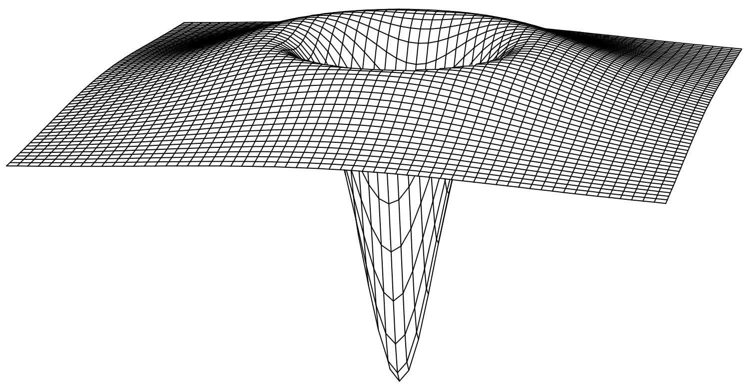
Vortices in Chern-Simons-Ginzburg-Landau Theory and the Fractional Quantum Hall Effect

Håkon Enger



Thesis submitted to the Degree of
Candidatus Scientiarum

Department of Physics
University of Oslo

September 1998

Acknowledgments

First of all I would like to give my thanks to my supervisor, Jon Magne Leinaas, who has introduced me to the exciting world of two-dimensional physics and the quantum Hall effect. My thanks also go to Susanne Viefers for giving me some very useful references on the Quantum Hall Effect, and for introducing me to numerical methods with Maple. Thanks to Serguei Isakov for some references on dispersion in magnetoactive plasmas, and special thanks to Heidi Kjønsberg for reading my manuscript and giving me some useful comments. Last but not least, thanks to Lina for supporting me at all times.

Contents

Chapter 1

Introduction

1.1 Motivation and Description

In 1980, three experimental physicists, K. von Klitzing, G. Dorda and M. Pepper made the unexpected discovery[1] that conductivity in a two-dimensional system sometimes is quantized, i.e. that it takes only discrete values. But while this was a surprising discovery, it did not take long before a theoretical understanding of the phenomenon was reached. It was therefore even more unexpected when three other physicists, D. C. Tsui, H. L. Störmer and A. C. Gossard in 1982[2] found that the conductivity was in fact quantized in fractional steps, a discovery that could not be explained by the theory.

This example shows that even if one believes that one has a good understanding of a phenomenon, it is not always correct. And there is another morale to the story as well: It was later found that the latter discovery could be explained by using a theoretical construction known as particles with intermediate statistics, which was found by the theorists J. M. Leinaas and J. Myrheim in 1977[3]. Although this construction at the time of discovery seemed to have no application, it was still convenient to explain an actual physical phenomenon. In other words, a theoretical construction might be useful in the future even though it does not seem so at the moment.

The main focus of this thesis is the *Chern-Simons-Ginzburg-Landau* (CSGL) theory. This theory is an attempt of a phenomenological description of the *fractional quantum Hall effect*, which is the name given to the discovery by Tsui et. al. mentioned above. It is therefore natural to consider the application of the CSGL theory to this phenomenon. However, in many parts of the thesis we will study the CSGL theory without considering the direct applications of the results. In fact, the CSGL theory might not be a very accurate description of this effect. Still, motivated by the example of how Leinaas' and Myrheim's discovery in 1977 was found to be useful several years later, we will study the theoretical aspects of the theory.

In the CSGL theory, a mathematical phenomenon called vortices also appears. These vortices are believed to be the analogue to the particles with intermediate statistics mentioned above. They are believed to be very important for the fractional quantum Hall effect, and are therefore certainly interesting objects for study. Most of this thesis will concentrate on the properties and effects of the vortices. The details of the vortices are studied both analytically and numerically in chapter 3, and we compare the analytical results to the numerical ones.

To give the reader the necessary theoretical background needed for the study of the CSGL theory, chapter 2 gives a review of those subjects the reader is likely to be unfamiliar with. We have not attempted to go into details of all the necessary subjects, as this could easily fill several large books. However, references are provided for the reader who wants more information on any subject.

In chapter 4, we study various extensions of the CSGL theory. These extensions are made by adding terms to the CSGL Lagrangian. Their motivation is to create a more accurate description of the fractional quantum Hall effect, but as with the pure CSGL theory we will mainly study their mathematical properties without considering physical implications. The extended theories are mainly studied numerically. An interesting part of this study is to compare the results for the extended theories with the pure CSGL theory results.

In chapter 5, we return to the vortices in pure CSGL theory. We show how these vortices may be understood as particles in another theoretical construction, the *Maxwell-Chern-Simons* theory. We also try to find out how these vortices will alter some properties of the CSGL theory through the connection to Maxwell-Chern-Simons theory.

1.2 Notation

This section gives a brief overview of the notation used in this thesis. We will here restrict ourselves to explaining some general rules and concepts of notation. A complete reference of symbols used in the thesis is provided in appendix A. The reader is advised to consult that appendix for explanation of symbols encountered in the thesis.

Although the CSGL theory is non-relativistic, relativistic notation is used everywhere in this thesis. This implies that we will differ between contravariant components of a vector, written as v^μ and covariant components, written as v_μ. However, it does not imply that quantities or equations will be Lorentz invariant in general. A flat space-time is always assumed, and we may in fact define the covariant vector components by $v_i \equiv -v^i$.

The quantum Hall effect occurs in the plane. Thus, in this thesis we will mainly be considered with two-dimensional vectors. Such vectors are denoted in a bold typeface, e.g. $\mathbf{v}$. For unit vectors, we will use the notation

$$\hat{\mathbf{e}}_x \equiv \hat{\mathbf{e}}_1 = (1,0), \tag{1.1}$$

$$\hat{\mathbf{e}}_y \equiv \hat{\mathbf{e}}_2 = (0,1). \tag{1.2}$$

In two dimensions, the result of a cross product between two vectors is a pseudo-scalar. The resulting vector of cross multiplying two three-dimensional vectors in the plane would be a vector normal to the plane. In two dimensions, the z-component of this three-dimensional vector will be the resulting pseudo-scalar. The definition is

$$\mathbf{v} \times \mathbf{w} = \epsilon^{ij} v^i w^j, \tag{1.3}$$

where we have introduced the antisymmetric tensor ϵ^{ij}, defined as $\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon^{11} = \epsilon^{22} = 2$. We will also need the totally antisymmetric tensor in three dimensions, $\epsilon^{\mu\nu\sigma}$, which is defined such that $\epsilon^{\mu\nu\sigma}$ has the sign of the permutation of the indices (μ, ν, σ), i.e. $\epsilon^{012} = 1$, $\epsilon^{102} = -1$ etc. The tensor component is zero if two indices are equal.

Sometimes we will need to rotate a vector by 90 degrees. For this operation we define the special vector $\hat{z}$, which may be understood as a unit vector in the z-direction. It will be used only in cross products such that $\hat{z} \times \mathbf{v}$ results the rotation of $\mathbf{v}$ 90 degrees in the clockwise direction.

In most of the thesis we will work in *natural units*, where $\hbar = c = 1$, however in chapter 2, we will for clarity show factors of $\hbar$ and c explicitly. Occationally, we will denote the differentiation with respect to time with a dot, i.e. $\dot{v} \equiv \frac{dv}{dt}$.

Chapter 2

Background

2.1 Phenomenology of the Hall Effect

2.1.1 The Hall Effect

The ordinary Hall effect is well known and completely explained by classical physics[4]. It was discovered in 1879 by E. C. Hall. The effect occurs when current carriers, usually electrons, in a conductor are subjected to a magnetic field perpendicular to the electric field. In such a system one will observe an electrical potential difference between opposite points on the conductor perpendicular to the direction of the current.

Let $\mathbf{E}$ be the electric field in the conductor, and $\mathbf{j}$ the linear current density. One defines the conductivity tensor σ^{ij} and the resistivity tensor ρ^{ij} as

$$E^i = \sigma^{ij} j^j, \tag{2.1}$$

$$j^i = \rho^{ij} E^j, \tag{2.2}$$

so that ρ^{ij} is the matrix inverse of σ^{ij}. One finds that in general, σ^{ij} and ρ^{ij} are quantities depending on the material of the conductor. If there was no Hall effect (e.g. if $B = 0$), one would have $\sigma^{ij} = \sigma \delta^i{}_j$, where σ is the (scalar) conductivity of the material. If the conductor is made from a homogeneous material, we must have $\sigma^{xy} = -\sigma^{yx}$ and $\sigma^{xx} = \sigma^{yy}$. It is thus natural to define a longitudinal conductivity σ_L and a Hall conductivity σ_H by

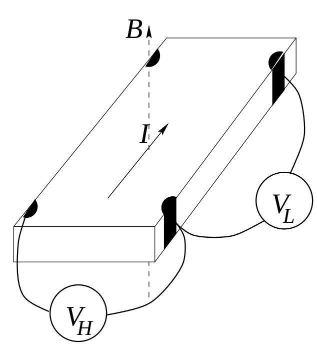


Figure 2.1: Measuring conductivity and resistivity

$$\sigma_H \equiv \sigma^{xy}, \tag{2.3}$$

$$\sigma_L \equiv \sigma^{xx}. \tag{2.4}$$

σ_H and σ_L are not measured directly. Rather, one measures the current I passing through a sample, and the longitudinal and normal voltages V_L and V_H, as shown in figure 2.1. We define the *Hall resistance* R_H by

$$V_H = R_H I. \tag{2.5}$$

The classical microscopic picture of the Hall effect is this: The current carriers are affected by the Lorentz force, $\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$, where q is the charge of the particles and v is the velocity. The magnetic field will force current carriers to gather on one side of the conductor until the electric field from these particles is strong enough to negate the effect of the magnetic field. We then have $\mathbf{E}_H = \mathbf{v}\times\mathbf{B}$, where $\mathbf{E}_H$ is the part of the electric field perpendicular to the current flow. Equation (2.5) then implies $R_H = \frac{vBx}{I} = \frac{B}{\rho e}$, where ρ is the planar density of current carriers. Thus, the Hall resistance is expected to be proportional to the applied magnetic field.

2.1.2 The Quantum Hall Effect

The *Quantum* Hall Effect (QHE) occurs at low temperatures ($\leq 4K$), in interfaces between two semiconductors or between a semiconductor and an insulator.

The effect is seen when adding a large transverse magnetic field ($\approx 1-30T$) to the setup described above. What von Klitzing, Dorda and Pepper discovered in 1980[1], was that the Hall conductivity σ_H is *quantized* as

$$\sigma_H = i\frac{e^2}{h} \qquad i = 1, 2, 3, \ldots \tag{2.6}$$

Moreover, this conductivity was *independent* of the materials used, the temperature (as long as it is low enough), and other variables that conductivity usually depends on. Because of this independence the Hall effect is convenient for defining a standard resistance, and it may also be applied as an unusually accurate measure of the fine structure constant $\alpha = \frac{e^2}{\hbar c}$.

The most used experimental setup for measuring the QHE is a setup for a DC transport experiment in an interface between two gallium-arsenic semiconductors (GaAs heterojunction), schematically set up as shown in figure 2.1. A steady current I transports electrons through the interface. The longitudinal voltage V_L in the figure and the Hall voltage V_H are then measured as well as the length L and width W of the sample. Assuming a uniform current density parallel to the y-axis in the figure, and a uniform electric field (these are good approximations if $L \gg W$ and the connectors for the voltage measurements are far from the edges of the sample), the current density $\mathbf{j}$ and the longitudinal component of the electric field E_L are given by

$$\mathbf{j} = \frac{I}{W}\hat{\mathbf{e}}_y, \qquad E_L = \frac{V_L}{L}. \tag{2.7}$$

The longitudinal and Hall resistances, R_L and R_H are given by

$$R_L = \frac{V_L}{I} = \rho_L \frac{L}{W}, \quad R_H = \frac{V_H}{I} = \rho_H. \tag{2.8}$$

So we see that for a 2D rectangular system, the Hall resistance and the Hall resistivity are in fact equal. When the resistances are measured, results such as shown in figure 2.2 are found, contrary to the expected linear dependence of R_H on B.

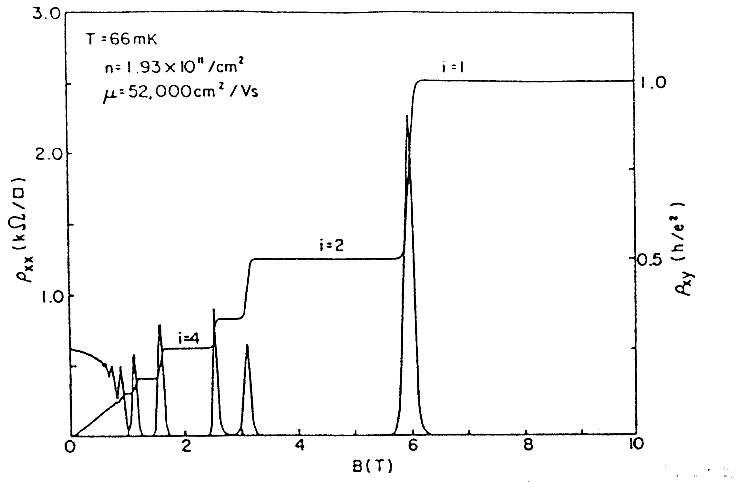


Figure 2.2: Typical experimental results from a DC transport experiment as described in the text. From [5].

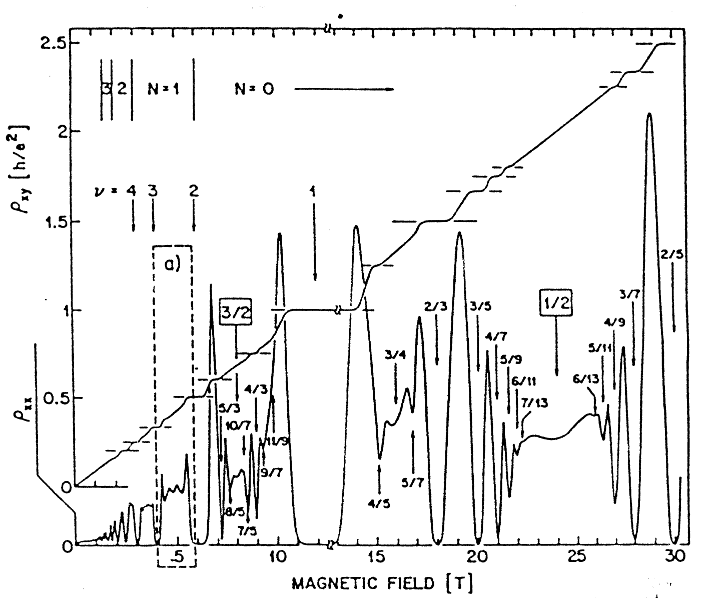


Figure 2.3: Experimental results from [5] showing the fractional quantum Hall effect.

2.1.3 The Fractional Quantum Hall Effect

While the quantum Hall effect was a surprising discovery in 1980, it was an even greater surprise when Tsui, Störmer and Gossard in 1982[2] found that not only was the Hall conductivity quantized in integer steps as in equation (2.6), but under certain conditions it was also quantized in *fractional* steps,

$$\sigma_H = \frac{p}{q}\frac{e^2}{h} \qquad p, q = 1, 2, 3, \dots , \tag{2.9}$$

as shown in figure 2.3. The surprise was great because while the Integer Quantum Hall Effect (IQHE) was well described by the independent electron model (see section 2.2.5), the newly discovered Fractional Quantum Hall Effect (FQHE) did not fit into this picture. The FQHE is only seen when the samples used in experiments are exceptionally clean.

2.2 Microscopic Theory

2.2.1 Freezing out the z-dimension

The system where the quantum Hall effect occurs is the interface between a semiconductor and an insulator or between two semiconductors. The mobility of the electrons in this system is much larger in the plane of the interface than in the transverse direction. We might then expect the dynamics of the electrons to be effectively 2-dimensional (2D), so that we can approximate the electrons as moving in a 2D world.

In fact, as a result of quantum mechanics, this approximation is almost exact. Consider a free particle in a (3-dimensional) box. We can separate the wave function in x, y and z-dependent parts:

$$\psi(\mathbf{r}) = X(x)Y(y)Z(z). \tag{2.10}$$

Each part will be an eigenstate of the one-dimensional Schrödinger equation for a particle in a box; for the function $Z(z)$ we get

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + V_z(z)\right\} Z(z) = E_z Z(z) \tag{2.11}$$

where E_z is the contribution from $Z(z)$ to the total energy $E = E_x + E_y + E_z$, and

$$V_z(z) = \begin{cases} 0 & 0 < z < H \\ \infty & \text{otherwise,} \end{cases} \tag{2.12}$$

where H is the height of the box.

The solution to this problem is

$$Z_n(z) = \sin(\frac{2\pi n z}{H}), \qquad n = 0, 1, 2, \dots , \tag{2.13}$$

and the energy is given by

$$E_n = \frac{h^2 n^2 z^2}{2mH^2}. \tag{2.14}$$

From this equation we see that if H is very small, the spacing between energy levels will be very large. If the temperature is low, the thermal energy in the system will be too small to allow excitations in the z-direction. Therefore, what we study is (effectively) a 2-dimensional electron gas (2DEG). Since the temperature has to be small enough to disallow thermal excitations in the z-direction, this is known as "freezing out" the z-dimension.

2.2.2 Electrons in a Magnetic Field

The fundamental image of the Hall effect is that of electrons moving in a magnetic field. It is therefore essential to understand the basics of such a system before we may start to describe the Hall effect.

The Schrödinger equation for an electron with charge $-e$ moving in an electromagnetic field may be written as

$$\frac{1}{2m}\left(\frac{1}{i}\hbar\boldsymbol{\nabla} + \frac{e}{c}\mathbf{A}\right)^2 \psi = E\psi. \tag{2.15}$$

For a constant magnetic field $B = \boldsymbol{\nabla} \times \mathbf{A}$, one possible choice of gauge (known as Landau gauge) is

$$A^x = -yB \qquad A^y = 0. \tag{2.16}$$

This gives the Schrödinger equation

$$\frac{1}{2m}\left[\left(\frac{1}{i}\hbar\partial_x - \frac{eB}{c}y\right)^2 + \left(\frac{1}{i}\hbar\partial_y\right)^2\right]\psi(x,y) = E\psi(x,y). \tag{2.17}$$

Substituting the ansatz $\psi(x,y) = e^{ikx}\phi(y)$, we arrive at

$$\left[-\frac{\hbar^2}{2m}(\partial_y)^2 + \frac{1}{2}m\omega_C^2\left(y - \frac{\hbar kc}{eB}\right)^2\right]\phi(y) = E\phi(y), \tag{2.18}$$

with the *cyclotron frequency* $\omega_C = \frac{eB}{mc}$, which we recognize as a harmonic oscillator (HO) with frequency ω_C centered at $\frac{\hbar kc}{eB}$. The solution to the HO problem is well known to be

$$\phi_n(y) \propto e^{\frac{1}{2\ell^2}(y-\ell^2 k)} H_n(\frac{y}{\ell} - \ell k), \tag{2.19}$$

where H_n are the Hermite polynomials and we have introduced the *magnetic length* $\ell = \sqrt{\frac{\hbar c}{eB}}$. The quantum number n denotes the *Landau level.* The energy depends only on n, not on k:

$$E_n = (n + \frac{1}{2})\hbar\omega_C. \tag{2.20}$$

The wave functions (2.19) are localized to a band about $y = \ell^2 k$ by the Gaussian factor. Ignoring boundary effects and confining the 2DEG to $0 < y < W$, we find that k is constrained by $0 < k < \frac{W}{\ell^2}$. Imposing periodic boundary conditions in the x direction $\psi(0, y) = \psi(L, y)$, we get the condition $k = \frac{2\pi p}{L}$, with $p = 0, \pm 1, \pm 2, \ldots$. Combining these, we see that $0 < p < \frac{LW}{2\pi\ell^2}$, so that the number of states for one Landau level is $N = \frac{LW}{2\pi\ell^2}$. Thus, the density of states in each Landau level is $\rho_B = \frac{N}{LW} = \frac{1}{2\pi\ell^2}$. This allows us to define the *filling factor* ν for a magnetic system with a density ρ of electrons as

$$\nu = \frac{\rho}{\rho_B} = \frac{2\pi\hbar c}{eB}\rho. \tag{2.21}$$

The above situation is easily extended to include an electric field $\mathbf{E} = \mathcal{E}\hat{\mathbf{e}}_y$. This field may be incorporated in the Schrödinger equation by adding the potential $V(x, y) = eEy$. Substituting as before, equation (2.18) becomes

$$\left[-\frac{\hbar^2}{2m}(\partial_y)^2 + \frac{1}{2}m\omega_C^2\left(y - \frac{\hbar kc}{eB}\right)^2 + e\mathcal{E}y\right]\phi = E\phi, \tag{2.22}$$

which we may transform into

$$\left[-\frac{\hbar^2}{2m}(\partial_y)^2 + \frac{1}{2}m\omega_C^2\left(y + y_0 - \frac{\hbar kc}{eB}\right)^2 - \frac{1}{2}m\omega_C^2 y_0^2 + \frac{\hbar kc\mathcal{E}}{B}\right]\phi = E\phi, \tag{2.23}$$

where $y_0 = \frac{2mc^2E}{2eB^2}$, i.e. still a shifted HO with essentially the same solutions as before, but with an energy varying with k so that the degeneracy of the Landau levels is broken.

To extend the model to include the effects of spin, we may add an interaction term

$$H_Z = \frac{g\mu_B}{2}B\sigma_z, \tag{2.24}$$

known as a *Zeeman term*. Here, μ_B is the *Bohr magneton*, $\mu_B = \frac{e\hbar}{2mc}$ and g is the gyromagnetic ratio, or the *Landé g-factor* for the electron. In practice, one would use an effective g-factor different from the vacuum value of $g = 2.0023\ldots$. σ_z is a Pauli matrix,

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.25}$$

and the wave function ϕ must be extended to a two component vector. We return to the concept of particles with spin in chapter 4.

