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Risk Indifference Pricing of Functional Claims of the Yield Surface in the Presence of Partial Information

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Abstract. In this paper we study the problem of risk indifference pricing of interest rate claims which are functionals of a bond yield surface under partial information. Our approach to solve this problem relies on a maximum principle for partial information control of stochastic differential games based on generalized bond portfolios. The latter method enables us to establish an explicit representation of the risk indifference price of such claims.

1 Introduction

In this paper we aim at analyzing the pricing (and hedging) of functional claims of the yield surface in the presence of partial information. To be more precise, we want to consider interest rate derivatives which are functions of the yield surface

$$((t,x) \mapsto R(t,t+x)), \tag{1}$$

where $R(t,T)$ denotes the interest rate at time t with time-to-maturity $x = T-t$. Here we assume that pricing of such claims is based on limited access to market information.

Examples - out of a vast variety of claims traded on fixed income or over-the-counter markets worldwide - are bond options, swaptions, floors or caps (see e.g. [20]). For example, a cap (or a caplet), which provides the

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holder with protection against rising interest rates, has the following payoff at time $T = t + x$:

$$Caplet_x(t) = N \cdot x \cdot \max(R(t, t+x) - K, 0), \tag{2}$$

where N is the notional amount and K the fixed cap rate.

Another type of a claim, which - in contrast to (2) - is a function of the whole yield surface (1) is the Asian option of a cap, with payoff given by

$$\frac{1}{(T_2 - T_1)(x_2 - x_1)} \int_{T_1}^{T_2} \int_{x_1}^{x_2} Caplet_x(t)\, dx\, dt. \tag{3}$$

We remark that due to its averaging property the latter claim exhibits the advantage of reducing the volatility risk inherent in the option.

Popular stochastic models for the dynamics of $R(t,T)$, $0 \leq t \leq T$ (T fixed), which can be found in the financial literature, are e.g. the Heath-Jarrow-Morton or the LIBOR model. See [16] or [25] and the references therein. Assuming full access to market information in such models, it is well known that replicating strategies with respect to bonds of a given maturity can be used to determine the fair price of the cap in (2). On the other hand, taking into account the existence of maturity-specific risk of bonds with different maturities, pricing of functional claims of the yield surface - such as the Asian option (3) - is in general impossible within the above mentioned models. A model that takes into account maturity-specific risk is e.g. the Musiela equation. See e.g. [5] or [10]. This model, which is based on a stochastic partial differential equation, describes the fluctuations of the entire yield surface. This approach leads to an infinite dimensional model, which has the attractive feature that hedging strategies of claims for generalized bond portfolios (i.e. portfolios of bonds of arbitrary maturities) are unique.

A deficiency of a bond market model based on the Musiela equation is that it is in general incomplete, even if there exists a unique martingale measure (see e.g. [5]). Thus, the determination of the arbitrage-free price of a claim based on exact replicating trading strategies is not always possible. Of course, if we in addition assume that the portfolio manager only has restricted access to market information, then pricing of both types of options (2), (3) converts into a pricing problem on incomplete markets.

One approach to option pricing on incomplete markets is e.g. utility indifference pricing. This method has been studied by many authors in literature from different points of view. See e.g. [18], where the authors consider a hedging problem under certain model constraints. Further, the

authors in [15] apply similar techniques to a stochastic volatility model. The work of [28] also deals with a financial application under incomplete information. See also [8], [6], [17], [24] and [23].

The utility indifference price of a claim is defined at a level which makes the issuer of the claim utility indifferent between the investment strategies of either selling the claim and entering the market with the collected initial payment, or entering the market without selling the contract. In contrast to that approach, in this paper we want to employ *risk indifference pricing* to address the problem of pricing (and hedging) of functional claims of the yield surface under incomplete market information. The latter pricing principle is related to utility indifference pricing but it is based on a risk measure instead of the utility function. For more information on risk measures the reader may consult [13] and the references therein. Regarding the topic of risk measure pricing we refer the reader to [29], [4] and [22].

The main result of our paper is a formula for the risk indifference price of an interest rate claim under partial information with respect to a certain class of risk measures. Our approach to deriving this formula rests on a stochastic maximum principle for differential games based on generalized bond portfolios, which are described by a stochastic evolution equation on a Hilbert space. This technique is inspired by [3], where the authors study a jump diffusion market modelled by an SDE. See also [2]. A paper related to the latter article is [27], which treats the case of Markovian controls in the framework of stochastic dynamic programming. Finally, we mention [7], where the authors analyze hedging of generalized bond portfolios in a Markovian setting by means of Hamilton-Jacobi-Bellman equations on Hilbert spaces.

Our paper is organized as follows: In Section 2 we introduce the mathematical tools we will use throughout the paper. Further, in Section 3 we give the precise statement of our pricing problem in the context of generalized bond portfolios. Sections 4 and 5 are devoted to establishing a stochastic maximum principle based on stochastic evolution equations, which is used in Section 6 to derive a formula for the risk indifference price of functional interest rate claims.

2 The general model

In this section we elaborate on some concepts essential for our further presentation. We begin by briefly recalling the classical Heath-Jarrow-Morton (HJM) framework for term structure modelling.

Let us denote by $P(t,T)$ the price at time t of a zero-coupon bond, that is a security that pays one unit of a given currency at maturity T. In the sequel the bond prices are modelled by non-negative adapted processes $\{P(t,T)\}_{0\le t\le T}$ for each $T>0$ on a filtered probability space

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P}), \tag{4}$$

where $\mathcal{F}_t$ is $\mathbb{P}$-completed and generated by independent one-dimensional Brownian motions $B_t^{(j)}$, $0\le t\le T$, $j=1,\dots,d$.

In the HJM model the bond prices $P(t,T)$ are modelled as

$$P(t,T) = exp\Big(-\int_t^T f(t,s)\,ds\Big), \tag{5}$$

where $f(t,T)$, $0\le t\le T<\infty$ are instantaneous forward rates described by the SDE

$$df(t,T) = \alpha(t,T)\,dt \;+\; \sum_{j=1}^{d}\sigma^{(j)}(t,T)dB_t^{(j)}, \tag{6}$$

where $\alpha(t,T)$, $\sigma^{(j)}(t,T)$, $0\le t\le T$ are predictable processes. In order to rule out arbitrage opportunities in this setting one has to impose the following restriction on the drift coefficient $\alpha(t,T)$ in (6):

$$\alpha(t,T) = \sum_{j=1}^{d}\sigma^{(j)}(t,T)\,\left(\int_t^T \sigma^{(j)}(t,s)ds + \lambda(t)\right), \tag{7}$$

where $\lambda(t)$ is a risk premium process.

A shortcoming of the HJM model is that the implied hedging strategies are not unique. This is a consequense of the finite dimensional character of the model, i.e. it assumes that the noise is driven by finitely many Brownian motions. This assumption leads to the situation that e.g. in the HJM model driven by 3 Brownian motions, an option writen on a 5-year bond can be hedged with bonds of maturities e.g. 20, 25 and 30 years - a rather unrealistic implication from the point of view of a fixed income trader.

One way to extend the HJM model is to incorporate the notion of a *maturity specific risk*. This is done by explicitly recognizing the infinite dimensional character of the term structure. The latter leads to the Musiela formulation of the HJM model, which is given by the following stochastic partial differential equation (SPDE).

$$df_t(x) = \left(\frac{d}{dx}f_t(x) + \alpha_t(x)\right)\,dt \;+\; \sum_{j=1}^{\infty}\sigma_t^{(j)}(x)dB_t^{(j)}, \tag{8}$$

where $B_t^{(j)}$, $j \geq 1$ are independent one-dimensional Brownian motions. Here we use the notation $f_t(x) := f(t, t+x)$ and $x := T-t$ is the time-to-maturity of the forward rate;$\alpha_t(x) := \alpha(t, t+x)$, $\sigma_t^{(j)}(x) := \sigma^{(j)}(t, t+x)$ for predictable processes $\sigma_t^{(j)}(T)$, $j \geq 1$, $0 \leq t \leq T$.

One can now look at the forward curve $x \mapsto f_t(x)$ as a single element of an appropriate function space $\mathcal{H}$. It is natural to require that this space has the property that the evaluation functionals

$$\delta_x : \mathcal{H} \to \mathbb{R},\ f \to f(x) \tag{9}$$

are continuous for all x. In addition we shall assume that the generator $A := \frac{d}{dx}$ in (8) has a strongly continuous semigroup S_t on $\mathcal{H}$. The semigroup S_t is the left shift operator given by

$$\big(S_t f\big)(x) = f(t+x) \tag{10}$$

An example of a suitable function space on which one can properly describe the evolution of forward curves is the Hilbert space of Sobolev type:

$$\mathcal{H} := \left\{ f \,:\, [0,\infty) \to \mathbb{R} \,:\, f \text{ is absolutely continuous and } \int_0^\infty \left(\frac{d}{dx} f(x)\right)^2 w(x)\, dx < \infty \right\} \tag{11}$$

with the scalar product given by

$$\langle f, g \rangle_{\mathcal{H}} := f(0) \cdot g(0) + \int_0^\infty \frac{d}{dx} f(x) \cdot \frac{d}{dx} g(x)\, w(x)\, dx \tag{12}$$

The function $w : [0,\infty) \to (0,\infty)$ is required to be increasing and to satisfy the following condition

$$\int_0^\infty \frac{1}{w(x)}\, dx < \infty \tag{13}$$

See e.g. [5] for details.

