{\rho t}{\gamma-1}}(t)}{X(t) + Y(t)e^{\rho\delta}\beta}. \quad (3.23)$$

Hence, to ensure that

$$p_1(t) = \frac{e^{-\rho\delta}}{\beta} p_2(t)$$

we need that

$$\alpha = e^{\rho\delta}\beta(\mu + \lambda + e^{\rho\delta}\beta).$$

So

$$\begin{aligned} dp_1(t) &= -[e^{-\rho t}(u(t))^\gamma(X(t) + Y(t)e^{\rho\delta}\beta)^{\gamma-1}dt + (\mu - u(t))p_1(t) + e^{\rho\delta}\beta p_1(t)]dt, \\ &= -[\mu p_1(t) + e^{\rho\delta}\beta p_1(t)]dt, \end{aligned}$$

which gives us that

$$p_1(t) = p_1(0)e^{-(\mu + e^{\rho\delta}\beta)t},$$

for some constant  $p_1(0)$ . Hence by (3.23) we get

$$u(t) = u_{p_1(0)} = \frac{p_1(0)^{\frac{1}{\gamma-1}}}{(X(t) + Y(t)e^{\rho\delta}\beta)} e^{\frac{1}{\gamma-1}(\rho t - (\mu + e^{\rho\delta}\beta)t)},$$

for all  $t > 0$  and some  $p_1(0)$ . In analogy with Example 3.4 it is natural to conjecture that the optimal value,  $K$ , of  $p_1(0)$  is given by

$$K = \inf\{p_1(0) : X^{p_1(0)}(t) + Y^{p_1(0)}(t)e^{\lambda\delta}\beta > 0, \text{ for all } t > 0\},$$

see [9]. So, the optimal control is given by

$$u(t) = \frac{K^{\frac{1}{\gamma-1}}}{(X(t) + Y(t)e^{\rho\delta}\beta)} e^{\frac{1}{\gamma-1}(\rho t - (\mu + e^{\rho\delta}\beta)t)}.$$

From this we get that  $\lim p_1(t) = \lim p_2(t) = 0$ , so that we have

$$E[\lim \hat{p}_1(t)(X(t) - \hat{X}(t))] \geq 0,$$

and

$$E[\lim \hat{p}_2(t)(Y(t) - \hat{Y}(t))] \geq 0.$$

This tells us that  $\hat{u}$  is an optimal control.

## 4 A necessary maximum principle

In addition to the assumptions in the previous section, we now assume the following.

(A<sub>1</sub>) For all  $u \in \mathcal{A}_{\mathcal{E}}$  and all  $\beta \in \mathcal{A}_{\mathcal{E}}$  bounded, there exists  $\epsilon > 0$  such that

$$u + s\beta \in \mathcal{A}_{\mathcal{E}} \quad \text{for all } s \in (-\epsilon, \epsilon).$$

(A<sub>2</sub>) For all  $t_0, h$  and all bounded  $\mathcal{E}_{t_0}$ -mesurable random variables  $\alpha$ , the control process  $\beta(t)$  defined by

$$\beta(t) = \alpha 1_{[s, s+h]}(t) \tag{4.1}$$

belongs to  $\mathcal{A}_{\mathcal{E}}$ .

(A<sub>3</sub>) For all bounded  $\beta \in \mathcal{A}_{\mathcal{E}}$ , the derivative process

$$\xi(t) := \frac{d}{ds} X^{u+s\beta}(t) \big|_{s=0} \tag{4.2}$$

exists and belongs to  $L^2(\lambda \times P)$ .

It follows from (2.1) that

$$\begin{aligned} d\xi(t) = & \left\{ \frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) + \frac{\partial b}{\partial a}(t) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) dr + \frac{\partial b}{\partial u}(t)\beta(t) \right\} dt \\ & + \left\{ \frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial \sigma}{\partial a}(t) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) dr + \frac{\partial \sigma}{\partial u}(t)\beta(t) \right\} dB(t) \\ & + \int_{\mathbb{R}_0} \left\{ \frac{\partial \theta}{\partial x}(t, z)\xi(t) + \frac{\partial \theta}{\partial y}(t, z)\xi(t-\delta) + \frac{\partial \theta}{\partial a}(t, z) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) dr + \frac{\partial \theta}{\partial u}(t, z)\beta(t) \right\} \tilde{N}(dt, dz), \end{aligned}$$

where, for simplicity of notation, we define

$$\frac{\partial}{\partial x} b(t) := \frac{\partial}{\partial x} b(t, X(t), X(t-\delta), A(t), u(t)),$$

and used that

$$\frac{d}{ds} Y^{u+s\beta}(t) \big|_{s=0} = \frac{d}{ds} X^{u+s\beta}(t-\delta) \big|_{s=0} = \xi(t-\delta)$$

and



$$\begin{aligned}
\frac{d}{ds} A^{u+s\beta}(t) \big|_{s=0} &= \frac{d}{ds} \left( \int_{t-\delta}^t e^{-\rho(t-r)} X^{u+s\beta}(r) dr \right) \big|_{s=0} \\
&= \left( \int_{t-\delta}^t e^{-\rho(t-r)} \frac{d}{ds} X^{u+s\beta}(r) \right) \big|_{s=0} dt \\
&= \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) dr.
\end{aligned}$$

