The Topos-theoretical Approach to Quantum Physics

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Preface

The aim of this thesis is to apply concepts and methods from category theory and, in particular, topos theory within quantum physics. This research program, called *topos physics* by some, has come to fruition over the last ten years. The original contributions were made by Chris Isham, Jeremy Butterfield and Andreas Döring. Later, Chris Heuen, Bas Spitters, Nicolaas P. Landsman and Sander Wolters have developed an alternative version of the theory in which the topos-theoretical foundations of the approach have been further clarified. For these authors, topos theory (the theory of generalized universes of sets and generalizes spaces) is a tool with which quantum physics may be constructed by "gluing together" classical perspectives or "snapshots". Thus, in words which would please the classicist, topos physics builds the "modern" quantum world from fragments of the old. So far, the theory has led to a new, intuitionistic interpretation of quantum logic. It is also thought that topos physics may be a useful tool for building models of quantum gravity.

In the present thesis, we review the basic method of the topos-theoretical approach to quantum physics: the construction of a state space by means of (covariant or contravariant) functors on a category of commutative operator algebras. We then study how a certain theory of quantum gravity, so-called loop quantum gravity (LQG), may be interpreted within topos physics by appeal to the Weyl algebra version of LQG due to Christian Fleischhack. We investigate the topological properties of the state space of LQG within the topos model and show how to interpret the gauge and diffeomorphism invariance requirements within the theory. Finally, using a little-known technique for quantising on general structures (due to Isham), we extend the apparatus of topos physics to a larger category-theoretical framework. In order to achieve this, we define a measure theory on categories and study the basic entities, the arrow fields on a category, in the category of their representations. It is suggested how this may be applied to the theory of causal sets, and, more radically, to a theory of quantised logic.

Full descriptions of the contents of the thesis are given in the introduction to the each chapter. Beware, however, of the amount of theory necessary in order to get this subject off the ground. The fields under consideration, topos theory and quantum physics, normally target very different audiences, so we have tried to give compact introductions to both subjects (completeness is out of the question). The first section of chapter 1 addresses the problems of quantum physics from a logical point of view, and may be read in isolation from the rest. The remaining sections of the chapter are an attempt to fill in some of the details which are needed in order to appreciate the physical motivation behind the constructions to follow. In chapter 2, we present the fundamentals of category theory and topos theory as a prelude to the review of the concepts and methods of topos physics. Both schools, the "neo-realism" of Isham and the "Bohrification" approach of the Dutchmen, are given in some detail, and, in chapter 3, the latter scheme is applied to the theory of LQG. This chapter therefore contains a brief summary of the main results of LQG. Likewise, in chapter 4 we state the essentials of Isham's quantisation on categories, the notion of an arrow field. As a general rule, proofs which may be found elsewhere, are not included.

Thus, an important part of this long work is taken up with the arrangement and preparation for use of existing theories: original work is only to be found in the sections 3.2 and 4.2–4.7. I have tried to give complete credits and references wherever possible, but in chapter 1 and section 2.1, where the common lore of the topics, quantum physics and topoi, is dealt with, only a few suggestions for reading are given. Parts of the story can be found, for quantum physics, in chapter 1 of Weinberg (1995) and, for topos theory, in the prologue to Mac Lane and Moerdijk (1992). No summary can give full justice to the masterful contributions of Chris Isham. Also, the present thesis can only hint, in the manner of a snapshot, at the range and difficulty of the themes involved. This should be remedied: in foundational work, particularly in quantum gravity, one-sidedness is not a long-term option. There is little space (nor, indeed, space-time) for more advanced applications of the theory. These must be a subject for further study.

The present thesis brings together results from arenas as diverse as quantum gravity and topos theory. This makes for difficult reading. Part of the difficulty resides in the subjects themselves, another part is due to the clumsiness of the eager expositor. I would like to thank my thesis advisor, Dag Normann at the Department of mathematics, for his helpful assistance during the completion of this work. Also, I owe thanks to him and Herman Ruge Jervell at the Department of informatics for allowing me to present parts of the thesis in the friendly atmosphere of their logic seminar. I also thank my parents for their kindness and support. Finally, I take the opportunity to warn my great friends Rita and Cornelia against sleeping dragons and witchcraft.

CHAPTER 1

Quantisation, Space and Gravity

In this chapter, we shall outline a fairly standard description of physical theory, based on genuinely physical, but initially vague, notions such as *state, space, observable* and *probability.* Although the assumptions we make may seem incontrovertible, the plausibility we ascribe to them should be based on the strength of the empirical support of the formal theory, not on the almost tautological character of our first probings. Also, when the step to quantum physics is taken, plausibility is soon replaced by non-arbitrariness and mere correspondences, often bold, with classical physics. As we broaden our description, so as to include the possible formulations of a quantum theory of gravity, even analogy and the negative notion of lack of arbitrariness will fail us. This may be discouraging. But along the way there will be directions not taken, generalizations only hinted at. These will be our cues for the constructions undertaken in chapter 3 and 4 of this treatise. Our main tool, the theory of categories and topoi, will be the subject of the next chapter.

Section 1.1, which is a kind of *pons asinorum* to the description of physical systems for logicians, will give the reader sufficient background to follow the presentation of topos physics in chapter 2. However, in order to gain some understanding of the motivation behind the models there, the remaining sections of the present chapter should also be read. Section 1.2 focuses on the standard probabilistic description of quantum physics (as found e.g. in Mackey (1963), or, for the algebraic viewpoint, the first two chapters of Araki (1999)). Our reconstruction of physical theory is piecemeal, but some steps differ from the rest by their steepness or importance. In particular, this applies to the extension from a finite formalism to an infinite one, the transition from a deterministic to a probabilistic theory, the quantisation of classical theories, and the

choice between discrete and continuum concepts. In section 1.3 we recapitulate the mathematical theory of quantisation in some detail. We discuss the sense of this procedure outside the standard context (of infinite dimensional Hilbert spaces), and ask how quantum structure may arise from a more fundamental theory. In the next section (1.4), we turn to the non-standard approaches to space and time. We consider three cases: the possibility of a discrete notion of space or space-time, the generalized spaces of noncommutative geometry, and synthetic differential geometry, the study of smoothness without classical logic and ZF set theory. Finally (sec. 1.5), the major problems confronting a consistent quantum theory of gravity are outlined, and we try to show how the different approaches to a full theory derive from choices made at the critical junctions encountered in the preceding sections.

1.1. The Logic of Physical Systems

1.1.1. The Logic of Classical Mechanics

In this section, we give a quick introduction to the description of physical systems, both classical and quantum-mechanical. We unify the presentation by choosing the logician's point of view. In particular, we ask if the logic governing these descriptions is classical, or if a more unusual framework should be preferred, perhaps a non-distributive logic *à la* Birkhoff - von Neumann. The discussion will be simplified, but a more detailed review of the conceptual problems facing a complete theory of quantum physics can be found in the remaining sections (1.2-1.5).

We model physical systems (classical or quantum-mechanical) by making three fundamental assumptions:

* There is a set $\mathcal{S}$ of *states* for the system to be in;
* There is a set $\mathcal{O}$ of *observables* associated with the system;
* There is a value set $\mathcal{R}$ which contains the possible results of a *measurement* of an observable for a system in a given state.

If we like, we may regard the observables as classes of measuring apparatuses (Araki (1999)). Two apparatuses belong to the same class if they record the same measurement, regardless of the state of the system. The other way round, we say that two states are equal if there exists no measuring apparatus (and, hence, no observable) to distinguish between them. The availability of a choice between states and observables - which is the most fundamental? - is an important starting point for *algebraic quantum field theory*, and it is also useful for the setup of observational contexts in topos physics (chapter 2). Notice that we could have introduced *probability* as a fourth fundamental notion. This is certainly necessary when we are dealing with a system for which our information is incomplete, or if measurements performed on the system in a given state do not give the same result each time. But for the moment we shall consider only small systems (that is, systems with a low number of particles) from classical mechanics, a deterministic theory.

It is easy to identify candidates for the sets $\mathcal{S}$, $\mathcal{O}$ and $\mathcal{R}$ in newtonian physics. Assume, for simplicity, that our system consists of only one particle (or point mass) moving in a one-dimensional space. This space will be called the *configuration space*, and we identify it with $\mathbb{R}$, the set of real numbers. We introduce the notation

q - the position of the particle in space;
p - the momentum of the particle.

The product space $\mathbb{R}^2$, which consists of all possible pairs (q, p), is the collection of all possible states of the system. This will be called the *phase space*. (For a larger system, say n particles moving in three-dimensional space, the phase space is $\mathbb{R}^{6n}$.) The following principle is fundamental:

Newton's principle of determinacy The initial state (q_0, p_0) in the phase space of the system at time t_0 uniquely determines the states (q_t, p_t) for all $t > t_0$.

Particles also have acceleration. If the principle of determinacy is true, even the acceleration a of the particle in our simple system should be given as a function F of q and p. So we have *the equation of motion*

$$ma = \dot{p} = F(q, p). \tag{1.1}$$

This is the classical picture. Future positions and velocities of the particle are found by integration of second-order differential equations.

Example 1.1 Consider a particle or mass (M) attached to a spring (S), as illustrated in the figure below.

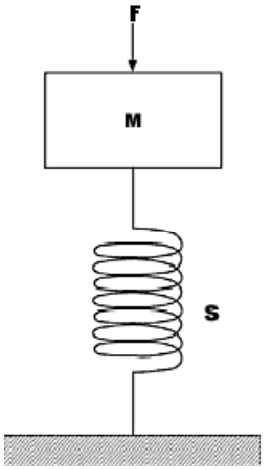


Figure 1.1. The oscillator.

It is known from experiments that $F(q, p) = -kq$. Here, F is the negative gradient $-\partial U/\partial q$ of the function $U(q, p) = kq^2/2$. U is the *potential energy* of the system. The *kinetic energy* of the mass is defined as $T = p^2/2m$. We may now introduce a new observable, the *total energy* of the system, or the *Hamiltonian function*: $E = H(q, p) = T + U = p^2/2m + kq^2/2$. From (1.1), it is easily shown that $dE/dt = 0$, so total energy is conserved. This system is a *harmonic oscillator*, a fundamental paradigm in physics, indispensable even within advanced subjects such as quantum field theory.

We shall demand that *all* observables are functions from the phase space to the real numbers. For an n-particle system, the following picture emerges:

$\mathcal{S}$ = the phase space $\mathbb{R}^{6n}$;
$\mathcal{O}$ = all functions $f : S \to R$;
$\mathcal{R}$ = the real number set $\mathbb{R}$.

We may now sketch a small language $\mathcal{L}_{\text{cl}}$ for this theory. For an observable o and $\Delta \subset \mathbb{R}$ we say that *elementary sentences* are of the form

$$\Delta(o)$$

"the value of the observable o belongs to the subset Δ of the real numbers".

Sentences in $\mathcal{L}_{\text{cl}}$ are interpreted as subsets of the phase space $\mathcal{S}$:

$$[\Delta(o)] = o^{-1}(\Delta).$$

We say that $\Delta(o)$ is *true* for the state s of the system if $o(s) \in \Delta$, equivalently if $s \in [\Delta(o)] = o^{-1}(\Delta)$. Thus, $\Delta(o)$ is interpreted as the set of states for which o is observed in the number set Δ (e.g. an interval). The logical connectives $\neg$, $\vee$ and $\wedge$ are interpreted as the corresponding operations – (complement), $\bigcup$ and $\bigcap$ on sets. The set-theoretical operations S form a boolean algebra. The associated logic for $\mathcal{L}_{\text{cl}}$ is *classical*.

1.1.2. The Logic of Quantum Mechanics

The construction of $\mathcal{S}$, $\mathcal{O}$ and $\mathcal{R}$ above seems very natural. Nevertheless, the phase space $\mathcal{S}$ is unsuitable as an exact model of ultimate physical reality. In order to recognize this, we shall consider a system which consists of a microscopic particle. Our observables q and p will again be the position and momentum of the particle (that is, the projections onto the first and second coordinate in the classical phase space). It is well-known that the standard deviations Δp $(= \langle (p - \langle p \rangle)^2 \rangle^{1/2})$ and Δq for measurements of p and q will be given by

Heisenberg's uncertainty principle $\Delta p \cdot \Delta q \geq \hbar/2$ ($\hbar$ is Planck's constant).

Increased precision in the determination of the position of the particle corresponds to greater imprecision in the determination of momentum, and vice versa. Hence, the temporal order of our experiments is no longer a matter of indifference. The physical magnitudes p and q *do not commute,* $pq \neq qp$. Observables in $\mathcal{O}$ are therefore no longer functions inot the reals. This fact enforces a new model choice for $\mathcal{S}$, $\mathcal{O}$ and $\mathcal{R}$. In the quantum-mechanical description, the triple $\mathcal{S}$, $\mathcal{O}$ and $\mathcal{R}$ comes in the following guise:

$\mathcal{S}$ = a complex Hilbert space H where the states are normalized vectors ψ;
$\mathcal{O}$ = the self-adjoint operators $A : H \to H$ represent observables;
$\mathcal{R}$ = the value set $\mathbb{R}$ (just as before).

Self-adjoint operators have real eigenvalues. These eigenvalues are the possible results of a measurement of the corresponding observable A. For a system in a given state, experience tells us that A has some probability of assuming each of these values. Accordingly, we must allow *probability* as a fourth fundamental notion. The probability P that an observable (represented by the operator) A upon measurement of a system in the state ψ assumes the eigenvalue a, is given by

$P(a) = |\langle a|\psi\rangle|^2$, with $\langle a|$ the eigenvector corresponding to the eigenvalue a.

The eigenvectors of a self-adjoint operator A span *subspaces* (not just *subsets*) of the state space $\mathcal{S} = H$. The closed linear subspaces of a Hilbert space form a lattice with a partial order given by inclusion. From this point of departure, Birkhoff and von Neumann suggested their famous *quantum logic*. Again we may define a simple language, $\mathcal{L}_{\text{qm}}$, with an associated interpretation. For $\Delta \subset \mathbb{R}$ and $A \in \mathcal{O}$, we construct the elementary sentences

$$\Delta(A)$$

"the value of the observable A belongs to the subset Δ of the real numbers".

And, once more, we shall interpret sentences by appeal to the states for which A is observed in the number set Δ:

$[\Delta(A)]$ = the linear subspace spanned by the eigenvectors ψ of A for an eigenvalue $a \in \Delta$

(i.e. $\psi \in \text{Im}(E(\Delta))$, with $E(\Delta)$ the spectral projection defined by A and Δ).

The lattice structure $L(H)$ of the set of subspaces of H now provides an interpretation of the logical connectives. We get

$$[\Delta_1(A_1) \bigvee \Delta_2(A_2)] = [\Delta_1(A_1)] \dot{+} [\Delta_2(A_2)] \text{ (i.e. Span}([\Delta_1(A_1)], [\Delta_2(A_2)])),$$
$$[\Delta_1(A_1) \bigwedge \Delta_2(A_2)] = [\Delta_1(A_1)] \bigcap [\Delta_2(A_2)] \text{ (also a subspace),}$$
$$[\neg\, \Delta_1(A_1)] = [\Delta_1(A_1)]^{\perp} \text{ (the orthocomplement of } [\Delta_1(A_1)]).$$

However, the lattice of subspaces is not boolean. The distributive law does not hold:

$$\Delta_1(A_1) \wedge (\Delta_2(A_2) \vee \Delta_3(A_3)) \Leftrightarrow (\Delta_1(A_1) \wedge \Delta_2(A_2)) \vee (\Delta_1(A_1) \wedge \Delta_3(A_3))$$

(Instead, a limited kind of distribution holds, based on the relation $U \subseteq V \Rightarrow U \bigcup (U^{\perp} \bigcap V) = V$ for subspaces U and V. The lattice is *orthomodular*.) On the other hand, the principle of contradiction holds:

$$\neg(\Delta_1(A_1) \wedge \neg\Delta_1(A_1))$$

Likewise, we have *tertium non datur*:

$$\Delta_1(A_1) \bigvee \neg\, \Delta_1(A_1).$$

(This is easily shown. E.g. $[\Delta_1(A_1) \bigvee \neg\, \Delta_1(A_1)] = [\Delta_1(A_1)] \dot{+} [\Delta_1(A_1)]^{\perp} = H = \top$.)

Quantum logic, therefore, has rather unusual characteristics. Birkhoff and von Neumann described it as essentially quasi-physical and opposed it favourably to intuitionistic logic, which, in their eyes, was guided by "introspective and philosophical considerations" (Birkhoff and von Neumann (1936), p. 837). Others, for similar reasons, have concluded that logic is, or ought to be, an empirical science. (E.g., according to Hilary Putnam the true logic should be "read off" from the Hilbert space (Putnam (1975), p. 179).) In practice, the development of quantum logic has encountered severe difficulties. A natural implication operator seems to be lacking, and no construction of a predicate calculus has succceded.

Example 1.2 It seems reasonable to demand that implication, $\Rightarrow$, fulfills

$$\frac{[p \bigwedge q] \subseteq [r]}{[p] \subseteq [q \Rightarrow r]}$$

in the lattice of subspaces. ($\wedge$ and $\Rightarrow$ are adjoints in the category-theoretical sense.) For the interpretation of the classical language $\mathcal{L}_{\mathrm{cl}}$, ordinary material implication, defined by $p \Rightarrow q := \neg p \vee q$, suffices. Assume that this definition is also the correct one for the language $\mathcal{L}_{\mathrm{qm}}$, interpret the sentences of $\mathcal{L}_{\mathrm{qm}}$ in the lattice $L(H)$ of subspaces of H, and consider the observables S_x and S_z, the spin of an electron in the x- and z-directions, both with eigenvalues in the set $\{-1/2, 1/2\}$ (for a choice of units with $\hbar = 1$). S_x and S_z are not simultaneously observable. Hence, $[S_x = 1/2] \bigcap [S_z = 1/2] = \emptyset$. Now let $p = (S_x = 1/2)$, $q = (S_z = 1/2)$ and $r = (S_z = -1/2)$. Then

$$\emptyset = [S_x = 1/2] \bigcap [S_z = 1/2] = [p \wedge q] \subseteq [r],$$

but it is not the case that

$$[S_x = 1/2] = [p] \subseteq [q \Rightarrow r] = [\neg q \vee r] = \mathrm{Span}([S_z = 1/2]^{\perp}, [S_z = -1/2]) = \mathrm{Span}([S_z = -1/2]).$$

(The last equality follows because the Hilbert space is spanned by the eigenvectors of S_z.) As remarked above, it is not possible to measure spin in the x- and z-directions at the same time.

In chapter 2, we shall pay close attention to the way in which Isham and Döring's theory of topos physics modifies the present account of quantum logic. Meanwhile, let us end this introductory sketch by noting that the link between observables in classical physics and in quantum mechanics is deeper than we have suggested so far. Consider the following example:

Example 1.3 In example 1.1 we gazed briefly at the Hamiltonian function $H(q, p) = p^2/2m + kq^2/2$ for a classical harmonic oscillator. It is postulated in quantum mechanics that classical observables shall be represented by corresponding operators. These will be the quantum-mechanical observables. For position and momentum, we have

$$q \mapsto \hat{q} = x \times \text{-} \text{ (multiplication by } x)$$

$$p \mapsto -i\hbar \frac{\partial}{\partial x}.$$

The Hamiltonian function may now be represented by the Hamiltonian operator

$$H(q, p) \mapsto \hat{H} = \hat{p}^2/2m + k\,\hat{q}^2/2 = -\frac{\hbar}{2\,m}\frac{\partial^2}{\partial x^2} + \frac{k}{2}x^2 \times \text{-}.$$

Above, we said that the state space will be a complex Hilbert space. We now choose $L^2(\mathbb{R})$ for this. Let $\psi(t) \in L^2(\mathbb{R})$ be the state of the harmonic oscillator at time t. We postulate that the dynamical evolution of the system is given by the equation

$$i\hbar \frac{\partial \psi}{\partial x} = \hat{H}\psi.$$

Insertion of $\hat{H}$ gives *Schrödinger's equation* for the system under consideration. The solution will be a curve in the Hilbert space.

The quantisation procedure above is neither complete nor unambiguous. Firstly, there exist quantum mechanical degrees of freedom (spin is an example of this) with no counterpart in classical physics. Secondly, we may ask if a classical observable ab ought to be represented as $\hat{a}\hat{b}$ or $\hat{b}\hat{a}$ when the corresponding quantum-machanical observables do not commute (cf. section 1.3 below).

The quantum-mechanical perspective is limited in yet another way. A complete physical theory should be consistent with the special theory of relativity. In special relativity, simultaneous interaction between separate particles is excluded, and "action at a distance" must be replaced by some kind of field theory. The physical fields will be dynamical objects in their own right, and changes in the fields must propagate with a finite velocity. For the law of conservation of energy to hold, at least locally, the fields ought also to be carriers of energy. In the final quantisation of physical theory, these field must themselves be quantised. This is done in *quantum field theory.* In a nutshell: Start from example 1.3 above, and build the physical field by stationing harmonic oscillators at each point of space. Then quantise these in such a manner that invariance under Lorentz transformations is preserved, as demanded by special relativity (cf. subsection 1.2.5 below).

Gravity apart, succesful quantum field theories have been found for all known interactions in nature. Quantisation of the metric tensor, which codes information about the curvature of space-time, seems to be an immensely difficult task. We shall have more to say about quantisation and gravitation theory in subsection 1.2.6 and section 1.5 below. Also, in section 3.1, we give a more detailed review of a particular theory of quantum gravity. The topos models presented in chapter 2 are in part motivated by a wish to develop general methods for a future theory of quantum gravity. Let us now have a closer look at the central notions in the brief sketch so far given.

1.2. The Description of Physical Systems

1.2.1. States, Observables and Probability

The fundamental notions are present at the very start of our description of a physical system. In physics, a measurement is made on a system in a certain state, and the result of the measurement is recorded as a measured value. Recall from section 1.1 that different measuring apparatuses record the same value for identically prepared states of a system, we shall say that they measure the same *classical observable.* We denote classical observables by letters *P, Q,* Q_1, Q_2, ... and the set of classical observables by the letter $\mathcal{O}_{\mathrm{Cl}}$. *Classical states*, on the other hand, are identical when their recorded values are identical for all classical observables. We denote classical states by letters α, α_1, α_2, ... and the set of classical states by the letter $\mathcal{S}_{\mathrm{Cl}}$. Also, all recorded *values r,* r_1, r_2, ... are members of a value set $\mathcal{R}_{\mathrm{Cl}}$. (In classical mechanics, a state would be a point in the phase space of the system, an observable would be a function defined on the phase space, and $\mathcal{R}_{\mathrm{Cl}}$ would be the real numbers.) Our protocol will then consist of series of statements of the form $Q(\alpha) = r$.

So far, we have made no decision as to the final interpretation of our states and observables. Our letters may be the primitive signs of a formal language for physics. But the present vocabulary is too narrow in several respects. Firstly, information about the state of a physical system may be incomplete. (This is certainly the case when we are dealing with a system with a large number of particles.) Secondly, it may be that the measurement of a system in a certain state has different results at different times.

We shall deal with the first difficulty by the introduction of a partially ordered set $\mathcal{P}$. We then consider functions μ, perhaps partial, from $\mathrm{P}(\mathcal{S}_{\mathrm{Cl}})$, the power set of $\mathcal{S}_{\mathrm{Cl}}$, to the set $\mathcal{P}$. We denote the members of $\mathcal{P}$ by *p,* p_1, p_2, ... Intuitively, μ records our information about a system by ordering sets of classical states according to their probability. It seems natural, perhaps unavoidable, to suppose that $\mathcal{P}$ contains members $0_{\mathcal{P}}$ and $1_{\mathcal{P}}$ such that $p \geq 0_{\mathcal{P}}$ and $p \leq 1_{\mathcal{P}}$ for all $p \in \mathcal{P}$. (In statistical mechanics, the domain of μ would be a σ-algebra, and μ would be a countably additive measure with values in the unit interval [0, 1].) Tentatively, the functions μ are introduced as our next construction step. We shall call them the *probabilistic states*, denote them by letters μ, μ_1, μ_2, ..., and collect them in the set $\mathcal{S}_{\mathrm{Pr}}$. It is simple to see how the classical states may be represented in the new set $\mathcal{S}_{\mathrm{Pr}}$. We shall map the classical state α_1 to the probabilistic state μ_1 with the property that $\mu_1(A) = 0_{\mathcal{P}}$ and $\mu_1(B) = 1_{\mathcal{P}}$ whenever $\alpha_1 \notin A$ and $\alpha_1 \in B$, where A and B are sets of classical states, and $\leq$ is the ordering on $\mathcal{P}$. In general, a representative state μ_1 may not always exist, or it may not be unique.

At present, there is no reason why the incompleteness of information should compel us to assume the existence of new observables, so we keep the old set $\mathcal{O}_{\mathrm{Cl}}$ until further notice. We also retain the value set $\mathcal{R}_{\mathrm{Cl}}$. However, we still want to record our measurements in a protocol, and statements of the form *"*$Q(\mu) = r$" does not make sense for the probabilistic states μ. The problem can be solved by combining the maps $Q^{-1}: \mathcal{R}_{\mathrm{Cl}} \to \mathcal{S}_{\mathrm{Cl}}$ and $\mu : \mathrm{P}(\Sigma_{\mathrm{Cl}}) \to \mathcal{P}$ into the map $w_\mu^Q : \mathrm{P}(\mathcal{R}_{\mathrm{Cl}}) \to \mathcal{P}$ defined by $w_\mu^Q(E) = \mu(Q^{-1}(E))$ for $E \subset \mathcal{R}_{\mathrm{Cl}}$. The record *"*$w_\mu^Q(E) = p$*"* then states that "when the system is in the probabilistic state μ, the estimate that the value of the observable Q is to be found in the set E should be ranked as p in our probability order".

If there is only a finite number of classical states, we may define the *expectation value* of the observable Q in the state μ as

$$\overline{Q}(\mu) = \sum_{\alpha \in \mathcal{S}_{\mathrm{Cl}}} \mu(\{\alpha\}) \cdot Q(\alpha) \tag{1.2}$$

This presupposes some structure on the value set $\mathcal{R}_{\mathrm{Cl}}$ and the probability set $\mathcal{P}$. At the very least, addition should be defined on $\mathcal{R}_{\mathrm{Cl}}$, and $\mathcal{R}_{\mathrm{Cl}}$ should have a zero element $0_{\mathcal{R}}$. We shall also need a map $\cdot : \mathcal{P} \times \mathcal{R}_{\mathrm{Cl}} \to \mathcal{R}_{\mathrm{Cl}}$ satisfying $0_{\mathcal{P}} \cdot r = 0_{\mathcal{R}}$ and $1_{\mathcal{P}} \cdot r = r$ for $r \in \mathcal{R}_{\mathrm{Cl}}$. We then have $Q(\alpha) = \overline{Q}(\mu)$ whenever a classical state α_1 is identified with a probabilistic state μ_1. In the infinite case, the family $\mathcal{B}$ of subsets E of the value set $\mathcal{R}_{\mathrm{Cl}}$ would be a σ-algebra, and w_μ^Q would be a measure defined on $\mathcal{B}$. The expectation value is then defined as $\overline{Q}(\mu) = \int r \, dw_\mu^Q$.

The extension of our formal apparatus solves the problem of incomplete information. In doing this, it seems that we have also taken care of the second problem, the possibility that the repetition of a measurement may give different results for the same system state. We still have a wide range of representations at our disposal, including systems based on the phase space of classical mechanics. The next problem, however, will narrow our freedom of choice considerably.

1.2.2. Simultaneous Measurability and Quantisation

Above, we had to adjust our explication of a physical state, while the notion of an observable remained intact. We do not want any unneccesary restrictions on the set $\mathcal{O}$ (hereafter, we omit the subscript) of observables. That is, if f is any function on the set $(\mathcal{R}_{\mathrm{Cl}})^n$ of n-tuples of values of observables such that $\mathrm{Rn}(f) \subset \mathcal{R}_{\mathrm{Cl}}$, we would like to have an observable $f(Q_1, ..., Q_n)$, defined as the composition of f with $Q_1, ..., Q_n$. This surely makes sense if the observables $Q_1, ..., Q_n$ are functions $f_i(Q)$ of a single observable Q. We say that they are *simultaneosly measurable*. Then the observable $f(Q_1, ..., Q_n) = f(f_1(Q), ..., f_n(Q)) = g(Q)$ and we define $w_\mu^{f(Q_1, ..., Q_n)}(E) = w_\mu^Q(g^{-1}(E))$.

It is well-known from quantum mechanics that certain observables, such as position and momentum, are not simultaneously measurable. It behoves us therefore to choose the entities of physical theory with some care. Our first step will be to identify the set of states $\mathcal{S}$ with a complex Hilbert space. The letters $\mathcal{H}(\mathcal{S})$ will denote this space, while the *vector states*, the vectors with norm 1 that are unique up to multiplication by a complex number of unit modulus, are denoted by letters ϕ, ψ, ψ_1, ψ_2, ... Observables P, Q, Q_1, ...$\in \mathcal{O}$ then correspond to *self-adjoint operators* on $\mathcal{H}(\mathcal{S})$. The basic definitions of probability and expectation value now take the form

$$w_{\psi}^{Q}(E) = (\psi, \chi_E(Q)\,\psi) \tag{1.3}$$

$$\overline{Q}\,(\psi) = (\psi, Q\psi) \tag{1.4}$$

The brackets on the right denote the inner product on the Hilbert space, E is a (Borel) subset of the value set $\mathcal{R}$ of complex numbers (we omit the subscript on $\mathcal{R}$), and χ_E is the characteristic function of $E \subset \mathcal{R}$. On Hilbert spaces, two self-adjoint operators commute, $PQ = QP$, if there is a self-adjoint operator Q_1 and (Borel) functions f and g such that $P = f(Q_1)$ and $Q = g(Q_1)$. Two observables that are not simultaneously measurable, will then have a *commutator*, $[P, Q] = PQ - QP$, different from 0.

The representation of states and observables by means of vectors in a Hilbert space and operators on them may seem arbitrary, but it is amply born out by successful predictions in quantum physics. If we consider $\mathcal{Q}$, the subset of *questions* of the observables $\mathcal{O}$, we shall find that there is also a strong theoretical justification for this choice (e.g. Mackey (1963), p. 72f). In the notation of subsection 1.2.1, a question Q is an observable such that $w_{\mu}^{Q}(\{0,1\}) = 1_{\mathcal{P}}$, where 0 and 1 are mutually exclusive outcomes ("yes" and "no") of the measurement of Q in the state μ. By switching the answers, we have the complementary question 1-Q. It is also possible to define a partial ordering on $\mathcal{Q}$ in this way:

$$Q_1 \leq Q_2 \text{ i ff } w_{\mu}^{Q_1}(1) \leq w_{\mu}^{Q_2}(1) \text{ for all states } \mu. \tag{1.5}$$

For an observable P, Q_1 may be the question $\chi_{E_1}(P)$, "Is the value of P in the set E_1?", and Q_2 may be the question $\chi_{E_2}(P)$, "Is the value of P in the set E_2?" If $E_1 \subset E_2$, then $Q_1 \leq Q_2$. Also, if $Q_1 \leq 1 - Q_2$, Q_1 and Q_2 are *disjoint* questions. If the partially ordered set $\mathcal{P}$ of probability values is the unit interval [0, 1], we naturally expect that

$$w_{\mu}^{Q_1+Q_2+\cdots}(1) = w_{\mu}^{Q_1}(1) + w_{\mu}^{Q_2}(1) + \ldots \text{ for disjoint questions } Q_i. \tag{1.6}$$

$w_{\mu}^{Q}(1)$ is then a probability measure on $\mathcal{Q}$. If, as above, we identify the observables with self-adjoint operators on a Hilbert space, we see that the structure of complementation and the probability measure (interpreted as in (1.1)) are present in the subset of $\mathcal{O}$ of self-adjoint operators Q such that Q has eigenvalues 1 and 0 and $QQ = Q$. The two sets have the same "logic". We will return to the problem of the appropriate quantum logic in chapter 2.

With the definition of the expectation value of an observable (a self-adjoint operator) Q at hand, we are able to define its *uncertainty* ΔQ by $(\Delta Q)^2 = \overline{Q^2} - \overline{Q}^2$. The consequences of assuming the existence of observables that are not simultaneously measurable are then expressed as the *uncertainty principle* (for a proof, see e.g. Jordan (1969))

$$(\Delta P)\,(\Delta Q) \geq \frac{1}{2}\left|\overline{PQ - QP}\right|. \tag{1.7}$$

When Q_i and P_i are the position and momentum variables ($i = 1, 2, 3$) the commutation relations $[Q_i, P_j] = i\hbar\delta^i_j$ allow us to derive the *Heisenberg uncertainty relations*

$$(\Delta P)\,(\Delta Q) \geq \frac{\hbar}{2}\,\delta^i_j. \tag{1.8}$$

The canonical relations $[Q_i, P_j] = i\hbar\delta^i_j$, together with the trivial relations $[Q_i, Q_j] = 0$ and $[P_i, P_j] = 0$, specify the *quantisation algebra* of the system. We shall explore quantisation schemes for different systems in section 1.3.

1.2.3. The Symmetries of Time and Space

Although we spoke of simultaneous measurability in the last subsection, time itself did not enter the description. Is it a formal parameter, a background structure, or a construction within the theory? For the system sketched above, we may first try the standard approach. We therefore leave the domain of statics and introduce a set $\mathcal{U}$ of transformations of the vector states ϕ in the state space $\mathcal{H}(\mathcal{S})$. The members U_t of $\mathcal{U}$ are parameterized by $t \in \mathbb{R}$, and we shall assume that U_0 is the identity, and $U_{t_1} \cdot U_{t_2} = U_{t_1+t_2}$. $\mathcal{U}$ is *the dynamical one-parameter group*, and we identify the formal parameter t as classical time. We shall also postulate *strict causality:* If the system is in the state ϕ with probability p then, t moments later, the system is in the state $U_t\phi$ with the same probability.

The notion of *continuous development* belongs to the same family. Formally, we have the condition

$w^Q_{U_t\psi}(E)$ is a continuous function of t for all $\psi \in \Sigma$, $Q \in \mathcal{O}$ and Borel sets $E \in \mathbb{R}$.

In $\mathcal{H}(\mathcal{S})$, the Hilbert space representation of $\mathcal{S}$ and $\mathcal{O}$, the states and observables, U_t is a *unitary operator.* By Stone's theorem, U_t may be written as e^{-iHt}, where H is a self-adjoint operator (this is analogous to $u = e^{i\theta}$ for θ a real number and u a complex number of unit modulus). Then, if $\psi(0)$ is the state at time zero, the state at time t is $\psi(t) = e^{-iHt}\psi(t)$, and the *Schrödinger equation* may be derived:

$$\frac{d}{dt}\psi(t) = -iH\psi(t) \tag{1.9}$$

We note in passing that the choice of a *complex* Hilbert space in subsection 1.2.2 is founded on the presence of i in Stone's theorem. The self-adjoint operator H must be an observable. We shall call it the *Hamiltonian* of the system. (For the time being, we identify observables and self-adjoint operators. Restrictions of the operators representing physical quantities are known as *superselection rules*.) It may be proven that observables Q that are *conserved* in time (so that $w^{Q}_{U_t\psi}\left(E\right)$ is independent of t for given ψ, E) commute with H. These observables, including H itself, will be called *symmetry operations* of the system. When Q is not conserved, the new observable $\dot{Q} = i[H, Q]$ has as its expectation value the time derivative of the expectation value of the observable Q with respect to a given state.

Because the Hamiltonian H is the generator of the dynamical development of the states ψ (in the Schrödinger picture) and the observables Q (in the Heisenberg picture), its concrete form will depend on the physical system under consideration. Normally, this is done by *canonical quantisation*. Starting from a classical system $(\Sigma_{\mathrm{Cl}}, \mathcal{O}_{\mathrm{Cl}})$, we find suitable analogues (quantisations) of the classical observables in $\mathcal{O}_{\mathrm{Cl}}$ among the operators in $\mathcal{O}$. The quantisation of the classical Hamiltonian observable H_{Cl} will then be our Hamiltonian H. We will look further into quantisation methods in section 1.3 below and (for some very general models) in chapter 4.

If we like, we may now define transformations (rotations and translations) of classical space that, together with the Galilei transformations $\mathbf{x} \to \mathbf{x} + \mathbf{v}t$, extend the construction of $\mathcal{U}$ in the present subsection. The generators for these transformations form the ten-parameter Lie group called the *Galilei group* (Jordan (1969), p. 107ff). Instead of pursuing this track, we shall turn directly to the transformations associated with the space-time of special relativity.

1.2.4. The Symmetries of Special Relativity

Above, time and space were present in our description as separate structures. From the theory of special relativity, we know that this will not do. The invariance of the laws of physics in all inertial frames forces us to continue our search for a complete theory within the framework of the four-dimensional *Minkowski space-time* $(\mathcal{M}, \eta)$. Here, η defines the Lorentz-invariant scalar product $(x, y) = \eta_{\mu\nu}x^{\mu}y^{\nu}$, where $x = (x^{\mu})$ is a four-vector in $\mathcal{M}$ and $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$ ($\eta_{\mu\nu}$ has signature +2). The four-vectors in $\mathcal{M}$ are classified as *timelike, lightlike* and *spacelike*. We now introduce the *group of Poincaré transformations* $\mathrm{T}(\Lambda, a)$:

$$x^{\mu} \to \Lambda^{\mu}_{\nu}\, x^{\nu} + a^{\mu}. \tag{1.10}$$

As in the case of time evolution above, the transformations $T(\Lambda, a)$ induce unitary linear transformations $U(\Lambda, a)$ on the states in the Hilbert space $\mathcal{H}(\Sigma)$. The composition rule $U(\hat{\Lambda}, \hat{a})U(\Lambda, a) = U(\hat{\Lambda}\Lambda, \hat{\Lambda}a+\hat{a})$ is then satisfied. (There are also anti-unitary transformations, such as the PCT symmetry, of great importance. See e.g. Streater and Wightman (1980), p. 17.) The transformations $U(\Lambda, a)$ acting on states Ψ and observables Q are

$$\Psi \to U(\Lambda,\ a)\,\Psi \tag{1.11}$$

$$Q \to UQU^{-1}. \tag{1.12}$$

Then $(\Psi, Q\Psi) = (U\Psi, UQ\Psi) = (U\Psi, (UQU^{-1})U\Psi)$, so the inner product on the Hilbert space respects the special relativistic symmetry of the Poincaré group.

We may now easily incorporate some standard features of physical theories in our description. As the *vacuum state* Ψ_0 has zero energy, momentum and angular momentum for all observers, we naturally expect $U(\Lambda, a)\Psi_0 = \Psi_0$ (up to a phase factor) for all $U(\Lambda, a)$. *One-particle states* with zero or positive-definite mass may be introduced, where the details of the representation depend on the spin of the particle. For positive-definite masses, this will be an irreducible unitary representation of the three-dimensional rotation group *SO(3)* (Weinberg (1995), p. 68ff). *Multi-particle states* with no interactions between the particles may be handled as direct products of one-particle states. The complete space for states with an arbitrary number of particles, the direct product of the n-fold tensor product spaces for all n, is called the *Fock space* (Araki (1999), p. 72).

Because particles are effectively non-interacting before and after a collision, the description can be applied to physical experiments in scattering theory. Also, we may label the states $\Psi_{p,\sigma}$ with the spin z-component σ and momentum parameter p. (Here, momentum is defined group-theoretically. $P = (P^\mu)$ is the generator of the translation $U(1, a)$, and $P^\mu\Psi = p^\mu\Psi$.) One observable feature of classical mechanics, the particle, has therefore survived in our quantisation of the theory.

1.2.5. Quantum Field Theory

The quantisation of another fundamental notion of physics, the field, classically understood as an observable physical magnitude defined on the points of space-time, is the subject of quantum field theory. If we apply the quantisation scheme suggested above, quantum fields should be sets of self-adjoint operators (observables) on space-time points, and appropriate representations of the transformations $T(\Lambda, a)$ (and other symmetries, such as parity P, time inversion T, and charge conjugation C) should be found. Standardly, they will be built from the *annihilation* and *creation operators* $a(p, \sigma)$ and $a^\dagger(p, \sigma)$ of quantum mechanics. As operators on the Fock space, a and $a^\dagger$ take n-particle states to the particle level $n-1$ or $n+1$. The *annihilation field* is of the form (the *creation field* is defined in a corresponding manner)

$$\psi_\ell^+(x) = \int d^3\, p\, u_\ell(x;\, p)\, a(p,\ \sigma). \tag{1.13}$$

The index ℓ runs over the components of the representation $D(\Lambda)$ of the Lorentz transformations Λ. The coefficient functions u_ℓ must satisfy the Lorentz transformation rules for this representation (Weinberg (1995), p. 192). The four-vector representation is particularly interesting, as it gives us a first glimpse of the existence of *antiparticles*, particles with the same mass and spin as the familar ones, but with opposite charge (op. cit., p. 211). Also, the connection between the spin and the statistics of a particle emerges at this level.

The family of *nonabelian gauge transformations*, transformations that vary with place in space-time, is needed for the representation of electroweak theory and quantum chromodynamics (QCD). We shall not enter into the details. One of the formulations of nonabelian gauge theory, Wilson's *lattice gauge theory*, has inspired the LQG approach to quantum gravity presented in chapter 3 (cf. Smit (2002)).

Quantum field theories, sophisticated as they are, may still be only *effective* theories. The descriptions provided by a quantum field theory may not be valid above a certain energy level Λ. Above this level, calculations may diverge. In the theories known as *renormalizable*, the calculations are halted at this level, and the apparent dependence of the result on the cutoff level Λ is made to disappear by the introduction of *coupling constants* (these are redefinitions of the physical constants, such as masses and charges (Zee (2003), p. 150)). This solves the famous problem of "taming the infinities" for the renormalizable case. While the nonabelian gauge theories are renormalizable, the theory of gravitation is not. Even such theories, where the dependence on Λ is not eliminable, may be useful to some degree.

1.2.6. The Curved Space-time Background

The quantum theory above is formulated in flat (Minkowski) space-time. In order to complete our sketch, two further additions to the description should be considered. First, we should admit the possibility that the quantum fields may be defined on a curved space-time background. Second, the curvature of the background may be included in the dynamics of our theory. This ideal, yet unrealized, completes the former descriptions and is known as the physics of *quantum gravity*. In this subsection, we give a brief outline of the first option, the theory of quantum fields in curved space-time.

In curved space-time, the Minkowski metric $\boldsymbol{\eta}$ of flat space-time is supplanted by the Lorentz metric $\mathbf{g}$ on the four-dimensional manifold $\mathcal{M}$, and it is assumed that *Einstein's field equations*, governing the dynamics of $\mathbf{g}$, hold on $\mathcal{M}$ (Hawking and Ellis (1973), sec. 3.4). If we ignore the dynamics of $\mathbf{g}$, the theory of quantum fields may be developed along the same lines as in the flat case. In order to have a unique solution of the field equations for given initial conditions, it is useful to consider only *globally hyperbolic* space-times $(\mathcal{M}, \mathbf{g})$. On these manifolds there is a *Cauchy surface* Σ, a closed subset of $\mathcal{M}$ such that every point in $\mathcal{M}$ has a causal curve that passes through Σ. The construction of the Fock space is then unhindered, and the classical observables may be represented as operators on the space (Wald (1994), p. 59).

The transition to curved space-time raises problems that were not present in the flat background. The possibility of the definition of a vacuum state and a natural particle interpretation was mentioned in subsection 1.2.4. In fact, in many approaches to quantum field theory on the Minkowski space (notably Weinberg (1995)), the particle interpretation is built into the theory at the outset, and the fields are derived from the postulate of Lorentz invariance. In physics on a curved space-time, this option is no longer available. So it may be that the notion of a particle in general space-times is only approximate, and that the notion of a field is the more fundamental one.

The algebraic approach to quantum field theory initiated by Haag and others (overview in Araki (1999)) has been suggested as a natural framework for a field theory on a curved space-time background. Here, instead of identifying the states of the system with the unit vectors of a Hilbert space $\mathcal{H}$, we start with observables $\mathcal{O}(D)$ defined on a limited space-time region D and given the structure of a C^*-algebra. The states are then objects that act on the algebra by mapping an observable onto its expectation value in the state. Possibly, this approach may be combined with the topos methods found in chapter 2 and 3 below.

Quantum field theory on curved space-time has been successfully applied to the derivation of some quantum theoretical phenomena, most notably Hawking's analysis of particle creation by black holes. We noticed above that the gravitational field is non-renormalizable, so divergences in calculations of basic physical experiments are not removable. Still, certain low-energy calculations may be carried out in the theory, and the *graviton*, the interaction particle of the gravitational field may be associated with the perturbation of a stationary background Lorentz metric. This is analogous to the way the photon is introduced in electrodynamics.

The full quantisation of all physical fields, including the gravitational field, is the subject of the theories of quantum gravity. One may then try to sort out the ingredients of a theory that are affected by quantisation. Do they include the manifold itself, perhaps even the topology of the manifold (cf. Kiefer (2004), p. 21)? Accordingly, before we discuss the theory of quantum gravity (section 1.5), we shall look more closely at two of the intermediate constructions above, the quantisation step from classical dynamics to quantum theory (1.3), and the manifold structure of space-time (1.4).

1.3. Schemes of Quantisation

1.3.1. The Harmonic Oscillator in One Dimension

In subsection 1.2.3, we mentioned that there exists a recipe, canonical quantisation, that allows us to find quantum mechanical representatives of the classical observables, such as position and momentum. Strangely, as we pass from the poorer classical system to quantum physics by this purely formal step, we seem to gain some extra information for free. But it is well known from most textbooks in quantum mechanics (see e.g. Shankar (1994), p. 120ff) that quantisation is neither complete nor wholly unambiguous. It is incomplete because there may be quantum degrees of freedom, such as spin, which have no classical counterpart. Also, the classical observables are ambiguous with respect to order. Should the classical observable qp be represented by the operator QP or PQ? Because Q and P do not commute, the choice matters. In fact, according to the Groenewold-Van Hove theorem, it is *impossible* to quantise all classical observables in p and q on $\mathbb{R}^{2n}$.

We shall first have a look at a guiding example from the mathematical theory of quantisation. For a simple system, say the harmonic oscillator with basic observables given by the position coordinate q and momentum coordinate p, we construct the Poisson brackets

$$\{q, q\} = \{p, p\} = 0, \{q, p\} = 1. \tag{1.14}$$

Quantisation of this system is deceptively simple. We switch from Poisson brackets to commutator brackets, and find self-adjoint operators Q and P on the Hilbert space L^2 that fulfill the corresponding relations (where $\hbar I$ is the identity operator multiplied by Planck's constant)

$$[Q, Q] = [P, P] = 0, [Q, P] = i\hbar I. \tag{1.15}$$

The operators Q and P defined by $Q(\psi)(q) = q\psi(q)$ and $P = -i\hbar\frac{\partial}{\partial q}$ satisfy (1.14), and, by substitution in the classical Hamiltonian $H_{cl} = p^2/2m + kq^2/2$ of the system, the Hamiltonian operator H is found. The Schrödinger equation (cf. (1.7) above) for the harmonic oscillator in one dimension may then be expressed as

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial q^2} + \frac{k}{2} q^2 \psi. \tag{1.16}$$

1.3.2. The Semi-Classical Solution

Following Bates and Weinstein (1997), we now outline how approximate solutions can be found in the slightly more general case of a 1-dimensional potential $V(x)$. (The results presented in this section are purely expository, and will not be needed for the rest of the thesis.) Suppose the classical Hamiltonian of the system under consideration is

$$H_{\text{cl}}(q, p) = \frac{p^2}{2m} + V_{\text{cl}}(q). \tag{1.17}$$

We are looking for the stationary state-solutions $\psi(x,t) = \phi(x)e^{-i\omega t}$ of the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \tag{1.18}$$

A solution that is accurate to order $O(\hbar^2)$ is then given by (ignoring the factor $e^{-i\omega t}$)

$$\phi(x) = e^{iS(x)/\hbar} a(x) \tag{1.19}$$

Here, a is the *amplitude function* and S is the real-valued *phase function*. S must then satisfy the *Hamilton-Jacobi equation*

$$H_{\text{cl}}(x, S'(x)) = \frac{(S'(x))^2}{2m} + V_{\text{cl}}(x) = E \,(\text{constant}). \tag{1.20}$$

The amplitude function a must satisfy the *homogeneous transport equation* (where Δ is the Laplacian)

$$\mathrm{a}\Delta S + 2 \sum_j \frac{\partial a}{\partial x_j} \frac{\partial S}{\partial x_j} = 0. \tag{1.21}$$

This procedure is known as the Wentzel-Kramers-Brillouin method (see Shankar (1994), p. 435ff, for an elementary description).

For the harmonic oscillator, the solution curves to the classical equation $H_{\text{cl}}(q, p) = E$ are ellipses in the phase space $\mathbb{R}^2$. As usual, we identify $\mathbb{R}^2$ with the cotangent space $T^*\mathbb{R}$ and consider the differential $dS : \mathbb{R} \to T^*\mathbb{R}$ of the phase function S. It is an immediate consequence of the Hamilton-Jacobi equation that $L = \text{im}(dS) \subset H^{-1}(E)$. That is, the 1-dimensional manifold L is contained in the classical solution curve for a constant energy E. Ignoring the amplitude a for the moment, it seems natural to identify the state ϕ with the manifold L. This is known as the *geometric* viewpoint on quantisation.

The amplitude function a can also be given a geometric interpretation. We rewrite the homogeneous transport equation as

$$\nabla\left(a^2\,\nabla S\right) = 0. \tag{1.22}$$

Equivalently, this may be expressed as

$$\mathcal{L}_{a^2\,\nabla S}(\,|\,dx_1 \wedge \ldots \wedge dx_n\,|) = 0. \tag{1.23}$$

$\mathcal{L}$ is the Lie derivative. The surrounding bars turn the n-form into a canonical *density*. It can then be shown that the pull-back of the density $a^2\,|\,dx_1 \wedge \ldots \wedge dx_n\,|$ from $\mathbb{R}^n$ to the manifold L under the usual projection π: $T^*\,\mathbb{R}^n \to \mathbb{R}^n$ is invariant along the Hamiltonian vector field $X_H = (\frac{\partial H}{\partial p},\ -\frac{\partial H}{\partial q})$. Accordingly, we identify the geometric states as the pull-back of *half-densities* $a\,|\,dx_1 \wedge \ldots \wedge dx_n\,|^{1/2}$ on $L = \mathrm{im}(dS)$ that are invariant along X_H. (Densities and half-densities are explained in Bates and Weinstein (1997), app. A.)