In what follows suppose that

$$\alpha_t(\cdot),\ \sigma_t^{(j)}(\cdot) \in \mathcal{H},\ \text{a.e.}, \quad \forall t \geq 0$$

Now we want to rewrite Equation (8) as a stochastic evolution equation on the Hilbert space $\mathcal{H}$. For that purpose consider a Q-Wiener process W_t, where Q is a symmetric non-negative operator on a separable Hilbert space U with $\text{Trace}(Q) < \infty$. Define the Hilbert space $U_0 = Q^{1/2}(U)$, with norm

$$\|h\|_0 := \|Q^{-1/2}(h)\|, \quad h \in U_0$$

Further, we shall denote by $L_2(U, \mathcal{H})$ the space of Hilbert-Schmidt operators from U to $\mathcal{H}$ with the norm $\|\cdot\|_{L_2}$. Let u_j, $j \geq 1$, be an orthonormal basis of U, and suppose that there exists a Borel-measurable function

$$\sigma\colon [0,T] \longrightarrow L(U_0, \mathcal{H})$$

such that

$$\sigma_t \left[Q^{1/2}(u_j) \right] = \sigma_t^{(j)}(\cdot)$$

and

$$\sigma_t \circ Q^{1/2} \in L_2(U, \mathcal{H})$$

for all t, j in Equation (8), where $\circ$ refers to the composition of mappings. Then $\left\{ B_t^{(k)} \right\}_{0 \leq t \leq T}$, $k \geq 1$, in Equation (8) can be regarded as a Wiener process B_t cylindrically defined on U, and Equation (8) can be recast as

$$\mathrm{d}f_t = \Big(Af_t + \alpha_t \Big)\, \mathrm{d}t + \sigma_t\, \mathrm{d}B_t \tag{14}$$

In the following we assume that there is a predictable unique mild solution

$$\big(t \longmapsto f_t(\cdot) \big) \in C([0,T]; \mathcal{H})$$

to the SPDE (14). As for sufficient criteria for the existence and uniqueness of mild, weak or even strong solutions of SPDE's we refer the reader to [21].

In order to rule out arbitrage opportunities with respect to our forward curve model (14) we shall require that the forward curves f_t satisfy the generalized HJM no-arbitrage condition:

$$\alpha_t(x) = \sum_{j \geq 1} \sigma_t^{(j)}(x) \big(J_x(\sigma_t^{(j)}) + \lambda_t^{(j)} \big), \tag{15}$$

where J_x is a continuous linear functional on $\mathcal{H}$ defined by

$$J_x(f) := \int_0^x f(u)\, du$$

and where the processes $\lambda_t^{(j)}$, $j \geq 1$ are the components of the $\mathcal{H}$-valued process

$$\lambda_t = \sum_{j \geq 1} \lambda_t^{(j)}\, v_j \tag{16}$$

Here v_j, $j \geq 1$ is an orthonormal basis of $\mathcal{H}$. The processes $\lambda_t^{(j)}$, $j \geq 1$ can be financially interpreted as risk premiums with respect to different times-to-maturity, that is these premiums entice investors to bear the volatility risk of bonds of different maturities.

3 The risk indifference price of an interest rate claim as a solution of a stochastic differential game

This Section explains the concept of risk indifference pricing. In simple words, this pricing technique relies on minimization of a chosen risk measure. We need to resort to this pricing method because of incompleteness of the infinite dimensional bond market that we are studying. Our approach involves reformulating the risk indifference pricing problem into a stochastic differential game and then using available mathematical tools to obtain a simplified pricing formula. The particular choice of a benchmark risk measure is unimportant. Instead, in our derivations we use a general representation formula for a convex risk measure. In accordance with that representation formula, we choose a risk measure that will enable us to obtain closed-form results.

We begin by describing the market and the problem faced by the investor. Assume that the filtration $\{\mathcal{F}_t\}$ in (4) is generated by the Wiener process B_t in (14). Define $P_t(x) := P(t, t+x)$ to be the bond price at time t with constant time to maturity x. Further, let $m : [0,\infty) \times \mathcal{H} \to \mathbb{R}$ and $g : \mathcal{H} \to \mathbb{R}$ be Borel measurable functions, where $\mathcal{H} \subseteq C([0,\infty))$ is a Hilbert space as in Section 2. Our objective is to price an option of the following form:

$$G_\tau := \int_0^\tau m\left(t, P_t(\cdot)\right)\, dt \,+\, g(P_\tau(\cdot)) \tag{17}$$

where τ is the time at which the option expires. All prices are measured in the units of the bank account, so we consider discounted quantities. We assume that there are the following investment possibilities:

- Bank account: $B_t^0 = 1$, $\forall t \in [0,\tau]$
- Bonds with date of maturity $T < \infty$, $P(t,T)$.

In the sequel let us assume that the conditions

$$\mathbb{E}\Big[exp\Big\{\int_0^t \langle \lambda_s, dB_s\rangle_0 - \frac{1}{2}\int_0^t \|\lambda_s\|_0^2\, ds\Big\}\Big] = 1 \tag{18}$$

and

$$\int_0^t \Big(\int_0^s \|\delta_{s-u} \circ \sigma_s\|_{L_0^2}^2\, du\Big)^{\frac{1}{2}} ds < \infty \tag{19}$$

hold for all $t \geq 0$, where $\|L\|_{L_0^2} := \|L \circ Q^{\frac{1}{2}}\|_{L_2}$ for each $L \in L_2(U_0, \mathcal{H})$. Then in our HJM framework one can show by Itô's formula and Girsanov's theorem that

$$P(t,T) \;=\; P(0,T) - \int_0^t P(s,T)\, J_{T-s} \circ \sigma_s \, d\tilde{B}_s, \tag{20}$$

where $\tilde{B}_t = B_t - \int_0^t \lambda_s \, ds$ is a Wiener process under a local martingale measure $\tilde{\mathbb{P}}$. Further, let us require that $\tilde{\sigma}$ given by

$$\tilde{\sigma}_t(\omega, x) := P_t(x)\, J_x \circ \sigma_t \tag{21}$$

is a predictable $L_2(U_0, \mathcal{H})$-valued process, such that $\int_0^T \|\tilde{\sigma}_s\|_{L_0^2}^2 \, ds < \infty$ a.e. Then the bond price curves P_t are $\mathcal{H}$-valued and fulfil

$$dP_t \;=\; AP_t dt - \tilde{\sigma}_t \, d\tilde{B}_t \tag{22}$$

or

$$dP_t \;=\; \Big(AP_t + \tilde{\sigma}_t(\lambda_t)\Big)dt - \tilde{\sigma}_t \, dB_t \tag{23}$$

in the mild sense, where as before $A = \frac{d}{dx}$.

Using our notation in Section 2, Equation (23) can be equivalently written as

$$\begin{aligned} dP_t(x) \;&=\; (AP_t(x) + P_t(x) \cdot b_t(x))\; dt \\ &-\; \sum_{j \geq 1} P_t(x)\, \delta_t^{(j)}(x)\, dB_t^{(j)}, \end{aligned} \tag{24}$$

where $\delta_t^{(j)}(x) := J_x(\sigma_t^{(j)})$ and $b_t(x) := \sum_{j \geq 1} J_x(\sigma_t^{(j)}) \lambda_t^{(j)}$.

In the sequel we assume (the rather strong condition) that there exists a unique strong solution $P_t \in \mathcal{H}$ to Equation (22). See [21] for sufficient criteria.

In this paper we aim at using *risk indifference pricing* to price options of the form (17) in the presence of partial information. We are now going to explain the idea behind this pricing concept, but first we introduce the concept of a convex risk measure. Let $\mathbb{F}$ be the space of all equivalence classes of real-valued random variables defined on Ω.

Definition 3.1. ([11], [14]) A *convex risk measure* $\rho : \mathbb{F} \to \mathbb{R} \cup \{\infty\}$ is a mapping satisfying the following properties, for $X, Y \in \mathbb{F}$,

(i) (convexity): $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \qquad \lambda \in (0,1);$

(ii) (monotonicity): If $X \leq Y$, then $\rho(X) \geq \rho(Y)$;

(iii) (translation invariance): $\rho(X+m) = \rho(X) - m, \qquad m \in \mathbb{R}.$

As its name suggests, a risk measure serves to evaluate the risk exposure associated with a certain financial asset or a project. The defining properties of the risk measure have concrete economic interpretations. Thus, the latter property in the above definition means that adding an amount of cash m to the portfolio reduces the portfolio's risk by the same amount, while the second property implies that a financial project Y, which generates higher profits than another project X, must have a lower risk measure. The first property, which is a relaxation of a stronger sub-additivity property, i.e. $\rho(X+Y) \leq \rho(X)+\rho(Y)$, that characterizes coherent risk measures, demonstrates the virtue of diversification. It can be illustrated as follows. The risk measure associated with e.g. financial operations of a bank must not exceed the sum of risk measures associated with the work of its individual departments. Had it been otherwise, it would have made more sense to split the bank and operate its departments as separate entities.