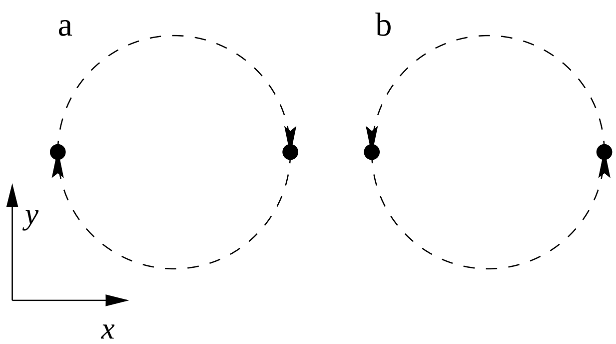


Figure 2.4: Exchanging particles

2.2.3 Statistics in Two Dimensions—Anyons

It is well known that in our three dimensional world there exist two fundamentally different kinds of particles, bosons and fermions, and that these are the only kinds of particles that can exist. Whether a particle is a fermion or boson depends on what happens to the wave function of two identical particles when they are exchanged. Fermions gain a phase factor of -1, while the wave function for bosons is unchanged under this operation. The reason why only these two types of particles are allowed, is of geometrical nature: In general, the wave function might pick up a phase factor $e^{i\theta}$, where $0 \leq \theta \leq 2\pi$ when the particles are exchanged in, say, a clockwise fashion as shown in figure 2.4a. However, in three dimensions this clockwise exchange is equivalent to the counter-clockwise exchange of figure 2.4b, since the two are related by a rotation of the plane containing the particles around the x- or y-axis. In effect, any rotation will be observed as a clockwise rotation by an observer located on one side of the rotation plane and as an anti-clockwise rotation by an observer on the other side. Therefore, the phase factor $e^{i\theta}$ must be the same as its inverse, and this only allows θ to be 0 or π. θ is called the statistics parameter of the particles.

In two dimensions, the picture is different, as realized by Leinaas and Myrheim in 1977[3]. There is no possibility to rotate around the x- or y-axis, and therefore there is no reason why the phase factor should be constrained to be 1 or -1. Observers who are constrained to move in the plane of the rotation will always agree on whether the rotation is clockwise or anti-clockwise. In fact, all possible values of θ are allowed. Particles with arbitrary statistics have been named *anyons* by Wilczek[6].

Reality, of course, is three dimensional, and therefore this discovery of fractional statistics might seem to have little application in real world physics. However, in systems where particles are constrained to move effectively only in two dimensions, such as the interfaces where the quantum Hall effect occurs, the particles might create collective excitations known as quasiparticles, which might behave as anyons. In fact, we will see that there indeed exist such particles in a quantum Hall system.

2.2.4 The Aharonov-Bohm Phase Factor

Consider a magnetic field confined to a relatively small area (e.g. the inside of an infinitely long solenoid) and a charged particle free to move in the region outside, where there is no magnetic field.

Classically, the particle would not be affected by the field since $B = 0$ where the particle is and the particle only feels the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Quantum mechanically, however, the particle will in fact be affected by the field, since the vector field $\mathbf{A}$ is non-zero also on the outside of the solenoid. (This is clear because the flux $\int Bd^2r = \oint \mathbf{A} \cdot d\mathbf{r}$ is non-zero.) If the particle moves one complete rotation around the solenoid (fig. 2.5), it will pick up a phase factor[1] $e^{iq\oint \mathbf{A}\cdot d\mathbf{r}}$ relative to the phase it would pick up if the solenoid was not there. Since $\oint \mathbf{A} \cdot d\mathbf{r} = \int Bd^2r = \Phi$, the total flux, the phase factor will be $e^{iq\Phi}$. This is the Aharonov-Bohm effect[7]. The effect was first observed experimentally by Chambers[8], but although the effect acquired the names of Aharonov and Bohm, Ehrenberg and Siday[9] were actually the first to discuss the effect.

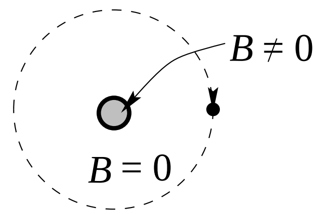


Figure 2.5: The Aharonov-Bohm Effect

2.2.5 The Integer Effect

We will now give a short introduction to the microscopical picture of the quantum Hall effect. This introduction is mainly based on Prange and Girvin[10] and Karlhede et. al.[11]. We do not intend to give a full description, and refer the reader to e.g. those references for complete reviews.

To understand the integer quantum Hall effect, it is useful first to consider a 2DEG subjected to a transverse magnetic field B in a translatorially invariant geometry. To find the conductivity tensor of this system, we subject it to an electric field $\mathbf{E}$. A Lorentz transformation of this system gives the value of the field in a frame moving with a speed $\boldsymbol{\beta}$ relative to the laboratory frame,

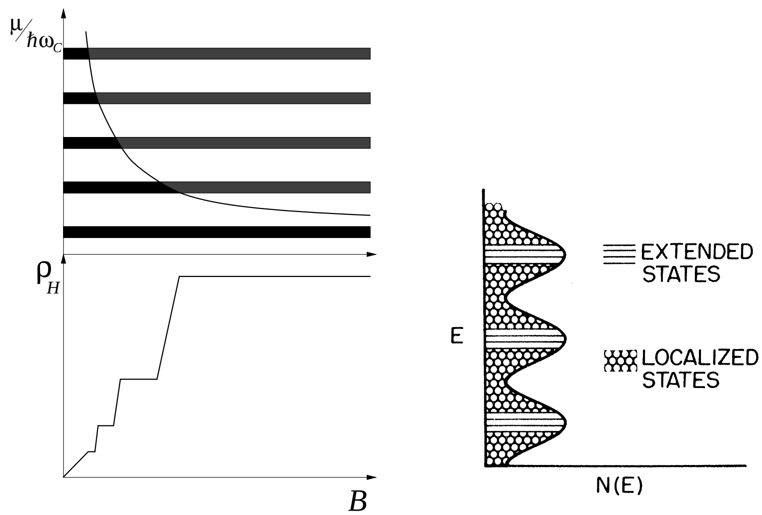
$$E'^i = \gamma E^i - \frac{\gamma^2}{\gamma + 1}\beta^i \beta^j E^j + \gamma B \epsilon^{ij} \beta^j, \tag{2.26}$$

where $\gamma = \frac{1}{1-\beta^2}$. In a frame moving with relative velocity $\beta_E^i = \frac{1}{B}\epsilon^{ij} E^j$ there is no electric field, and thus there is (by symmetry) no current. Therefore, the current in the laboratory frame is

$$j^i = -\rho e \beta_E^i = -\frac{\rho e}{B}\epsilon^{ij} E^j, \tag{2.27}$$

where ρ is the density of electrons, and the conductivity tensor is $\sigma^{ij} = -\frac{\rho e}{B}\epsilon^{ij} = -\nu \frac{e^2}{h}\epsilon^{ij}$. One would expect this result to be a good approximation also for less than

[1] See appendix C for an explanation of how the particle picks up a phase factor



(a) Gaps in the spectrum creates the QHE

(b) Disorder creates localized states. From [10].

Figure 2.6: Energy states and the quantum Hall effect

ideal systems. This would imply that the Hall resistivity is linear in the magnetic field strength, which is not what is observed.

The explanation of the integer effect makes use of two more concepts[11]: A fixing of the chemical potential and a gap in the excitation spectrum for the electrons. If it was possible to fix the chemical potential for the electrons at a value μ, and if there was a gap in the spectrum for some value ν of the filling factor, the filling factor would be constant equal to ν for all values of B such that μ is in the gap. Thus, according to equation (2.27), we would observe a quantized Hall effect for such values of ν. This point is illustrated in figure 2.6a.

Fixing the chemical potential μ seems to be difficult since the number of particles (electrons) in our system is given by the background charge, and thus is approximately constant. However, if we include disorder in the model, there seems to be a solution. In a system with localized disorder, we would expect there to be localized states that do not contribute to the conductivity. These states would lie between the Landau levels as shown in figure 2.6b and would in effect behave as an internal particle reservoir allowing us to fix the chemical potential as desired[11].

From the discussion in section 2.2.2, we know that the ideal system has highly degenerate energy levels where each energy level corresponds to a Landau level. Introducing disorder into the system, we expect that the degeneracy will break, but the gaps in the spectrum for *current-carrying* excitations will still be at the boundaries between Landau levels, corresponding to $\nu = 1, 2, \ldots$, just where the IQHE is observed.

Thus, we have successfully obtained a microscopical description of the IQHE. This description is called the *independent electron model*, since it does not include interactions between the electrons.

2.2.6 The Fractional Effect

The microscopical workings of the fractional quantum Hall effect are complicated and still not fully understood. We will in this section give a very brief summary of some of the concepts involved in the microscopical theory. A more complete summary of the microscopical theory may be found in the review by Karlhede, Kivelson and Sondhi[11], while a detailed (but somewhat dated) review of the microscopical picture is available in the book by Prange and Girvin[10].

It is clear that the FQHE can not be described without including the effect of interaction between the electrons, since as we saw in the previous section, the independent electron model predicts that there will be no gaps in the spectrum other than for completely filled Landau levels. The arguments for a gap at filling fractions $\frac{1}{m}$ where m is an odd integer rely on the fact that the ground state of the interacting electron system may be approximated very well by the multi-particle wave function

$$\Psi = \prod_{j<k} (z_j - z_k)^m e^{-\frac{1}{2}\sum_j |z_j|^2}, \tag{2.28}$$

where $z_j = \frac{1}{\sqrt{2}\ell}(x + iy)$ is the complex coordinate of particle number j. This wave function was studied as a variational function by Laughlin[12], who found it to be an exceptionally accurate approximation. The arguments for a gap are rather subtle, and we will not enter the subject here.

From the microscopical theory, another important implication also follows. There must exist quasiparticle excitations in the FQHE system that have fractional charge. These fractionally charged quasiparticles have further been shown theoretically to exhibit fractional statistics[13, 14], with statistics parameter

$$\theta = \frac{\pi}{m}. \tag{2.29}$$

Quantized Hall conductance at other fractions than the “Laughlin fillings” $\nu = \frac{1}{m}$ are produced by a *hierarchy construction*[13, 15]. The quantization of these filling fractions appear when the quasiparticles of a related filling fraction *condense* into a homogeneous fluid.

2.3 Ginzburg-Landau Theory

2.3.1 The Mean Field Method

When studying a system with many degrees of freedom (e.g. a large number of particles), the equations of motion quickly get too complicated to be useful. It is necessary to use an approximation to be able to get useful results. One old and simple approximation that often is accurate enough is the *mean field method*[2].

The main idea behind a mean field approximation is to approximate the detailed structure of a system with a mean field. For example, in a gas of interacting particles, it is impossible to keep track of the interactions between all particles. In a gas of N particles, there will be $N(N-1)$ such interactions, and it is obvious that such a system can not be treated in full detail when N is a macroscopic number. The mean field method involves assuming that each particle feels a total force proportional to the mean density of particles. It is this approximation that gives us the well known van der Waals equation of state, which is known to be accurate enough for most purposes in thermodynamics.

2.3.2 Landau Mean Field Theory

Several mean field approximations were used to study phase transitions in magnetic systems. The fact that all these different models gave the same results, lead Landau to propose a general theory of 2nd order phase transitions in magnetic systems. The theory was also found to be useful for other types of systems.

Landau realized that most 2nd order phase transitions may be looked at as a transition from a disordered to an ordered phase. One may therefore assign an *order parameter* ϕ which is zero in the disordered phase an nonzero in the ordered phase. If we assume that the free energy F is a regular function of the order parameter at least near the critical point, since the order parameter will be small here, we may expand F in orders of ϕ:

$$F = F_0 + \alpha\phi^2 + \frac{1}{2}\beta\phi^4. \tag{2.30}$$

Odd powers of ϕ may usually be ignored by symmetry considerations. The parameters α and β are in general temperature dependent.

2.3.3 The Ginzburg–Landau Equations

Extension of the Landau theory of 2nd order phase transitions to a superconductor or a superfluid involves treating the wave function as an order parameter, as realized by Ginzburg and Landau[18]. This is possible because in a superconductor, electrons will pair up in so called *Cooper pairs* in the superconducting phase. The wave function of such pairs will then be identically zero in the non-superconducting phase and may then be used as an order parameter. Similarly, in a superfluid, the number of particles in the superfluid state will be vanishingly small in the normal phase.

Treating the wave function as an order parameter implies that we have a complex order parameter that will also be a function of position. We may then expand the

[2] This introduction to the mean field method, Landau mean field theory, and the Ginzburg-Landau equations is based on Tilley and Tilley[16] and Ravndal[17]

free energy density $f(r)$ in the same way as in equation (2.30), but we will have to include a term covering the variation with position:

$$f(\mathbf{r}) = f_0 + \alpha|\phi(\mathbf{r})|^2 + \frac{\beta}{2}|\phi(\mathbf{r})|^4 + \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\phi(\mathbf{r})|^2. \tag{2.31}$$

The coefficient $\frac{\hbar^2}{2m}$ of the "kinetic energy" term is a conventional normalization of the free energy.

Ginzburg and Pitaevskii[19] applied the Ginzburg–Landau (GL) theory to liquid helium. Minimizing the total free energy $F = \int f(\mathbf{r})dV$ with respect to the wave function ϕ gives

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\phi + \alpha\phi + \beta|\phi|^2\phi = 0. \tag{2.32}$$

For the case of a superconductor, it is necessary to include the effect of an applied field in the theory. This is done by the minimal coupling

$$-i\hbar\boldsymbol{\nabla} \to -i\hbar\mathbf{D} \equiv -i\hbar\boldsymbol{\nabla} + 2e\mathbf{A}. \tag{2.33}$$

(Note that the Cooper pairs have charge $-2e$.) In addition, to study the system at a constant applied magnetic field H_0, one must minimize the Gibbs free energy instead of the Helmholz free energy. The Gibbs free energy density becomes

$$g(\mathbf{r}) = f_0 + \alpha|\phi(\mathbf{r})|^2 + \frac{\beta}{2}|\phi(\mathbf{r})|^4 + \frac{1}{2m}|(-i\hbar\boldsymbol{\nabla} + 2e\mathbf{A})\phi|^2 + \frac{B^2}{2\mu_0} - H_0 \cdot B + \frac{1}{2}\mu_0 H_0^2. \tag{2.34}$$

Minimizing the total Gibbs energy $G = \int g(\mathbf{r})dV$ with respect to ϕ and $\mathbf{A}$ gives

$$\frac{1}{2m}(-i\hbar\boldsymbol{\nabla} + 2e\mathbf{A})^2\phi + \alpha\phi + \beta|\phi|^2\phi = 0, \tag{2.35}$$

$$\frac{1}{\mu_0}\boldsymbol{\nabla} \times \mathbf{B} = \frac{ie\hbar}{m}(\phi^*\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\phi^*) - \frac{4e^2}{m}\phi^*\phi\mathbf{A}. \tag{2.36}$$

These are the Ginzburg-Landau equations.

2.3.4 The GPG Equation—Relation to Microscopic Theory

The derivation of the Ginzburg-Landau equations in the previous chapter had a purely phenomenological approach. This is a powerful approach relying only on the assumption that the free energy may be expanded in powers of ϕ. However, it leaves something to be desired when it comes to the connection with microscopic theory, and it is somewhat unclear when the mentioned assumption is actually valid.

A different approach to the construction of an effective theory starts with the microscopical description. We will here follow Nozieres and Pines[20] in describing the derivation of an effective theory first made by Gross[21] and Pitaevskii[22].

The derivation considers Helium-II only, but a similar derivation from a microscopic theory might be possible for a superconductor as well.

A superfluid condensate is characterized by having a macroscopic number of particles in one quantum state (the ground state.) Such a system is denoted by $|\phi(N)\rangle$. We single out an off-diagonal matrix element of the destruction operator $\psi(\mathbf{r})$ by

$$\phi(\mathbf{r}) = \langle\phi(N-1)|\, \psi(\mathbf{r})\, |\phi(N)\rangle\,, \tag{2.37}$$

where the state $|\phi(N-1)\rangle$ is obtained by removing one particle from the condensate. When applied to the state $|\phi(N)\rangle$, the operator $\psi(\mathbf{r})$ satisfies the Heisenberg equation of motion

$$i\frac{\partial\psi(\mathbf{r})}{\partial t} = [\psi(\mathbf{r}), H]\,, \tag{2.38}$$

where the Hamiltonian H is given by

$$H = -\int d^2r\psi^\dagger(\mathbf{r})\frac{\boldsymbol{\nabla}^2}{2m}\psi(\mathbf{r}) + \frac{1}{2}\int d^2rd^2r'V(\mathbf{r}-\mathbf{r}')\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})\psi(\mathbf{r}'), \tag{2.39}$$

where $V(\mathbf{r}-\mathbf{r}')$ is the interaction potential of the particles. We will assume that this potential may be simplified as

$$V(\mathbf{r}-\mathbf{r}') = \lambda\delta^2(\mathbf{r}-\mathbf{r}'), \tag{2.40}$$

an assumption that is expected to be valid as long as the typical spacing of the particles is larger than the typical range of the interaction. Equation (2.38) now becomes

$$i\frac{\partial\psi(\mathbf{r})}{\partial t} = -\frac{\boldsymbol{\nabla}^2}{2m}\psi(\mathbf{r}) + \lambda\psi^\dagger(\mathbf{r})\psi(\mathbf{r})\psi(\mathbf{r}), \tag{2.41}$$

and for the *condensate wave function* $\phi(\mathbf{r})$, we obtain

$$i\frac{\partial\phi(\mathbf{r},t)}{\partial t} = -\frac{\boldsymbol{\nabla}^2}{2m}\phi(\mathbf{r},t) + \lambda\langle\psi^\dagger(\mathbf{r})\psi(\mathbf{r})\psi(\mathbf{r})\rangle, \tag{2.42}$$

by taking the "expectation value" $\langle\cdot\rangle \equiv \langle\phi(N-1)|\cdot|\phi(N)\rangle$ of equation (2.41). To solve this equation, Gross[21] and Pitaevskii[22] proposed using the following *factorization approximation*:

$$\langle\psi^\dagger(\mathbf{r})\psi(\mathbf{r})\psi(\mathbf{r})\rangle \approx \langle\psi^\dagger(\mathbf{r})\rangle\langle\psi(\mathbf{r})\rangle\langle\psi(\mathbf{r})\rangle, \tag{2.43}$$

which results in the equation

$$i\frac{\partial\phi(\mathbf{r},t)}{\partial t} = -\frac{\boldsymbol{\nabla}^2}{2m}\phi(\mathbf{r},t) + \lambda|\phi(\mathbf{r},t)|^2\phi(\mathbf{r},t) \tag{2.44}$$

for the condensate wave function $\phi(\mathbf{r},t)$. This is the *Ginzburg–Pitaevskii–Gross* (GPG) equation. This equation is also often called the non-linear Schrödinger equation, for its similarity with the (linear) Schrödinger equation.

The wave function may be separated into a space independent and a time independent part $\phi(\mathbf{r},t) = \phi(\mathbf{r})e^{-i\mu t}$ to produce the following eigenvalue equation for the time independent field $\phi(\mathbf{r})$:

$$-\frac{\boldsymbol{\nabla}^2}{2m}\phi(\mathbf{r}) + \lambda|\phi(\mathbf{r})|^2\phi(\mathbf{r}) = \mu\phi(\mathbf{r}), \tag{2.45}$$

where the energy eigenvalue μ now corresponds to the *chemical potential* of the particles in the condensate. It is easily seen from the definition of $\phi(\mathbf{r},t)$ that the eigenvalue of the Hamiltonian in this case is the chemical potential. Taking the expectation value on both sides of equation (2.38), we obtain

$$i\frac{\partial\phi(\mathbf{r},t)}{\partial t} = \langle\phi(N-1)|\,[\psi(\mathbf{r}),H]\,|\phi(N)\rangle = (E_N - E_{N-1})\phi(\mathbf{r},t), \tag{2.46}$$

where E_N is the energy of the system with N particles in the condensate, so that μ is given by $\mu = E_N - E_{N-1}$, which is the increase in energy when one particle is added to the condensate. This is the definition of the chemical potential.

Inserting a constant ground state $\phi(\mathbf{r}) = \sqrt{\rho_0}$ into (2.45) gives us that $\mu = \lambda\rho_0$. This inspires us to write (2.45) on the form

$$-\frac{\boldsymbol{\nabla}^2}{2m}\phi(\mathbf{r}) + \lambda(|\phi(\mathbf{r})|^2 - \rho_0)\phi(\mathbf{r}) = 0, \tag{2.47}$$

where the constant ρ_0 replaces the chemical potential μ as the eigenvalue. By comparing this equation to (2.32), we observe that these two equations are equal if we let $\alpha = \lambda\rho_0$ and $\beta = \lambda$. In other words, we have derived the Ginzburg–Landau equation for liquid Helium from microscopical principles, and the GPG equation (2.44) is nothing but the Ginzburg–Landau equation (2.32) extended to time dependent fields. When we later refer to Ginzburg–Landau (GL) theory, we shall be meaning equation (2.47), or the corresponding equations for a superconductor, obtained by inserting $\alpha = \lambda\rho_0$ and $\beta = \lambda$ into equations (2.35) and (2.36):

$$\frac{1}{2m}(-i\hbar\boldsymbol{\nabla} + 2e\mathbf{A})^2\phi + \lambda(|\phi|^2 - \rho_0)\phi = 0, \tag{2.48}$$

$$\frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{B} = \frac{ie\hbar}{m}(\phi^*\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\phi^*) - \frac{4e^2}{m}\phi^*\phi\mathbf{A}. \tag{2.49}$$

It is customary to introduce the constant

$$\xi = \frac{1}{\sqrt{2\lambda m\rho_0}}, \tag{2.50}$$

called the *coherence length*[16]. The coherence length is the typical length scale of variations in the solution of equation (2.47).

Equation (2.44) may conveniently be described in dimensionless quantities. Rescaling the parameters and fields according to

$$t = \frac{\hat{t}}{\lambda\rho_0} \qquad \mathbf{r} = \frac{\hat{\mathbf{r}}}{\sqrt{\lambda m\rho_0}} \qquad \phi = \sqrt{\rho_0}\hat{\phi}, \tag{2.51}$$

the dimensionless field $\hat{\phi}(\hat{\mathbf{r}}, \hat{t})$ satisfies the equation

$$i\frac{\partial \hat{\phi}(\hat{\mathbf{r}}, \hat{t})}{\partial \hat{t}} = -\frac{\hat{\boldsymbol{\nabla}}^2}{2}\hat{\phi}(\hat{\mathbf{r}}, \hat{t}) + |\hat{\phi}(\hat{\mathbf{r}}, \hat{t})|^2\hat{\phi}(\hat{\mathbf{r}}, \hat{t}). \tag{2.52}$$

and the time independent field $\hat{\phi}(\hat{\mathbf{r}})$ satisfies

$$-\frac{\hat{\boldsymbol{\nabla}}^2}{2}\hat{\phi}(\hat{\mathbf{r}}, \hat{t}) + (|\hat{\phi}(\hat{\mathbf{r}}, \hat{t})|^2 - 1)\hat{\phi}(\hat{\mathbf{r}}, \hat{t}) = 0. \tag{2.53}$$

Equations (2.48) and (2.49) may also be rescaled to dimensionless quantities, but we will not get rid of all constants as in equations (2.52) and (2.53). Making the same rescaling as above, and also rescaling the Maxwell field $\mathbf{A}$ according to

$$2e\mathbf{A} = \sqrt{\lambda m \rho_0}\hat{\mathbf{A}}, \tag{2.54}$$

we find

$$\frac{1}{2}(-i\hat{\boldsymbol{\nabla}} + 2e\hat{\mathbf{A}})^2\hat{\phi} + (|\hat{\phi}|^2 - 1)\hat{\phi} = 0, \tag{2.55}$$

$$\frac{\kappa^2}{2}\hat{\boldsymbol{\nabla}} \times \hat{B} = \frac{i}{2}(\hat{\phi}^*\hat{\boldsymbol{\nabla}}\hat{\phi} - \hat{\phi}\hat{\boldsymbol{\nabla}}\hat{\phi}^*) - \hat{\phi}^*\hat{\phi}\hat{\mathbf{A}}, \tag{2.56}$$

where $\kappa^2 = \frac{m^2\lambda}{2\mu_0 e^2}$.