The naturalness of the geometric approach should be noted. As defined by the functions S and a, the approximate solution $\phi(x) = e^{iS(x)/\hbar}\,a(x)$ still has some undesirable properties. The amplitude $a(x)$ may have singularities on the configuration space $\mathbb{R}^n$, and there may be no suitable function $S(x)$ such that $L = \mathrm{im}(dS)$. The geomtric solution $(L,\ a)$ does not have these weaknesses. The half-density a is smooth on L, and the projection of X_H on the configuration space coincides with ∇S, so there is no need to mention S at all. Before we turn to quantisation in 1.3.4, we shall give an outline of the semi-classical picture in a more general setting.

1.3.3. Symplectic Geometry

A natural framework for further analysis of the quantisation procedure is found in symplectic geometry. A *symplectic manifold* N is a manifold with a *symplectic structure* (a closed, nondegenerate *2*-form ω) defined on it. Above, $T^*\,\mathbb{R}^n$ was a $2n$-dimensional symplectic manifold with $\omega = \Sigma\ dq_j \wedge dp_j$. For $\tilde{\omega}(x) = \omega(x,\ \text{-})$, Hamilton's equations of motion imply that the Hamiltonian vector field X_H is given by

$$X_H = \tilde{\omega}^{-1}(dH). \tag{1.24}$$

The Hamiltonian H and the symplectic form ω thus suffice for finding the integral curves of the motion in the phase space. Also, it can be shown that H and ω are invariant along X_H. We now note that the characteristics of a symplectic structure allow us to reproduce these results in the general case.

(1) We rely on the *nondegeneracy* of ω and *define* the Hamiltonian vector field X_H by means of the equation (1.24).

(2) The invariance of H along X_H, $\mathcal{L}_{X_H}\,H = \omega(X_H,\ X_H) = 0$, is a consequence of the *skew-symmetry* of the *2*-form ω.

(3) The invariance of ω along X_H, $\mathcal{L}_{X_H}\,\omega = 0$, is a consequence of the *closedness* of ω.

We shall concentrate on the case $N = T^*M$, the cotangent bundle of a smooth (riemannian) manifold M. In this case, the *2*-form $\omega = -d\alpha$, where the primitive α is the intrinsically defined *Liouville form* (here, $\pi : T^*M \to M$ is the projection and $< , >$ is the usual *1*-form action on a vector):

$$\alpha\,(x,\, b)\,(v) \;=\; < b,\, \pi_*\, v > . \tag{1.25}$$

Above, the submanifold $L \subset T^* \mathbb{R}^n$ was of particular interest to us. L is an example of a *lagrangian submanifold.* For these submanifolds, the tangent space T_pL equals $\left(T_p\, L\right)^{\perp}$, the orthogonal to T_pL in the tangent space at $p \in T^*M$ under the symplectic form ω. There is a bijection between the closed *1*-forms ϕ of M and the projectable lagrangian submanifolds of T^*M (proof in Bates and Weinstein (1997), p. 29). When the form ϕ is exact, so that $\phi = dS$ for some function S on M, we call ϕ an *exact lagrangian submanifold*. The function S is called the *generalized phase function*.

It is now possible to find the geometric states when the basic space is the contangent bundle T^*M. The condition on a lagrangian embedding $\iota : L \to T^*M$ corresponding to the Hamilton-Jacobi equation (1.20) for the phase function S is

$$\iota \;=\; H^{-1}(E) \text{ for smooth functions } H \,:\, T^*\, M \;\to\; \mathbb{R}. \tag{1.26}$$

When $X_{H,\iota}$ is the nonsingular vector field on L induced by the Hamiltonian vector field X_H, the homogeneous transport equation takes the following form for a half-density a on L (cf. equations (1.21) - (1.23)):

$$\mathcal{L}_{X_{H,\iota}}\, a = 0. \tag{1.27}$$

If the embedding ι is exact, there is a generalized phase function ϕ on L such that ι^* maps the tautological Liouville form α to $d\phi$. We then have a half-density $e^{\mathrm{i}S(x)/\hbar}a$ on L that can be pushed forward to M, where it is a stationary state $\left(\iota^{-1} \circ \pi^{-1}\right)^* e^{\mathrm{i}\phi/\hbar}a$ for the time-independent Schrödinger equation $H\phi = E\phi$. Because M is our configuration space, we identify the pull-back operation along $L \to M$ as *prequantisation*. (Note that ϕ, as a smooth half-density, is a state in the *intrinsic* Hilbert space of M. See Bates and Weinstein (1997) for details.)

1.3.4. Prequantisation and Maslov Quantisation

Let us review what has been achieved so far. In subsection 1.3.2, we found semi-classical approximate solutions $\phi_{(x)} = e^{iS(x)/\hbar}\, a(x)$ in $C^\infty(\mathbb{R}^n)$ to the Schrödinger equation $(H - E)\,\phi = 0$. Although these solutions had asymptotic singularities, we were able to define a smooth half-density a on the lagrangian submanifold $L = \mathrm{im}(dS)$ that mimicked the behaviour of the functions S and a. That is, the corresponding versions of the Hamilton-Jacobi equation for S (conservation of energy) and the homogeneous transport equation for a (stationary character of the solution under the flow of the hamiltonian vector field X_H) were satisfied.

We found that these results could be generalized to arbitrary riemannian manifolds M (subsection 1.3.3). The pull-back along $M \to L$ then completed the quantisation of the semi-classical solutions by letting us pass from half-densities on the lagrangian submanifold L to the intrinsic Hilbert space on the configuration space M.

It must be noted that the procedure has certain limitations, well described in Bates and Weinstein (1997, ch. 4). Firstly, the embedding $\iota : L \to T^*M$ may not be exact. But in the absence of a global primitive ϕ such that $d\phi = \iota^*\alpha$, we can still find a cover $\{L_j\}$ of L and functions $\phi : L_j \to \mathbb{R}$ with $d\phi_j = \iota^*\alpha|_{L_j}$. The half-densities (for $a_j = a|_{L_j}$)

$$I_j = \left(\iota^{-1} \circ \pi^{-1}\right)^* e^{i\phi_j/\hbar}\, a_j \tag{1.28}$$

can then be patched together, provided that the condition

$$\phi_j - \phi_k \in 2\pi\hbar \cdot \mathbb{Z} \tag{1.29}$$

is fulfilled on all intersections $L_j \cap L_k$. If the mapping $\pi_L = \pi \circ \iota : L \to M$ is a diffeomorphism, we shall say that the langrangian submanifold (L, ι) is *prequantisable*.

We may now imagine cases where π_L is not a diffeomorphism. The embedding $\iota : L \to T^*\mathbb{R}$ defined by $\iota(x) = (x^2, x)$ is an example (Bates and Weinstein (1997), p. 38). Here, we may still find a quantisation by piecing together the quantisations on the upper and lower components of the parabola $\iota(L)$. The solution is, however, not defined for $q \leq 0$ in the configuration space $\mathbb{R}$.

This necessitates a more general formulation of the theory, the *Maslov quantisation*. In the simplest case, a lagrangian immersion $\iota : L \to T^*\mathbb{R}$ such that π_L is not a diffeomorphism onto $\mathbb{R}$, we may switch the roles of the coordinates (q, p) on the phase space and use the form $-qdp$ instead of $\alpha = pdq$. We then find a new phase function $\tau = \iota^*(-qdp)$ om L such that the mapping π_p from L to the p-coordinate is a diffeomorphism. The case is then similar to quantisation for diffeomorphisms π_L on the q-coordinate, except that the operation in (1.27) gives us a half-density B on p-space when the function τ is used instead of ϕ. B is then transferred to q-space by means of an asymptotic Fourier transform $\mathcal{F}_\hbar$. The Maslov quantisation is

$$J_\hbar = \mathcal{F}_\hbar^{-1}(B)\, |\, dq\, |^{1/2} \tag{1.30}$$

By finding a cover $\{L_j\}$ of the lagrangian submanifold $\iota : L \to T^*\mathbb{R}$ with the properties that each L_j is diffeomorhic in the p- or q-coordinate and π_L is a diffeomorphism when restricted to intersections $L_j \cap L_k$. A partition of unity $\{h_j\}$ is defined for $\{L_j\}$, and half-densities a on L are quantised by (pre- or Maslov) quantisation on each $(L_j, \iota, a \cdot h_j)$. The half-densities I_j are then summed to the final quantisation of L:

$$I_j(L, \iota, a) = \sum_j I_j \tag{1.31}$$

A consistency criterion is imposed to make this definition coincide with the quantisation in (1.28) (see Bates and Weinstein (1997), p. 45).

1.3.5. Generalizations of the Theory of Quantisation

The theory of quantisation presented in the last subsection can be continued to cover lagrangian submanifolds of any cotangent bundle T^*M and, in the general case, any symplectic manifold P. There is also an alternative approach, *algebraic quantisation* or *deformation quantisation*, where the noncommutative algebraic structure of the quantum observables is derived by the construction of a family of algebras $\mathcal{A}_\hbar$ such that $\mathcal{A}_0$ is commutative in the classical limit $\hbar \to 0$.

Generalizations in other directions are possible. At the far end of the scale, there is Isham's work, starting with group-theoretical quantisation and culminating in a series of articles on quantisation in a general category (Isham (2003), (2004)). In the latter framework, quantisation is carried out on a configuration space consisting of the object set Ob(**C**) in a category **C**. Exponentiated versions of the canonical commutation relations $[Q_i, P_j] = i\hbar\delta^i_j$ are then found for certain "arrow operators" on appropriate function spaces on Ob(**C**).

In chapter 4, we shall try to situate the issue of quantisation within a category-theoretical approach to quantum theory. The choice of the correct level of quantisation becomes particularly urgent in quantum gravity. Any structural level of space-time may be regarded as a possible candidate for quantisation, from the point set structure via topological and causal structure and up to the (Lorentz) metric (Isham (1993)).

On consideration, the issue of quantum gravity may be even more involved than this. At the deepest level, we confront the choices between discrete and continuum versions of space-time, and between space-time as a set elements (points) or as some more general concept.

For a discrete space-time, as used e.g. in the common interpretation of loop quantum gravity, Isham's technique for quantising on categories may, at present, be the sole option available. We shall see how these intuitions can be formalized in chapter 4.

As a formal theory, loop quantum gravity is built on a differentiable manifold, and the theory is developed within the standard set-theoretical framework (the category of sets, **Sets**). But even for continuum theories of space-time there exists a well-known non-standard approach, the models with infinitesimal (nilpotent) numbers. We present the fundamentals of this theory, known as synthetic differential geometry, in the next section. Certain aspects of Hamiltonian mechanics (cf. 1.3.2 - 1.3.3 above), such as the Liouville form, have been transferred to synthetic differential geometry by Lavendhomme (1996). This opens the possibilty of a geometric quantisation in the synthetic context.

1.4. Non-standard Versions of Space

General relativity, also known as *geometrodynamics*, is a theory of the causal connections between events in space-time, standardly modelled as points of a connected Hausdorff manifold ($\mathcal{M}$, **g**), where **g** is a Lorentz metric. The metric **g** is a dynamical object governed by Einstein's equations. Before we turn to the competing perspectives on quantum gravity, we shall widen our range of options by presenting various alternative conceptions of space or space-time.

1.4.1. Synthetic Differential Geometry

(a) Basics of smooth infinitesimal analysis. The possibility of a consistent way of reasoning with infintesimal magnitudes in classical logic surfaced again in the work of a logician, Robinson. The work of Weil and Grothendieck also suggests that alternatives to the Newtonian calculus may be useful, or even necessary, in algebraic geometry. This led to the development of *synthetic differential geometry (SDG)* by Lawvere and Kock. The point of departure for this theory is the following axiom (where the *geometric line R*, a $\mathbb{Q}$-algebra, is the synthetic substitute for the classical field $\mathbb{R}$ of real numbers):

Kock-Lawvere axiom *Let D be the set of square-zero elements in R. For every* $f : D \to R$, *there exists one and only one* $b \in R$ *such that, for every* $d \in D$, $f(d) = f(0) + d \cdot b$.

Using classical logic, the axiom is content-free, as we quickly derive the disturbing proposition $R = \{0\}$. (For all results in this section, see Kock (2006) and Lavendhomme (1996).) In the naïve formulation of the theory, classical logic is then abandoned, and synthetic calculus is developed from axioms, using only the rules of intuitionistic logic. The *derivative f'(a)* of a function $f : R \to R$ at a can be defined at once as the unique b in R such that

$$\forall\, d \in D\; f(a + d) = f(a) + d \cdot b \tag{1.32}$$

Consequently, all functions are infinitely differentiable. Proofs of the algebraic properties of the derivative and a version of Taylor's formula are natural and simple in this context. By extension to several variables, directional derivatives are also definable. The notion of an interval requires the introduction of the reflexive and transitive *preorder relation* $\leq$. This relation is assumed to be compatible with the algebraic properties of R, but it is not antisymmetric, due to the further unusual assumption

$$d \in D \Rightarrow 0 \leq d \bigwedge d \leq 0 \tag{1.33}$$

Antisymmetry would collapse the set of nilpotent elements to $D = \{0\}$. It is now possible to define the *closed interval from a to b* as

$$[a, b] = \{x \in R \mid a \leq x \leq b\} \tag{1.34}$$

The nilpotent elements are then members of the infinitesimally short interval [0, 0]. In order to define integration, a new axiom is necessary:

Integration axiom (Kock-Reyes) *For any function* $f : [0, 1] \to R$*, there exists a unique function* $g : [0, 1] \to R$ *such that* $g' = f$ *and* $g(0) = 0$.

$\int_0^x f(t)\, dt$ (for $x \in [0, 1]$) is then simply the value $g(x)$. The definition is extended to any interval $[a, b]$ by an application of Hadamard's lemma, and the usual properties of the integral hold.

(b) Manifolds as microlinear objects. (The rest of this subsection is a tool box for future work in topos physics, but the material will not be needed in the rest of this thesis.) It is now tempting to commence the subject of differential geometry by introducing tangency in the following manner:

A *tangent vector* to a manifold M with base point m is a mapping $t : D \to M$ such that $t(0) = m$.

Indeed, this is the correct definition, but we need to be clearer about the notion of a manifold, or a microlinear object. A *Weil algebra* is finitely presented as $W \simeq R[X_1, X_2, ..., X_r] / I$, where $R[X_1, X_2, ..., X_r]$ is the free R-algebra with r generators, and I is an ideal generated by a finite set of polynomials. For an R-algebra C, we define the *spectrum* of W in C as

$$Spec_C(W) = \{(a_1, ..., a_r) \in C^r \mid P(a_1, ..., a_r) = 0 \text{ for all generators of } I\}. \tag{1.35}$$

The spectra are the "small" objects of SDG. Note that the set D is simply $Spec_R(W)$ for $W = R/(X^2)$. Several other small objects can now be defined, such as

$$D_k = Spec_R\left(R[X] \Big/ \left(X^{k+1}\right)\right) = \{d \in R \mid d^{k+1} = 0\} \tag{1.36}$$

$$D(k) = Spec_R\left(R[X_1, X_2, ..., X_k] \Big/ \left(X_i X_j\right)\right) = \left\{(d_1, d_2, ..., d_k) \,\middle|\, d_i d_j = 0\right\} \tag{1.37}$$

$$D^2 = D \times D = Spec_R\left(R[X, Y] \Big/ \left(X^2, Y^2\right)\right) = \left\{(d_1, d_2) \,\middle|\, {d_1}^2 = {d_2}^2 = 0\right\}. \tag{1.38}$$

The geometric line R possesses a certain "blindness" with regard to these small objects. E.g. if $f(d_1 \cdot d_2) = 0$ for all $d_1, d_2 \in D$, then $f(d) = 0$ for all $d \in D$. This idea can be given a formal expression by means of category theory. By applying the functor $Spec_R$ to a limit $(L \xrightarrow{\lambda_i} A_i)$ in the category of R-algebras, we shall say that the resulting diagram is a *finite quasi colimit* when L itself is a Weil algebra and λ_i are homomorphisms of Weil algebras. The property of R that we want to capture, can then be expressed by saying that an object M is *microlinear* if M perceives (that is, M factors uniquely through L) finite quasi colimits L of small objects as colimits. R is microlinear in this sense (if we assume a general version of the Kock-Lawvere axiom), and so are R^n and R^X (for X a set). Even the infinite-dimensional R^R is a microlinear object, and, in general, so is M^M.

With the notion of a "manifold" (a microlinear object) and a tangent vector in place, the *tangent vector set* T_mM at $m \in M$ and the *tangent bundle* TM $(\tau : M^D \to M)$ on M can be defined in the natural way. A *vector field* on M is a section of the tangent bundle on M. A vector field may be seen as a mapping $X : D \to M^M$, the image X_d being an infinitesimal transformation. $\mathcal{X}(M)$ will be the R-module of vector fields, another microlinear object. From the microlinearity of M^M it can be proven that there is a unique vector field $[X, Y]$, the *commutator* or *Lie bracket* of vector fields X and Y such that (for $d_1, d_2 \in D$)

$$[X, Y](d_1 \cdot d_2) = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1} \tag{1.39}$$

In a natural manner, $[X, Y]$ attempts to close the microcurve starting at a point $m \in M$. The Jacobi identity is satisfied by the Lie bracket, so $\mathcal{X}(M)$ is an R-Lie algebra. For a vector field X, the *Lie derivative* of a function $f : M \to R$ is the function $L_X(f) : M \to R$ given by the equation

$$f(X(m, d)) = f(m) + d \cdot L_X(f) \tag{1.40}$$

By introducing *n-microcubes* on M as mappings $\gamma : D^n \to M$, we may define *singular differential n-forms* on M with value in the R-module E as mappings $\omega : M^{(D^n)} \to E$ that are alternated (they give the same value on microcubes γ that are permutated with respect to their arguments $(d_1, d_2, \ldots, d_n)$, multiplied with the sign σ of the permutation) and n-homogeneous $(\omega(\gamma)(d_1, \ldots, \alpha \cdot d_k, \ldots, d_n) = \alpha \cdot \omega(\gamma)(d_1, d_2, \ldots, d_n)$ for every k). The microlinear R-module of differential n-forms is denoted by $\Omega_n(M; E)$.

We note that the notion of a singular differential form is more inclusive than the classical notion. The latter is defined as a mapping from $D(n)$ to E, and, as may easily be seen from (1.36)-(1.37) above, there exists a canonical injection $i : D(n) \to D^n$. In case M is an R-module, the flat connection $\nabla : M^{(D(2))} \to M^{(D \times D)}$ then allows us to construct a bijection from the classical to the singular 2-forms.

The exterior differential in SDG is constructed by modifying the n-microcubes into the classically strange *marked n-microcubes*. These are pairs $(\gamma, \overline{e})$ of a microcube γ and $\overline{e} \in D^n$, where the element $\overline{e}$ identifies the edge of the microcube. The free R-module $C_n(M)$ generated by the marked microcubes then contains the *inifinitesimal n-chains*. The boundary operator ∂ can be defined, and it is easy to prove that the boundary of a boundary is zero. The *integral* of a differential n-form ω on the marked microcube $(\gamma, \overline{e})$ has a particularly simple form:

$$\int_{(\gamma, \overline{e})} \omega = e \cdot e \cdot \cdots \cdot e \cdot \omega(\gamma). \tag{1.41}$$

It is possible to prove the existence and uniqueness of a differential $(n+1)$-form $d\omega$ with the property that

$$\int_{\partial(\gamma, \overline{e})} \omega = \int_{(\gamma, \overline{e})} d\omega. \tag{1.42}$$

The form $d\omega$ is the *exterior derivative* of ω. For microlinear objects M and $E = R$, the definition of integration can be extended beyond the infinitesimal case, and a synthetic version of Stoke's theorem is provable. Common constructions of differential geometry, such as an exterior algebra and Lie derivatives of differential forms, are also within reach. The definition of a *connection* has a strong intuitive motivation. As mentioned above, it is given as a map $\nabla : M^{(D(2))} \to M^{(D \times D)}$, and takes a pair of infinitesimal line segments (t_1, t_2) at the point m to a microsquare at m, thereby capturing the notion of parallel transport.

(c) Symplectic structure on T^*M and Hamiltonian mechanics. In subsection 1.3.3 we discussed possible applications of symplectic geometry in the mathematical theory of quantisation. Let us have a look at how the notion of symplectic structure fares in SDG. The elements of the theory have been sketched in Lavendhomme ((1997), ch. 7).

In classical mechanics, our standard example of a $2n$-dimensional symplectic manifold is $T^* \mathbb{R}^n$ with symplectic structure $\omega = \Sigma dq_j \wedge dp_j$. Instead of the manifold $\mathbb{R}^n$, we shall use the microlinear object R^n. As we have seen, the tangent bundle TR^n $(\tau : (R^n)^D \to R^n)$ is constructed with the SDG tangents, maps from the infinitesimal geometric line D into R^n, as members. The fibres of the *cotangent bundle* $\pi : T^*R^n \to R^n$ are then the duals $(T_x R^n)^*$. We recall the definitions of singular n-forms and exterior derivation given above and introduce the *Liouville 1-form* θ and the *Liouville 2-form* ω on π as

$$\theta : TT^* R^n \to R^n \text{ given by } \theta(v) = v(0)\,(\pi \circ v) \text{ for tangent } v \tag{1.43}$$

$$\omega = -\mathrm{d}\theta. \tag{1.44}$$

If we work through the definition in (1.42), we find that the tangent vector $w : D \to T^*R^n$ such that $w(d) = (x, u) + d\cdot(y, v)$ at the point (x, u) in T^*R^n is mapped to $\theta(w) = w(0)(\pi\circ w) = (x, u)(x + d\cdot y) = u(y) = \Sigma u_i y_i$. If we use standard notation $(q, p) = (x, u)$ for the position and momentum coordinates in the phase space T^*R^n, we derive the usual definitions $\theta = \Sigma p_i dq_i$ and $\omega = \Sigma dq_j \wedge dp_j$. The 2-form ω is a symplectic structure in the sense explained in subsection 1.3.3 (closed and nondegenerate). T^*R^n is a symplectic manifold.

The Hamiltonian H is a map from T^*R^n to the smoth line R, e.g. $H = p^2/2m + kq^2/2$ for the harmonic oscillator. We use the *1*-form $dH : TT^* R^n \to R$ and define the Hamiltonian vector field $X_H : T^* R^n \to TT^* R^n$ as the unique solution of the equation

$$dH(Y) = \omega(X_H, Y). \tag{1.45}$$

It can then be shown that an integral curve $(q_i(t), p_i(t))$ in $T^* R^n$ for the Hamiltonian field X_H satifies the equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \text{ and } \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{1.46}$$

These are the *canonical equations of Hamilton*. The symplectic structure discussed in this section is the special case $(R^{(T^* R^n)}, \mathcal{X}(T^* R^n))$ in the synthetic theory of *Lie objects* (A, L), where A is an R-algebra and L is a Lie algebra over R.

(d) The model theory of SDG. The theory above has been developed using intuitionistic logic in a naïve set-theoretical context. There exists, however, a family of sophisticated models of SDG, the *topos models*. In these structures, the notion of an element x as a member of a set A is understood in a different manner. One may speak of the "snapshot" of an object at a certain stage, of an object varying over or parametrized by a domain of variation (such as a time line), or regard the commutative rings $\mathbb{R}$, $\mathbb{R}[\epsilon] = \mathbb{R}[x]/(x^2)$ and $\mathbb{C}$ as the "stages of definition" of a geometric object. The *well-adapted* models are then the topoi in which the smooth manifolds are contained as a subcategory.

Most of these models are of the form $\mathbf{Sets}^C$, where **Sets** is the category of sets and C is the category varied over. In the SDG case C may be a small category of rings. As an example (from Lavendhomme (1996)), we may let C be the category $\mathbb{R}\text{-}alg_f$ of $\mathbb{R}$-algebras of finite type (here $\mathbb{R}$ is the standard set of reals). The forgetful functor R, the functor that disregards the structure of the objects in $\mathbb{R}\text{-}alg_f$ and knows the $\mathbb{R}\text{-}alg_f$-homomorphisms only as functions in the set-theoretical sense, will then represent the real line in the topos $\mathbf{Sets}^{\mathbb{R}-alg_f}$. In fact, R is representable in the topos, and the same holds for the object D of square-zeros. Also, the object R is isomorphic to the functor $A \to$ (the set of elements of $A[X]$). This points to the limitations of the model: all functions from R to R are polynomials.

Other models will have to be considered in order to compare the classical and the synthetic theory of manifolds. Such are the topoi $\mathcal{S}$ (the topos with C^∞-Alg_{fg}, the finitely generated C^∞-algebras, as the category varied over) and $\mathcal{G}$ (the topos of sheaves on the dual of the category of C^∞-algebras with germ-determined ideals). Topoi are also useful for modelling other aspects of quantum physics, as demonstrated by the work of Döring and Isham (2007). We shall study topos models more thoroughly in chapter 2.

1.4.2. Discrete Models of Space-time

At the other extreme from the smooth spaces of synthetic differential geometry, we find research programs that attempt to build physical theory on a discrete conception of space-time. These may either take the idea of a discrete structure as fundamental (Sorkin's causal sets) and try to embed discreteness into standard manifolds $(\mathcal{M}, \mathbf{g})$, or, if the theory is formulated within standard differential geometry, one may attempt to prove the discreteness of the spectra of certain geometric entities, such as the area and volume operators (loop quantum gravity).

(a) The causal set programme. The idea of a discrete space is already present in Riemann's famous *Habilitationsschrift* (see Spivak (1979)). The systematic study of discrete structure as a serious candidate for a theory of physical space-time was initiated by Sorkin. (We shall follow the review in Henson (2006).) As befits a discrete theory, the basic construction is strikingly simple. A *causal set* or *causet* is a partially ordered set C with the order relation $\prec$ ("to the past of") and satisfying the condition of *local finiteness*:

$$\forall\, x, z \in C : \mathbf{card}\,\{y \in C \,|\, x \prec y \prec z\} < \infty. \tag{1.47}$$

The diagram in figure 1 represents a simple causet C_1. This diagram, known as the "crown", cannot be embedded in Minkowski 1+1-dimensional space $\mathcal{M}_2$ (the elements 2 and 5 are not causally related in C_1, but their light cones intersect in $\mathcal{M}_2$). Yet, unfolding the diagram in 2+1-dimensional Minkowski space, we find that the causet can be embedded in $\mathcal{M}_3$.

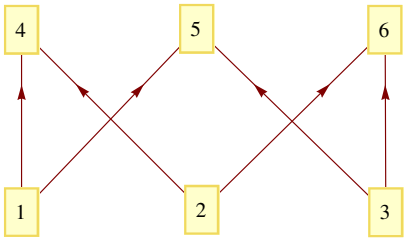


Figure 1.2. The "crown" causet.

As it is expected that sums of causets, properly weighted, should be approximated by a Lorentzian manifold $\mathcal{M}$ near classical solutions, this observation places an important condition on the available manifolds. But in order to have useful information about the manifold, it is also necessary that the elements of C have a certain density ρ in $\mathcal{M}$. The concept of a *sprinkling*, a random selection of points in $\mathcal{M}$, is introduced for this purpose. For a causet C embedded in a manifold $\mathcal{M}$, we say that the embedding is *faithful* if C comes from a sprinkling of $\mathcal{M}$ with high probability. An unproven intuition (the *"Hauptvermutung"*) that a causet should not be faithfully embeddable in "dissimilar" manifolds, guides this research.

Given a causet C, it may be possible to extract information about the corresponding manifold $\mathcal{M}$. Estimators of dimension, distance and volume have been constructed. The volume V is approximated by the number of elements sprinkled into the region, and timelike distance (in four dimensions) can then be found from the formula (where $I(x, y)$ is the interval between timelike points x and y)

$$V(I(x, y)) = \frac{\pi}{24} d(x, y)^4. \tag{1.48}$$

In our description of physical theories above, we introduced the strongly supported condition of Lorentz invariance (subsection 1.2.4). Due to the random character of the sprinkling process, no preferred directions can be identified in causal set structures, so the theory is Lorentz invariant by default. This contrasts with theories where discreteness is modelled using lattices.

In the last subsection, we saw that the microlinear objects of synthetic differential geometry replaced the standard manifolds as mathematical structures suitable for constructions in dynamics. The picture in causal set dynamics is less clear. One early idea, inspired by the path integral formulation in quantum physics, was to extract the equations of motion by finding the stationary value of the sum over histories $\Sigma\exp(i\text{-}(S(C))$, where $S(C)$ is the action of the causal set C. Apart from the problem of defining an appropriate action for causal sets, there is no obvious bound to the terms of the sum. Should causal sets that are not embeddable in a manifold be included?

Another approach to causal set dynamics relies on the intuitive idea of growing causal sets by adding future or spacelike points to finite causets. The probability of each such transition is then constrained by physical laws. These spaces, the models of *classical sequential growth*, are still not fully developed for the quantum case (cf. 1.4.4 below).

It should be noted that a particle in the causal set picture is naturally modelled as moving along a branch in the set. In order to determine the velocity of the particle, points in the past of the particle must be examined, as the set of points that are spacelike to the position of the particle at a time are unstructured as a causal subset, and carry no information. Postulating a principle of locality for casual sets, it then follows that the velocity of the particle can only be determined with a degree of uncertainty. This phenomenon is referred to as a *swerving of the particle* or *diffusion*.

Causal sets have been used by Isham (2003) as a simple illustration of quantisation on a category. As they are *bon à penser*, we will return to them in chapter 4.

(b) Loop quantum gravity. Causal set theory is still incomplete in many respects. There is, however, another approach to the structure of space and time that does not take discreteness for granted, but instead tries to derive it within the theory. This is the program of loop quantum gravity (LQG). We shall sketch an outline of the geometric implications of LQG and return to the full theory in chapter 3.

The radical space-time structures of LQG are embedded in a standard differential manifold, and physics is implemented by representing the observables of the theory as operators on a Hilbert space. Therefore, the cumbersome name "modern canonical quantum general relativity" is sometimes used. The real geometrical structure of the theory emerges because the operators of the theory are constructed in accordance with the principle of *diffeomorphism invariance*. This is the mathematical reflection of the background independence of general relativity: at the outset, there is no space-time structure, flat or curved, on which the states of the theory may be defined.

The operators corresponding to the geometric properties of the system are then defined, and the attempt is made to deduce the spectral properties of the geometric entities. In the case of the volume operator **V**, the spectrum is claimed to be discrete, leading to the non-classical picture of a space composed of finite, indivisible "grains" or "quanta of gravity". The eigenvectors $|s\rangle$ of the geometric operator form a basis of the corresponding Hilbert space $\mathcal{K}_{\text{Diff}}$, and the radical picture of space as a superposition of "spin networks" emerges, with the nodes representing volume grains and the links representing the adjacent areas. This is the real quantum geometry which underlies the arbitrary coordinatization of the manifold we started with.

The brief description above relies on the canonical formulation of LQG. There exists an alternative version, inspired by Feynman's path integral recipe. In this formalism, the spin networks are allowed to grow into "histories", known as *spinfoams*, and an analogue of the path integral in quantum mechanics is found by summing over histories.

1.4.3. Noncommutative Geometry

The incompatibility of the pictures presented in the last two sections, the point as a nucleus with an aura of infinitesimal magnitudes (synthetic differential geometry) or as an isolated node in a discrete network tagged by a probability distribution (causal sets), may lead one to doubt the coherence and usefulness of the notion of space-time as a point-filled structure. As a final step, therefore, we should examine the "pointless" spaces investigated in *noncommutative geometry*. (The main reference in this area is Connes (1994). A comprehensive introduction is found in Gracía-Bondia, Várilly and Figueroa (2001).)

The starting point of the theory is the important correspondence between algebras and spaces stated in the following theorem:

First theorem of Gelfand and Naimark. *Every commutative C*-algebra A with a unit is isomorphic to C(X) for some compact Hausdorff space X.*

The algebraic operations in $C(X)$ are pointwise, and involution is given as $f^*(x) = \overline{f(x)}$. The theorem is proved by identifying the Hausdorff space X with the Gelfand spectrum $\Delta(A)$ of characters ω of the C^*-algebra A. The points x of X are thus recovered as algebra homomorphisms $\omega : A \to \mathbb{C}$.

One may now translate other geometric concepts into their algebraic counterparts, creating a series of correspondences between the two realms. In this way, it is shown that the group of automorphisms of a commutative C^*-algebra A is isomorphic to the group of homeomorphisms of the space $\Delta(A)$. Also, there is a duality between probability measures μ defined on a metrizable space X and positive normalized linear functionals ρ on A. The group structure of a compact topological group is represented as a *-homomorphism $\phi : A \to A \otimes A$ of the algebra A. The table of relationships can be extended to include a wide range of examples from topology and differential geometry.

The key idea of noncommutative geometry is to generalize the correspondences observed in the case of a commutative C^*-algebra to an arbitrary C^*-algebra A. By the second theorem of Gelfand and Naimark, A is isomorphic to a subalgebra of the bounded operators $B(H)$ that is norm-closed and involution-closed. These algebras are in general associated with *quantum spaces* that may not have a standard topological or manifold structure. The space of leaves of foliation of a smooth manifold exemplifies this. Another beautiful example is the space of equivalence classes of Penrose tilings (Connes (1994)).

Standard symplectic structures were the starting point for the quantisation scheme described in section 1.3. Then, in subsection 1.4.1, we noted that the basics of classical symplectic mechanics were readily reproduced in the framework of synthetic differential geometry. The difficulties of doing canonical quantum mechanics in the framework of synthetic differential geometry is an area yet to be opened for research.

This contrasts with the close symbiosis between quantum mechanical concepts and the constructions of noncommutative geometry. Noncommutative geometry offers us a quantum counterpart to the classical phase spaces of symplectic geometry, the quantum space (or noncommtative algebra) $A[\nu]$ with the Fedosov product $\star$. This is the procedure of deformation quantisation mentioned briefly in 1.3.5. There is thus a deep connection between the programme of noncommutative geometry and quantum mechanical theory.

Recently, Banaschewski and Mulvey (2006) have proven that the result of Gelfand and Naimark can be extended to a duality between the category of commutative *C**-algebras and the category of certain locales in a Grothendieck topos (*the constructive Gelfand duality theorem*). We shall look into the construction of the state space of Heunen, Landsman and Spitters (2008) with the help of Gelfand duality when we turn to the topos models of quantum physics in the next chapter. Because of the importance of topoi as models for synthetic differential geometry, we shall also consider the implications of constructive Gelfand duality in this context.

Likewise we may point out some common ground between the study of discrete structures and noncommutative geomtry. A causet *C* corresponds to a noncoummatative *C**-algebra of operator valued functions (see Criscuolo and Waelbroeck (1998)). We also suggest that there may be a connection via the concept of diffusion mentioned above.

1.5. Quantum Gravity

In subsection 1.2.6, we briefly explored the possibility of doing quantum field theory (QFT) on a fixed, non-dynamical space-time background. We interrupted our discussion of physical theory at the threshold of quantum gravity, the attempt to build a consistent theory of quantum gravitational fields. From here, the road branches off in several directions. There seems to be no general agreement on the classification of these programs, but two main lines of research are distinguished by most observers (see e.g. Kiefer (2004) and Rovelli (2003), app. C). These are the covariant and the canonical approaches to quantum gravity (QG).

1.5.1. The Covariant Approach

Like QFT on a curved space-time, this program keeps a non-dynamical background, and introduces gravitational waves as perturbations of the space-time metric on the manifold. To this extent, it represents the least radical departure from the physical theory presented in section 1.2. It then tries to quantise the linear part of the gravitational field, preserving Poincaré invariance in the process (cf. subsection 1.2.4). For this reason, it is known as the *covariant* apporach.

We shall begin, therefore, by assuming that the metric **g** on our model of space-time, the differentiable manifold $\mathcal{M}$, can be divided into two components, the Minkowski metric $\boldsymbol{\eta}$ and a small perturbation **f**:

$$g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}. \tag{1.49}$$

Redefining the perturbation field **f** as $\overline{f}_{\mu\nu} = f_{\mu\nu} - 1/2\,\eta_{\mu\nu} f^{\rho}{}_{\rho}$, Einstein's equations can be written as the wave equation

$$\Box \overline{f}_{\mu\nu} = -16\,\pi G T_{\mu\nu}. \tag{1.50}$$

The energy-momentum tensor T acts as a source of the field **f**. The solutions for the vacuum are given as plane waves with the polarization tensor $e_{\mu\nu}$:

$$f_{\mu\nu}(x) = e_{\mu\nu}\, \mathrm{e}^{ikx} + (e^*)_{\mu\nu}\, \mathrm{e}^{-ikx}. \tag{1.51}$$

Under natural assumptions, the plane wave has two independent polarization states transverse to the direction of propagation. Let us consider the states of right and left circular polarization, e_R and e_L. For a rotation with an angle θ, the plane wave $f_{\mu\nu}$ transforms as

$$f_{\mu\nu} \to \mathrm{e}^{\pm i2\theta}\, f_{\mu\nu}. \tag{1.52}$$

We say that $f_{\mu\nu}$ has the *helicity* ±2. If this theory is developed into a full quantum field theory on flat space-time in the manner explained in subsection 1.2.6, we are lead back to non-quantised general relativity (Kiefer (2004), p. 27f). There we also hinted at the *approximate* nature of the particle interpretation of the theory. This point may now be clarified. We fix the Minkowski metric η and consider the unitary transformations $U(\Lambda, 0)$ on the states Ψ of the Hilbert space $\mathcal{H}(\Sigma)$, where (Λ, a) is a Poincaré transformation with rotation θ on $\mathcal{M}$ (see 1.2.4). As usual, one-particle states are introduced, and the Lorentz transformation for a massless particle of helicity 2 is given by (Weinberg (1995), p. 72):

$$U(\Lambda)\, \Psi_{p,\sigma} = \mathrm{N}\exp(\mathrm{i}2\theta(\Lambda, p))\, \Psi_{p,\sigma}. \tag{1.53}$$

This particle, definable in the context of equation (1.49), is called the *graviton*. The step to quantum field theory (cf. 1.2.5) can now be taken by quantising the solution in (1.50). The result is

$$f_{\mu\nu}(x) = \sum_{\sigma} \int \frac{d^3k}{\sqrt{2\,|\,\mathbf{k}\,|}} \left[a(\mathbf{k}, \sigma)\, e_{\mu\nu}(\mathbf{k}, \sigma)\, e^{ikx} + a^{\dagger}(\mathbf{k}, \sigma)\, e^*{}_{\mu\nu}(\mathbf{k}, \sigma)\, \mathrm{e}^{-ikx} \right]. \tag{1.54}$$

As noted in subsection 1.2.5, the perturbation theory of this formalism faces major obstacles. When we calculate terms corresponding to the more elaborate Feynman diagrams associated with a scattering situation, divergences show up. Analysis shows that the degree of divergence is tied to the dimensionality Δ of the coupling constant for the interaction in question. In general, Δ is found from

$$\Delta = 4 - d - \sum_f n_f \left(s_f + 1 \right). \tag{1.55}$$

Here, d is the number of derivatives of the interaction formula, n_f is the number of fields of type f, and s_f is a number dependent on the particle type $\left(s_f = 0 \text{ for gravitons}\right)$. When $\Delta < 0$, as in the case of the self-interaction $Gf(\partial f)(\partial f)$ of the gravitational field, the calculation diverges, and the theory is therefore non-renormalizable. In subection 1.5.3, we shall see how work along these lines culminated in the theory of strings.

1.5.2. The Canonical Approach

Like the covariant approach, the *canonical* approach is an attempt to quantise a classical theory. Unlike the covariant line of research, it does not preserve covariance along the way. The canonical approach quantises a Hamiltonian system of constraints for general relativity (GR). The abstract scheme for quantising such systems was given in Dirac's *Lectures on Quantum Mechanics* (Dirac (1964)).

In non-relativistic quantum theory, *time* has a particular status. It is not an operator of its own, but remains in the classical background during the quantisation of the other parameters. The time-dependent Schrödinger equation (cf. 1.9) is thus manifestly non-relativistic. In quantum field theory, the symmetries of special relativity are dealt with by demanding that the Poincaré group has a unitary representation. Written as a condition on the Hamiltonian density $\mathcal{H}$, this amounts to a demand for *microcausality* (for space-time points x, x' with spacelike separation; see Weinberg (1995), p. 191):

$$[\mathcal{H}(x), \mathcal{H}(x')] = 0. \tag{1.56}$$

This requirement, along with the transformation properties of the scalar $\mathcal{H}$, implies the Lorentz-invariance of the *S-matrix*, the main tool for describing scattering experiments in quantum physics.

The absolute character of time in this conception does not accord very well with the dynamical notion of time known from GR. One may try to hide the absoluteness of time by treating it as a dynamical variable. Formally, this can be done by transforming absolute time t into a formal parameter $\tau(t)$. In the simple case of a one-particle system, the Lagrangian $L(q, dq/dt)$ can then be rewritten as a Lagrangian $\overline{L}(q, dq/d\tau, dt/d\tau)$ that is homogeneous in the velocities. For the new dynamical variable t, we have the canonical momentum (using the Legendre transformation)

$$p_t = \frac{\partial \overline{L}}{\partial \dot{t}} = -H. \tag{1.57}$$

H is the Hamiltonian of L, not $\overline{L}$. Defining the *total Hamiltonian* as $H_T = H + p_t$, we have the constraint

$$H_T \approx 0. \tag{1.58}$$

The notion of *weak equality* ('≈' in the equation above) was introduced by Dirac (1964). We are only allowed to use the dependency of the dynamical variables expressed by (1.57) after working out the Poisson brackets under consideration. Fundamentally, this is due to the non-uniqueness of the Hamiltonian for a system with dependent variables.

Turning to general relativity, the steps to quantum gravity can be outlined in the following manner (cf. Kiefer (2004), p. 118ff). Firstly, we identify the constraints corresponding to (1.57) for GR. Secondly, using Dirac's abstract scheme of quantisation, we switch from Poisson brackets for the dynamical variables of GR to commutator brackets for the associated operators (cf. 1.13-1.14 above). Thirdly, the representation space for the operators is found. Commonly, this will be a space of functionals Ψ on the space of 3-metrics. Finally, the GR constraints are quantised as

$$\mathcal{H}_{\perp}\,\Psi = 0 \bigwedge \mathcal{H}_{a}\,\Psi = 0. \tag{1.59}$$

The first of these constraints is known as the *Wheeler-De Witt equation*. Only states for which (1.58) holds, qualify as physical. Later, in section 3.1, we shall see how the quantisation of the constraints and the restriction to physical space is carried out in one version of the canonical approach, loop quantum gravity (cf. subsection 1.4.2(b) for the space-time interpretation associated with this program).

1.5.3. Quantum Gravity in String Theory

The covariant and the canonical camps start with a classical theory of gravity and try to apply quantisation rules. String theory, introduced in the 1960s as an attempt within covariant physics to explain the proliferation of strongly interacting particles, developed into something different. The aims of string theory are more comprehensive than the approaches we have considered so far. String theory is, or should be, a unified theory of all physical interactions, among them gravity. In it, quantum gravity appears as a consequence of a larger quantum framework. Below, we give only a considerably simplified review of this deduction. More details are found in the standard accounts of Green, Schwarz, Witten (1987) and Polchinski (1998). (Another, very accessible account of the fundamentals of string theory is found in Zwiebach (2004).)

The *relativistic string* sweeps out a two-dimensional world-sheet in Minkowski space. Its surface $X^{\mu}(\tau, \sigma)$ is described by two parameters. The equations of motion for the string is obtained by defining the *Nambu-Goto action* S_{NG} as (where M is the world-sheet and h_{ab} is the induced metric)

$$S_{\mathrm{NG}} = -\frac{1}{2\,\pi\alpha'}\int_{M} d\tau\, d\sigma(-\det h_{ab})^{1/2}. \tag{1.60}$$

The Nambu-Goto action is Poincaré invariant. An alternative version, the Polyakov action, quickly gives us a useful wave equation for the case of an open string:

$$\left(\frac{\partial}{\partial\tau^{2}} - \frac{\partial}{\partial\sigma^{2}}\right) X^{\mu}(\tau, \sigma) = 0. \tag{1.61}$$

The solution for a string with freely moving endpoints (with x^{μ} and p^{μ} the position and momentum of the centre of mass) is

$$X^{\mu}(\tau, \sigma) = x^{\mu} + 2\,\alpha'\,p^{\mu} + i\,\sqrt{2\,\alpha'}\sum_{n\neq 0}\frac{\alpha_{n}^{\mu}}{n}\,e^{-\mathrm{in}\tau}\,\mathrm{cosn}\sigma. \tag{1.62}$$

Instead of using the commutation relations for X^μ and the conjugate canonical momentum P^μ, we quantise x^μ, p^μ and the discrete modes α_n^μ. For the case of the closed string, we must also distinguish between rightmoving modes α_n^μ and leftmoving modes $\overline{\alpha}_n^\mu$. Quantising the Poisson brackets for these variables, we have the commutator brackets for the associated operators, e.g.

$$[\alpha_n^\mu, \alpha_n^\nu] = \mathrm{m}\delta_{m,-n}\,\eta^{\mu\nu}. \tag{1.63}$$

The Fock space (cf. 1.2.4) may now be built from a ground state with the help of the annihilation and creation operators α_n^μ and $\alpha_n^{\mu\dagger}$. For the closed string, the first excited state is massless with three irreducible parts. One of these, the symmetric tensor representation, is a spin-2 particle in four dimensions. In subsection 1.5.1, we identified the massless spin-2 particle as the graviton, the particle of the gravitational interaction.

It would seem that the richness of structure in string theory allows the development of a quantum gravity formalism. Yet we are in a strange landscape. The theory only lives in 26-dimensional space-time, and there are no half-spin particles. *Supersymmetry*, the pairing of bosons with corresponding fermions, is therefore added to string theory (cf. Dine (2007)). This is the theory of *superstrings*.

1.5.4. Other Approaches to Quantum Gravity

Among the other current approaches to quantum gravity we mention the *sum-over-histories* approach and *lattice quantum gravity* (of which *Regge calculus* and *dynamical triangulations* are varieties). There are also the lesser schools of QG associated with the non-commutative program, the causal set programme of Sorkin, and Isham's topos ideas.

In subsection 1.4.2(a) we looked at the discrete space of causal sets. The properties of a causal set that are relevant for the reconstruction of space-time geometry are its *order* and *number*. It is known that the order relation by itself suffices if we want to recover a conformal metric on four-dimensional Minkowski space-time $\mathcal{M}_4$. The conformal factor, or the volume element $\sqrt{-g}\;d^n x$, is then found by counting the finitely many causet elements in the region. As an example, a light ray of $\mathcal{M}_4$ is given as the maximal chain such that an interval between members of the chain is also a chain.

This takes care of the kinematics of the theory. Let us now try to state the implications for a discrete quantum gravity, or *quantum causet dynamics* (cf. Sorkin (2003)). In the sequential growth models (cf. 1.4.2), the causets are seen as developing in time under a *law of growth*, an assignment of probability to the birth of a new element. The assignments are limited by two principles, *Bell causality* (the exclusion of superluminal influence) and a particular brand of covariance, *discrete general covariance* (the probability of reaching a given causet is independent of the order of birth of its elements). With these restrictions, Rideout and Sorkin (1999) have given formulae for the transition probability $C \to C'$, where C' is the birth of a new element. In chapter 4, we will return to the discrete approaches of the causal set program in the context of category quantisation.

We will be guided in our search by recent ideas on the connections between quantum physics, category theory and the theory of topoi. Indeed, the relationship between these fields is the main subject of this dissertation.

CHAPTER TWO

Topoi for Quantum Physics

In chapter 1, the description of a physical system was organized around the concepts "state", "observable" and "value". In particular, we had a glimpse at the ordinary model for this triple in quantum mechanics. Let us consider briefly the setup of a typical experimental situation. The Copenhagen interpretation of quantum mechanics presents us with the following picture:

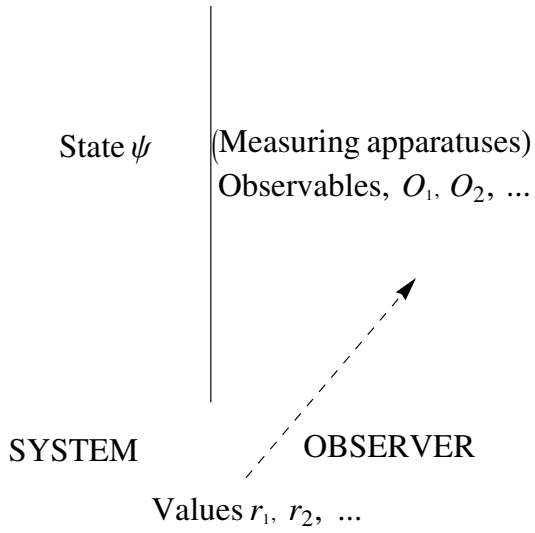


Figure 2.1. Physical descriptions.

The Kochen-Specker theorem in quantum mechanics tells us that we may not assume that the real values "exist" on the left-hand side above. The physically meaningful real value is not an entity on the system side, it arises on the side of the observer as the result of a measurement. This picture is conceptually troublesome: if everything in the universe, the observer included, is moved over to the system side, there are no physically meaningful quantities left either.

Let this problem act as a first motivation for the models of topos physics to be presented in this chapter. They are meant to remedy this defect, the paradox of quantum cosmology. In the foundational approach of Chris Isham and Andreas Döring, physical magnitudes have values independent of observation. We shall see that this value does not have to be a real number. In section 2.1, we review the backgrouond from category theory and topos theory which is necessary in order to understand these models. Then, in section 2.2, we turn to the latest version of the topos models, developed by Isham and Döring in a series of articles in 2007. A summary of the closely related work of Heunen, Landsman and Spitters (and several others) is also given.