A popular example of a convex risk measure is the *Expected Shortfall*, which has the following interpretation. The expected shortfall at a q % confidence level is the expected loss of the portfolio in the worst $(1-q)$ % of the cases. This risk measure is computed according to the folrmula

$$ES_q(X) := \mathbb{E}\left[x | x < \mu\right], \tag{25}$$

where μ is the $(1-q)\,\%$-quantile of the distribution of X. Another risk measure routinely used in practice is *Value at Risk*. However, there is a lot of criticism against the use of this risk measure. In particular, it is not convex as it often violates the convexity requirement.

Coming back to our issue at hand, if an investor sells a liability to pay out the amount G_τ at the time moment τ and receives an initial payment p for such a contract, then the minimal risk involved for the seller is

$$\Phi_G(v+p) = \inf_{\varphi \in \mathcal{P}} \rho\left(V_\tau^{v+p}(\varphi) - G_\tau\right), \tag{26}$$

where $V_\tau^{v+p}(\varphi)$ denotes a replicating portfolio at the time moment τ under a self-financing strategy φ with initial wealth being equal to v, and $\mathcal{P}$ is the

set of self-financing strategies such that $V_t^v(\varphi) \geq c$, for some finite constant c and for $0 \leq t \leq \tau$.

If the investor does not issue a claim (and hence no initial payment p is received), then the minimal risk for the investor is

$$\Phi_0(v) = \inf_{\varphi \in \mathcal{P}} \rho(V_\tau^v(\varphi)). \tag{27}$$

We formulate the risk indifference pricing principle in the form of the following definition.

Definition 3.2. The seller's *risk indifference price*, $p = p_{risk}^{seller}$, of the claim G is the solution p of the equation:

$$\Phi_G(v + p) = \Phi_0(v). \tag{28}$$

Thus p_{risk}^{seller} is the initial payment p that makes an investor risk indifferent between selling the contract with liability payoff G and not selling the contract.

We are now going to recast the risk indifference pricing problem in the context of stochastic differential games. For that purpose we are going to need the following representation formula for a convex risk measure, suggested in [12].

Theorem 3.3. (Representation Theorem [12], [11], [14]) *A map $\rho : \mathbb{F} \to \mathbb{R}$ is a convex risk measure if and only if there exists a family $\mathcal{L}$ of measures $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_\tau$ and a convex "penalty" function $\zeta : \mathcal{L} \to (-\infty, +\infty)$ with $\inf_{\mathbb{Q} \in \mathcal{L}} \zeta(\mathbb{Q}) = 0$ such that*

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{L}} \{\mathbb{E}_{\mathbb{Q}}[-X] - \zeta(\mathbb{Q})\}, \qquad X \in \mathbb{F}. \tag{29}$$

This representation shows that every convex risk measure ρ is defined by the corresponding family of measures $\mathcal{L}$, and the penalty function ζ. Equalities (26) and (27) now look as follows:

$$\Phi_G(v + p) = \inf_{\varphi \in \mathcal{P}} \Big(\sup_{\mathbb{Q} \in \mathcal{L}} \{\mathbb{E}_{\mathbb{Q}}[-V_\tau^{v+p}(\varphi) + G_\tau] - \zeta(\mathbb{Q})\} \Big), \tag{30}$$

and

$$\Phi_0(v) = \inf_{\varphi \in \mathcal{P}} \Big(\sup_{\mathbb{Q} \in \mathcal{L}} \{\mathbb{E}_{\mathbb{Q}}[-V_\tau^{v}(\varphi)] - \zeta(\mathbb{Q})\} \Big), \tag{31}$$

for a given penalty function ζ and the family of measures $\mathcal{L}$.

In the case of (local) martingale measures $\mathbb{Q} \in \mathcal{L}$, these equalities can be seen as two stochastic differential games, in which Player 1 - the trader - wants to minimize his risk exposure by choosing an appropriate trading strategy φ; while Player 2 - the market - seeks to maximize the corresponding expectation defining the risk measure ρ, by choosing the optimal measure $\mathbb{Q}$. As we will show in the following sections, one can use the tools, such as the stochastic maximum principle, available in the field of stochastic differential games to simplify these problems in a way that will enable us to give a simplified formula for the risk indifference price of an interest rate claim.

4 Modelling framework

We consider the situation in which the investor is able to construct a replicating portfolio only by holding traditional bonds, i.e. bonds with fixed *dates of maturity*, $T \in (0, \infty)$. In such a situation, to replicate the payoff of an option written on bonds with constant *time to maturity* will in general require an infinite dimensional portfolio, i.e. the one containing infinitely many bonds with different dates of maturity. In order to better explain the construction of such an infinite dimensional portfolio we begin with a simple case. Suppose there are just 2 bonds with dates of maturity T_1 and T_2. Then the portfolio value will be given by:

$$V_t(\pi) := \pi_t^0 \cdot 1 + \pi_t^1 \cdot P(t, T_1) + \pi_t^2 \cdot P(t, T_2), \tag{32}$$

where π_t^0 is the number of units of the bank account held in the portfolio; and π_t^i, $i = 1, 2$ are the number of units of bonds with dates of maturity T_1 and T_2 correspondingly.

The dynamics of the portfolio value will look as follows:

$$\begin{aligned} dV_t(\pi) &:= \pi_t^1 \cdot dP(t, T_1) + \pi_t^2 \cdot dP(t, T_2) \qquad (33)\\ &= \pi_t^1 \cdot [P(t, T_1)\, b_t(T_1 - t)]\, dt - \\ &\qquad -\pi_t^1 \cdot \sum_{j \geq 1} P(t, T_1)\, \delta_t^{(j)}(T_1 - t)\, dB_t^{(j)} \\ &\qquad +\pi_t^2 \cdot [P(t, T_2)\, b_t(T_2 - t)]\, dt - \\ &\qquad -\pi_t^2 \cdot \sum_{j \geq 1} P(t, T_2)\, \delta_t^{(j)}(T_2 - t)\, dB_t^{(j)} = \\ &= \left[\pi_t^1 \cdot P(t, T_1) \cdot b_t(T_1 - t) + \pi_t^2 \cdot P(t, T_2) \cdot b_t(T_2 - t)\right] dt \\ &\qquad - \sum_{j \geq 1} \left[\pi_t^1 \cdot P(t, T_1)\, \delta_t^{(j)}(T_1 - t) + \pi_t^2 \cdot P(t, T_2)\, \delta_t^{(j)}(T_2 - t)\right] dB_t^{(j)} \end{aligned}$$

Consider an $\mathcal{H}^*$-valued process φ_t given by

$$\varphi_t := \beta_1 \cdot \delta_{T_1 - t} + \beta_2 \cdot \delta_{T_2 - t}, \tag{34}$$

where δ_x is the evaluation functional and $\beta_i(t) := \frac{\pi_t^i \cdot P(t,T_i)}{V_t(\pi)}$, if $V_t(\pi) \neq 0$, is a fraction of wealth invested in the bond with date of maturity T_i, $i = 1, 2$. Then equation (33) becomes

$$\begin{aligned} dV_t(\varphi) &= V_t(\varphi) \cdot \varphi_t(b_t(\cdot))\, dt \\ &\quad - V_t(\varphi) \cdot \sum_{j \geq 1} \varphi_t(\delta_t^{(j)}(\cdot))\, dB_t^{(j)} \end{aligned} \tag{35}$$

We can view the process φ_t in (35) as representing a generalized portfolio strategy, which can now be infinite dimensional.

In the sequel we say that an $\mathcal{H}^*$-valued process φ_t is a *self-financing strategy* if the risk-neutral evolution of the discounted portfolio value is given by

$$dV_t(\varphi) = -V_t(\varphi) \cdot \sum_{j \geq 1} \varphi_t(\delta^{(j)}(\cdot))\, d\tilde{B}_t^{(j)}, \tag{36}$$

where $\tilde{B}_t^{(j)} = B_t^{(j)} - \int_0^t \lambda_s^{(j)}\, ds$, $j \geq 1$ are Brownian motions under a martingale measure and $\lambda_t^{(j)}$, $j \geq 1$ are the risk premium processes.

Let $\mathcal{P}$ be the class of such self-financing strategies. In what follows we want to consider hedging strategies $\varphi \in \mathcal{P}$ of traders with limited access to market information, i.e. we assume that $\varphi \in \mathcal{P}$ is $\mathcal{E}_t$-predictable, where $\mathcal{E}_t \subseteq \mathcal{F}_t$. We shall also call a strategy $\varphi \in \mathcal{P}$ *admissible* if φ is $\mathcal{E}_t$-predictable, solves (36) in the strong sense and satisfies

$$\int_0^\tau \Big\{ |V_t(\varphi) \cdot \varphi_t(b_t(\cdot))| + \sum_{j \geq 1} V_t(\varphi)^2 \cdot \varphi_t(\delta_t^{(j)}(\cdot))^2 \Big\} dt < \infty.$$

The collection of such strategies is denoted by Π.