That the GL theory of superconductors contains a dimensionless constant has an important implication. It is found that there are two regions of values for κ which lead to very different properties for a superconductor. If $\kappa < \frac{1}{\sqrt{2}}$, the superconductor is said to be of type I, while if $\kappa > \frac{1}{\sqrt{2}}$, the superconductor is of type II. Type-II superconductors allows the magnetic field to penetrate through the superconductor in quantized vortices[23], while in a type-I superconductor the magnetic field cannot penetrate the superconductor at all.

The Ginzburg-Landau theory may also be studied by means of a Lagrangian density function. The form of a Lagrangian is justified only by its ability to reproduce the equations of motion. There may thus be several possibilities for this function. One possible form is

$$\mathcal{L} = i\phi^* D_0\phi + \frac{1}{2m}\phi^*\mathbf{D}^2\phi - \frac{\lambda}{2}(|\phi|^2 - \rho_0)^2 - \frac{1}{2\mu_0 e^2}\left(B - B^{\text{ext}}\right)^2, \tag{2.57}$$

where $D_0 = \partial_0 - 2eA_0$, $\mathbf{D} = \boldsymbol{\nabla} + 2e\mathbf{A}$, B^{ext} is the externally imposed magnetic field and μ_0 is the magnetic permeability. This Lagrangian reproduces the equations

$$i\frac{\partial\phi}{\partial t} = \frac{1}{2m}(-i\hbar\boldsymbol{\nabla} + 2e\mathbf{A})^2\phi + \lambda(|\phi|^2 - \rho_0)\phi, \tag{2.58}$$

$$\frac{1}{\mu_0}\boldsymbol{\nabla} \times \mathbf{B} = \frac{ie\hbar}{m}(\phi^*\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\phi^*) - \frac{4e^2}{m}\phi^*\phi\mathbf{A}, \tag{2.59}$$

which are the time-dependent versions of the Ginzburg-Landau equations (2.48) and (2.49).

2.3.5 Hydrodynamical Analogy

We will review briefly the well known analogy between hydrodynamics and quantum mechanics. The analogy, which shows that one may treat the condensate wave function of e.g. a superfluid as a function describing an ensemble of classical particles subject to classical forces, was first made by Madelung[24] in 1927. The quantum effects are introduced by a "quantum force" (or "quantum potential"), with some uncommon effects. This brief review is mainly based on a review given in the thesis by Myklebust[25].

In general, the Schrödinger equation (linear or non-linear) for particles in an electromagnetic field may be written

$$i\hbar\dot{\psi} = eA_0\psi - \frac{\hbar^2}{2m}\mathbf{D}^2\psi + V\psi, \tag{2.60}$$

where $\mathbf{D} = \nabla - i\frac{e}{\hbar}\mathbf{A}$ and $V = V(\psi, \mathbf{r})$ is some operator for a potential. The only assumption about the potential is that it is only a function of ψ and $\mathbf{r}$.

By defining

$$\psi = \sqrt{\rho}e^{iS}, \tag{2.61}$$

where ρ and S are real functions of $\mathbf{r}$ and by inserting this expression into the Schrödinger equation and separating in imaginary and real parts, one obtains the following equations:

$$-\dot{S} = eA_0 - \frac{\hbar^2}{2m}\left(\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{\sqrt{\rho}} - (\boldsymbol{\nabla}S - \frac{e}{\hbar}\mathbf{A})^2\right) + V(\rho, S, \mathbf{r}) \tag{2.62}$$

$$\dot{\rho} = -\frac{\hbar}{m}\boldsymbol{\nabla}\cdot\left[\rho(\boldsymbol{\nabla}S - \frac{e}{\hbar}\mathbf{A})\right] \tag{2.63}$$

The hydrodynamical analogy is then given by identifying the gradient of the phase of the wave function $\boldsymbol{\nabla}S$ with the momentum[3] $\mathbf{p}$. Since we have a coupling to an electromagnetic field, this means

$$m\mathbf{v} = \mathbf{p} - e\mathbf{A} = \hbar\boldsymbol{\nabla}S - e\mathbf{A}. \tag{2.64}$$

Substituting this into (2.63) gives a continuity equation:

$$\dot{\rho} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0, \tag{2.65}$$

and by taking the gradient of equation (2.62), we get

$$m\dot{\mathbf{v}} + \frac{1}{2}m\boldsymbol{\nabla}(\mathbf{v}^2) + e\left(\dot{\mathbf{A}} + \boldsymbol{\nabla}A_0\right) + \boldsymbol{\nabla}(V + Q) = 0, \tag{2.66}$$

[3] Note that the vector field $\mathbf{p}(\mathbf{r})$ gives the momentum of a particle at $\mathbf{r}$, not the momentum density

where we have introduced the "quantum potential" $Q = -\frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{\sqrt{\rho}}$. We may rewrite this using that $\boldsymbol{\nabla}(\mathbf{v}^2) = 2(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} + 2\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v})$ and using (2.64) again:

$$m\dot{\mathbf{v}} + m(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} + \boldsymbol{\nabla}(V+Q) - \mathbf{F} = 0, \tag{2.67}$$

where $\mathbf{F}$ is the Lorentz force: $\mathbf{F} = e(\mathbf{E}+\mathbf{v}\times\hat{z}B)$. This equation is identical to Euler's equation for a fluid influenced by a force $-\boldsymbol{\nabla}(V+Q)+\mathbf{F}$.

2.4 Vortices

2.4.1 Brief Background on Hydrodynamics

To better understand the properties of vortices in the Ginzburg-Landau theory and in the Chern-Simons-Ginzburg-Landau theory which is the main topic of this thesis, it is necessary to have some knowledge of the vortices found in classical hydrodynamics. We will therefore give a very brief review of the needed concepts before we go on to discuss quantum vortices. This review is based mainly on the review in the thesis of Myklebust[25], who studied vortices in the Ginzburg-Landau theory of liquid Helium in detail.

An *ideal fluid* is characterized by having no viscosity and an infinite thermal conductivity, so that the temperature is constant throughout the fluid. The fundamental equations of hydrodynamics for an ideal fluid are the equation of continuity,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0, \tag{2.68}$$

and Euler's equation[4],

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla}p}{\rho}, \tag{2.69}$$

where ρ is the density of the fluid, $\mathbf{v}$ is the velocity field (i.e. the velocity of the fluid at each point) and p is the pressure.

If the fluid is not only ideal but also isentropic, i.e. the entropy is constant throughout the fluid, the thermodynamical identity

$$dh = Tds + Vdp, \tag{2.70}$$

where h is the enthalpy and $V = \frac{1}{\rho}$ is the specific volume, reduces to

$$dh = \frac{1}{\rho}dp, \tag{2.71}$$

so that Euler's equation (2.69) becomes

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}h, \tag{2.72}$$

[4]Obtained in 1755 by L. Euler

and we obtain the identity

$$\frac{\partial \omega}{\partial t} - \mathbf{\nabla} \cdot (\omega \mathbf{v}) = 0 \tag{2.73}$$

by defining the *vorticity* of the fluid $\omega = \mathbf{\nabla} \times \mathbf{v}$ and taking the curl of (2.72). Equation (2.73) now has the form of a continuity equation and states that vorticity is a conserved quantity, i.e. that the total vorticity in the fluid is constant in time.

If $\omega = 0$ at all points of the fluid, the fluid is said to be *irrotational*. In this case we may write

$$\mathbf{v} = \mathbf{\nabla} \phi \tag{2.74}$$

since $\mathbf{\nabla} \times \mathbf{v} = \omega = 0$, which defines (up to a constant) the *velocity potential* ϕ of the fluid. An irrotational fluid is said to have *potential flow* since it allows this definition.

2.4.2 Definition of a Vortex in Classical Hydrodynamics

We now have the background to define what we mean by a vortex in a 2-dimensional fluid. The definition relies on the observation that in physical fluids, the vorticity is localized in small areas, and most of the fluid is irrotational. A *vortex* (or *vortex filament*) is now defined as such an enclosed filament of the fluid. Outside of vortices, the fluid is irrotational, and describes potential flow. In a 3-dimensional fluid, a vortex is actually a tube. Defining the *strength* of the vortex κ as $\kappa = \int \omega dS$, there are three properties fundamental to vortex motion in an ideal, isentropic fluid[25]:

Theorem 2.1 ***a)*** *The same fluid particles constitute a vortex at all times.* ***b)*** *The strength $\kappa = \int \omega dS$ of a vortex is constant in time.* ***c)*** *The strength of a vortex is constant throughout the tube.*

Only the first two parts of the theorem make sense for a 2D fluid, of course. These are known as Helmholtz' theorems. However, the third part tells us that it may make sense to consider a 3D fluid as effectively 2-dimensional.

An important idealization of a vortex is the *point vortex*, a vortex with zero area. A velocity field generating one such vortex is

$$\mathbf{v} = \frac{\kappa}{2\pi r} \hat{\mathbf{e}}_\theta. \tag{2.75}$$

This field has the properties that $\mathbf{\nabla} \times \mathbf{v} = 0$ except at the origin, and $\int \mathbf{\nabla} \times \mathbf{v} d^2 r = \lim_{r \to \infty} \int \frac{\kappa}{2\pi r} r d\theta = \kappa$. A fluid containing only point vortices will have potential flow almost everywhere.

2.4.3 Energy and Movement of Classical Vortices

For a classical system of (point) vortices in an incompressible liquid, it is possible to show that the energy is given by[25]

$$E = \frac{\rho}{4\pi} \left(\sum_{i=1}^{N} \kappa_i \right)^2 \ln \frac{R}{a} - \frac{\rho}{2\pi} \sum_{i<j} \kappa_i \kappa_j \ln \frac{|\mathbf{R}_i - \mathbf{R}_j|}{a}, \tag{2.76}$$

where R is the length scale of the whole system and a is a cutoff of the order of the vortex core radius. The first term in this expression is the kinetic energy from the rotating fluid associated with each vortex, summed, while the second term may be interpreted as the interaction of the vortices. Note that this interaction does not come from an electrical charge of the vortices (which are neutral), but has the same form as a Coloumb interaction in two dimensions.

Using this value as a Hamiltonian, one may deduce the equations of motion for a system of vortices. For a pair of vortices it is useful to define the guiding center and relative coordinates

$$\mathbf{R}_{\text{gc}} = \mathbf{R}_1 + \mathbf{R}_2 \qquad \mathbf{R}_{\text{rel}} = \mathbf{R}_1 - \mathbf{R}_2 \tag{2.77}$$

For a pair of vortices with equal strength κ one then finds the following equations of motion:

$$\dot{X}_{\text{gc}} = 0 \qquad \dot{Y}_{\text{gc}} = 0 \tag{2.78}$$

$$\dot{X}_{\text{rel}} = -\frac{\kappa}{\pi}\frac{Y_{\text{rel}}}{\mathbf{R}_{\text{rel}}^2} \qquad \dot{Y}_{\text{rel}} = \frac{\kappa}{\pi}\frac{X_{\text{rel}}}{\mathbf{R}_{\text{rel}}^2} \tag{2.79}$$

These equations describe circular motion around a fixed point (the stationary guiding center) with an angular velocity of

$$\Omega = \frac{\kappa}{\pi r^2}, \tag{2.80}$$

where $r = |\mathbf{R}_{\text{rel}}|$ is the constant separation of the vortices.

For a pair of vortices with opposite vorticity (a vortex–anti-vortex pair) the equations of motion will be

$$\dot{X}_{\text{gc}} = \frac{\kappa}{\pi}\frac{Y_{\text{rel}}}{\mathbf{R}_{\text{rel}}^2} \qquad \dot{Y}_{\text{gc}} = -\frac{\kappa}{\pi}\frac{X_{\text{rel}}}{\mathbf{R}_{\text{rel}}^2} \tag{2.81}$$

$$\dot{X}_{\text{rel}} = 0 \qquad \dot{Y}_{\text{rel}} = 0 \tag{2.82}$$

The vortices will not move relative to each other, but will follow a straight line perpendicular to the line connecting the two vortices.

2.4.4 Vortices in Ginzburg-Landau Theory

In 1949, L. Onsager[26] proposed that circulation in (superfluid) Helium II was *quantized* with the quantum of circulation being $\frac{h}{m}$. Quantization of vorticity was also proposed and discussed by Feynman[27] and a quantized line was observed by Vinen[28, 29] in 1958.

The quantization of vorticity in He-II is easy to understand if we apply the Ginzburg–Landau theory[5]. We know from section 2.3.5 that the velocity field of

[5] The introduction in this section is based on Nozières and Pines[20], Tilley and Tilley[16] and Myklebust[25]

a superfluid described by a GL wave function $\phi = \sqrt{\rho}e^{iS}$ can be written

$$\mathbf{v} = \frac{\hbar}{m}\boldsymbol{\nabla}S, \tag{2.83}$$

so that the circulation around a closed path C becomes

$$\kappa = \oint_C \mathbf{v}\cdot d\mathbf{l} = \frac{\hbar}{m}\oint_C \boldsymbol{\nabla}S\cdot d\mathbf{l} = \frac{\hbar}{m}\Delta S, \tag{2.84}$$

where ΔS is the change in the phase of the wave function as one moves around the closed path C. The wave function ϕ must be single valued, so that ΔS must be an integer multiply of 2π. We thus find

$$\kappa = \frac{h}{m}s, \qquad s = 0, \pm1, \pm2, \ldots. \tag{2.85}$$

For quantized vortices, the integer s is usually referred to as the vorticity. Note that this definition of vorticity is not exactly the same as the vorticity used in classical hydrodynamics, $\omega = \boldsymbol{\nabla}\times\mathbf{v}$.

The only rotationally invariant wave function having the property (2.85) is

$$\phi(\mathbf{r}) = f(r)e^{is\theta}, \tag{2.86}$$

where $f(r)$ is an undetermined function, and so we expect a basic quantized vortex to have this form. This wave function produces the same velocity field as the classical point vortex (equation (2.75)):

$$\mathbf{v} = \frac{\kappa}{2\pi r}\hat{\mathbf{e}}_\theta. \tag{2.87}$$

One can easily find the energy for a single vortex with this velocity field if ignoring the energy associated with the interactions between fluid particles. The energy is then given by the kinetic energy[20],

$$E = \int \frac{1}{2}m\mathbf{v}^2 d^2r = \frac{\hbar^2\pi}{m}s^2\ln\frac{R}{\xi}, \tag{2.88}$$

where we use the coherence length ξ as a cutoff to avoid the logarithmic divergence near the vortex core, and R is the radius of the fluid system. The coherence length is a natural cutoff since it describes the typical length of variations in the fluid.

Inserting the expected vortex form (2.86) into the dimensionless GPG equation (2.53), we find the following equation for the function $f(r)$:[25]

$$\frac{d^2f}{dr^2} + \frac{df}{rdr} + \left(2 - \frac{s^2}{r^2}\right)f - 2f^3 = 0. \tag{2.89}$$

This equation can only be solved numerically. A graphical presentation of the result is given in figure 2.7. From equation (2.89) we note that in Ginzburg-Landau theory, the form of the vortex depends only on s^2, i.e. not on the sign of s.

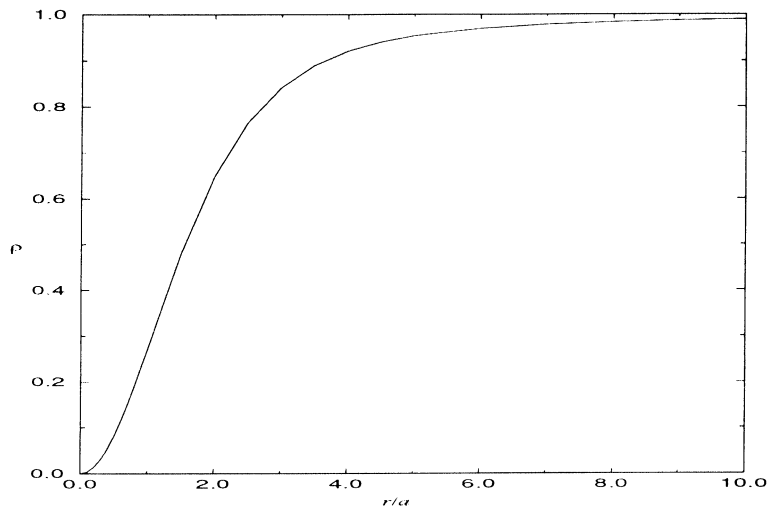


Figure 2.7: Density profile $\rho = f^2$ for a Ginzburg-Landau $s = 1$ vortex. From Myklebust[25], based on data by Kawatra and Pathria[30]. a is the coherence length.

A related phenomenon is the occurrence of quantized flux lines in superconductors. Superconductors are described by a GL theory with nonzero electromagnetic field $\mathbf{A}$. Instead of equation (2.83) above, we therefore have

$$\hbar \boldsymbol{\nabla} S = m\mathbf{v} + q\mathbf{A}, \tag{2.90}$$

giving the quantization condition

$$\oint m\mathbf{v} \cdot d\mathbf{l} - \oint q\mathbf{A} d\mathbf{l} = nh. \tag{2.91}$$

By using the Maxwell equation (2.49), which we may write as

$$-\hat{z} \times \boldsymbol{\nabla} B = \mu_0 \mathbf{j} = \mu_0 q |\psi|^2 \mathbf{v}, \tag{2.92}$$

and by taking the curl of equation (2.90), we find the following equation for B:

$$\lambda^2 \frac{1}{r} \frac{d}{dr} \left(r \frac{dB}{dr} \right) - B = 0, \tag{2.93}$$

where $\lambda = \sqrt{\frac{m}{\mu_0 q^2 \rho}}$ is the *London penetration depth*[16][6] (Note that this λ is not related to the parameter λ defined previously.) This equation may be solved analyt-

[6] The London penetration depth is the typical length of variations in the magnetic field, as opposed to the correlation length, which is the typical length of variations in the particle density

ically, giving

$$B = \frac{n\hbar}{\lambda q} K_0(\frac{r}{\lambda}), \tag{2.94}$$

$$j = -\frac{n\hbar}{\lambda^3 q \mu_0} K_1(\frac{r}{\lambda}), \tag{2.95}$$

where K_1 is a modified Bessel function of the second kind. The coefficient of B has been determined by the boundary condition (2.91). Since the modified Bessel functions decay exponentially[7], $K_1(\frac{r}{\lambda}) \propto e^{-\frac{r}{\lambda}}$, equation (2.91) may be simplified for a path with a large radius to

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int B dA = \frac{nh}{q}, \tag{2.96}$$

giving that the magnetic flux through the vortex is quantized with the *flux quantum* being $\phi_0 = \frac{h}{q}$.

The realization that flux lines are quantized has been very important for the understanding of type-II superconductivity. A. A. Abrikosov[23] showed that vortices may arrange themselves in a regular lattice (Abrikosov considered a square lattice, while W. H. Kleiner et. al.[31] considered the later observed triangular lattice), a behavior that has been experimentally confirmed[32].

2.5 Maxwell-Chern-Simons Theory

Maxwell-Chern-Simons (MCS) theory is an extension to regular Maxwell theory (i.e. electromagnetism) that is only possible in two dimensions. The theory describes anyons as sources for a field somewhat similar to the regular electromagnetic field, in the same way as charged particles are sources for the electromagnetic field in regular Maxwell theory. The theory got its name from a paper by Chern and Simons[33] from 1971. MCS theory has been extensively studied in the thesis by Løvvik[34], and we will take the results we need from there.

The relation between MCS theory and Maxwell theory is very much the same as the relation between Chern-Simons-Ginzburg-Landau theory and Ginzburg-Landau theory. This will be further established in the following chapters. Since MCS theory is considered useful for describing a system of anyons, and since the vortices in CSGL theory are indeed anyons, it might be interesting and useful to have a look at the relations between these two theories.

The theory is generated by adding a *Chern Simons term* $\frac{\mu}{2}\epsilon^{\mu\nu\sigma}A_\mu\partial_\nu A_\sigma$ to the regular Maxwell Lagrangian. Thus the Lagrangian for MCS theory is[34]

$$\mathcal{L}_{\text{MCS}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu + \frac{\mu}{2}\epsilon^{\mu\nu\sigma}A_\mu\partial_\nu A_\sigma. \tag{2.97}$$

[7] Properties of the modified Bessel functions are summarized in appendix B

Here, j^μ is an external current field and μ is a free parameter of dimension length. The current j^μ is the current of the anyons in the theory.

From this Lagrangian, one obtains the following field equations:

$$\partial_i E^i - \mu B = \rho \tag{2.98}$$
$$\epsilon^{ij}\partial_j B - \partial_0 E^i - \mu\epsilon^{ij}E^j = j^i, \tag{2.99}$$

where we have introduced the "physical" fields $E^i = -\partial_i A^0 - \partial_0 A^i$ and $B = \epsilon^{ij}\partial_i A^j$. These definitions also lead to a Faraday's law,

$$\epsilon^{ij}\partial_i E^j = -\partial_0 B. \tag{2.100}$$

These equations are quite similar to the equations of electromagnetism, we see that in the limit $\mu \to 0$, we regain Maxwell's equations. However, for finite μ it is found that the solutions of the equations are quite different from the solutions of Maxwell's equations.

The most important solutions for us will be the fields from point particles. It is easy to find the fields from a static point particle by letting $j^0 = q\delta(\mathbf{r})$ in the equations above. The solution is found[34] to be

$$B(\mathbf{r}) = -\frac{\mu q}{2\pi}K_0(\mu r) \tag{2.101}$$
$$\mathbf{E}(\mathbf{r}) = \frac{\mu q}{2\pi}K_1(\mu r)\hat{\mathbf{e}}_r, \tag{2.102}$$

where K_0 and K_1 are modified Bessel functions and $\hat{\mathbf{e}}_r$ is a unit vector in the radial direction. There are two major differences between this solution and the corresponding solution to Maxwell's equations, which in two dimensions is[34]

$$B(\mathbf{r}) = 0 \tag{2.103}$$
$$\mathbf{E}(\mathbf{r}) = \frac{q}{2\pi r}\hat{\mathbf{e}}_r. \tag{2.104}$$

First, we see that while the Maxwell B-field vanishes for a point particle, the MCS solution has both fields B and $\mathbf{E}$ non-zero. Second, the MCS fields are exponentially damped (some properties of the Bessel functions are given in appendix B), while the electric Maxwell field goes as $\frac{1}{r}$ as $r \to \infty$. The exponential damping means that two MCS particles separated far from each other do not interact.