2.1. A Short Introduction to Topoi

2.1.1. Some Basic Category Theory

The standard introduction to category theory is Mac Lane (1997). We summarize the basic definitions and results that will be needed in the sequel. A *category* **C** is a collection Ob(**C**) of objects $A, A', ..., X, ...$ and a collection of morphisms $f, g, ...$ The morphisms (arrows) are assigned objects $\mathrm{dom}(f) = X$ and $\mathrm{cod}(f) = Y$. We write this as $f : X \to Y$. From $f : X \to Y$ and $g : Y \to Z$ we form the composition $g \circ f : X \to Z$, for which the associative law $h \circ (g \circ f) = (h \circ g) \circ f$ holds. For any X, there is also an identity arrow 1_X fulfilling the identity laws $f \circ 1_X = f$ and $1_Y \circ g = g$. We write $\mathrm{Hom}(A, A')$ for the set of arrows f with $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = A$. Hom(**C**) is the total set of arrows in **C**. For a category **C**, the *opposite category* $\mathbf{C}^{\mathrm{op}}$ has the same objects and arrows as **C**, but dom(f) and cod(f) are interchanged for all f, and $g \circ f$ in **C** equals $f \circ g$ in $\mathbf{C}^{\mathrm{op}}$. **Sets** is the category of sets.

A *terminal object* 1 of a category **C** is an object such that for any object X there is a unique arrow $X \to A$. (The terminal objects of **Sets** are the singleton sets $\{*\}$.)

The *limit* C for a diagram D in the category **C** is a cone above the diagram such that any other cone above D factorizes through C (C has the *universal property*).

An *equalizer* of a pair $f, g : C \to D$ is a limit for the diagram

$$C \underset{g}{\overset{f}{\rightrightarrows}} D.$$

Explicitly, an equalizer is an arrow such that the diagram below commutes for a unique k:

$$\begin{array}{ccccc} B & \xrightarrow{e} & C & \underset{g}{\overset{f}{\rightrightarrows}} & D \\ {\scriptstyle k}\uparrow & \nearrow_{h} & & & \\ A & & & & \end{array}$$

Vi may also say that e is a pullback. This is a special case of a more general construction. We say that the limit of the diagram $B \xrightarrow{f} D \xleftarrow{g} C$ is a *pullback*.

A *product* $A \times B$ is a limit for the diagram

$$A \qquad B.$$

We have dual constructions for cones *under* a diagram. These are are called *colimits*, *coequalizers*, *pushouts* and *coproducts*.

A morphism f *is monic* if, for any pair g and h, $f \circ g = f \circ h$ implies $g = h$. f is *epic* if, for any pair g and h, $g \circ f = h \circ f$ implies $g = h$. Also, we write $A \simeq B$ (the objects A and B are *isomorphic*) if there is an *invertible* arrow $f : A \to B$, that is, there exists an arrow g such that $f \circ g = 1_B$ and $g \circ f = 1_A$. A *subobject* of an object C in a category $\mathbf{C}$ is a monic $f : A \rightarrowtail C$. We do not distinguish between subobjects $f : A \rightarrowtail C$ and $g : B \rightarrowtail C$ if $A \simeq B$.

A *functor* between categories $\mathbf{C}$ and $\mathbf{D}$ is an operator which assigns an object $F(C)$ in $\mathbf{D}$ to each object C in $\mathbf{C}$, and an arrow $F(f)$ in $\mathbf{D}$ to each arrow f in $\mathbf{C}$, such that $F(g \circ f) = F(g) \circ F(f)$ and $F(1_X) = 1_{F(X)}$. We write $F : \mathbf{C} \to \mathbf{D}$. If F and G are two functors, a *natural transformation* α from F to G is an operation that associates an arrow $\alpha_A : F(A) \to G(A)$ with each A in $\mathbf{C}$, such that the following diagram commutes for all f in $\mathbf{C}$:

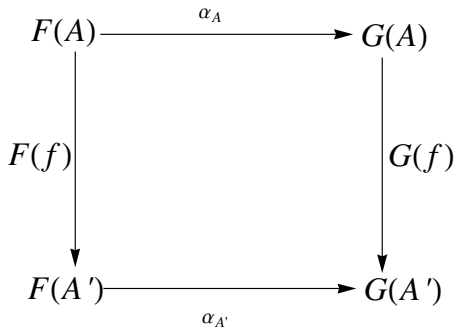


Figure 2.2. A natural transformation.

The composition $\beta \circ \alpha$ of two natural transformations $\alpha : F \to G$ and $\beta : G \to H$ defined by $(\beta \circ \alpha)_C = \beta_{G(C)} \circ \alpha_C$ then allows us to form the *functor category* $\mathbf{D}^{\mathbf{C}}$, with the functors $F : \mathbf{C} \to \mathbf{D}$ as objects and the natural transformations as morphisms. The case $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is of particular interest below. An object P (a contravariant set-valued functor) in this category is called a *presheaf* on $\mathbf{C}$. A *global element* of a presheaf P in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is an arrow $s : 1 \to P$, where 1 is the terminal object in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ defined by $1(A) = \{*\}$ for all $A \in \mathbf{C}$ and $1(f)$ is the map $1(f)(*) = *$. For a given presheaf P, an arrow $f : D \to C$ in $\mathbf{C}$, and $x \in P(C)$, we shall often use the notation $x \cdot f \equiv P(f)(x)$ ("the restriction of x along f").

2.1.2. Elements of the Theory of Topoi

We now turn to the characteristics of topoi. (Useful references for this subsection are Mac Lane and Moerdijk (1992), Goldblatt (1984) and Bell (1988).) Consider first the following list of categories:

List 1. Topoi

Category	*Collection of objects*	*Collection of morphisms*
Sets	all sets	the set − theoretical functions
BG (or G-**Sets**)	representations (X, μ), (Y, ν) ... of the group G	the equivariant functions f between representations ($f(x\cdot g) = f(x)\cdot g)\cdot g$)
$\mathbf{Sets}^{\mathbf{C}^{op}}$	presheaves P over a given category **C** (the functors from $\mathbf{C}^{op}$ to **Sets**)	the natural transformations between functors
Sh(X)	sheaves F over a topological space X (presheaves over $O(X)$ which can be "glued together")	the natural transformations between functors

These categories have something in common: we may regard them as *(generalized) universes of sets*. Note first that they all have a terminal object. For example, all singleton sets $\{*\}$ in **Sets** are terminal, and they are all isomorphic. In $\mathbf{Sets}^{\mathbf{C}^{op}}$, the terminal object 1 (a presheaf) is defined by $1(C) = \{*\}$ and $1(f)(*) = *$.

We may also take products. For **Sets**, this is well-known: the product $A \times B$ of two sets A and B is the set of ordered pairs (a, b), with $a \in A$ and $b \in B$. For **B**G, the product $(X, \mu) \times (Y, \nu)$ will be the representation $(X \times Y, \mu \times \nu)$. In the category $\mathbf{Sets}^{\mathbf{C}^{op}}$, the product $F \times G$ of two presheaves F and G must be a presheaf too (i.e. a contravariant functor over **C**). We construct $F \times G$ pointwise (let A, B be objects in **C**, $f : A \to B$ an arrow in **C**):

$$(F \times G)(A) = F(A) \times G(A) \text{ (defined because Sets allows products),}$$
$$(F \times G)(f)(x, y) = Ff(x) \times Gf(y) \text{ (with } x \in F(B) \text{ and } y \in G(B)).$$

(In the last line, $(F \times G)(f)$ is a set-theoretical function from $(F \times G)(B)$ to $(F \times G)(A)$. In order to show that $F \times G$ is a product, we must find natural transformations π_1 og π_2 as in the figure below, and if H is a cone above F and G, there must be a unique natural transformation ρ (the dotted line):

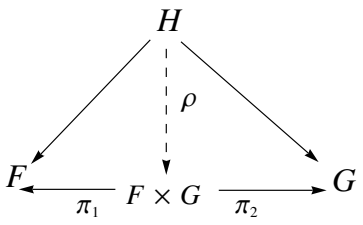


Figure 2.3. A product of presheaves.

This is easy. We let $\pi_1(A) : (F \times G)(A) = F(A) \times G(A) \to F(A)$ be the projection on the first component, and correspondingly for π_2. The procedure illustrates a general trait of topoi of functors (like $\mathbf{Sets}^{\mathbf{C}^{op}}$): limits are taken pointwise.

We now turn to another, more uncommon property of these categories. In **Sets** there is a correspondence between

the subsets $S \subset X$ of a set X, and

the characteristic functions χ_S defined on X by $\chi_S = 0$ for $x \in S$, $\chi_S = 1$ for $x \notin S$.

Category-theoretically, this may be expressed by the following diagram:

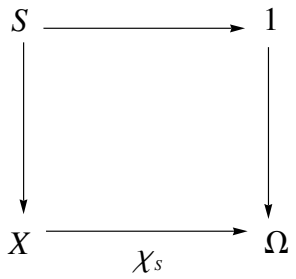


Figure 2.4. The subobject classifier.

We introduce the notation Ω for the set $\{0,1\}$, which is called the *truth object* in **Sets**. The arrow from S to X is set inclusion (hence, S is a subobject of X in **Sets**), and 1 is the terminal object (a singleton set). We say that the arrow $1 \to \Omega$ is the subobject classifier in **Sets**. In general, a category **C** with all finite limits has a *subobject classifier,* a monic $true : 1 \to \Omega$, if all subobjects are unique pullbacks of *true.*

It turns out that this construction is available in the other categories in the list above. Let us concentrate on the case $\mathbf{Sets}^{\mathbf{C}^{op}}$. The truth object Ω must be a functor from $\mathbf{C}^{op}$ to **Sets**. A *sieve* on the object C in the category **C** is a set S of arrows with codomain C such that $f \in S$ implies that $f \circ h \in S$ for all h such that $f \circ h$ is defined. S is "downwards closed". The *maximal sieve* on C, $t(C)$, consists of all arrows with C as their codomain. The truth object Ω of $\mathbf{Sets}^{\mathbf{C}^{op}}$ is defined as follows:

$$\Omega(C) = \{S \mid S \text{ is a sieve on } C\},$$

$$\Omega(g) : \Omega(C) \to \Omega(C') \text{ is given by } S \mapsto S \cdot g = \{h \mid g \circ h \in S\} \text{ (for } g : C' \to C).$$

The subobject classifier $true : 1 \to \Omega$ is defined by

$$true_C : 1_C \to \Omega_C \text{ given by } * \mapsto t(C).$$

It is clear that *true* is a natural transformation. Now Ω is a subobject classifier in $\mathbf{Sets}^{\mathbf{C}^{op}}$ if subobjects Q of functors P in $\mathbf{Sets}^{\mathbf{C}^{op}}$ are unique pullbacks of *true*. This is indeed the case. Let us outline the mode of reasoning by proving the pullback property. For all arrows $f : A \to C$ in $\mathbf{C}$, there is a function $P(f) : P(C) \to P(A)$. Let $x \in P(C)$ be given. Then either $x \cdot f \in Q(A)$ or not. (We use the notation $x \cdot f$ introduced in the preceding subsection). Define $\phi_C(x)$ to be the set of f for which $x \cdot f \in Q(A)$. Then $\phi_C(x)$ is a sieve, and ϕ is a natural transformation from P to Ω. Suppose $x \in Q(C)$. Then $x \cdot f \in Q(A)$ for all f with codomain C (because Q is a subobject, and hence a subfunctor of P), so $\phi_C(x) = t(C)$, the maximal sieve. Also, if $x \cdot f \in Q(A)$ for all f with codomain C, then $x \in Q(C)$ (Let $A = C$, $f = 1_C$.) ϕ will be the characteristic function of Q. We have shown that the diagram below commutes:

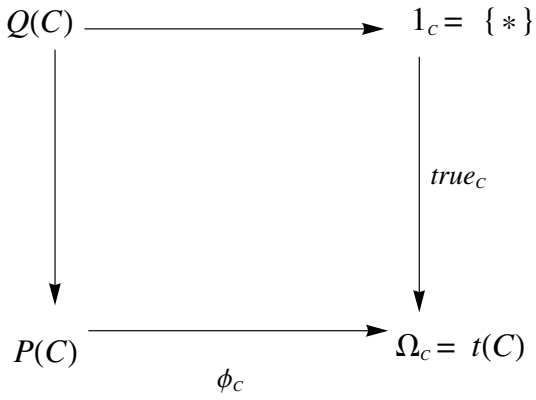


Figure 2.5. The pullback property of *true*.

For presheaves, pointwise pullback implies full pullback, so we are through. Uniqueness may also be proven, so Ω is a subobject classifier.

A final important property is *exponentiation*. Recall that sets Z and X in **Sets** allow us to form a set Z^X of all functions $f : X \to Z$. Phrased differently, there is a bijection

$$\frac{Y \times X \to Z}{Y \to Z^X}.$$

We say that the functor $- \times X$ is *left adjoint* to the functor $(-)^X$ (which is *right adjoint*). The categories in our list all have this property.

We may now define an *elementary topos* as a category with a terminal object, pullbacks, exponentiation and a subobject classifier. The categories above are topoi in this sense. Usually, the qualification "elementary" is omitted. An important group of examples is provided by the so-called Grothendieck topoi, which generalize the topos **Sh**(X) of sheaves on a topological space X. Here, we simply note that **Sh**(X) is a subcategory of $\mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}}$, where $\mathcal{O}(X)$ is the collection of open sets in X, partially ordered by inclusion. The objects F in **Sh**(X) are those presheaves for which, given any covering $\bigcup V_i$ of the open set U such that $f_i \in F(V_i)$ and $f_j \in F(V_j)$ coincide on the intersection of V_i and V_j, there is an $f \in F(U)$ which restricts to f_i and f_j. (Thus, the *continuous* functions on U form a sheaf, whereas the *bounded* functions do not.)

2.1.3. The Internal Language of a Topos

In this section, we shall prove a theorem in topos theory which will be needed in subsection 2.2.2. (Also, def. 2.1 will surface again in the final section of chapter 4.) Basically, there are two ways of deriving results for topoi. Mostly, we shall treat the sets of objects and arrows of our category as ordinary set-theoretical constructions, and our reasoning shall be classically valid. When we follow this path, we say that we choose an *external* viewpoint on the topos. Alternatively, we can collect the properties of the topos under consideration in a list of axioms, including both rules that hold generally for topoi and assertions which are endemic to the set-theoretical substrate of the topos. By picking a suitable first-order language for this task, we then explore the topos from the *internal* viewpoint. In a further refinement of this approach, we now note that a topos $\mathcal{E}$ is associated with a certain typed language, the *Mitchell-Bénabou language* of the topos. We proceed to define this language, along with its interpretation in $\mathcal{E}$. Each term in the language is interpreted as an arrow in the topos.

Definition 2.1 (Cf. Mac Lane and Moerdijk (1992), p. 298f, and Bell (1988), p. 92.) *The terms of the* Mitchell-Bénabou language $\mathcal{L}$ (or $\mathcal{L}(\mathcal{E})$) *of the topos* $\mathcal{E}$, *and their interpretations, are given inductively by the following conditions:*

(a) Each object X in $\mathcal{E}$ *is a type in* $\mathcal{L}$, *and each variable x of type X is interpreted as the identity arrow on X,* $\iota : X \to X$.

(b) If σ *is a term of type X and* τ *is a term of type Y, then* $\langle \sigma, \tau \rangle$ *is a term of type* $X \times Y$. *If* σ *and* τ *are interpreted as arrows* $\sigma : U_1 \times \cdots \times U_n \to X$ *and* $\tau : V_1 \times \cdots \times V_n \to Y$, *then* $\langle \sigma, \tau \rangle$ *is interpreted as the arrow* $\langle \sigma p, \tau q \rangle : W \to X \times Y$, *where W is a product of the types corresponding to the variables in* $\langle \sigma, \tau \rangle$ *(each variable counted only once), p and q are the projections* $p : W \to U_1 \times \cdots \times U_n$ *and* $q : W \to V_1 \times \cdots \times V_n$, *and* $\langle\ ,\ \rangle$ *is the product map in a topos.*

(c) If σ *and* τ *are terms of type X, then* $\sigma = \tau$ *is a term of type* Ω *(the subobject classifier). If* σ *and* τ *are interpreted as in (b), then* $\sigma = \tau$ *is interpreted as the arrow* $(\sigma = \tau) : W \xrightarrow{\langle \sigma p, \tau q \rangle} X \times X \xrightarrow{\delta_X} \Omega$. *Here, W, p and q are given as in (b), and* δ_X *is the characteristic map of the diagonal* $\Delta : X \to X \times X$ *(because* $\mathcal{E}$ *is a topos, charcteristic maps always exist).*

(d) If σ *is a term of type X, then, for each arrow* $f : X \to Y$ *in* $\mathcal{E}$, *there is a term* $f \circ \sigma$ *of type Y. If* σ *is interpreted as in (b), then* $f \circ \sigma$ *is interpreted as the arrow* $f \circ \sigma : U \xrightarrow{\sigma} X \xrightarrow{f} Y$. *Here,* $\circ$ *is composition in the category* $\mathcal{E}$.

(e) If σ *is a term of type X and* θ *is a term of type* Y^X, *then* $\theta(\sigma)$ *is a term of type Y. If* σ *is interpreted as in (b) and* θ *is interpreted as* $\theta : V_1 \times \cdots \times V_n \to Y^X$, *then* $\theta(\sigma)$ *is interpreted as the arrow* $\theta(\sigma) : W \xrightarrow{\langle \theta q, \sigma p \rangle} Y^X \times X \xrightarrow{e} Y$, *where W, p and q are as in (b), and e is the evaluation map in the topos.*

(f) If σ *is a term of type X and* τ *is a term of type* Ω^X, *then* $\sigma \in \tau$ *is a term of type* Ω. *If* σ *is interpreted as in (b) and* τ *is interpreted as* $\tau : V_1 \times \cdots \times V_n \to \Omega^X$, *then* $\sigma \in \tau$ *is interpreted as the arrow* $\sigma \in \tau : W \xrightarrow{\langle \sigma p, \tau q \rangle} X \times \Omega^X \xrightarrow{e} \Omega$, *with W, p, q and e as above.*

(g) If x is a variable of type X and σ *is a term of type Z, then* $\lambda x \sigma$ *is a term of type* Z^X. *If x is interpreted as in (a) and* $\sigma : X \times U_1 \times \cdots \times U_n \to Z$ *(that is, x occurs in* σ*), then* $\lambda x \sigma$ *is interpreted as the transpose of* σ, $\lambda x \sigma : U_1 \times \cdots \times U_n \to Z^X$ *(because* $\mathcal{E}$ *is a topos, the transpose can always be found). If x does not occur in* σ, *interpreted as* $\sigma : U_1 \times \cdots \times U_n \to Z$, *then* $\lambda x \sigma$ *is interpreted as the arrow* $\lambda x \sigma : U_1 \times \cdots \times U_n \xrightarrow{\sigma} Z \xrightarrow{\iota} Z^X$, *where the monomorphism* ι *is the image of Z in* Z^X.

(The last clause allows the formation of λ-terms from variable-free terms and differs a little from the standard definition.) Terms of type Ω will be called *formulae*. Now recall that, in a general topos, we always have morphisms corresponding to the logical operations $\vee$, $\wedge$, $\Rightarrow$ and $\neg$. Let us consider $\wedge$ as an example. In lemma 3.1, we noted that the subobjects of an object in a topos are a Heyting algebra. For subobjects X and Y of an object A, we form the meet-subobject $X \bigcap Y$. The characteristic function of $X \bigcap Y$ may then be factored as $\chi_{X \cap Y} : A \xrightarrow{\langle \chi_X, \chi_Y \rangle} \Omega \times \Omega \to \Omega$. The last arrow will be the internal meet, $\wedge : \Omega \times \Omega \to \Omega$. By using clause (d) above, the standard connectives are added to $\mathcal{L}$.

In a similar manner, quantifiers have been present in the topos $\mathcal{E}$ from the outset. By definition, each object A in a topos has a power object PA. In fact, it can be shown that P is a contravariant functor in $\mathcal{E}$. For any arrow $f : X \to 1$, the arrow $P(f) : P(1) \to P(X)$ then belongs to $\mathcal{E}$. As usual in a topos, PA is identified with the exponential Ω^A, so we have $P(f) : \Omega \to \Omega^X$. (For $\mathcal{E}$ = **Sets**, this is the operation which picks the characteristic function of X considered as a subset of itself.) It is well-known (Mac Lane and Moerdijk (1992), p. 206, 209) that $P(f)$ has a left adjoint $\exists_f : P(X) \to P(1)$ and a right adjoint $\forall_f : P(X) \to P(1)$. (For $\mathcal{E}$ = **Sets**, $\exists_f$ maps non-empty subsets of X to $\top$, whereas $\forall_f$ maps X to $\top$ and all proper subsets of X to $\bot$.) Again invoking clause (d), $\exists x\, \phi := \exists_f \circ \lambda x\phi$ and $\forall x\, \phi := \forall_f \circ \lambda x\phi$ are formulae of $\mathcal{L}$.

Finally, the notation $\{x \mid \phi(x)\}$ is available in $\mathcal{L}$. For $\phi(x) : X \to \Omega$ a formula, we simply interpret $\{x|\phi(x)\}$ as the subobject of X with $\phi(x)$ as its "characteristic function".

We shall say that the formula $\phi(x_1, ..., x_n) : U_1 \times \cdots \times U_n \to \Omega$ is *valid* in $\mathcal{E}$ when $\phi(x_1, ..., x_n)$ factors through $\top : 1 \to \Omega$, the truth arrow. For $\phi : 1 \to \Omega$ variable-free, we say that ϕ is *true* in $\mathcal{E}$. Using the interpretation above, we may now prove results about the validity and truth of formulae in $\mathcal{L}$. This task may be considerably simplified if we formulate our proofs by means of the forcing relation of *Kripke-Joyal semantics*. Recall that the familiar notion of an element a of a set s has the following equivalent in a topos $\mathcal{E}$: the arrow $\alpha : U \to X$ is a *generalized element* of X defined at stage U.

Definition 2.2 (Mac Lane and Moerdijk (1992), p. 303f.) *The formula $\phi(x) : X \to \Omega$ is* forced *to hold for the generalized element $\alpha : U \to X$ if and only if α factors through the subobject $\{x \mid \phi(x)\}$ of X. We say that $\phi(\alpha)$ is* forced *at stage U (or U* forces *$\phi(\alpha)$) and write*

$$U \Vdash \phi(\alpha).$$

One notable trait of '$\Vdash$' above is that clauses similar to the standard definition of a forcing relation hold as *theorems* about $\mathcal{E}$. E.g. it can be shown that $U \Vdash \exists y\, \phi(\alpha, y))$ (for y a variable of type Y) if and only if there exist an epimorphism $p : V \to U$ and a generalized element $\beta : V \to Y$ such that $V \Vdash \phi(\alpha p, \beta)$. Validity of a formula ϕ can be expressed as the demand that $1 \Vdash \phi$. For a topos of presheaves, $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ (considered as a category of sheaves with the trivial Grothendieck topology), these clauses simplify further. In this case, they are identical with the rules of Kripke semantics (Mac Lane and Moerdijk (1992), p. 317f). Thus, for formulae $\phi(x)$ and $\psi(x)$, x a variable of type X, $C \in \mathbf{C}$ and $\alpha \in X(C)$, we have

(i) $C \Vdash \phi(\alpha) \wedge \psi(\alpha)$ iff $C \Vdash \phi(\alpha)$ and $C \Vdash \psi(\alpha)$;

(ii) $C \Vdash \phi(\alpha) \vee \psi(\alpha)$ iff $C \Vdash \phi(\alpha)$ or $C \Vdash \psi(\alpha)$;

(iii) $C \Vdash \phi(\alpha) \Rightarrow \psi(\alpha)$ iff for all $f : D \to C$ in $\mathbf{C}$, $D \Vdash \phi(\alpha f)$ implies $D \Vdash \psi(\alpha f)$ (where $\alpha f = X(f)(\alpha)$ is the restriction of α along f in the presheaf X);

(iv) $C \Vdash \neg\phi(\alpha)$ iff there is no $f : D \to C$ such that $D \Vdash \phi(\alpha f)$;

(v) $C \Vdash \exists y\, \phi(\alpha, y)$) (for y a variable of type Y) iff there exists $\beta \in Y(C)$ such that $C \Vdash \phi(\alpha, \beta)$;

(vi) $C \Vdash \forall y\, \phi(\alpha, y)$) (for y a variable of type Y) iff, for all $f : D \to C$ in $\mathbf{C}$ and all $\beta \in Y(D)$, it holds that $D \Vdash \phi(\alpha f, \beta)$.

We shall also need a clause for identity statements:

(vii) $C \Vdash \sigma(\alpha) = \tau(\alpha)$ for $\sigma(x)$ and $\tau(x)$ terms of type Y iff, for all $f : D \to C$ in $\mathbf{C}$, $\sigma_D(\alpha f) = \tau_D(\alpha f)$, where σ_D and τ_D are the components of σ and τ at D (and therefore morphisms $X(D) \to Y(D)$).

In general, the base category $\mathbf{C}$ may not have a terminal object 1. In these cases, we say that a formula ϕ is valid if and only if $C \Vdash \phi$ at all stages C.

As an illustration of the semantical method, we shall prove a familiar fact, the validity of the axiom of dependent choice (**DC**) in presheaf topoi. (Thus, in particular, it holds in the topos $\mathbf{BAF_C}$, which we shall meet in chapter 4.) For X an object in the topos $\mathcal{E}$, R a subobject of $X \times X$, N the n.n.o. object, r the characteristic function of R, x and y variables of type X and f a variable of type X^N, **DC** is the formula

$$\forall x\, \exists y\, r(x, y) \Rightarrow \forall x\, \exists f\, (f(0) = x \wedge \forall n\, r(f(n), f(n+1))).$$

In the proof, we shall need a basic technical lemma from category theory, the *Yoneda lemma* (e.g. Mac Lane (1997), p. 61), which states that, for C an object in the category $\mathbf{C}$ and P a presheaf in $\hat{\mathbf{C}} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection θ between natural transformations $\mathbf{y}(C) \to P$ and elements of the set $P(C)$, where $\mathbf{y}(C)$ denotes the representable presheaf $\mathrm{Hom}_{\mathbf{C}}(-, C)$:

$$\theta : \mathrm{Hom}_{\hat{\mathbf{C}}}(\mathbf{y}(C), P) \cong P(C).$$

The bijection θ is given by $\eta \mapsto \eta_C(1_C)$ for a natural transformation η.

Theorem 2.1 *The axiom of dependent choice* (**DC**) *holds in all presheaf topoi* $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$.

Proof The proof idea is simple. **DC** holds in set theory, so for each object C in $\mathbf{C}$, we can find a sequence f in the set $X(C)$ for a relation fulfilling the antecedent of **DC**. We then just need to make sure that f picks the same object at different stages in $\mathbf{C}$.

Let $\mathcal{E} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ be a presheaf topos. We must prove that $C \Vdash \mathbf{DC}$ for all stages C. By clauses (iii) and (vi) above, this means that if,

for all $g : D \to C$ in $\mathbf{C}$ and all $\beta \in X(D)$, it holds that $D \Vdash \exists y\, r(\beta, y)$,

then,

for all $h : D \to C$ and all $\gamma \in X(D)$, it holds that $D \Vdash \exists f\, (f(0) = \gamma \wedge \forall n\, r(f(n), f(n+1)))$.

By clause (v), we know that, for all β,

$D \Vdash \exists y\, r(\beta, y)$ iff there exists $\delta \in X(D)$ such that $D \Vdash r(\beta, \delta)$, (*)

and, again by (v), we want to prove that the right-hand side of this implies that

there exists $\phi \in X^N(D)$ such that $D \Vdash \phi(0) = \gamma \wedge \forall n\, r(\phi(n), \phi(n+1))$. (**)

By the Yoneda lemma, we have the isomorphism

$$X^N(D) \cong \mathrm{Hom}_{\hat{\mathbf{C}}}(\mathbf{y}(D), X^N),$$

and, by the definition of exponentiation in a topos,

$$\mathrm{Hom}_{\hat{\mathbf{C}}}(\mathbf{y}(D),\, X^N) \cong \mathrm{Hom}_{\hat{\mathbf{C}}}(\mathbf{y}(D) \times N,\, X).$$

Recall that the n.n.o. N in a presheaf topos is the constant sheaf $\Delta(\mathbb{N})$, where $\mathbb{N}$ is the ordinary natural numbers, so $N(C) = \mathbb{N}$ for all objects C in $\mathbf{C}$. We also need the identification $(\mathbf{y}(C))(A) = \mathrm{Hom}_{\mathbf{C}}(A,\, D)$. The claim in (**) is, then, that there exists a natural transformation ϕ between the presheaves $\mathbf{y}(D) \times N$ and X such that

(1) $\phi_A : \mathrm{Hom}_{\mathbf{C}}(A,\, D) \times \mathbb{N} \to X(A)$ is a set-theoretical function for all A in $\mathbf{C}$,

(2) for all $h : A \to D$ in $\mathbf{C}$, $\gamma \in X(D)$, $\phi h(0) = e_A(\phi h,\, 0) = (\phi h)_A(1_A,\, 0) = \gamma h \in X(A)$,

(3) for all $h : A \to D$ in $\mathbf{C}$ and all $n \in N$, it holds that $A \Vdash (rh)_A((\phi h)_A(1_A,\, n),\, (\phi h)_A(1_A,\, n + 1))$.

Here, the second line uses condition (e) in the definition of the language $\mathcal{L}$, the forcing condition (vii) for identity, and the meaning of evaluation e in a presheaf topos (cf. Mac Lane and Moerdijk (1992), p. 46). The last line comes from the forcing condition (vi) and the constancy of the n.n.o. N.

By assumption of the right-hand side in (*), for all β in the set $X(D)$, there exists $\delta \in X(D)$ such that $D \Vdash r(\beta,\, \delta)$. However, **DP** holds in ordinary set theory, so, for any $\gamma \in X(D)$, we can find a set-theoretic function $s_D : \mathbb{N} \to X(D)$ with $s_D(0) = \gamma$ and $\forall n\ (D \Vdash r(s_D(n),\, s_D(n + 1)))$. By the constancy of the n.n.o. and (vi) above, it follows that $D \Vdash \forall n\ (r(s_D(n),\, s_D(n + 1)))$. Applying the Yoneda lemma once more, we find the bijection

$$X(D) \cong \mathrm{Hom}_{\hat{\mathbf{C}}}(\mathbf{y}(D),\, X).$$

So, corresponding to s_D and fulfilling the same conditions, we have a set-theoretic function $\tilde{s}_D : \mathbb{N} \to \mathrm{Hom}_{\hat{\mathbf{C}}}(\mathbf{y}(D),\, X)$ with $\gamma = s_D(0)$ corresponding to $(\tilde{s}_D(0))_D(1_D)$ under the bijection, and $s_D(n)$ corresponding to $(\tilde{s}_D(n))_D(1_D)$. ($\tilde{s}_D$ is the sequence f mentioned at the beginning of the proof.) But then the following definition of the ϕ will do: for any object A in $\mathbf{C}$ and arrows $h : A \to D$ (so $h \in \mathbf{y}(D)(A) = \mathrm{Hom}_{\mathbf{C}}(A,\, D)$), we define

$$\phi_A(h,\, n) := (\tilde{s}_D(n))_A(h) \in X(A).$$

Note first that ϕ, with ϕ_A as its components, is a natural transformation. Indeed, for $f : B \to A$ and $h : A \to D$ morphisms in $\mathbf{C}$, we have

$$\begin{aligned}
\phi_B((\mathbf{y}\,(D)\,f) \times 1)\,(h,\, n) &= \phi_B(h \circ f,\, n) = (\tilde{s}_D(n))_B\,(h \circ f) \\
&= (\tilde{s}_D(n))_B\,(\mathbf{y}(D)\,(h \circ f)\,(1_D)) \\
&= X(h \circ f)\,((\tilde{s}_D(n))_D\,(1_D)) \\
&= (Xf \circ Xh)\,((\tilde{s}_D(n))_D\,(1_D)) = (Xf)\,(\tilde{s}_D(n))_A\,(h).
\end{aligned}$$

The second line is a consequence of the definition of the functor $\mathbf{y}(D)$, whereas the third line is obtained because $\tilde{s}_D(n)$ is a natural transformation between $\mathbf{y}(D)$ and X for any n. Finally, the functorial nature of X is invoked, and the procedure is reversed. We also have, by the definition of ϕ,

$$(Xf \circ \phi_A)\,(h,\, n) = Xf\,((\tilde{s}_D(n))_A\,(h)).$$

Then the following diagram commutes, so ϕ is a natural transformation:

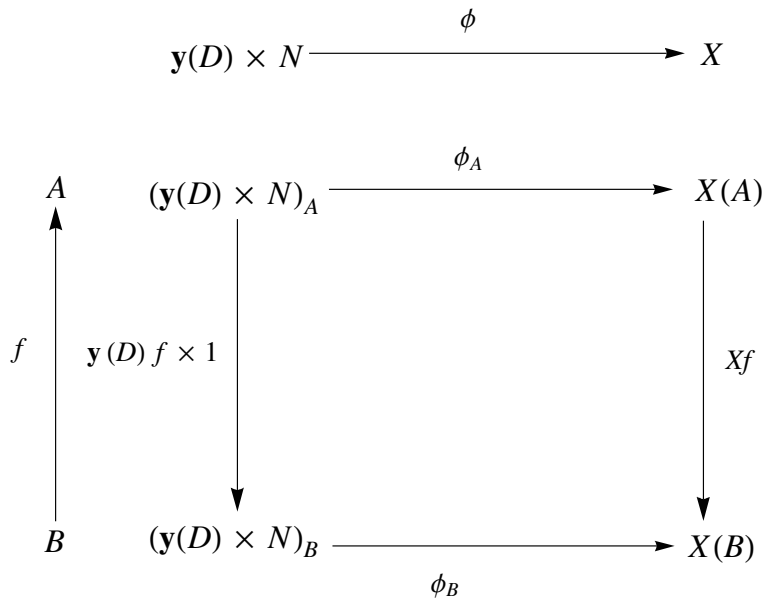


Figure 2.6. ϕ is a natural transformation.

It is evident that ϕ satisfies (1) above. (2) is more troublesome, but note that $\phi h = X^N(h)(\phi)$, the restriction of ϕ along $h : A \to D$ which takes $\phi \in \mathrm{Hom}_{\hat{\mathbf{C}}}(\mathbf{y}(D) \times N, X)$ to ϕh in $\mathrm{Hom}_{\hat{\mathbf{C}}}(\mathbf{y}(A) \times N, X)$. Here, $(\phi h)_B(g, n) = \phi_B(h \circ g, n)$ for all $g : B \to A$ in $\mathbf{C}$, by the definition of exponentiation in a topos. Then $(\phi h)_A(1_A, 0) = \phi_A(h, 0)$. As $(\tilde{s}_D(0))_D(1_D)$ corresponds to $\gamma = s_D(0)$, we must show that $(\tilde{s}_D(0))_D(1_D)h = (\phi h)_A(1_A, 0)$ for all $h : A \to D$ in $\mathbf{C}$. But we have

$$(\tilde{s}_D(0))_D(1_D)\, h = \phi_D(1_D, 0)\, h = X(h)(\phi_D(1_D, 0))$$

$$= \phi_A((\mathbf{y}(D) h) \times 1)(1_D, 0) = \phi_A(1_D \circ h, 0) = \phi_A(h, 0).$$

The last step uses commutativity of a diagram similar to 2.5, but with $h : A \to D$ instead of f. This proves (2).

Finally, (3) can be proven. We noted above that D forces $r((\tilde{s}_D(n))_D(1_D), (\tilde{s}_D(n + 1))_D(1_D))$ for all n. By definition of ϕ, this means that $D \Vdash \forall n\, r_D(\phi_D(1_D, n), \phi_D(1_D, n + 1))$. (As in clause (vii) above, the subscript D on r here becomes necessary, because r is really the "characteristic function" of the relation R, and, therefore, a natural transformation from $X \times X$ to Ω.) By clause (vi) of the semantics, for any $h : A \to D$ in $\mathbf{C}$ and all $n \in N$, it then holds that $A \Vdash (rh)_A((\phi h)_A(1_A, n), (\phi h)_A(1_A, n + 1))$. This ends the proof. □

The axiom of dependent choice is quite powerful. Whereas the full version of the Hahn-Banach theorem for normed spaces presupposes Zorn's lemma, **DC** suffices for the case of a separable space (see Bridges and Richman (1987), ch. 2, for the development of analysis in a constructive setting). With the tools of this section in hand, we may now approach the models of topos physics.

2.2. Neo-realism and Bohrification

In a series of articles, Chris Isham and Andreas Döring (2008a, 2008b, 2008c, 2009d) have proposed a set of new models for quantum physics, dubbed as *neo-realism*. Neo-realism is conceived as an alternative to the well-known *Copenhagen interpretation*, which introduces a separation of the measurement process for a physical magnitude into two components, a quantum system S and a classical observer V. Here, the possible states of S are wave functions Ψ from a configuration space into the set of complex numbers, whereas the observer V always registers a real value as the outcome of his experiment. In Copenhagen terminology, the wave function "collapses" onto the registered value with probability

$$P(r) = \ |\langle r \mid \Psi \rangle|^2.$$

The physically meaningful (real) value r is not a value of the physical quantity before the measurement is made. The interpretation breaks down for closed systems where no "outside" observer is to be found, such as quantum cosmology. In the topos scheme suggested by Isham and Döring, physical quantities *does* have a value independent of any observer V. The scheme relies on non-standard representations of the states and quantity values of physics. It also turns out that a new, intuitionistic quantum logic supplants the familiar non-distributive logic of Birkhoff and van Neumann. (It should be noted that the choice of the tag "neo-realism" would be protested by philosophers and logicians, such as Michael Dummett, who regard acceptance of the law of excluded middle as the hallmark of philosophical *realism* (cf. Dummett (2010), p. 130ff.))

In subsection 2.2.1, we only give a brief outline of the topos scheme introduced by Döring and Isham, including sketches of some central proofs. In order to appreciate the full scope of the models, it is necessary to read the original articles. Then, in subsection 2.2.2, we sketch the alternative, mathematically sophisticated version of the topos-theoretical approach found in the work of Heunen, Landsman and Spitters (2007, 2009). Appealing to the related work of Banaschewski and Mulvey (2006), where the Gelfand duality for C*-algebras is extended to arbitrary topoi, these authors stress the connection between quantum logic and quantum space. In chapter 3, we shall try to apply the topos model of these authors to the theory of loop quantum gravity.

2.2.1. The Döring-Isham Scheme for Quantum Mechanics

Following earlier work by Isham and Butterfield, Döring and Isham (2008b) start their approach to quantum systems by assuming that the physical quantities A of a system S are represented by self-adjoint operators $\hat{A}$ in the non-commutative von Neumann algebra, $\mathcal{B}(\mathcal{H})$, of all bounded operators on the separable Hilbert space $\mathcal{H}$ of the states of S. The unital, commutative subalgebras V of $\mathcal{B}(\mathcal{H})$ are then considered as *classical contexts* or *perspectives* on the system S, and the *context category* $\mathcal{V}(\mathcal{H})$ is defined with $\mathrm{Ob}(\mathcal{V}(\mathcal{H}))$ as the set of contexts V and $\mathrm{Hom}(\mathcal{V}(\mathcal{H}))$ given by the inclusions $i_{V'V} : V' \to V$. (The reader may prefer to regard the contexts as experimental situations, that is, an observer and his measuring apparatuses.)

A self-adjoint operator $\hat{A}$, if present in a subalgebra V, makes the physical quantity A fully measurable from the classical perspective represented by the subalgebra. The possible values of A are contained in the spectrum of $\hat{A}$, $\sigma(\hat{A})$, a subset of the real numbers. It is possible to form propositions of the form '$A \in \Delta$', where Δ is a Borel set, and, by the spectral theorem, each proposition of this form is represented by a projection operator $\hat{P}$ (the "yes-no" questions of subsection 1.1.2).

In general, a context V will exclude many operators. But, in a certain sense, excluded operators still have "proxys" in V. For we note that $\hat{P}$, even if not present in the context V, may be approximated by the set (where $\mathcal{P}(V)$ is the complete lattice of projections in V, and the ordering $\gtrsim$ is defined as $\hat{Q} \gtrsim \hat{P}$ if and only if $\mathrm{Im}\hat{P} \subseteq \mathrm{Im}\hat{Q}$, or, equivalently, $\hat{P}\hat{Q} = \hat{P}$)

$$\delta(\hat{P})_V := \bigwedge \{\hat{Q} \in \mathcal{P}(V) \,\big|\, \hat{Q} \gtrsim \hat{P}\}. \tag{2.1}$$

This is a first step towards the definition of states, observables and values as entities in a presheaf topos. The set of approximations of $\hat{P}$, one for each context V, may now supplant $\hat{P}$ as the interpretation of the proposition '$A \in \Delta$' in the model to be constructed. We try

$$\text{"}[\![A \in \Delta]\!] := \delta(\hat{P}) := \{\delta(\hat{P})_V |\ V \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}))\}\text{"}.$$

However, the logical structure of these sets is limited, so this will not quite do. Note now that truth values may be assigned the projectors in each context V by using the Gelfand spectrum Σ_V. This is the set

$$\Sigma_V := \{\lambda : V \to \mathbb{C} \mid \lambda \text{ is a positive multiplicative linear functional of norm 1}\}. \tag{2.2}$$

Lemma 2.2 *When $\hat{P}$ is a projection, the value $\lambda(\hat{P})$ is either* 0 *(false) or* 1 *(true).*
(Proof: $\lambda(\hat{P}) = \lambda(\hat{P}\hat{P}) = \lambda(\hat{P})\lambda(\hat{P})$.)

So λ behaves like a "local" state for V: it answers "yes" or "no" to the "question" $\hat{P}$. The construction of the state object $\underline{\Sigma}$, the representation of the physical state space in the topos scheme, may now be undertaken. (Hereafter, objects of the topos will be underlined.) The state object will be an element in the class of objects of the topos of presheaves over the context category $\mathcal{V}(\mathcal{H})$,

$$\tau := \mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}. \tag{2.3}$$

Definition 2.3 (Döring and Isham (2008b)) *The* spectral presheaf *(or* state object*),* $\underline{\Sigma}$*, is an element in* $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$ *(that is, a contravariant functor from* $\mathcal{V}(\mathcal{H})$ *to* **Sets**) *such that* $\underline{\Sigma}_V := \Sigma_V$ *and, for morphisms* $i_{V'V} : V' \to V$ *in* $\mathcal{V}(\mathcal{H})$*,* $\underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V \to \underline{\Sigma}_{V'}$ *is defined by* $\underline{\Sigma}(i_{V'V})(\lambda) := \lambda|_{V'}$ *(the restriction of* $\lambda : V \to \mathbb{C}$ *to* $V' \subseteq V$*).*

In classical physics, the states of the system are given as points of the state space. By analogy, one might try to define the states in the topos model as global elements of the state space, that is, as morphisms $\underline{1} \to \underline{\Sigma}$ in τ (where $\underline{1}$ is the terminal object in τ). It has been shown by Butterfield and Isham that the existence of global elements in τ is inconsistent with the Kochen-Specker theorem, the famous "no-hidden-variable" result from quantum physics. Instead, interest focuses on the entities known as the 'clopen' subobjects of $\underline{\Sigma}$:

Definition 2.4 (Döring and Isham (2008b)) *A contravariant functor* $\underline{S}$ *from* $\mathcal{V}(\mathcal{H})$ *to* **Sets** *is a* clopen subobject *of* $\underline{\Sigma}$ *if* $\underline{S}$ *is a subfunctor of* $\underline{\Sigma}$ *(in the standard sense) and the set* $\underline{S}_V$ *is both open and closed as a subset of the compact Hausdorff space* $\underline{\Sigma}_V$*.*

There is now, from Gelfand spectral theory, a lattice isomorphism between the lattice $\mathcal{P}(V)$ of projections in V and the lattice of closed and open *subsets*, $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}_V)$, of $\underline{\Sigma}_V$:

$$\alpha : \mathcal{P}(V) \to \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}_V) \text{ where } \alpha(\hat{P}) := \left\{ \lambda \in \underline{\Sigma}_V \,\middle|\, \lambda(\hat{P}) = 1 \right\} \equiv S_{\hat{P}}. \tag{2.4}$$

Lemma 2.3 $\mathcal{P}(V)$ and $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}_V)$ *are Boolean algebras.*

Proof Firstly, it must be shown that $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}_V)$ is a *Heyting algebra,* i.e. a distributive lattice with an implication operator which satisfies the adjunction scheme

$$\frac{a \cap b \subseteq c}{a \subseteq (b \Rightarrow c)}.$$

This can be done by defining implication $a \Rightarrow b$ as $a^c \bigcup b$. It is clear that $\mathrm{Sub_{cl}}(\underline{\Sigma}_V)$ is a distributive lattice because the usual set-theoretical operations preserve clopenness; we shall show that the adjunction property holds. It is clear that $a^c \bigcup b$ is clopen. Assume that $a \bigcap b \subseteq c$. Then $a \subseteq (b^c \bigcup c)$; this follows from $a = (a \bigcap b) \bigcup (a \bigcap b^c)$. Now assume that $a \subseteq (b^c \bigcup c)$, and let $x \in a \bigcap b$. It follows that $x \in c$. $a^c \bigcup b$ is therefore a suitable implication operator. It now suffices to point out that a Boolean algebra is defined by adding just the condition of material implication

$$a \Rightarrow b = \neg\, a \bigcup b \text{ (with } \neg\, a \equiv a^c).$$

Hence, $\mathrm{Sub_{cl}}(\underline{\Sigma}_V)$ (and, by the isomorphism, $\mathcal{P}(V)$) is a Boolean algebra. □

We also note for use in chapter 4 that a Heyting algebra is Boolean if and only if the law of excluded third holds (Mac Lane and Moerdijk (1992), p. 55). The commutative algebras V are classical contexts within the theory, so it is certainly proper that these lattices are Boolean. Certainly, extraordinary logic is the last thing we would expect to find when we are engaged in experimental physics. (Call this the "principle of charity" in our interrogation of physical reality.) This construction can now be extended to the total context category $\mathcal{V}(\mathcal{H})$. Döring and Isham ((2008b), th. 2.4) prove that, for each projection $\hat{P} \in \mathcal{B}(\mathcal{H})$, there is a clopen sub-object $\underline{S}_{\hat{P}}$ of the spectral presheaf $\underline{\Sigma}$ given by

$$\underline{S}_{\hat{P}} := \left\{ S_{\delta(\hat{P})_V} \subseteq \underline{\Sigma}_V \;\middle|\; V \in \mathrm{Ob}(\mathcal{V}(\mathcal{H})) \right\}. \tag{2.5}$$

Lemma 2.4 *$\underline{S}_{\hat{P}}$ is a clopen subobject of $\underline{\Sigma}$.*

Sketch of proof If the context V' is poor compared to V, the representation $\delta(\hat{P})_{V'}$ will be a less fine-grained operator than $\delta(\hat{P})_V$, i.e. it projects onto a greater subspace. So we may define an operator $\hat{Q} = \delta(\hat{P})_{V'} - \delta(\hat{P})_V$. We now pick $\lambda \in S_{\delta(\hat{P})_V}$. Then $\lambda(\delta(\hat{P})_V) = 1$, so $\lambda(\delta(\hat{P})_{V'}) = \lambda(\delta(\hat{P})_V) + \lambda(\hat{Q}) = 1$ (because $\hat{Q}$ is an operator with vaules in $\{0, 1\}$). Then the restriction of λ to V' belongs in $S_{\delta(\hat{P})_{V'}}$. But restrictions along $V' \to V$ is just how the functor $\underline{\Sigma}$ handles morphisms, so the subobject $\underline{S}_{\hat{P}}$ is a *clopen* subobject of $\underline{\Sigma}$. □

This leads to the cornerstone of the theory, the concept of *daseinisation*, the map which "throws" the observable into a world of classical perspectives:

Definition 2.5 (Döring and Isham (2008b)) *The* daseinisation *δ of projection operators $\hat{P} \in \mathcal{P}(\mathcal{H})$ is the mapping*

$$\delta : \mathcal{P}(\mathcal{H}) \to \mathrm{Sub_{cl}}(\underline{\Sigma})$$

$$\hat{P} \mapsto \underline{S}_{\hat{P}}.$$

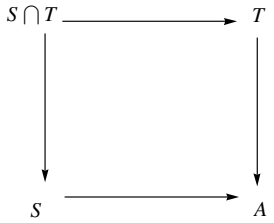
The daseinisation map is injective. In chapter 1, we interpreted quantum mechanical sentences $\Delta(A)$ as $[\Delta(A)]$, the linear subspace spanned by vectors ψ, the eigenvectors of the observable A for an eigenvalue $a \in \Delta$. This amounts to an identification of $[\Delta(A)]$ with the projection onto this subspace. Recall also that we interpreted classical sentences as a subspace of the classical state space. The importance of the definition of daseinisation rests on the mapping between a projection, which in *quantum* physics is the representative of a proposition of the theory, and a sub-object of the 'state object' $\underline{\Sigma}$, the topos analogue of a subset of the state space, which is the *classical* notion of a proposition in physics:

$$\begin{array}{c} \text{projection } \hat{P}\text{ ["a quantum mechanical statement"]} \overset{\delta}{\rightarrow} \\ \text{subobject } S_{\hat{P}}\text{ [topos analogue of a subspace or a "classical statement"]} \end{array} .$$

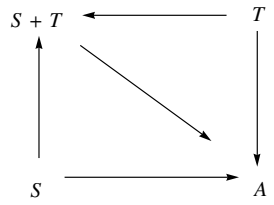
The above constructions now allow us infer the logic appropriate for quantum physics in topoi. For an arbitrary topos it is known that

Lemma 2.5 *The subobjects of an object in a topos is a Heyting algebra.*

Sketch of proof The complete proof is found in Mac Lane and Moerdijk ((1992), p. 186ff, 198ff). Firstly, it must be shown that the partially ordered set Sub A of subobjects of A forms a lattice. Let $S >\!\!\rightarrow A$ and $T >\!\!\rightarrow A$ be subobjects of A. Then meet, $S \bigcap T$, is a pullback (which, in a topos, always can be found):



Join (union) is a little more demanding:



We start by forming the coproduct (sum) $S + T$. Then, by definition of the coproduct, there is a unique arrow from $S + T$ to A. Using another result from topos theory, this arrow may be split into an epic $S + T \rightarrow> M$ and a monic $M >\rightarrow A$, aand it can be shown that M is the union $S \bigcup T$. $0 >\rightarrow A$ and $1_A : A \rightarrow A$ will be top and bottom in the lattice. To prove that this lattice is also a Heyting algebra, the exponentiation property in topoi must be used. □

Theorem 2.6 $\mathrm{Sub_{cl}}(\underline{\Sigma})$, *the clopen subobjects of* $\underline{\Sigma}$*, is a Heyting algebra.*

Sketch of proof For S and T subobjects of $\underline{\Sigma}$, and V a context, we define

$$(S \vee T)_V = S_V \bigcup T_V$$
$$(S \wedge T)_V = S_V \bigcap T_V.$$

If S and T are clopen, it may be shown that $S \vee T$ are $S \wedge T$ clopen also. $0 = \{\emptyset_V\}$ and $1 = \underline{\Sigma}$ are easily defined, and clopen. Negation is more complicated:

$$(\neg S)_V = \mathrm{int} \bigcap_{V' \subseteq V} \{\lambda \in \Sigma_V \mid \lambda \mid_{V'} \in S^c_{V'}\}.$$

The interior of the set is used to guarantee openness of the meet, which may be infinite. Implication may also be defined (but not as $\neg S \vee T$). □

Suppose that $\hat{P}$ is the projection operator which answers "yes" or "no" to the question "Is the observable A in the real number set Δ?". We then have the following interpretation of the quantum physical sentences $\Delta(A)$:

$$\Delta(A) \mapsto \hat{P} \mapsto \delta(\hat{P}) = S_{\hat{P}}.$$

The truth-functional connectives are to be interpreted as the corresponding operations in the Heyting algebra $\mathrm{Sub_{cl}}(\underline{\Sigma})$. The distributive law holds in all Heyting algebras:

$$x \wedge (y \vee z) \leftrightarrow (x \wedge y) \vee (x \wedge z)$$

Hence, it is valid in or propositional logic. The well-known laws below, however, do not hold:

$$x \vee \neg x \text{ og } \neg\neg x \rightarrow x.$$

The logic of the quantum-theoretical propositional calculus is *intuitionistic*.