Let us consider the case of unrestricted access to market information. Then a market with respect to our model is referred to as *complete* if each contingent claim can be replicated. This means that for all square-integrable (non-negative) $\mathcal{F}_\tau$-measurable random variables h there exists an admissible strategy φ such that

$$V_\tau(\varphi) = h$$

An advantage of our generalized bond model (24) is that replicating strategies are unique (under certain conditions on $\tilde{\sigma}_t$ in (21)). See [5]. Furthermore, this model satisfies the intuitive requirement that bond maturities

used in the hedging strategies do correspond to those of the underlying of the claim. These natural properties, however, cannot be captured by finite-rank models, such as (6). In such models replicating hedging strategies are not unique in general and call options written on a 5-year bond can be hedged by e.g. a 30-year bond. This is a shortcoming that contradicts market practice.

On the other hand, a deficiency of our infinite-dimensional HJM framework is that the existence of the unique martingale measure does not in general imply the completeness of our bond market model. This is actually a property not exhibited by finite rank models. However, one can show that if the kernel of $\tilde{\sigma}_t$ in (21) is zero (t,ω)-a.e. then our bond market is *approximately* complete, that is for all contingent claims h and all $\epsilon > 0$ there is an admissible strategy ϕ^ϵ such that

$$\mathbb{E}_{\tilde{\mathbb{P}}}\Big[\Big(\mathbb{E}_{\tilde{\mathbb{P}}}(h) + \int_0^\tau \phi_t^\epsilon \circ \tilde{\sigma}_t \, d\tilde{B}_s - h\Big)^2\Big] < \epsilon$$

See [5].

Now we define the measures $\mathbb{Q}_q$ parametrized by given $\mathcal{E}_t$-predictable processes $q_t := \left\{q_t^{(j)}\right\}_{j\geq 1}$ such that

$$d\mathbb{Q}_q(\omega) := K_\tau \cdot d\mathbb{P}(\omega) \qquad \text{on } \mathcal{F}_\tau, \tag{37}$$

where $\mathbb{P}$ is the objective probability measure and $\mathbb{Q}_q$ is a measure absolutely continuous with respect to $\mathbb{P}$. The Radon-Nikodym derivative K_τ is defined as follows:

$$dK_t := \sum_{j\geq 1} K_t \, q_t^{(j)} \, dB_t^{(j)}, \qquad K_0 = k \tag{38}$$

We say that the control q is *admissible*, and write $q \in \Theta$, if $q_t^{(j)}$ is adapted to the sub-filtration $\mathcal{E}_t$ for all j, such that

$$\int_0^\tau \sum_{j\geq 1} \left(q_t^{(j)}\right)^2 dt < \infty$$

and

$$\mathbb{E}[K_\tau] = k > 0. \tag{39}$$

Further, we define $\mathcal{L}$ in Theorem 3.3 to be the class of measures given by

$$\mathcal{L} := \{\mathbb{Q}_q \,:\, q \in \Theta\} \tag{40}$$

Thus, the control process - denoted by u_t - in our stochastic control problems (30) and (31) consists of the processes $\left\{q_t^{(j)}\right\}_{j\geq 1}$ determining the risk measure, chosen by the market, and the portfolio strategy φ_t chosen by the investor:

$$u_t = \begin{bmatrix} \left\{q_t^{(j)}\right\}_{j\geq 1} \\ \varphi_t \end{bmatrix} \tag{41}$$

Our state process is given by

$$Y_t = \begin{bmatrix} K_t \\ P_t(\cdot) \\ V_t(\varphi) \end{bmatrix} := \begin{bmatrix} \tilde{Y}_t \\ V_t(\varphi) \end{bmatrix}, \qquad y := Y_0 = \begin{bmatrix} k \\ P_0(\cdot) \\ V_0(\varphi) \end{bmatrix} \tag{42}$$

Its dynamics is described by the following SPDE:

$$\begin{aligned} dY_t = & \begin{bmatrix} 0 \\ AP_t(\cdot) + P_t(\cdot)\cdot b_t(\cdot) \\ V_t(\varphi)\cdot\varphi_t(b_t(\cdot)) \end{bmatrix} dt + \\ & + \begin{bmatrix} K_t\, q_t^{(1)} & K_t\, q_t^{(2)} & \dots \\ -P_t(\cdot)\,\delta_t^{(1)}(\cdot) & -P_t(\cdot)\,\delta_t^{(2)}(\cdot) & \dots \\ -V_t(\varphi)\cdot\varphi_t(\delta_t^{(1)}(\cdot)) & -V_t(\varphi)\cdot\varphi_t(\delta_t^{(2)}(\cdot)) & \dots \end{bmatrix} \cdot \begin{bmatrix} dB_t^{(1)} \\ dB_t^{(2)} \\ \vdots \end{bmatrix} \end{aligned} \tag{43}$$

We now define another set $\mathcal{M}$ of measures as follows:

$$\mathcal{M} := \{\mathbb{Q}_q;\, q \in \mathbb{M}\}, \tag{44}$$

where

$$\mathbb{M} := \{q \in \Theta \,:\, \mathbb{E}[b_t(x) - \sum_{j\geq 1} \delta_t^{(j)}(x) q_t^{(j)} | \mathcal{E}_t] = 0,\ \forall t, x\}. \tag{45}$$

Thus, if $k = 1$ in (39) then the measures $\mathbb{Q}_q$ in $\mathcal{M}$ become equivalent martingale measures with respect to bond prices given by

$$\begin{aligned} d\overline{P}_t(x) &= \left(A\overline{P}_t(x) + \overline{P}_t(x)\mathbb{E}[b_t(x)|\mathcal{E}_t]\right) dt \\ &\quad + \overline{P}_t(x) \sum_{j\geq 1} \mathbb{E}[\delta_t^{(j)}(x)|\mathcal{E}_t]\, dB_t^{(j)} \end{aligned} \tag{46}$$

To complete the definition of our benchmark risk measure, as given in (29), we require that the penalty function ζ takes the form

$$\zeta(\mathbb{Q}_q) := \mathbb{E}_{\mathbb{P}}\left[\int_0^\tau \Lambda\left(t, q_t, \tilde{Y}_t\right)\, dt \,+\, h(\tilde{Y}_\tau)\right] \tag{47}$$

for some convex functions $\Lambda : [0,\infty)\times\mathcal{H}\times\mathbb{R}\times\mathcal{H}\to\mathbb{R}$ and $h : \mathbb{R}\times\mathcal{H}\to\mathbb{R}$, such that

$$\mathbb{E}\Big[\int_0^\tau |\Lambda\left(t, q_t, \tilde{Y}_t\right)|dt + |h(\tilde{Y}_\tau)|\Big] < \infty,$$

for all $q = (q_j)_{j\geq 1} \in \Theta$. Thus, the risk measure ρ, which we are going to use, is given in Equation (29) with $\mathcal{L}$ defined in (40) and $\zeta(\mathbb{Q})$ as given above, in Equation (47).

Now we formulate our stochastic differential game problem corresponding to equation (30), incorporating the form of the option payoff (17) and the representation formula (29) for our benchmark risk measure ρ.

Problem A: Determine $\Phi_G^{A,\mathcal{E}}(t,y)$ and $(q^*,\varphi^*) \in \Theta\times\Pi$, such that

$$\Phi_G^{A,\mathcal{E}}(t,y) = \inf_{\varphi\in\Pi}\Big(\sup_{q\in\Theta} J_A^{q,\varphi}(t,y)\Big) = J_A^{q^*,\varphi^*}(t,y), \tag{48}$$

where

$$\begin{aligned} J_A^{q,\varphi}(t,y) \;&:=\; \mathbb{E}_{\mathbb{P}}^y\left[\int_0^\tau -\Lambda\left(s, q_s, \tilde{Y}_s\right)\, ds \,-\, h(\tilde{Y}_\tau) \,+\, \int_0^\tau K_s\cdot m(s, P_s(\cdot))\, ds +\right. \\ &+\; \left. K_\tau\cdot g(P_\tau(\cdot)) - K_\tau\cdot V_\tau(\varphi)\right] \\ &=\; \mathbb{E}_{\mathbb{P}}^y\left[\int_t^\tau -\tilde{\Lambda}\left(s, q_s, \tilde{Y}_s\right)\, ds \,+\, \Psi(Y_\tau)\right], \end{aligned} \tag{49}$$

where the functions $\tilde{\Lambda} : [0,\infty)\times\mathcal{H}\times\mathbb{R}\times\mathcal{H}\to\mathbb{R}$ and $\Psi : \mathbb{R}\times\mathcal{H}\times\mathbb{R}\to\mathbb{R}$ are defined as

$$\Psi(K_t, P_t(\cdot), V_t(\varphi)) := -h(K_t, P_t(\cdot)) \,+\, K_t\cdot g(P_t(\cdot)) - K_t\cdot V_t(\varphi) \tag{50}$$

and

$$\tilde{\Lambda}\left(t, q_t, K_t, P_t(\cdot)\right) := \Lambda\left(t, q_t, K_t, P_t(\cdot)\right) - K_t\cdot m(t, P_t(\cdot)). \tag{51}$$

Here we assume that $\tilde{\Lambda} \in C_b^{1,1}\big([0,\infty)\times\mathcal{H}\times\tilde{\mathcal{H}}\big)$ for $\tilde{\mathcal{H}} := \mathbb{R}\times\mathcal{H}$, i.e. $\tilde{\Lambda}$ is continuously Fréchet differentiable w.r.t. $(t, q_t) \in (0,\infty)\times\mathcal{H}$ and

$(K_t, P_t(\cdot)) \in \tilde{\mathcal{H}}$, $\forall t$, with bounded partial derivatives, which have continuous extensions to $[0,\infty) \times \mathcal{H} \times \tilde{\mathcal{H}}$. Further, suppose that $\Psi \in C_b^1(X)$, where $X := \mathbb{R} \times \mathcal{H} \times \mathbb{R}$.