Note that

$$\xi(t) = 0 \text{ for } t \in [-\delta, \infty) .$$

**Theorem 4.1 (Necessary maximum principle)** *Suppose that  $\hat{u} \in \mathcal{A}_{\mathcal{E}}$  with corresponding solutions  $\hat{X}(t)$  of (2.1)-(2.2) and  $\hat{p}(t)$ ,  $\hat{q}(t)$ , and  $\hat{r}(t, z)$  of (3.2)-(3.3), and corresponding derivative process  $\hat{\xi}(t)$  given by (4.2).*

*Assume that for all  $u \in \mathcal{A}_{\mathcal{E}}$  the following hold:*

$$\begin{aligned}
&E \left[ \int_0^\infty \hat{p}^2(t) \left\{ \left( \frac{\partial \sigma}{\partial x} \right)^2(t) \hat{\xi}^2(t) + \left( \frac{\partial \sigma}{\partial y} \right)^2(t) \hat{\xi}^2(t-\delta) + \left( \frac{\partial \sigma}{\partial a} \right)^2(t) \left( \int_{t-\delta}^t e^{-\rho(t-r)} \hat{\xi}(r) dr \right)^2 + \left( \frac{\partial \sigma}{\partial u} \right)^2(t) \right. \right. \\
&+ \int_{\mathbb{R}_0} \left\{ \left( \frac{\partial \theta}{\partial x} \right)^2(t, z) \hat{\xi}^2(t) + \left( \frac{\partial \theta}{\partial y} \right)^2(t, z) \hat{\xi}^2(t-\delta) + \left( \frac{\partial \theta}{\partial a} \right)^2(t, z) \left( \int_{t-\delta}^t e^{-\rho(t-r)} \hat{\xi}(r) dr \right)^2 + \left( \frac{\partial \theta}{\partial u} \right)^2(t, z) \right\} \nu(dz) \\
&\left. \left. + \int_0^\infty \hat{\xi}^2(t) \left\{ \hat{q}(t) + \int_{\mathbb{R}_0} \hat{r}^2(t, z) \nu(dz) \right\} dt \right] < \infty.
\end{aligned}$$

and

$$E \left[ \overline{\lim_{t \rightarrow \infty}} \hat{p}(t) (X(t) - \hat{X}(t)) \right] \geq 0.$$

Then the following assertions are equivalent.

(i) For all bounded  $\beta \in \mathcal{A}_{\mathcal{E}}$ ,

$$\frac{d}{ds} J(\hat{u} + s\beta) \big|_{s=0} = 0.$$

(ii) For all  $t \in [0, \infty)$ ,

$$E \left[ H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot) \mid \mathcal{E}_t \right]_{u=\hat{u}(t)} = 0 \text{ a.s.}$$

**Proof.** Suppose that assertion (i) holds. Then

$$\begin{aligned}
0 &= \frac{d}{ds} J(\hat{u} + s\beta) \big|_{s=0} \\
&= \frac{d}{ds} E \left[ \int_0^\infty f(t, X^{\hat{u}+s\beta}(t), Y^{\hat{u}+s\beta}(t), A^{\hat{u}+s\beta}(t), \hat{u}(t) + s\beta(t)) dt \right] \bigg|_{s=0} \\
&= E \left[ \int_0^\infty \left\{ \frac{\partial f}{\partial x}(t) \xi(t) + \frac{\partial f}{\partial y}(t) \xi(t - \delta) + \frac{\partial f}{\partial a}(t) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) dr + \frac{\partial f}{\partial u}(t) \beta(t) \right\} dt \right]
\end{aligned}$$

We know by the definition of  $H$  that

$$\frac{\partial f}{\partial x}(t) = \frac{\partial H}{\partial x}(t) - \frac{\partial b}{\partial x}(t)p(t) - \frac{\partial \sigma}{\partial x}(t)q(t) - \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(t, z)r(t, z)\nu(dz)$$

and the same for  $\frac{\partial f}{\partial y}(t)$ ,  $\frac{\partial f}{\partial a}(t)$  and  $\frac{\partial f}{\partial u}(t)$ .

We have

$$E \left[ \overline{\lim}_{t \rightarrow \infty} \hat{p}(t)(X(t) - \hat{X}(t)) \right] \geq 0$$

So

$$E \left[ \overline{\lim}_{t \rightarrow \infty} (\hat{p}(t)X^{\hat{u}+s\beta}(t)) \right] \geq E \left[ \overline{\lim}_{t \rightarrow \infty} (\hat{p}(t)X^{\hat{u}}(t)) \right]$$

for all  $\beta \in \mathcal{A}_{\mathcal{E}}$  and all  $s \in (-\epsilon, \epsilon)$ .

Hence

$$\frac{d}{ds} \left[ E \left\{ \overline{\lim}_{t \rightarrow \infty} (\hat{p}(t)X^{\hat{u}+s\beta}(t)) \right\} \right] \bigg|_{s=0} = 0$$

If  $\frac{d}{ds} \left| \left( \overline{\lim}_{t \rightarrow \infty} \hat{p}(t)X^{\hat{u}+s\beta}(t) \right) \right| \big|_{s=0} < g(w)$ , where  $g(w)$  is some integrable function. From the uniform limits with uniform convergence of the derivative, we can interchange the derivative and integration, and get

$$\begin{aligned}
0 &= \frac{d}{ds} \left[ E \left\{ \overline{\lim}_{t \rightarrow \infty} (\hat{p}(t)X^{\hat{u}+s\beta}(t)) \right\} \right] \bigg|_{s=0} \\
&= E \left[ \frac{d}{ds} \left\{ \overline{\lim}_{t \rightarrow \infty} (\hat{p}(t)X^{\hat{u}+s\beta}(t)) \right\} \right] \bigg|_{s=0} \\
&= E \left[ \overline{\lim}_{t \rightarrow \infty} \left\{ \hat{p}(t) \frac{d}{ds} (X^{\hat{u}+s\beta}(t)) \right\} \right] \bigg|_{s=0} .
\end{aligned}$$