With the topos equivalent of a state space in hand, the next step is to consider a topos representation of the physical quantities A (the self-adjoint operators $\hat{A}$).The final aim is to represent each $\hat{A}$ as an arrow between the state object, $\underline{\Sigma}$, and a value object still to be defined. This is done by appeal to the spectral theorem, which associates a spectral family of projection operators, $\{\hat{E}^A{}_\lambda\}_{\lambda \in \mathbb{R}}$, with each self-adjoint operator $\hat{A}$. Then

$$\hat{A} = \int_{\mathbb{R}} \lambda \, d\hat{E}^A{}_\lambda. \tag{2.6}$$

The construction in (2.1) above is now very useful.

Definition 2.6 (Döring and Isham (2008 *c*)) *For $\hat{P}$ a projection operator and V a context in $\mathcal{V}(\mathcal{H})$, the* outer daseinisation *operator is*

$$\delta^o(\hat{P})_V := \bigwedge \{\hat{Q} \in \mathcal{P}(V) \,\big|\, \hat{Q} \gtrsim \hat{P}\}.$$

The inner daseinisation *operator is*

$$\delta^i(\hat{P})_V := \bigvee \{\hat{Q} \in \mathcal{P}(V) \,\big|\, \hat{Q} \lesssim \hat{P}\}.$$

$\delta^o(\hat{P})_V$ approximates $\hat{P}$ from above, while $\delta^i(\hat{P})_V$ approximates it from below. The extension of inner and outer daseinisation to the self-adjoint operators is then close at hand.

Definition 2.7 (Döring and Isham (2008c)) *For $\hat{A}$ a self-adjoint operator and V a context in $\mathcal{V}(\mathcal{H})$, the* outer *and* inner daseinisation *of $\hat{A}$ are*

$$\delta^o(\hat{A})_V := \int_{\mathbb{R}} \lambda \, d\left(\delta^i(\hat{E}^A{}_\lambda)_V\right),$$

$$\delta^i(\hat{A})_V := \int_{\mathbb{R}} \lambda \, d\left(\bigwedge_{\mu>\lambda} \delta^o(\hat{E}^A{}_\lambda)_V\right).$$

The operators $\delta^o(\hat{A})_V$ and $\delta^i(\hat{A})_V$ are self-adjoint operators in the context V. They can be represented by a Gel'fand transform with values in the spectrum, e.g.

$$\overline{\delta^o(\hat{A})_V} : \underline{\Sigma}_V \to \mathrm{sp}\left(\delta^o(\hat{A})_V\right). \tag{2.7}$$

Döring and Isham therefore introduce the functions (where $\lambda \in \underline{\Sigma}_V$ and $\downarrow V$ is the set of unital von Neumann subalgebras of V)

$$\check{\delta}^o(\hat{A})_V(\lambda) : \downarrow V \to \mathrm{sp}(\hat{A}) \tag{2.8}$$

$$\left(\check{\delta}^o(\hat{A})_V(\lambda)\right)(V') \mapsto \lambda\left(\delta^o(\hat{A})_{V'}\right). \tag{2.9}$$

The density of the notation may cause some confusion. But note that for V'' a smaller subalgebra of $V' \subseteq V$, it must be the case that (in the ordering of the operators)

$$\delta^o(\hat{A})_{V'} \lesssim \delta^o(\hat{A})_{V''}. \tag{2.10}$$

The functional $\lambda : V \to \mathbb{C}$ will take a greater or equal real value on the coarser of the two operators, so $\check{\delta}^o(\hat{A})_V(\lambda)$ is an *order-reversing* function. If, in general, we had equality in (2.10), we could use the standard real number object $\underline{\mathbb{R}}$ (the constant functor from $\mathcal{V}(\mathcal{H})$ to $\mathbb{R}$) and define a morphism from $\underline{\Sigma}$ to $\underline{\mathbb{R}}$ by means of the prescription

$$"\overline{\delta^o(\hat{A})_V} : \underline{\Sigma}_V \to \mathrm{sp}(\delta^o(\hat{A})_V) \to \mathbb{R}".$$

When we switch from a "rich" observational context to a "poorer" one, we should not expect the representative of the observable $\hat{A}$ to give the same measurement outcome. The daseinisation of $\hat{A}$ will change as we move between classical perspectives V, so instead of using the constant presheaf $\underline{\mathbb{R}}$, a more elaborate construction is needed:

Definition 2.8 (Döring and Isham (2008c)) *The* value object $\underline{\mathbb{R}}^{\succeq}$ *in* $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$ *is is an object in* $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$ *(that is, a contravariant functor from* $\mathcal{V}(\mathcal{H})$ *to* **Sets**) *such that* $\underline{\mathbb{R}}^{\succeq}_V := \{\mu : \downarrow V \to \mathbb{R} \mid \mu \text{ is an order-reversing function}\}$ *and, for morphisms* $i_{V'V} : V' \to V$ *in* $\mathcal{V}(\mathcal{H})$, $\underline{\mathbb{R}}^{\succeq}(i_{V'V}) : \underline{\mathbb{R}}^{\succeq}{}_V \to \underline{\mathbb{R}}^{\succeq}{}_{V'}$ *is defined by* $\underline{\mathbb{R}}^{\succeq}(i_{V'V})(\mu) := \mu|_{V'}$ *(the restriction of* $\mu : V \to \mathbb{R}$ *to* $V' \subseteq V$*).*

This, finally, leads to a satisfactory interpretation of a physical quantity as a morphism in the topos. From (2.8) above, it is clear that $\check{\delta}^{o}(\hat{A})_V(\lambda)$ is an order-reversing function from $\downarrow V$ to sp $(\hat{A}) \subseteq \mathbb{R}$ (for any given λ), so the connection between the spectral presheaf, $\underline{\Sigma}$, and the value object, $\underline{\mathbb{R}}^{\succeq}$, can be made:

Theorem 2.2 (Döring and Isham (2008c)) *The mappings* $\check{\delta}^{o}(\hat{A})_V$, $V \in \mathcal{V}(\mathcal{H})$, *given by*

$$\check{\delta}^{o}(\hat{A})_V : \underline{\Sigma}_V \to \underline{\mathbb{R}}^{\succeq}_V$$
$$\lambda \mapsto \check{\delta}^{o}(\hat{A})_V(\lambda),$$

are the components of a natural transformation (a morphism in the topos)

$$\check{\delta}^{o}(\hat{A}) : \underline{\Sigma} \to \underline{\mathbb{R}}^{\succeq}.$$

Again, the formulae may be rather unfamiliar, but the physical interpretation is close at hand. Recall that λ is a functional in the Gel'fand spectrum of the classical (abelian) context V, and, as such takes a value in the spectrum of $\delta^{o}(\hat{A})_V$, $\mathrm{sp}(\delta^{o}(\hat{A})_V) \subseteq \mathbb{R}$, the representation of $\hat{A}$ from the perspective of V. As we approach V from the subalgebras $V' \subset V$, we pick decreasing values $\lambda(\delta^{o}(\hat{A})_{V'}) \in \mathbb{R}$ because the representations $\delta^{o}(\hat{A})_V$ grow smaller in the operator order. That is, we approach the supremum of the value range of $\hat{A}$ from above.

The mapping $\check{\delta}^{o}(\hat{A})$ may be combined with a twin mapping, $\check{\delta}^{i}(\hat{A})$, which corresponds to the inner daseinisation, to give the context-dependent 'spread' of the operator $\hat{A}$. If this is done, the components of the new value object, $\underline{\mathbb{R}}^{\leftrightarrow}$, will be sets of pairings of order-reversing and order-preserving functions. This definition of the reals as *de facto* intervals which are refined as we switch to richer contexts, is closely related to the intuition of "what a topos really is". Brouwer's construction of the real numbers as Cauchy sequences of rational numbers also comes to mind.

The value object $\underline{\mathbb{R}}^{\succeq}$ still has palpable weaknesses. As it stands, the presheaf $\underline{\mathbb{R}}^{\succeq}$ is only a monoid object in the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$, a far cry from the field structure of the "real" reals in $\mathbb{R}$. However, using a construction of Grothendieck, Döring and Isham show that there is a completion of $\underline{\mathbb{R}}^{\succeq}$, called $k(\underline{\mathbb{R}}^{\succeq})$, which is an abelian group object in the topos. We shall not enter into this.

The relationship between classical mechanics, quantum mechanics and the topos models is summarized in the table below:

State space (S)	Observables (O)	Value space (R)	Logic
$\mathbb{R}^{6n}$	$f : \mathbb{R}^{6n} \to \mathbb{R}$	$\mathbb{R}$	Classical
$\mathcal{H}$	self-adjoint operators $A : H \to H$	$\mathbb{R}$	Birkhoff-von Neumann
state object $\underline{\Sigma}$	natural transformations $\check{\delta}^{O}(\hat{A}) : \underline{\Sigma} \to \underline{\mathbb{R}}^{\preceq}$	value object $\underline{\mathbb{R}}^{\preceq}$	Intuitionistic

Note that the constructions of Döring and Isham provide an answer to the challlenge posed by the Copenhagen interpretation: *In the topos τ, all physical magnitudes A have a value independently of observation, namely* $\check{\delta}^{O}(\hat{A})(S)$. Here, $\hat{A}$ is a self-adjoint operator, and S is a subobject of the state space $\underline{\Sigma}$. The value $\check{\delta}^{O}(\hat{A})(S)$ is a subobject of the value space $\underline{\mathbb{R}}^{\preceq}$.

2.2.2. Bohrification

An alternative, mathematically sophisticated version of the topos-theoretical approach is found in the work of Heunen, Landsman and Spitters (2008, 2009). The HLS alternative, known as 'Bohrification', utilizes the topos-theoretical generalisation of the notion of space, *locales*. The quantum logic is then read off from the Heyting algebra structure of the open subsets of a locale L (or, strictly, the frame $\mathcal{O}(L)$), identified as the state space of the system. Because locales will be important to us when we formulate loop quantum gravity in topos physics in chapter 3, we review the main characteristics of Bohrification in this subsection.

Above, the context category was given by a family of commutative subalgebras V of a non-commutative von Neumann algebra. The state object $\underline{\Sigma}$ was a functor in the topos, with $\underline{\Sigma}_V$ the Gelfand spectrum in the context V. The states were then defined as clopen subobjects of $\underline{\Sigma}$. The last construction relied on the rich supply of projections available in von Neumann algebras. In the HLS approach, a family of commutative subalgebras of a non-commutative C*-algebra is used instead of the von Neumann algebras. These algebras are generally poor in projections, special cases (such as Rickart algebras or von Neumann algebras) excepted, so the former notion of a state is no longer useful.

HLS topos physics starts from the topos $\mathbf{Sets}^{C(A)}$ of *covariant* functors, where $C(A)$ is the set of commutative C*-subalgebras of a C*-algebra A. The tautological functor $\underline{A} : C(A) \to \mathbf{Sets}$, which acts on objects as $\underline{A}(C) = C$, and on morphisms $C \subseteq D$ as the inclusion $\underline{A}(C) \to \underline{A}(D)$, is called the *Bohrification* of A.

Now consider the functor $\mathcal{T}$: **CStar** → **Topos**, where **CStar** is the category of unital C*-algebras (with arrows defined as linear multiplicative functions which preserve the identity and the *-operation), and **Topos** is the category of topoi (with geometric morphisms as arrows). $\mathcal{T}$ is defined by $\mathcal{T}(A) = \mathbf{Sets}^{C(A)}$ on objects and $\mathcal{T}(f)^*(\underline{T})(D) = \underline{T}(f(D))$ on morphisms $f : A \to B$, with $\underline{T} \in \mathbf{Sets}^{C(B)}$ and $D \in C(A)$. ($\mathcal{T}(f)^*$ is the inverse image part of the geometric morphism $\mathcal{T}(f)$.) It can then be shown that $\underline{A}$ is a commutative C*-algebra in the topos $\mathcal{T}(A) = \mathbf{Sets}^{C(A)}$. This crucial result rests upon a general fact from topos theory:

Fact. *If* **Model**($\mathbb{T}$, **T**) *denotes the category of models of a geometric theory* $\mathbb{T}$ *in the topos* **T***, there is an isomorphism of categories*

$$\mathbf{Model}(\mathbb{T}, \mathbf{Sets}^{C(A)}) \approx \mathbf{Model}(\mathbb{T}, \mathbf{Sets})^{C(A)}.$$

This is a special case of lemma 3.13 in the HLS article. (For a proof, see cor. D1.2.14 of the *Elephant,* Johnstone (2002).)

The proof of the commutativity of $\underline{A}$ (Heunen, Landsman and Spitters (2009), p. 19) appeals to Kripke-Joyal semantics for Kripke topoi (cf. subsection 2.1.3 above). It also makes use of the axiom of dependent choice (**DC**), which holds in $\mathbf{Sets}^{C(A)}$ (see subsection 2.1.3 for the statement and proof of **DC** in a similar context). Commutativity of $\underline{A}$ in $\mathbf{Sets}^{C(A)}$ is proved by exploiting the proximity of the theory of C*-algebras to a *geometric* theory. In these theories, all statements have the form

$$\forall(\vec{x})[\psi(\vec{x}) \to \phi(\vec{x})].$$

Here, ψ and ϕ are positive formulae; i.e. formulae built by means of finite conjunctions and existential quantifiers. Thus, geometric theories are formulae with "finite verification" (see Mac Lane and Moerdijk (1997), ch. X for more about this notion). If the theory of abelian C*-algebras (Banach algebras with involution, and satisfying $\|a^*a\| = \|a\|^2$) had been a geometric theory, we could start from the following piece of information about $\underline{A}$:

$$\underline{A} \in \mathbf{Model}(\text{The theory of abelian C*-algebras}, \mathbf{Sets})^{C(A)}.$$

This is true by the definition of $\underline{A}$ as the tautological functor, and because $C(A)$ contains only commutative subalgebras. By the fact stated above, it would then seem follow that

$$\underline{A} \in \mathbf{Model}(\text{The theory of abelian C*-algebras}, \mathbf{Sets}^{C(A)}).$$

That is, $\underline{A}$ is an internal C*-algebra in the topos $\mathbf{Sets}^{C(A)}$. However, the theory of abelian C*-algebras is not a geometric theory: the axiom of completeness (the convergence of any Cauchy sequence in the algebra) fails us. In order to circumvent this difficulty, the authors introduce the notion of a "pre-semi-C*-algebra". All C*-algebras are "pre-semi", and the theory of these algebras *is* geometric. Again, by appeal to the fact above, $\underline{A}$ is an internal abelian "pre-semi". It is then shown "by hand" that $\underline{A}$ is, in fact, an internal abelian C*-algebra.

Now recall that, in the topos **Sets**, there is an equivalence (the *Gelfand duality*) between the categories **cCStar** (the commutative C*-algebras) and **KHausTop** (the compact Hausdorff topological spaces). In turn, **KHausTop** is equivalent to the category **KRegLoc** of *compact regular locales* in **Sets**. (We shall apply the notion of "pointfree" spaces, or locales, in section 3.2, and give the precise definitions there.) As mentioned in subsection 1.4.3, Banaschewski and Mulvey (2006) have shown that the equivalence **cCStar** ⇄ **KRegLoc** holds in any topos. We shall not give the details of this beautiful, but demanding construction, which recently has been improved by Coquand and Spitters.

Let $\underline{\Sigma}$ be the morphism from **cCStar** to $\mathbf{KRegLoc}^{\mathrm{op}}$ in $\mathbf{Sets}^{C(A)}$. (The underlining, also of $\underline{\Sigma}$, is a reminder that objects and morphisms between them are now internal to this topos.) Consider the locale $\underline{\Sigma}(\underline{A})$, the *Gelfand spectrum* of $\underline{A}$ (which, as we noted, is commutative in $\mathbf{Sets}^{C(A)}$). $\underline{\Sigma}(\underline{A})$ is the state space of HLS topos physics, corresponding to the state object $\underline{\Sigma}$ in the Isham-Döring model (definition 2.3 above). Interestingly, the locale $\underline{\Sigma}(\underline{A})$ is pointfree for $A = \mathbf{Hilb}(H, H)$, with H a Hilbert space of dimension greater than 2 (Heunen, Landsman and Spitters (2009), theorem 4.10), and also for more general classes of C*-algebras. This is the HLS version of the Kochen-Specker theorem, which was formulated for topos physics by Isham and Butterfield.

The construction of $\underline{\Sigma}(\underline{A})$ is done by means of formal symbols for each self-adjoint element a of A, but we shall not need this. As usual for entities in topoi, the Gelfand spectrum $\underline{\Sigma}(\underline{A})$ may alternatively be given an *external* description, and it can be shown that $\underline{\Sigma}(\underline{A})$ is determined by the value taken at $\mathbb{C}$ (the algebra of complex numbers is the least member of $C(A)$). $\underline{\Sigma}(\underline{A})(\mathbb{C})$, denoted by Σ_A, is known as the *Bohrified state space* of A. We shall study a concrete example of an external state space when we apply topos methods within loop quantum gravity in the next chapter.

Finally, the frame (or Heyting algebra) $\mathcal{O}(\Sigma_A)$ provides a new quantum logic, which may be compared with the old Birkhoff-von Neumann logic when the C*-algebra has enough projections. This is the case when A is a Rickart C*-algebra (see Heunen, Landsman and Spitters (2009), sec. 5, for definitions and results). The atomic propositions of the theory are identified with elements of $\mathcal{O}(\Sigma_A)$, and the resulting logic is intuitionistic.

CHAPTER 3

A Topos Model for Loop Quantum Gravity

One of the motivations behind topos physics is to provide a framework for new theories of quantum gravity. We shall not search for such theories presently. Instead, we ask how some of the known candidates for a final theory fit into the approach sketched in the foregoing chapter. As a first application of the methods developed, we shall try to represent a particular version of quantum gravity, the theory of loop quantum gravity (LQG), within the topos-theoretical framework. We had a glimpse of this theory when we discussed discrete space-time in subsection 1.4.2 above. LQG generalizes the canonical methods from standard quantum mechanics, so it seems to be a natural choice for a topos model. In section 3.1, a *very* dense overview of the main formulae of LQG is given, not an introduction to loop theory as such. An excellent one, Rovelli's *Quantum Gravity* (Rovelli (2004), is already available. Rovelli's book is informal on some points, but many of the technical details and proofs can be found in Thiemann (2007). We then (section 3.2) give a fairly detailed presentation of a topos-theoretical version of LQG, relying on the methods of Heunen, Landsman and Spitters. The description will supplement the sketch of the Bohrification approach given in subsection 2.2.2 above. The construction relies on the C*-algebra version of loop quantum gravity introduced in Fleischhack (2004). We bring together LQG results and methods from topos physics in a proof of the non-sobriety of the external "state space" Σ of the "Bohrified" LQG theory, and show that the construction obeys the standard requirements of diffeomorphism and gauge invariance.

3.1. The Basic Structure of Loop Quantum Gravity

Loop quantum gravity has been described as the "attempt to construct a mathematically rigorous, non-perturbative, background independent quantum theory of four-dimensional, Lorentzian general relativity plus all known matter in the continuum" (Thiemann (2007), p. 16). While the theory does not assume a background space-time, a standard differentiable manifold is still presupposed for the construction of states and operators. Contracting the description above even further, we may say that LQG is the *quantisation* of the *canonical (hamiltonian)* formulation of *general relativity*.

It is common practice to write the equations of LQG in the framwork of Hamiltonian mechanics, and we will do so too. In the Lagrangian formulation, one starts from the *Einstein-Hilbert action*

$$S = \kappa^{-1} \int_M d^4 x \sqrt{-g}\ R. \tag{3.1}$$

The Euler-Langrange equations for this functional then allow us to derive the familar Einstein field equations (for a proof, see e.g. Wald (1984), p. 450ff). The Einstein-Hilbert action is covariant, but the covariance is lost if we turn from the Lagrangian to the Hamiltonian formulation. This, however, we must do, because we want to apply standard quantisation to canonical position and momentum variables. We therefore foliate the 4-dimensional space-time M into Cauchy surfaces Σ_t. This amounts to giving M the topology

$$M = \mathbb{R} \times \Sigma. \tag{3.2}$$

A new form of the action principle, *the ADM action*, is now available:

$$S = \kappa^{-1} \int_{\mathbb{R}} dt \int_{\Sigma} d^3 x \mathcal{L}. \tag{3.3}$$

A time coordinate is thereby introduced, and we can define canonical momenta in the usual manner by deriving the Lagrangian density $\mathcal{L}$ with respect to the velocities. In particular, we have a momentum p^{ab} conjugate to the three-metric h^{ab} induced by $g^{\mu\nu}$ on the surface Σ (so $a, b = 1, 2, 3$):

$$p^{ab}(t, x) = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ab}}. \tag{3.4}$$

We may picture the choice of h^{ab} and p^{ab} as dynamical variables as the evolution of a three-metric on a given 3-dimensional manifold Σ. The configuration space is then the set of three-metrics h^{ab} defined on Σ, and the state space of the corresponding quantum theory will consist of functionals defined on this configuration space.

This is a reasonable starting point, but the new formalism of loop quantum gravity invites us to mix these variables in a different manner. LQG uses the *tetrad field* (or *Vierbein*) *formalism.* We shall see that these variables are particularly suitable for quantisation. (Kiefer (2004), sec. 4.1-4.2, has a nice review of the older approach. Below, we follow mainly Rovelli (2004).)

3.1.1. The Choice of Variables

In Einstein's theory of general relativity (GR), it will always be possible to pick coordinates such that the metric is locally flat. GR may therefore be seen as a set of rules for the transformation of an arbitrary reference frame $x = (x^\mu)$ to a locally inertial reference frame $X = (X^I)$ (here the index I labels components in Minkowski space or Euclidean space). Assuming that an event A has coordinates $X^I(A) = 0$ in the inertial frame, we have (by Taylor expansion to the first non-vanishing term)

$$X^I(x) = \left(\frac{\partial X^I(x)}{\partial x^\mu}\right)_{x=x(A)} \cdot x^\mu \tag{3.5}$$

We may then define the *gravitational field* $e^I_\mu(x)$ (also known as the *Vierbein* field) as the Jacobian of the transformation to locally inertial coordinates:

$$e^I_\mu(x) \equiv \left(\frac{\partial X^I(x)}{\partial x^\mu}\right)_{x=x(A)} \tag{3.6}$$

In the usual metric field formalism, the metric tensor may be used to introduce a unique *connection*, that is, a prescription for a directional derivative depending only on a direction at a given point. This can also be done with the tetrad field $e^I(x) = e^I_\mu(x)\, dx^\mu$. The *spin connection* ω (with values in the Lie algebra of the Lorentz group $SO(3, 1)$) will be of the form

$$\omega^I_J(x) = \omega^I_{\mu J}(x)\, \mathrm{d}x^\mu. \tag{3.7}$$

$SO(3, 1)$ is the group preserving the Minkowski metric η_{IJ}, so we can use it to correct the deviaton of the coordinate derivative with respect to parallel translation of vectors v^I with local Minkowski labels. This gives the covariant derivative

$$D_\mu v^I = \partial_\mu v^I + \omega^I_{\mu J}\, v^I. \tag{3.8}$$

A *complex self-dual connection* A is given by (with $i = 1, 2, 3$ and $\omega^i = 1/2\, \epsilon^i_{jk}\, \omega^{jk}$)

$$A^i = \omega^i + \mathrm{i}\omega^{0\,i}. \tag{3.9}$$

This last step, introduced by Ashtekar in 1986, is crucial, as it simplifies the constraints to follow. Note that the six components of $\omega^I_{\mu J}$ simply are collected in the three components of the complex connection A. The discovery that the connection A and (a version of) the field e may be treated as the configuration and momentum variables with classical Poisson brackets is the starting point for LQG theory. In fact, further simplification may be achieved (Rovelli (2004), section 4.2): we may choose to use the *Euclidean connection*

$$A^i = \omega^i + \omega^{0\,i}. \tag{3.10}$$

This time, however, simplification comes at a price. The Hamiltonian (see below) will be more complicated.

Let us now consider the phase space of LQG. In the "old" canonical formalism h^{ab} and p^{ab} were dynamical variables on a given 3-dimensional manifold Σ. We now choose the space of three-dimensional forms $A^i(\vec{\tau}) = A^i_a(\vec{\tau})\, \mathrm{d}\tau^a$ induced by the four-dimensional form A^i as our configuration space Γ. The momentum variables will be

$$E^a_i(\vec{\tau}) = \frac{1}{2}\, \epsilon_{ijk}\, \epsilon^{abc}\, e^j_b\left(\vec{\tau}\right) e^k_c\left(\vec{\tau}\right). \tag{3.11}$$

The small indices are a reminder that we consider the three-dimensional forms (the *triads*) of the tetrads e^I_μ. It follows that $E^{ai} = (\det e)\, e^{ai}$, so this is simply e densitized. Importantly, we now have the Poisson bracket

$$\left\{A^i_a(\vec{\tau}),\, E^b_i(\vec{\tau}')\right\} = (8\,\pi \mathrm{i} G)\,\delta^b_a\,\delta^i_j\,\delta^3(\vec{\tau},\,\vec{\tau}'). \tag{3.12}$$

(For the Euclidean connection, i is omitted on the right.)

3.1.2. The Quantization of Euclidean General Relativity

Heuristically, quantisation of a classical theory is the translation of Poisson brackets into commutator relations. The Poisson bracket of the variables A^i_a and E^b_i above thus becomes

$$\left[\hat{A}^{\,i}_a(\vec{\tau}),\, \hat{E}^b_i(\vec{\tau}')\right] = (8\,\pi \mathrm{i} G\hbar)\,\delta^b_a\,\delta^i_j\,\delta^3(\vec{\tau},\,\vec{\tau}'). \tag{3.13}$$

The connection $\hat{A}^{\,i}_a$ gives the intrinsic three-geometry on the Cauchy surface Σ, and the momentum $\hat{E}^b_i$ can be shown to be related to the extrinsic curvature, the embedding of Σ in space-time. By making measurements of the gravitational field more precise, measurements of the momentum ("the time-rate of change of the field") will grow correspondingly imprecise, and our picture of the three-surface as embedded in the 4-dimensional manifold will grow "blurry". To put it bluntly, in quantum gravity space-time does not exist.

Still at the formal stage, the commutation relations hold if we let the operators act on functionals in the manner below:

$$\hat{A}^{\,i}_a(\vec{\tau})\,\Psi[A] = A^i_a(\vec{\tau})\,\Psi[A] \tag{3.14}$$

$$\hat{E}^b_i(\vec{\tau}')\,\Psi[A] = -\mathrm{i}\hbar\,\frac{\delta}{\delta\, A^i_a(\vec{\tau})}\,\Psi[A]. \tag{3.15}$$

A state space of complex-valued functionals $\Psi[A]$ can now be defined. Just as in standard quantum mechanics, the configuration space Σ of the classical theory becomes the domain of the wave functions (functionals in our case) of the quantum theory. The state space $\mathcal{S}$ (which also will include unphysical states) is obtained as the linear space of functionals

$$\Psi_{\Gamma,f}[A] = f(U(A,\,\gamma_1),\ \ldots,\ U(A,\,\gamma_L)). \tag{3.16}$$

Here, $f(U_1, \ldots, U_L)$ is a smooth function of the group elements of the connection. Γ is a collection of L oriented paths γ_i on the 3d space Σ, and $U(A, \gamma)$ is the *holonomy* of the connection A along the curve γ, that is, the rule for parallel transport along γ. Look at the holonomy as a recipe for computation: given a vector v^B, we calculate $U^A_B(A, \gamma)\, v^B$ to find the parallel transport of v^B at the endpoint of γ. The functional $\Psi_{\Gamma,f}$ therefore tests the connection A at the endpoints of Γ. The *cylindrical functional* Ψ has *support* on the graph Γ.

(A short remark on the foundations in differential geometry of the last paragraph: A is a connection on the fiber bundle over the base manifold Σ with structural group $SO(3)$. The Lie algebra of $SO(3)$ is identical with $\mathfrak{su}(2) = \{B \in \mathfrak{gl}(n) \mid B + B^* = 0 \wedge tr\, B = 0\}$, the Lie algebra of $SU(2)$, so it is common in the LQG literature to regard A as an $\mathfrak{su}(2)$ connection over Σ. Note also that we have suppressed the dependency of U on the local trivialization of the fibre bundle over Σ with structure group $SU(2)$.)

In the special case where $\gamma = \alpha$, a closed curve or loop, and $f = \mathrm{tr}$, the trace of the matrix $U(A, \alpha)$, one usually writes

$$\mathbf{\Psi}_{\alpha,\,\mathrm{tr}}[A] = \langle A \mid \alpha \rangle. \tag{3.17}$$

If we have a scalar product it is then possible, using the Riesz-Fréchet theorem, to define functionals on the loop space as $\Psi[\alpha] = \langle \mathbf{\Psi}_{\alpha,\,\mathrm{tr}} | \Psi \rangle$ (hence the name "loop quantum gravity"). And we do have one. We may define the *scalar product* on the space $\mathcal{S}$ (where L is the number of paths in $\Gamma' \bigcup \Gamma''$) as

$$\left\langle \mathbf{\Psi}_{\Gamma',f'} \mid \mathbf{\Psi}_{\Gamma'',f''} \right\rangle \equiv \int \overline{f'(U_1,\ ...,\ U_L)}\ f''(U_1,\ ...,\ U_L)\, dU_1 \ldots dU_L. \tag{3.18}$$

(The Haar measure is used on the group $SU(2)$.) The completion of $\mathcal{S}$ in the scalar product above will be the *state space* $\mathcal{K}$. Label irreducible representations R of $SU(2)$ by their half-integer spin (quantum number) j and write the matrix elements as

$$R^{(j)\,\alpha}{}_{\beta}\,(U) \equiv \langle U \mid j,\, \alpha,\, \beta \rangle. \tag{3.19}$$

According to the Peter-Weyl theorem, we may span the space $\mathcal{K}$ with the vectors $|\Gamma,\ j_1, ...,\ j_L, \alpha_1, ...,\ \alpha_L, \beta_1, ...,\ \beta_L\rangle \equiv |\Gamma,\ j_l, \alpha_l, \beta_l\rangle$ given by

$$\langle A \mid \Gamma,\ j_l, \alpha_l, \beta_l \rangle \equiv R^{(j_1)\,\alpha_1}{}_{\beta_1}\,(U\,(A,\ \gamma_1)) \ldots R^{(j_L)\,\alpha_L}{}_{\beta_L}\,(U\,(A,\ \gamma_L)). \tag{3.20}$$

It must be noted that different paths may have different quantum numbers associated with them, corresponding to the different varieties of matter that make up the states. In order to fulfill the demands of invariance under gauge transformations and diffeomorphisms in LQG, we are interested in the projection of the space $\mathcal{K}$ onto the space $\mathcal{K}_{Diff}$ of invariant functionals $\mathbf{\Psi}$. A first step towards this, is the construction of the space $\mathcal{K}_0$, the space of *spin network states.* A spin network is a set $\mathrm{S} = \langle \Gamma,\ j_l,\ i_n \rangle$ of L links and N nodes, where i_n is the intertwiner of the tensor product of the $SU(2)$ representations on links starting or ending on the node n. The state $|\mathrm{S}\rangle$ in $\mathcal{K}_0$ is then found by contracting the state $|\Gamma,\ j_l,\ \alpha_l,\ \beta_l\rangle$ with the intertwiners. States $|\mathrm{S}\rangle$ are invariant under local $SU(2)$ gauge transformations.

Finally, the *kinematical state space* $\mathcal{K}_{Diff}$ of diffeomorphism invariant states is found as a subspace of the *space of linear functionals* on the states $\mathbf{\Psi}$**.** A state in $\mathcal{K}_{Diff}$ is an image under a map P_{Diff} defined by

$$(P_{\mathrm{Diff}}\,\mathbf{\Psi})\,(\mathbf{\Psi}') = \sum_{\mathbf{\Psi}''=U_\phi\,\mathbf{\Psi}} \langle \mathbf{\Psi}'' \mid \mathbf{\Psi}' \rangle. \tag{3.21}$$

The (finite) sum is taken over the states Ψ'' equivalent to Ψ under a representation U_ϕ of a diffeomorphism ϕ on the 3d manifold σ. The state $P_{\text{Diff}}\,\Psi$ is then diffeomorphism invariant by definition. One may also regard the diffeomorphism invariant states as equivalence classes of colored (c) knots K with nodes. The denotation $|s\rangle = |K,c\rangle$ will be used for the *spin-knot states.* In contrast to the space $\mathcal{K}$ above, the space $\mathcal{K}_{Diff}$ is separable.

Let us hint at the interpretation of these states. The spin-knot states $|s\rangle = |K,c\rangle$ are to be understood as orientation-free versions of the spin network states $|S\rangle = |\Gamma, j_l, i_n\rangle$ of L links and N nodes. No information about the localization of the state $|S\rangle$ on the manifold is preserved in $|s\rangle$. The links are, however, still colored with half-integer spin values j_l. Let the set Γ of links signify relations of contiguity between the nodes. Following Rovelli (2003, p. 190) we associate a surface l with the link between two nodes, and interpret j_l as the area of the surface. We interpret the intertwiner i_n as the volume of the node n. A very tagged simple network (with two nodes and three links) is shown in figure 3.2 below (from Rovelli (2003), p. 13).

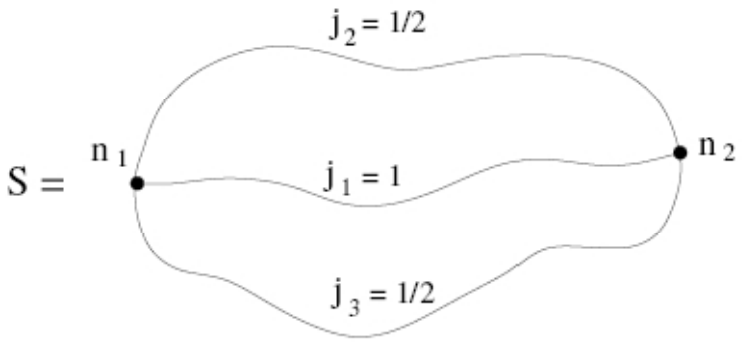


Figure 3.1. A spin network [From Rovelli (2004), p. 235].

3.1.3. The Quantum Operators

The basic operators may now be defined. In order to stay in the state space $\mathcal{S}$, we use the multiplication operator corresponding to the holonomy U instead of the connection operator A:

$$\left(U^A{}_B\,(A,\,\gamma)\,\Psi\right)[A] = U^A{}_B(-,\,\gamma)\,\Psi[A]. \tag{3.22}$$

For the vectors $|\Gamma,\, j_l,\, \alpha_l,\, \beta_l\rangle$ (and choosing a representation $R^{(j)}$ of $SU(2)$) this gives

$$R^{(j)\,\alpha}{}_{\beta}(U\,(A,\,\gamma))\left|\Gamma,\, j_l,\, \alpha_l,\, \beta_l\right\rangle = \left|\Gamma \bigcup \{\gamma\},\, j,\, j_l,\, \alpha,\, \alpha_l,\, \beta,\, \beta_l\right\rangle. \tag{3.23}$$

We may therefore think of the holonomy operator as adding a link γ in the j representation to Γ.

The momentum operator E is given as (Rovelli (2003), p. 174)

$$E_i(S) \equiv -i\hbar \int_S d\sigma^1\, d\sigma^2\, n_a(\vec{\sigma})\, \frac{\delta}{\delta A_a^i\left(x(\vec{\sigma})\right)}. \tag{3.24}$$

Here, $\vec{\sigma} = (\sigma^1, \sigma^2)$ are coordinates on the surface S in the 3d manifold Σ. $n_a(\vec{\sigma})$ is the normal 1-form on S. By letting the curve γ of the holonomy cross the surface S, we derive the action of the momentum operator on the holonomy in any representation j:

$$E_i(S)\, R^j(U(A,\ \gamma)) = \sum_{p\in(S\bigcap\gamma)} \pm i\hbar\, R^j\left(U\left(A,\ \gamma_1^p\right)\right)\, {}^{(j)}\tau_i\, R^j\left(U\left(A,\ \gamma_2^p\right)\right). \tag{3.25}$$

In this formula, the curve γ is divided into two parts γ_1^p and γ_2^p by the intersection point p, and the generator ${}^{(j)}\tau_i$ in the j representation of $SU(2)$ is inserted at the intersection. The operator $E_i(S)$ *grasps* γ. A peculiar feature of the momentum operator $E_i(S)$ may already be noted: When acting on a spin network state, the operator adds a node in each point of intersection of the links l and the surface S. Likewise, the holonomy U (the "configuration variable" in our quantising procedure) adds a new link to the state. (The trace operator of U adds a closed loop.)

This operator is *not* invariant under the internal gauge transformations. However, by means of E we may define the *geometric operators* **A** (area) and **V** (volume). This is done by partitioning the surface S in N smaller surfaces S_n:

$$\mathbf{A}(S) \equiv \lim_{N\to\infty} \sum_n \sqrt{E_i(S)\, E_i(S)}\ . \tag{3.26}$$

The operator **A** is gauge invariant and self-adjoint, and it has the discrete spectrum given by

$$\mathbf{A}(S)\left|\mathrm{S}\right\rangle = \hbar \sum_{p\in(S\bigcap\gamma)} \sqrt{j_p\left(j_{p+1}+1\right)}\ |\mathrm{S}\rangle. \tag{3.27}$$

This formula, predicting the quantisation of physical area, may be the most important result of the theory. It is not yet strictly proven. Indeed, there are indications that a proof is still some way off (Dittrich and Thiemann (2008)): such observables have not been defined for the *physical* space, that is, the area operator above does not commute with the Hamiltonian operator.

Similar difficulties surround the volume operator $\mathbf{V}(\mathcal{R})$. This operator is defined by taking the limit on small cubes, and it is shown that the action of **V** on the spin network states $|\mathrm{S}\rangle$ is restricted to the intertwiners (Rovelli (2004), p. 259). **V** is also self-adjoint and has a discrete spectrum. Neither **A** nor **V** are physical observables in the full sense demanded by LQG. Although $SU(2)$-invariant, they are not diffeomorphism-invariant (see Kiefer (2004), p. 175).

3.1.4. The Dynamics

The central dynamical equations of general relativity are the *Einstein equations,* providing six independent differential equations for the metric in terms of the energy-momentum tensor. The equations may be derived by requiring that the Einstein-Hilbert action is stationary under variation of the metric components. An even more general approach would be to define the Hamilton-Jacobi equation for the Euclidean system above as (Rovelli (2003), section 4.1)

$$F^{ij}_{ab}(\vec{\tau})\,\frac{\delta S[A]}{\delta A^i_a(\vec{\tau})}\,\frac{\delta S[A]}{\delta A^j_b(\vec{\tau})} = 0. \tag{3.28}$$

Here, $S[A]$ is the principal Hamilton functional, and F is the curvature of A. The complex field A is defined on a 3-dimensional space σ without boundaries and coordinatized by $\vec{\tau}$. (On σ the value of A will be an element of the Lie algebra $\mathfrak{su}(2) \cong \mathfrak{so}(3)$.) We may regard A as the collection of measurements of A on σ. The functional $S[A]$ is defined on the space *Riem* Σ of the 3d connections A, and it is invariant under internal gauge transformations and 3d diffeomorphisms.

The quantisation of the *physical* theory now proceeds in the following manner. We obtain the quantum dynamical *Wheeler-DeWitt equation* by interpreting $S[A]$ as $\hbar \times$ the phase of $\mathbf{\Psi}[A]$, simultaneously replacing the derivatives in the Hamilton-Jacobi equation above with derivative operators:

$$F^{ij}_{ab}(\vec{\tau})\,\frac{\delta}{\delta A^i_a(\vec{\tau})}\,\frac{\delta}{\delta A^j_b(\vec{\tau})}\,\mathbf{\Psi}[A] = 0. \tag{3.29}$$

Or, defining the hamiltonian operator H,

$$H\mathbf{\Psi} = 0. \tag{3.30}$$

In this way, a *classical* constraint has been turned into a restriction on the state space of the *quantum* theory: the physically allowable states must obey the Wheeler-DeWitt equation.

The states in physical space $\mathcal{H}$ are found as solutions of the Wheeler-De Witt equation. The hamiltonian operator $\mathbf{H}$ of the equation is given as the limit when $\epsilon \to 0$ of the operator

$$\begin{aligned} H_\epsilon\,|\,\mathrm{S}\rangle = &-\frac{i}{\hbar}\sum_{n\in S} N_n \sum_{l,l',l''} \epsilon_{ll'l''} \\ &Tr\Big(U\big(A,\ \gamma^{-1}_{x(n),l}\big)\,U\big(A,\ \alpha_{x(n),l',l''}\big)\Big[\mathbf{V}(\mathcal{R}_n),\ U\big(A,\ \gamma_{x(n),l}\big)\Big]\Big)\,|\,\mathrm{S}\,\rangle\,. \end{aligned} \tag{3.31}$$

Here, $x(n)$ is the position of the node n, $\gamma_{x(n),l}$ is a path of coordinate length ϵ along the link l, and $\alpha_{x(n),l',l''}$ is the triangle along the links l' and l'' with side lengths ϵ closed by a line connecting the end points. $\mathcal{R}_n$ is the small region around the node n. The limit can be shown to exist on diffeomorphism invariant states. Also, on these states, the operator is quantised.

3.2. The Bohrification of Loop Quantum Gravity

Quantum physics, even including quantum gravity, should satisfy Bohr's principle of classical observation. Interpreted in a topos model, this implies that all measurements on observables derived from the gravitational field should be contained in the family of classical contexts. As indirect justification for this doctrine in the quantum gravitational case, one may perhaps cite evidence pointing to the difficulty of quantising a system without also quantising any other system to which it is coupled (Kiefer (2004), p. 19).

Our goal is to find a certain generalized space (to be precise, a locale) which may be identified as the state space of quantum gravity in the topos model. This space will turn out to be the Gelfand spectrum of the *commutative* (in the topos) algebra $\underline{\mathcal{W}}$ of observables in the topos $\mathbf{Sets}^{C(\mathcal{W})}$. The Bohrification procedure, as outlined in subsection 2.2.2, suggests that we should proceed in the following manner:

1. Find a suitable non-commutative C*-algebra $\mathcal{W}$ which will represent a (complete) set of (kinematical) observables of LQG.
2. Delineate the family of commutative subalgebras of $\mathcal{W}$.
3. Compute the (external) description $\Sigma_{\mathcal{W}}$ (the *Bohrified state space* of $\mathcal{W}$) of the locale $\underline{\Sigma}(\underline{\mathcal{W}})$ (the *Gelfand spectrum* of $\underline{\mathcal{W}}$).
4. Investigate the topological properties of $\Sigma_{\mathcal{W}}$.
5. Are the requirements of gauge and diffeomorphism invariance satisfied in the "Bohrified" version of LQG?

Below, we give just the basic construction of the topos model of LQG, and a first discussion of its elementary properties. In a more advanced exposition, one should ask whether the model is relevant for attacking some of the remaining issues in LQG theory, such as the problem of the classical limit (see Rovelli (2004), p. 292, for a list). Can we start from an algebra $\mathcal{D}$ of physical ('Dirac') observables, and is the corresponding state space $\Sigma_{\mathcal{D}}$ the true dynamical space of the theory? Is it possible to calculate transition amplitudes in this model?

We refer the reader to section 3.1 for some of the physical justification of the LQG constructions to follow. A compact reference for C*-algbras is de Faria and de Melo ((2010), appendix B).

3.2.1. The Choice of a C*-Algebra

In this subsection, we sketch some of the difficulties connected with the choice of a suitable operator algebra for a topos model of LQG. We shall simplify matters considerably by considering only the non-dynamical case. Thus, we will not enter into the problems connected with the (physical) Hamiltonian constraint here (cf. subsection 3.1.4). For the moment, this is not unduly restricting. As mentioned in section 3.1, the gauge and diffeomorphism constraints already give us a glimpse of the fine-scale structure of space-time in LQG. In fact, the space $\mathcal{K}_0$ of states which are invariant under the local $SU(2)$ gauge transformations suffices for this.

In subsection 3.1.3, the canonical variables of LQG, $U^A{}_B$ (the configuration variables) and $E_i(S)$ (the momentum variables) were promoted to operators. We also include the identity operator, thus completing the set of "fundamental" variables. We saw that these operators could be represented on a space $\mathcal{S}$. A scalar product was defined for $\mathcal{S}$, and, completing $\mathcal{S}$ in the associated norm, we found the state space $\mathcal{K}$, a Hilbert space. Can we build a concrete C*-algebra from these materials? (Recall that a set $\mathfrak{A}$ of bounded linear operators on a Hilbert space H is called a *concrete C*-algebra* if it is a *-subalgebra of $B(H)$ and closed in the norm topology, i.e. if a sequence A_n of operators satisfies $\lim \|A_n - A\| = 0$, then $A \in \mathfrak{A}$.) Briefly, we outline a result which shows that we must be careful with our construction.

Lemma 3.1 (i) *All multiplicative operators defined from cylindrical functionals (cf. 3.16 above) are bounded. In particular, the cylindrical operators $U^A{}_B$ are bounded.* (ii) *The flux operators $E_i(S)$ are unbounded and the area operators $\mathbf{A}(S)$ are unbounded and positive.*

Sketch of proof (i) Multiplicative operators given by cylindrical functionals are smooth functions $f : SU(2)^N \to \mathbb{C}$. But $SU(2)^N$ has the product topology built from the compact Lie group $SU(2)$. Therefore, $SU(2)^N$ is a compact space, so f is bounded. The operator norm then coincides with the sup norm of f.

(ii) In order to prove that an operator $E_i(S)$ is unbounded, it suffices to consider its action on states which are of the form $|\Gamma, j_l, \alpha_l, \beta_l\rangle$, where $\Gamma = \{\gamma_1, ..., \gamma_N\}$ is a graph in the 3d space Σ, j_l is a representation of $SU(2)$ and α_l, β_l are matrix components of $R^{(j)\,\alpha}{}_{\beta}(U(A, \gamma_l))$ (cf. (3.20) above). Assume there is an N such that $\|E_i(S)\| < N\hbar$. Recall from (3.25) above that (cf. also Rovelli (2007), p. 245)

$$E_i(S)\,(U(A,\ \gamma)) = \sum_{p \in (S \cap \Gamma)} \pm i\hbar\, U\left(A,\ \gamma_{l1}^p\right) \tau_i\, U\left(A,\ \gamma_{l2}^p\right). \tag{3.32}$$

Each path γ_l is cut in two halves, γ_{l1}^p and γ_{l2}^p, at the intersection points in $S \cap \Gamma$. For simplicity, we have chosen the representation $j = 1/2$. It is possible to pick paths γ_l and a connection A such that the graph Γ intersects the surface S in N points and

$$\tau_i = U^*\left(A,\ \gamma_{l1}^p\right) U^*\left(A,\ \gamma_{l2}^p\right). \tag{3.33}$$

It then follows that

$$E_i(S)\,|\,\Gamma,\ j_l,\ \alpha_l,\ \beta_l\rangle = \sum_{p\in(S\bigcap\gamma)} \pm i\hbar\,. \tag{3.34}$$

$U(A, \gamma) \in SU(2)$, so the last result implies that

$$\|E_i(S)\| \geq \left|\sum_{p\in(S\bigcap\gamma)} \pm i\hbar\right| = N\hbar. \tag{3.35}$$

This contradicts our assumption, hence $E_i(\mathcal{S})$ is unbounded.