Later in this paper we want to exploit a certain connection between Problem A and the following stochastic control problem:

$$\Phi_G^{B,\mathcal{E}} = \sup_{\mathbb{Q}\in\mathcal{M}} \{\mathbb{E}_{\mathbb{Q}}[G_\tau] - \zeta(\mathbb{Q})\} \tag{52}$$

The latter will enable us to simplify the problem setting by removing one of the controls, namely the trading strategy φ. Using our notation for $\widetilde{Y}_t$, this new problem can be stated as follows:

Problem B: Search for $\Phi_G^{B,\mathcal{E}}(t,\tilde{y})$ and $\check{q} \in \mathbb{M}$, such that

$$\Phi_G^{B,\mathcal{E}}(t,\tilde{y}) = \sup_{q\in\mathbb{M}} J_B^q(t,\tilde{y}) = J_B^{\check{q}}(t,\tilde{y}), \tag{53}$$

where

$$\tilde{y}_t = \begin{bmatrix} k \\ P_0(\cdot) \end{bmatrix} \tag{54}$$

and

$$\begin{aligned} J_B^q(t,\tilde{y}) &:= \mathbb{E}_{\mathbb{P}}^{\tilde{y}}\left[\int_t^\tau -\Lambda\left(s,q_s,\tilde{Y}_s\right)\,ds \,-\, h(\tilde{Y}_\tau) \,+\, \int_t^\tau K_s \cdot m(s,P_s(\cdot))\,ds + \right. \\ &+ \left. K_\tau \cdot g(P_\tau(\cdot))\right] \\ &= \mathbb{E}_{\mathbb{P}}^{\tilde{y}}\left[\int_t^\tau -\tilde{\Lambda}\left(s,q_s,\tilde{Y}_s\right)\,ds \,+ \Phi(\tilde{Y}_\tau)\right], \end{aligned} \tag{55}$$

where the function $\Phi : \mathbb{R} \times \mathcal{H} \to \mathbb{R}$ is given by

$$\Phi(K_t, P_t(\cdot)) := -h(K_t, P_t(\cdot)) \,+\, K_t \cdot g(P_t(\cdot)). \tag{56}$$

We require here that $\Phi \in C_b^1(V)$, for $V := \mathbb{R} \times \mathcal{H}$.

As for Problem A, we aim at introducing the following *Hamiltonian* $H^A : [0,\infty) \times \mathbb{R} \times \mathcal{H} \times \mathbb{R} \times \mathcal{H} \times \mathcal{H}^* \times (\mathbb{R} \times \mathcal{H} \times \mathbb{R}) \times (\mathcal{H} \times L_2(U,\mathcal{H}) \times \mathcal{H}) \to \mathbb{R}$ given by

$$\begin{aligned} H^A(t,K_t,P_t(\cdot),V_t(\varphi),q_t,\varphi_t,\mathbf{p}^A,\mathbf{q}^A) &:= -\tilde{\Lambda}\left(t,q_t,K_t,P_t(\cdot)\right) + \langle (P_t\, b_t)(\cdot), p_2^A\rangle_K \\ &+ V_t(\varphi)\cdot\varphi(b_t(\cdot))\cdot p_3^A \\ &+ K_t \cdot \langle q_t, q_1^A\rangle_K \,-\, \sum_{j\geq 1}\langle (P_t\cdot\delta_t^{(j)}(\cdot), q_2^{A,(j)}\rangle_K \\ &- \sum_{j\geq 1} V_t(\varphi)\cdot\varphi(\delta_t^{(j)}(\cdot))\cdot q_3^{A,(j)}, \end{aligned} \tag{57}$$

where

$$\mathbf{p}^A = \begin{bmatrix} p_1^A \\ p_2^A \\ p_3^A \end{bmatrix}, \qquad \text{and} \qquad \mathbf{q}^A = \begin{bmatrix} q_1^A \\ q_2^A \\ q_3^A \end{bmatrix}, \tag{58}$$

with $q_i^A = \sum_{j\geq 1} q_i^{A,(j)} u_j$, $i = 1, 3$, $q_2^A = \{q_2^{A,(j)}\}_{j\geq 1}$ for an orthonormal basis u_j, $j \geq 1$ of $\mathcal{H}$.

On the other hand, we can define the Hamiltonian for Problem B as a map $H^B : [0,\infty) \times \mathbb{R} \times \mathcal{H} \times \mathcal{H} \times (\mathbb{R} \times \mathcal{H}) \times (\mathcal{H} \times L_2(U, \mathcal{H})) \to \mathbb{R}$ given by

$$\begin{aligned} H^B(t, K_t, P_t(\cdot), q_t, \mathbf{p}^B, \mathbf{q}^B) &:= -\tilde{\Lambda}\left(t, q_t, K_t, P_t(\cdot)\right) + \langle (P_t(\cdot)\, b_t)(\cdot), p_2^B \rangle_K \\ &+ K_t \cdot \langle q_t, q_1^B \rangle_K - \sum_{j\geq 1} \langle (P_t \cdot \delta_t^{(j)}(\cdot), q_2^{B,(j)} \rangle_K \end{aligned} \tag{59}$$

where

$$\mathbf{p}^B = \begin{bmatrix} p_1^B \\ p_2^B \end{bmatrix}, \qquad \text{and} \qquad \mathbf{q}^B = \begin{bmatrix} q_1^B \\ \{q_2^{B,(j)}\}_{j\geq 1} \end{bmatrix}, \tag{60}$$

Let us require that H^A and H^B are Fréchet differentiable with respect to $(K_t, P_t(\cdot), V_t(\varphi)) \in \mathbb{R} \times \mathcal{H} \times \mathbb{R}$ and $(K_t, P_t(\cdot)) \in \mathbb{R} \times \mathcal{H}$, respectively. In the sequel we denote by ∇g the gradient of a function $g : Z \to Z$ on a Hilbert space Z. We recall that $\nabla g : Z \to Z$ is a function characterized by the equation

$$\langle (\nabla g)(x),\, h \rangle_Z = (Dg)(x)(h), \tag{61}$$

for all $x, h \in Z$, where $(Dg)(x)(h)$ is the directional derivative at point x in the direction of h.

The adjoint equations with respect to H^A are given by the following backward stochastic (partial) differential equations:

$$\begin{cases} dp_1^A(t) &= \left[\nabla_{K_t} \tilde{\Lambda}\left(t, q_t, \tilde{Y}_t\right) - \langle q_t, q_1^A(t) \rangle_K\right] dt + \sum_{j\geq 1} q_1^{A,(j)}(t)\, dB_t^{(j)} \\ p_1^A(\tau) &= -\nabla_{K_t} h(\tilde{Y}_\tau) - g(P_\tau(\cdot)) - V_\tau(\varphi) \end{cases} \tag{62}$$

$$\begin{cases} dp_2^A(t,x) &= \Big[- \nabla_{P_t(\cdot)} F(t, q_t, \tilde{Y}_t, p_t^*, q_t^*) - A^* p_2^A(t,x) \Big]\, dt \\ & \quad + \sum_{j\geq 1} q_2^{A,(j)}(t,x)\, dB_t^{(j)} \\ p_2^A(\tau,x) &= -\nabla_{P_t(\cdot)} h(\tilde{Y}_\tau) + K_\tau \cdot \nabla_{P_t(\cdot)} g(P_\tau(\cdot)), \end{cases} \tag{63}$$

where A^* is the adjoint operator for the differential operator A in (22) and F is a function given by

$$F(t,q_t,K_t,P_t(\cdot),\mathbf{p}^A,\mathbf{q}^A) := -\tilde{\Lambda}\left(t,q_t,K_t,P_t(\cdot)\right)+\langle(P_t\cdot b_t)(\cdot),p_2^A\rangle_K-\sum_{j\geq 1}\langle(P_t\cdot\delta_t^{(j)})(\cdot),q_2^{A,(j)}\rangle_K \tag{64}$$

$$\begin{cases} dp_3^A(t) &= \Big[-\varphi_t(b_t(\cdot))\cdot p_3^A(t)+\sum_{j\geq 1}\varphi_t(\delta_t^{(j)}(\cdot))\cdot q_3^{A,(j)}(t)\Big]\,dt \\ & \quad +\sum_{j\geq 1} q_3^{A,(j)}(t)\,dB_t^{(j)} \\ p_3^A(\tau) &= -K_\tau \end{cases} \tag{65}$$