Applying the Itô formula to  $\hat{p}(t) \frac{d}{ds} (X^{\hat{u}+s\beta}(t))$  we obtain

$$\begin{aligned}
0 &= E \left[ \overline{\lim}_{T \rightarrow \infty} \left\{ \hat{p}(T) \frac{d}{ds} (X^{\hat{u}+s\beta}(T)) \mid_{s=0} \right\} \right] = E \left[ \overline{\lim}_{T \rightarrow \infty} \left\{ \hat{p}(T) \xi(T) \right\} \right] \\
&= E \left[ \int_0^\infty \hat{p}(t) \left\{ \frac{\partial b}{\partial x}(t) \xi(t) + \frac{\partial b}{\partial y}(t) \xi(t-\delta) + \frac{\partial b}{\partial a}(t) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) dr + \frac{\partial b}{\partial u}(t) \beta(t) \right\} dt \right. \\
&\quad + \int_0^\infty \xi(t) E(\mu(t) \mid \mathcal{F}_t) dt + \int_0^\infty q(t) \left\{ \frac{\partial \sigma}{\partial x}(t) \xi(t) + \frac{\partial \sigma}{\partial y}(t) \xi(t-\delta) + \frac{\partial \sigma}{\partial a}(t) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) dr + \frac{\partial \sigma}{\partial u}(t) \beta(t) \right\} dt \\
&\quad + \left. \int_0^\infty \int_{\mathbb{R}_0} r(t, z) \left\{ \frac{\partial \theta}{\partial x}(t, z) \xi(t) + \frac{\partial \theta}{\partial y}(t, z) \xi(t-\delta) + \frac{\partial \theta}{\partial a}(t, z) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) dr + \frac{\partial \theta}{\partial u}(t, z) \beta(t) \right\} \nu(dz) dt \right] \\
&= -\frac{d}{ds} J(\hat{u} + s\beta) \mid_{s=0} + E \left( \int_0^\infty \frac{\partial H}{\partial u}(t) \beta(t) dt \right).
\end{aligned}$$

Therefore

$$E \left( \int_0^\infty \frac{\partial H}{\partial u}(t) \beta(t) dt \right) = 0.$$

Use

$$\beta(t) = \alpha 1_{[s, s+h]}(t)$$

where  $\alpha(\omega)$  is bounded and  $\mathcal{E}_{t_0}$ -mesurable,  $s \geq t_0$  and get

$$E \left( \int_s^{s+h} \frac{\partial H}{\partial u}(s) ds \alpha \right) = 0$$

Differentiating with respect to  $h$  at 0, we have

$$E \left( \frac{\partial H}{\partial u}(s) \alpha \right) = 0$$

This holds for all  $s \geq t_0$  and all  $\alpha$ , we obtain that

$$E \left( \frac{\partial H}{\partial u}(t_0) \mid \mathcal{E}_{t_0} \right) = 0.$$

This proves that assertion (i) implies (ii).

To complete the proof, we need to prove the converse implication; which is obtained since every bounded  $\beta \in \mathcal{A}_{\mathcal{E}}$  can be approximated by linear combinations of controls  $\beta$  of the form (4.1). ■

## 5 Existence and uniqueness of the time-advanced BSDEs on infinite horizon

The main result in this section refer to the existence and uniqueness for (3.3) – (3.4) where the coefficients satisfy a Lipschitz condition.

We now study time-advanced backward stochastic differential equations driven by a Brownian motion  $B(t)$ , and a compensated Poisson random measure  $\tilde{N}(dt, d\zeta)$ .

Let  $B(t)$  be a Brownian motion and  $\tilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)dt$ , where  $\nu$  is the Lévy measure of the jump measure  $N(\cdot, \cdot)$ , be an independent compensated Poisson random measure on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t < \infty})$ .

Given a positive constant  $\delta$ , denote by  $D([0, \delta], \mathbb{R})$  the space of all càdlàg paths from  $[0, \delta]$  into  $\mathbb{R}$ . For a path  $X(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ ,  $X_t$  will denote the function defined by  $X_t(s) = X(t + s)$  for  $s \in [0, \delta]$ . Put  $\mathcal{H} = L^2(\nu)$ . Consider the  $L^2$  space  $V_1 := L^2([0, \delta] \rightarrow \mathbb{R}; ds)$  and  $V_2 := L^2([0, \delta] \rightarrow \mathcal{H}; ds)$ . Let

$$F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times V_1 \times \mathbb{R} \times \mathbb{R} \times V_1 \times \mathcal{H} \times \mathcal{H} \times V_2 \times \Omega \rightarrow \mathbb{R}$$

be a function satisfying the following Lipschitz condition: There exists a constant  $C$  such that

$$\begin{aligned} & |F(t, p_1, p_2, p, q_1, q_2, q, r_1, r_2, r, \omega) - F(t, \bar{p}_1, \bar{p}_2, \bar{p}, \bar{q}_1, \bar{q}_2, \bar{q}, \bar{r}_1, \bar{r}_2, \bar{r}, \omega)| \\ & \leq C(|p_1 - \bar{p}_1| + |p_2 - \bar{p}_2| + |p - \bar{p}|_{V_1} + |q_1 - \bar{q}_1| + |q_2 - \bar{q}_2| + |q - \bar{q}|_{V_1} \\ & + |r_1 - \bar{r}_1|_{\mathcal{H}} + |r_2 - \bar{r}_2|_{\mathcal{H}} + |r - \bar{r}|_{V_2}). \end{aligned} \quad (5.1)$$