It can also be seen that the area operators $\mathbf{A}(S)$ are unbounded and positive. The spin network states $|\mathrm{S}\rangle$ form an orthonormal basis in $\mathcal{K}_0$ (Rovelli (2004), p. 236). The properties then follow from the action of $\mathbf{A}(S)$ on $|\mathrm{S}\rangle$ as given in (3.27). □

The operators $E_i(S)$ are neither bounded nor self-adjoint. Now, selfadjointness can be had if we switch to the area operators $\mathbf{A}(S)$, but these operators are still unbounded. If we want to use the C*-algebra formalism of the Bohrification approach, we must find a way to tame these operators. At this point, it might be of interest to introduce Weyl's device for avoiding unbounded operators. If we stick to the unbounded operators $\mathbf{A}(S)$, we may try to define the one-parameter group of *Weyl operators*

$$V(t) = e^{it\mathbf{A}(S)}. \tag{3.36}$$

These will be bounded operators, in fact, they are unitary. Naively, we may then try to create a C*-algebra in the following manner. Consider holonomies $U(A, \alpha)$ with α a single closed curve. The *Wilson loops* $T_\alpha[A] = \frac{1}{2}\mathrm{tr}\ U(A, \alpha)$ are multiplicative operators on the state space $\mathcal{S}$. (The algebra generated by these functionals is known as the *holonomy algebra.*) Generally, the holonomy fulfills $U(A, \gamma)U(A, -\gamma) = I$ for any curve γ. For the gauge group $SU(2)$, this implies that $U(A, \gamma)^\dagger = U(A, -\gamma)$. From the definition of the scalar product (3.18), we see that $T_\alpha^*[A] = \overline{T_\alpha[A]}$ (here, * is the adjoint operation). We deduce the elementary properties

$$T_\alpha[A] = T_\alpha^*[A], \tag{3.37}$$

$$T_\alpha^*[A] = \frac{1}{2}\overline{\mathrm{tr}U(A, \alpha)} = \frac{1}{2}\mathrm{tr}U(A, \alpha)^\dagger = \frac{1}{2}\mathrm{tr}U(A, -\alpha) = T_{-\alpha}[A]. \tag{3.38}$$

By the first property, the operators $T_\alpha[A]$ are self-adjoint. Also, the operator norm coincides with the supremum norm on the Wilson loops, so we note that $\|T_\alpha\| = 1$ because the trace of an element in the group $SU(2)$ is less than or equal to 2. Let us assume for simplicity that $|\mathrm{S}\rangle$ is a state such that the loop α does not intersect it. Then $T_\alpha[A]|\mathrm{S}\rangle = |\mathrm{S}\bigcup\alpha\rangle$. Together with the unbounded operators $\mathbf{A}(S)$, the $T_\alpha[A]$'s then form canonical commutation relations

$$[T_\alpha[A], \mathbf{A}(S)] = -\mathrm{i}\hbar \sum_{p\in(S\bigcap\alpha)} \sqrt{j_p\left(j_{p+1}+1\right)}\ T_\alpha[A]. \tag{3.39}$$

For the usual construction of a Weyl C*-algebra, we expect a relation such as "$[T_\alpha[A], \mathbf{A}(S)] = -i\hbar I$" in order to apply the Baker-Hausdorff formula, so the above result is not very useful. If we want to proceed within the topos approach, we shall have to appeal to a deeper analysis of the operators involved. Several such analyses are available in the litterature on LQG. The best known is probably the so-called holonomy-flux algebra, a *-algebra (Thiemann (2007)). Below, we shall use the Weyl C*-algebra developed in Fleischhack (2004).

Before we turn to the construction, we remark that the procedure outlined above suggests the following supplement to the Bohrification method of section 2.2.2. Accord ing to Bohr's thesis, observation is always filtered through classical concepts. However, one may argue that it is not the observables themselves that are of primary importance, but rather their evolution. Of main interest in particle physics is the calculation of transition probabilities when one or several particles approach an interaction region from infinitely far off, and leave again at infinity. In fact, these probability distributions (the cross-sections) are all that is measured. For example, suppose that Ψ_{in} is the incoming particle state, and we want to find the probability that the outgoing state is Ψ_{out}. The amplitude for development from Ψ_{in} to Ψ_{out} will then be given by the quantity $\langle \Psi_{out} \mid e^{-iHt}\, \Psi_{in} \rangle$, where $U \equiv e^{-iHt}$ is the unitary Weyl operator corresponding to the Hamiltonian H.

So the contexts of our topos model ought to be subalgebras of a Weyl algebra A_W (to be named $\mathcal{W}$ in the LQG case below). The functor $\underline{A}_W$, the counterpart of A_W in the topos, may then be called the *Weylification* of the original, untamed algebra A generated by the "position" and "momentum" operators of A. Quite apart from the present topic, quantum gravity, it would be of interest to see to if Weylification modifies the constructions of the topos-theoretical approach to quantum physics. Presently, we shall pursue a variant of this scheme. Figure 3.2 below sums up the path that we shall follow in the rest of this chapter.

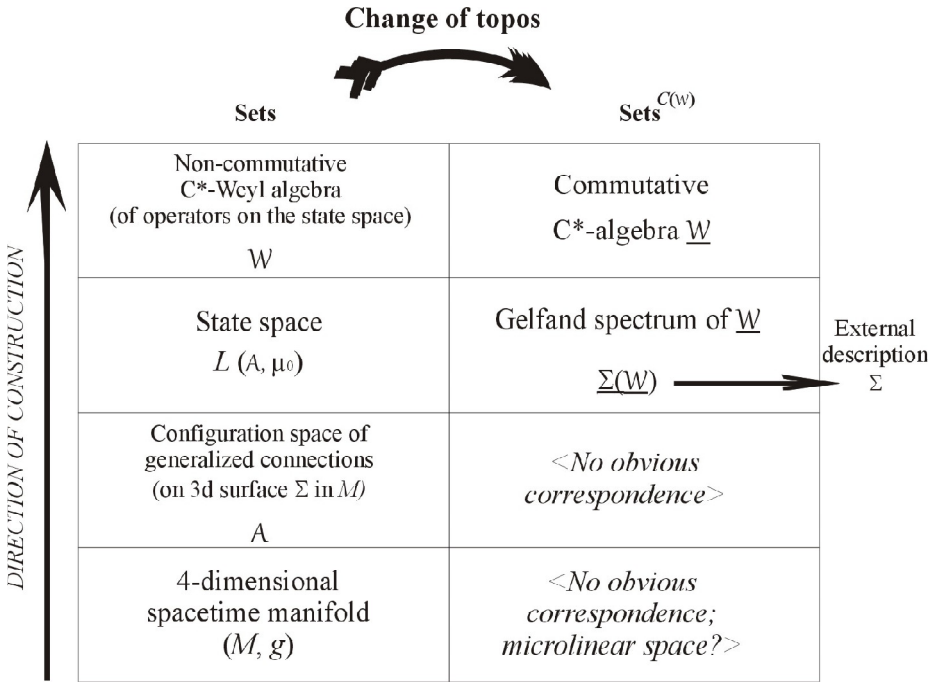


Figure 3.2. The Bohrification of LQG.

3.2.2. The Configuration Space $\overline{\mathcal{A}}$

We have already spoken of the quantum states as defined on a configuration space $\mathcal{A}$ consisting of the "position" observables A, the connections defined on a Cauchy surface Σ. Let us now be a bit more precise. (Variants of the definitions to follow can be found e.g. in Thiemann (2007), p. 162-175, and Fleischhack (2004), p. 14-34.) For technical reasons (e.g. Fleischhack (2004), p. 2), it is useful to extend the space $\mathcal{A}$ to include generalized connections.

Below, Σ is a 3d manifold, but we leave it open whether Σ is differentiable, analytic or even, for some purposes, semi-analytic (see Thiemann (2007), p. 162, for an enumeration of demands on Σ). The reader may prefer to think of Σ as the Cauchy surface on which we collect our physical data.

The set of all *paths* in Σ (equivalent up to endpoints, orientation-preserving reparametrizations and the deletion of retraced curves $c \circ c^{-1}$) is denoted by $\mathcal{P}$. We may regard $\mathcal{P}$ as a groupoid under composition of paths. (We shall only be able to compose paths when the second path takes off from the end point of the first, so the operation is only partial.) Informally, we say that *graphs* v are finite collections of independent edges, where an *edge* is a path with no crosses (but possibly closed). All paths are finite combinations of edges. A collection of edges is *independent* if the edges meet each other at most in the beginning and final points. Then $\mathcal{P}_v$ is the subgroupoid in $\mathcal{P}$ generated by the edges in the graph v.

Definition 3.1 (i) Hom($\mathcal{P}$, $SO(3)$) *is the set of groupoid morphisms from the set of paths in* Σ *into* $SO(3)$. (ii) Hom($\mathcal{P}_v$, $SO(3)$) *is the set of groupoid morphisms from the subgroupoid* $\mathcal{P}_v$ *to* $SO(3)$. *(Homomorphisms in this set will be denoted by* x_v.)

In section 3.1, paths γ in Σ and connections A in $\mathcal{A}$ were associated with matrices in $SO(3)$ (or $SU(2)$), the holonomies $h_\gamma(A) = U(A, \gamma)(1)$. The holonomy is defined as the unique solution of the equation

$$\frac{d}{ds} U(A, \gamma)(s) = U(A, \gamma)(s)\, A_i(\gamma(s))\, \dot{\gamma}^i(s). \tag{3.40}$$

Formally, one also writes

$$h_\gamma(A) = P \int_\gamma A. \tag{3.41}$$

For holonomies, it holds that $h_{\gamma_1 \circ \gamma_2}(A) = h_{\gamma_1}(A) h_{\gamma_2}(A)$ and $h_{\gamma^{-1}}(A) = h_\gamma(A)^{-1}$ (this simply says that parallel transport forth and back along a curve does not change the vector), so any connection A may be identified with its groupoid morphism $\gamma \mapsto h_\gamma(A)$. Thus $\mathcal{A} \subset \mathrm{Hom}(\mathcal{P}, SO(3))$.

Definition 3.2 (i) $\overline{\mathcal{A}} = \mathrm{Hom}(\mathcal{P}, SO(3))$ *is the* set of generalized connections *(on a principal* $SO(3)$ *bundle with base manifold* Σ*).* (ii) $\overline{\mathcal{A}_v} = \mathrm{Hom}(\mathcal{P}_v, SO(3))$.

It has been pointed out (e.g. Thiemann (2007), p. 169) that $\overline{\mathcal{A}}$ also contains distributional elements, so it is very large. (This is what we expect to find in a quantum field theory: in the Wightman axiomatization (Streater and Wightman (1980)) the fields are operator-valued distributions.)

The next step is to find a topology for the space $\overline{\mathcal{A}}$. Note that the sets $\overline{\mathcal{A}_v}$ may be identified with $SO(3)^{\#v}$ (where $\#v$ is the number of edges in v). $SO(3)$ is a compact Hausdorff space, so $\overline{\mathcal{A}_v}$ is compact Hausdorff too. Recall that the *Tychonov topology* on a direct product X_∞ (of any cardinality) of topological spaces X_v is the weakest topology such that the projections onto the component spaces are continuous. Also, by Tychonov's theorem, the direct product space of compact topological spaces is a compact topological space (in the Tychonov topology). Accordingly, the direct product space $\mathcal{A}_\infty := \prod_{v \subset \mathcal{P}} \overline{\mathcal{A}_v}$ is compact (it is also Hausdorff).

In order to situate $\overline{\mathcal{A}}$ within $\mathcal{A}_\infty$, we appeal to the notion of a projective limit. We let $\mathcal{L}$ be the set of subgroupoids of $\mathcal{P}$, and say that $\mathcal{P}_v \prec \mathcal{P}_{v'}$ (or simply $v \prec v'$) iff $\mathcal{P}_v$ is a subgroupoid of $\mathcal{P}_{v'}$. It can be shown that $\mathcal{L}$ is a partially ordered and directed set (the last point is not entirely trivial; see Thiemann (2007), th. 6.2.13). The set $\mathcal{L}$ is associated with a *projective family* $(\overline{\mathcal{A}_v},\ p_{v'v})_{v\prec v'\in\mathcal{L}}$, where $p_{v'v} : \overline{\mathcal{A}_{v'}}\ (= \mathrm{Hom}(\mathcal{P}_{v'}, SO(3))) \to \overline{\mathcal{A}_v}\ (= \mathrm{Hom}(\mathcal{P}_v, SO(3)))$ is the projection of the groupoid $\overline{\mathcal{A}_{v'}}$ onto its subgroupoid $\overline{\mathcal{A}_v}$. The *projective limit* $\overline{\mathcal{A}}$ of the projective family $(\overline{\mathcal{A}_v},\ p_{v'v})_{v\prec v'\in\mathcal{L}}$ is the subset of $\mathcal{A}_\infty$ given by

$$\overline{\mathcal{A}} = \left\{(x_v)_{v\in\mathcal{L}} \;\middle|\; \forall\, v \prec v'\,(p_{v'v}(x_{v'}) = x_v)\right\}. \tag{3.42}$$

The elements $(x_v) \equiv (x_v)_{v\in\mathcal{L}}$ are called *nets*. Note that the sign $\overline{\mathcal{A}}$ here makes a second entrance. This is justified by the following lemma.

Lemma 3.2 [cf. Thiemann (2007), th. 6.2.22] *The set of generalized connections $\overline{\mathcal{A}}$ is the projective limit of the projective family $(X_l,\ p_{l'l})_{l\prec l'\in\mathcal{L}}$.*

If we let $\pi_\gamma : \overline{\mathcal{A}} \to \overline{\mathcal{A}_v} \simeq SO(3)^{\sharp\, v}$ be the projection of a generalized connection $\overline{A}$ onto its family member in $\overline{\mathcal{A}_v}$, we may extend the domain of the holonomies h_γ to include all generalized connections $\overline{A}$ by the stipulation

$$h_\gamma(\overline{A}) := \pi_\gamma(\overline{A}). \tag{3.43}$$

Below, we shall also switch to the notation $h_{\overline{A}}(\gamma) = h_\gamma(\overline{A})$, or even $\overline{A}(\gamma)$, whenever it is more convenient (e.g. in proposition 3.4). We now define the topology on $\overline{\mathcal{A}}$ as the subspace topology of $\overline{\mathcal{A}}$ with respect to the Tychonov topology on $\mathcal{A}_\infty$.

Lemma 3.3 [cf. Thiemann (2007), th. 6.2.19] *$\overline{\mathcal{A}}$ is a closed subset of $\mathcal{A}_\infty$.*

Because $\mathcal{A}_\infty$ is compact, it follows from the last lemma that $\overline{\mathcal{A}}$ is a compact. $\overline{\mathcal{A}}$ is the configuration space upon which the states of the theory are to be defined.

3.2.3. The State Space $\mathcal{H}$

The next step on the construction ladder (fig. 3.2) will be to identify the state space $\mathcal{H}$ as $L_2(\overline{\mathcal{A}},\ \mu_0)$, the square-integrable functions on the space of generalized connections. To do this, we must define a measure μ_0 on $\overline{\mathcal{A}}$. Note that $SO(3)$, as a compact Lie group, has a unique Borel measure, the Haar measure dg. The Haar measure is invariant under left and right translation by g in $SO(3)$:

$$\int_{SO(3)} f(g)\,dg = \int_{SO(3)} f(gh)\,dg = \int_{SO(3)} f(hg)\,dg. \tag{3.44}$$

By Fubini's theorem, integration over $\overline{\mathcal{A}_v} \approx SO(3)^{\#v}$ may be reduced to iterated integration over $SO(3)$ (e.g. Sepanski (2007)). Denoting the projection of $\overline{\mathcal{A}}$ onto $\overline{\mathcal{A}_v}$ by π_γ, we demand that a measure μ_0 on $\overline{\mathcal{A}}$ shall satisfy the condition

$$\mu_0 = dg_1\, dg_2 \cdots dg_{\#v} \circ \pi_\gamma. \tag{3.45}$$

There is a unique (regular) Borel measure which fulfills this requirement, the *Ashtekar-Lewandowski measure* μ_0 (for a precise definition, see Thiemann (2007), def. 8.2.4). (It has been proven that the state space $\mathcal{K}$ introduced in subsection 3.1.2 can be identified with $L_2(\overline{\mathcal{A}}, \mu_0)$.) As usual, we have the norm on $L_2(\overline{\mathcal{A}}, \mu_0)$ induced by the standard inner product.

Definition 3.3 *The* state space $\mathcal{H}$ *is the Hilbert space* $L_2(\overline{\mathcal{A}}, \mu_0)$ *of measurable square-integrable functions over the space* $\overline{\mathcal{A}}$ *of generalized connections, where* μ_0 *is the Ashtekar-Lewandowski measure.*

3.2.4. The Operators and Their Interpretation

We now have the set of bounded operators on the state space, $B(L_2(\overline{\mathcal{A}}, \mu_0))$, at our disposal, and may proceed with the construction of the C*-algebra by picking the appropriate operators within this set. Firstly, we shall need representatives of the holonomy (or "configuration") operators U^A_B. We remarked in the proof of lemma 3.1 that these are multiplicative operators. But if f is any continuous function on $\overline{\mathcal{A}}$, we define the operator $T_f \in B(L_2(\overline{\mathcal{A}}, \mu_0))$ by

$$\left(T_f\, \phi\right)(A) = f(A)\, \phi(A),\ \ \phi \in L_2\left(\overline{\mathcal{A}}, \mu_0\right). \tag{3.46}$$

We therefore identify the set of configuration operators with the set of multiplicative operators corresponding to the continuous functions on the configuration space $\overline{\mathcal{A}}$.

Definition 3.4 $T = \{T_f \in B(L_2(\overline{\mathcal{A}}, \mu_0)) \mid f \in C(\overline{\mathcal{A}})\}$ *is the* set of configuration operators *on the state space* $(L_2(\overline{\mathcal{A}}, \mu_0)$.

Above, we considered two kinds of flux (or "momentum") operators, $E_i(S)$ and $\mathbf{A}(S)$. We now concentrate on $E_i(S)$, which was defined relative to a surface S in the 3d manifold Σ (the Cauchy surface). Recall (from (1.24)) that the full expression for $E_i(S)$ was

$$E_i(S) = -i\hbar \int_S d\sigma^1\, d\sigma^2\, n_a(\vec{\sigma})\, \frac{\delta}{\delta A^i_a\left(x(\vec{\sigma})\right)}. \tag{3.47}$$

We first focus on the notion of a surface. There are several options available in the LQG literature for the definition of a surface, and we choose to follow Fleischhack ((2004), p. 22). We say that a subset S of Σ is a *quasi-surface* iff every edge γ can be decomposed into a finite set of segments $\{\gamma_1, ..., \gamma_n\}$ such that the interior of any segment γ_i is either included in S or has no points in common with S. As we have not yet commited ourselves to a particular choice of smoothness properties for Σ and the edges in Σ, this flexibility carries over into the definition of S. We shall, however, suppose that the surfaces have an orientation. For this purpose, we say that a quasi-surface S is *oriented* if there exists a function σ_S from the set of all parameterized paths to the set $\{-1, 0, 1\}$ such that

$$\sigma_S(\gamma) = \begin{cases} \pm 1 & \text{if } \gamma(0) \in S \text{ and } \gamma \text{ does not have an initial segment included in } S \\ 0 & \text{if } \gamma(0) \notin S \text{ or } \gamma \text{ has an initial segment included in } S. \end{cases}$$

We also demand that, if γ_1 and γ_2 are paths such that γ_1 ends where γ_2 begins, then $\sigma_S(\gamma_1 \circ \gamma_2) = \sigma_S(\gamma_1)$ unless γ_1 ends on S. For a γ which starts and ends on S without crossing it, we demand $\sigma_S(\gamma) = \sigma_S(\gamma)$. This assures that $\sigma_S(\gamma_1) = \sigma_S(\gamma_2)$ for paths γ_1 and γ_2 which start on S and have an initial segment in common. Another reasonable requirement is to set $\sigma_S(\gamma) = (-)^n \sigma_S(\gamma)$ for paths which start and end on S, after crossing it n times. The function σ_S is called the *intersection function.*

The second ingredient in the definition of $E_i(S)$ is the functional differential $\delta/\delta A_a^i\big(x(\vec{\sigma})\big)$, which is smeared across the surface S. From quantum mechanics, we expect the momentum operator, as the conjugate of the configuration operator, to be the generator of (infinitesimal) translations in configuration space. Indeed, for a particle moving in the one-dimensional space $\mathbb{R}$, we can define the translation operator T_ϵ by stipulating that

$$(T_\epsilon \psi)(x) = \psi(x - \epsilon). \tag{3.48}$$

Expanding both sides to first order in ϵ, we have

$$\left(\left(I - \frac{i\epsilon}{\hbar} P\right)\psi\right)(x) = \psi(x) - \frac{d\psi}{dx}\epsilon. \tag{3.49}$$

It follows at once that $P = -i\hbar d/dx$, so the momentum operator generates a translation by ϵ. Analogously, we should define the quantum gravity momentum as the generator of small translations in our configuration space, the generalized connection space $\overline{\mathcal{A}}$. The next result is therefore important:

Proposition 3.4 [Fleischhack (2004), prop. 3.19] *Given a quasi-surface S and an intersection function σ_S, and let γ be a path in Σ which does not traverse the surface S. There is a unique map $\Theta^{S,\sigma_S} : \overline{\mathcal{A}} \times \mathrm{Maps}(\Sigma, SO(3)) \to \overline{\mathcal{A}}$ such that*

$$h_{\Theta^{S,\sigma_S}(\overline{A},d)}(\gamma) = \begin{cases} d(\gamma(0))^{\sigma_S(\gamma)}\, h_{\overline{A}}(\gamma)\, d(\gamma(1))^{-\sigma_S(\gamma^{-1})} & \text{if } int\,(\gamma) \text{ is not included in } S \\ h_{\overline{A}}(\gamma) & \text{if } int\,(\gamma) \text{ is included in } S. \end{cases}$$

Θ^{S,σ_S} is continuous if Maps(Σ, $SO(3)$) *is given the product topology. We define* $\Theta_d^{S,\sigma_S} : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$ *by*

$$\Theta_d^{S,\sigma_S}(\overline{A}) = \Theta^{S,\sigma_S}\left(\overline{A},\, d\right).$$

Then Θ_d^{S,σ_S} is a homeomorphism which preserves the measure μ_0 on $\overline{\mathcal{A}}$.

Θ_d^{S,σ_S} is the sought-for momentum operator (or rather, class of operators). We shall give a heuristic motivation of its definition. For simplicity, we shall consider the case when the triple (S, σ_S, d_g) is given, with d_g the constant mapping of Σ to the element g in $SO(3)$, and where S is a quasi-surface with orientation σ_S. We then denote the corresponding operator simply by Θ_g $(= \Theta_{d_g}^{S,\sigma_S})$. By the second clause of the definition, $h_{\Theta_g(\overline{A})}(\gamma) = h_{\overline{A}}(\gamma)$ when int(γ) is on the surface, so Θ_g leaves the holonomy unchanged for such paths. (This corresponds to case (a) in figure 3.3.) For a path γ_1 which leaves the surface at $\gamma_1(0)$ on the upper (positive) side, we have $h_{\Theta_g(\overline{A})}(\gamma_1) = g^{\sigma_S(\gamma_1)} h_{\overline{A}}(\gamma_1) = g h_{\overline{A}}(\gamma_1)$ (case (b)), whereas a path γ_2 entering on the upper side gives $h_{\Theta_g(\overline{A})}(\gamma_2) = h_{\overline{A}}(\gamma_2) g^{-\sigma_S(\gamma_2^{-1})} = h_{\overline{A}}(\gamma_2) g^{-1}$, because $\sigma_S(\gamma_1) = \sigma_S(\gamma_2^{-1}) = 1$ (case (c)). For a path γ_3 which starts on the surface and then returns to it, we have $h_{\Theta_g(\overline{A})}(\gamma_3) = g^{\sigma_S(\gamma_1)} h_{\overline{A}}(\gamma_3) g^{-\sigma_S(\gamma_3^{-1})} = g h_{\overline{A}}(\gamma_3) g^{-1}$ (case (d)). Similar results hold for paths entering or leaving the surface from below (negative side). The cases (e) - (g) in the figure are excluded.

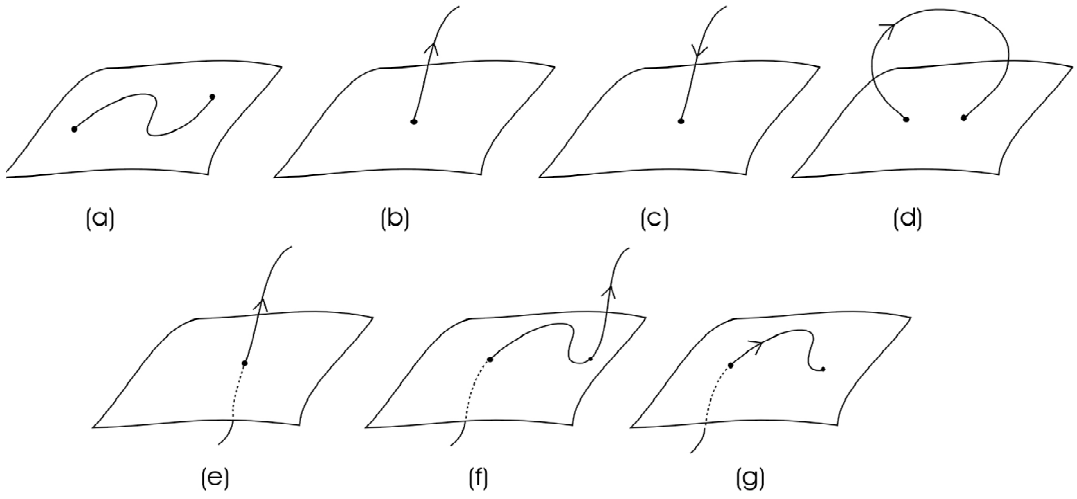


Figure 3.3. Some combinations of surfaces and paths.

The holonomy $h_{\overline{A}}(\gamma)$ smears the connection $\overline{A}$ along the path γ. We now see that the operators Θ_g act upon the holonomy where the path γ hits the surface S. E.g. in case (b) we saw that

$$h_{\Theta_g(\overline{A})}(\gamma) = g\, h_{\overline{A}}(\gamma). \tag{3.50}$$

For paths γ of small length ϵ, multiplication by an element g in $SO(3)$ reduces to addition by a linear combination $\epsilon^i \tau_i$ in the corresponding Lie algebra $\mathfrak{so}(3) \approx \mathfrak{su}(2)$. So Θ_g (or, generally, Θ_d^{S,σ_S} when d is non-constant) is a generator of infinitesimal translations. Briefly,

$$"\Theta = I + \epsilon^i \tau_i".$$

In most cases, Θ_d^{S,σ_S} will be unbounded, but this difficulty is quickly removed. Again, we may pursue an analogy with quantum theory. Here, it is demanded that symmetry transformations (such as the Galilean or the Lorentz transformation) should be represented by unitary operators on the state space. It then turns out (Weinberg (1995), p. 59) that the first-order term of the unitary operator U of the Lorentz transformation can be identified as the momentum operator P. The construction in Fleischhack ((2004), p. 26) achieves this by means of the following result:

Proposition 3.5 *For (X, μ) a compact Hausdorff space, μ a regular Borel measure on X and $\psi : X \to X$ a continuous surjective map which leaves μ invariant, the pull-back map $\psi^* : C(X) \to C(X)$ can be extended to a unitary operator on $L_2(X, \mu)$.*

Proof The proof, though simple, is omitted from most text books on functional analysis, so we include it here. It suffices to prove that ψ^* is an isometry of $L_2(X, \mu)$ onto $L_2(X, \mu)$. Note first that ψ^* is an isometry on the linear subspace $C(X)$ of $L_2(X, \mu)$:

$$|\psi^* f|^2 = \int_X \overline{\psi^* f}\, \psi^* f \, d\mu = \int_X \overline{f}\, f \circ \psi \, d\mu = \int_X \overline{f}\, f \, d\mu = |f|^2.$$

By the Stone-Weierstrass theorem, $C(X)$ is dense in $L_2(X, \mu)$, so ψ^* can be extended to $L_2(X, \mu)$. By continuity of ψ, ψ^* is an isometry on $L_2(X, \mu)$. It is also onto, for given $f \in L_2(X, \mu)$, there is a sequence $\{f_n\}$ in $C(X)$ such that $f_n \to f$. But ψ is surjective, so there is a convergent sequence $\{f_n'\}$ in $C(X)$ with $\psi^* f_n' = f_n' \circ \psi = f_n$ for each n. Let $f' = \lim f_n'$. Then $\psi^* f' = \psi^*(\lim f_n') = \lim \psi f_n' = \lim (f_n' \circ \psi) = \lim f_n = f$. □

We also have the following useful lemma:

Lemma 3.6 *For $f \in C(X)$ and $\psi : X \to X$, the corresponding operators T_f and $w \equiv \psi^*$ in $B(L_2(X, \mu))$ satisfy $w \circ T_f \circ w^{-1} = T_{w(f)}$.*

Proof Assume $h \in C(X)$. Then $(T_{w(f)} \circ w)h = T_{w(f)}(w(h)) = T_{w(f)}(h \circ \psi) = w(f)\,(h \circ \psi) = (f \circ \psi)\,(h \circ \psi) = (f\,h) \circ \psi = w(f\,h) = w(T_f h) = (w \circ T_f)h$. But $C(X)$ is dense in $L_2(X, \mu)$, so the relation holds also for $h \in L_2(X, \mu)$. □

The configuration space $\overline{\mathcal{A}}$ with measure μ_0 fulfills the conditions in proposition 3.5. Also, Θ_d^{S,σ_S} is a homeomorphism, hence surjective. Application of the proposition to the momentum operators now allows us to define the Weyl operators:

Definition 3.5 [Fleischhack (2004), def. 3.21] *Let $\Theta_d^{S,\sigma_S} : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$ be a momentum operator as in proposition 3.4. The Weyl operator $w_d^{S,\sigma_S} : L_2(\overline{\mathcal{A}}, \mu_0) \to L_2(\overline{\mathcal{A}}, \mu_0)$ is defined as the pull-back of the momentum operator,*

$$w_d^{S,\sigma_S} := (\Theta_d^{S,\sigma_S})^*.$$

We may regard the first-order terms of the operators w above as generators of translations in the space of generalized connections $\overline{\mathcal{A}}$.

3.2.5. The Weyl C*-algebra $\mathcal{W}$

The discussion has prepared us for the next, central definition, which gives us the C*-algebra needed for the toposification of loop quantum gravity:

Definition 3.6 [Fleischhack (2004), def. 4.1] *The* loop quantum gravity C*-algebra $\mathcal{W}$ is the subalgebra of bounded operators in $B(L_2(\overline{\mathcal{A}}, \mu_0))$ generated by $T = \{T_f \in B(L_2(\overline{\mathcal{A}}, \mu_0)) \mid f \in C(\overline{\mathcal{A}})\}$ (where T_f is the multiplicative operator associated with f) and the Weyl operators w_d^{S,σ_S}.*

We say that T is the set of *position* (or *configuration*) *operators*. The choice of $C(\overline{\mathcal{A}})$ for this purpose is analogous to the definition of position operators in quantum mechanics (cf. Fleischhack (2004), p. 15 and Rovelli (2004), p. 199). $\mathcal{W}$, as defined above, fulfills the demands on a (concrete) C*-algebra. (The norm of the algebra is simply the operator norm in $B(L_2(\overline{\mathcal{A}}, \mu_0)$, which satisfies the additional norm condition $\|A^*A\| = \|A\|^2$.) It is also clear that $\mathcal{W}$ has non-trivial commutative subalgebras. Below, we mention some of them.

Example 3.1 Let W_T be the subalgebra of $\mathcal{W}$ generated by the set of configuration operators $T = \{T_f \in B(L_2(\overline{\mathcal{A}}, \mu_0)) \mid f \in C(\overline{\mathcal{A}})\}$. Then W_T is a commutative C*-subalgebra of $\mathcal{W}$.

Example 3.2 [Cf. Fleischhack (2004), cor. 3.23] Let (S, σ_S) be an oriented quasi-surface, and let D be a set of functions $d : \Sigma \to SO(3)$ such that $d_1 d_2 = d_2 d_1$ for all d_1, $d_2 \in D$. Define $W_{S,D}$ as the subalgebra of $\mathcal{W}$ generated by the set of all operators w_d^{S,σ_S} with $d \in D$. Then $w_{d_1}^{S,\sigma_S} w_{d_2}^{S,\sigma_S} = w_{d_1 d_2}^{S,\sigma_S} = w_{d_2}^{S,\sigma_S} w_{d_1}^{S,\sigma_S}$. Indeed, assume $f \in L_2(\overline{\mathcal{A}}, \mu_0)$ and let $\{f_n\}$ be a sequence in $C(X)$ such that $f_n \to f$. We have, for each n,

$$(w_{d_1}^{S,\sigma_S} w_{d_2}^{S,\sigma_S}) f_n = w_{d_1}^{S,\sigma_S}(f_n \circ \Theta_{d_2}^{S,\sigma_S}) = (f_n \circ \Theta_{d_2}^{S,\sigma_S}) \circ \Theta_{d_1}^{S,\sigma_S} = f_n \circ \Theta_{d_1 d_2}^{S,\sigma_S} = w_{d_1 d_2}^{S,\sigma_S} f_n.$$

By taking the limit, we find that these operators are commutative over all of $L_2(\overline{\mathcal{A}}, \mu_0)$. Above, we used the relation

$$\Theta_{d_2}^{S,\sigma_S} \circ \Theta_{d_1}^{S,\sigma_S} = \Theta_{d_1 d_2}^{S,\sigma_S}.$$

This can easily be derived from the definition (in prop. 3.4) of Θ. Consider e.g. the case where a path γ leaves the surface S in the positive direction without return. Then

$$h_{\Theta_{d_1 d_2}^{S,\sigma_S}(\overline{A})}(\gamma) = d_1(\gamma(0))\, d_2(\gamma(0))\; h_{\overline{A}}(\gamma) = d_1(\gamma(0))\; h_{\Theta_{d_2}^{S,\sigma_S}(\overline{A})}(\gamma) = h_{\Theta_{d_1}^{S,\sigma_S}\left(\Theta_{d_2}^{S,\sigma_S}(\overline{A})\right)}(\gamma) = h_{\Theta_{d_1}^{S,\sigma_S} \circ \Theta_{d_2}^{S,\sigma_S}(\overline{A})}(\gamma).$$

This shows that $W_{S,D}$-algebras are commutative subalgebras of $\mathcal{W}$. The proof depended crucially on the commutativity of the "translator functions" d. For a given surface S, the algebra W_S generated by the set $\bigcup_{D \text{ commutative}} W_{S,D}$ will in general not be commutative.

Example 3.3 The algebras W_T and $W_{S,D}$ belong to the configuration operator and momentum operator region, respectively. We might wonder if there are commutative subalgebras of $\mathcal{W}$ which combine these regions. Let S be a given quasi-surface. If we apply lemma 3.6 to the operators w_d^{S,σ_S} and T_f, we see that the relation $w_d^{S,\sigma_S} \circ T_f \circ \left(w_d^{S,\sigma_S}\right)^{-1} = T_{w_d^{S,\sigma_S}(f)}$ holds in $\mathcal{W}$. Therefore, whenever $f = w_d^{S,\sigma_S}(f)$, this reduces to a commutative relation

$$w_d^{S,\sigma_S} \circ T_f = T_f \circ w_d^{S,\sigma_S}. \tag{3.51}$$

Writing $f \equiv T_f$ for the multiplicative operator and calculating with $h \in L_2(\overline{\mathcal{A}}, \mu_0)$, this amounts to the demand that

$$w_d^{S,\sigma_S}(fh) = f \cdot w_d^{S,\sigma_S}(h). \tag{3.52}$$

Now consider the relation

$$f\left(\Theta_d^{S,\sigma_S}\left(A\right)\right) = \left(w_d^{S,\sigma_S}(f)\right)(A) = f(A) \text{ for all } A \in \overline{\mathcal{A}}. \tag{3.53}$$

We say that f is *θ-invariant* when relation (3.53) holds for a momentum operator θ. Given a set V of commuting Weyl operators, f is said to be *V-invariant* if the relation holds for all momentum operators corresponding to the Weyl operators in V. If we like, we may also consider the subalgebra generated by V and the set of all multiplicative operators for which the relation holds. In particular, given the set $W_{S,D}$ as in example 3.2.2, we define $F(S, D) = \{T_f \in C(\overline{\mathcal{A}}) \mid f \text{ is } W_{S,D}\text{-invariant}\}$ and let $W_{F(S,D)}$ be the subalgebra generated by $W_{S,D} \bigcup F(S, D)$.

Lemma 3.7 *For f a θ-invariant function in* $C(\overline{\mathcal{A}})$, $w_d^{S,\sigma_S} = (\Theta_d^{S,\sigma_S})^*$ *commutes with* T_f.

Proof First, note that, for $A \in \overline{\mathcal{A}}$,

$$[w_d^{S,\sigma_S}(fh)](A) = [(fh) \circ \Theta_d^{S,\sigma_S}](A) = f(\Theta_d^{S,\sigma_S}(A)) \cdot h(\Theta_d^{S,\sigma_S}(A)).$$

We also have

$$[f \cdot w_d^{S,\sigma_S}(h)](A) = f(A) \cdot [w_d^{S,\sigma_S}(h)](A)$$
$$= f(A) \cdot h(\Theta_d^{S,\sigma_S}(A) = f(\Theta_d^{S,\sigma_S}(A)) \cdot h(\Theta_d^{S,\sigma_S}(A)).$$

The last step uses the θ-invariance of f. This shows that (3.52) holds, from which commutativity follows. □

One may inquire further about the circumstances under which θ-invariance occurs. For $d(x) = I$ (the identity in $SO(3)$), we have the trivial case, $w_d^{S,\sigma_S} = w_I^{S,\sigma_S} = I$ (the identity operator on $L_2(\overline{\mathcal{A}}, \mu_0)$). If f is Θ_d^{S,σ_S}-invariant for all operators Θ_d^{S,σ_S} defined with respect to a given surface S, we say that f is *S-invariant.* We also say that f is *S-invariant around* $\overline{A}$ if $f(\overline{B}) = f(\overline{A})$ for all connections $\overline{B}$ such that $\overline{B}(\gamma) \equiv h_{\overline{B}}\gamma = g(\gamma(0))\, h_{\overline{A}}\gamma\, g(\gamma(1))^{-1}$ for some function $g : S \to SO(3)$ and all loops γ starting and ending on S.

Lemma 3.8 *f is S-invariant if f is S-invariant around* $\overline{A}$ *for all generalized connections* $\overline{A}$.

Proof Immediate from the definition of the operators Θ_d^{S,σ_S}. Let $\overline{A}$ be any connection. Then, for the connection $\Theta_d^{S,\sigma_S}(\overline{A})$, we may take $g = d\,|_S$ (the restriction of d to S). From the S-invariance of f around $\overline{A}$, it follows that $f(\Theta_d^{S,\sigma_S}(\overline{A})) = f(\overline{A})$, so f is S-invariant. □

In quantum theory, position and momentum operators do not commute when there is a time-like or light-like separation between the points at which the operators are defined. Likewise, operators T_f and Θ_d^{S,σ_S} will not, in general, commute. There are still other results for surfaces, functions and operators which one may exploit in order to harvest abelian relations inside $\mathcal{W}$ (e.g. Fleischhack (2004), cor. 3.26). The above examples suggest that a complete classification of the commutative subalgebras of $\mathcal{W}$ should be possible, but we will leave this task aside.

Note that the operators in $\mathcal{W}$ may also have commutative relations with operators *outside* the algebra. Thus, lemma 3.9 below gives the commutative instances of Fleischhack's constructions of "graphomorphisms". Assume that $\phi : \Sigma \to \Sigma$ is a bijective function such that $\phi(S) = S$ and that ϕ does not switch the orientation of the surface S. The smoothness properties of ϕ should correspond to those of the paths (which we have left undecided). Now ϕ induces a map $\phi_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$ on $\mathcal{P}$, namely $\phi_{\mathcal{P}}(\gamma) = \phi \circ \gamma$. Also, we may define still another map $\phi_{\overline{\mathcal{A}}}$, this time on the connections in $\overline{\mathcal{A}}$:

$$\phi_{\overline{\mathcal{A}}}(\overline{A})(\gamma) := h_{\overline{A}}\left(\phi^{-1} \circ \gamma\right). \tag{3.54}$$

It can been shown (by a proof similar to Fleischhack (2004), prop. 3.31) that $\phi_{\overline{\mathcal{A}}}$ is a homeomorphism. The final step is to define the "external" operator $\alpha_\phi : C(\overline{\mathcal{A}}) \to C(\overline{\mathcal{A}})$ by

$$\alpha_\phi(f) := f \circ \phi_{\overline{\mathcal{A}}}^{-1}. \tag{3.55}$$

Again, the domain of α_ϕ may be extended to all of $L_2(\overline{\mathcal{A}}, \mu_0)$ by prop. 3.5.

Proposition 3.9 [Cf. Fleischhack (2004), prop. 3.34] *For a function* $d : \Sigma \to SO(3)$ *such that* $d = d \circ \phi^{-1}$*, we have the commutative relation*

$$w_d^{S,\sigma_S} \circ \alpha_\phi = \alpha_\phi \circ w_d^{S,\sigma_S}. \tag{3.56}$$

Proof For $f \in C(\overline{\mathcal{A}})$, $\overline{A} \in \overline{\mathcal{A}}$, we have

$$\{[w_d^{S,\sigma_S} \circ \alpha_\phi](f)\}(\overline{A}) = \{w_d^{S,\sigma_S}(f \circ \phi_{\overline{\mathcal{A}}}^{-1})\}(\overline{A}) = [f \circ \phi_{\overline{\mathcal{A}}}^{-1} \circ \Theta_d^{S,\sigma_S}](\overline{A}) \quad (*)$$

and

$$\{[\alpha_\phi \circ w_d^{S,\sigma_S}](f)\}(\overline{A}) = \{\alpha_\phi(f \circ \Theta_d^{S,\sigma_S})\}(\overline{A}) = [f \circ \Theta_d^{S,\sigma_S} \circ \phi_{\overline{\mathcal{A}}}^{-1}](\overline{A}). \quad (**)$$

There are now several instances to consider. As in example 3.2, we give the proof for the case where a path γ leaves the surface S in the positive direction without return. Then, by our assumption on d,

$$\begin{aligned} h_{\Theta_d^{S,\sigma_S}(\overline{A})}(\gamma) &= d(\gamma(0))\, h_{\overline{A}}(\gamma) = (d \circ \phi^{-1})(\gamma(0))\, h_{\overline{A}}(\gamma) \\ &= d(\phi^{-1}(\gamma(0))\, h_{\overline{A}}(\gamma) = d(\phi^{-1}(\gamma(0)) \cdot h_{\phi_{\overline{\mathcal{A}}}^{-1}(\overline{A})}(\phi^{-1} \circ \gamma). \end{aligned}$$

The last step follows because ϕ maps S to S and because, by (3.54),

$$h_{\phi_{\overline{\mathcal{A}}}^{-1}(\overline{A})}(\phi^{-1} \circ \gamma) = \phi_{\overline{\mathcal{A}}}\left(\phi_{\overline{\mathcal{A}}}^{-1}(\overline{A})\right)(\gamma) = h_{\overline{A}}(\gamma).$$

From prop. 3.4 we then have

$$h_{\Theta_d^{S,\sigma_S}(\overline{A})}(\gamma) = h_{\Theta_d^{S,\sigma_S}[\phi_{\overline{\mathcal{A}}}^{-1}(\overline{A})]}(\phi^{-1} \circ \gamma).$$

But, by the definition of the map $\phi_{\overline{\mathcal{A}}}$,

$$h_{\overline{A}}\left(\phi^{-1} \circ \gamma\right) = h_{\phi_{\overline{\mathcal{A}}}(\overline{A})}(\gamma).$$

Putting the last two steps together, we have

$$h_{\Theta_d^{S,\sigma_S}(\overline{A})}(\gamma) = h_{\phi_{\overline{\mathcal{A}}}(\overline{A})[\Theta_d^{S,\sigma_S}[\phi_{\overline{\mathcal{A}}}^{-1}(\overline{A})]]}(\gamma).$$

This shows, finally, that

$$\Theta_d^{S,\sigma_S} = \phi_{\overline{\mathcal{A}}}(\overline{A}) \circ \Theta_d^{S,\sigma_S} \circ \phi_{\overline{\mathcal{A}}}^{-1}.$$

The remaining cases for paths γ are similar. By (*) and (**) above, this suffices to prove commutativity of w_d^{S,σ_S} and α_ϕ for $f \in C(\overline{\mathcal{A}})$. The general result then follows by the density of $C(\overline{\mathcal{A}})$ in $L_2(\overline{\mathcal{A}}, \mu_0)$. □

Ought we to include operators like the α_ϕ's in our algebra? We will return to this question in subsection 3.2.8. This ends the construction in the familiar topos **Sets**. We now switch to another topos and proceed with the investigation in the less explored surroundings of $\mathbf{Sets}^{C(\mathcal{W})}$.

3.2.6. The Commutative Algebra $\underline{\mathcal{W}}$

In subsection 3.2.1, we argued that Fleischhack's non-commutative C*-algebra $\mathcal{W}$ has an appropriate format if we want to apply topos-theoretical methods to loop quantum gravity. We shall use the Bohrification method sketched in chapter 2, and our first step is the construction of the *commutative* algebra $\underline{\mathcal{W}}$ in a certain topos, namely

Definition 3.7 *Let $C(\mathcal{W})$ be the partially ordered set of commutative* C*-*subalgebras of* $\mathcal{W}$*. Then* $\tau_{\mathcal{W}} :=$ **Sets**$^{C(\mathcal{W})}$ *(or* $[C(\mathcal{W}),$ **Sets**$]$*) is the topos of covariant functors from the category* $C(\mathcal{W})$ *to the category* **Sets**.

(Note that the category structure $C(\mathcal{W})$ stems from the partial order on $C(\mathcal{W})$ given by inclusion: there is a morphism $C \to D$ iff $C \subset D$.)

Definition 3.8 $\underline{\mathcal{W}}$ *is the tauological functor* $\underline{\mathcal{W}} : C(\mathcal{W}) \to$ **Sets** *such that* $C \mapsto C$*, and* $C \subset_{C(\mathcal{W})} D \mapsto C \subset_{\mathbf{Sets}} D$ *for morphisms.*

As a special case of the result proven by Heunen, Landsman and Spitters (2008), it holds that $\underline{\mathcal{W}}$ is a commutative C*-algebra in the topos $[C(\mathcal{W}),$ **Sets**$]$. The same authors then apply the constructive Gelfand duality of Banaschewski and Mulvey in order to find the Gelfand spectrum $\underline{\Sigma}(\underline{A})$ of a commutative algebra $\underline{A}$ in the topos $[C(A),$ **Sets**$]$ (cf. subsection 2.2.2). The computation of this spectrum has been greatly clarified in the general case in Wolters (2010). We will seek out its consequences for the Gelfand spectrum $\underline{\Sigma}(\underline{\mathcal{W}})$ (hereafter denoted by $\underline{\Sigma}$) of the LQG algebra $\underline{\mathcal{W}}$ in $[C(\mathcal{W}),$ **Sets**$]$. Our aim is to deduce the sobriety properties of the external description of this functor.

In the long run, the attempt should be made to sort out if a non-standard topos (that is, a topos different from **Sets**) is the most natural setting for LQG. Recall that, in LQG, the discrete nature of space-time emerges as a calculation within the theory. One may hope that, if the theory is stripped of non-physical content in the manner suggested by the topos approach, the auxilliary apparatus of standard differential geometry may be overcome, and the radical geometric structure of the theory may be founded on sound empiricist principles. It is unclear to what extent this program may be carried out, and we will not go far towards it in the present thesis. A less ambitious task is to work out toy examples to show what the physics of quantum gravity looks like from a stance within the topos $\tau_{\mathcal{W}}$ above.

3.2.7. The Topological Properties of the External Gelfand Spectrum Σ

In order to prove results about the sobriety of the Gelfand spectrum, it will be advantageous to rely on the *external* description of $\underline{\Sigma}(\underline{\mathcal{W}})$.

Definition 3.9 [cf. Wolters (2010)] *The* external Gelfand spectrum Σ of $\mathcal{W}$ *is the set* $\{(C, \lambda) \mid C \in C(\mathcal{W}), \lambda \in \Sigma_C$ *(the Gelfand spectrum of the commutative subalgebra* $C)\}$ *with topology* $\mathcal{O}\Sigma$ *such that*

$U \in \mathcal{O}\Sigma$ *iff (1)* $U_C \equiv \{\lambda \in \Sigma_C \mid (C, \lambda) \in U\}$ *is open (in the weak*-topology of* Σ_C*), and (2) if* $\lambda \in U_C$*,* $C \subseteq C'$ *and* $\lambda'|_C = \lambda$ *for* $\lambda' \in \Sigma_{C'}$*, then* $\lambda' \in U_{C'}$.

As an easy consequence of the definition, we may characterize the closed sets:

Lemma 3.10 *A set V is closed in Σ iff (1) V_C is closed (in the weak*-topology of Σ_C) for all $C \in C(\mathcal{W})$ and (2) if $\lambda \in V_C$ and $D \subseteq C$ then $\lambda|_D \in V_D$.*

Proof Assume first that V is closed in Σ. Then $\Sigma\backslash V$ is open, so $(\Sigma\backslash V)_C = \Sigma_C \setminus V_C$ is open in Σ_C. Hence, V_C is closed in Σ_C for all C. Let $\lambda \in V_C$ and $D \subseteq C$, and assume that $\lambda|_D \notin V_D$. Then $\lambda|_D \in (\Sigma\backslash V)_D$, which is open in Σ_D. But $\Sigma\backslash V$ is open in Σ, so $\lambda \in V_C \subseteq \Sigma_C$ implies that $\lambda \in (\Sigma\backslash V)_C = \Sigma_C \setminus V_C$, which contradicts $\lambda \in V_C$. So closedness implies condition (2) also. Implication in the opposite direction may be proven in a similar manner. □

Now, Σ is the external description of the Gelfand spectrum $\underline{\Sigma}(\underline{\mathcal{W}})$:

Proposition 3.11 [cf. Wolters (2010), cor. 2.18] *The projection $\pi : \Sigma \to C(\mathcal{W})$ given by $\pi(C, \lambda) = C$ is isomorphic to $\underline{\Sigma}(\underline{\mathcal{W}})$ as a locale.*

The proof was completed recently by Wolters (2010), and we shall not repeat it here. We should, however, use this opportunity to clarify a few points with respect to the internal/external distinction in topoi (see also subsection 2.1.3 above). Recall (cf. Mac Lane and Moerdijk (1992), ch. IX) that a *locale* is an object of the category **Locales**, the opposite of the category of frames, **Frames**. A *frame* is a lattice with all finite meets and all joins which satisfies the infinite distribution law

$$U \wedge \left(\bigvee_i V_i \right) = \bigvee_i (U \wedge V_i). \tag{3.57}$$

If X is a locale, one usually denotes the corresponding frame by $\mathcal{O}(X)$. A map $f : X \to Y$ between locales corresponds to a frame map denoted by $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$. A *point* p^* in a frame F is a map $p^* : F \to \{0, 1\} = \mathcal{O}(*)$. (Hence, for $F = \mathcal{O}(X)$, the open sets of a space X, $p \in X$ defines a point p^* in f if we set

$$p^*(U) = 1 \text{ iff } p \in U.$$

We also say that $\mathrm{Pt}(F)$ are the points in F with open sets $\mathrm{Pt}(U) \equiv \{p^* \mid p^*(U) = 1\}$. A frame F is *spatial* if it is isomorphic (as a frame) to $\mathcal{O}(\mathrm{Pt}(F))$. Dually, a topological space X is *sober* if it is homeomorphic to $\mathrm{Pt}(\mathcal{O}(X))$.