For particles in motion in regular electromagnetism, the solution to Maxwell's equations is the *retarded potential* well known from classical physics. An analogous solution is MCS theory has been found by Løvvik[34]. This is the solution to the above equations with $j^\mu = q\beta^\mu(t)\delta^{(2)}((r) - \mathbf{R}(t))$, where $\mathbf{R}(t)$ is the path of the particle and β^μ is the 3-velocity given by $\beta^\mu = (1, \dot{\mathbf{R}}(t))$. The solution is found by using the propagator of MCS theory, which is given by[34]

$$D^{\mu\nu}_{\text{MCS}}(k) = \frac{g^{\mu\nu}}{-k^2+\mu^2} - \frac{k^\mu k^\nu}{k^2(-k^2+\mu^2)} + \frac{i\mu\epsilon^{\mu\nu\lambda}k_\lambda}{k^2(-k^2+\mu^2)} - \frac{\alpha k^2}{(k^2)^2}, \tag{2.105}$$

where α is a gauge-fixing constant. The "retarded" potential is found[34] to be

$$A_\nu(\mathbf{r}) = \mathfrak{I}_\nu(\mu) + \frac{1}{\mu}\epsilon_{\mu\nu\lambda}\partial^\mu \left(\mathfrak{I}^\lambda(0) - \mathfrak{I}^\lambda(\mu)\right), \tag{2.106}$$

where

$$\mathfrak{I}^\lambda(\mu) = \int_{-\infty}^{\infty} \frac{\theta(t-t'-|\mathbf{r}-\mathbf{R}(t')|)\cos\left(\mu\sqrt{(t-t')^2-|\mathbf{r}-\mathbf{R}(t')|^2}\right)}{2\pi\sqrt{(t-t')^2-|\mathbf{r}-\mathbf{R}(t')|^2}} q\beta^\lambda(t')dt'. \tag{2.107}$$

The Maxwell-Chern-Simons theory may be rescaled to dimensionless variables, and is found to contain no free dimensionless parameters. One possible rescaling is

$$x^\nu = \frac{\hat{x}^\nu}{\mu} \quad j^\nu = \mu^2 \hat{\jmath}^\nu \quad A^\nu = \hat{A}^\nu. \tag{2.108}$$

If we also rescale the Lagrangian according to $\mathcal{L} = \mu^2\hat{\mathcal{L}}$, we find

$$\hat{\mathcal{L}}_{\mathrm{MCS}} = -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \hat{\jmath}^\mu\hat{A}_\mu + \frac{1}{2}\epsilon^{\mu\nu\sigma}\hat{A}_\mu\hat{\partial}_\nu\hat{A}_\sigma. \tag{2.109}$$

We see that the rescaling is equivalent to setting $\mu = 1$. The rescaled static point particle fields become

$$\hat{B}(\hat{\mathbf{r}}) = -\frac{q}{2\pi}K_0(\hat{r}) \tag{2.110}$$

$$\hat{\mathbf{E}}(\hat{\mathbf{r}}) = \frac{q}{2\pi}K_1(\hat{r})\hat{\mathbf{e}}_{\hat{r}}. \tag{2.111}$$

Chapter 3

Chern-Simons-Ginzburg-Landau Theory

3.1 Background

3.1.1 The Need for an Effective Theory

The fractional quantum Hall effect is a remarkable example of quantum effects observable on a macroscopic level. Other examples include superfluidity and superconductivity. The latter phenomena have been successfully described by the Ginzburg–Landau effective field theory, and there are many similarities between superfluidity/-conductivity and the FQHE. All these systems have a ground state with a constant non-zero density of particles, interaction between particles is important to describe the full system, and there are quasiparticle excitations (vortices). If one could describe the FQHE by an effective field theory, one might hope to produce a better understanding and description of the quantum Hall phenomenon.

There are also some very important aspects of the FQHE which are not present in the GL theories. The most important ones[10] are these: There is a gap in the spectrum which leads to the *incompressibility* of the FQHE system; vortices (quasiparticles) have *finite energy*, as opposed to the vortices in He-II; and vortices have *fractional charge*.

In 1988, Zhang, Hansson and Kivelson[35] and Read[36] proposed a Ginzburg–Landau theory for the fractional quantum Hall effect based on the introduction of a Chern–Simons term into the Lagrangian for GL theory. The theory has been quite successful in describing various properties of the FQHE (for a review, see e.g. [37]).

3.1.2 Motivation

In this section we will make an attempt to motivate the form of the Lagrangian of the Chern Simons Ginzburg Landau (CSGL) theory. It is not the purpose to show a derivation from microscopical principles, but to aid the understanding of the theory by giving a brief overview of the background.

We will start this motivation by showing how one with a singular gauge transformation can turn a fermionic (indeed, anyonic) wave function into a bosonic one: Imagine a system of two particles, such that interchanging them multiplies the wave function by a phase factor $e^{i\theta}$. Working in complex relative coordinates, where $z = x + iy$, this may be expressed as

$$\psi(e^{i\pi}z) = e^{i\theta}\psi(z). \tag{3.1}$$

(The notation does not imply that ψ is an analytical function of z. Indeed, for anyons, the wave function will in general be multi-valued.) We want to describe the same system with a bosonic wave function $\phi(z) = \phi(e^{i\pi}z)$. By assuming that the two wave functions are related by a gauge transformation $\phi(z) = e^{i\eta(z)}\psi(z)$, we get

$$e^{i\eta(z)}\psi(z) = \phi(z) = \phi(e^{i\pi}z) = e^{i\eta(e^{i\pi}z)}e^{i\theta}\psi(z), \tag{3.2}$$

$$\eta(z) - \eta(e^{i\pi}z) = \theta. \tag{3.3}$$

It may be easily verified that the solution to this equation is

$$\eta(z) = -\frac{\theta}{\pi}\mathrm{Arg}(z). \tag{3.4}$$

When we perform a gauge transformation of the wave function of the form $\psi \to e^{i\eta}\psi$, we must also transform the electromagnetic field according to $eA^{\mu} \to eA^{\mu} + \partial^{\mu}\eta$. The gauge field corresponding to this transformation is given by

$$\mathbf{a}_{(\mathrm{rel})}(\mathbf{r}) = -\frac{1}{e}\boldsymbol{\nabla}\eta(\mathbf{r}) = \frac{\theta}{\pi e}\frac{\hat{z}\times\mathbf{r}}{r^2}, \tag{3.5}$$

giving a "magnetic field"

$$b_{(\mathrm{rel})}(\mathbf{r}) = -\frac{\theta}{\pi e}\boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{r^2} = -\frac{2\theta}{e}\delta^{(2)}(\mathbf{r}). \tag{3.6}$$

The label '(rel)' is added here to show that these fields are given in relative coordinates. We would, however, like to express the fields in absolute coordinates. For two particles, we would then want the field from one particle at the position of the second to equal $\mathbf{a}_{(\mathrm{rel})}(\mathbf{r}_1 - \mathbf{r}_2)$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the particles. The field we want is given by

$$\mathbf{a}(\mathbf{r}) = \mathbf{a}_{(\mathrm{rel})}(\mathbf{r} - \mathbf{r}_1) + \mathbf{a}_{(\mathrm{rel})}(\mathbf{r} - \mathbf{r}_2), \tag{3.7}$$

when we ignore "self-interaction" terms of the kind $\mathbf{a}_{(\mathrm{rel})}(0)$. The magnetic field becomes

$$b(\mathbf{r}) = -\frac{2\theta}{e}\left(\delta^{(2)}(\mathbf{r} - \mathbf{r}_1) + \delta^{(2)}(\mathbf{r} - \mathbf{r}_2)\right). \tag{3.8}$$

This formula may be generalized to any number of particles in the form

$$b(\mathbf{r}) = -\frac{2\theta}{e}\sum_{i=1}^{N}\delta^{(2)}(\mathbf{r} - \mathbf{r}_i) = -\frac{2\theta}{e}\rho(\mathbf{r}), \tag{3.9}$$

where $\rho(\mathbf{r})$ is the density of the particles. In the following section, we will show that the CSGL Lagrangian produces exactly this result.

3.1.3 Field Equations

The CSGL Lagrangian density is given by[35]

$$\mathcal{L} = i\phi^*(\partial_0 + ieA_0 + iea_0)\phi + \frac{1}{2m}\phi^*(\partial_i + ieA_i + iea_i)^2\phi - \frac{\lambda}{2}(\phi^*\phi - \rho_0)^2 + \frac{\mu}{2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma, \quad (3.10)$$

where $A^\mu = (A^0, \mathbf{A})$. A^μ represents an external field, and a^μ is a "statistical" gauge field. From this expression we see that there are five parameters to the theory: e, m, λ, ρ_0 and μ. The first two of these are the effective charge and mass, respectively, in the interface where the effect occurs. Not that in particular, the mass parameter m is here to be understood as an effective mass not equal to the electron mass[1] m_e. The effective electron charge e is not an important parameter, since it may be "absorbed" by a redefinition of the electromagnetic field A_μ. The parameters λ and ρ_0 arise in the same fashion as the parameters with the same names in Ginzburg–Landau theory, as discussed in sections 2.3.3 and 2.3.4. These parameters therefore reflect that the CSGL theory is an effective theory, and they must be determined on a phenomenological basis. One could imagine a more "realistic" model by replacing the third term in 3.10 by a term including a potential $V(\mathbf{r}-\mathbf{r}')$, but it is not obvious what this potential should look like. The last parameter, μ will be shown below to be connected to the fact that the electrons described by CSGL theory are fermions, as opposed to the bosons described by GL theory.

Variation of (3.10) with respect to ϕ^* gives

$$-(i\partial_0 - eA_0 - ea_0)\phi = \frac{1}{2m}(\partial_i + ieA_i + iea_i)^2\phi - \lambda(\phi^*\phi - \rho_0)\phi. \quad (3.11)$$

This is just the Ginzburg–Pitaevskii–Gross field equation, or the nonlinear Schrödinger equation (see section 2.3.4), for a system of bosons in an electromagnetic field $A^\mu + a^\mu$.

Variation with respect to a_0 gives

$$\mu\epsilon^{ij}\partial_i a_j = e\phi^*\phi = e\rho. \quad (3.12)$$

As announced, this is exactly equation (3.9), which we wanted for a system of anyons, when we define

$$\mu = \frac{e^2}{2\theta}. \quad (3.13)$$

Equation (3.12) together with (3.11) has an interesting interpretation: In two dimensions we may look at anyons as bosons with an attached magnetic flux equal to $-\frac{e}{\mu}$. For fermions, we have $\mu = \frac{e^2}{2\pi(1+2n)}$, $n = 1, 2, 3, \ldots$, and the magnetic flux attached to each particle becomes $-(1+2n)\frac{2\pi}{e}$.

[1] In fact, is has been argued[38, 35, 36] that the parameter m is not connected to the electron mass at all, but depend only on the Coloumb interaction between the electrons.

Finally, variation with respect to a_i gives,

$$\mu\epsilon^{ij}(\partial_j a_0 - \partial_0 a_j) = ej^i, \tag{3.14}$$

where $j^i = \frac{1}{2mi}\{\phi^*(\partial_i + iea_i + ieA_i)\phi - [(\partial_i + iea_i + ieA_i)\phi]^*\phi\}$. This equation states that the statistical "electric" field $e^i = -\partial_i a_0 + \partial_0 a_i$ may be identified with the particle current density j^i.

3.1.4 Energy

The energy density of a CSGL system may be derived in the standard fashion from the Lagrangian density (3.10):

$$\mathcal{E} = \tilde{\phi}\frac{d\phi}{dt} + \tilde{a}^\mu\frac{da_\mu}{dt} - \mathcal{L}. \tag{3.15}$$

where the canonical conjugate fields $\tilde{\phi}$ and $\tilde{a}_\mu$ are given by

$$\tilde{\phi} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = i\phi^*, \tag{3.16}$$

$$\tilde{a}^\mu = \frac{\partial\mathcal{L}}{\partial a_{\mu,0}} = -\frac{\mu}{2}\epsilon^{\nu 0\mu}a_\nu. \tag{3.17}$$

After some algebra, we end up with

$$\mathcal{E} = \phi^*(eA_0 + ea_0)\phi - \frac{1}{2m}\phi^*\mathbf{D}^2\phi + \lambda(\rho-\rho_0)^2 + \mu\epsilon^{ij}a_0 a_{j,i}. \tag{3.18}$$

where we have used that $\epsilon^{\mu\nu\sigma}a_\mu a_{\sigma,\nu} = \epsilon^{ij}(a_i a_{0,j} - a_i a_{j,0} + a_0 a_{j,i})$. By ignoring a surface term, we may rewrite the above as

$$\mathcal{E} = \frac{1}{2m}|\mathbf{D}\phi|^2 + a_0(e\rho + \mu\epsilon^{ij}a_{j,i}) + eA_0\rho + \frac{\lambda}{2}(\rho-\rho_0)^2. \tag{3.19}$$

From (3.12) we see that the term proportional to a_0 vanishes and we are left with

$$\mathcal{E} = \frac{1}{2m}|\mathbf{D}\phi|^2 + eA_0\rho + \frac{\lambda}{2}(\rho-\rho_0)^2. \tag{3.20}$$

Note that this is just the energy from a Ginzburg-Landau system in an electromagnetic field $A^\mu + a^\mu$, but that a^0 does not contribute to the energy. Also note that for the QHE, A^0 is normally 0 since the external field is purely magnetic and static.

3.2 Simple Solutions

3.2.1 Ground State

From (3.20) we see that if we could find a solution such that $\mathbf{D}\phi = 0$ and $\rho = \rho_0$, then this would surely be the ground state of the system. By letting $\phi(x) = \sqrt{\rho_0}e^{iS(x)}$, we find that this requirement gives $\partial_i S + eA_i + ea_i = 0$. For constant solutions, we must then have $a_i = -A_i$. However, equation (3.12) tells us that $\epsilon^{ij}\partial_i a_j = \frac{e}{\mu}\rho_0$. Thus, a constant ground state is only possible for external fields such that

$$B^{\text{ext}} = -\epsilon^{ij}\partial_i A_j = \frac{e}{\mu}\rho_0. \tag{3.21}$$

This may be understood as follows: Each particle carries a magnetic flux of strength $\frac{e}{\mu}$, according to (3.12). In this ground state, the density of particles is constant and exactly the right size to cancel the external field A^i. Thus, the particles feel no net magnetic field and behave as bosons interacting with each other via the ϕ^4-type interaction.

When (3.21) does not hold, it is still possible to find a constant solution

$$\rho = \frac{\mu}{e}B^{\text{ext}}. \tag{3.22}$$

This solution must have $ea_0 = \lambda(\rho - \rho_0)$, as can be readily seen from equation (3.11). As noted by Curnoe and Weiss[39], a change in ρ_0 (or equivalently B^{ext}) can always be cancelled by a constant change in a_0, so that in this sense solutions of the field equations are *independent* of B^{ext}. The ground state, however, need not be the same for different values of B^{ext}. But in fact, there will be a "window" where B^{ext} can change around $\frac{e}{\mu}\rho_0$, where the ground state of the system is the constant solution (3.22)[39, 40]. The width of this window is determined by the creation energy of a vortex and an anti-vortex by[40]

$$\Delta\rho_0 = \frac{\epsilon^{\text{v}}(\rho_0) + \epsilon^{\text{av}}(\rho_0)}{2\pi\lambda}, \tag{3.23}$$

where $\epsilon^{\text{v}}(\rho_0)$ and $\epsilon^{\text{av}}(\rho_0)$ are the creation energies for a vortex and an anti-vortex, respectively.

3.2.2 Plane Waves

By using the hydrodynamical analogy of section 2.3.5, it is possible to find a dispersion relation for plane waves in a CSGL system. To find plane wave solutions, we proceed by expanding around the ground state $\rho = \rho_0, \mathbf{v} = 0$, assuming the deviation from the ground state is small:

$$\rho = \rho_0 + \delta\rho, \tag{3.24}$$

$$\mathbf{v} = \delta\mathbf{v}. \tag{3.25}$$

We also insert the potential $V = \lambda(\rho - \rho_0)$ and equation (3.21) into (2.67) and linearize (2.65) and (2.67) in $\delta\rho$ and $\delta\mathbf{v}$, arriving at [2]

$$\frac{\partial}{\partial t}\delta\rho + \rho_0 \boldsymbol{\nabla}\delta\mathbf{v} = 0, \tag{3.26}$$

$$m\frac{\partial}{\partial t}\delta\mathbf{v} + \lambda\boldsymbol{\nabla}\delta\rho - \frac{1}{4m\rho_0}\boldsymbol{\nabla}\boldsymbol{\nabla}^2\delta\rho - \frac{e^2\rho_0}{\mu}\delta\mathbf{v}\times\hat{z} = 0. \tag{3.27}$$

We now make the assumption that $\delta\rho$ and $\delta\mathbf{v}$ describe plane waves,

$$\delta\rho = \delta\rho_0 e^{i(\mathbf{kx}-\omega t)}, \tag{3.28}$$

$$\delta\mathbf{v} = \delta\mathbf{v}_0 e^{i(\mathbf{kx}-\omega t)}, \tag{3.29}$$

where a phase difference between the components of $\delta\mathbf{v}$ and between $\delta\mathbf{v}$ and $\delta\rho$ are allowed through $\delta\mathbf{v}_0$, which is assumed to be complex. We also introduce the longitudinal and transverse components of $\delta\mathbf{v}$,

$$\mathbf{k}\cdot\delta\mathbf{v} \equiv k\delta v_{\parallel}, \tag{3.30}$$

$$\mathbf{k}\times\delta\mathbf{v} \equiv k\delta v_{\perp}. \tag{3.31}$$

Inserting these relations into equation (3.27) and taking the transverse component, we find the relation

$$-im\omega\delta v_{\perp} + \frac{e^2\rho_0}{\mu}\delta v_{\parallel} = 0, \tag{3.32}$$

which tells us that the longitudinal and transverse components of $\delta\mathbf{v}$ in general have a phase difference of 90°. In addition, as $\frac{\mu}{me^2\rho_0} \to \infty$, the transverse component $\delta v_{\perp}$ disappears.

Inserting our plane waves into equation (3.26), we find a relation between the density and velocity of the particles creating the waves:

$$\rho_0 k\delta v_{\parallel} = \omega\delta\rho. \tag{3.33}$$

Inserting the two preceding relations into the longitudinal component of equation (3.27), we find the dispersion relation,

$$\omega^2 = \frac{\lambda\rho_0}{m}k^2 + \frac{1}{4m^2}k^4 + \frac{e^4{\rho_0}^2}{m^2\mu^2}. \tag{3.34}$$

If we omit the k^4-term (i.e. going to a long wavelength limit), this dispersion relation tells us that the plane waves act like a field of particles with mass $\frac{e^2\rho_0}{m\mu}$. Again using equation (3.21), we see that the mass is identical to the cyclotron frequency $\omega_C = \frac{eB^{\text{ext}}}{m}$ defined in section 2.2.2.

This mass is associated with the gap in the spectrum that was discussed in the introduction. The fact that the plane wave solutions have a mass, shows that there is a gap, so that a finite energy is needed to create excitations to the system. This is also the same as saying that the electrons describe an incompressible quantum liquid, since the liquid can not be deformed by small perturbations.

[2] We have used equations (3.12) and (3.14) to write the Lorentz force as $\mathbf{F} = e(\boldsymbol{\mathcal{E}} + \mathbf{v}\times\hat{z}\boldsymbol{\mathcal{B}}) \approx \frac{e^2\rho_0}{\mu}\delta\mathbf{v}\times\hat{z}$.

3.3 Rescaling to Dimensionless Variables

The Lagrangian (3.10) allows one to perform a rescaling of the fields $\phi(x)$, $A_{\mu}(x)$, and $a_\mu(x)$ together with x to produce a new Lagrangian only dependent on one dimensionless parameter. Substituting

$$\begin{array}{ll} t = \frac{\hat{t}}{\lambda\rho_0} & \mathcal{A}_0 = \frac{\lambda\rho_0}{e}\hat{\mathcal{A}}_0 \\ r^i = \frac{\hat{r}^i}{\sqrt{\lambda m\rho_0}} & \mathcal{A}_i = \frac{\sqrt{\lambda m\rho_0}}{e}\hat{\mathcal{A}}_i \\ \mu = \frac{e^2}{m\lambda}\hat{\mu} & \phi = \sqrt{\rho_0}\hat{\phi} \end{array} \tag{3.35}$$

into (3.10) and rescaling the Lagrangian according to $\hat{\mathcal{L}} = \frac{\mathcal{L}}{\rho_0^2\lambda}$, gives us a new Lagrangian

$$\hat{\mathcal{L}} = i\hat{\phi}^*(\hat{\partial}_0 + i\hat{A}_0 + i\hat{a}_0)\hat{\phi} + \frac{1}{2}\hat{\phi}^*(\hat{\partial}_i + i\hat{A}_i + i\hat{a}_i)^2\hat{\phi} - \frac{1}{2}(\hat{\phi}^*\hat{\phi} - 1)^2 + \frac{\hat{\mu}}{2}\epsilon^{\mu\nu\sigma}\hat{a}_\mu\hat{\partial}_\nu\hat{a}_\sigma, \tag{3.36}$$

which will be used in the rest of this chapter. We will omit the marks on the fields except when necessary to distinguish dimensional and dimensionless quantities. With the Lagrangian (3.36), the field equations become

$$iD_0\phi + \frac{1}{2}D_i^2\phi - (\phi^*\phi - 1)\phi = 0 \tag{3.37}$$

$$\mu\epsilon^{ij}\partial_i a_j = \phi^*\phi \tag{3.38}$$

$$\mu\epsilon^{ij}(\partial_j a_0 - \partial_0 a_j) = j^i. \tag{3.39}$$

The dimensionless parameter $\hat{\mu}$ is given in terms of the statistics parameter θ as

$$\hat{\mu} = \frac{m\lambda}{2\theta}, \tag{3.40}$$

by using equation (3.13). It can also be formulated in terms of the external magnetic field by using equation (3.21),

$$\hat{\mu} = \frac{\lambda m\rho_0}{eB^{\text{ext}}}. \tag{3.41}$$

The rescaling (3.35) is not unique in respect to creating dimensionless variables. It is obvious that one could define another rescaling $\tilde{\partial}_0 = \alpha\hat{\partial}_0$ etc., where α etc. are dimensionless constants, that would be just as good. Furthermore, since we have a dimensionless parameter (μ) in this system, the constants α etc. might depend on this parameter. As an example, consider applying the following dimensionless rescaling after (3.35):

$$\begin{array}{ll} \hat{t} = \hat{\mu}\tilde{t} & \hat{\mathcal{A}}_0 = \frac{\tilde{\mathcal{A}}_0}{\hat{\mu}} \\ \hat{r}^i = \sqrt{\hat{\mu}}\tilde{r}^i & \hat{\mathcal{A}}_i = \frac{\tilde{\mathcal{A}}_i}{\sqrt{\hat{\mu}}} \end{array} \tag{3.42}$$

If we also scale the Lagrangian, $\tilde{\mathcal{L}} = \hat{\mu}\hat{\mathcal{L}}$, we obtain

$$\tilde{\mathcal{L}} = i\tilde{\phi}^*(\tilde{\partial}_0 + i\tilde{A}_0 + i\tilde{a}_0)\tilde{\phi} + \frac{1}{2}\tilde{\phi}^*(\tilde{\partial}_i + i\tilde{A}_i + i\tilde{a}_i)^2\tilde{\phi} - \frac{\hat{\mu}}{2}(\tilde{\phi}^*\tilde{\phi} - 1)^2 + \frac{1}{2}\epsilon^{\mu\nu\sigma}\tilde{a}_\mu\tilde{\partial}_\nu\tilde{a}_\sigma, \quad (3.43)$$

where we observe that the parameter $\hat{\mu}$ has now moved to another term in the Lagrangian. In the original units, the scaling (3.42) equals $t = \frac{\mu m}{e^2\rho}\tilde{t}$ and $r^i = \sqrt{\frac{\mu}{e^2\rho}}\tilde{r}^i$. Using (3.21), we see that this may be understood as measuring length in units of the magnetic length $\ell = \frac{1}{\sqrt{eB^{\text{ext}}}}$, while we see that the rescaling we will use, (3.35), is equivalent to measuring length in units of $\sqrt{2}\xi$, where $\xi = \frac{1}{\sqrt{2\lambda m\rho_0}}$ is the coherence length defined in section 2.3.4 for Ginzburg–Landau theory. The alternative rescaling (3.42) was e.g. used by Tafelmayer[41].