Assume that  $(t, \omega) \rightarrow F(t, p_1, p_2, p, q_1, q_2, q, r_1, r_2, r, \omega)$  is predictable for all  $p_1, p_2, p, q_1, q_2, q, r_1, r_2, r$ . Further we assume the following:

$$E \int_0^\infty e^{\lambda t} |F(t, 0, 0, 0, 0, 0, 0, 0, 0, 0)|^2 dt < \infty,$$

for all  $\lambda \in \mathbb{R}$ . We now consider the following backward stochastic differential equation in the unknown  $\mathcal{F}_t$ -adapted processes  $(p(t), q(t), r(t, z)) \in H \times H \times \mathcal{H}$ :

$$\begin{aligned} dp(t) &= E[F(t, p(t), p(t + \delta, p_t, q(t + \delta), q_t, r(t + \delta), r_t) | \mathcal{F}_t] dt \\ &+ q(t) dB(t) + \int_{\mathbb{R}_0} r(t, z) \tilde{N}(dz, dt), \end{aligned} \quad (5.2)$$

where

$$E \left[ \int_0^\infty e^{\lambda t} |p(t)|^2 dt \right] < \infty, \quad (5.3)$$

for all  $\lambda \in \mathbb{R}$ .

**Theorem 5.1 (Existence and uniqueness)** *Assume the condition (5.1) is fulfilled. Then the backward stochastic partial differential equation (5.2) - (5.3) admits a unique solution  $(p(t), q(t), r(t, z))$  such that*

$$E \left[ \int_0^\infty e^{\lambda t} \{ |p(t)|^2 + |q(t)|^2 + \int_{\mathbb{R}_0} |r(t, z)|^2 \nu(dz) \} dt \right] < \infty,$$

for all  $\lambda \in \mathbb{R}$ .

**Proof.**

**Step 1:**

Assume  $F$  is independent of its second, third and fourth parameter.

Set  $q^0(t) := 0$ ,  $r^0(t, z) := 0$ . For  $n \geq 1$ , define  $(p^n(t), q^n(t), r^n(t, z))$  to be the unique solution of the following BSDE:

$$\begin{aligned} dp^n(t) = & E \left[ F(t, q^{n-1}(t), q^{n-1}(t + \delta), q_t^{n-1}, r^{n-1}(t, \cdot), r^{n-1}(t + \delta, \cdot), r_t^{n-1}(\cdot)) \mid \mathcal{F}_t \right] dt \\ & + q^n(t) dB(t) + \int_{\mathbb{R}_0} r^n(t, z) \tilde{N}(dt, dz); \end{aligned} \quad (5.4)$$

for  $t \in [0, \infty)$  such that

$$E \left[ \int_0^\infty e^{\lambda t} |p^n(t)|^2 dt \right] < \infty$$

This exists by Theorem 3.1 in [4].

Our goal is to show that  $(p^n(t), q^n(t), r^n(t, z))$  forms a Cauchy sequence.

By Itô's formula we get that

$$\begin{aligned} 0 = & |e^{\lambda t} p^{n+1}(t) - p^n(t)|^2 + \int_t^\infty \lambda e^{\lambda s} |p^{n+1}(s) - p^n(s)|^2 ds \\ & + \int_t^\infty e^{\lambda s} |q^{n+1}(s) - q^n(s)|^2 ds \\ & + \int_t^\infty e^{\lambda s} \int_{\mathbb{R}_0} |(r^{n+1}(s, z) - r^n(s, z))|^2 d\nu(dz) \\ & + 2 \int_t^\infty e^{\lambda s} \langle (p^{n+1}(s) - p^n(s)), E[F^n - F^{n-1} \mid \mathcal{F}_s] \rangle ds \\ & + 2 \int_t^\infty e^{\lambda s} (p^{n+1}(s) - p^n(s)) (q^{n+1}(s) - q^n(s)) dB_s \\ & + \int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s} (|r^{n+1}(s, z) - r^n(s, z)|^2 \\ & + 2(p^{n+1}(s^-) - p^n(s^-)) (r^{n+1}(s, z) - r^n(s, z))) \tilde{N}(ds, dz). \end{aligned}$$

Rearranging, using that for all  $\epsilon > 0$ ,  $ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$  we have by the Lipschitz

requirement (5.1)