An *internal frame* (or a *frame object*) of a topos is an object F in the topos together with arrows $\bigwedge : F \times F \to F$ and $\bigvee : F \times F \to F$ such that the usual lattice identities and the distribution law (1.57) can be translated into commutative diagrams. In this sense, $\underline{\Sigma}(\underline{\mathcal{W}})$ is an internal locale of the topos $[C(\mathcal{W}), \mathbf{Sets}]$. Now note that the projection $\pi : \Sigma \to C(\mathcal{W})$ is a map between locales; that is, $\pi^{-1} : \mathcal{O}C(\mathcal{W}) \to \mathcal{O}\Sigma$ is a map between frames. Thus, it is claimed in prop. 3.11 that π^{-1} is a *frame isomorphism*.

Relationship with the Döring-Isham formalism. The reader may have noticed that the external spectrum Σ looks quite similar to the state object $\underline{\Sigma}$ as defined in the Döring-Isham approach presented in chapter 2. (This presupposes that we replace the von Neumann algebra $\mathcal{V}(\mathcal{H})$ with a C*-algebra A, so the daseinisation procedure is no longer available.) Indeed, the connection can be made precise by the following piece of category theory:

Lemma 3.12 Σ, *regarded as a category with morphisms* $(C, \lambda) \to (C', \lambda|_{C'})$ *for* $C' \subseteq C$, *is the category of elements of the state object* $\underline{\Sigma}$; *briefly,*

$$\Sigma = \int_{C(\mathcal{W})} \underline{\Sigma}.$$

Proof This follows directly from the definition of a category of elements (Mac Lane and Moerdijk (1993), p. 41). According to the definition, the objects of $\int_{C(\mathcal{W})} \underline{\Sigma}$ are all pairs (C, λ) with $C \in A$ and $\lambda \in \underline{\Sigma}(C) = \Sigma_C$, the Gelfand spectrum of C, and the morphisms are $(C, \lambda) \to (C', \underline{\Sigma}(C' \to C)(\lambda)) = (C', \lambda|_{C'})$ for $C' \subseteq C$. □

The sobriety of a topological space may also be characterized in the following manner: we shall say that a topological space S is *sober* iff every nonempty irreducible closed subset $V \subseteq S$ is the closure of a unique point s; explicitly, $V = \overline{\{s\}}$ (Mac Lane and Moerdijk (1992), p. 477); recall that a closed set is *irreducible* if it is not the union of two smaller closed subsets).

In the next theorem, the 3d *surface* Σ should not be confused with the *Gelfand spectrum* Σ. We also assume, as before, that surfaces and paths have matching smoothness properties. Recall from the start of this chapter, that in geometrodynamics we assume that space-time $(\mathcal{M}, \mathbf{g})$ is globally hyperbolic. By a theorem of Geroch (cf. Wald (1984), th. 8.3.14) it follows that $\mathcal{M}$ may be given the spatial topology $\mathcal{M} \simeq \mathbb{R} \times \Sigma$, with arbitrary topology on the 3d manifold Σ, a Cauchy surface. This topology is exemplified by de Sitter and anti-de Sitter space-times of constant curvature, $R > 0$ or $R < 0$ (see Hawking and Ellis (1973), p. 124, 131). For the following theorem, we also choose one among several reasonable ways of simplying the structure of Σ.

Theorem 3.13 *For* $\Sigma \simeq \mathbb{R} \times S$, *the external Gelfand spectrum* $\Sigma \equiv \Sigma(\mathcal{W})$ *is not sober.*

Proof First note that if (C, λ) is any point in Σ, its closure is

$$\overline{\{(C, \lambda)\}} = \{(D, \lambda') \mid D \subseteq C \wedge \lambda' = \lambda|_D\}. \qquad (*)$$

Indeed, if we write $X = \{(D, \lambda') \mid D \subseteq C \wedge \lambda' = \lambda|_D\}$, the singleton set $X_D = X \bigcap \Sigma_D = \{(D, \lambda')\}$ is trivially closed in Σ_D (under the weak*-topology), and, if $\lambda' \in X_D$ and $E \subseteq D$, then $\lambda' = \lambda|_D$, so $\lambda'|_E = (\lambda|_D)|_E = \lambda|_E \in X_E$, so X is closed by lemma 3.10. But if Y is an arbitrary closed set which contains (C, λ), by closedness Y contains all (D, λ') such that $D \subseteq C$ and $\lambda' = \lambda|_D$, so $X \subseteq Y$. That is, X is the closure of $\{(C, \lambda)\}$.

We shall construct an irreducible closed subset X^* of $\Sigma(\mathcal{W})$ which is not of the form (*), thereby proving that $\Sigma(\mathcal{W})$ is not sober. By Wolters ((2010), lemma 2.24), irreducibility of a closed subset V of $\Sigma(\mathcal{W})$ is equivalent to the following conditions:

1. For all $C \in C(\mathcal{W})$, $V_C \equiv V \bigcap \Sigma_C$ is either empty or singleton, and
2. For all nonempty V_C and $V_{C'}$, there exists C'' such that $C, C' \subseteq C''$ and $V_{C''}$ is nonempty.

We shall now simplify the structure of Σ by assuming that it has the topology $\Sigma = \mathbb{R} \times S$ for an arbitrary 2d manifold S. We may then use a sequence $\{S_i\}_{i \in \mathbb{N}}$ of non-intersecting surfaces in Σ as the basis for our construction of X^*.

For each i, we now pick a Weyl operator $w_d^{S_i, \sigma_{S_i}}$ (with $d \in \text{Maps}(\Sigma, SU(2))$ an arbitrary mapping), and define a sequence $\{V_i\}$ of subalgebras of $\mathcal{W}$:

$$V_i = \text{the C*-algebra generated by } \{1, w_d^{S_1, \sigma_{S_1}}, \ldots, w_d^{S_i, \sigma_{S_i}}\}.$$

For a given i, assume that $w_d^{S_j, \sigma_{S_j}}, w_d^{S_k, \sigma_{S_k}} \in V_i$. Then

$$w_d^{S_j, \sigma_{S_j}} w_d^{S_k, \sigma_{S_k}} = w_d^{S_k, \sigma_{S_k}} w_d^{S_j, \sigma_{S_j}}.$$

(This is lemma 3.26 in Fleischhack (2004).) Thus V_i is commutative, so $V_i \in C(\mathcal{W})$ for each i.

Now assume that a character λ_n has been constructed on V_n for a given n (that is, a multiplicative linear map $\lambda_n : V_n \to \mathbb{C}$). We may then extend λ_n to a character on V_{n+1} by defining $\lambda_{n+1}(w_d^{S_{n+1}, \sigma_{S_{n+1}}}) = c$ for some complex number c. (As multiplicative linear maps on a C*-algebra have norm 1 and the Weyl operators are unitary, we must demand that c is on the unit circle.) Consider the set

$$X^* = \{(C, \lambda) \mid \text{there is some } n \in \mathbb{N} \text{ such that } V_n \subseteq C \subseteq V_{n+1} \wedge \lambda = \lambda_{n+1}|_C\}.$$

It follows that X_C^* is either a singleton or the empty set, hence closed in Σ_C. Also, if X_C^* is nonempty then $X_C^* = \{(C, \lambda)\}$, so $\lambda = \lambda_{n+1}|_C$ for some n. But then $D \subseteq C$ implies $\lambda|_D = (\lambda_{n+1}|_C)|_D = \lambda_{n+1}|_D \in X_D^*$. By lemma 3.10 again, X^* is closed. Irreducibility of X^* is likewise an easy consequence. We just noted that condition 1 holds, and for nonempty X_C^* and $X_{C'}^*$ there are n, n' such that $C \subseteq V_n$ and $C' \subseteq V_{n'}$. If we let $n^* = \max\{n, n'\}$, we have $C, C' \subseteq V_{n^*}$, and V_{n^*} contains λ_{n^*}, hence is nonempty. Condition 2 above is thus satisfied.

Finally, we see that X^* is not the closure of a unique point in $\Sigma(\mathcal{W})$. Assume, to the contrary, that $(F, \lambda|_F)$ is a point such that $X^* = \overline{\{(F, \lambda|_F)\}} = \{(D, \lambda') \mid D \subseteq C \wedge \lambda' = (\lambda|_F)|_D\}$. Note that $\bigcup_i V_i$ is a commutative subalgebra of $\mathcal{W}$, with a character $\lambda_\cup$ given by stipulating that $\lambda_\cup(w_d^{S_i,\sigma_{S_i}}) = \lambda_i(w_d^{S_i,\sigma_{S_i}})$ for each i. By definition of X^*, $X^*_{\bigcup_i V_i} = X^* \bigcap \Sigma_{\bigcup_i V_i} = \emptyset$, so $(\bigcup_i V_i, \lambda_\cup) \notin X^*$. From the assumption that $X^* = \overline{\{(F, \lambda)\}}$ we also know that there is no n such that $V_n \subseteq F \subseteq V_{n+1}$. Yet $(V_i, \lambda_i) \in \overline{\{(F, \lambda)\}}$ for all i, which implies that $V_i \subset F$ and $\lambda_i = \lambda_\cup|_{V_i}$ for all i. Hence, $\bigcup_i V_i \subseteq F$ and $\lambda_\cup = \lambda|_F|_{\bigcup_i V_i}$, from which it immediately follows that $(\bigcup_i V_i, \lambda_\cup) \in \overline{\{(F, \lambda|_F)\}}$.

We have a contradiction, so the irreducible closed subset X^* is not the closure of a unique point, and, accordingly, Σ is not a sober space. □

We say that $C(\mathcal{W})$ satisfies the *ascending chain condition* iff for every chain $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots$ of contexts in $C(\mathcal{W})$, there is an n such that $C_{m+1} = C_m$ for all $m \geq n$ [Wolters (2010), th. 2.25].

Corollary 3.14 *The algebra $\mathcal{W}$ does not satisfy the ascending chain condition.*

Proof Immediate from the construction in the proof of theorem 3.13, or by noting that soberness is a consequence of the ascending chain condition (Wolters (2010), th. 2.25). □

Theorem 3.13, then, brings together concepts from the arenas of loop quantum gravity and topos physics. Let us comment briefly on the status of this result. As a property of topological spaces, sobriety is situated between T_0 (the Kolmogorov condition) and T_2 (the Hausdorff condition) (Mac Lane and Moerdijk (1993), p. 477). Intuitively, the soberness of a space implies that if we continue to split a closed set into closed proper subsets, the process will only terminate at sets which are the closures of singleton sets. The L_2 state spaces familiar from quantum mechanics are Hausdorff spaces, and therefore sober. The non-sober state space Σ above may be seen as a generalized (pointfree) space by noting that the functors $X \mapsto \mathcal{O}(X)$ and $\mathrm{Pt}(F) \hookleftarrow F$ give an equivalence between the categories

Sober spaces ≃ Spatial frames^{op}.

Above we defined the category of pointfree spaces, **Locales**, as the opposite of the category of frames (Heunen, Landsman, Spitters, Wolters (2010)):

Locales := **Frames^{op}**.

From Mac Lane and Moerdijk ((1993), p. 480f) it now follows that, even if the external Gelfand spectrum Σ is non-sober, the associated frame $\mathcal{O}(\Sigma)$ is spatial, or, differently phrased, the locale has "enough points". Thus, the scarcity of points (non-sobriety) in the external space does not rule out the availability of corresponding spatial frames (locales) in the topos, where the notion of a locale emerges as a proxy for the notion of a space. The duality of external non-sobriety and internal sobriety seems to be a topological counterpart to the algebraic duality between the non-commutative algebra $\mathcal{W}$ in **Sets** and the commutativity of its representative $\underline{\mathcal{W}}$ in the topos $\tau_{\mathcal{W}}$. It should be explored further.

3.2.8. Gauge Invariance and Diffeomorphism Invariance

As a quantised theory of general relativity, LQG should fulfill the requirements of gauge invariance (under the Poincaré group for the full-blown theory) and diffemomorphism invariance (full freedom of coordinate choice). We must now ask how these invariance types are to be understood within topos physics. Focusing on the diffeomorphism case, the following interpretation is suggested. Note first that we may associate any diffeomorphism $\phi : \Sigma \to \Sigma$ with the *-morphism $A_\phi : \mathcal{W} \to \mathcal{W}$ defined by the following action on all generators T_f and $w_d^{\sigma_S}$ of $\mathcal{W}$:

$$\left(A_\phi\left(T_f\right)\right)(\overline{A}) = T_f\left(\phi(\overline{A})\right) \quad \left(\text{for } \overline{A} \in \overline{\mathcal{A}}\right), \tag{3.58}$$

$$A_\phi\left(w_d^{S,\sigma_S}\right) = w_{\phi(d)}^{\phi(S),\phi(\sigma_S)}. \tag{3.59}$$

Note that here, as in subsection 3.2.5, we use the lifting of ϕ to a map $\phi \equiv \phi_{\overline{\mathcal{A}}} : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$ by stipulating that $\phi(\overline{A})(\gamma) \equiv \pi_{\phi^{-1}\circ\gamma}(\overline{A})$, with $\pi_{\phi^{-1}\circ\gamma}$ the projection onto the component space $\overline{\mathcal{A}_{\phi^{-1}\circ\gamma}}$ of the path $\phi^{-1}\circ\gamma$ (that is, $\phi_{\overline{\mathcal{A}}}(\overline{A})(\gamma) = h_{\overline{A}}^{-}\left(\phi^{-1}\circ\gamma\right)$). We also write $\phi(S) \equiv \phi \circ S$; $\phi(d) = d \circ \phi^{-1}$; and $\phi(\sigma_S)(\gamma) = \sigma_{\phi^{-1}(S)}(\phi^{-1}\circ\gamma)$ (cf. Fleischhack (2004), def. 3.25).

Fleischhack (2004) shows that we can associate the diffeomorphisms $\phi : \Sigma \to \Sigma$ with operators α_ϕ on $L_2(\overline{\mathcal{A}}, \mu_0)$ in a natural manner (cf. subsection 3.2.5): for any $f \in C(\overline{\mathcal{A}})$, we let $\alpha_\phi : C(\overline{\mathcal{A}}) \to C(\overline{\mathcal{A}})$ be given by $\alpha_\phi\,(f\,) \equiv f \circ \phi^{-1}_{\overline{\mathcal{A}}} \equiv f \circ \phi^{-1}$, the pullback of ϕ^{-1} (the lifting of ϕ^{-1} to $\overline{\mathcal{A}}$). Applying proposition 3.5, we extend α_ϕ to a unitary operator on $L_2(\overline{\mathcal{A}}, \mu_0)$. As the operators α_ϕ merely reflect a switch of coordinates and have no observational content, we have chosen not to include them in the Weyl algebra $\mathcal{W}$.

For a subalgebra C, we denote by ϕC the algebra generated by the image $A_\phi(C)$.

Definition 3.10 *A context (that is, a commutative subalgebra) $C \in C(\mathcal{W})$ is* diffeomorphism invariant *if, for any diffeomorphism $\phi : \Sigma \to \Sigma$, the algebra ϕC is a commutative subalgebra in $C(\mathcal{W})$. If this holds for all contexts C, we say that $C(\mathcal{W})$ itself is* diffeomorphism invariant.

Keep in mind that, just as when we were discussing paths and surfaces, we do not want to commit ourselves to a particular choice of diffeomorphism type. (Fleischhack (2004) considers the "stratified analytic diffeomorphisms".) The definition is intended to capture the intuition that an observer who, perhaps in order to ease his calculations, chooses to change his coordinates, should still be able to conduct his investigation within a classical (commutative) context. The following lemma shows that this is indeed possible:

Theorem 3.15 *$C(\mathcal{W})$ is diffeomorphism invariant.*

Proof Let C be any context in $C(\mathcal{W})$. Then any Weyl operators $w_{d_1}^{S_1,\sigma_{S_1}}$ and $w_{d_2}^{S_2,\sigma_{S_2}}$ in C commute:

$$w_{d_1}^{S_1,\sigma_{S_1}} \circ w_{d_2}^{S_2,\sigma_{S_2}} = w_{d_2}^{S_2,\sigma_{S_2}} \circ w_{d_1}^{S_1,\sigma_{S_1}}.$$

For any diffeomorphism $\phi : \Sigma \to \Sigma$ and $f \in C(\overline{\mathcal{A}})$, we have

$$A_\phi\left(w_d^{S,\sigma_S}\right)(f\,) = w_{\phi(d)}^{\phi(S),\phi(\sigma_S)}\,(f\,) = f \circ \Theta_{\phi(d)}^{\phi(S),\phi(\sigma_S)}.$$

Let γ be a path which starts from $\phi(S)$ without returning. (The proof for the remaining choices of γ is similar.) By the same steps as in in prop. 3.9, it follows that (for any generalized connection $\overline{A} \in \overline{\mathcal{A}}$)

$$h_{\Theta_{\phi(d)}^{\phi(S),\phi(\sigma_S)}(\overline{A}\,)}(\gamma) = d(\phi^{-1}(\gamma(0))\, h_{\overline{A}}(\gamma)$$
$$= d(\phi^{-1}(\gamma(0)) \cdot h_{\phi^{-1}_{\overline{\mathcal{A}}}(\overline{A})}(\phi^{-1} \circ \gamma) = h_{\phi_{\overline{\mathcal{A}}}\left[\Theta_d^{S,\sigma_S}\left[\phi^{-1}_{\overline{\mathcal{A}}}\,(\overline{A})\right]\right]}(\gamma).$$

This establishes that

$$\Theta_{\phi(d)}^{\phi(S),\phi(\sigma_S)} = \phi_{\overline{\mathcal{A}}} \circ \Theta_d^{S,\sigma_S} \circ \phi^{-1}_{\overline{\mathcal{A}}}. \quad (*)$$

Using (*) and the definition $\alpha_\phi\,(f\,) \equiv\ f \circ \phi^{-1}_{\overline{\mathcal{A}}}$ repeatedly, we may now reason in the following manner:

$$\{\alpha_\phi \circ w_d^{S,\sigma_S} \circ \alpha_\phi^{-1}\}(f\,) = \alpha_\phi(w_d^{S,\sigma_S}(\alpha_\phi^{-1}(f\,))) = \alpha_\phi(\{\alpha_\phi^{-1}(f\,)\} \circ \Theta_d^{S,\sigma_S})$$
$$= \alpha_\phi(\{\alpha_\phi^{-1}(f\,)\} \circ \left(\phi^{-1}_{\overline{\mathcal{A}}} \circ \Theta_{\phi(d)}^{\phi(S),\phi(\sigma_S)} \circ \phi_{\overline{\mathcal{A}}}\right)) = (\{\alpha_\phi^{-1}(f\,)\} \circ \left(\phi^{-1}_{\overline{\mathcal{A}}} \circ \Theta_{\phi(d)}^{\phi(S),\phi(\sigma_S)} \circ \phi_{\overline{\mathcal{A}}}\right)) \circ \phi^{-1}_{\overline{\mathcal{A}}}$$

$$= \{\alpha_\phi^{-1}(f)\} \circ \phi_{\mathcal{A}}^{-1} \circ \Theta_{\phi(d)}^{\phi(S),\phi(\sigma_S)} = \alpha_\phi(\alpha_\phi^{-1}(f)) \circ \Theta_{\phi(d)}^{\phi(S),\phi(\sigma_S)} = f \circ \Theta_{\phi(d)}^{\phi(S),\phi(\sigma_S)}.$$

This proves that (cf. proposition 3.34 in Fleischhack (2004))

$$A_\phi\left(w_d^{S,\sigma_S}\right) = \alpha_\phi \circ w_d^{S,\sigma_S} \circ \alpha_\phi^{-1}.$$

It then follows by trivial steps that A_ϕ preserves commutativity of Weyl operators:

$$A_\phi(w_{d_1}^{S_1,\sigma_{S_1}}) \circ A_\phi(w_{d_2}^{S_2,\sigma_{S_2}}) = (\alpha_\phi \circ w_{d_1}^{S_1,\sigma_{S_1}} \circ \alpha_\phi^{-1}) \circ (\alpha_\phi \circ w_{d_2}^{S_2,\sigma_{S_2}} \circ \alpha_\phi^{-1})$$
$$= \alpha_\phi \circ (w_{d_1}^{S_1,\sigma_{S_1}} \circ w_{d_2}^{S_2,\sigma_{S_2}}) \circ \alpha_\phi^{-1} = \alpha_\phi \circ (w_{d_2}^{S_2,\sigma_{S_2}} \circ w_{d_1}^{S_1,\sigma_{S_1}}) \circ \alpha_\phi^{-1}$$
$$= (\alpha_\phi \circ w_{d_2}^{S_2,\sigma_{S_2}} \circ \alpha_\phi^{-1}) \circ (\alpha_\phi \circ w_{d_1}^{S_1,\sigma_{S_1}} \circ \alpha_\phi^{-1}) = A_\phi(w_{d_2}^{S_2,\sigma_{S_2}}) \circ A_\phi(w_{d_1}^{S_1,\sigma_{S_1}}).$$

The two remaining cases, involving the multiplicative operators by themselves and the mixed case of multiplicative operators and Weyl operators, are far simpler. □

Let us, briefly, consider the corresponding definition and result for gauge invariance of $C(\mathcal{W})$. Following, in part, Fleischhack ((2004), def. 3.26), we define the *generalized gauge transformations* $\mathcal{G}$ as the set of maps $g : \Sigma \to SO(3)$. For each g, we say that $\beta_g(f\,)(\overline{A}) := f\,(\overline{A}_g)$, where $\overline{A}_g$ is given by $h_{\overline{A}_g}\gamma := g\,(\gamma(0))^{-1}\; h_{\overline{A}}\gamma\; g(\gamma(1))$. Then $\left(B_g\left(T_f\right)\right)(\overline{A}) := T_f(\overline{A}_g)$ and $B_g\left(w_d^{S,\sigma_S}\right) := w_{g\cdot d\cdot g^{-1}}^{S,\sigma_S}$ define the transformations corresponding to (3.58) and (3.59) above.

Definition 3.11 *A context $C \in C(\mathcal{W})$ is* gauge invariant *if, for any gauge transformation $g : \Sigma \to SO(3)$, the algebra generated by the image $B_g(C)$ is a commutative subalgebra in $C(\mathcal{W})$. If this holds for all contexts C, we say that $C(\mathcal{W})$ itself is* gauge invariant.

Theorem 3.16 *$C(\mathcal{W})$ is gauge invariant.*

Proof Similar to theorem 3.15. □

The approach to quantum gravity outlined above has been strictly limited to a globally hyperbolic space-time $\mathcal{M} \simeq \mathbb{R} \times \Sigma$. This foliation into separate entities "time" and "space" is dependent on the choice of an observer, and the requirement of 4-dimensional diffeomorphism invariance in general relativity is not fulfilled. Thus, the diffeomorphism invariance established in lemma 3.15 does not hold for the general case, but solely for the restricted group of diffeomorphisms on the 3-dimensional hypersurface Σ. In a fully coordinate-free description of the laws of physics, we should expect the observational contexts in $C(\mathcal{W})$ to respect the full diffeomorphism group. This would dispel the implicit notion of a "meta-observer", slicing space-time from some arbitrary perspective. More precisely, in a complete topos theory on gravity, the unrealistic kinematical observational contexts would be replaced by dynamical contexts with physically realistic observables. This is an important task, but we will not enter into it in this thesis.

CHAPTER FOUR

Quantisation in Generalized Contexts

In the preceding chapters, quantum theory has been modelled on the basis of a set of partially ordered classical observational contexts. The elements of this ordering were the commutative subalgebras of the operator algebra of a quantum physical system. The operators were realized as operators on Hilbert spaces by canonical quantisation of classical variables. There are several alternatives available if we want to relax one or more of these assumptions. In section 4.1, we present a broader picture of the observational structure of quantum physics, relying for an example on the systems-processes scheme of so-called *categorical quantum mechanics*. While differing from this approach, the construction of a mathematical basis for quantum theory in the rest of the chapter will be undertaken within the framework of category theory. We shall, in section 4.2, concentrate on the task of quantisation, generalized by Isham to a category-theoretic setting. We introduce Isham's notion of an arrow field on a category and study the operator relations for representations of these entities. In particular, we construct a representation that takes the shape of a "particle fable", the **C**-particle representation (section 4.3). We also try to develop a measure theory for a certain σ-algebra, the algebra of double cones, on a general category (section 4.4). Admittedly, our objects of study are far removed from the standard structures of classical or quantum physics. It is an important task for further study to investigate how far physical principles and methods can be seen as analogues, or even special cases, of the more abstract scheme. (E.g. one may ask in what sense refined approaches to quantisation, such as the geometric method of section 1.3, can be implemented at the categorial level.) We choose instead to continue in a more speculative, or more playful, manner. Firstly, as an illustration (section 4.5), we turn to causal sets. Attempts to develop a quantum theory for causal sets have so far been unsuccessful. We suggest how this can be done with the methods introduced in the preceding sections. Thereafter, we situate quantisation of arrow fields within a wider theory of representations (section 4.6), and extend the topos formalism presented in chapters 2 and 3 to the more general setting of quantised categories. Finally, we speculate on the possibility of a new quantum (or rather, a *quantised*) logic based on the arrow field approach.

4.1. Categories for Quantum Physics

The constructions in topos physics in chapter 2 and 3 have all been carried out against the background of a given quantisation of physical theory. To wit, an *algebraic* viewpoint has been assumed, and classicality has been reintroduced in the form of commutative subalgebras, "snapshots" of a von Neumann or C*-algebra, representing the quantum world. Of course, the procedure of "toposification", in whatever shape, can hardly be expected to eliminate all weaknesses from the background quantum theory. Thus, in chapter 3, the topos approach was only applied to the kinematical fragment of loop quantum gravity, and the problems of the full dynamical theory, such as the mysterious behaviour of the geometric operators at this level, were put aside. This lacuna should certainly be filled. Also, the main motivation behind the topos-theoretical approach is not to bottle up existing theories of quantum physics, but rather to provide tools for the construction of new theories.

There are several directions to explore if we seek to develop topos physics from the present state. Firstly, one may stay with the starting point, a context category based on an underlying operator algebra, and then try to develop quantum field theory within the algebraic program of Haag and Kastner. Here, a set of observables $\mathcal{O}(D)$ will be locally defined on a space-time region D, and observables from different C*-algebras $\mathfrak{a}(D_1)$ and $\mathfrak{a}(D_2)$ will commute for regions D_1 and D_2 with spacelike separation. The completion of this picture is a fascinating task. No less fascinating is the idea that this may be combined with a non-standard approach to the geometry of space. The natural setting for smooth geometry in topoi is not standard differential geometry, but the microlinear spaces of synthetic differential geometry (cf. subsection 1.4.1). (Note that the category of C^∞-manifolds in not cartesian closed; that is, the space of C^∞-maps between manifolds does not have to be a manifold (cf. Moerdijk and Reyes (1991).) This possibility is at the opposite extreme of the proposal explored in chapter 3 above. Instead of the finite chunks of elementary volume implied by the theory of loop quantum gravity, we are led to postulate "crowded" points, surrounded by infinitesimal structure.

Perhaps the boldest suggestion on record is to invert the usual procedure of starting from a classical context and then quantising the system. Thus, quantum systems or processes (axiomatized or modelled in the appropriate way) will be primary notions, and the classical processes will be characterized as special cases within the framwork. In fact, this is the path chosen by the adherents of *categorical quantum mechanics* (CQM hereafter; for an introduction, see Coecke (2010)). Within this subject, at the crossroads between quantum theory, category theory, computer science and topological quantum field theory, the notion of a *process* is more central than that of a *system*: the *internal* structure of a system is disregarded, and focus is shifted to the *external* connections with other systems, as mediated by interacting processes.

From this point of view, it seems natural to mimic physical structures within category theory, representing physical systems as the objects of a certain category, and modelling processes (evolutions, measurements, preparations, etc.) by means of morphisms. There will be two kinds of composition of processes. The first of these, *sequential composition*, is modelled in the category by the usual (partial) binary operation $\circ$, so the processes $f : A \to B$ and $g : B \to C$ will relate the input system A to the output system C by the sequential composition $g \circ f : A \to C$. On the other hand, *independent* processes $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ may always be glued together in the *separate composition* $f_1 \otimes f_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$. Consequently, the systems A_1 and A_2 (and, likewise, B_1 and B_2) must also be independent. If we liken the physical processes to derivation rules of a logical system, with sequential composition $\otimes$ corresponding to conjunction, we note that processes

$$A \to A \otimes A \text{ and } A \otimes B \to A$$

may not always be available. That is, it may not always be possible to clone a system or "lose" a system into the enviroment. The logical parallel is given by the failure of the rules

$$A \vdash A \otimes A \text{ and } A \otimes B \vdash A$$

in Girard's linear logic. (Here, $\otimes$ is linear conjunction.) The relation between sequential and separate composition is captured by the *interaction rule:*

$$(g_1 \circ f_1) \otimes (g_2 \circ f_2) = (g_1 \otimes g_2) \circ (f_1 \otimes f_2).$$

The mathematical structures which reflect these properties are known as *strict symmetric monoidal categories*. The classical (i.e. non-quantum) processes are now identified *within* this framework as the processes for which the process of cloning, $A \to A \otimes A$, (or "sharing of information by copying") is available. The very special substructures which match the classicality concept, are the *commutative Frobenius algebras* (see Kock (2003)).

Although we shall not pursue the matter in the present thesis, the Frobenius algebras seem eminently worthy of further study, for the following reason. The category of such algebras (**cFA**) is equivalent to the category of 2-dimensional topological quantum fields (**2TQFT**). The n-dimensional TQFTs, on the other hand, may be characterized as the functors conserving monoidal structure from the category of n-cobordisms (**nCob**) to the category of vector spaces over a field k ($\mathbf{Vect}_k$). By regarding the closed manifolds (the objects of **nCob)** as representing space and the cobordisms (the arrows of **nCob**, oriented manifolds with boundaries) as representing time, one may then try to build miniature models of quantum gravity.

The development of CQM has affinities with our subject, topos physics. In fact, Coecke ((2010), sec. 1.2) claims that the CQM framework is broad enough to accomodate the topos-theoretical perspective. As noted, CQM enters physics from the quantum end, and classical objects are introduced as citizens of the quantum community. We shall not be so daring in the present thesis. Our point of departure will be the quantisation of a "classical" structure (albeit a very general one, namely a category).

4.2. Quantisation on a Category

We now turn to Isham's proposal for a scheme of quantisation at the most general level, the theory of categories. By choosing this rather abstract approach, our geometrical and physical commitments are minimal. On the other hand, it may prove difficult to interpret the theory unless we introduce further principles along the way. The theory may then be useful for suggesting procedures of quantisation in quantum gravity. The basic definitions 4.1-4.6 are all found in Isham (2003a). We introduce the simple states of a representation, and investigate analogues of the commutation relations known from quantum mechanics.

4.2.1 Isham's Arrow Fields

In chapter 1, we noted that the configuration and momentum variables of classical physics are quantised as operators which satisfy certain commutation relations. Equivalently, the commutation relations can be implemented as conditions on an appropriate representation space, in most cases a Hilbert space $\mathcal{H}$. For variables q and p, the commutation relation for the corresponding operators, $[Q, P] = i\hbar I$, can be realized as $Q\,\Psi = q\Psi$ and $P\,\Psi = -i\hbar(\partial/\partial q)\,\Psi$ for $\Psi \in \mathcal{H}$. In fact, all of the familiar quantum field theories satisfy canonical commutation (or anticommutation) relations. (One might wonder if the procedure can be generalized to include some, or all, of the non-standard spaces we considered in the first chapter.)

A scheme for quantisation in a very general context has been suggested by Isham. Loosely, the idea is to view the objects of a category as the configuration space of a physical theory, and to interpret the arrows as momentum variables. Below, we give a short review of Isham's quantisation on a category. A fuller account is found in Isham (2004), with details in Isham (2003a, 2003b, 2003c).

Let **C** be a category of objects $A, A', \ldots$ Pursuing the analogy of the canonical representation $Q\,\Psi = q\Psi$, we identify the *configuration variables* as the real-valued functions β on Ob(**C**).

What are the momentum variables in this setting? In relativistic mechanics on Minkowski space, we expect the (exponential of) the representation of the momentum variable, $\exp(-iP^{\mu}\,a_{\mu})$, to handle finite translations on the space $\mathcal{H}$. Translations act uniformly on the configuration space, but we should not expect this to be a general trait of momentum transformations. If we want our quantisation scheme to be of relevance for non-standard versions of space-time such as discrete sets, we should leave open the possibility that the transformations between two objects A and A' may depend on the internal structure of the objects. The set of arrows Hom(A, A') is thus a candidate for the transformations between A and A'.

Upon closer inspection, this will not quite do. By the definition of a category, the composition $g \circ f$ of the arrows f and g may be undefined, and yet it seems perfectly reasonable that the representation $U(g \circ f)$ on a space $\mathcal{H}$ exists, as it should equal $U(g)U(f)$. Isham therefore introduces the notion of an *arrow field*, defined in the following manner:

Definition 4.1 [Isham (2004)] *An* arrow field X *is an assignment* $X : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Hom}(\mathbf{C})$ *such that* $\mathrm{dom}(X(A)) = A$ *for* $A \in \mathrm{Ob}(\mathbf{C})$.

Arrow fields X_1 and X_2 are combined in the arrow field $(X_1 \& X_2)(A) := X_2(\mathrm{cod}\, X_1(A)) \circ X_1(A)$ for A in $\mathbf{C}$. Also, we define $\iota(A) := 1_A$.

Definition 4.2 [Isham (2004)] $\mathrm{AF}(\mathbf{C})$ *is the monoid of arrow fields on the category* $\mathbf{C}$ *equipped with the combination operation* & *and the two-sided identity* ι.

The arrow fields X are Isham's analogues of the *momentum variables* in classical physics. The following arrow fields are of particular interest:

Definition 4.3 [Isham (2004)] *If* $f : A \to B$, *then* X_f *is the arrow field* $X_f(C) := f$ *if* $C = A$, $X_f(C) := 1_A$ *otherwise.*

We shall use the notation $\ell_X(A) := \mathrm{cod}X(A)$. The utility of the last definition emerges when we try to find a representation of the configuration variables β and the momentum variables X. Note that we shall also say that $A \in \ell_X^{-1}\{B\}$ iff $\ell_X(A) = B$ and $X(A) \neq 1_A$. (We "divide" X over the physically uninteresting identity.) The simplest choice will be to represent β (configuration variable), X (momentum variable) as operators $\hat{\beta}$, $\hat{a}(X)$ on the vector space $\mathcal{H}$ of complex-valued functions $\psi : \mathrm{Ob}(\mathbf{C}) \to \mathbb{C}$ and define

$$(\hat{a}(X)\,\psi)(A) := \psi(\ell_X(A)) \tag{4.1}$$

$$\left(\hat{\beta}\,\psi\right)(A) := \beta(A)\,\psi(A). \tag{4.2}$$

However, if $f \neq g$, we want $\hat{a}(X_f) \neq \hat{a}(X_g)$. According to (4.1), this does not hold when f and g have the same domain and codomain. The arrows are not *separated*. The definitions of the operators are therefore amended by the introduction of a presheaf $\kappa \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$. As a presheaf, κ fulfills the contravariant functor condition

$$\kappa(f) \circ \kappa(g) = \kappa(g \circ f) \text{ for all } f,\ g \text{ such that } \mathrm{dom}(g) = \mathrm{cod}(f). \tag{4.3}$$

We also generalize the definition of the quantum states in order to capture the inner structure of the objects A of $\mathbf{C}$. This is done by associating (possibly different) Hilbert spaces $\kappa(A)$ with each A, and letting a state ψ be a section of the bundle $\bigcup_{A\in \mathrm{Ob}(\boldsymbol{C})}\kappa(A)$. In the notation from chapter 1, $\mathcal{S}$ will be the state space of these *section states*. If we now demand that the arrow $f : A \to B$ is assigned the linear map $\kappa(f) : \kappa(B) \to \kappa(A)$ by the presheaf κ, we have the following faithful representation of the configuration and momentum variables:

Definition 4.4 [Isham (2004)] *For κ a presheaf over* $\mathbf{C}$*, the operators $\hat{a}(X)$ and $\hat{\beta}$ are*

$(\hat{a}(X)\psi)(A) := \kappa(X(A))\psi(\ell_X(A))$ *and*

$\left(\hat{\beta}\,\psi\right)(A) := \beta(A)\,\psi(A)$.

We note that the former identification of ψ with complex-valued functions is based on the presheaf $\kappa(A) = \mathbb{C}$ (for each A) and $\kappa(f) = \mathrm{Id}_{\mathbb{C}}$ (for each f). The complex-valued states are problematic for all linear maps $\kappa(f)$, as the following example shows:

Example 4.1 Assume that $\kappa(A) = \mathbb{C}$ (for each A) and that $\kappa(f)$ is multiplication by some constant $c_f \in \mathbb{C}$ (for each f). This implies $\kappa(g \circ f) = \kappa(f \circ g)$ for all arrows f, g by (4.3). If $f \circ g \neq g \circ f$, the representation will be unfaithful. Of course, such f and g abound. The case for a set A and maps $f, g : A \to A$ is shown in the figure below.

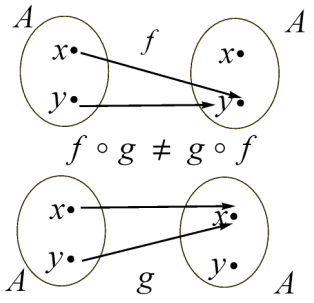


Figure 4.1. Noncommutativity of arrows.

Note also that if ψ had been defined as a section of the presheaf (a global element), the definition of $\hat{a}(X)$ would have been reduced to the identity operator I. (Isham (2004a), p. 356.)

The quantum states have a natural inner product, at least in the finite case:

Definition 4.5 [Isham (2004)] *For* $\mathrm{Ob}(\mathbf{C})$ *finite, the* inner product *on $\mathcal{S}$ is*

$\langle\psi,\phi\rangle := \sum_{A\in \mathrm{Ob}(\boldsymbol{C})} \langle\psi(A), \phi(A)\rangle_{\kappa(A)}$,

where $\langle\ ,\ \rangle_{\kappa(A)}$ is the inner product on the Hilbert space $\kappa(A)$.

Early on, in subsection 1.2.2, it was pointed out that we need self-adjoint operators for the representation of the observables in quantum mechanics. In order to define adjointness, we use the *covariant* functors $\kappa^\dagger \in \mathbf{Sets}^{\mathbf{C}}$.

Definition 4.6 [Isham (2004)] *The* adjoint $\hat{a}(X)^\dagger$ *of the operator* $\hat{a}(X)$ *is given by*

$(\hat{a}(X)^\dagger \psi)(B) := \sum_{A \in \ell_X^{-1}\{B\}} \kappa^\dagger(X(A))\, \psi(A)$,

where $\kappa^\dagger(f) : \kappa(A) \to \kappa(B)$ *for* $f : A \to B$ *is the adjoint of the linear operator* $\kappa(f)$ *in definition 2.4. (And so* $\langle \Phi, \kappa(f)\Psi\rangle_{\kappa(A)} = \langle \kappa(f)^\dagger \Phi, \Psi\rangle_{\kappa(B)}$ *for* $\Phi \in \kappa(A)$ *and* $\Psi \in \kappa(B)$.*)*

Lemma 4.1 *For a finite category* **C***, the definition 4.6 of the adjoint implies the standard adjointness condition* $\langle \phi, \hat{a}(X)\psi\rangle = \langle \hat{a}(X)^\dagger \phi, \psi\rangle$.

Proof We let $\ell_X(A) = B$. Then, using definition 4.5 of the inner product on $\mathcal{S}$,

$$\langle \phi,\, \hat{a}(X)\,\psi\rangle = \sum_{A \in \mathrm{Ob}(\boldsymbol{C})} \langle \phi(A),\, (\hat{a}(X)\,\psi)\,(A)\rangle_{\kappa(A)}$$

$$= \sum_{A \in \mathrm{Ob}(\boldsymbol{C})} \langle \phi(A),\, \kappa(X(A))\,\psi\,(\ell_X\,(A))\rangle_{\kappa(A)}$$

$$= \sum_{A \in \mathrm{Ob}(\boldsymbol{C})} \left\langle \kappa(X(A))^\dagger\, \phi(A),\, \psi\,(\ell_X(A))\right\rangle_{\kappa(\ell_X(A))}$$

$$= \sum_{B \in \mathrm{Ob(C)}} \sum_{A \in \ell_X^{-1}\{B\}} \left\langle \kappa(X(A))^\dagger\, \phi(A),\, \psi\,(B)\right\rangle_{\kappa(B)}$$

$$= \sum_{B \in \mathrm{Ob(C)}} \left\langle \sum_{A \in \ell_X^{-1}\{B\}} \kappa(X(A))^\dagger\, \phi(A),\, \psi\,(B)\right\rangle_{\kappa(B)} =$$

$$\sum_{B \in \mathrm{Ob(C)}} \left\langle \left(\hat{a}(X)^\dagger\, \psi\right)(B),\, \psi\,(B)\right\rangle_{\kappa(B)} = \left\langle \hat{a}(X)^\dagger\, \phi,\, \psi\right\rangle. \quad \square$$

By calculation, we also find the operator relations

$$\left(\left[\hat{a}(X),\, \hat{a}(X)^\dagger\right]\psi\right)(A) := \left(\left(\hat{a}(X)\,\hat{a}(X)^\dagger - \hat{a}(X)^\dagger\,\hat{a}(X)\right)\psi\right)(A) =$$

$$\sum_{C \in \ell_X^{-1}\{\ell_X\, A\}} \kappa(X(A))\,\kappa(X(C))^\dagger\,\psi(C) - \sum_{C \in \ell_X^{-1}\{A\}} \kappa(X(C))^\dagger\,\kappa(X(C))\,\psi(A) \tag{4.4}$$

$$\left(\left\{\hat{a}(X),\, \hat{a}(X)^\dagger\right\}\psi\right)(A) := \left(\left(\hat{a}(X)\,\hat{a}(X)^\dagger + \hat{a}(X)^\dagger\,\hat{a}(X)\right)\psi\right)(A) =$$

$$\sum_{C \in \ell_X^{-1}\{\ell_X\, A\}} \kappa(X(A))\,\kappa(X(C))^\dagger\,\psi(C) + \sum_{C \in \ell_X^{-1}\{A\}} \kappa(X(C))^\dagger\,\kappa(X(C))\,\psi(A). \tag{4.5}$$

Here, [,] are the usual *commutator* brackets. {,} is the *anticommutator*. The utility of these expressions will appear more clearly below.

4.2.2. Creation and Annihilation

At this stage, we would like to introduce Dirac's bracket notation and write $\langle A|\psi\rangle$ for $\psi(A)$. If the states ψ are complex-valued functions $\psi : \mathrm{Ob}(\mathbf{C}) \to \mathbb{C}$, this is unproblematic. We may then identify the state vector $|A\rangle$ with the function ϕ_A that takes the value $\phi_A(A) = 1$ and is zero everywhere else. Then $\langle A|\psi\rangle = \langle \phi_A,\psi\rangle = \sum_{B\in \mathrm{Ob}(C)} \phi_A(B)^* \psi(B) = \psi(A)$, using the definition of the inner product for the special case of complex-valued state functions. However, we saw above that this representation does not separate arrows, and, hence, is unfaithful.

For the general case of section states, we also want to use the notation $\langle A|\psi\rangle := \psi(A)$. From definition 4.4 we may try

$$\begin{aligned}&\langle A \mid \hat{a}(X)\,\psi\rangle \\ &\quad = \ (\hat{a}\,(X)\,\psi)\,(A) = \ \kappa\,(X\ (A))\,\psi\,(\ell_X(A)) = \ \langle \ell_X(A) \mid \kappa\,(X\ (A))\circ\psi\rangle.\end{aligned} \tag{4.6}$$

Using definition 2.6, we find that

$$\begin{aligned}&\left\langle B \mid \hat{a}^\dagger\left(X\right)\psi\right\rangle = \ \left(\hat{a}(X)^\dagger\,\psi\right)(B) \\ &\quad = \ \sum_{A\in \ell_X^{-1}\{B\}} \kappa(X(A))^\dagger\,\psi\,(A) = \ \sum_{A\in \ell_X^{-1}\{B\}} \left\langle A \mid \kappa(X(A))^\dagger\circ\psi\right\rangle.\end{aligned} \tag{4.7}$$

But note that, for section spaces $\mathcal{S}$, the condition "$\langle \phi_A, \psi\rangle = \langle A|\psi\rangle = \psi(A)$ for all $\psi \in \mathcal{S}$" is senseless, as $\psi(A)$ is, in general, not a complex number. We replace it by the demand that

$$\begin{aligned}&\text{for each } A \in \mathrm{Ob}(\mathbf{C}),\ \text{there is a section state } \phi_A \text{ such that } \langle \phi_A, \psi\rangle \\ &\quad = \ \langle \phi_A(A), \psi(A)\rangle_{\kappa(A)} \text{ for all } \psi \in \mathcal{S}.\end{aligned} \tag{4.8}$$

Thus, the inner product depends only on the value of ϕ_A at A. This reduces to the usual condition for complex-valued state vectors. For section states, it is a consequence of definition 4.5 that $\phi_A(B)$ is the zero-vector in $\kappa(A)$ whenever $A \neq B$.

In preparation for the probability interpretation of quantum physics, we also require that the states ϕ_A are *normalized*. That is,

$$\langle \phi_A, \phi_A\rangle = 1 \text{ for all } A \in \mathrm{Ob}(\mathbf{C}). \tag{4.9}$$

There may be several section states ϕ_A that fulfill the condition (4.8). Our choice of a particular state corresponding to the configuration space element A should thus be informed by the physical system under consideration. For a given realization, we shall write $|A\rangle := \phi_A$. These are the *simple states* in $\mathcal{S}$. One must be aware that the notations $\langle \phi_A,\psi\rangle$ and $\langle A|\psi\rangle$ are not exchangable in the general case. The first expression denotes a complex number, the second a vector in the Hilbert space $\mathcal{K}(A)$.

We would certainly like to keep as many of the standard bracket relations as possible. For complex-valued states, it is easily found that $\hat{a}(X)^\dagger|A\rangle = |\ell_X A\rangle$ (see Isham (2003a), p. 357). We may also work out the corresponding formula $(\hat{a}(X)^\dagger\phi_A)(B) = \phi_{\ell_X A}(B)$, so the identification $|A\rangle := \phi_A$ is justified in the simplest case. The bundle-valued case is more complicated. We shall assume that the presheaf adjoint $\kappa^\dagger$ satisfies the condition

$$\phi_B(B) = \kappa^\dagger(f)\,\phi_A(A) \text{ for } f : A \to B. \tag{4.10}$$

Trivially, (4.10) is consistent with the covariant functor condition $\kappa^\dagger(g) \circ \kappa^\dagger(f) = \kappa^\dagger(g \circ f)$ on $\kappa^\dagger$. Equivalently, we may demand of the presheaf κ that it satifies

$$\kappa(f)\,\phi_B(B) = \phi_A(A) \text{ for } f : A \to B. \tag{4.11}$$

If we prefer the bracket notation, we may write $|B\rangle = \kappa(f)^\dagger|A\rangle$ and $\kappa(f)|B\rangle = |A\rangle$ for these relations. The correct interpretation of κ and $\kappa^\dagger$ should probably be judged case by case. The choice of κ for causal sets will occupy us in section 4.5.

Lemma 4.2 *If $\kappa^\dagger$ satisfies (4.10), the equality $(\hat{a}(X)^\dagger\phi_A)(B) = \phi_{\ell_X A}(B)$ holds for A, $B \in \mathrm{Ob}(\mathbf{C})$.*

Proof Using definition 4.6, we find that

$(\hat{a}(X)^\dagger\phi_A)(\ell_X A) = \sum_{C\in\ell_X^{-1}\{\ell_X A\}} \kappa^\dagger(X(C))\,\phi_A(C) = \kappa(X(A))^\dagger\phi_A(A).$

The last equation follows from $A \in \ell_X^{-1}\{\ell_X A\}$ and $\phi_A(B) = 0$ for $B \neq A$. Also, for $B \neq \ell_X A$, we have

$(\hat{a}(X)^\dagger\phi_A)(B) = \sum_{C\in\ell_X^{-1}\{B\}} \kappa^\dagger(X(C))\,\phi_A(C) = 0.$

Here, the expression reduces to zero because $B \neq \ell_X A \Rightarrow A \notin \ell_X^{-1}\{B\}$ and $\phi_A(C) = 0$ when $C \neq A$. From condition (4.10) it then follows that $(\hat{a}(X)^\dagger\phi_A)(\ell_X A) = \phi_{\ell_X A}(\ell_X A)$ and, for $B \neq \ell_X A$, $(\hat{a}(X)^\dagger\phi_A)(B) = 0 = \phi_{\ell_X A}(B)$. This proves the lemma. □

The notation $\hat{a}(X)^\dagger|A\rangle = |\ell_X A\rangle$ is then justfied in the general case.

The operators $\hat{a}(X)^\dagger$ and $\hat{a}(X)$ have important affinities with the correspondingly named operators in quantum mechanics. Using the bracket notation, we know from lemma 4.2 that

$$\hat{a}(X)^\dagger\,|A\rangle = |\ell_X\,A\rangle. \tag{4.12}$$

Similarly, for $\hat{a}(X)$ we derive the relation

$$\hat{a}(X)\,|A\rangle = \sum_{B\in\ell_X^{-1}\{A\}} |B\rangle. \tag{4.13}$$

These expressions are the generalized versions of the formulae for the complex-valued states (as given in Isham (2004a)). The latter expression can be derived by noting that $(\hat{a}(X)\phi_A)(C) = \kappa(X(C))\,\phi_A(\ell_X\, C) = \kappa(X(C))\,\phi_A(A) = \phi_C(C)$ for $C \in \ell_X^{-1}\{A\}$, where the last equality follows from (4.11). When A is not the codomain of any $f = X(B)$ for $B \in \mathrm{Ob}(\mathbf{C})$, we have

$$\hat{a}(X)\,|A\rangle = 0. \tag{4.14}$$

Due to this property, we shall say that $\hat{a}(X)$ is an *annihilation operator* on $\mathcal{S}$. Because the arrow field X is defined on all objects in $\mathrm{Ob}(\mathbf{C})$, $\hat{a}(X)^\dagger$ is never zero on the normalized states $|A\rangle$. We say that $\hat{a}(X)^\dagger$ is the *creation operator* on $\mathcal{S}$. We shall need the creation and annihilation operators for our particle interpretation of quantisation on categories in the next section. Let us first pursue some analogies with the operator relations in quantum mechanics.