On the other hand, the adjoint equations with respect to the Hamiltonian H^B take the form

$$\begin{cases} dp_1^B(t) &= \Big[\nabla_{K_t}\tilde{\Lambda}\left(t,q_t,\tilde{Y}_t\right)-\langle q_t,q_1^B(t)\rangle_K\Big]\,dt+\sum_{j\geq 1} q_1^{B,(j)}(t)\cdot dB_t^{(j)} \\ p_1^B(\tau) &= -\nabla_{K_t}h(K_\tau,P_\tau(\cdot))+g(P_\tau(\cdot)) \end{cases} \tag{66}$$

$$\begin{cases} dp_2^B(t,x) &= \Big[-\nabla_{P(\cdot)}\tilde{F}(t,q_t,\tilde{Y}_t,\mathbf{p}_t^B,\mathbf{q}_t^B)-A^*p_2^B(t,x)\Big]\,dt \\ & \quad +\sum_{j\geq 1} q_2^{B,(j)}(t,x)\cdot dB_t^{(j)} \\ p_2^B(\tau,x) &= -\nabla_{P(\cdot)}h(\tilde{Y}_\tau)+K_\tau\cdot\nabla_{P(\cdot)}g(P_\tau(\cdot)), \end{cases} \tag{67}$$

where $\tilde{F}$ is a function defined by

$$\tilde{F}(t,q_t,K_t,P_t(\cdot),\mathbf{p}^B,\mathbf{q}^B) := -\tilde{\Lambda}\left(t,q_t,K_t,P_t(\cdot)\right)+\langle(P_t\cdot b_t)(\cdot),p_2^B\rangle_K-\sum_{j\geq 1}\langle(P_t\cdot\delta_t^{(j)})(\cdot),q_2^{B,(j)}\rangle_K \tag{68}$$

Regarding the conditions ensuring the existence and uniqueness of (strong) solutions of such B(S)PDEs the reader may consult e.g. [19], [26] and the references therein.

The next auxiliary result gives a link between the solutions of the adjoint equations (62), (63) and (65) for Problem A and (66) and (67) for Problem B, as well as the relation between Hamiltonians H^A and H^B in Problems A and B, respectively.

Lemma 4.1. *Choose $\forall q\in\Theta$ and $\forall\varphi\in\Pi$. If the chosen $q\in\mathbb{M}$, then the solutions of the adjoint equations for Problem A and Problem B are connected as follows:*

$$p_1^A(t) := p_1^B(t)-V_t(\varphi) \tag{69}$$

$$p_2^A(t,x) = p_2^B(t,x) \tag{70}$$

$$p_3^A(t) = -K_t \tag{71}$$

where $\mathbf{p}^B(t) = (p_1^B(t), p_2^B(t))$ *is a (strong) solution of the corresponding adjoint equations* (66) *and* (67) *for Problem B, and* $\mathbf{p}^A(t) = (p_1^A(t), p_2^A(t), p_3^A(t))$ *is a (strong) solution of the adjoint equations* (62), (63) *and* (65) *for Problem A. Moreover, the Hamiltonians in Problem A and Problem B are related to each other as follows:*

$$\begin{aligned} & H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A) = H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) \\ + \quad & K_t \cdot V_t(\varphi) \left[\varphi_t \left(2 \sum_{j \geq 1} q_t^{(j)} \cdot \delta_t^{(j)}(\cdot) - b_t(\cdot) \right) \right] \end{aligned} \tag{72}$$

Proof. Our proof closely follows the arguments in [3], Lemma 3.1, where the finite dimensional case was treated. Using the dynamics of $p_1^A(t)$, $p_1^B(t)$ and $V_t(\varphi)$ we find that

$$\begin{aligned} & dp_1^A(t) = dp_1^B(t) - dV_t(\varphi) \\ & = \left[\nabla_{K_t} \tilde{\Lambda} \left(t, q_t, \tilde{Y}_t \right) - \sum_{j \geq 1} q_t^{(j)} \cdot q_1^{B,(j)}(t) \right] dt + \sum_{j \geq 1} q_1^{B,(j)}(t)\, dB_t^{(j)} \\ & \quad - V_t(\varphi) \cdot \varphi_t(b_t(\cdot))\, dt + V_t(\varphi) \cdot \sum_{j \geq 1} \varphi_t(\delta_t^{(j)}(\cdot))\, dB_t^{(j)} \\ & = \left[\nabla_{K_t} \tilde{\Lambda} \left(t, q_t, \tilde{Y}_t \right) - \sum_{j \geq 1} q_t^{(j)} \cdot q_1^{B,(j)}(t) - V_t(\varphi) \cdot \varphi_t(b_t(\cdot)) \right] dt \\ & \quad + \sum_{j \geq 1} \left[q_1^{B,(j)}(t) + V_t(\varphi) \cdot \varphi_t(\delta_t^{(j)}(\cdot)) \right] dB_t^{(j)} \end{aligned} \tag{73}$$

So, it follows from (62) that

$$- \sum_{j \geq 1} q_t^{(j)} \cdot q_1^{A,(j)}(t) = - \sum_{j \geq 1} q_t^{(j)} \cdot q_1^{B,(j)}(t) - V_t(\varphi) \cdot \varphi_t(b_t(\cdot)), \tag{74}$$

and

$$q_1^{A,(j)}(t) = q_1^{B,(j)}(t) + V_t(\varphi) \cdot \varphi_t(\delta_t^{(j)}(\cdot)) \tag{75}$$

One can see that (74) holds, provided that $\varphi_t(\sum_{j \geq 1} \delta_t^{(j)}(\cdot)\, q_t^{(j)}) = \varphi_t(b_t(\cdot))$. Since the latter equality must be satisfied for every admissible strategy φ_t one concludes that $\sum_{j \geq 1} \delta_t^{(j)}(x)\, q_t^{(j)} = b_t(x)$, for all x, which also implies that $q \in \mathbb{M}$, as claimed.

Doing the same thing for equation (71) we observe that

$$-\varphi_t(b_t(\cdot)) \cdot p_3^A(t) + \sum_{j\geq 1} \varphi_t(\delta_t^{(j)}(\cdot)) \cdot q_3^{A,(j)}(t) = 0 \tag{76}$$

and

$$q_3^{A,(j)}(t) = -K_t\, q_t^{(j)}. \tag{77}$$

Substituting (77) into (76) we see that the latter is satisfied provided that $p_3^A(t) = -K_t$ and $\sum_{j\geq 1} \delta_t^{(j)}(x)\, q_t^{(j)} = b_t(x)$, for all x, as claimed.

Now, the Hamiltonian in Problem A and the one in Problem B are related to each other as follows:

$$\begin{aligned} H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A) &= H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^A, \mathbf{q}^A) + \\ &+ V_t(\varphi) \cdot \varphi_t(b_t(\cdot)) \cdot p_3^A(t) - \\ &- \sum_{j\geq 1} V_t(\varphi) \cdot \varphi_t(\delta_t^{(j)}(\cdot)) \cdot q_3^{A,(j)}(t) \end{aligned} \tag{78}$$

Using (69), (70), (71) and (75), as well as assuming that $q_t \in \mathbb{M}$, $\forall t \in [0, \tau)$, we obtain

$$\begin{aligned} &H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A) = H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) \\ &\quad + \sum_{j\geq 1} K_t\, q_t^{(j)} \cdot V_t(\varphi) \cdot \varphi_t(\delta_t^{(j)}(\cdot)) - V_t(\varphi) \cdot \varphi_t(b_t(\cdot)) \cdot K_t \\ &\quad + \sum_{j\geq 1} V_t(\varphi) \cdot \varphi_t(\delta_t^{(j)}(\cdot)) \cdot K_t\, q_t^{(j)} \\ &= H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) + K_t \cdot V_t(\varphi) \left[\varphi_t \left(2 \sum_{j\geq 1} q_t^{(j)} \cdot \delta_t^{(j)}(\cdot) - b_t(\cdot) \right) \right] \end{aligned} \tag{79}$$

□

Thus, Lemma 4.1 claims that the Hamiltonians, as well as the solutions to adjoint equations for Problems A and B are connected in the above stated way, provided that $q \in \mathbb{M}$. The following Lemma states the connection between Problems A and B working in the opposite direction. Namely, if Equations (69), (70) and (71) hold and certain optimum conditions are satisfied, then indeed $q \in \mathbb{M}$.