$$\begin{aligned}
& E[e^{\lambda t} |p^{n+1}(t) - p^n(t)|^2] \\
& + \int_t^\infty \lambda e^{\lambda s} |p^{n+1}(s) - p^n(s)|^2 ds \\
& + E[\int_t^\infty e^{\lambda s} |q^{n+1}(s) - q^n(s)|^2 ds] \\
& + E[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s} |(r^{n+1}(s, z) - r^n(s, z))|^2 \nu(dz) ds] \\
& \leq C_\epsilon E[\int_t^\infty e^{\lambda s} |p^{n+1}(s) - p^n(s)|^2 ds] \\
& + \epsilon 6 E[\int_t^\infty e^{\lambda s} |q^n(s) - q^{n-1}(s)|^2 ds] \\
& + \epsilon 6 E[\int_t^\infty e^{\lambda s} |q^n(s + \delta) - q^{n-1}(s + \delta)|^2 ds] \\
& + \epsilon 6 E[\int_t^\infty e^{\lambda s} \int_s^{s+\delta} |q^n(u) - q^{n-1}(u)|^2 du ds] \\
& + \epsilon 6 E[\int_t^\infty e^{\lambda s} |r^n(s) - r^{n-1}(s)|_{\mathcal{H}}^2 ds] \\
& + \epsilon 6 E[\int_t^\infty e^{\lambda s} |r^n(s + \delta) - r^{n-1}(s + \delta)|_{\mathcal{H}}^2 ds] \\
& + \epsilon 6 E[\int_t^\infty e^{\lambda s} \int_s^{s+\delta} |r^n(u) - r^{n-1}(u)|_{\mathcal{H}}^2 du ds]
\end{aligned}$$

where  $C_\epsilon = \frac{C^2}{\epsilon}$  and we used the abbreviation

$$F^n(t) := F(t, q^n(t), q^n(t + \delta), q_t^n, r^n(t, \cdot), r^n(t + \delta, \cdot), r_t^n(\cdot)).$$

Note that

$$\begin{aligned}
& E[\int_t^\infty e^{\lambda s} |q^n(s + \delta) - q^{n-1}(s + \delta)|^2 ds] \\
& \leq e^{-\lambda \delta} E[\int_t^\infty e^{\lambda s} |q^n(s) - q^{n-1}(s)|^2 ds].
\end{aligned}$$

Using Fubini

$$\begin{aligned}
& E\left[\int_t^\infty \int_s^{s+\delta} e^{\lambda s} |q^n(u) - q^{n-1}(u)|^2 du ds\right] \\
& \leq E\left[\int_t^\infty \int_{u-\delta}^u e^{\lambda s} |q^n(u) - q^{n-1}(u)|^2 ds du\right] \\
& \leq \left(\frac{1}{\lambda}(1 - e^{-\lambda\delta})\right) E\left[\int_t^\infty e^{\lambda s} |q^n(s) - q^{n-1}(s)|^2 ds\right] \\
& \leq E\left[\int_t^\infty e^{\lambda s} |q^n(s) - q^{n-1}(s)|^2 ds\right].
\end{aligned}$$

Similiar for  $r^n - r^{n-1}$ . It now follows that

$$\begin{aligned}
& Ee^{\lambda t} |p^{n+1}(t) - p^n(t)|^2 \\
& + E\left[\int_t^\infty e^{\lambda s} |q^{n+1}(s) - q^n(s)|^2 ds\right] \\
& + E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s} |(r^{n+1}(s, z) - r^n(s, z))|^2 \nu(dz) ds\right] \\
& \leq (C_\epsilon - \lambda) E\left[\int_t^\infty e^{\lambda s} |p^{n+1}(s) - p^n(s)|^2 ds\right] \\
& + \epsilon 6(2 + e^{-\lambda\delta}) E\left[\int_t^\infty e^{\lambda s} |q^n(s) - q^{n-1}(s)|^2 ds\right] \\
& + \epsilon 6(2 + e^{-\lambda\delta}) E\left[\int_t^\infty e^{\lambda s} |r^n(s) - r^{n-1}(s)|_{\mathcal{H}}^2 ds\right]. \tag{5.5}
\end{aligned}$$

Choosing  $\epsilon = \frac{1}{12(2+e^{-\lambda\delta})}$  so that

$$\begin{aligned}
& Ee^{\lambda t} |p^{n+1}(t) - p^n(t)|^2 \\
& + E\left[\int_t^\infty e^{\lambda s} |q^{n+1}(s) - q^n(s)|^2 ds\right] \\
& + E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s} |(r^{n+1}(s, z) - r^n(s, z))|^2 \nu(dz) ds\right] \\
& \leq (C_\epsilon - \lambda) E\left[\int_t^\infty e^{\lambda s} |p^{n+1}(s) - p^n(s)|^2 ds\right] + \frac{1}{2} E\left[\int_t^\infty e^{\lambda s} |q^n(s) - q^{n-1}(s)|^2 ds\right] \\
& + \frac{1}{2} E\left[\int_t^\infty e^{\lambda s} |r^n(s) - r^{n-1}(s)|_{\mathcal{H}}^2 ds\right]
\end{aligned}$$

This implies that

$$\begin{aligned}
& -\frac{\partial}{\partial t}(e^{(C_\epsilon-\lambda)t}E[\int_0^\infty e^{\lambda s}|p^{n+1}(t)-p^n(t)|^2]) \\
& + e^{(C_\epsilon-\lambda)t}E[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2]ds \\
& + e^{(C_\epsilon-\lambda)t}E[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|(r^{n+1}(s,z)-r^n(s,z))|^2\nu(dz)ds] \\
& \leq \frac{1}{2}e^{(C_\epsilon-\lambda)t}E[\int_t^\infty e^{\lambda s}|q^n(s)-q^{n-1}(s)|^2ds \\
& + \frac{1}{2}e^{(C_\epsilon-\lambda)t}E\int_t^\infty e^{\lambda s}|r^n(s)-r^{n-1}(s)|_{\mathcal{H}}^2ds
\end{aligned}$$