4.3. The C-Particle Representation

In this section, we define the particle states of the representation, thereby imitating the construction of the Fock space in the context of categories. These developments are built into the definition of a **C**-particle representation of a causal category.

Definition 4.7 *The* number operator $\hat{n}(X)$ *is given as* $\hat{n}(X) := \hat{a}(X)^\dagger\,\hat{a}(X)$.

In quantum mechanics, the number operator $N = a^\dagger a$ satisfies the relation $N|n\rangle = n|n\rangle$, where n is the number of quanta of the system. The operator $\hat{n}(X)$ is analogous to N because of the following property (here, $|\ell_X^{-1}\{A\}|$ is the number of objects in $\ell_X^{-1}\{A\}$):

$$\hat{a}(X)^\dagger\,\hat{a}(X)\,|A\rangle = \left|\ell_X^{-1}\{A\}\right|\,|A\rangle. \tag{4.15}$$

When the arrow field has no arrows with codomain A, this reduces to the zero vector. Note how the commutation relations (4.4) and (4.5) fare in the context of the normalized states $|A\rangle$:

$$\begin{aligned}\left[\hat{a}(X), \hat{a}(X)^\dagger\right]|A\rangle &= \hat{a}(X)\,\hat{a}(X)^\dagger\,|A\rangle - \hat{a}(X)^\dagger\,\hat{a}(X)\big)\,|A\rangle \\ &= \sum_{B\in\ell_X^{-1}\{\ell_X\, A\}} |B\rangle - \sum_{B\in\ell_X^{-1}\{A\}} |A\rangle = \sum_{B\in\ell_X^{-1}\{\ell_X\, A\}} |B\rangle - \left|\ell_X^{-1}\{A\}\right|\,|A\rangle\end{aligned} \tag{4.16}$$

$$\begin{aligned}\left\{\hat{a}(X), \hat{a}(X)^\dagger\right\}|A\rangle &= \hat{a}(X)\,\hat{a}(X)^\dagger\,|A\rangle + \hat{a}(X)^\dagger\,\hat{a}(X)\big)\,|A\rangle = \\ &\sum_{B\in\ell_X^{-1}\{\ell_X\, A\}} |B\rangle + \sum_{B\in\ell_X^{-1}\{A\}} |A\rangle = \sum_{B\in\ell_X^{-1}\{\ell_X\, A\}} |B\rangle + \left|\ell_X^{-1}\{A\}\right|\,|A\rangle.\end{aligned} \tag{4.17}$$

The above relations are more complicated than e.g. the familiar commutation relation "$[a, a^\dagger] = 1$" known from the case of the quantised harmonic oscillator. As these relations play a crucial role for many constructions in quantum physics, we must find a way to circumvent this difficulty. This will be the subject of the present subsection. For the arrow fields X_f which correspond to single arrows $f : A \to B$ in **C** (def. 4.3), the relations simplify in the expected manner, e.g.:

$$\left[\hat{a}(f),\, \hat{a}(f)^\dagger\right]|A\rangle = \sum_{C \in \ell_{X_f}^{-1}\{\ell_{X_f} A\}} |C\rangle - |\ell_X^{-1}\{A\}|\,|A\rangle = |A\rangle - 0\,|A\rangle = |A\rangle. \tag{4.18}$$

Note that we write $\hat{a}(f)$ and $\hat{a}(f)^\dagger$ instead of $\hat{a}\left(X_f\right)$ and $\hat{a}\left(X_f\right)^\dagger$. Also note that the "division by identity" in the definition of ℓ_X^{-1} plays a crucial role in this derivation.

In subsection 1.2.6 we commented on the choice between particles and fields as the fundamental notion of quantum physics. We shall now try to transfer the standard *particle* interpretation (cf. Weinberg (1995), ch. 4, for quantum field theory) to the context of category quantisation. For this purpose, we need the commutation relations between the number operator $\hat{n}(X)$ and the creation and annihilation operators $\hat{a}(X)^\dagger$ and $\hat{a}(X)$.

Lemma 4.3 *The operators $\hat{n}(X)$, $\hat{a}(X)$ and $\hat{a}(X)^\dagger$ satisfy the commutation relations*

$$[\hat{a}(X), \hat{n}(X)]|A\rangle = \sum_{B \in \ell_X^{-1}\{A\}} s_{A,B}\,|B\rangle \text{ and}$$

$$[\hat{a}(X)^\dagger, \hat{n}(X)]|A\rangle = -\,d_A|\ell_X\, A\rangle,$$

where $s_{A,B}(X) := |\ell_X^{-1}\{A\}| - |\ell_X^{-1}\{B\}|$ for $B \in \ell_X^{-1}\{A\}$ (otherwise $s_{A,B}(X) := |\ell_X^{-1}\{A\}|$); and $d_A(X) := |\ell_X^{-1}\{\ell_X A\}| - |\ell_X^{-1}\{A\}|$.

Proof This is a straightforward calculation, using (4.15). We have

$$[\hat{a}(X), \hat{n}(X)]|A\rangle = [\hat{a}(X), \hat{a}(X)^\dagger \hat{a}(X)]|A\rangle = \hat{a}(X)\hat{a}(X)^\dagger\hat{a}(X)|A\rangle - \hat{a}(X)^\dagger\hat{a}(X)\hat{a}(X)|A\rangle$$

$$= \hat{a}(X)\left(\left|\ell_X^{-1}\{A\}\right||A\rangle\right) - \hat{a}(X)^\dagger\left(\sum_{C \in \ell_X^{-1}\left(\ell_X^{-1}\{A\}\right)} |C\rangle\right)$$

$$= \left|\ell_X^{-1}\{A\}\right|\sum_{B \in \ell_X^{-1}\{A\}} |B\rangle - \sum_{C \in \ell_X^{-1}\left(\ell_X^{-1}\{A\}\right)} |\ell_X\, C\rangle = \left|\ell_X^{-1}\{A\}\right|\sum_{B \in \ell_X^{-1}\{A\}} |B\rangle - \sum_{B \in \ell_X^{-1}\{A\}} \left|\ell_X^{-1}\{B\}\right| |B\rangle$$

$$= \sum_{B \in \ell_X^{-1}\{A\}} \left(\left|\ell_X^{-1}\{A\}\right| - \left|\ell_X^{-1}\{B\}\right|\right)|B\rangle.$$

The second equation is proven in a similar manner. □

When the arrow field X is given, we often omit the argument X in $s_{A,B}(X)$ and $d_A(X)$. We shall call $s_{A,B}$ the *causal surplus* of A over B. d_A is the *causal deficit* of A with regard to its immediate successor.

The relations above are more similar to the commutation rules of quantum physics than may appear at first sight. In fact, if we assume that $s_{A,B} = 1$ and $d_A = 1$ for all $A, B \in \mathrm{Ob}(\mathbf{C})$, we find that $[\hat{a}(X), \hat{n}(X)]|A\rangle = \sum_{B \in \ell_X^{-1}\{A\}} |B\rangle = \hat{a}(X)|A\rangle$ and $[\hat{a}(X)^\dagger, \hat{n}(X)]|A\rangle = -|\ell_X\, A\rangle = -\hat{a}(X)^\dagger|A\rangle$. This means that the ordinary commutation rules for the construction of states of non-interacting particles are at our disposal (see e.g. Shankar (1994)). Including (4.15), we have

$$\hat{n}(X)\,|A\rangle = \left|\ell_X^{-1}\{A\}\right|\,|A\rangle \tag{4.19}$$

$$\left[\hat{a}(X), \hat{n}\,(X)\right] = \hat{a}(X) \tag{4.20}$$

$$\left[\hat{a}(X)^\dagger, \hat{n}(X)\right] = -\hat{a}(X)^\dagger. \tag{4.21}$$

Before we move on, we should have a closer look at the conditions for this reduction to the canonical case. The condition for $s_{A,B}$ says that $|\ell_X^{-1}\{B\}| = |\ell_X^{-1}\{A\}| - 1$ for all $B \in \ell_X^{-1}\{A\}$. If we envisage the arrow field X as a graph G_X over $\mathrm{Ob}(\mathbf{C})$, this means that all immediate predecessors B of A in G_X have one immediate predecessor less than A.

The physical interpretation of this demand will depend on the category under consideration. In a finite category of causal sets (subsection 1.4.2), the condition on the causal surplus $s_{A,B}$ says, roughly, that a causet is influenced by a larger past than any of its causal predecessors. The demand that the causal deficit d_A equals 1 has a related meaning: The past of a causet is properly included in the past of its causal successors. The two conditions are not equivalent, as $s_{A,B} = 0$ and $d_A \geq 1$ when A has no predecessors in G_X. We state the following trivial consequences without proof:

Lemma 4.4 *Given $A \in \mathrm{Ob}(\mathbf{C})$, the following statements hold for all arrow fields X such that $|\ell_X^{-1}\{A\}| \neq 0$:*

a) $s_{A,B} = 1$ for all $B \in \ell_X^{-1}\{A\}$ if and only if $d_A = 1$.

b) If $d_A = 1$, then $|\ell_X^{-1}\{B\}| = |\ell_X^{-1}\{A\}| - 1$ for all $B \in \ell_X^{-1}\{A\}$.

It is now easy to verify, using (4.19) - (4.21), that $\hat{a}(X)^\dagger$ and $\hat{a}(X)$ behave like the creation and annihilation operators familiar from quantum mechanics. We have $\hat{n}(X)\hat{a}(X)|A\rangle = (\hat{a}(X)\hat{n}(X) - [\hat{a}(X), \hat{n}(X)])|A\rangle = (\hat{a}(X)\hat{n}(X) - \hat{a}(X))|A\rangle = (|\ell_X^{-1}\{A\}| - 1)\hat{a}(X)|A\rangle$, and, likewise, $\hat{n}(X)\hat{a}(X)^\dagger|A\rangle = (|\ell_X^{-1}\{A\}| + 1)\hat{a}(X)^\dagger|A\rangle$. Writing ϵ_A for the eigenvalue $|\ell_X^{-1}\{A\}|$ corresponding to the eigenstate $|\epsilon_A\rangle := |A\rangle$ of the number operator $\hat{n}(X)$, we have found that $|\epsilon_A - 1\rangle := \hat{a}(X)|\epsilon_A\rangle = \hat{a}(X)|A\rangle = \sum_{B \in \ell_X^{-1}\{A\}} |B\rangle$ and $|\epsilon_A + 1\rangle := \hat{a}(X)^\dagger|\epsilon_A\rangle = \hat{a}(X)^\dagger|A\rangle = |\ell_X\, A\rangle$ are eigenstates for which

$$\hat{n}\,(X)\,|\epsilon_A - 1\rangle = (\epsilon_A - 1)\,|\epsilon_A - 1\rangle \tag{4.22}$$

$$\hat{n}\,(X)\,|\epsilon_A + 1\rangle = (\epsilon_A + 1)\,|\epsilon_A + 1\rangle. \tag{4.23}$$

This looks very much like particle creation and annihilation. In (4.15) we noticed that the simple states $|A\rangle$ have a "particle number" for each counter $\hat{n}(X)$. For arbitrary states $\psi \in \mathcal{S}$ we have

$$\left(\hat{n}\,(X)\,\psi\right)(A) = \sum_{B \in \ell_X^{-1}\{A\}} \kappa(X(B))^{\dagger}\,\kappa(X(B))\,\psi(A) \tag{4.24}$$

This reduces to (4.15) if we assume that $\kappa(X(B))^{\dagger}\kappa(X(B)) = 1$ for all B. The following definition will be useful:

Definition 4.8 *The presheaf $\kappa \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ such that $f : A \to B$ is assigned the linear map $\kappa(f) : \kappa(B) \to \kappa(A)$ is* unitary *if the linear map $\kappa(f)$ is unitary for all $f \in \mathrm{Hom}(\mathbf{C})$.*

For a unitary representation κ of C we then have

$$\left(\hat{n}\,(X)\,\psi\right)(A) = \left|\ell_X^{-1}\{A\}\right|\,\psi(A). \tag{4.25}$$

For the simple states $|B\rangle = \psi_B$, the equality above is not dependent on the argument A. In general, we shall speak of such states as **C**-*particle states*.

Definition 4.9 *A state $\psi \in \mathcal{S}$ is called an X-*particle state *if $\left(\hat{n}\,(X)\,\psi\right)(A) = n_X\psi(A)$ for the arrow field X and all $A \in \mathrm{Ob}(\mathbf{C})$, where $n_X \in \mathbb{N}$ depends only on X. In general, if ψ is an X-particle state for some arrow field X defined on* **C***, we say that ψ is a* **C**-particle state*. If ψ is an X-particle state for all arrow fields X on* **C***, and if $n_X = n_Y$ for all arrow fields X and Y, ψ is a* fundamental **C**-particle state*.*

The last clause mimics the distinction between fundamental particles (e.g. electrons) and particle-conglomerates (e.g. atoms) in particle physics. All simple states are **C**-particle states. In (4.22) and (4.23) we found other **C**-particle states from the simple states by applying the creation and annihilation operators. There may exist categories **C** with a unitary representation κ for which the set of **C**-particle states is even larger than the closure of the set of simple states under $\hat{a}(X)^{\dagger}$ and $\hat{a}(X)$. In general, this will depend on the arrow structure and the connectedness of the category **C**.

Example 4.2 Let us assume, for simplicity, that $\kappa(A) = \mathbb{C}$ for all $A \in \mathrm{Ob}(\mathbf{C})$ and $\kappa(f) = 1_{\mathbb{C}}$ for all $f \in \mathrm{Hom}(\mathbf{C})$. If there are no arrows f pointing into the two elements A and B, and if ψ is a state such that $\psi(A) = \psi(B) = 1$ (otherwise $\psi_k(C) = 0$), it is easy to show that ψ is a **C**-particle state with $n_X = 0$ for all X. In fact, ψ is a *ground state* of **C**.

Definition 4.10 *A* **C***-particle state $\psi \in \mathcal{S}$ is a* ground state *of the category* **C** *if $n_X = 0$ for all arrow fields X on* **C***.*

Trivially, $\hat{a}(X)\psi = 0$ if ψ is a ground state. This reproduces the usual definition from physics. There may exist several ground states in a category **C**. If Ob(**C**) is infinite, the number of ground states may also be infinite. (This should, perhaps, remind us of the existence of an infinite number of vacua for certain potentials in quantum field theory (see e.g. Zee (2003)). The choice of a particular ground state would be a sort of "symmetry breaking".)

The operation $X_1 \,\&\, X_2$ (*arrow field composition*) was introduced in Isham (2003a) (see the remark preceding def. 4.2 above). Note, however, that even if $d_A(X_i) = 1$ for $X_\mathrm{i} \in \mathbf{X}$ (i = 1, 2), it will often be the case that $d_A(X_1 \,\&\, X_2) \neq 1$. With this in mind, we collect some of the conditions we may want to impose upon a category and its representation.

Definition 4.11 *If* $\boldsymbol{X}$ *is a subset of the monoid* AF(**C**) *which is closed under arrow field composition* $X_1 \,\&\, X_2$*, we shall say that the category* $\mathbb{C}$ *(or* $\mathbb{C}_{\boldsymbol{X}}$ *for precision) with* Ob($\mathbb{C}$) = {∗}, Hom($\mathbb{C}$) = $\boldsymbol{X} \bigcup \{\iota\}$ *and composition* $X_2 \circ X_1 = X_1 \,\&\, X_2$ *is a* pre-causal category.

In particular, AF(**C**) is pre-causal. In def. 4.11, we left out the condition $d_A(X) = 1$ on the causal deficit d_A. We reintroduce it below, tentatively, in order to keep the usual commutation rules. Using def. 4.4 and 4.6, we may represent pre-causal categories on the following structures:

Definition 4.12 *The pair* $\mathcal{S}_\kappa := (\mathcal{S}, \kappa)$ *of a state space* $\mathcal{S}$ *and a presheaf* $\kappa \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ *is a* **C-**representation (of **C**) *if*

(a) $\kappa(A)$ *is a Hilbert space for each* $A \in$ Ob(**C**),

(b) κ *is unitary,*

(c) the states $\psi \in \mathcal{S}$ *are sections of the bundle* $\bigcup_{A \in \mathrm{Ob}(\mathbf{C})} \kappa(A)$,

(d) there is an inner product $\langle\ ,\ \rangle$ *on* $\mathcal{S}$ *such that* $\mathcal{S}$ *is a Hilbert space (i.e. is complete) in the associated norm,*

(e) for each $A \in$ Ob(**C**)*, there is a normalized state* ϕ_A *such that* $\langle \phi_A, \psi \rangle = \langle \phi_A(A), \psi(A) \rangle_{\kappa(A)}$ *for all* $\psi \in \mathcal{S}$*, and for all* ϕ_A and ϕ_B, $\kappa(f)\,\phi_B(B) = \phi_A(A)$ *for* $f : A \to B$.

If, in addition, $d_A(X) = 1$ *for some* X *in* $\mathbb{C}$ *and all* $A \in \mathbf{C}$*, we say that* $\mathcal{S}_\kappa$ *is an* X-particle representation *(or simply a* **C**-particle representation*).*

As introduced above, the state space $\mathcal{S}$ of a **C**-particle representation imitates the Fock space familiar from quantum theory. The Fock space formalizes the idea of a system of free (non-interacting) particles. We may now, if we wish, try to repeat the constructions of boson and fermion spaces in the context of pre-causal categories, and introduce representations of arbitrary spin.

Another task would be to look for instances of pre-causal categories, and, if possible, examine the rôle of the **C**-particles available. We shall turn to this task in section 4.5. Each choice of a category for quantisation will introduce a different perspective on the **C**-particles. (One should, at each new step, question if the difficulties we encounter point to weaknesses of the category-theoretical approach to quantisation, or whether they are limitations inherent in any particle representation of physics.)

4.4. Measure Theory on a Category

4.4.1. The ⋄-Measure

We now investigate various definitions of a measure on a category. We focus on the notion of a σ-algebra $\sigma_\diamond$ of "double cones" as particularly natural for the set of objects in a general category. With this in hand, we define the inner product for measures on $\sigma_\diamond$ and demonstrate the simplicity and order of these notions by proving the boundedness of an important class of operators on the state space.

We shall, firstly, try to remove one of the obstacles to a category theory of quantisation. We have spoken freely of categories of any size (albeit "small" in the set-theoretical sense), and yet the inner product on the state space $\mathcal{S}$ of the category **C** has only been given (definition 4.5) for finite Ob(**C**). The inner product chosen there seems quite natural, and may be extended to the countably infinite case if we make certain restrictions on the states available:

Definition 4.13 [Isham (2004), p. 23] *The space* $\ell^2(\mathbf{C})$ *consists of the state vectors* ψ *such that*

$\langle\psi, \psi\rangle := \sum_{A\in \mathrm{Ob}(\boldsymbol{C})} \langle\psi(A), \psi(A)\rangle_{\kappa(A)} < \infty.$

It is a consequence of the Cauchy-Schwarz inequality for the Hilbert spaces $\kappa(A)$ that $|\langle\psi, \phi\rangle| = \left|\sum_{A\in \mathrm{Ob}(\boldsymbol{C})} \langle\psi(A), \phi(A)\rangle_{\kappa(A)}\right| \leq \sum_{A\in \mathrm{Ob}(\boldsymbol{C})} \left|\langle\psi(A), \phi(A)\rangle_{\kappa(A)}\right| \leq \sum_{A\in \mathrm{Ob}(\boldsymbol{C})} \|\psi(A)\|_{\kappa(A)}\|\phi(A)\|_{\kappa(A)} < \infty$. (Here, $\|\psi(A)\|_{\kappa(A)} := \left|\langle\psi(A), \psi(A)\rangle_{\kappa(A)}\right|^{1/2}$ is the norm on $\kappa(A)$.) The inner product is therefore well-defined on $\ell^2(\mathbf{C})$, and completeness of $\ell^2(\mathbf{C})$ with respect to the norm $\|\psi\| := |\langle\psi, \psi\rangle|^{1/2}$ is easily proven.

We will try to extend this inner product to a general category. Let us, at first, see how far we can get without assuming the definition of a measure μ on the object set Ob(**C**) of the category **C**. As we may want to sum over an uncountable number of objects, we shall need the definition (where the supremum is taken over all finite sets $\mathrm{X} \subset \mathrm{Ob}(\mathbf{C})$) below:

$$\sum_{A\in \mathrm{Ob}(\boldsymbol{C})} \|\psi(A)\|_{\kappa(A)} := \sup_{\mathrm{X}\subset \mathrm{Ob}(\boldsymbol{C})} \sum_{A\in \mathrm{X}} \|\psi(A)\|_{\kappa(A)}. \tag{4.26}$$

This is infinite for states ψ such that $\psi(A) \neq 0$ for an uncountable set of objects. We introduce the state space

$$L^2(\mathbf{C}) := \left\{ \psi \in \mathcal{S} \,\middle|\, \sum_{A \in \mathrm{Ob}(\boldsymbol{C})} \|\psi(A)\|_{\kappa(A)} < \infty \right\}. \tag{4.27}$$

Finally, we define the inner product on $L^2(\mathbf{C})$ in the same manner (with X finite):

$$\langle \psi, \phi \rangle := \sum_{A \in \mathrm{Ob}(\boldsymbol{C})} \langle \psi(A), \phi(A) \rangle_{\kappa(A)} := \sup_{\mathrm{X} \subset \mathrm{Ob}(\boldsymbol{C})} \sum_{A \in \mathrm{X}} \langle \psi(A), \phi(A) \rangle_{\kappa(A)}. \tag{4.28}$$

Lemma 4.5 *$\langle \psi, \phi \rangle < \infty$ for all $\psi, \phi \in L^2(\mathbf{C})$, and $L^2(\mathbf{C})$ is complete in the inner product.*

Outline of proof We omit the proof of finiteness. As to completeness, assume that $\{\psi_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbf{C})$. Because the spaces $\kappa(A)$ are complete for any $A \in \mathrm{Ob}(\mathbf{C})$, the sequences $\{\psi_n(A)\}_{n \in \mathbb{N}}$ are Cauchy sequences in each $\kappa(A)$. Therefore, for each A, there exists $\psi_A \in \kappa(A)$ such that $\lim \psi_n(A) = \psi_A$ when $n \to \infty$. We may then define ψ by $\psi(A) := \psi_A$ for $A \in \mathrm{Ob}(\mathbf{C})$, prove that $\psi \in L^2(\mathbf{C})$, and that $\lim \psi_n = \psi$ when $n \to \infty$. □

The inner product above reduces to the former one in the finite or countable case. If we decide to use it, we will forego the possibility of incorporating into our definition any physically relevant traits the category under consideration may have. On the other hand, if the category comes equipped with a natural measure μ, we would surely want to include it in our definition of the inner product. The definition for a general category $\mathbf{C}$ with state space $\mathcal{S}$ will then be (Isham (1984a), p. 352):

Definition 4.14 [Isham (2004), p. 352] *For a (small) category $\mathbf{C}$ with state space $\mathcal{S}$, the* inner product *on $\mathcal{S}$ is $\langle \psi, \phi \rangle := \int \langle \psi(A), \phi(A) \rangle_{\kappa(A)} \, d\mu(A)$, where $\langle \, , \, \rangle_{\kappa(A)}$ is the inner product on the Hilbert space $\kappa(A)$, and the measurability of $f(A) := \langle \psi(A), \phi(A) \rangle_{\kappa(A)}$ is assumed.*

A proposal for a (probability) measure on a space of causal sets has been given in Brightwell et al. (2002). While suggestive, the measure considered there has features that are intrinsic to the causal set approach (cf. subsection 1.4.2; also section 4.5 below). The measurable space X is, in this case, the infinte causets (i), and the generators of the σ-algebra are the "cylinder sets" $\mathrm{cyl}(\overline{b})$ of infinite causets starting from the finite causet $\overline{b}$, together with their isomorphic copies in X (ii). The probabilities are then calculated from the *dynamics* of the theory. In the so-called sequential growth approach (cf. subsection 4.5 below), the dynamics is fully determined by the transition probabilities governing the addition (or "birth") of a new element to a given structure (iii).

Here, points (i) and (iii) are obviously endemic to the causal set programme, but even (ii) may prove too narrow in a general category-theoretical context. If there is, in any sense, a natural measure on a category **C**, it should probably be based on the arrow structure of the category. The simplest choice seems to be to start from the following σ-algebra:

Definition 4.15 *(a) If A and B are elements in* Ob(**C**)*, the set* $\diamond_{AB}$ *consists of all C* $\in$ Ob(**C**) *such that there exists f,* $g \in$ Hom(**C**) *with* $f : A \to C$ *and* $g : C \to B$*. We call* $\diamond_{AB}$ *the* double cone *between A and B.*

(b) The collection $\sigma_\diamond$ *of subsets of* Ob(**C**) *such that* (i) *the set* $\diamond_{AB}$ *is contained in* $\sigma_\diamond$ *for all A,* $B \in$ Ob(**C**), (ii) $\sigma_\diamond$ *is closed under complementation and countable union, and* (iii) $\sigma_\diamond$ *is the smallest set satisfying* (i) *and* (ii)*. We call* $\sigma_\diamond$ *the* σ-algebra of double cones.

The σ-algebra $\sigma_\diamond$ is named after the double cones or "diamonds" $V_a^b := \{x \in \mathbb{R}^4 |\ a–x \in V_+ \wedge x–b \in V_+\}$ in Minkowski space, where V_+ is the future cone at 0. (The double cones are useful e.g. as the domains of the local observables in algebraic quantum field theory.) Due to this analogy, we can inspect the consistency of our assumptions by interpreting them in the "category of space-time points", where the "arrow set" consists of all lightlike and timelike vectors at each point.

There is also a notable overlap with the terminology from category theory. For a double cone $\diamond_{AB}$, A is a *cone over* $\diamond_{AB}$ and B is a *cone under* $\diamond_{AB}$. Because of the identity arrows 1_A and 1_B, we have $A, B \in \diamond_{AB}$ whenever $\diamond_{AB}$ is non-empty, so in this sense the double cones are "closed". Note that $\diamond_{AA}$ may be larger than $\{A\}$, also in the case where 1_A is the only arrow in Hom(A, A). The definition of the σ-algebra of double cones is the first step towards a definition of a measure on a category. The second step is standard:

Definition 4.16 *A* measure μ *on a category* **C** *is a nonnegative set function defined for all sets of a* σ*-algebra such that* $\mu(\emptyset) = 0$ *and* $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu A_i$ *for any sequence of disjoint sets* A_i *in* σ*. A* measure category *is a triple* (**C**, σ, μ)*. If* $\sigma = \sigma_\diamond$*, the* σ*-algebra of double cones, we say that* μ *is a* $\diamond$-measure*. A* category with a $\diamond$-measure *is a triple* (**C**, $\sigma_\diamond$, μ).

Example 4.3 An immediate choice of a measure seems to be the "volume" of the double cones $\diamond_{AB}$: that is, the product of the number of elements in $\diamond_{AB}$ with the number of arrows from A and the number of arrows into B. Basic category-theoretic notions may come into play when we pick the appropriate measure for a particular category. E.g. if **C** is a cocomplete category, we recognize that if B is the colimit of the diagram associated with the double cone $\diamond_{AB}$, the addition of a cone C under $\diamond_{AB}$ to form the double cone $\diamond_{AC}$ introduces a unique arrow $B \to C$, so in this case we get the "law" $\mu(\diamond_{AC}) = \mu(\diamond_{AB}) + \mu(\diamond_{CC})$. For a category where the objects carry equal weight, this means that the arrow count drops out of the measure. This is the simple counting measure that was implicit in definition 4.5 (see also def. 4.22 below).

For categories with one object, such as the monoid category **M**, the arrows must be counted, or promoted to the status of objects, perhaps as in a 2-category. We will give some more details on properties of a natural choice of a $\diamond$-measure in the section on the category of causal sets below.

4.4.2. Boundedness of the Operators $\hat{a}(X)$

Let us, meanwhile, consider some general characteristics of the $\diamond$-measures. In particular, we are interested in the boundedness properties of our operators under these measures. We will prove under what conditions the operators $\hat{a}(X)$ are bounded for a category **C** with Ob(**C**) infinite. For this purpose, we need a few definitions and results. Due to the novelty or strangeness of some of the notions involved, we will spell out the proofs in some detail. The definition of boundedness is, of course, the familiar one from operator theory. A linear operator L on a Hilbert space $\mathcal{H}$ is *bounded* if there is a positive number b such that $||L\psi|| \leq b||\psi||$ for every vector $\psi \in \mathcal{H}$. The smallest number b with this property is called the *operator norm* and is denoted by $||L||$.

Definition 4.17 *An arrow field X on a category* **C** *is* weakly translatable *if* $\ell_X \diamond_{AB} = \diamond_{\ell_X A \ell_X B}$ *for all A,* $B \in$ Ob(**C**). *(In other words, we demand the closure of the dotted lines in the diagram below.) If, in addition,* $\ell_X A = \ell_X B$ *implies* $A = B$ *for all A and B, we say that X is* strongly translatable. *If X is strongly translatable and, for each A, there exists B such that* $B \xrightarrow{X(B)} A$, *X is called a* Minkowski arrow field.

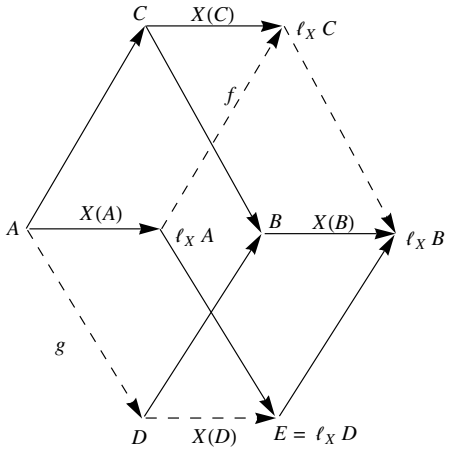


Figure 4.2. Weak translatability of an arrow field.

The definition can be broken down into a few simpler conditions. We say that an arrow field X on a category $\mathbf{C}$ is *completable* if, whenever $A \xrightarrow{f} B$ is an arrow in Hom($\mathbf{C}$), there exists an arrow $\ell_X A \xrightarrow{g} \ell_X B$ in Hom($\mathbf{C}$) (the diagram below closes). An arrow field X is *commutable* if, whenever $A \xrightarrow{X(A)} \ell_X A \xrightarrow{f} E$ is an arrow chain in Hom($\mathbf{C}$), there exists an object $D \in \mathrm{Ob}(\mathbf{C})$ such that $A \xrightarrow{g} D \xrightarrow{X(D)} E = \ell_X D$ in Hom($\mathbf{C}$). If X is completable and commutable it will also be weakly translatable. We shall not need this finer structure now.

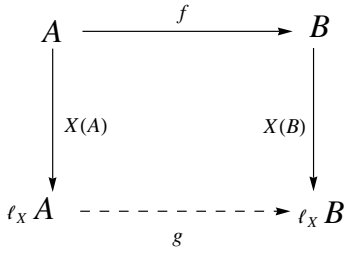


Figure 4.3. Completability of an arrow field.

Lemma 4.6 *If X is a Minkowski arrow field on* $\mathbf{C}$ *and* $U \in \sigma_\diamond$ (*that is, U is a* $\diamond$-measurable *set*) *then* $\ell_X U \in \sigma_\diamond$.

Proof By induction. Assume first that $U = \diamond_{AB}$ for some $A, B \in \mathrm{Ob}(\mathbf{C})$. Then $\ell_X U = \ell_X \diamond_{AB} = \diamond_{\ell_X A \ell_X B} \in \sigma_\diamond$ by the translatability of X.

For the induction step, assume first that $U = \overline{V} \in \sigma_\diamond$ (where $\overline{V}$ is the complement of V in Ob(**C**)), with $V \in \sigma_\diamond$ such that $\ell_X V \in \sigma_\diamond$. We must prove that $\ell_X \overline{V} \in \sigma_\diamond$. Because the σ-algebra $\sigma_\diamond$ is closed under complementation, it suffices to prove that $\ell_X \overline{V} = \overline{\ell_X V}$. Assume $A \in \ell_X \overline{V}$. Then we have $B \xrightarrow{X(B)} A$ for some B $\in \overline{V}$. If now A $\in \ell_X V$ we have $C \xrightarrow{X(C)} A$ for some C $\in V$, but then $\ell_X B = A = \ell_X C$, so $B = C$ by strong translatability of X. We have a contradiction, hence A $\in \overline{\ell_X V}$. On the other hand, if A $\in \overline{\ell_X V}$ there exists no arrow $B \xrightarrow{X(B)} A$ for any B in V. Because X is a Minkowski arrow field, there must, however, be an arrow B in $\overline{V}$ with this property, so A $\in \ell_X \overline{V}$.

Finally, assume that $U = \bigcup V \in \sigma_\diamond$ for a countable union of sets V, with $V \in \sigma_\diamond$ such that $\ell_X V \in \sigma_\diamond$. We must prove that $\ell_X(U) = \ell_X(\bigcup V) \in \sigma_\diamond$. We find, trivially, that $\ell_X(\bigcup V) = \bigcup(\ell_X V)$. But then $\ell_X(\bigcup V)$ is a countable union of sets in $\sigma_\diamond$, hence itself a member of $\sigma_\diamond$. □

Definition 4.18 *If μ is a measure on a category* **C** *and X is a Minkowski arrow field such that $\mu(\ell_X U) = \mu(U)$ for all U in the σ-algebra, we say that X* conserves μ *on* **C**.

The notion of conservation is well-defined for $\sigma_\diamond$ by lemma 4.6. Lemma 4.6 is also needed for the definition of a new measure on $\sigma_\diamond$:

Definition 4.19 *If* (**C**, $\sigma_\diamond$, μ) *is a $\diamond$-measure category then, for a Minkowski arrow field X which conserves μ on* **C**, $\ell_X * \mu$ *is the X-*induced measure *on* **C** *and σ defined by* $(\ell_X * \mu)(U) := \mu(\ell_X U)$ *for all* $U \in \sigma_\diamond$.

As usual, we shall say that a measure ν is *absolutely continuous* with respect to the measure μ if $\nu(A) = 0$ for all $A \in \sigma_\diamond$ for which $\mu(A) = 0$. We use standard notation $\nu \ll \mu$ to symbolize this. Also, we say that a category **C** is *σ-finite* for a given σ-algebra and a measure μ if Ob(**C**) is the union of a countable collection of sets U in the σ-algebra with measure $\mu(U) < \infty$. The next lemma is trivial.

Lemma 4.7 *If X is a Minkowski arrow field which conserves μ then $\ell_X * \mu \ll \mu$.*

Proposition 4.8 *If* $\overline{\mathbf{C}} = (\mathbf{C}, \sigma_\diamond, \mu)$ *is a $\diamond$-measure category such that* **C** *is σ-finite for $\sigma_\diamond$, and X is a Minkowski arrow field which conserves μ, then the operators $\hat{a}(X)$ are bounded for unitary representations $\mathcal{S}_\kappa = (\mathcal{S}, \kappa)$ whenever the Radon-Nikodym derivative of $\ell_X * \mu$ with respect to μ is bounded on* Ob(**C**).

Proof The operators $\hat{a}(X)$ are linear on $\mathcal{S}$. In order to prove boundedness, we must show that there is a positive number b such that $\|\hat{a}(X)\psi\| \leq b\|\psi\|$ for every vector $\psi \in \mathcal{S}$. Here, $\| \ \|$ is the norm on $\mathcal{S}$ given by the inner product $\langle \ , \ \rangle$ in definition 4.14 for the $\diamond$-measure μ in $\overline{\mathbf{C}}$. By definition,

$$\|\psi\| = |\langle\psi, \psi\rangle|^{1/2} = \left|\int \langle\psi(A), \psi(A)\rangle_{\kappa(A)} \, d\mu(A)\right|^{1/2} \text{ and}$$

$$\|\hat{a}(X)\psi\| = |\langle\hat{a}(X)\psi, \hat{a}(X)\psi\rangle|^{1/2} = \left|\int \langle(\hat{a}(X)\psi)(A), (\hat{a}(X)\psi)(A)\rangle_{\kappa(A)} \, d\mu(A)\right|^{1/2}$$

$$= \left|\int \langle\kappa(X(A))\psi(\ell_X A), \kappa(X(A))\psi(\ell_X A)\rangle_{\kappa(A)} \, d\mu(A)\right|^{1/2}$$

$$= \left|\int \langle\psi(\ell_X A), \psi(\ell_X A)\rangle_{\kappa(\ell_X A)} \, d\mu(A)\right|^{1/2}.$$

The last step follows because κ is unitary in the **C**-particle representation $\mathcal{S}_\kappa$. We have seen that the X-induced measure $\ell_X{}^*\mu$ is another $\diamond$-measure on **C**. Due to the σ-finiteness of **C** and the absolute continuity of $\ell_X * \mu$ with respect to μ (lemma 4.7), we may invoke the Radon-Nikodym theorem. Hence, for each $U \in \sigma_\diamond$ we have

$\ell_X * \mu(U) = \int_U \left[\frac{d\ell_X * \mu}{d\mu}\right] d\mu(A)$, where $\left[\frac{d\ell_X * \mu}{d\mu}\right]$ is the Radon-Nikodym derivative of $\ell_X * \mu$ with respect to μ.

We continue the derivation above and find that

$$\|\hat{a}(X)\psi\| = \left|\int \langle\psi(\ell_X A), \psi(\ell_X A)\rangle_{\kappa(\ell_X A)} \, d\mu(A)\right|^{1/2} =$$

$$\left|\int \langle\psi(A), \psi(A)\rangle_{\kappa(A)} \, d(\ell_X * \mu)(A)\right|^{1/2} = \left|\int \langle\psi(A), \psi(A)\rangle_{\kappa(A)} \left[\frac{d\ell_X * \mu}{d\mu}\right] d\mu(A)\right|^{1/2}$$

$$\leq C^{1/2} \cdot \left|\int \langle\psi(A), \psi(A)\rangle_{\kappa(A)} \, d\mu(A)\right|^{1/2} = C^{1/2} \cdot \|\psi\|.$$

The second step uses the definition of $\ell_X * \mu$. The third step uses a general property of the Radon-Nikodym derivative (cf. Royden (1967), p. 241, and note that the integrand on the left is non-negative because $\langle \ , \ \rangle_{\kappa(A)}$ is the inner product on $\kappa(A)$). For the fourth step, the assumption of the boundedness of the Radon-Nikodym derivative on Ob(**C**) is needed, with C a positive constant such that $\left|\left[\frac{d\ell_X * \mu}{d\mu}\right](A)\right| \leq C$ for all $A \in$ Ob(**C**). This proves the proposition. □

This result provides an example of the application of the ideas of category quantisation. Algebras of bounded operators were important for the topos models in chapters 2 and 3. We should note that there are, still, several lacunae in the presentation above. Among other things, the theory of operators and representations in sections 4.2 and 4.3 should be developed, from the "$\diamond$-point of view", for the non-finite case. One would also like to know when a **C**-particle representation $\mathcal{S}_\kappa = (\mathcal{S}, \kappa)$ of a given $\diamond$-measure category $\overline{\mathbf{C}} = (\mathbf{C}, \sigma_\diamond, \mu)$ is irreducible (cf. Isham (2003a), p. 363). We should also try to gain some intuition for the theory by studying the incarnations of its entities in "concrete" categories. As an illustration, we now look at the category of causal sets.

4.5. Quantisation on a Category of Causal Sets

We encountered the causal set program in section 1.4.2 above. The proponents of this approach attempt to build a theory of quantum gravity by imposing simple causality conditions on a discrete space-time. A somewhat unusual *classical* dynamics has been developed on the basis of stochastical growth processes on the causal sets, and it is hoped that this model may be extended to a full quantum theory, e.g. by means of the theory of "quantum measures" (Sorkin (1994)). Below, we shall use some of the ideas from this chapter in order to formulate a different quantum theory of casual sets. Our first step (subsection 4.5.1) will be to define a category with a $\diamond$-measure, $\mathcal{CCS}$ = (**CS**, $\sigma_\diamond$, μ), which captures the central assumptions of the (classical) causal set program, such as local finiteness, general covariance and Bell causality. Then, in subsection 4.5.2, we explore a quantised theory for causal sets by means of the arrow fields on **CS** and their associated "position" and "momentum" operators. We also explain the notion of a **CS**-particle for the theory.

4.5.1. The Category of Causal Sets

Recall that the *causal sets* are partially ordered sets that are locally finite in the sense explained in subsection 1.4.2. The order-relation $\prec$ is irreflexive: an element does not precede itself. Also beware that causal sets are insensitive to labelling: thus, we shall identify order-isomorphic sets. These will be the objects of the category **CS** below.

The terminology of the theory is very suggestive ("event", "universe", "big bang"). We shall use it in some places, but it must be kept in mind that the underlying structure (oriented graphs) is very elementary. If C is a causal set and e is an element (or "event") in C, the "past" of e is defined as $\text{Past}(e) = \{a \in C \mid a \prec e\}$. We will also speak of the past of a subset D of C, $\text{Past}(D)$. The *n-antichain* $A(n)$ is the completely unordered set with n elements.

The clause on order-preserving in part (a) of the definition below was suggested by Isham (2004). Part (c) is based on the notion of Bell causality (see Rideout and Sorkin (1999), sec. 3). Hereafter, we simplify our notation by writing μ_{AB} for $\mu(\diamond_{AB})$, the value of the $\diamond$-measure μ on the double cone $\diamond_{AB}$. We say that the causal set D *(immediately) succeeds* the causal set C (for C, D different) if $C \to D$ and $C \to A \to D$ implies $A = C$ or $A = D$. (We also call such morphisms $C \to D$ *successions.*)

Definition 4.20 *A* classical category of causal sets with a $\diamond$-measure *is a triple* $\mathcal{CCS}$ = (**CS**, $\sigma_\diamond$, μ)*, where*

(a) **CS** *is the* category of causal sets*, with* Ob(**CS**) *the collection of causal sets and* Hom(**CS**) *the collection of functions* $f : A \to B$ *for* $A, B \in$ Ob(**CS**)*, such that, if* $a \prec b$ *for* $a, b \in A$*, then* $f(a) \prec f(b)$ *in* B (preservation of order)*, and, if* $b \in B$ *but* $b \notin Im(f)$*, there is no* $a \in A$ *such that* $b \prec f(a)$ (internal temporality).

(b) $\sigma_\diamond$ *is the* σ*-algebra of double cones on* **CS***, and*
(c) μ *is a* $\diamond$*-measure which satisfies the condition*

if the causal sets B *and* C *succeed the causal set* A*, and if there are monics (i.e. injective functions)* $f : A \to A'$*,* $g : B \to B'$ *and* $h : C \to C'$ *in* Hom(**CS**) *such that there exist successions* $A' \to B'$ *and* $A' \to C'$*, it holds that*

$$\frac{\mu_{AB}}{\mu_{AC}} = \frac{\mu_{A'B'}}{\mu_{A'C'}}.$$

Thus, the objects of the causal category **CS** are not discrete points, but ordered sets of such points. One may think of the arrows $f : A \to B$ as possible transitions of the universe from stage A to stage B. The condition on μ, an attempt to capture the notion of Bell causality within the present framework, will be explained later in this section. The set of morphism may be restricted further, e.g. by demanding injectivity. (Yet another possibility would be to define the objects of the category as triples of causal sets with a σ-measure (C, σ, μ), and extend the morphisms so that they also represent changes in the measure theory of the causets. In this way, σ and μ become dynamical objects of the theory.)

Above, the elementary properties of the causal sets (such as local finiteness) are mirrored in the *internal* structure of the objects of the category **CS**. In addition, collections of causets satisfy the *condition of internal temporality:* that is, the condition that no new elements in the causal growth process are introduced to the past of existing elements. In our definition, this is reflected in the temporality condition on the morphisms *f*. The usual dynamical conditions on the causets (e.g. Rideout and Sorkin (1999)) will be implemented as requirements on the *external* structure. Thus, the *principle of discrete general covariance* and the *Bell causality condition* are refound at the level of the σ-algebra $\sigma_\diamond$ and the $\diamond$-measure μ. We examine covariance and Bell causality in this subsection. In subsection 4.5.2 we quantise $\mathcal{CCS}$ by means of arrow fields.

(a) Discrete general covariance. Invariance principles are central in physics (see e.g. subsection 1.2.4 above) because they narrow the field of possible solutions. The principle of discrete general covariance has been coined to do the same work in the theory of causal sets. A causal set C grows by the addition of a new element (event) c. Dynamics is introduced by the assignment of a probability P to the subset of $b \in C$ such that $b \prec c$, the past of c. There should be no elements b in C such that $c \prec b$. The covariance principle then states that the probability of reaching a given finite causet B from another causet A (perhaps the empty causet 0, "the big bang") should be independent of the path taken from A to B. For example, the two paths to the top causet in figure 4.4 are physically indistinguishable. As each path corresponds to a natural labelling of the causets where the order of birth of the elements is indicated, the covariance principle may be formulated in an alternative manner: the labels of a causet have no physical significance.

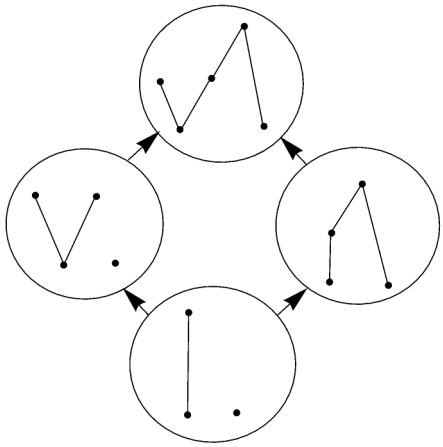


Figure 4.4. The principle of discrete general covariance.

It has been shown that this requirement is fulfilled in classical sequential growth (CSG) models, the most common approach to causal set dynamics. Essentially, this is done by proving that the probability of transition, α_n, from a given causet with n elements to another causet with $n + 1$ elements, depends on a countable set of parameters $q_0 \geq q_1 \geq q_2 \geq ...$ (in the terminology of the theory, this is the "physical" coupling constants). The parameter q_n is the probability of transition from the totally unordered set with n elements, the n-antichain $A(n)$, to the $(n + 1)$-antichain $A(n + 1)$. The probability of growth from a causet A with k elements to a causet B with $k + 1$ elements turns out to be independent of the particular path chosen between the two causets (details are found in Rideout and Sorkin (1999)).

In our model $\mathcal{CCS}$, this trait is captured at the level of the σ-algebra $\sigma_{\diamond}$. The steps in the growth process of CSG are simply morphisms in the category **CS**. The double cone $\diamond_{AB}$ contains the causal sets found along all possible paths from A to B, so a probability measure defined on this structure will not distinguish between different paths between the two sets.

(b) Bell causality. In CSG, the choice of a non-increasing sequence of constants q_k determines the dynamics completely. The proof relies on the Bell causality condition, which may be interpreted as a causal set-equivalent of the ban on superluminal influence known from relativity theory. We now ask about the relationship between Bell causality and condition (c) on $\mathcal{CCS}$ above.

Bell causality is usually stated as the requirement that, given the size of the causet C ("the development of the universe so far"), the probability of a new event e should only depend on the subset past(e). (The qualification on the size of C is important: events with the same past, but occuring against the background of different causets (universes) will not generally have the same probability.) In CSG, this condition is expressed as an equality between ratios of transition probabilities *prob* between causal sets:

$$\frac{prob(\gamma_{AB})}{prob(\gamma_{AC})} = \frac{prob(\gamma_{A'B'})}{prob(\gamma_{A'C'})}. \tag{4.29}$$

We use the notation γ_{AB} for a path (or "causal growth sequence") from causet A to causet B. It is demanded that the probability ratio of *single-step processes* (that is, processes for which a single element is added to the causet) $\gamma_{A'B'}$ and $\gamma_{A'C'}$ starting from the causet A' may just as well be calculated from the (often) simpler processes starting from the union A of the pasts of B' and C', and completed in single-step processes γ_{AB} and γ_{AC}, with a new element added in the same manner as in B' and C', respectively.

In our model, we frame Bell causality as a condition on the $\diamond$-measure μ. When μ is a probability measure *prob* as in (4.29), the requirement stated in the definition of a classical causal set category follows. To see this, we define:

Definition 4.21 *For prob a probability measure which satisfies general covariance and Bell causality, the* classical measure μ_{cl} *is given by* $\mu_{\mathrm{cl}}(\diamond_{AB}) := prob(\gamma_{AB})$.

Lemma 4.9 μ_{cl} *is a* $\diamond$*-measure which satisfies the Bell causality condition for* **CS** *(condition (c) of def. 4.20).*

Proof All paths γ_{AB} from A to B are assigned the same probability because *prob* satisfies general covariance, so $\mu_{\mathrm{cl}}(\diamond_{AB})$ is well-defined as a $\diamond$-measure. Assume that we have triples of causets A, B, C and A', B', C' related by succession, and a triple of monics f, g, h for which we can find successions $j : A' \to B'$ and $k : A' \to C'$.

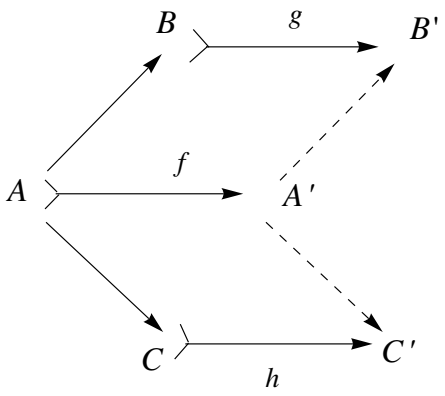


Figure 4.5. Bell causality.