3.4 Self-Dual Point

A mathematical model such as the one we are discussing here, may sometimes have the property that the equations of motion may be reduced to a simpler set of equations for special values of the parameters. In certain cases, the equations may be reduced to a set of "self-dual" equations, e.g. equations on the form $v_i = i\epsilon^{ij}v_j$ for some quantity v. We will show that for one particular choice of the dimensionless parameter of CSGL theory, equation (3.11) reduces to such an equation.

To find the self-dual point and the self-dual equations for CSGL theory, we will look at the energy density (3.20). This quantity may be rewritten in the following way[41, 42]:

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}|\mathbf{D}\phi|^2 + \frac{1}{2}(|\phi|^2 - 1)^2 \\
&= \frac{1}{2}|(D_1 + iD_2)\phi|^2 + \frac{1}{2}i\epsilon^{ij}(D_i\phi)(D_j\phi)^* + \frac{1}{2}(|\phi|^2 - 1)^2 \\
&= \frac{1}{2}|(D_1 + iD_2)\phi|^2 + \left(\frac{1}{2} - \frac{1}{2\mu}\right)(|\phi|^2 - 1)^2 \\
&\qquad + \frac{1}{2}\epsilon^{ij}\partial_i\left(\frac{1}{2i}(\phi^*\partial_j\phi - \phi\partial_j\phi^*) - \mathcal{A}^j(|\phi|^2 - 1)\right),
\end{aligned} \quad (3.44)$$

where we have used equation (3.12). Integrating to find the total energy, we may use Green's theorem on the last term and find

$$E = \int \frac{1}{2}|(D_1 + iD_2)\phi|^2 d^2x + \frac{\Phi}{2} + \lim_{R\to\infty}\oint_{C_R}\frac{1}{2}j^i dl^i, \quad (3.45)$$

where Φ is the total magnetic flux and C_R is a circular path with radius R. Since we may assume that the current vanishes at infinity, the last integral vanishes. Furthermore, since the total flux by equations (3.12) and (3.21) is given by the charge in the system,

$$\Phi = \int d^2x \epsilon^{ij} \partial_i (A^j + a^j) = \frac{1}{\mu} \int d^2x (1 - |\phi|^2), \tag{3.46}$$

we may assume that this is given, and so, for $\mu = 1$ the energy is minimized for solutions of the equation

$$(D_1 + iD_2)\phi = 0, \tag{3.47}$$

which is the self-duality equation for this system. This equation is also known as the Bogomoln'yi equation.

3.5 Vortex Solutions in CSGL Theory

3.5.1 Single Vortex

We will look for time invariant solutions to (3.37) that have the form of a rotationally invariant vortex located at the origin,

$$\phi(r, \theta) = f(r)e^{is\theta}, \tag{3.48}$$

where $f(r)$ is a real positive function and s is an integer, the *vorticity* of the vortex. Furthermore, we will assume that the external field A^μ is a pure magnetic field as discussed in section 3.2.1. We will choose gauge such that $A^0 = 0$.

The energy of a vortex, given by equation (3.20), should be finite. In dimensionless variables, the energy is given by

$$\mathcal{E} = \frac{1}{2}|\mathbf{D}\phi|^2 + \frac{1}{2}(\rho - 1)^2, \tag{3.49}$$

where we have set $A_0 = 0$. We would like the integral over all space of this expression to be finite, so we must demand[41] that $r^2|\mathbf{D}\phi|^2 \to 0$ and $r^2[(f(r))^2 - 1]^2 \to 0$ as $r \to \infty$. Hence, we get the condition that the total vector field must satisfy

$$\mathbf{A}(\mathbf{r}) + \mathbf{a}(\mathbf{r}) \underset{r\to\infty}{\to} \frac{s}{r}\mathbf{e}_\theta \tag{3.50}$$

more quickly than $\frac{1}{r}$. Here, $\mathbf{e}_\theta$ is a unit vector in the θ direction.

In polar coordinates, equation (3.38) becomes, when adding the external field $B = \frac{1}{\mu}$,

$$\frac{\mu}{r}\left[\frac{\partial}{\partial r}(rA^\theta + ra^\theta) - \frac{\partial(A^r + a^r)}{\partial\theta}\right] = 1 - [f(r)]^2. \tag{3.51}$$

It is obvious that we may choose $a^r(r,\theta) = -A^r(r,\theta)$ and $A^\theta(r,\theta) + a^\theta(r,\theta) = \frac{\alpha(r)}{r}$. We then have to solve the equation for α,

$$\mu\alpha'(r) + r\left[(f(r))^2 - 1\right] = 0. \tag{3.52}$$

The equation for a_0, (3.39), contains the current $\mathbf{j}$, which is defined by

$$\mathbf{j} = \frac{1}{2i}\left[\phi^*\mathbf{D}\phi - (\mathbf{D}\phi)^*\phi\right]. \tag{3.53}$$

In polar coordinates, we have $D_r = \frac{\partial}{\partial r} - iA^r - ia^r = \frac{\partial}{\partial r}$ and $D_\theta = \frac{1}{r}\frac{\partial}{\partial\theta} - iA^\theta - ia^\theta$. The current becomes

$$\begin{aligned} j^r &= 0 \\ j^\theta &= \frac{s-\alpha}{r}(f(r))^2. \end{aligned}$$

Using this and the expression for the curl in polar coordinates, (3.39) becomes

$$\frac{\mu}{r}\frac{\partial a_0}{\partial\theta} = j^r = 0 \tag{3.54}$$

$$-\mu\frac{\partial a_0}{\partial r} = j^\theta = \frac{s-\alpha(r)}{r}[f(r)]^2, \tag{3.55}$$

so we see that a_0 is a function of r only: $a_0(r,\theta) = a_0(r)$ Inserting (3.48) in (3.37), we obtain

$$f''(r) + \frac{1}{r}f'(r) - \left(\frac{s-\alpha}{r}\right)^2 f(r) - 2a_0(r)f(r) + 2\left[1-(f(r))^2\right]f(r) = 0. \tag{3.56}$$

Equations (3.52), (3.55) and (3.56) must be solved to solve the vortex problem (3.48).

3.5.2 Topological Invariants

From elementary quantum mechanics, we are aware of the fact that most quantum numbers follow from the symmetry of a system. E.g. in a rotationally invariant system one knows that angular momentum is quantized. From the rotation group and its generators, one can also deduce the degeneracy of the energy levels in a rotationally invariant system.

But in the case of a quantum vortex, the vorticity is not quantized because of some symmetry but simply because we require that the wave function should be single valued (see the discussion in section 2.4.4.) This requirement causes the vorticity s to be an integer. Furthermore, total vorticity is a topological invariant, since no continuous transformations on the wave function will alter the total vorticity. Vorticity is therefore called a *topological charge*.

For a vortex, there are some quantities that depend only on the value of s. These quantities are thus dictated by the *topology* of the vortex. We know from section 2.4.4 that in the Ginzburg Landau theory for superconductors, the magnetic flux is quantized. By using equation (3.50), we find that the same is true for the *total* magnetic flux in the CSGL theory:

$$\Phi = \int d^2x\epsilon^{ij}\partial_i(a^j + A^j) = 2\pi s. \tag{3.57}$$

In CSGL theory this equation has an implication not present in GL theory. Since there is a simple relation between the statistical electromagnetic field a_μ and the electron density given by equation (3.38), and since the external magnetic field is constant equal to $\epsilon^{ij}\partial_i A^j = \frac{1}{\mu}$ in the rescaled units, we find for the charge of the CSGL vortex

$$Q = \int d^2x(|\phi|^2 - 1) = \mu \int d^2x \epsilon^{ij}\partial_i(a_j + A_j) = -2\pi s\mu. \tag{3.58}$$

Equations (3.58) and (3.57) are true for all vortices, i.e. no matter the detailed form of the vortex, the charge is always the same, and determined by the vorticity s. In dimensional units, the charge translates to[3] $Q = -\frac{\mu}{e}2\pi s = -\frac{e}{1+2n}s$, when we use the fact that the electrons are fermions with $\mu = \frac{e^2}{2\pi(1+2n)}$. Note that the charge of a vortex ($s > 0$) is positive, since the electrons are pushed out to infinity. The charge of an anti-vortex ($s < 0$) is negative, so for an anti-vortex there is an abundance of electrons near the vortex. However, we know that the density of electrons must vanish in the center of the anti-vortex as well.

There is also another important quantity that is topological invariant in the CSGL theory. The angular momentum is given by[43, 44, 41]

$$J = 2\pi \int_0^\infty drr(s - \alpha(r))(1 - |\phi|^2) = -\pi s^2 \tag{3.59}$$

or, in the original dimensional units, $J = \frac{\pi\mu}{e^2} = \frac{1}{2(2n+1)}s^2$. Since the quasiparticles of the quantum Hall effect are believed to have a statistics parameter of $\theta = \frac{\pi}{2n+1}$ (see section 2.2.6), this corresponds to the picture of the $s = \pm 1$ vortices in CSGL theory as the Laughlin quasiparticles assuming that the generalized spin-statistics theorem $J = \frac{\theta}{2\pi}$ holds for anyons[6].

3.5.3 Asymptotic Form of Vortex Outside the Core

The set (3.52), (3.55) and (3.56) of nonlinear differential equations can only be solved numerically. However, in the limit $r \to \infty$, we may approximate the solution by expanding around the values of the fields in this limit. We define

$$f(r) = 1 - \chi(r) \tag{3.60}$$

$$a_0(r) = \psi(r) \tag{3.61}$$

$$\alpha = s - \omega(r), \tag{3.62}$$

where the new functions χ, ψ and ω are all assumed to be small far from the vortex core Inserting these definitions into equations (3.52), (3.55) and (3.56) and linearizing,

[3] In dimensional units, $Q = e\int d^2x(|\phi|^2 - \rho_0) = \frac{\mu}{e}\frac{\hat{Q}}{\hat{\mu}}$, where $\hat{Q}$ is the dimensionless quantity.

we get a new set of equations,

$$\chi'' + \frac{1}{r}\chi' = 4\chi - 2\psi \tag{3.63}$$

$$\mu\omega' + 2r\chi = 0 \tag{3.64}$$

$$\mu\psi' + \frac{\omega}{r} = 0, \tag{3.65}$$

which has the set of solutions[41]

$$\chi = CK_0(\frac{r}{\zeta}) \tag{3.66}$$

$$\omega = \frac{2\zeta Cr}{\mu}K_1(\frac{r}{\zeta}) \tag{3.67}$$

$$\psi = \frac{2C\zeta^2}{\mu^2}K_0(\frac{r}{\zeta}), \tag{3.68}$$

where K_0 and K_1 are modified Bessel functions of the second kind, C is an unknown constant, and ζ is a solution to the equation

$$4\zeta^2(1 - \frac{\zeta^2}{\mu^2}) = 1. \tag{3.69}$$

In the limit $r \to \infty$, this gives $\chi(r) = 1 - f(r)$ the asymptotic form of

$$\chi(r) \underset{r\to\infty}{\to} C\sqrt{\frac{\zeta\pi}{2r}}e^{-\frac{r}{\zeta}}. \tag{3.70}$$

Equation (3.69) for the length scale ζ of the exponential decay of χ is of fourth degree in ζ and therefore has four solutions given by

$$\zeta^2 = \frac{\mu^2}{2}\left(1 \pm \sqrt{1 - \frac{1}{\mu^2}}\right). \tag{3.71}$$

When $|\mu| > 1$, ζ^2 is positive, so ζ may take one of four real values, two of which are positive and two of which are negative. From (3.70) we see that we can only allow positive ζ, or else χ would blow up as $r \to \infty$. We therefore have, for $|\mu| > 1$, two solutions for ζ given by

$$\zeta_1 = \mu\sqrt{\frac{1}{2}\left(1 + \sqrt{1 - \frac{1}{\mu^2}}\right)} \qquad \zeta_2 = \mu\sqrt{\frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{\mu^2}}\right)}. \tag{3.72}$$

In the limit $\mu \gg 1$, we find that these two solutions have quite different character:

$$\zeta_1 \underset{\mu\to\infty}{\to} \mu, \qquad \zeta_2 \underset{\mu\to\infty}{\to} \frac{1}{2}. \tag{3.73}$$

The total solution will be

$$f(r) = 1 - \sqrt{\frac{\pi}{2r}} \left(C_1 \sqrt{\zeta_1} e^{-\frac{r}{\zeta_1}} + C_2 \sqrt{\zeta_2} e^{-\frac{r}{\zeta_2}} \right). \tag{3.74}$$

For large μ, we see from (3.73) that the first exponential will be dominating, since the second one will be small for $r > \frac{1}{4}$. In this limit, we can make an approximation for the asymptotic form of $f(r)$ by

$$f(r) \underset{r\to\infty}{\to} 1 - C\sqrt{\frac{\pi\mu}{2r}} e^{-\frac{r}{\mu}}. \tag{3.75}$$

3.5.4 Inner Core

In the inner core of the vortex, all the fields may be assumed to be small, and the equations may be linearized. We are left with a decoupled set of equations:

$$f'' + \frac{1}{r} f' + \left(2 - \left(\frac{s}{r}\right)^2 \right) f = 0 \tag{3.76}$$

$$\mu\alpha' - r = 0 \tag{3.77}$$

$$\mu a_0' = 0 \tag{3.78}$$

which has the solution

$$f(r) = CJ_s(r) \tag{3.79}$$

$$\alpha(r) = \frac{1}{2\mu} r^2 \tag{3.80}$$

$$a_0(r) = a_0. \tag{3.81}$$

where $J_s(r)$ is the Bessel function of order s, and C and a_0 are unknown constants.

3.5.5 Comparing CSGL and GL Theory

Equation (3.56) is the same as the corresponding equation in Ginzburg–Landau theory, except for the fields a_0 and α. From equations (3.55) and (3.52) one may write down expressions for these fields,

$$\alpha(r) = -\frac{1}{\mu} \int_0^r dr' r' (\rho(r') - 1) \tag{3.82}$$

$$a_0(r) = \frac{1}{\mu} \int_r^\infty dr' \frac{s - \alpha(r')}{r'} \rho(r'). \tag{3.83}$$

These functions are clearly seen to vanish as $\mu \to \infty$, so we would expect the results from GL theory to coincide with those of CSGL theory in this limit.

This correspondence is particularly visible inside the vortex core, where the solutions are identical in CSGL and GL theory. For larger r, the correspondence is not so obvious. A feature of the vortices in the Ginzburg-Landau theory for Helium-II is that the density is not decaying exponentially as $r \to \infty$. This is a major difference of CSGL and GL vortices, but we know from equation (3.75) that the exponential decay of $f(r)$ is given as $e^{-\frac{r}{\mu}}$, so that when $\mu \to \infty$, the exponential decay will vanish.

It is easy to see from equation (3.56) why we have this behavior: we know that $\alpha \to 0$ for $r \to 0$ and that $\alpha \to s$ for $r \to \infty$. Therefore, equation (3.56) will be equal to the GL equation for a vortex with vorticity s for small r, while it will be equal to that of no vortex for large r (i.e. the solution will be the constant ground state). Ergo, we would assume that the wave functions of GL and CSGL vortices were approximately equal as $r \to 0$, but that the CSGL vortex density approach the asymptotic constant value more quickly than the GL vortex density.

This behavior allows us to make a simple estimate of the energy of a CSGL vortex as a function of μ, since we know the energy for a GL vortex (equation (2.88).) If we as a first approximation say that the energy density for a CSGL vortex is equal to the energy density for a GL vortex for $r < \mu$, and that it is equal to 0 for $r > \mu$, we get that the energy of a CSGL vortex is equal to that of a GL vortex of radius μ, which when $\mu \gg \xi$ is

$$E \approx \frac{\hbar^2 \pi}{m} \kappa^2 \ln \frac{\mu}{\xi}. \tag{3.84}$$

3.5.6 Comparing MCS and CSGL Theory

Maxwell-Chern-Simons theory, which was briefly reviewed in section 2.5, is a description of anyons. Therefore, it would be interesting to see if there was any connection between this theory and the vortices in CSGL theory, which are believed to be anyons. In this section we will compare the MCS B-field from a static point particle, for which the dimensionless expression is given by equation (2.110), with the total magnetic field from a CSGL vortex. The CSGL vortex has a finite core size, so we would not expect the point particle approximation to be exact. We might however expect the MCS and CSGL fields to be equal in the limit $r \to \infty$.

The dimensionless B-field of MCS theory for a point particle is given by equation (2.110):

$$B_{\text{MCS}}(\mathbf{r}) = -\frac{q}{2\pi} K_0(r), \tag{3.85}$$

where q is the MCS charge. The total magnetic field from a CSGL vortex may be found in the limit $r \to \infty$ by using equations (3.38) and the asymptotic form for $\chi(r) = 1 - f(r)$, which was found in equation (3.66):

$$\mathcal{B}(r) = \frac{1}{\mu} \left\{1 - [f(r)]^2\right\} \approx \frac{2}{\mu} \chi(r) \approx \frac{2C}{\mu} K_0(\frac{r}{\mu}), \tag{3.86}$$

where we have used that the length scale ζ may be approximated by $\zeta \approx \mu$ for large μ, as we discussed in section 3.5.3.

It is evident from these expressions that if we rescale the coordinate according to

$$\mathbf{r}_{\text{MCS}} = \frac{\mathbf{r}}{\mu}, \tag{3.87}$$

and let $r \to \infty$ while keeping $\frac{r}{\mu}$ constant[4], the MCS magnetic field B_{MCS} is connected to the CSGL magnetic field $\mathcal{B}$ by

$$B_{\text{MCS}}(\mathbf{r}_{\text{MCS}}) \approx -\frac{q\mu}{4\pi C}\mathcal{B}(\mathbf{r}). \tag{3.88}$$

Unfortunately, the constant C is not determined. The value to use for the MCS charge q is not obvious either at this point. However, it seems clear that the CSGL theory is equivalent to the MCS theory (at least for the magnetic fields) in the limit $r, \mu \to \infty$, $\frac{r}{\mu}$ constant.

In chapter 5 we will make a more thorough study of the connection between MCS and CSGL theory.

3.5.7 Units and Length Scales[5]

In the dimensionless units we have been using, length is measured in terms of $\sqrt{2}\xi$, where ξ is the coherence length. This quantity is the typical length scale for variations in the electron density, so it may be taken as a measure of the core size of the vortices. We will write this as $R_{\text{core}} \approx \sqrt{2}\xi$. However, a vortex in CSGL theory also has a second length scale associated with it—the length scale of the exponential damping of the vortex "tail". This is the size of the total vortex, R. In section 3.5.3, we found two solutions for the length scale of the exponential damping, ζ_1 and ζ_2, given in equation (3.73). We also commented that ζ_1 would be dominating, so we will assume that this is the scale of the vortex size. When $\hat{\mu}$ is large, we may use equation (3.73) and the vortex size is given in dimensional units by

$$R = \zeta_1 = \frac{\hat{\mu}}{\sqrt{\lambda m \rho_0}} = \frac{\mu}{e^2}\sqrt{\frac{\lambda m}{\rho_0}} = \frac{\ell^2}{\rho_{\text{core}}}, \tag{3.89}$$

where ℓ is the magnetic length, $\ell = \frac{1}{\sqrt{eB^{\text{ext}}}}$.

Interestingly, demanding $R = R_{\text{core}}$, is equivalent to setting $\ell = \sqrt{2}\xi$, which again is the same as saying that $\hat{\mu} = 1$ in the dimensionless description. This is exactly where we find the "self-dual" solutions! This is analogous to the case of a superconductor, where the self-dual point is found when $\xi = \sqrt{2}\lambda$, where λ here is the London penetration depth (cf. section 2.4.4.)

We have also seen that the CSGL theory is equivalent to the Ginzburg-Landau theory of Helium-II in the limit $\hat{\mu} \to \infty$. In dimensional units, this limit becomes $\frac{m\lambda\mu}{e^2} \to \infty$, or in terms of the statistics parameter θ (using equation (3.40)),

$$\frac{m\lambda}{2\theta} \to \infty. \tag{3.90}$$

[4] Or, equivalently let $\mu \to \infty$ while keeping $\frac{r}{\mu}$ constant

[5] In this section we will temporarily return to dimensional units

We observe that when $\theta = 0$, we return to the Ginzburg-Landau theory. This is not surprising, since it means that the particles we are describing are bosons. Furthermore, when the interaction between the particles (which is given by λ) is strong, we approach the Ginzburg-Landau limit.

In the limit $\hat{r}, \hat{\mu} \to \infty$, $\frac{\hat{r}}{\hat{\mu}}$ constant, we found that the CSGL theory approaches the MCS theory. In terms of the dimensional parameters, we find that

$$\frac{\hat{r}}{\hat{\mu}} = 2\theta\sqrt{\frac{\rho_0}{\lambda m}}r, \tag{3.91}$$

so the condition $\frac{\hat{r}}{\hat{\mu}}$ constant becomes $\frac{\rho_0}{\lambda}$ constant.