Integrating the last inequality we get that

$$\begin{aligned}
& E[\int_0^\infty e^{\lambda s}|p^{n+1}(t)-p^n(t)|^2dt] \\
& + \int_0^\infty e^{(C_\epsilon-\lambda)t}E[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2ds] \\
& + \int_0^\infty e^{(C_\epsilon-\lambda)t}E[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|(r^{n+1}(s,z)-r^n(s,z))|^2\nu(dz)ds]dt \\
& \leq \frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E[\int_t^\infty e^{\lambda s}|q^n(s)-q^{n-1}(s)|^2dsdt \\
& + \frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\int_t^\infty e^{\lambda s}|r^n(s)-r^{n-1}(s)|_{\mathcal{H}}^2dsdt. \tag{5.6}
\end{aligned}$$

So that

$$\begin{aligned}
& \int_0^\infty e^{(C_\epsilon-\lambda)t}E[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2ds] \\
& + \int_0^\infty e^{(C_\epsilon-\lambda)t}E[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|(r^{n+1}(s,z)-r^n(s,z))|^2\nu(dz)ds]dt \\
& \leq \frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E[\int_t^\infty e^{\lambda s}|q^n(s)-q^{n-1}(s)|^2dsdt \\
& + \frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\int_t^\infty e^{\lambda s}|r^n(s)-r^{n-1}(s)|_{\mathcal{H}}^2dsdt
\end{aligned}$$



This gives that

$$\begin{aligned}
& \int_0^\infty e^{(C_\epsilon - \lambda)t} E \left[ \int_t^\infty e^{\lambda s} |q^{n+1}(s) - q^n(s)|^2 ds \right] \\
& + \int_0^\infty e^{(C_\epsilon - \lambda)t} E \left[ \int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s} |(r^{n+1}(s, z) - r^n(s, z))|^2 \nu(dz) ds \right] dt \\
& \leq \frac{1}{2^n} C_3,
\end{aligned}$$

if  $\lambda > \frac{C}{\epsilon}$ . It then follows from (5.6) that

$$E \left[ \int_0^\infty e^{\lambda t} |p^{n+1}(t) - p^n(t)|^2 dt \right] \leq \frac{1}{2^n} C_3.$$

From (5.5) and (5.6), we now get

$$\begin{aligned}
& E \left[ \int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s} |(r^{n+1}(s, z) - r^n(s, z))|^2 \nu(dz) ds \right] dt \\
& + E \left[ \int_t^\infty e^{\lambda s} |q^{n+1}(s) - q^n(s)|^2 ds \right] \leq \frac{1}{2^n} C_3 n C_\epsilon.
\end{aligned}$$

From this we conclude that there exist progressively measurable processes  $(p(t), q(t), r(t, z))$ , such that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} E[e^{\lambda t} |p^n(t) - p(t)|^2 dt] = 0, \\
& \lim_{n \rightarrow \infty} E \left[ \int_0^\infty e^{\lambda s} |p^n(t) - p(t)|^2 dt \right] = 0, \\
& \lim_{n \rightarrow \infty} E \left[ \int_0^\infty e^{\lambda s} |p^n(t) - p(t)|^2 dt \right] = 0, \\
& \lim_{n \rightarrow \infty} E \left[ \int_0^\infty e^{\lambda s} |p^n(t) - p(t)|^2 dt \right] = 0, \\
& \lim_{n \rightarrow \infty} E \left[ \int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s} |(r^n(s, z) - r(s, z))|^2 \nu(dz) ds \right] dt = 0.
\end{aligned}$$

Letting  $n \rightarrow \infty$  in (5.4) we see that  $(p(t), q(t), r(t, z))$  satisfies

$$\begin{aligned}
dp(t) &= E[F(t, q(t), q(t + \delta), q_t, r(t, \cdot), r(t + \delta, \cdot), r_t(\cdot)) \mid \mathcal{F}_t] dt \\
&+ q(t) dB(t) + \int_{\mathbb{R}_0} r(t, z) \tilde{N}(dt, dz),
\end{aligned}$$

for all  $t > 0$ .

**Step 2:**  
General F.

Let  $p^0(t) = 0$ . For  $n \geq 1$  define  $(p^n(t), q^n(t), r^n(t, z))$  to be the unique solution of the following BSDE:

$$\begin{aligned} dp^n(t) = & E[F(t, p^{n-1}(t), p^{n-1}(t + \delta), p_t^{n-1}, q^n(t), q^n(t + \delta), q_t^n, r^n(t), r^n(t + \delta), r_t^n) | \mathcal{F}_t] dt \\ & + q^n(t) dB(t) + \int_{\mathbb{R}_0} r^n(t, z) \tilde{N}(dz, dt), \end{aligned}$$

for  $t \in [0, \infty)$ . The existence of  $(p^n(t), q^n(t), r^n(t, z))$  was proved in Step 1.