Both B and C succeed A, so the number of elements ("events") in B and C must be one greater than the number of elements in A (otherwise, we could find $D \neq A, B$ such that $A \to D \to B$, and likewise for C). Also, the monics g and h in the definition must respect temporality by clause (a) of def. 4.20, so no elements are added in B' (or C') to the past of the element b (or c) that was "born" in B (or C). Because morphisms preserve order, it follows that the past of the image b' of b added in B' is isomorphic to the past of b, and likewise for the image c' of c in C'. We now define the triple $A'' = \text{Past}(b) \bigcup \text{Past}(c)$, $B'' = A'' \bigcup \{b\}$, $C'' = A'' \bigcup \{c\}$. From (4.29) and past isomorphism we have

$$\frac{prob(\gamma_{AB})}{prob(\gamma_{AC})} = \frac{prob(\gamma_{A''B''})}{prob(\gamma_{A''B''})} = \frac{prob(\gamma_{A'B'})}{prob(\gamma_{A'C'})}. \tag{4.30}$$

Condition (c) then follows by definition 4.21. □

Thus, if we think of μ as the "causal weight" of the double cones $\diamond_{AB}$, the relative value of the weights should be unchanged by the monic mappings. As another example, we may look at

Definition 4.22 *The* counting measure μ_0 *is the* $\diamond$*-measure defined on* $\sigma_\diamond$ *by setting* $(\mu_0)_{AB} = |\diamond_{AB}|$, *the number of objects in* $\diamond_{AB}$.

Lemma 4.10 *If* μ_0 *is the counting measure on the category of causal sets* **CS**, *it is the case that*

(i) $(\mu_0)_{AA} = 1$;

(ii) $(\mu_0)_{AB} = 2$ *for successions* $A \to B$;

(iii) in general, $(\mu_0)_{AB} \geq$ #(events in B) – #(events in A);

(iv) Condition (c) of def. 4.20 holds for μ_0.

Proof (i) Beacuse order-isomorphic causal sets are identified, there can be only one morphism $A \to A$, namely the identity 1_A, so $A \in \diamond_{AB}$. Also, there is no $B \neq A$ such that $A \to B \to A$. So $|\diamond_{AB}| = 1$. (ii) and (iii) are also trivial, and (iv) follows from (ii) because the morphisms in def. 4.20(c) are successions. □

The counting measure satisfies Bell causality, but in a non-informative manner: all succession ratios are equal to 1. In general, the arrow structure should be taken into account.

4.5.2. Arrow Fields and Operators on Causal Sets

We start this subsection with a quotation: "When all one has to work with is a discrete set and a partial order, even the notion of what we should mean by a dynamics is not obvious" (Rideout, Sorkin (2004), p. 1). In CSG, this problem is solved by the assignment of values to the coupling constants q_n: the dynamical evolution of the theory is wholly contained in the transition probabilities between the antichains. In the category approach, on the other hand, arrow fields are the carriers of dynamical content. Also, the techniques from sections 4.2 and 4.3 make these fields immediately available for quantisation. In fact, quantisation has not been achieved within the CSG program, which remains a purely classical construction.

In both approaches, a complete "physical" dynamics will require the equivalent of a "Hamiltonian" within the theory. (This is inexact: the causets may belong to different time stages, so a Hamiltonian formulation is out of the question.) So far, this is not within reach. The table below sums up the implementation of standard notions from causal set theory within arrow field theory for $\mathcal{QCS}$, the quantum category of causal sets.

Table 4.1.

Causal sets	**Arrow field theory in** $\mathcal{QCS}$
Partial ordering, local finiteness	Internal structure of objects in **CS**
Temporality	Morphisms in **CS**
General covariance	σ – algebra $\sigma_\diamond$
Bell causality and classical dynamics	$\diamond$ –measure μ
Quantum dynamics	Operator repr. of arrow fields X on **CS**

We now consider the monoid AF(**CS**) of arrow fields X on the causal set category **CS**. This is a pre-causal category in the sense of def. 4.11. We shall deal only with the finite case, so the analysis in section 4.3 can be applied. This corresponds to the choice of μ_0 as our $\diamond$-measure, so Bell causality is trivially fulfilled.

Definition 4.23 *A* quantum representation of causal sets $\mathcal{QCS}$ *is a* **CS**-*representation* $(\mathcal{S}, \kappa)$ *with a state space* $\mathcal{S}$ *and a presheaf* $\kappa \in \mathbf{Sets}^{\mathbf{CS}^{op}}$ *which satisfies the conditions (a)-(e) of def. 4.12. (In particular, the states* $\psi \in \mathcal{S}$ *are sections of the bundle* $\bigcup_{A \in \mathrm{Ob}(\mathbf{CS})} \kappa(A)$*, where* $\kappa(A)$ *is a Hilbert space for each causet* $A \in \mathrm{Ob}(\mathbf{CS})$*.)*

In subsection 4.2.1, the presheaf κ was introduced in order to represent arrow fields X faithfully as operators $\hat{a}(X)$. It was noted that presheaves κ for which $\kappa(A) = \mathbb{C}$ (with A now a causet) were unsatisfactory because they do not capture any inner structure of the objects A. The nearest alternative of any interest is to try a contravariant functor κ such that

$$\kappa(A) = \mathbb{C}^{|A|} \text{ for } A \in \mathrm{Ob}(\mathbf{CS}) \text{ and } \kappa(f) \text{ some linear map } \kappa_f : \mathbb{C}^{|B|} \to \mathbb{C}^{|A|} \text{ for } f : A \to B. \tag{4.31}$$

For the choice of the linear maps $\kappa(f)$, there are, in effect, three special cases to consider, as exemplified below.

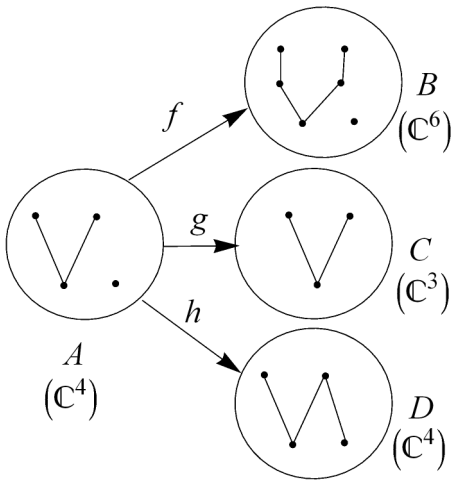


Figure 4.6. Object and arrow domain of the multiplier κ.

(i) For arrows $f : A \to B$ where the causet B introduces new events, we let $\kappa(f)$ be the projection of $\mathbb{C}^{|B|}$ onto $\mathbb{C}^{|A|}$ which erases the components of $(c_1, ..., c_{|B|})$ corresponding to the new elements.

(ii) For arrows $g : A \to C$ where the causet C collapses unrelated events, $\kappa(g)$ will be the proper inclusion of $\mathbb{C}^{|C|}$ in $\mathbb{C}^{|A|}$, but we must choose a unique source in A for the collapsed event in C. If there is a causally related source e in A (which must be unique, due to order-preservation) then e is the natural choice for the mapping of components. Otherwise, for event sources which are causally unrelated in A, the choice is arbitrary. (Note that case (ii) is ruled out if we restrict further the set of morphisms, as in the remark following def. 4.20.)

(iii) There may also be arrows $h : A \to D$ which introduce new causal structure, but for which A and D have the same event set. Then $\kappa(h)$ will be the identity on $\mathbb{C}^{|D|} = \mathbb{C}^{|A|}$.

Other cases are combinations of (i)-(iii). Trivially, the unitarity requirement of def. 4.12 is not satisfied, so this is not a **CS**-representation in the full sense. There is, nevertheless, a lesson to be had about the interpretation of the multiplier κ. In the light of the remarks above, it seems possible to regard κ as a sort of non-physical (gauge) transformation, locally dependent on the causet A. Recall def. 4.4 of the operator $\hat{a}(X)$ for an arrow field X. As shown in figure 4.7 for $X(A) = f$ as above, $\hat{a}(X)\psi$ is a translation of the state ψ such that, from the point of view of particular causet A, the components associated with new events in the causet B is superfluous.

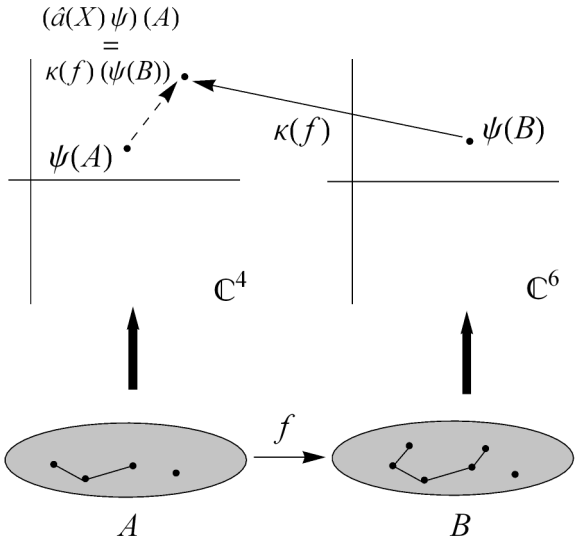


Figure 4.7. Representation of arrows by κ.

Let us now outline how the elementary theory of causal sets within quantised categories may be further developed:

(1) More examples involving different choices of state spaces $\mathcal{S}$, modifier κ and precausal categories $\boldsymbol{X}$ should be worked out to see if the definition of qunatum representation above is the natural one, or whether it should be modified.

(2) In a more advanced development, the counting measure should be replaced by a general $\diamond$-measure.

(3) The question of Bell causality in the quantised theory should be investigated. Should we really expect (some analog of) Bell causality to hold in the quantum case too, as suggested by Rideout and Sorkin ((1999), p. 24)? Or will this allow us to infer Bell's inequalities, thus violating well-known quantum entanglement effects (see Penrose (2004), p. 582ff, for an introduction to the subject)?

(4) The "points" in a causet A are interpreted as events. But what is a *particle*? Can we define a particle representation for **CS**, perhaps along the lines suggested in section 4.3? Can we have a particle interpretation which hints to the presence of gravity in the model? Ideally, the canonical representation on the causal sets should be structurally similar to the spin-2 particle of gravitation theory, the graviton. At present, it remains a mystery how this can be done.

(5) Connected with the last point, a full dynamical account should complete the "kinematical" sketch above.

4.6. Representations of Arrow Fields

We have seen that arrow fields can be represented as operators on certain state spaces $\mathcal{S}$. In this section, we take a step further away from the paradigm of Hilbert spaces, still prominent in the approach above. We shall try to extend the topos formalism to a more general setting, where operators on Hilbert spaces are no longer the sole option. The basic idea is to assume that operators are really representations of arrow fields. To start with, in subsection 4.6.1 we collect all representations of arrow fields on a category **C** in a category $\mathbf{BAF_C}$ of monoid representations, and derive the elementary properties of this category. There is a technical issue here, due to the fact that some of the operators we have to deal with are really anti-representations (subsection 4.6.2). Finally, we unify quantisation on categories and the topos scheme described in chapters 2 and 3. A presheaf topos $\boldsymbol{\tau}(\mathbf{C})$ for a quantised category **C** may then be constructed (subsection 4.6.3).

4.6.1. The Category of Representations $\mathbf{BAF_C}$

In section 4.1 we introduced arrow fields X on a category **C** and defined operators $\hat{a}(X)$ and $\hat{a}(X)^\dagger$ by means of them. Where do these entities belong? The arrow fields have a simple structure indeed: there is a unit arrow field, combination of arrow fields is associative, and, in contrast with the arrows in **C**, combination is always defined. The set of arrow fields with combination on **C** is a *monoid*, AF(**C**) (def. 4.2). From elementary category theory, any monoid may be regarded as a category with one object. We therefore define

Definition 4.24 *The* category of arrow fields *over a category* **C**, $\mathbf{AF_C}$, *is determined by*

(i) $\mathrm{Ob}(\mathbf{AF_C})$ *has exactly one element, the set* $* := \mathrm{Ob}(\mathbf{C})$ *of objects in* **C**;

(ii) $\mathrm{Hom}(\mathbf{AF_C}) = \mathrm{AF}(\mathbf{C})$, *the arrows in* $\mathbf{AF_C}$ *are the arrow fields over* **C**;

(iii) *the (unique) identity arrow* 1_* *is the arrow field* ι *defined by* $\iota(A) := \mathrm{id}_A$, *with* id_A *the identity arrow on* $A \in \mathrm{Ob}(\mathbf{C})$;

(iv) *composition* $\circ$ *is defined by* $X_2 \circ X_1 := X_1 \& X_2$ *for* $X_1, X_2 \in \mathrm{Hom}(\mathbf{AF_C})$, *where* $X_1 \& X_2$ *is the arrow field combination (defined in section 4.1 as)* $(X_1 \& X_2)(A) := X_2(\mathrm{Cod}(X_1(A))) \circ_{\mathbf{C}} X_1(A)$, *where* '$\circ_{\mathbf{C}}$' *denotes composition in the category* **C**.

Note the inverted order of the arrow fields in '$\circ$' and '&' (see remark in example 4.8 below). It is easy to show that associativity of $\circ$ holds. In addition, $\mathrm{Dom}(X) = \mathrm{Cod}(X) = *$, and $X \circ Y$ is defined for all arrows (arrow fields) X, Y. In general, commutativity of composition, $X \circ Y = Y \circ X$, will not hold. From now on, we shall stick to the one-object category $\mathbf{AF_C}$, not the monoid $\langle \mathrm{AF}(\mathbf{C}), \iota, \& \rangle$.

Arrow fields are quite abstract entities, so we are interested in their representations. It is a basic fact in topos theory that the set of all representations of any monoid M (or group G) forms a category M-**Sets** or **B**M (G-**Sets** or **B**G for groups), moreover, it can be shown that **B**M (**B**G) is a *topos* (lemma 4.11 below). Let us spell out what this means for $M = \mathbf{AF_C}$.

Definition 4.25 *The* category of all representations of $\mathbf{AF_C}$*, denoted as* $\mathbf{BAF_C}$ *or* $\mathbf{AF_C}$**-Sets**, *is determined by*

(i) $\mathrm{Ob}(\mathbf{BAF_C}) = \{\langle S, \mu \rangle \mid S \text{ is a set and } \mu : S \times AF(\mathbf{C}) \to S\}$, *where we denote* $\mu(a, X)$ *by* $a \cdot X$ *for* $a \in S$ *and* $X \in AF(\mathbf{C})$. *We also demand that* $a \cdot \iota = a$ *and* $(a \cdot X) \cdot Y = a \cdot (X \circ Y) = a \cdot (Y \& X)$ *for all* $a \in S$ *and* $X, Y \in AF(\mathbf{C})$ (μ *is the* right action *of* $\mathbf{AF_C}$ *on* S);

(ii) $\mathrm{Hom}(\mathbf{BAF_C}) = \{f \mid f : S \to T$ *is a function and* $f(a \cdot X) = f(a) \cdot X$ *for* $a \in S$ *and* $X \in AF(\mathbf{C})\}$ (*that is, morphisms* f *between representations* $\langle S, \mu \rangle$ *and* $\langle T, \nu \rangle$ *respect the action; we also say that* f *is* equivariant);

(iii) *the identity arrow* $1_{\langle S, \mu \rangle}$ *is the identity function* $i_S : S \to S$ *for all sets* S;

(iv) *composition* $\circ$ *is functional,* $(g \circ f)(a) := g(f(a))$ *for* $a \in S$, $f : S \to T$ *and* $g : T \to U$.

If, instead, we demand that $(a \cdot X) \cdot Y = a \cdot (Y \circ X)$ *for all* $a \in S$ *and* $X, Y \in AF(\mathbf{C})$, *we say that we have the* category of anti-representations of $\mathbf{AF_C}$*, denoted by* $\mathbf{BAF_C^*}$.

Example 4.4 It is obvious that the *translational* representation $S = \mathrm{Ob}(\mathbf{C})$, $A \cdot X = \mu(A, X) = \mathrm{Cod}(X(A))$ (for A an object in $\mathbf{C}$) is an element in $\mathbf{BAF_C}$. This representation moves elements A in $\mathbf{C}$ one step along arrow fields X. (In section 4.1, $A \cdot X$ was denoted as $\ell_X(A)$ or $\ell_X A$, and we will use this notation freely.) If $\mathbf{C}$ is a subcategory of a category $\mathbf{D}$, we may form an *extended* translational representation of $\mathbf{AF_C}$ by choosing, for each arrow field X over $\mathbf{C}$, an arrow field X' which coincides with X on $\mathrm{Ob}(\mathbf{C})$.

Example 4.5 In general, the elements of an algebraic structure may be associated with vectors in a vector space. The *regular* (or *adjoint*) representation of the structure then consists of mappings on the associated vector space, and these are defined from a combinatorial operation in the algebraic structure. For the monoid $\mathbf{AF_C}$, we let S be a vector space with basis $\{e_X \mid X \in AF(\mathbf{C})\}$ and define the linear mapping $e_Y \cdot X := X(e_Y) := e_{XY}$ as the representation of the element (an arrow field) X in the monoid.

Example 4.6 We now consider the operator representations of $\mathbf{AF_C}$ constructed in section 4.1. Recall that the vector space (state space) $\mathcal{S}_\kappa$ there had vectors ψ that were sections of the bundle $\bigcup_{A\in \mathrm{Ob}(\mathbf{C})}\kappa(A)$, where $\kappa(A)$ was a Hilbert space for each $A \in \mathrm{Ob}(\mathbf{C})$. The action of the annihilation operator, $\hat{a}(X)$, associated with an arrow field X was defined as $(\hat{a}(X)\psi)(A) := \kappa(X(A))\psi(\mathrm{Cod}(X(A)))$, where κ was a *contravariant* functor on $\mathbf{C}$, $\kappa(f) : \kappa(B) \to \kappa(A)$ for $f : A \to B$ in $\mathrm{Hom}(\mathbf{C})$. We may therefore represent annihilation in $\mathbf{BAF_C}$ as $\hat{a} := \langle \mathcal{S}_\kappa, \mu\rangle$, with $\psi \cdot X = \mu(\psi, X) := \hat{a}(X)\psi$. Then

$$\begin{aligned}(\psi\cdot X)\cdot Y &= \mu(\mu(\psi, X), Y) = \\ &= \hat{a}(Y \,\&\, X)\,\psi = \psi\cdot(Y \,\&\, X) = \psi\cdot(X\circ Y).\end{aligned} \tag{4.32}$$

So condition (i) on representations (objects in $\mathbf{BAF_C}$) is satisfied. (The third equation is the only non-trivial step: note that

$$\begin{aligned}&[\hat{a}(Y)(\hat{a}(X)\psi)](A) = \kappa(Y(A))[\hat{a}(X)\psi](\ell_Y A)\\ &= \kappa(Y(A))[\kappa(X(\ell_Y A))\psi(\ell_X(\ell_Y A))]\\ &= \kappa((Y \,\&\, X)(A))\psi(\ell_{Y\&X} A) = (\hat{a}(Y \,\&\, X)\psi)(A).\end{aligned} \tag{4.33}$$

We have thus derived the representation $\hat{a}(Y)\hat{a}(X) = \hat{a}(Y\&X)$ of the arrow field combination '&'.)

Example 4.7 One may ask if the creation operator, $\hat{a}^\dagger$, furnishes another representation of $\mathbf{AF_C}$. We consider only the finite case. By definition 4.6, $(\hat{a}(X)^\dagger\psi)(B) := \sum_{A\in\ell_X^{-1}\{B\}}\kappa^\dagger(X(A))\psi(A)$, where $\kappa^\dagger$ is a *covariant* functor. By a calculation similar to the above, it can be shown that $\hat{a}(Y)^\dagger\hat{a}(X)^\dagger = \hat{a}(X\&Y)^\dagger$. If we define $\hat{a}^\dagger = \langle \mathcal{S}_\kappa, \mu_\dagger\rangle$, with $\psi \cdot X = \mu_\dagger(\psi, X) := \hat{a}(X)^\dagger\psi$, it is clear that

$$\begin{aligned}(\psi\cdot X)\cdot Y &= \mu_\dagger\left(\mu_\dagger(\psi, X), Y\right)\\ &= \hat{a}(Y)^\dagger\left(\hat{a}(X)^\dagger\psi\right) = \hat{a}(X \,\&\, Y)^\dagger\psi = \psi\cdot(X \,\&\, Y) = \psi\cdot(Y\circ X).\end{aligned} \tag{4.34}$$

Whereas $\hat{a}$ is a representation of $\mathbf{AF_C}$, $\hat{a}^\dagger$ is an *anti-representation* of the monoid of arrow fields. Thus, $\hat{a}^\dagger$ is an object of $\mathbf{BAF_C^*}$. The choice of $\hat{a}^\dagger$ as an anti-representation is of course conventional, and may be changed by inverting the order of the composition '$\circ$' in definition 4.25. But it is *not* a matter of convention that $\hat{a}$ and $\hat{a}^\dagger$ do not belong in the same category.

Let us collect some simple characteristics of the category $\mathbf{BAF_C}$. In general, we may identify categories $\mathbf{B}M$ with the category of presheaves $\mathbf{Sets}^{\mathbf{M}^{\mathrm{op}}}$, where $\mathbf{M}$ is the one-object category corresponding to the monoid M.

Lemma 4.11 $\mathbf{BAF_C}$ *is a topos of presheaves (namely,* $\mathbf{BAF_C} \simeq \mathbf{Sets}^{\mathbf{AF_C^{op}}}$*).*

Proof Assume the representation $\langle S, \mu\rangle$ is an object in $\mathbf{BAF_C}$. The presheaf P corresponding to $\langle S, \mu\rangle$ will assign the set S to the sole object * in $\mathbf{AF_C}$, and to each arrow field X in $\mathrm{Hom}(\mathbf{AF_C})$ an arrow $P(X) : S \to S$ given by $P(X)(a) = \mu(a, X)$. We have

$$P(X \circ Y)(a) = \mu(a, X \circ Y) = a \cdot (X \circ Y) = (a \cdot X) \cdot Y = \mu(a, X) \cdot Y$$
$$= \; P(X)(a) \cdot Y = \mu(P(X)(a), Y) = P(Y)(P(X)(a)) = (P(Y) \circ P(X))(a).$$

Thus, the functors are contravariant, so the tag 'op' is justified. Also, we may identify the morphisms f between representations $\langle S, \mu \rangle$ and $\langle T, \nu \rangle$ as natural transformations θ between presheaves Q and P by setting $\theta_* = f$ for $*$, the sole object in $\mathbf{AF_C}$. □

What are the subobjects of the representations in $\mathbf{BAF_C}$? If $f : S \rightarrowtail T$ is a monomorphism (that is, injective) and $a \in S$, we see that $f(a) \cdot X = f(a \cdot X)$ by equivariance of f. So $f(S)$ is closed under the action of $\mathbf{AF_C}$. Since S and $f(S)$ are the same (up to isomorphism), it follows that subobjects of T are subsets closed under the right action. We shall say that a subobject S of a representation T is a *subrepresentation*. For groups, it will also be the case that the complement of S in T is closed, and hence $\sim S$ is a subobject too. For monoids, such as $\mathbf{AF_C}$, this will, generally, not be the case.

Example 4.8 Let X be an arrow field over a category $\mathbf{C}$, and assume that f is a morphism between the translational representation C over $\mathbf{C}$ and an extended representation D over the category $\mathbf{D}$, where $\mathrm{Ob}(\mathbf{C}) \subset \mathrm{Ob}(\mathbf{D})$. C will then be a subobject of D, but if X' is the extension in $\mathbf{AF_D}$ of an arrow field X over $\mathbf{C}$, it may well happen that X' takes $\mathrm{Ob}(\mathbf{D}) \backslash \mathrm{Ob}(\mathbf{C})$ into $\mathrm{Ob}(\mathbf{C})$, that is, for some $A \in \mathrm{Ob}(\mathbf{D}) \backslash \mathrm{Ob}(\mathbf{C})$, $A \cdot X' = \ell_{X'} A \in \mathrm{Ob}(\mathbf{C})$.

The lack of invariance on complements makes $\mathbf{BAF_C}$ a little more complicated than the category **Sets** (or even $\mathbf{B}G$ for G a group). Some basic facts are gathered below. (Heyting algebras were defined in subsection 2.2.1.)

Lemma 4.12 *The set* Sub S *of subrepresentations of a representation* S *in* $\mathbf{BAF_C}$ *is a Heyting algebra.*

Proof This holds for the set of subobjects of an object in any topos (Mac Lane and Moerdijk (1992), p. 201), and a subrepresentation in $\mathbf{BAF_C}$ is just a subobject, so the statement follows immediately. Let us spell out some of the details in the case under consideration. As usual, a Heyting algebra is a distributive lattice with implication satisfying $x \leq (y \rightarrow z)$ if and only if $x \wedge y \leq z$. The ordering '$\leq$' is given by the subob-

ject relation. Now let S be a representation of $\mathbf{AF_C}$, and consider the subrepresentations V and W of S. Denoting the intersection (the greatest lower bound) as $V \wedge W$, we observe that if the action does not take us outside V and W for any arrow field X, then $V \wedge W$ is also closed under the action. ($\mathbf{BAF_C}$ is a topos, so $V \wedge W$ is the pullback of $V \to S$ along $W \to S$.) Similarly, it is easy to define intersection (least upper bound) $V \vee W$ in the manner of lemma 2.5 (join). Because $\mathbf{AF_C}$ is a one-object category, the obvious definition of implication also works: the representation $V \to W$ is the set of $a \in S$ such that for all arrow fields X on $\mathbf{B}$, if $a \cdot X \in V$, then $a \cdot X \in W$. Negation, $\neg V$, is defined as $V \to 0$ (0 is the initial object of the topos, the empty representation). $\neg V$ therefore consists of the elements in the complement of V that are not sent into V by the action in S. It is then easy to show that the laws of a Heyting algebra are satisfied. (In fact, we already met the required proof of the law of implication in lemma 2.3.) □

We define *a right ideal L of* $\mathbf{AF_C}$ as a set of arrow fields such that, for $X \in L$, $X \circ Y \in L$ for all arrow fields Y in $\mathbf{AF_C}$.

Lemma 4.13 *The subobject classifier* $\Omega = \Omega_{\mathbf{BAF_C}}$ *in* $\mathbf{BAF_C}$ *is the pair* $\langle S, \mu \rangle$, *where* S *is the set of right ideals of* $\mathbf{AF_C}$, *and the action* μ *is defined as* $\mu(L, X) := L \cdot X := \{Y \in AF(\mathbf{C}) \mid X \circ Y \in L\}$ for all $X \in AF(\mathbf{C})$..

Proof This is the instance $\mathbf{C} = \mathbf{AF_C}$ of the general case $\mathbf{Sets}^{\mathbf{C}^{op}}$, for which a proof can be found e.g. in Bell (1988), p. 62. We sketch the special case. For V a subrepresentation of T, the characteristic function $\phi_V : T \to \Omega$ is defined for all $a \in T$ by $\phi_V(a) :=$ the set of arrow fields X on $\mathbf{C}$ such that $a \cdot X \in V$. That is, we include the arrow fields whose action representation sends everything into V. V is invariant, so this is a right ideal. Also, if $a \in V$, $\phi_V(a) = AF(\mathbf{C})$, which is consistent with the definition of the truth arrow $\top : 1 \to \Omega$ as $\top(*) := AF(\mathbf{C})$, the maximal right ideal. (Recall that the terminal object 1 in $\mathbf{BAF_C}$ is the trivial representation $\{*\}$.) □

We say that a topos $\mathcal{E}$ is *two-valued* if its subobject classifier Ω has only two global sections $1 \to \Omega$, or, equivalently, if Ω has only two subobjects ("true" and "false"). For the case $\mathcal{E} = \mathbf{Sets}$, these are the familiar truth values 0 (the initial object) and 1 (the terminal object). Perhaps surprisingly, this does not guarantee that the topos is *Boolean*, i.e. that the Heyting algebra of subobjects is Boolean for every object in the topos (cf. lemma 2.5):

Lemma 4.14 *The topos* $\mathbf{BAF_C}$ *is two-valued (but* $\mathbf{BAF_C}$ *is Boolean if only if* $\mathbf{AF_C}$ *is a group).*

Proof Again, the proof is standard (e.g. Mac Lane and Moerdijk (1992), p. 274). Suppose $\gamma : 1 \to \Omega$ is a global section ("global truth-value"), that is, $L = \gamma(*) = \gamma(*) \cdot X = L \cdot X$, where L is a right ideal and X an arrow field. Then, if $L \neq \emptyset$, there exists $X \in L$ such that for any $Y \in AF(\mathbf{C})$, we have $X \circ Y \in L$, but then, by the definition of action in Ω, $Y \in L \cdot X = L$, so $L = AF(\mathbf{C})$. Hence, there are only two global truth-values, corresponding to the choices $\emptyset$ and $AF(\mathbf{C})$ of right ideals in Ω.

Because this is reminiscent of the two-valued semantics of classical logic, it may appear that $\mathbf{BAF_C}$ is a Boolean topos (that is, $S \vee \neg S = T$ for any subrepresentation S of T), but such is not generally the case. We noted above that the complement $\sim S$ of S in T is closed only when $\mathbf{AF_C}$ is a group. On the other hand, we defined the negation of S in the Heyting algebra Sub T by

$$\neg S = \{\text{the elements in } \sim S \text{ that are not sent into } S \text{ by the action } \nu \text{ in } \langle T, \nu \rangle\}.$$

Therefore, generally $\neg S$ is a proper subset of $\sim S$ (the complement of S as a set), and $S \vee \neg S \neq T$. If $\mathbf{AF_C}$ is a group, consider an element a in $\sim S$ such that $\nu(a, X) = b \in S$ for some arrow field X. Then X^{-1} exists, and $\nu(b, X^{-1}) = a \notin S$. But S is a representation, hence closed under the action, so we have a contradiction. It follows that $\neg S = \sim S$ when $\mathbf{AF_C}$ is a group. Then $S \vee \neg S = T$ holds, so $\mathbf{BAF_C}$ is Boolean in this case. □

The non-boolean character of $\mathbf{BAF_C}$ is therefore tied to the monoid structure of $\mathbf{AF_C}$. This implies some more weirdness: an object G *generates* a category $\mathbf{C}$ if $f \neq g : A \to B$ implies that there exists $u : G \to A$ such that $fu \neq gu$. We say that $\mathbf{C}$ is *well-pointed* if the terminal object 1 generates $\mathbf{C}$. Thus e.g. **Sets** is a well-pointed category.

Lemma 4.15 $\mathbf{BAF_C}$ *is, in general, not well-pointed.*

Proof A well-pointed topos is two-valued and Boolean (Mac Lane and Moerdijk (1992), p. 276). It follows from lemma 4.14 that $\mathbf{BAF_C}$ is well-pointed only when $\mathbf{AF_C}$ is a group. In more detail, consider the terminal object 1 in $\mathbf{BAF_C}$, given by $1(*) = \{*\}$ and $1(X) =$ the map $* \to *$. Now note that there are not that many points $s : 1 \to S$ (with some action μ) available in a representation S: by def. 4.25, s is really an equivariant morphism, and, for simplicity, we write $s(*) = s$, identifying the arrow s with the element picked by it. But then $s = s(*) = s(* \cdot X) = s(*) \cdot X = s \cdot X$, so these are the members of S which are invariant under the action in S. Morphisms f and g between representations S and T may agree on these points, and yet be different elsewhere. □

All presheaf topoi have a natural numbers object (n.n.o.), namely the the presheaf with constant value $\mathbb{N}$ (Mac Lane and Moerdijk (1992), p. 269), so $\mathbf{BAF}_{\mathbf{C}}$ has one too, to be identified with $\mathbb{N}$. The natural numbers are a modest starting point for representing physics, but by invoking standard constructions for topoi, a lot of construction work may be done. In subsection 2.1.3, we defined the Mitchell-Bénabou language for topoi such as $\mathbf{BAF}_{\mathbf{C}}$. Objects Q (the object of rational numbers) and R (the object of real numbers) may then be constructed. For the reals, this construction applies the usual definition by means of Dedekind cuts. For presheaf topoi $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, the real number object R coincides with the constant presheaf, $\Delta(\mathbb{R})$, which assigns the set of ordinary reals $\mathbb{R}$ to every $C \in \mathbf{C}$. (For one-object categories we simply identify R with the ordinary reals.)

Using the Mitchell-Bénabou language, we may derive valid formulae by using *intuitionistic* predicate calculus (as noted in lemma 4.14, we may not appeal to the rule of excluded middle: $\mathbf{BAF}_{\mathbf{C}}$ is not Boolean). Our set-theoretical machinery is also limited, in particular, the axiom of choice (**AC**) does not hold for all topoi. However, from theorem 2.1 and lemma 4.11 it follows that a restricted version of **AC**, the axiom of dependent choice (**DC**), does hold in $\mathbf{BAF}_{\mathbf{C}}$.

The statements above are also valid (appropriately modified, by using left ideals and so on) for the category of anti-representations $\mathbf{BAF}^{*}_{\mathbf{C}}$.

4.6.2. Tensor Products and Adjoints

Above we saw that representations and anti-representations of the arrow fields on a category belong in *different* topoi. E.g. $\hat{a}$ belongs to $\mathbf{BAF}_{\mathbf{C}}$ and $\hat{a}^{\dagger}$ belongs to $\mathbf{BAF}^{*}_{\mathbf{C}}$. Here, $\hat{a} = \langle \mathcal{S}_{\kappa}, \mu \rangle$ and $\hat{a}^{\dagger} = \langle \mathcal{S}_{\kappa}, \mu_{\dagger} \rangle$, where the maps μ and $\mu_{\dagger}$ are given by the operations of def. 4.4 and 4.6 and $\mathcal{S}_{\kappa}$ is the usual state space. We shall try to extend what was done earlier in this chapter for the Hilbert space representations on $\mathcal{S}_{\kappa}$ to the more general setting provided by these categories.

Generally, for a given (physical) system, we consider only representations and anti-representations which act on the same set S, the *set of states* of the system. (But beware, hereafter we often write S for representations $\langle S, \mu \rangle$ and S^{*} for anti-representations $\langle S, \mu_{*} \rangle$. The actions will, equivocally, be denoted by "$\cdot$".) As before, we shall expect the adjoint of any representation (in $\mathbf{BAF}_{\mathbf{C}}$) to be an anti-representation in $\mathbf{BAF}^{*}_{\mathbf{C}}$, but it is less than clear how this is to be implemented.

Let us start by inquiring into the meaning of adjointness in the present context. A minimal demand, given a function $\langle\ ,\ \rangle : S \times S \to k$ ($\mathbb{C}$ or $\mathbb{R}$) identified as the "inner product", is that (for a given arrow field X)

$$\langle aS(X), b \rangle = \langle a,\ S^{*}(X)\, b \rangle \text{ (for all states } a,\ b \text{ in } S). \tag{4.36}$$

This holds for the special case $S = \hat{a}$, $S^* = \hat{a}^\dagger$, as shown in lemma 4.1. Other requirements, such as symmetry and bilinearity/sesquilinearity may or may not be meaningful, depending on the state space under consideration. The condition would hold if we were able to factorize $\langle\ ,\ \rangle$ like this:

$$S \times S \to S \otimes S \to k. \tag{4.37}$$

Here, "$\otimes$" denotes some kind of product such that

$$a \cdot S(X) \otimes b = a \otimes S^*(X) \cdot b. \tag{4.38}$$

Because $a \cdot S(X) = \mu(a, X) = a \cdot X$ and $S^*(X) \cdot b = \mu_*(b, X) = X \cdot b$, this may be written as (now also dropping "$\cdot$")

$$aX \otimes b = a \otimes Xb. \tag{4.39}$$

This looks very much like a tensor product, except that we have no additivity condition in the general case. Analogously to the construction in module theory, it is possible to build tensor products also in a category-theoretical context (Mac Lane and Moerdijk (1992), p. 351). Below, we go through the construction in the present case. In fact, we shall see that the product may be taken in such a manner that *all* representations and anti-representations fulfill condition (4.39).

Let S be a given representation in $\mathbf{BAF_C}$. That is, S is a contravariant functor (a presheaf) over the one-object category $\mathbf{AF_C}$:

$$S : \mathbf{AF_C^{op}} \to \mathbf{Sets}. \tag{4.40}$$

Likewise, T will be a given anti-representation in $\mathbf{BAF_C^*}$. So T is a covariant functor over $\mathbf{AF_C}$:

$$T : \mathbf{AF_C} \to \mathbf{Sets}. \tag{4.41}$$

Recall that $\mathbf{AF_C}$ is the category-theoretical version of AF($\mathbf{C}$) $\equiv M$, the monoid of arrow fields (def. 4.2). Working in **Sets**, we form the cartesian products $S \times T$ and $S \times M \times T$ (with S, T the domains of the (anti-)representations S, T) and define mappings

$$\theta(a, X, b) = (a \cdot S(X), b) = (aX, b), \tag{4.42}$$

$$\tau(a, X, b) = (a, T(X)\, b) = (a, Xb). \tag{4.43}$$

This gives the following diagram:

$$S \times M \times T \underset{\tau}{\overset{\theta}{\rightrightarrows}} S \times T. \tag{4.44}$$

In **Sets**, this diagram has a coequalizer C (that is, there is an arrow $S \times T \overset{\phi}{\to} C$ such that $\phi \circ \theta = \phi \circ \tau$, and, for all arrows $S \times T \overset{\psi}{\to} C'$ with $\psi \circ \theta = \psi \circ \tau$, there is a unique arrow $C \overset{u}{\to} C'$ such that $\psi = u \circ \phi$). We denote the coequalizer of (1.44) as $S \otimes_M T$ and compose:

$$S \times M \times T \underset{\tau}{\overset{\theta}{\rightrightarrows}} S \times T \overset{\phi}{\longrightarrow} S \otimes_M T. \tag{4.45}$$

Again, in **Sets** the tensor product $S \otimes_M T$ is simply the quotient of $S \times T$ by the least equivalence relation identifying $\theta\,(a, X, b)$ and $\tau\,(a, X, b)$. Introducing the notation $\phi\,(a, b) \equiv a \otimes b$, we have found that $aX \otimes b = \phi\,(aX, b) = \phi \circ \theta\,(a, X, b) = \phi \circ \tau\,(a, X, b) = \phi\,(a, Xb) = a \otimes Xb$. This proves the following proposition:

Proposition 4.16 *For all representations S in $\mathbf{BAF_C}$, anti-representations T in $\mathbf{BAF_C^*}$ and arrow fields X over the category $\mathbf{C}$, there is a tensor product - $\otimes_M$ - : $\mathbf{BAF_C} \times \mathbf{BAF_C^*} \to \mathbf{Sets}$, given by $S \times T \mapsto S \otimes_M T$, for which we have the equality*

$$aX \otimes b = a \otimes Xb.$$

(Here, $aX = \mu(a, X)$ for $S = \langle S, \mu\rangle$ and $Xb = \mu_*(b, X)$ for $T = \langle T, \mu_*\rangle$.)

Note that for the special case $S = \hat{a}$, $T = \hat{a}^\dagger$, both defined on the space $\mathcal{S}_\kappa$, this relation takes the form $(\hat{a}(X))\psi \otimes \phi = \psi X \otimes \phi = \psi \otimes X\phi = \psi \otimes \hat{a}(X)^\dagger\phi$ (for $\psi, \phi \in \mathcal{S}_\kappa$). The significance of the result above seems to be this: given a physical system in which the constructions above are interpreted (no mean task), the choice of an adjoint (anti-)representation is not arbitrary, but depends on the "inner product", and this should reflect a real constraint of the system.

4.6.3. Arrow Fields in Topos Physics

By means of the development in the last two subsections, we may try to extend the topos formalism of chapter 2 to the case of quantised categories. Let **C** be a given category, and consider a set $\boldsymbol{X}$ of arrow fields $X, Y, \ldots$ in AF(**C**) closed under composition. From def. 4.11, this set determines a pre-causal category $\mathbf{C}_{\boldsymbol{X}}$. Above, we defined the category $\mathbf{BAF_C}$ as the set of representations of the monoid $\mathbf{AF_C}$. Similarly, we may define $\mathbf{B}\boldsymbol{X}$ as the representations of the submonoid $\boldsymbol{X}$ (or the one-object category $\mathbf{C}_{\boldsymbol{X}}$). Full subcategories of $\mathbf{B}\boldsymbol{X}$ will contain a choice of representations of $\boldsymbol{X}$ and all equivariant maps between them. Subcategories of $\mathbf{B}\boldsymbol{X}$ for which all representations are defined over the same domain S, are denoted by **S**. Let †, the "adjoint selector" be a map † : $\mathbf{BAF_C} \to \mathbf{BAF_C^*}$ given by $S \mapsto \dagger(S) \equiv S^\dagger$. We now define:

Definition 4.26 *The* operator set $O(\mathbf{S})$ of a subcategory **S** (of $\mathbf{B}\boldsymbol{X}$) *is the set of maps*

$$S(X) : S \to S \text{ given by } a \mapsto \mu(a, X) \equiv (S(X))(a)$$

such that $S = \langle S, \mu\rangle \in \mathbf{S}$ and $X \in \boldsymbol{X}$. Also, $O(\mathbf{S})$ is closed under adjunction: that is, if $S(X) \in O(\mathbf{S})$ then $S^\dagger(X) \in O(\mathbf{S})$, and $S^\dagger$ has the same domain of representation as S.

Presently, a collection $C(\mathbf{S})$ of operator sets which are subsets of a given $O(\mathbf{S})$ will be the context category of the topos of the system under consideration. The arrow structure of $C(\mathbf{S})$ is partial ordering by inclusion. We shall assume, as in chapter 2, that commutativity is the criterion which settles the classicality of a given context.

Definition 4.27 *A* representational topos $\tau_{\mathbf{S}}(X, \mathbf{C})$ *(or simply* $\tau_{\mathbf{S}}(\mathbf{C})$ *if* $X = \mathrm{AF}(\mathbf{C})$*) is a topos of presheaves*

$$\tau_{\mathbf{S}}(X, \mathbf{C}) := \mathbf{Sets}^{C(S)^{op}},$$

where $C(\mathbf{S})$ *is a commutative collection of operator subsets of* $O(\mathbf{S})$*; i.e. if* $S(X)$ *and* $T(Y)$ *are maps in* $C(\mathbf{S})$ *(with T, S (anti-)representations in* $\mathbf{BAF}_{\mathbf{C}}$ ($\mathbf{BAF}^{*}_{\mathbf{C}}$) *and X, Y arrow fields over* $\mathbf{C}$*), then* $S(X)T(Y)(a) = T(Y)S(X)(a)$ *for all* $a \in S$*, the common space of the (anti-)representations.*

The choice of contravariant functors is in accordance with the Isham-Döring approach, but the covariant functors of Bohrification scheme may also be used. Note that the construction of a presheaf topos in chapter 2 falls into place as a rather degenerate case of the present scheme. Indeed, choose as the intial category $\mathbf{C} = \mathbf{Hilb}_k$, the category of Hilbert spaces (for a field k) with linear maps as morphisms, and let X be the set of all arrow fields X_B, where the B's are bounded linear maps over a fixed Hilbert space H. Now let $\mathbf{S}$ be the set of representations which take the fields X_B back to the linear maps B, and define the collection $C(\mathbf{S})$ as the commutative von Neumann subalgebras of the full algebra of bounded linear maps over H.

4.7. Prolegomenon to a Quantised Logic

Let us end by suggesting, very briefly, another application of the ideas above. We have noted earlier (e.g. in subsection 1.3.5) that one may ask at which structural level a procedure of quantisation (superposition of states) is meaningful. It has been suggested, e.g. by Isham, that even the deeper levels, such as the topological structure, should be treated according to the precepts of quantum theory.

In fact, using the method of quantisation on categories in this chapter, one step further may be taken: logic itself may be involved. Now note that the quantum logics spoken of earlier (Birkhoff-von Neumann logic in section 1.1; the intuitionistic logic of topos physics in section 2.2) really are *"fixed background logics"*; that is, there is no "superposition of different logics" within these frameworks. What would a "quantised" (as opposed to a quantum) logic mean?

In quantum mechanics, the states of the system are functions defined on the classical configuration space. What, in (propositional) logic, corresponds to the notion of a space? One suggestion which comes to mind is the following: *points in logic are types*. The quantum-mechanical smearing of the position of a particle over a region in space thus should have an analogue in the smearing of a "logical" state over a space of types. Recall that a construction which is relevant to the present purpose was presented in section 2.1. There, we spoke of the internal logic associated with a class of categories, the topoi, and we identified the types of a certain language, the Mitchell-Bénabou language, as the objects of the topos under consideration. This suggests the following strategy for a quantisation of logic:

1. As in section 4.2, all constructions are to be carried out on a category. We demand, in addition, that the category is a topos $\mathcal{E}$ (not necessarily Boolean, cf. subsection 4.6.1 or Mac Lane and Moerdijk (1992), p. 270).

2. Within the topos $\mathcal{E}$, the set of objects of $\mathcal{E}$, Ob($\mathcal{E}$), will be the *types A, B, ...* of the language $\mathcal{L}(\mathcal{E})$ (cf. subsection 2.1.3). Earlier, the terms of $\mathcal{L}(\mathcal{E})$ were defined inductively as arrows in Hom($\mathcal{E}$). This time, we shall identify "arrow terms" $\sigma : A \to B$ with their arrow fields X_σ (cf. definition 4.2), the arrow field whose only non-trival arrow starts from the type (object) A. We shall speak of arrow fields X_σ as *proto-terms* of $\mathcal{L}(\mathcal{E})$. Proto-terms X_ψ such that cod(ψ) = Ω, the truth object, are *proto-formulae* of $\mathcal{L}(\mathcal{E})$. For each pair of proto-terms X_σ, X_τ, there is a *pairing* $X_{\langle\sigma,\tau\rangle}$. The *logical connectives* $\vee$, $\wedge$, $\Rightarrow$, $\neg$ will also show up as proto-terms $X_\vee$, $X_\wedge$, $X_\Rightarrow$, $X_\neg$ in $\mathcal{L}(\mathcal{E})$: note e.g. that $\wedge$ is the arrow $\wedge : \Omega \times \Omega \to \Omega$ defined in terms of the internal Heyting algebra of Ω (cf. comment after definition 2.1 above), so $X_\wedge$ is the corresponding arrow field.

3. The set of types, Ob($\mathcal{E}$), is the configuration space on which we define the *logical states* Ψ, Φ, ... In this chapter, we have mainly considered such states as sections on a bundle $\bigcup_{A\in \mathrm{Ob}(\boldsymbol{C})}\kappa(A)$, where κ is a contravariant functor from the category **C** to be quantised into a category with Hilbert spaces. For the quantisation of logic, **C** must be a topos $\mathcal{E}$. So far in this chapter, we have restricted attention to the case where κ is a functor with $\kappa(A)$ a Hilbert space for all types A, but there seems to be less reason to do so now (but see pt. 5 below). Probably, we should demand that $\kappa(\Omega)$ "extracts" the logical structure of the truth object in some way. If we do choose the Hilbert space functor, the logical structure will be given by the lattice of subspaces of Hilbert space. The interpretation of κ on types other than Ω is still open.

4. The main vehicles of quantisation are the representations $\hat{a}(X_\sigma)$ of proto-terms X_σ, where $\hat{a}(X_\sigma)$ is given in definition 4.4:

$$(\hat{a}\,(X_\sigma)\,\Phi)\,(A) = \kappa\,(X_\sigma\,(A))\,\Phi\,(\ell_X\,(A)) = \quad \kappa\,(\sigma)\,\Phi\,(\ell_X\,(A)). \tag{4.46}$$

The operators $\hat{a}(X_\sigma)$ are the *terms* of $\mathcal{L}(\mathcal{E})$. For a proto-formula X_ψ, $\hat{a}(X_\psi) \equiv [\psi]$ is a *formula*. Because ψ is an arrow from a logical type A to the truth object Ω (i.e. $\ell_X(A) = \Omega$), $[\psi]\Phi$ really maps the logical state Φ by looking into its value at the truth object, $\Phi(\Omega)$. Then

$$([\psi]\,\Phi)\,(A) = \kappa\,(\psi)\,\Phi\,(\Omega). \tag{4.47}$$

It is natural, perhaps, to interpret $\kappa(\psi)$ as the "truth content" of the type A with respect to the formula $[\psi]$. Consider the simple case where $\kappa(\Omega) = \{0, 1\}$ and let Φ be a state which corresponds to some the set of individuals of type A, i.e. $\Phi(A) = \{a_1, a_2, ...\}$ for $a_1, a_2, ... : A$, but otherwise zero. (As before, we write $\Phi = |A\rangle$ for such states.) Informally, $\kappa(\psi)$ should then be given by stipulating that $\kappa(\psi)(0)$ is the (n-tuples of) individuals of type A for which ψ holds *classically*.

5. The advantage of picking a Hilbert space functor is that the former definition 4.14 of the scalar product is then at our disposal. We may use this to define truth values for formulae $[\psi]$. Appropriately for a quantised logic, this value is a complex number. Tentatively, *the truth value* $[\psi]_\Phi$ *of a (closed) formula* $[\psi]$ *with respect to the state* Φ is given by (where the denominator equals 1 for normalized states Ψ)

$$[\psi]_\Phi = \frac{\int \langle \Phi(A), [\psi]\,\Phi(A)\rangle_{\kappa(A)}\, d\!\!\!/\,\mu(A)}{\int \langle \Phi(A), \Phi(A)\rangle_{\kappa(A)}\, d\!\!\!/\,\mu(A)}. \tag{4.48}$$

If we focus on the countable case, which seems reasonable for a space of types, we get

$$[\psi]_\Phi = \frac{\sum_{A\in \mathrm{Ob}(\boldsymbol{C})} \langle \Phi(A), [\psi]\,\Phi(A)\rangle_{\kappa(A)}}{\sum_{A\in \mathrm{Ob}(\boldsymbol{C})} \langle \Phi(A), \Phi(A)\rangle_{\kappa(A)}}. \tag{4.49}$$

6. The proposal above raises several questions. What are the valid formulae of $\mathcal{L}(\mathcal{E})$? In what sense is the logic complete? Does it make sense to ask about the classical limit of quantised logic (Boolean in a Boolean topos, intuitionistic otherwise)? No less difficult are questions of the possible interpretation of the theory (the notion of a "smearing" of types seems particularly recalcitrant). Is it possible to apply a quantised logic within quantum physics, or is it just a formal possibility created by the theory of arrow fields? We shall reserve the full elaboration of pts. 1-6 for future work.

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