3.6 Self-dual Vortices

As discussed in section 3.4, the CSGL system has a "self-dual point" at $\mu = 1$. The energy is then given by equation (3.45). When we insert that the current vanishes at infinity this equation reduces to

$$E = \int \frac{1}{2}|(D_1 + iD_2)\phi|^2 d^2x + \frac{\Phi}{2}, \tag{3.92}$$

where Φ is the total flux. Since we now know that the flux is quantized by equation (3.57), we find that

$$E = \int \frac{1}{2}|(D_1 + iD_2)\phi|^2 d^2x + \pi s. \tag{3.93}$$

When the Bogomol'nyi equation (3.47) is satisfied, the energy is simplified further to

$$E = \pi s, \tag{3.94}$$

so that it is linear in the vorticity of the vortex. For a vortex (3.48), the Bogomol'nyi equation becomes

$$\frac{df}{dr} = \frac{s - \alpha}{r} f. \tag{3.95}$$

This gives a simple expression for the current,

$$j^\theta = \frac{s - \alpha}{r} f^2 = f\frac{df}{dr}, \tag{3.96}$$

which may be inserted into (3.55), solving this equation:

$$-\frac{da_0}{dr} = f\frac{df}{dr} = \frac{1}{2}\frac{d}{dr}f^2,$$
$$a_0(r) = C - \frac{1}{2}(f(r))^2.$$

By using this answer with (3.95), we find that

$$f''(r) + \frac{1}{r}f'(r) - \left(\frac{s-\alpha}{r}\right)^2 f(r) - 2a_0(r)f(r) + 2\left[1-(f(r))^2\right]f(r) = (1-2C)f. \tag{3.97}$$

Comparing this to (3.56), we see that we must have $C = \frac{1}{2}$ for consistency. This means that a_0 is completely solved in terms of f:

$$a_0(r) = \frac{1}{2}\left(1-(f(r))^2\right). \tag{3.98}$$

From equation (3.95) we find

$$r\frac{d\ln f}{dr} = s - \alpha. \tag{3.99}$$

Differentiating and inserting (3.52), we arrive at

$$\begin{aligned} \frac{d}{dr}r\frac{d\ln f}{dr} &= r(f^2-1), \\ \boldsymbol{\nabla}^2 \ln f^2 &= 2(f^2-1). \end{aligned} \tag{3.100}$$

Equation (3.100) has a strong resemblance to the Liouville equation,

$$\boldsymbol{\nabla}^2 \ln \rho(\mathbf{r}) = 2\gamma\rho(\mathbf{r}), \tag{3.101}$$

which has the general solution $\rho(\mathbf{r}) = \frac{4}{\gamma}\frac{|g'(z)|^2}{(1+|g(z)|^2)^2}$, where $g(z+iy)$ is an arbitrary function. The most general radially symmetric and nonsingular solution to the Liouville equation is $\rho(\mathbf{r}) = \frac{4n^2}{\gamma r^2}\left[(\frac{r_0}{r})^n + (\frac{r}{r_0})^n\right]^{-2}$[45]. Unfortunately, no solution has been found to equation (3.100).

As shown in section 3.4, the energy takes a particular simple form at the self dual point. Tafelmayer[41] has shown that for vortices this point signifies an important change in the properties of the system. For $\mu < 1$, the energy of n vortices with unit vorticity is *larger* than the energy of one vortex with vorticity n. For $\mu > 1$, the situation is reversed. The effect is shown in figure 3.1. This means that for $\mu < 1$, a system with several "small" vortices would be unstable, as the vortices would tend to join into one "large" vortex. We consider this to imply that the physically significant (for the sake of the QHE) region for μ is $\mu > 1$.

3.7 Vortices in Motion

3.7.1 Adiabatically Moving Vortex

To find the wave function for a vortex moving in a "CSGL fluid" at rest at infinity, we will make an approach similar to that used by Duan[46] to compute the mass of a

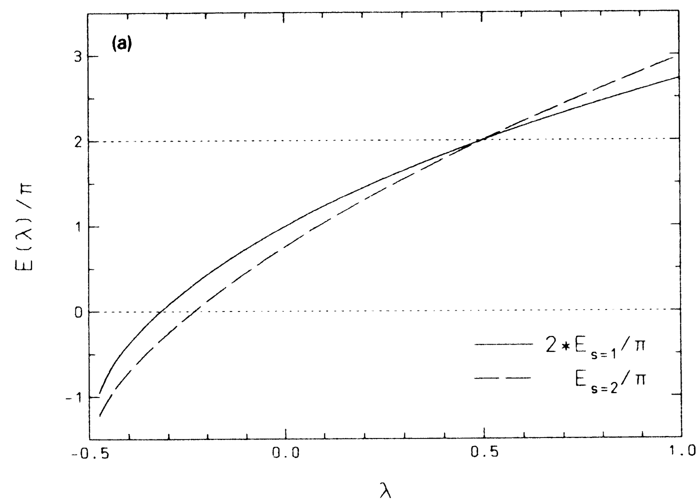


Figure 3.1: The crossing point for $s = 1$ and $s = 2$ vortices. From [41]. Note that in these units, $\lambda = 2\mu$.

vortex in superfluid ^{4}He. This approach was also used by Myklebust[25] to compute the wave function for an adiabatically moving vortex in a superfluid. We will assume that the time dependence of the problem only appears as

$$\phi(\mathbf{r}, t) = \phi(\mathbf{r} - \mathbf{v}t), \tag{3.102}$$

where $\phi(\mathbf{r})$ may be expressed as

$$\phi(r, \theta) = f(r)[1 + \lambda(r, \theta)]e^{is\theta}, \tag{3.103}$$

with $f(r)$ the solution of the vortex equations for a static vortex (Equations 3.52—3.56). We will also assume as a first approximation that the vector field is unchanged apart from a Galilean transformation:

$$\mathcal{A}_0(\mathbf{r}, t) = \mathcal{A}_0^{\mathrm{st}}(\mathbf{r} - \mathbf{v}t) + \mathbf{v} \cdot \mathcal{A}^{\mathrm{st}}(\mathbf{r} - \mathbf{v}t), \tag{3.104}$$

$$\mathcal{A}(\mathbf{r}, t) = \mathcal{A}^{\mathrm{st}}(\mathbf{r} - \mathbf{v}t), \tag{3.105}$$

where $\mathcal{A}_0^{\mathrm{st}}$ and $\mathcal{A}^{\mathrm{st}}$ are the fields from a static vortex. Inserting this into equation 3.11 and separating into real and imaginary parts gives the following equation from the real part when we linearize in λ:

$$\frac{s}{r}v_\theta = \mathbf{v} \cdot \mathcal{A}^{\mathrm{st}} + 2f^2\lambda - \frac{\boldsymbol{\nabla} f}{f} \cdot \boldsymbol{\nabla}\lambda - \frac{1}{2}\boldsymbol{\nabla}^2\lambda. \tag{3.106}$$

We have here used that the fields from the static vortex solves equation 3.11. v_θ is the θ component of the vortex velocity $\mathbf{v}$. Assuming further that λ is slowly varying at least far from the vortex core, this equation simply gives

$$\delta\rho = 2f^2\lambda = \frac{s-\alpha(r)}{r} v_\theta, \tag{3.107}$$

where $\alpha(r)$ is given by equation (3.52). As opposed to the situation in a superfluid, the deviation from the static wave function is exponentially damped[6]. However, the density change is still linear in the vortex velocity and dipole like, with the greatest deviation along a line perpendicular to the motion. We will come back to the problem of vortices in motion in the chapter 5, where the connection between MCS theory and CSGL vortices will be established.

3.7.2 Interaction Between Vortices

So far, the discussion has been for a single vortex in an infinitely large system. However, this is not a situation likely to appear in the real world. In a quantum Hall system, we indeed expect there to be so many vortices that they will condense into a homogeneous "fluid" under certain conditions. One might then wonder how these vortices will affect each other. We know that in a classical fluid, two vortices with the same vorticity will circle around each other and two vortices with opposite vorticity (i.e. one vortex and one anti-vortex) will be forced to move with a constant speed along a line perpendicular to the line between the vortices. Will this be the behavior of vortices in CSGL theory?

The answer is no. The reason for this is the exponential damping of the vortex tail in CSGL theory: A vortex situated far from another vortex will not sense the other vortex at all, except for the vector field **a**, which will give the vortices statistics according to the Aharonov–Bohm effect. In a more realistic model which might be made by including the (real) electric field caused by the fact that the vortices are charged or the Coloumb interaction between vortices instead of the ϕ^4-interaction in the CSGL theory, one would also expect an interaction between the vortices.

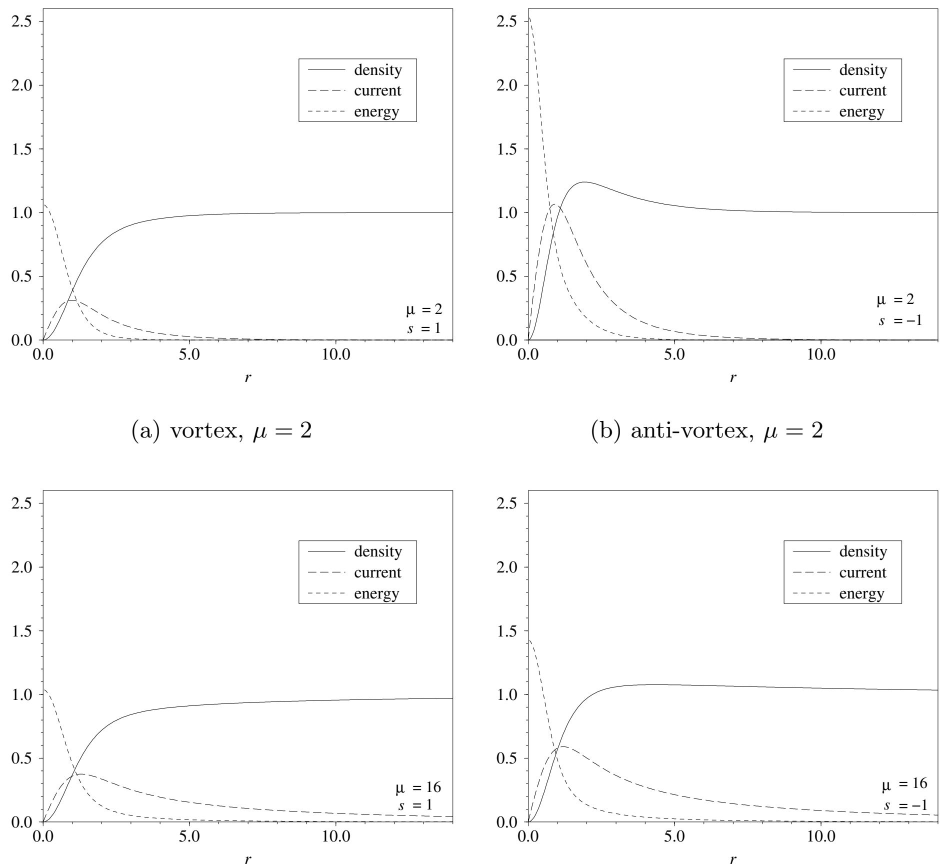
3.8 Numerical Solutions

3.8.1 Vortex Form

We have obtained numerical solutions to the dimensionless vortex equations (3.52), (3.55), and (3.56) by use of a relaxation method with a mesh spacing of 0.05. The relaxing method is described in appendix D. Solutions have mainly been studied for vortices with vorticity ± 1, and only for $\mu \geq 1$. For completeness, the solution for the Ginzburg–Pitavevskii–Gross equation without the electromagnetic fields has also been found.

Figure 3.2 shows the electron charge density and the current density for two vortices (fig. 3.2a,c) and two anti-vortices (fig. 3.2b,d) with the parameter choices $\mu = 2$ and $\mu = 16$. The density is seen to quickly attain its asymptotic value of

[6] Since $s - \alpha = \omega$ is exponentially damped by equation (3.67).



(a) vortex, $\mu = 2$ (b) anti-vortex, $\mu = 2$

(c) vortex, $\mu = 16$ (d) anti-vortex, $\mu = 16$

Figure 3.2: Typical vortices and anti-vortices

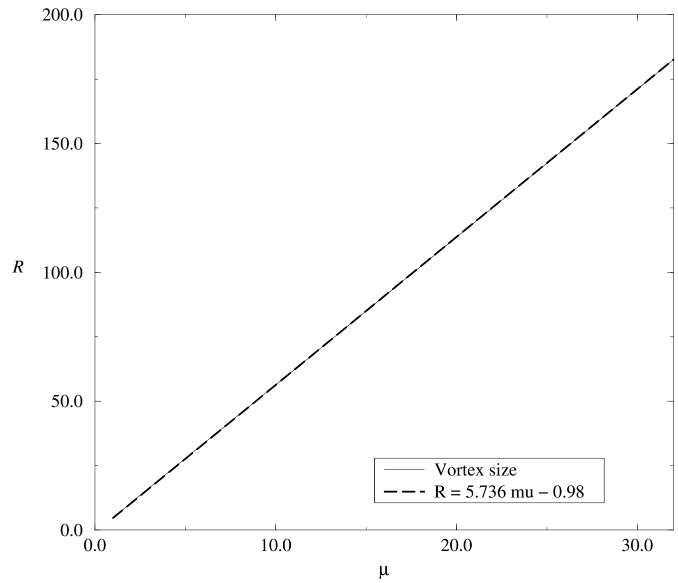


Figure 3.3: Vortex size as function of μ

1. Furthermore, one can see that the maximum of the current is at the edge of the vortex. The vortex clearly has a positive charge, while the anti-vortex has a (negative) charge surplus outside of the core. Based on the discussion in section 3.5.2, this is what we would expect.

The most staggering difference of $\mu = 2$ and $\mu = 16$ (anti-)vortices on figure 3.2, is for anti-vortices. Comparing the charge density on figures b and d, we see that the charge density for a $\mu = 16$ vortex is much less than that for a $\mu = 2$ vortex, even though we know that the total charge for a $\mu = 16$ vortex is 8 times that of a $\mu = 2$ vortex. This clearly shows how the size of a vortex grows when μ increases.

The size of a vortex was also calculated by defining it as the point where the integrated charge had reached 99% of its total value. The size of a vortex and anti-vortex was found to be equal within 1% for all values of μ, and was also found to be almost perfectly linear in μ for $\mu > 1$, as is shown in figure 3.3.

The size of the core was defined as the point with maximum current density, and is plotted in figure 3.4 for both a vortex and an anti-vortex. The core size is seen to be approximately constant for large values of μ, but to rapidly decrease for small μ-values.

3.8.2 Comparing CSGL and GL Vortices

As discussed in section 3.5.5, one would assume that the solution of equation (3.56) approaches that of the corresponding equation for GL theory as $\mu \to \infty$. In figure 3.5 the charge density for successive values of μ is plotted in the same graphs. Both vortices and anti-vortices are plotted. In figures c and d is also the density for a

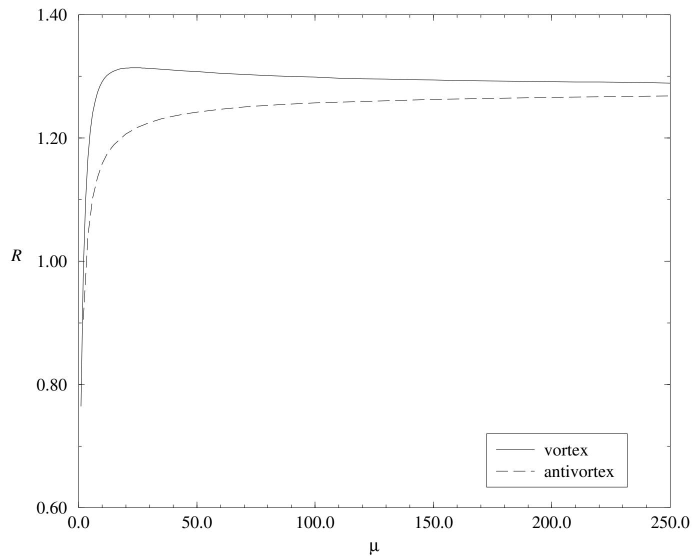


Figure 3.4: Core size of vortex and anti-vortex as function of μ

GL (anti-)vortex plotted[7]. It is evident that the solutions for CSGL theory really converge into that of GL theory for large μ, however μ must become quite large for the different theories to be approximately equivalent.

As one would expect, the core size as shown in figure 3.4 also converges to that of GL theory. The size of the whole vortex (figure 3.3) diverges. This is also to expect, since the vortices in GL theory are not exponentially damped, and thus in a sense has infinite size. The fact that the vortex size is linear in μ shows that there is not a uniform convergence of the CSGL vortex into the GL vortex, but rather a pointwise convergence. This is also expected since for all finite values of μ, the CSGL vortex is exponentially damped.

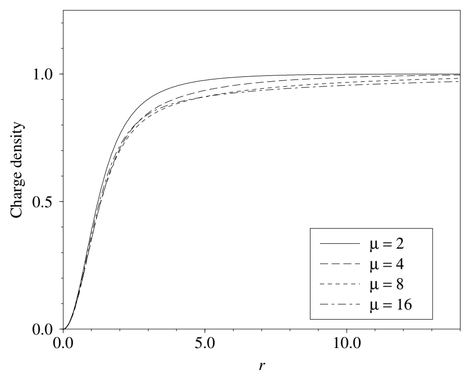
3.8.3 Vortex Energy

The behavior of the energy as μ varies, is shown in figure 3.6, in both a regular and semilogarithmic plot. This figure also shows a fit of the energy values for $\mu \geq 8$ to a logarithmic form $E = a \ln \mu + b$. One clearly sees that the logarithmic form postulated is a good approximation. The fit was made using linear regression on energy values for $\mu \geq 8$. The result of the fit was $a = 1.101 \pm 0.002$ and $b = 0.212 \pm 0.006$.

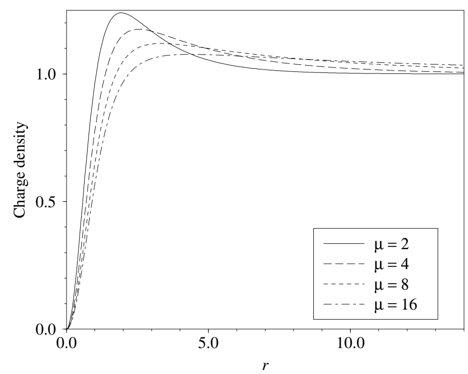
3.8.4 Comparing CSGL Vortices with MCS Fields

The connection between Maxwell-Chern-Simons theory and CSGL theory was discussed in section 3.5.6. The conclusion was that there is a connection in the limit

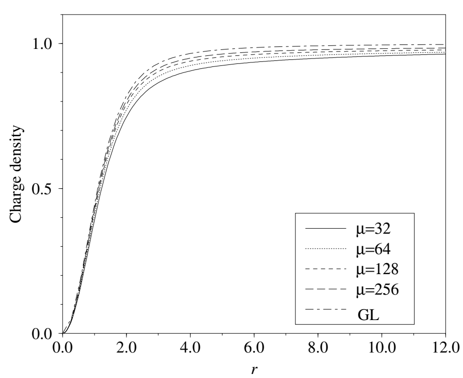
[7] Recall that a vortex is identical to an anti-vortex with the same vorticity in GL theory



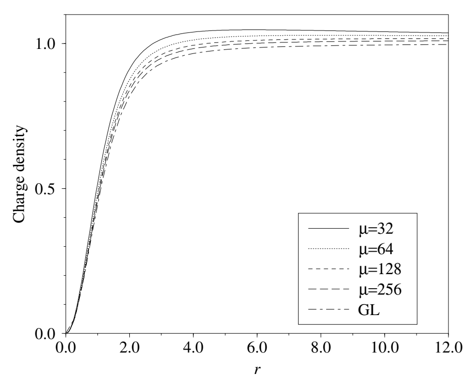
(a) vortices, $2 \leq \mu \leq 16$



(b) anti-vortices, $2 \leq \mu \leq 16$

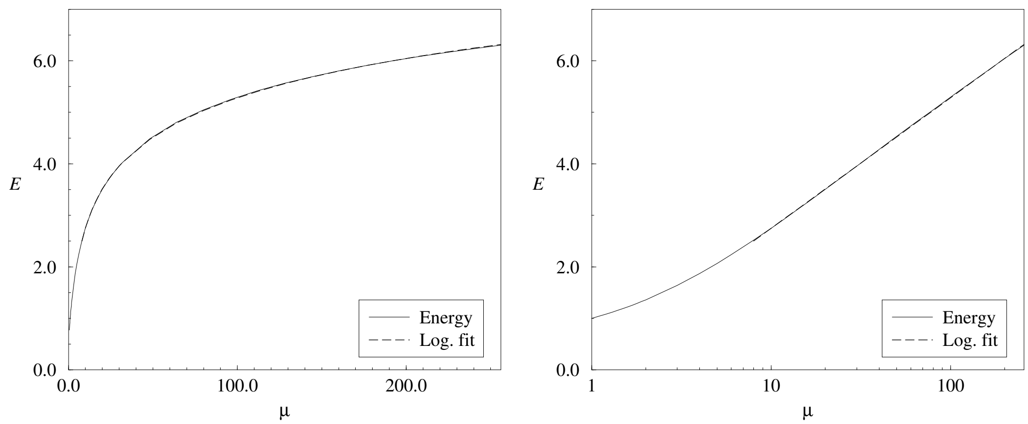


(c) vortices, $32 \leq \mu \leq 256$



(d) anti-vortices, $32 \leq \mu \leq 256$

Figure 3.5: Comparing CSGL and GL vortices and anti-vortices



(a) Regular plot (b) Semilogarithmic plot

Figure 3.6: Total vortex energy as a function of μ

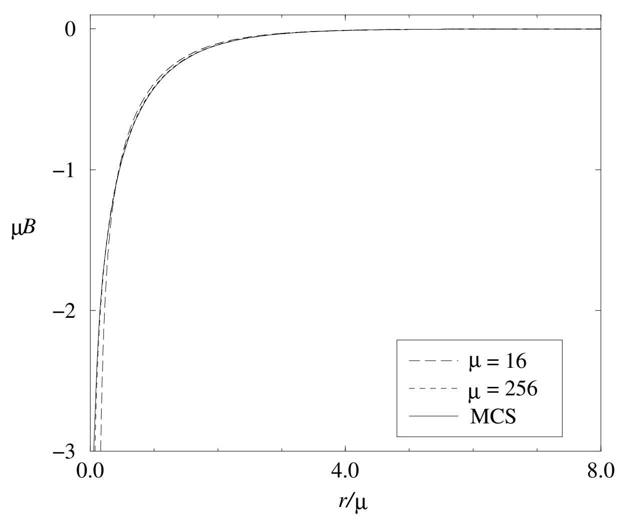


Figure 3.7: Comparing MCS and CSGL theory

$r, \mu \to \infty$, $\frac{r}{\mu}$ constant. In figure 3.7 we have plotted the dimensionless MCS B-field given by equation (2.110) with the charge q arbitrarily set to $q = 2\pi$. The total CSGL magnetic field as a function of $\frac{r}{\mu}$ multiplied with $-\mu^2$ is also plotted. It is evident from the figure, where the CSGL field is plotted for the two cases $\mu = 16$ and $\mu = 256$, that the MCS and CSGL solutions agree in the limit $\mu \to \infty$.