By using the same arguments as above, we deduce that

$$\begin{aligned} & Ee^{\lambda t} |p^{n+1}(t) - p^n(t)|^2 \\ & + E\left[\int_t^\infty e^{\lambda s} |q^{n+1}(s) - q^n(s)|^2 ds\right] \\ & + E\left[\int_t^\infty e^{\lambda s} \int_{\mathbb{R}_0} |(r^{n+1}(s, z) - r^n(s, z))|^2 \nu(dz) ds\right] \\ & \leq (C_\epsilon - \lambda) E\left[\int_t^\infty e^{\lambda s} |p^{n+1}(s) - p^n(s)|^2 ds\right] + \frac{1}{2} E\left[\int_t^\infty e^{\lambda s} |p^n(s) - p^{n-1}(s)|^2 ds\right] \end{aligned}$$

This implies that

$$-\frac{d}{dt}(e^{(C_\epsilon - \lambda)t} E\left[\int_t^\infty e^{\lambda s} |p^{n+1}(s) - p^n(s)|^2 ds\right]) \leq \frac{1}{2} e^{(C_\epsilon - \lambda)t} E\left[\int_t^\infty e^{\lambda s} |p^n(s) - p^{n-1}(s)|^2 ds\right].$$

Integrating from 0 to  $\infty$ , we get

$$E\left[\int_0^\infty e^{\lambda s} |p^{n+1}(s) - p^n(s)|^2 ds\right] \leq \frac{1}{2} \int_0^\infty e^{(C_\epsilon - \lambda)t} E\left[\int_t^\infty e^{\lambda s} |p^n(s) - p^{n-1}(s)|^2 ds\right] dt$$

So if  $\lambda \geq C_\epsilon$  then by iteration we see that

$$E\left[\int_0^\infty e^{\lambda s} |p^{n+1}(s) - p^n(s)|^2 ds\right] \leq \frac{K}{2^n(\lambda - C_\epsilon)^n},$$

for some constant  $K$ .

### Uniqueness:

In order to prove the uniqueness, we assume that there are two solutions  $(p^1(t), q^1(s), r^1(s, z))$  and  $(p^2(t), q^2(s), r^2(s, z))$  of the ABSDE

$$\begin{aligned} dp(t) = & E[F(t, p(t), p(t + \delta), p_t, q(t), q(t + \delta), q_t, r(t), r(t + \delta), r_t) | \mathcal{F}_t] dt \\ & + q(t) dB(t) + \int_{\mathbb{R}_0} r(t, z) \tilde{N}(dt, dz); \quad t \in [0, \infty) \end{aligned}$$

$$E\left[\int_0^\infty e^{\lambda t} |p(t)|^2 dt\right] < \infty; \quad \lambda \in \mathbb{R}.$$

By Itô's formula, we have

$$\begin{aligned}
& E \left[ e^{\lambda t} |p^1(t) - p^2(t)|^2 \right] + E \left[ \int_t^\infty \lambda e^{\lambda s} |p^1(s) - p^2(s)| ds \right] \\
& + E \left[ \int_t^\infty e^{\lambda s} |q^1(s) - q^2(s)|^2 ds \right] + E \left[ \int_t^\infty e^{\lambda s} \int_{\mathbb{R}_0} |r^1(s, z) - r^2(s, z)|^2 ds \nu(dz) \right] \\
& = 2E \int_t^\infty e^{\lambda s} \left[ |p^1(s) - p^2(s)| \right. \\
& \times \left( E \left[ F(s, p^1(s), p^1(s+\delta), p_s^1, q^1(s), q^1(s+\delta), q_s^1, r^1(s), r^1(s+\delta), r_s^1) \mid \mathcal{F}_s \right] \right. \\
& \left. \left. - E \left[ F(s, p^2(s), p^2(s+\delta), p_s^2, q^2(s), q^2(s+\delta), q_s^2, r^2(s), r^2(s+\delta), r_s^2) \mid \mathcal{F}_s \right] \right] \right) ds \\
& \leq 2E \int_t^\infty e^{\lambda s} \left[ |p^1(s) - p^2(s)| \right. \\
& \times C \left( |p^1(s) - p^2(s)| + |p^1(s+\delta) - p^2(s+\delta)| + \int_s^{s+\delta} |p^1(u) - p^2(u)| du \right. \\
& + |q^1(s) - q^2(s)| + |q^1(s+\delta) - q^2(s+\delta)| + \int_s^{s+\delta} |q^1(u) - q^2(u)| du \\
& \left. \left. + |r^1(s) - r^2(s)|_{\mathcal{H}}^2 + |r^1(s+\delta) - r^2(s+\delta)|_{\mathcal{H}}^2 + \int_s^{s+\delta} |r^1(u) - r^2(u)|_{\mathcal{H}}^2 du \right) \right] ds
\end{aligned}$$

By the above inequalities for  $(p, q, r)$  and the fact that  $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$ , we have that

$$\begin{aligned}
& E \left[ e^{\lambda t} |p^1(t) - p^2(t)|^2 \right] + E \left[ \int_t^\infty e^{\lambda s} |q^1(s) - q^2(s)|^2 ds \right] \\
& + E \left[ \int_t^\infty e^{\lambda s} \int_{\mathbb{R}_0} |r^1(s, z) - r^2(s, z)|^2 ds \nu(dz) \right] \\
& \leq \left( \frac{3C^2}{\epsilon} - \lambda \right) E \left[ \int_t^\infty e^{\lambda s} |p^1(s) - p^2(s)|^2 ds \right] \\
& + (2 + e^{-\lambda \delta}) \epsilon E \left[ \int_t^\infty e^{\lambda s} |p^1(s) - p^2(s)|^2 ds \right] \\
& + (2 + e^{-\lambda \delta}) \epsilon E \int_t^\infty e^{\lambda s} \left[ |q^1(s) - q^2(s)|^2 + |r^1(s, z) - r^2(s, z)|_{\mathcal{H}}^2 \right] ds
\end{aligned}$$