Lemma 4.2. *Suppose that $p_1^A(t)$, $p_2^A(t)$ and $p_3^A(t)$ are given by Equations (69), (70) and (71), with $\mathbf{p}^B(t) = (p_1^B(t), p_2^B(t))$ being a (strong) solution of*

the adjoint equations (66) *and* (67) *for Problem B, as in Lemma 4.1. Also, let the function*

$$q = \{q^{(j)}\}_{j\geq 1} \mapsto \mathbb{E}[H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)|\mathcal{E}_t], \qquad q \in \Theta,$$

have a maximum point at $\hat{q} = \{\hat{q}^{(j)}\}_{j\geq 1} = \{\hat{q}^{(j)}(\varphi)\}_{j\geq 1}$*, for all* $\varphi \in \Pi$*, and the function*

$$\varphi \mapsto \mathbb{E}[H^A(t, Y_t, \hat{q}_t(\varphi), \varphi_t, \mathbf{p}^A, \mathbf{q}^A)|\mathcal{E}_t], \qquad \varphi \in \Pi,$$

attain a minimum point at $\hat{\varphi} \in \Pi$*. Then,*

$$\hat{q}(\hat{\varphi}) \in \mathbb{M}. \tag{80}$$

Proof. In what follows we want to use the following notation: $q = \{q_j\}_{j\geq 1}$ and $\varphi = \{\varphi_i\}_{i\geq 1}$ if $q = \sum_{j\geq 1} q_j\, u_j$ and $\varphi = \sum_{j\geq 1} \varphi_i\, v_i$ for an orthonormal basis u_j and v_i in $\mathcal{H}$ and $\mathcal{H}^*$, respectively.

The assumption that the function $\mathbb{E}[H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)|\mathcal{E}_t]$ has a maximum at $\hat{q}^{(j)} = \hat{q}^{(j)}(\varphi)$ implies that

$$\mathbb{E}[\nabla_{q^{(j)}}(H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)_{q^{(j)}=\hat{q}^{(j)}(\varphi)}|\mathcal{E}_t] = 0, \qquad j \geq 1,\ \forall \varphi \in \Pi \tag{81}$$

Similarly, the necessary condition for the function $\mathbb{E}[H^A(t, Y_t, \hat{q}_t(\varphi), \varphi_t, \mathbf{p}^A, \mathbf{q}^A)|\mathcal{E}_t]$ to attain a minimum at $\hat{\varphi}$ is

$$\begin{aligned}\mathbb{E}\Big[\Big(\sum_{j\geq 1} \nabla_{q^{(j)}}(H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A) \cdot \nabla_{\varphi_i}\Big(\hat{q}^{(j)}(\varphi)\Big) \qquad (82)\\ + \nabla_{\varphi_i}\big(H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)\big)\Big)_{\substack{\varphi_i=\hat{\varphi}_i \\ q^{(j)}=\hat{q}^{(j)}(\hat{\varphi})}} \Big|\mathcal{E}_t\Big] = 0, \qquad i \geq 1,\end{aligned}$$

Choose $\varphi = \hat{\varphi}$. Then, by (81) and (82), we obtain

$$\mathbb{E}\Big[\nabla_{\varphi_i}\big(H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)\big)_{\substack{\varphi=\hat{\varphi} \\ q=\hat{q}(\hat{\varphi})}} \Big|\mathcal{E}_t\Big] = 0, \qquad i \geq 1 \tag{83}$$

Thus, after differentiating the Hamiltonian we obtain

$$\mathbb{E}\left[\left(\Big(V_t(\varphi)\cdot\nabla_{\varphi_i}\varphi(b_t(\cdot)\Big)\cdot p_3^A(t) - V_t(\varphi)\cdot\nabla_{\varphi_i}\Big(\sum_{j\geq 1}\varphi(\delta_t^{(j)}(\cdot)\Big)\cdot q_3^{A,(j)}(t)\Big)\right)_{\substack{\varphi=\hat{\varphi} \\ q=\hat{q}(\hat{\varphi})}} \Big|\mathcal{E}_t\right] = 0. \tag{84}$$

Combining this result with Lemma 4.1 yields

$$V_t(\varphi)\cdot K_t \cdot v_i\left(\mathbb{E}\left[\Big(b_t(\cdot) - \sum_{j\geq 1}\delta_t^{(j)}(\cdot)\cdot q_t^{(j)}\Big)\Big|\mathcal{E}_t\right]\right)_{\substack{\varphi=\hat{\varphi} \\ q=\hat{q}(\hat{\varphi})}} = 0, \qquad i \geq 1, \tag{85}$$

where we have used the fact that $v_i \in \mathcal{H}^*$. The condition that equality (85) holds for all $i \geq 1$ implies that for $\varphi = \hat{\varphi}$ and $q = \hat{q}(\hat{\varphi})$ $\mathbb{E}\Big[\Big(b_t(x) - \sum_{j\geq 1} \delta_t^{(j)}(x) \cdot q_t^{(j)}\Big)\Big|\mathcal{E}_t\Big] = 0$, for all $j \geq 1$ and any $x \in [0,\infty)$, i.e. if $\hat{q}(\hat{\varphi}) \in \mathbb{M}$, as claimed. □

5 Maximum principle for stochastic differential games on a generalized bond market

Analogues of Problem A were studied by a number of authors. See e.g. [2], [3] and [1]. Adapting their results to the present setting, we formulate the following result, which is an extension of Theorem 2.1 in [2].

Theorem 5.1. (Maximum principle for stochastic differential games [2, 9]) *For controls $(\hat{q}, \hat{\varphi}) \in \Theta \times \Pi$, suppose that the following partial information maximum principle holds*

$$\begin{aligned}
&\sup_{q\in\Theta} \mathbb{E}[H^A(t, Y_t, q_t, \hat{\varphi}_t, \hat{\mathbf{p}}^A, \hat{\mathbf{q}}^A)) \mid \mathcal{E}_t] \\
&= \mathbb{E}[H^A(t, Y_t, \hat{q}_t, \hat{\varphi}_t, \hat{\mathbf{p}}^A, \hat{\mathbf{q}}^A) \mid \mathcal{E}_t] \\
&= \inf_{\varphi\in\Pi} \mathbb{E}[H^A(t, Y_t, \hat{q}_t, \varphi_t, \hat{\mathbf{p}}^A, \hat{\mathbf{q}}^A) \mid \mathcal{E}_t]. \qquad (86)
\end{aligned}$$

for all $t \in [0,\tau]$, with $(\hat{\mathbf{p}}^A, \hat{\mathbf{q}}^A))$ being the strong solutions of the adjoint equations (62), (63) *and* (65) *in Problem A. Moreover, require that the function $q \mapsto J_A^{q,\varphi}(t,y)$ defined in* (49) *is concave, while $\varphi \mapsto J_A^{q,\varphi}(t,y)$ is convex. Then $(q^*, \varphi^*) := (\hat{q}, \hat{\varphi})$ is the optimal control and*

$$\begin{aligned}
\Phi_G^{A,\mathcal{E}}(t,y) &= \inf_{\varphi\in\Pi}\Big(\sup_{q\in\Theta} J_A^{q,\varphi}(t,y)\Big) = \sup_{q\in\Theta}\Big(\inf_{\varphi\in\Pi} J_A^{q,\varphi}(t,y)\Big) \\
&= \sup_{q\in\Theta} J_A^{q,\hat{\varphi}}(t,y) = \inf_{\varphi\in\Pi} J_A^{\hat{q},\varphi}(t,y) = J_A^{\hat{q},\hat{\varphi}}(t,y) \qquad (87)
\end{aligned}$$

We have come to the main theorem of the article. It provides the key result, which is used in the following Section to derive a formula for the risk indifference price of an interest rate claim. Its proof relies on the maximum principle stated above.

Theorem 5.2. *Let $p_1^B(t)$, $p_2^B(t,x)$ be strong solutions of the adjoint equations* (66) *and* (67) *of Problem B and $p_1^A(t)$, $p_2^A(t,x)$, $p_3^A(t)$ be defined*

by Equations (69), (70) *and* (71) *as in Lemma 4.1. Then, if the map* $q \mapsto H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B)$ *of Problem B is concave, then the optimal control* $\check{q}$ *for Problem B is the same as the corresponding optimal control* $\hat{q}(\hat{\varphi})$ *for Problem A, i.e.*

$$\check{q} = \hat{q}(\hat{\varphi}) \tag{88}$$

Proof. The proof is similar to that of Theorem 4.2 in [3]. Applying Theorem 5.1 to Problem B, one finds that $\check{q}$ is the optimal control, provided that

$$\begin{aligned} &\sup_{q \in \mathbb{M}} \mathbb{E}[H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) | \mathcal{E}_t] \\ &= \mathbb{E}[H^B(t, \tilde{Y}_t, \check{q}, \mathbf{p}^B, \mathbf{q}^B) | \mathcal{E}_t], \end{aligned} \tag{89}$$