Chapter 4

Extensions of the CSGL Theory

4.1 Maxwell Chern Simons Ginzburg Landau Theory

4.1.1 Dynamical Magnetic Field

For CSGL theory, we have treated the magnetic field purely as an external field, with no dynamics. Since the externally imposed field is very strong, it is assumed that the dynamics has little influence. The dynamics of the total electromagnetic field would be given by adding a Maxwell term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to the Lagrangian (3.10), giving a "Maxwell Chern Simons Ginzburg Landau" (MCSGL) theory.

In this section, we will include only the dynamics of the magnetic field, but still assume that the electric field is negligible. We will therefore not include the full Maxwell term, but only the magnetic field part proportional to B^2. The motivation for this is to find a connection between the CSGL theory and the Ginzburg–Landau theory of superconductors, which was discussed in section 2.3.3. In the GL theory the magnetic field is also the only field which is considered dynamical.

Adding the dynamical term for the magnetic field to the Lagrangian (3.10), we get

$$\mathcal{L} = i\phi^* D_0\phi + \frac{1}{2m}\phi^*\mathbf{D}^2\phi - \frac{\lambda}{2}(|\phi|^2 - \rho_0)^2 + \frac{\mu}{2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma - \frac{1}{2\mu_0}\left(B - B^{\text{ext}}\right)^2, \tag{4.1}$$

where μ_0 is the magnetic permeability. We have subtracted the external field B^{ext} in the dynamics term, since this field is assumed to be kept constant by external devices.

Performing the rescaling (3.35) to go to dimensionless variables, we find

$$\hat{\mathcal{L}} = i\hat{\phi}^*\hat{D}_0\hat{\phi} + \frac{1}{2}\hat{\phi}^*\hat{\mathbf{D}}^2\hat{\phi} - \frac{1}{2}(|\hat{\phi}|^2 - 1)^2 + \frac{\hat{\mu}}{2}\epsilon^{\mu\nu\sigma}\hat{a}_\mu\hat{\partial}_\nu\hat{a}_\sigma - \frac{\kappa^2}{4}\left(B - \frac{1}{\mu}\right)^2, \tag{4.2}$$

where $\kappa^2 = \frac{2\lambda m^2}{\mu_0 e^2}$. We have used (3.21), $B^{\text{ext}} = \frac{e}{\mu}\rho_0$, as the external field B^{ext}. The constant κ corresponds exactly to the κ used in the theory of superconductors[16].

Note that the definitions of κ in this section and in section 2.3.4 differs. This is because the Cooper-pairs in a superconductor consist of two electrons, and therefore have twice the mass and twice the charge charge of the electrons described by CSGL theory. For a superconductor, this parameter decides whether it is a type-I or type-II superconductor. Vortices only appear in type-II superconductors, for which $\kappa^2 > \frac{1}{2}$[42].

The introduction of κ into the Lagrangian shows that the CSGL theory with added magnetical dynamics has one more dimensionless parameter than the CSGL theory discussed in the previous chapter. This addition of one dimensionless parameter signals that there is one more length scale included in this theory than in pure CSGL theory. This length scale is associated with the electron charge e, which in the two-dimensional world has the dimension[1] of $\frac{1}{\sqrt{L}}$.

The Lagrangian (4.2) gives the same equation as our CSGL Lagrangian, equations (3.37)–(3.39), with the addition of one of Maxwell's equations. For easy reference, we repeat all the equations here:

$$iD_0\phi + \frac{1}{2}D_i^2\phi - (\phi^*\phi - 1)\phi = 0, \tag{4.3}$$

$$\mu\epsilon^{ij}\partial_i a_j = \phi^*\phi, \tag{4.4}$$

$$\mu\epsilon^{ij}(\partial_j a_0 - \partial_0 a_j) = j^i, \tag{4.5}$$

$$\epsilon^{ij}\partial_j BB = \frac{2}{\kappa^2}j^i. \tag{4.6}$$

From the last equation, it is evident that as $\kappa \to \infty$, B approaches a constant value, which must be equal to the externally imposed field $B_{\text{ext}} = \frac{1}{\mu}$. Thus, for large values of κ, the B^2-term has little influence and the MCSGL system reduces to the CSGL system. It is also evident that in the $\mu \to \infty$ limit, we are left with GL theory as the Chern Simons-term loses influence, parallel to the discussion in section 3.5.5. We have thus established the connection between CSGL theory with magnetical dynamics and the GL theory of superconductors. If we let both κ and μ approach infinity, it is evident that both the real and the statistical field will vanish, and we are left with the Ginzburg-Landau theory for Helium-II. If we call the Ginzburg-Landau theory for superconductors "MGL theory" to distinguish it from that of He-II (which has no Maxwell field), we can draw the following diagram to illustrate the connection between the theories:

$$\begin{array}{ccc} MCSGL & \xrightarrow{\mu\to\infty} & MGL \\ {\scriptstyle\kappa\to\infty}\big\downarrow & & \big\downarrow{\scriptstyle\kappa\to\infty} \\ CSGL & \xrightarrow{\mu\to\infty} & GL \end{array} \tag{4.7}$$

[1]The dimension of e may be seen by demanding that the action $S = \int \mathcal{L}d^3x$ is dimensionless.

4.1.2 Energy

We will return to dimensional variables in this section to compute the energy density for the MCSGL theory. From the Lagrangian (4.2), it is evident that the only change in the energy density from equation (3.20), will be the addition of an energy term for the dynamical magnetic field. Thus the energy density is given by

$$\mathcal{E} = \frac{1}{2m}|\mathbf{D}\phi|^2 + eA_0\rho + \frac{\lambda}{2}(\rho - \rho_0)^2 + \frac{1}{2\mu_0}\left(B - B^{\text{ext}}\right)^2. \tag{4.8}$$

From this expression one can see that if $B^{\text{ext}} = \frac{e}{\mu}\rho_0$, as in section 3.2.1, the ground state will be exactly the same as in the CSGL theory, i.e. a constant density $\rho = \rho_0$ and a constant magnetic field $B = B^{\text{ext}}$.

4.1.3 Vortices

We proceed straight to the discussion of vortices in MCSGL theory. As in section 3.5 we will look at rotationally invariant vortices located at the origin,

$$\phi(r,\theta) = f(r)e^{is\theta}. \tag{4.9}$$

From the energy expression (4.8), we see that we must have the same conditions on the total electromagnetic field and on the density field as in section 3.5.1. Assuming radial symmetry, we may choose $A^r(r,\theta) = 0$, $A^\theta(r,\theta) = A^\theta(r)$, giving $B = \frac{1}{r}\frac{\partial}{\partial r}(rA^\theta)$. Setting

$$\alpha = ra^\theta - \frac{r^2}{2\mu}, \tag{4.10}$$

$$\beta = rA^\theta + \frac{r^2}{2\mu}, \tag{4.11}$$

we find the following set of equations:

$$f''(r) + \frac{1}{r}f'(r) - \left(\frac{s-\alpha-\beta}{r}\right)^2 f(r) - 2a_0(r)f(r) + 2\left[1 - (f(r))^2\right]f(r) = 0, \tag{4.12}$$

$$-\mu a_0'(r) = \frac{s - \alpha(r) - \beta(r)}{r}(f(r))^2, \tag{4.13}$$

$$\mu\alpha'(r) - r + r(f(r))^2 = 0, \tag{4.14}$$

$$\frac{1}{r^2}\beta'(r) - \frac{1}{r}\beta''(r) = \frac{2}{\kappa^2 r}(s - \alpha(r) - \beta(r))(f(r))^2. \tag{4.15}$$

These equations are quite similar to the corresponding equations (3.52), (3.55) and (3.56). The difference is the addition of the field $\beta(\mathbf{r})$, which contains the dynamical magnetic field. As we discussed in section 4.1.1, and as is evident from equation (4.15), this field vanishes is the limit $\kappa \to \infty$, where we regain the CSGL theory discussed in chapter 3.

To find the asymptotic form of the vortices in the MCSGL theory, we might use the *ansatz*

$$f(r) - 1 = CK_0\left(\frac{r}{\zeta}\right), \tag{4.16}$$

where C is a constant and ζ is an unknown length scale, as we did in section 3.5.3 for the vortices in CSGL theory. If we insert equation (4.16) into the above equations for the MCSGL vortex, we find the following equation for the length scale ζ:

$$(8\mu^2 + 4\kappa^2)\zeta^4 - (4\kappa^2\mu^2 + 2\mu^2)\zeta^2 + \kappa^2\mu^2 = 0. \tag{4.17}$$

Dividing by κ^2 and letting $\kappa \to \infty$, we get

$$4\zeta^4 - 4\mu^2\zeta^2 + \mu^2 = 0, \tag{4.18}$$

which is excatly equivalent to equation (3.69) for the length scale of a CSGL vortex. Thus, we see again that we find the pure CSGL theory in the $\kappa \to \infty$ limit.

The solutions to (4.17) are

$$\zeta_1{}^2 = \frac{2\mu^2\kappa^2 + \mu^2 + \mu^2\sqrt{4\kappa^4\mu^2 - 4\kappa^2\mu^2 + \mu^2 - 4\kappa^4}}{4\kappa^2 + 8\mu^2}, \tag{4.19}$$

$$\zeta_2{}^2 = \frac{2\mu^2\kappa^2 + \mu^2 - \mu^2\sqrt{4\kappa^4\mu^2 - 4\kappa^2\mu^2 + \mu^2 - 4\kappa^4}}{4\kappa^2 + 8\mu^2}. \tag{4.20}$$

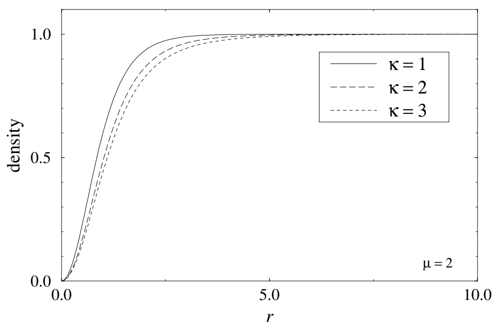
4.1.4 Topological Invariants

As we commented in the previous section, equation (3.50) in section 3.5.1 for the asymptotic behavior of the electromagnetic field still holds in MCSGL theory. Thus, the *total* magnetic flux is still quantized according to

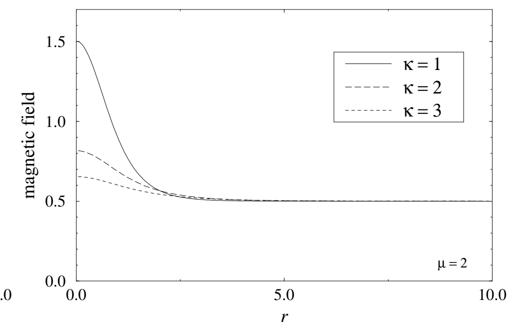
$$\int d^2x\epsilon^{ij}\partial_i(a_j + A_j) = 2\pi s. \tag{4.21}$$

However, this quantity is no longer equivalent to the charge as in equation (3.58). This is because the "real" field A_μ now includes a dynamic part and is no longer constant. Thus, the charge of the problem is no longer quantized. For the same reason, the angular momentum is also no longer a topological quantity.

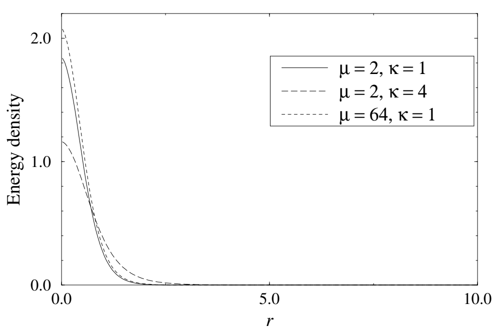
In the limit $\kappa \to \infty$, where the pure CSGL theory is found, the field A_μ approaches the constant external field and equation (4.21) again describes that the charge is quantized. In the limit $\mu \to \infty$, the statistical field a_μ vanishes, and equation (4.21) becomes the dimensionless equivalent of equation (2.96), saying that the magnetic flux is quantized.



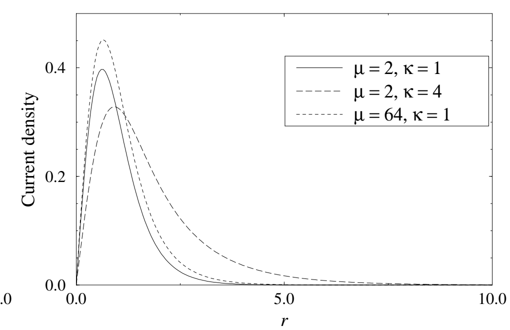
(a) Density



(b) Magnetic flux density



(c) Energy density



(d) Current density

Figure 4.1: MCSGL vortices

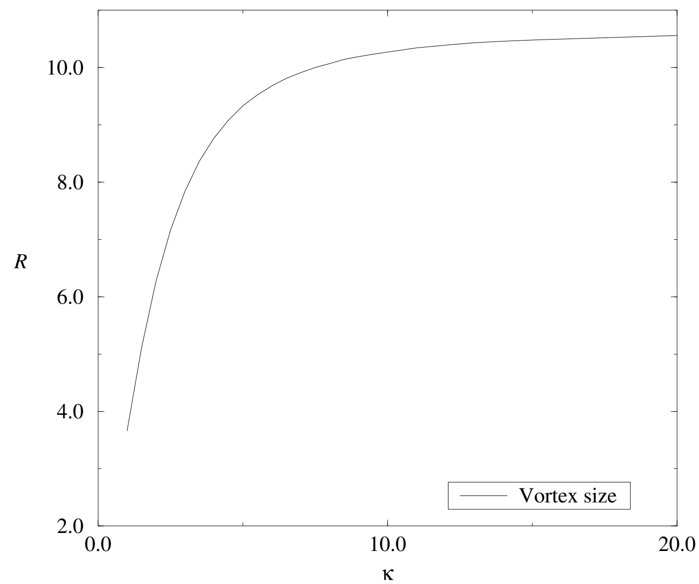


Figure 4.2: Size of MCSGL vortex with $\mu = 2$ as a function of κ.

4.1.5 Numerical Studies of Vortices

The equations (4.12)–(4.15) have been solved numerically for a selection of values for the constants κ and μ. In figure 4.1a we show how the value of κ modifies the structure of a vortex. For small values of κ, the vortex becomes much more localized, in the sense that the electron density more quickly approaches the constant value. This is further established in figure 4.2, where we show how the size of the vortex varies with κ. The size has been calculated by a procedure equivalent to that in figure 3.3, but since the charge is not quantized, we have used that the size if defined as the point where the total integrated magnetic field has reached 99% of its quantized value. From the figure it is evident that the size approaches a constant value as κ increases, and from the previous discussion we know that this must be the size of a corresponding CSGL vortex.

Figure 4.1b shows the magnetic field for the vortices shown in figure 4.1a. Figure 4.1c shows the energy density and figure 4.1d the current density for three different choices of values for κ and μ.

In figure 4.3 we show that it is not only the size of the vortices that diminishes as κ decreases. The same is true for the total energy, which is plotted in this figure as a function of κ for $\mu = 2$ and $\mu = 4$.

The charge of a vortex with $\kappa = 1$, $\kappa = 10$ and the charge of a CSGL vortex is plotted as a function of μ in figure 4.4. The figure shows that for small values of μ, $\mu \ll \kappa$, the charge is almost equal to that of a CSGL vortex, which is linear in μ. For larger values of μ, the charge decreases and approaches a constant value as $\mu \to \infty$.

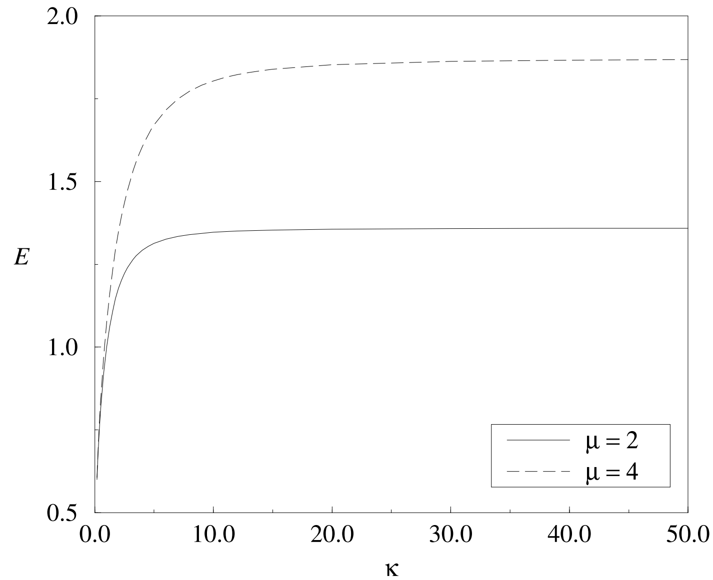


Figure 4.3: Total energy for MCSGL vortex for $\mu = 2$ and $\mu = 4$.

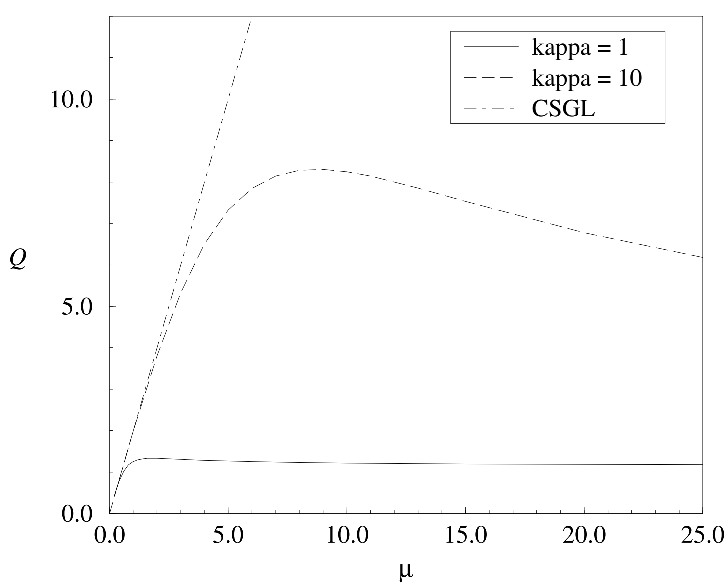


Figure 4.4: Charge of MCSGL vortex for $\kappa = 1$ and $\kappa = 10$, and charge of CSGL vortex.

4.1.6 Self-Dual Line

In the GL theory of superconductors, the self-dual point occurs at $\kappa^2 = \frac{1}{2}$, corresponding to the boundary between type-I and type-II superconductivity. For CSGL theory, we know from section 3.4 that there is a self-dual point at $\mu = 1$. For the combined MCSGL theory, we would then expect self-duality at a line in (μ, κ)-space. We can find this line by examining the dimensionless energy density as we did in section 3.4:

$$\begin{aligned}\mathcal{E} &= \frac{\kappa^2}{4}\left(B - \frac{1}{\mu}\right)^2 + \frac{1}{2}|\mathbf{D}\phi|^2 + \frac{1}{2}(|\phi|^2 - 1)^2 \\ &= \frac{\kappa^2}{4}\left[B - \frac{1}{\mu} - \frac{1}{\kappa^2}(1 - |\phi|^2)\right]^2 + \frac{1}{2}|(D_1 + iD_2)\phi|^2 \\ &\quad + \left(\frac{1}{2} - \frac{1}{2\mu} - \frac{1}{4\kappa^2}\right)(1 - |\phi|^2)^2 \\ &\quad + \frac{1}{2}\epsilon^{ij}\partial_i\left[\frac{1}{2i}(\phi^*\partial_j\phi - \phi\partial_j\phi^*) - (A^j + a^j)|\phi|^2 + A^j + a^j\right].\end{aligned} \tag{4.22}$$

The self-dual points are found when

$$\frac{1}{2} - \frac{1}{2\mu} - \frac{1}{4\kappa^2} = 0, \tag{4.23}$$

where the energy is minimized by

$$(D_1 + iD_2)\phi = 0, \tag{4.24}$$

$$B = \frac{1}{\mu} + \frac{1}{\kappa^2}(1 - |\phi|^2). \tag{4.25}$$

The line (4.23) marks the boundary between "type-I" and "type-II" states of the MCSGL system, as shown in figure 4.5.

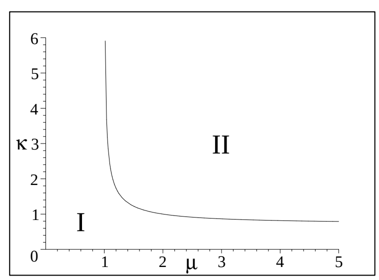


Figure 4.5: "phase diagram" of MCSGL theory

On the self–dual line, the energy of the ground state given by (4.24) and (4.25) is

$$E = \pi s \tag{4.26}$$

just as in the CSGL case (see section 3.4).

As an additional note, we see that the two solutions for the length scale, equations (4.19) and (4.20) are equal excactly at the self-dual line (4.23).

4.2 Two-Component CSGL Theory

4.2.1 Spin Effects in the Fractional Quantum Hall Effect

The discussion in chapters 2 and 3 did not take into account the fact that electrons have an intrinsic spin. We have, in effect, assumed that the electrons are fully polarized, i.e. all spins point in the same direction. This was originally assumed to be the case because one believed reversing spins was disallowed for energy reasons, since the external magnetic field is very strong. However, B. I. Halperin[47] pointed out that in GaAs the g-factor of the electron is in fact rather small, and thus gives a small Zeeman energy.

There are also other effects that are comparable with intrinsic spin effects. In multi-layered systems, electrons gain an additional quantum number that depend on which layer the electrons are in. In a two-layer system, electrons in the upper and lower layers would thus be analogous to electrons having spin up and down, respectively (for a review of double-layer quantum Hall systems, see [48].) Additionally, in certain systems (e.g. Silicon systems) there is an additional "valley" symmetry that can be treated as spin[11]. We will here concentrate our discussion on the intrinsic spin of electrons which couple to the magnetic field with a Zeeman term.

The effect of including electron spin in the microscopical model has been studied in numerical finite system size studies where the g-factor may be varied (see e.g. [49] for a review.) For large g, we know that the system is polarized, the question is what happens when $g \to 0$. The theoretical studies show that for *even integer* filling fractions, nothing happens, while for *odd integer* and *Laughlin* ($\frac{1}{m}$) filling fractions the ground state remains polarized, while the excitations change dramatically[11].

For odd or Laughlin fillings and small g, the *naïve quasiparticles* corresponding to (polarized) CSGL vortices and anti-vortices are no longer the lowest lying charged excitations in the quantum Hall system. As g is reduced, the lowest energy excitations increase their spin and size while the charge stays the same. In the limit $g \to 0$, the quasiparticles have a divergent size and a macroscopic spin with non-trivial spin order. These configurations are known as *skyrmions*.