Taking  $\epsilon$  such that  $(2 + e^{-\lambda \delta}) \epsilon = \frac{1}{2}$

$$\begin{aligned}
& E \left[ e^{\lambda t} |p^1(t) - p^2(t)|^2 \right] + E \left[ \int_t^\infty e^{\lambda s} |q^1(s) - q^2(s)|^2 ds \right] \\
& + E \left[ \int_t^\infty e^{\lambda s} \int_{\mathbb{R}_0} |r^1(s, z) - r^2(s, z)|^2 ds \nu(dz) \right] \\
& \leq \left( \frac{3C^2}{\epsilon} - \lambda + \frac{1}{2} \right) E \left[ \int_t^\infty e^{\lambda s} |p^1(s) - p^2(s)|^2 ds \right] \\
& + \frac{1}{2} E \left[ \int_t^\infty |q^1(s) - q^2(s)|^2 ds \right] \\
& + \frac{1}{2} E \left[ \int_t^\infty |r^1(s, z) - r^2(s, z)|^2_{\mathcal{H}} ds \right]
\end{aligned}$$

We get

$$\begin{aligned}
& E \left[ e^{\lambda t} |p^1(t) - p^2(t)|^2 \right] + \frac{1}{2} E \left[ e^{\lambda s} |q^1(s) - q^2(s)|^2 ds \right] \\
& + \frac{1}{2} E \left[ \int_t^\infty e^{\lambda s} \int_{\mathbb{R}_0} |r^1(s, z) - r^2(s, z)|^2 ds \nu(dz) \right] \\
& \leq \left( \frac{3C^2}{\epsilon} - \lambda + \frac{1}{2} \right) E \left[ \int_t^\infty e^{\lambda s} |p^1(s) - p^2(s)|^2 ds \right].
\end{aligned}$$

Using the fact that  $\lambda \geq \frac{3C^2}{\epsilon} + \frac{1}{2}$ , we obtain for all  $t \in [0, \infty)$ ,

$$E \left[ e^{\lambda t} |p^1(t) - p^2(t)|^2 \right] = 0,$$

which proves that  $p^1(t)$  and  $p^2(t)$  are indistinguishable. ■

**Acknowledgment 5.2** *We thank Salah-E.A. Mohammed for fruitful conversations.*

## References

- [1] G. Barles, R. Buckdahn, and E. Pardoux. Backward stochastic differential equations and integral-partial differential equations. *Stochastics and Stochastics Reports*, 60:57–83, 2009.
- [2] L. Chen and Z. Wu. Maximum principle for the stochastic optimal control problem with delay and application. *Automatica*, 46:1074–1080, 2010.
- [3] I. Elsanosi, B. Øksendal, and A. Sulem. Some solvable stochastic control problems with delay. *Stochastics and Stochastics Reports*, 71:69–89, 2000.
- [4] S. Haadem, B. Øksendal, and F. Proske. Maximum principles for jump diffusion processes with infinite horizon. *eprint arXiv:1206.1719*, 06, 2012.

- [5] H. Halkin. Necessary conditions for optimal control problems with infinite horizons. *Econometrica*, 42:267–272, 1974.
- [6] J. Li and S. Peng. Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton-Jacobi-Bellman equations. *Nonlinear Analysis*, 70:1779–1796, 2009.
- [7] B. Maslowski and P. Veverka. Infinite horizon maximum principle for the discounted control problem - incomplete version. *arXiv*, 2011.
- [8] S.-E.A. Mohammed. Stochastic differential systems with memory: Theory, examples and applications. In B. Øksendal L. Decreusefond, Jon Gjerde and A.S. Ustunel, editors, *Proceedings of The Sixth Workshop on Stochastic Analysis, Geilo, Norway, July 29-August 4, 1996, Stochastic Analysis and Related Topics VI. The Geilo Workshop*, pages 1–77. Progress in Probability, Birkhäuser, 1996.
- [9] S.-E.A. Mohammed and M.K.R. Scheutzow. Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales, part ii: Examples and case studies. *Ann. of Prob.*, 6:1210 – 1240, 1997.
- [10] B. Øksendal and A. Sulem. *A maximum principle for optimal control of stochastic systems with delay, with applications to finance: In J.M. Menaldi, E. Rofman and A. Sulem (editors): Optimal Control and Partial Differential Equations - Innovations and Applications*. IOS Press, Amsterdam, 2000.
- [11] B. Øksendal and A. Sulem. *Applied Stochastic Control of Jump Diffusions*. Springer, 2nd edition, 2007.
- [12] B. Øksendal, A. Sulem, and T. Zhang. Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations. *Adv. Appl. Prob.*, 43:572–596, 2011.
- [13] E. Pardoux. Bdses', weak convergence and homogenizations of semilinear pdes. In F.H. Clark and R.J. Stern, editors, *Nonlinear Analysis, Differential Equations and Control*, pages 503–549. Kluwer Academic, Dordrecht, 1999.
- [14] S. Peng and Y. Shi. Infinite horizon forward-backward stochastic differential equations. *Stoch. Proc. and their Appl.*, 85:75–92, 2000.
- [15] M. Royer. Backward stochastic differential equations with jumps and related non-linear expectations. *Stochastic Processes and Their Applications*, 116:1358–1376, 2006.
- [16] R. Situ. On solutions of backward stochastic differential equations with jumps and with non-Lipschitzian coefficients in Hilbert spaces and stochastic control. *Statistics and Probability Letters*, 60:279–288, 2002.
- [17] J. Yin. On solutions of a class of infinite horizon fbsdes. *Statistics and Probability Letters*, 78:2412–2419, 2008.