The corresponding first order conditions for the constrained maximization problem (89) imply that

$$\mathbb{E}\left[\bigtriangledown_{q^{(j)}} \left(H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) + C_t \cdot \left(\sum_{j \geq 1} \delta_t^{(j)}(x)\, q^{(j)} - b_t(x) \right) \right)_{q=\check{q}} | \mathcal{E}_t \right] = 0, \tag{90}$$

for $\forall j \geq 1$ and $x \in [0, \infty)$, with C_t being the corresponding Lagrange multiplier. Moreover,

$$\mathbb{E}\left[\left(\sum_{j \geq 1} \delta_t^{(j)}(x)\, q^{(j)} - b_t(x) \right)_{q=\check{q}} | \mathcal{E}_t \right] = 0, \, \forall x \in [0, \infty) \tag{91}$$

On the other hand, let $\hat{\varphi}$, $\hat{q}(\hat{\varphi})$ be the optimal controls for Problem A. Then,

$$\mathbb{E}[\bigtriangledown_{q^{(j)}} (H^A(t, Y_t, q_t, \hat{\varphi}_t, \mathbf{p}^A, \mathbf{q}^A)_{q=\hat{q}(\hat{\varphi}(t))} | \mathcal{E}_t] = 0, \; j \geq 1 \tag{92}$$

and by Lemma 4.2, $\hat{q}(\hat{\varphi}) \in \mathbb{M}$. Hence, using equality (72) in Lemma 4.1 yields

$$\begin{aligned} &\mathbb{E}\Big[\bigtriangledown_{q^{(j)}} \Big\{ H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) + \\ &+ \quad K_t \cdot V_t(\varphi) \left(\varphi_t \left(2 \sum_{j \geq 1} q_t^{(j)} \cdot \delta_t^{(j)}(\cdot) - b_t(\cdot) \right) \right) \Big\}_{q=\hat{q}(\hat{\varphi}(t))} \Big| \mathcal{E}_t \Big] = 0, \end{aligned} \tag{93}$$

for all $j \geq 1$ and all $\varphi \in \Pi$. Then, for any fixed $x \in [0, \infty)$ we can rewrite (93) as follows

$$\mathbb{E}\Big[\bigtriangledown_{q^{(j)}} \Big\{ H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) + \tag{94}$$

$$+ \quad K_t \cdot V_t(\varphi) \cdot \beta_t(x) \cdot \left(2 \sum_{j \geq 1} q_t^{(j)} \cdot \delta_t^{(j)}(x) - b_t(x) \right) \Big\}_{q = \hat{q}(\hat{\varphi}(t))} \Big| \mathcal{E}_t \Big] = 0,$$

where $\beta_t(x)$ is a fraction of wealth invested in $P_t(x)$ at the time moment t.

Since neither $b_t(\cdot)$ nor any of the terms outside of the brackets depend on $q^{(j)}$, we see that equation (94) is the same as equation (90), with $C_t(\cdot) = 2\, K_t \cdot V_t(\varphi)\, \beta_t(x)$. Moreover, by Lemma 4.2 the optimal market control in Problem A corresponds to a martingale measure, i.e. $\hat{q}(\hat{\varphi}) \in \mathbb{M}$, which implies that

$$\mathbb{E}\left[\left(\sum_{j \geq 1} \delta_t^{(j)}(x)\, q^{(j)} - b_t(x) \right)_{q = \hat{q}(\hat{\varphi})} | \mathcal{E}_t \right] = 0, \; \forall x \in [0, \infty) \tag{95}$$

We immediately observe that the optimal control $\hat{q}(\hat{\varphi})$ for Problem A also satisfies the first order conditions (90) and (91) corresponding to Problem B. Hence, by the uniqueness of the solution, we conclude that $\check{q} = \hat{q}(\hat{\varphi})$, as claimed. □

6 Risk indifference pricing of claims of the yield curve

In this section we aim at establishing a relation between the value function in Problem A and that in Problem B. Theorem 5.2 provides the key result needed for this purpose. Let $(q^*, \varphi^*) = (\check{q}, \hat{\varphi})$ be the optimal controls for Problem A with $\check{q}$ being optimal for Problem B, as in Theorem 5.2. Also, denote by $\tilde{Y}^* = \tilde{Y}^{q^*}$ the state process corresponding to the optimal control

$\check{q}$. The value function $\Phi_G^{A,\mathcal{E}}$ of Problem A then becomes

$$\begin{aligned}\Phi_G^{A,\mathcal{E}}(t,y) &= \inf_{\varphi\in\Pi}\Big(\sup_{q\in\Theta} J^{q,\varphi}(t,y)\Big) \qquad (96)\\ &= \inf_{\varphi\in\Pi}\Big(\sup_{q\in\Theta}\mathbb{E}_{\mathbb{P}}^{y}\Big[\int_t^\tau -\tilde{\Lambda}\left(s,q_s,\tilde{Y}_s\right)\,ds\,-\,h(\tilde{Y}_\tau)+\\ &\qquad +\,K_\tau\cdot g(P_\tau(\cdot))-K_\tau\cdot V_\tau(\varphi)]\Big)\\ &= \inf_{\varphi\in\Pi}\Big(\mathbb{E}_{\mathbb{P}}^{y}\Big[\int_t^\tau -\tilde{\Lambda}\left(s,q_s^*,\varphi_s,\tilde{Y}_s^*\right)\,ds\,-\,h(\tilde{Y}_\tau^*)+\\ &\qquad +\,K_\tau^*\cdot g(P_\tau(\cdot))-K_\tau^*\cdot V_\tau(\varphi)]\Big)\end{aligned}$$

Since the first part of equation (96) does not depend on the parameter φ, it can be rewritten as follows

$$\begin{aligned}\Phi_G^{A,\mathcal{E}}(t,y) &= \mathbb{E}_{\mathbb{P}}^{y}\Big[\int_t^\tau -\tilde{\Lambda}\left(s,q_s^*,\varphi_s,\tilde{Y}_s^*\right)\,ds\,-\,h(\tilde{Y}_\tau^*)+\\ &\qquad +\,K_\tau^*\cdot g(P_\tau(\cdot))]-\inf_{\varphi\in\Pi}\Big(\mathbb{E}_{\mathbb{P}}^{y}\left[K_\tau^*\cdot V_\tau(\varphi)\right]\Big) \qquad (97)\end{aligned}$$

Also, by the original assumption, φ^* is optimal for Problem A and by Theorem 5.2, $\check{q}=q^*$ is optimal for Problem B. Hence, by the formulation of Problem B $\check{q}\in\mathbb{M}$ and $\mathbb{Q}_{q^*}$ defined by the Radon-Nikodym derivative K_τ^* is a martingale measure. Therefore, $\mathbb{E}_{\mathbb{P}}^{y}[K_\tau^*\cdot V_\tau(\varphi)]=k\cdot V_0$, for all $\varphi\in\Pi$, and the previous expression becomes

$$\begin{aligned}\Phi_G^{A,\mathcal{E}}(t,y) &= \mathbb{E}_{\mathbb{P}}^{y}\Big[\int_t^\tau -\tilde{\Lambda}\left(s,q_s^*,\varphi_s,\tilde{Y}_s^*\right)\,ds\,-\,h(\tilde{Y}_\tau^*)+\\ &\qquad +\,K_\tau^*\cdot g(P_\tau(\cdot))]-k\cdot V_0\\ &= \sup_{q\in\mathbb{M}} J_B^q(t,\tilde{y})-k\cdot V_0\\ &= \tilde{\Phi}_G^{B,\mathcal{E}}(t,\tilde{y})-k\cdot V_0, \qquad (98)\end{aligned}$$

where we once again used the claim of Theorem 5.2. This result is analogous to the one stated in [1].

Coming back to our original problem, we want to find the risk indifference price $p=p_{risk}^{seller}$ of an interest rate claim, which is determined by the Equation (28):

$$\Phi_G^{A\mathcal{E}}(V_0+p)=\Phi_0^{A\mathcal{E}}(V_0). \qquad (99)$$

By the result in Equation (98), one can immediately see that the equality (28) becomes

$$\Phi_G^{B\mathcal{E}}(t,\tilde{y}) - k\cdot(V_0+p) = \Phi_0^{B\mathcal{E}}(t,\tilde{y}) - k\cdot V_0, \tag{100}$$

which implies that the risk indifference price is given by

$$p = p_{risk}^{seller} = k^{-1}\cdot\Big(\Phi_G^{B\mathcal{E}}(t,\tilde{y}) - \Phi_0^{B\mathcal{E}}(t,\tilde{y})\Big) \tag{101}$$

The latter expression provides the main result of this paper. For $k = 1$, we obtain the following representation for the risk indifference price of functional claims of the yield curve under partial information, which is similar to the one derived in [3]. We formulate it in the form of a theorem.

Theorem 6.1. (Risk indifference price of functional claims of the yield curve under partial information) *Given that the conditions of Theorem 5.2 hold, the risk indifference price $p_{risk}^{seller}(G_\tau,\mathcal{E})$ for the seller of an interest rate claim G_τ is given by*

$$p_{risk}^{seller}(G_\tau,\mathcal{E}) = \sup_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}_{\mathbb{Q}}^{\tilde{y}}[G_\tau] - \zeta(\mathbb{Q})\} - \sup_{\mathbb{Q}\in\mathcal{M}}\{-\zeta(\mathbb{Q})\}. \tag{102}$$

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