4.2.2 The Non-Linear σ-Model[2]

To understand the structure of the non-polarized excitations in spin-extended CSGL theory, it is educative first to consider a greatly simplified model for spin systems known as the *non-linear σ-model*. Approximating the electron spin as a classical vector field $\mathbf{n}(\mathbf{r})$, with $\mathbf{n}^2 = 1$, the Lagrangian for this (non-dynamical) model is

$$\mathcal{L} = -\frac{1}{2}\rho^s(\boldsymbol{\nabla} \cdot \mathbf{n})^2, \tag{4.27}$$

where ρ^s is the *spin stiffness*. It is possible to deduce this Lagrangian as the leading term of the CSGL Lagrangian extended with spin[51], when ignoring dynamics.

[2]This introduction is largely based on the review in the thesis by Lilliehöök[50].

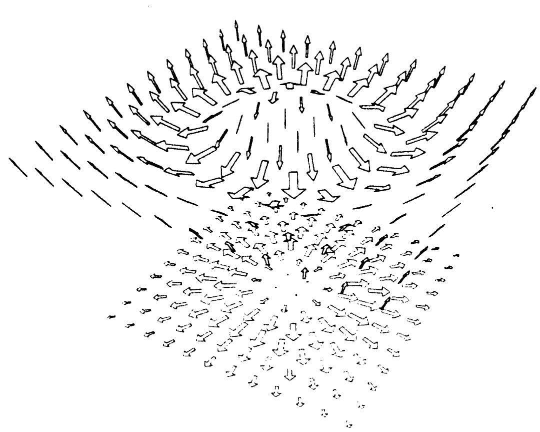


Figure 4.6: The spin configuration of a $Z = 1$ skyrmion. From [50].

The states of this system can be characterized by a topological invariant, the winding number Z. This number, which may be any integer, comes from the fact that the vector field $\mathbf{n}(\mathbf{r})$ is equivalent to the mapping of a sphere onto a sphere[50] [3]. The ground state with $\mathbf{n} = const$ has winding number $Z = 0$, but there may be excitations with any winding number. By solving the equations of motion that follow from the Lagrangian (4.27), one finds that simple solutions with winding number Z may be described by

$$\omega(z) = \left(\frac{z}{\lambda}\right)^Z, \tag{4.28}$$

where $z = x + iy$ and $\omega = \frac{2(n_x+in_y)}{1-n_z}$ (see e.g. [52] for a review of the non-linear sigma model.) A visualization of the state with $Z = 1$, taken from [50], may be seen in figure 4.6. The state has spin down in the middle and spin up at infinity. On a closed curve around the middle of the skyrmion, the spins will circle once around the z-axis. It is impossible to continuously transform this configuration into the ground state with all spins pointing up.

In the FQHE, $g \neq 0$ and this will give the excitations a finite size. We will use the term skyrmion for all excitations with a non-trivial spin order even though they have a finite size.

4.2.3 Adding Spin to CSGL Theory

To introduce a spin freedom into the CSGL system described by (3.10), it is necessary to make two changes: Replace the scalar field ϕ by a two-component field (ϕ_1, ϕ_2), and

[3] The spin field $\mathbf{n}(\mathbf{r})$ must align with the external magnetic field as $r \to \infty$, so the points at infinity may be identified with the north pole of a sphere

add a Zeeman term $-\frac{1}{2}g\mu_B B\phi^\dagger\sigma^z\phi$ for the interaction of the spin with the external magnetic field B. μ_B is the Bohr magneton, g is the (effective) gyromagnetic ratio for the electron, and σ^z is a Pauli matrix. The Zeeman term may also be interpreted as a difference in chemical potential for the upper and lower component[40]. We then end up with the following expression for the Lagrangian:

$$\mathcal{L} = \sum_{k=1}^{2} \phi_k^*(i\partial_0 - e\mathcal{A}_0 + \mu_k)\phi_k + \frac{1}{2m}\phi^\dagger(\boldsymbol{\nabla} - ie\boldsymbol{\mathcal{A}})^2\phi - \frac{\lambda}{2}(\phi^\dagger\phi)^2 + \frac{\mu}{2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma, \tag{4.29}$$

where μ_k, $k = 1, 2$ is the chemical potential for the upper and lower component, respectively.

4.2.4 Energy and Ground State

The energy density of a field configuration of the two-component system is

$$\mathcal{E} = \frac{1}{2m}|\mathbf{D}\phi|^2 + eA_0\rho + \frac{\lambda}{2}\rho^2 - \mu_1\rho_1 - \mu_2\rho_2. \tag{4.30}$$

For a constant solution, the energy is clearly lowest if all electrons are in the state with highest chemical potential which implies that the ground state is fully polarized and has a uniform density,

$$\begin{aligned} \phi_1 &= \sqrt{\frac{\mu_1}{\lambda}} \equiv \sqrt{\rho_0}, \\ \phi_2 &= 0, \end{aligned} \tag{4.31}$$

if $\mu_1 > \mu_2$, and if the magnetic field is at the plateau middle $B = \frac{1}{\mu}$.

With the above definition of ρ_0, the chemical potentials μ_1 and μ_2 may be written

$$\mu_1 = \lambda\rho_0, \tag{4.32}$$

$$\mu_2 = \lambda\rho_0 - g\mu_B B. \tag{4.33}$$

That the ground state is polarized independently of g is in agreement with the microscopical theory as discussed in section 4.2.1. If $g = 0$, the ground state is degenerate, as all SU(2) transformations of (4.31) have the same energy. The system would have to “choose” a specific ground state, which would lead to spontaneous symmetry breaking. The $g = 0$ case is not physical for the case of intrinsic spin, where there will always be a coupling between the spin and the magnetic field, however it might be physical for the other cases mentioned in section 4.2.1.

4.2.5 Rescaling

We perform a rescaling according to (3.35), and introduce a rescaled effective gyromagnetic ratio in terms of

$$\hat{g} = \frac{g\mu_B B}{\lambda\rho_0}. \tag{4.34}$$

This rescaling produces the Lagrangian, again suppressing the marks on the rescaled fields,

$$\mathcal{L} = \phi^\dagger(i\partial_0 - \mathcal{A}_0 + 1)\phi + \frac{1}{2}\phi^\dagger(\boldsymbol{\nabla} - i\boldsymbol{\mathcal{A}})^2\phi - \frac{1}{2}(\phi^\dagger\phi)^2 - g\phi_2^*\phi_2 + \frac{\mu}{2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma. \tag{4.35}$$

By extremizing this Lagrangian we find the following field equations:

$$-(i\partial_0 - A_0 - a_0)\phi_1 = \frac{1}{2}(\partial_i + iA_i + ia_i)^2\phi_1 - (\rho - 1)\phi_1, \tag{4.36}$$

$$-(i\partial_0 - A_0 - a_0)\phi_2 = \frac{1}{2}(\partial_i + iA_i + ia_i)^2\phi_2 - (\rho - 1 + g)\phi_2, \tag{4.37}$$

$$\mu\epsilon^{ij}\partial_i a_j = \rho, \tag{4.38}$$

$$\mu\epsilon^{ij}(\partial_j a_0 - \partial_0 a_j) = j^i, \tag{4.39}$$

where we now have the following expressions for the (rescaled) particle and current densities:

$$\rho(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_2(\mathbf{r}) = \phi_1(\mathbf{r})^*\phi_1(\mathbf{r}) + \phi_2(\mathbf{r})^*\phi_2(\mathbf{r}), \tag{4.40}$$

$$j^i(\mathbf{r}) = \frac{1}{2i}\left\{\phi^\dagger(\partial_i + ia_i + iA_i)\phi - [(\partial_i + iea_i + ieA_i)\phi]^\dagger\phi\right\}. \tag{4.41}$$

4.2.6 Self-Dual Point

Rewriting the energy as in equation (3.44), we find

$$\begin{aligned}\mathcal{E} = \frac{1}{2}|(D_1 + iD_2)\phi_1|^2 + \frac{1}{2}|(D_1 + iD_2)\phi_2|^2 + \left(\frac{1}{2} - \frac{1}{2\mu}\right)(|\phi_1|^2 + |\phi_2|^2 - 1)^2 + g|\phi_2|^2, \\ + \epsilon^{ij}\partial_i\left(\sum_{k=1}^{2}\frac{1}{2i}(\phi_k^*\partial_j\phi_k - \phi_k\partial_j\phi_k^*) - \mathcal{A}^j(|\phi_1|^2 + |\phi_2|^2 - 1)\right) - \frac{1}{2}.\end{aligned} \tag{4.42}$$

It is obvious that for $\mu = 1$ and $g > 0$, the energy is minimized for $\phi_2 = 0$ and $(D_1 + iD_2)\phi_1 = 0$, as in section 3.4. So at the self-dual point $\mu = 1$, the skyrmion solution that minimizes the energy is the fully polarized one.

4.2.7 Skyrmions

We will need an expression for the wave function of a skyrmion. As shown in figure 4.6, the spin should point up at infinity and down at the origin, and for $s = 1$, the spins will describe one turn around the z-axis when moving on a closed loop around the origin. In general, the spins will turn s times around the z axis on such a loop. The simplest wave function describing this configuration is

$$\begin{aligned}\phi_1(\mathbf{r}) &= f_1(r)e^{is\theta}, \\ \phi_2(\mathbf{r}) &= f_2(r),\end{aligned} \tag{4.43}$$

where we have made the gauge choice of keeping the phase of ϕ_2 constant. Inserting this form into the field equations for the two-component model, equations (4.36)–(4.39), we arrive at the following equations for a static skyrmion with winding number (topological charge) s:

$$f_1''(r) + \frac{1}{r}f_1'(r) - \left(\frac{s-\alpha(r)}{r}\right)^2 f_1(r) - 2a_0(r)f_1(r) - 2((f_1(r))^2 + (f_2(r))^2 - 1)f_1(r) = 0, \quad (4.44)$$

$$f_2''(r) + \frac{1}{r}f_2'(r) - \left(\frac{\alpha(r)}{r}\right)^2 f_2(r) - 2a_0(r)f_2(r) - 2((f_1(r))^2 + (f_2(r))^2 - 1 + g)f_2(r) = 0, \quad (4.45)$$

$$\mu a_0'(r) + \frac{s-\alpha(r)}{r}(f_1(r))^2 - \frac{\alpha(r)}{r}(f_2(r))^2 = 0, \quad (4.46)$$

$$\mu\alpha'(r) + r((f_1(r))^2 + (f_2(r))^2 - 1) = 0. \quad (4.47)$$

These equations reduce to the vortex case for $\phi_2 = 0$, so the polarized solution studied in the previous chapter is also a solution of these equations.

The equations (4.44)–(4.47) have been studied by solving them numerically for a range of the dimensionless parameters μ and g.

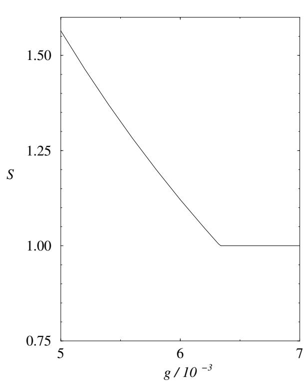
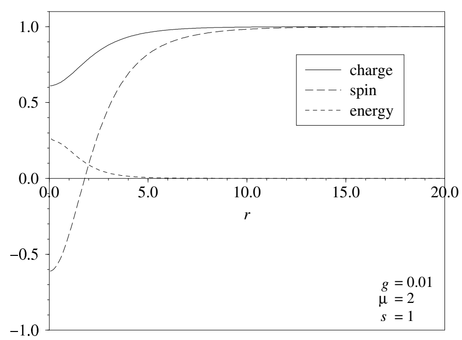


Figure 4.7: Spin per charge of skyrmion with $\mu = 1.1$

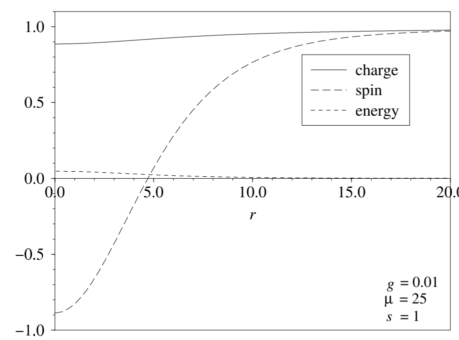
The qualitative behavior for skyrmions is shown in figure 4.8, where the electron density and spin density as well as the energy density for three skyrmions with different parameters are shown. When μ grows, the skyrmions are "smeared out" to a larger area. With g decreasing, the skyrmions also grow, but here it is mainly the spin density that is being spread out over a larger area. Figure 4.9 shows how the size of the skyrmion grows as g decreases. The size has been computed with the same definition as in section 3.8.1, i.e. as the point where the charge of the skyrmion has reached 99% of its total (quantized) value.

When g increases, the spin of the skyrmions decreases until finally, for large enough g, only fully polarized solutions are found, as shown in figure 4.7.

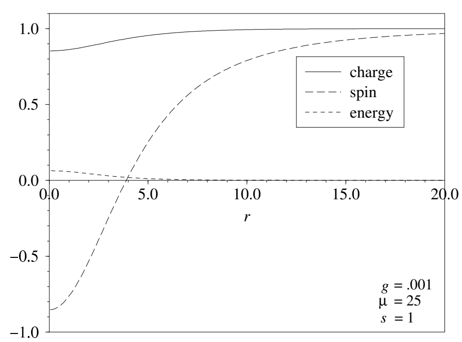
The differences of skyrmions with different topological charge are illuminated in figure 4.10. The figure shows that while the charge density is extremal at the center of the skyrmion for $|s| = 1$, the extremum of the charge density occurs in a ring around the center for $|s| > 1$.



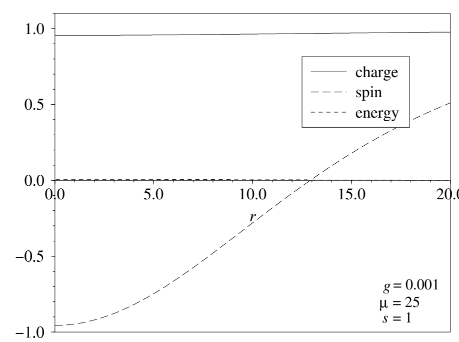
(a) $\mu = 2, g = 0.01$



(b) $\mu = 25, g = 0.01$



(c) $\mu = 2, g = 0.001$



(d) $\mu = 25, g = 0.001$

Figure 4.8: Four skyrmions with different parameters.

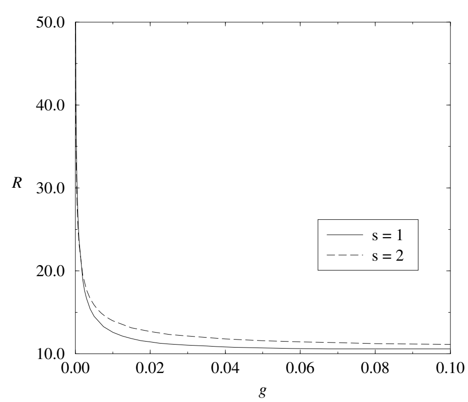
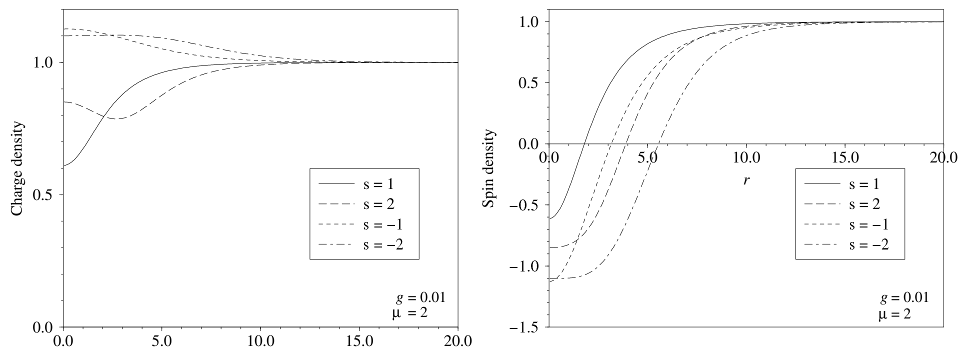
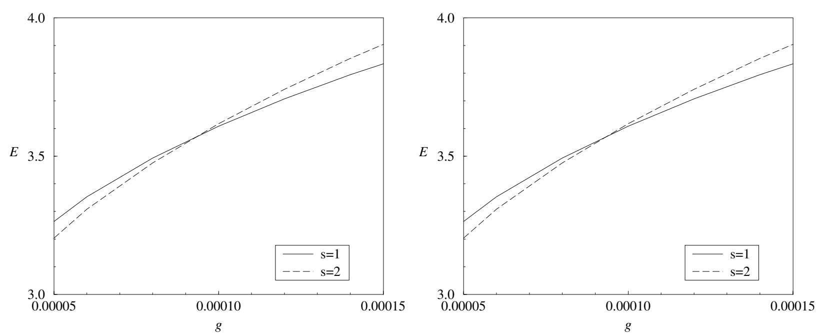


Figure 4.9: Size of skyrmion with $\mu = 2$ and $s = 1, 2$.



(a) Charge density (b) Spin density

Figure 4.10: Charge and spin density for skyrmions with $\mu = 2$ and $g = 0.01$ and $s = -2, -1, 1, 2$.



(a) $\mu = 2$ (b) $\mu = 50$

Figure 4.11: Crossing points for the energy of two charge one skyrmions and one charge two skyrmion.

4.2.8 Crossing of $s = 1$ and $s = 2$ Skyrmions

One important feature found by Lilliehöök et al.[53] is that a charge-2 skyrmion has lower energy than two charge-1 skyrmions for low enough values of the parameter g. The effective g-factor $\tilde{g}$ used by those authors is not exactly the same as the effective g-factor used in this thesis. Lilliehöök et. al. considered a non-linear σ-model with a Coloumb potential, with parameters adjusted to fit the microscopical theory for the fractional quantum Hall effect. Their effective g-factor is defined as

$$\tilde{g} = \frac{g\mu_B B}{e^2/\epsilon\ell}, \tag{4.48}$$

where ϵ is the electromagnetic permittivity. This is the ratio of the Zeeman energy to the characteristic Coloumb energy. In contrast, our definition (4.34) is the ratio of the Zeeman energy to the chemical potential. Lilliehöök et. al. found that when $\tilde{g} \lesssim 8.9 \cdot 10^{-5}$, the lowest lying charged excitations are charge-2 skyrmions.

This behavior has been qualitatively reproduced in our studies, as shown in figure 4.11. As is evident from this figure, the crossing point g_C varies with μ, and this is studied in figure 4.12. This figure shows a "phase diagram" for CSGL theory with spin, the phases shown are described in table 4.1.

The line given by g_C, which is the line separating the $\mathbf{S}_1$ and $\mathbf{S}_2$ phases, is seen to reach a maximum around $\mu = 2$, and approach zero as $\mu \to 1$. As μ approaches 1, the skyrmion solutions disappear. This displays the fact that the polarized solutions are the ones with lowest energy at the self-dual point (or line as viewed in (μ, g)-space) $\mu = 1$.

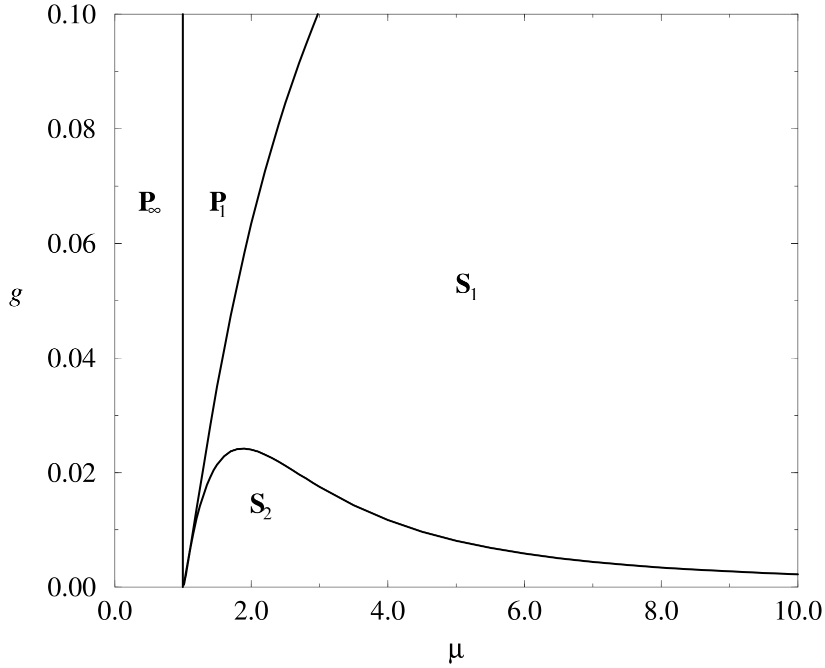


Figure 4.12: Phase diagram. The phases are explained in table 4.1.

Phase	Stable quasiparticles
$\mathbf{P}_1$	Spin polarized quasiparticles with charge 1
$\mathbf{P}_\infty$	No stable quasiparticles
$\mathbf{S}_1$	Skyrmions with charge 1
$\mathbf{S}_2$	Skyrmions with charge 2

Table 4.1: Description of phases displayed in figure 4.12

Chapter 5

Duality to Maxwell-Chern-Simons Theory

5.1 MCS and CSGL Theory

In section 2.5 we gave a short review of the Maxwell-Chern-Simons (MCS) theory. We stated there that it was a theory designed to model a system of anyons. We have also stated that the quasiparticles in the fractional quantum Hall effect are anyons. Thus, we might be able to describe them (possibly as an approximation) with the MCS theory or an extension of it. In section 3.5.6, we showed how the total magnetic field in CSGL theory approached that of MCS theory in the limit $r, \mu \to \infty$ when $\frac{r}{\mu}$ is held constant. However, the connection between MCS and CSGL theory is stronger than that. We will show that the CSGL equations of motion and the CSGL Lagrangian is indeed equivalent to an extension of the MCS equations and the MCS Lagrangian.

In field theory, the word duality is used to describe the connection between theories that equivalently describes the same phenomenon, but where topological excitations or quasiparticles in one theory are replaced by fundamental particles in the other. We will show that MCS theory can be derived as an approximation from CSGL theory, where the vortices in the latter become charged particles in the former theory. We can therefore say that there is an approximate duality between CSGL theory and MCS theory. There has been a considerable study of this duality in the literature (see e.g. [54, 55, 56]).

5.2 Simple Analogy

This section will be an introduction to the more rigorous treatment of the duality between CSGL and MCS theory in the following sections. We will show that one can find an analogy to MCS theory simply by making substitutions in the field equations for CSGL theory. We will in this section assume that the external field A^{μ} is a constant magnetic field such that equation (3.21), $B^{\text{ext}} = \frac{e}{\mu}\rho_0$, holds. In the context of the quantum Hall effect, this implies that the system is on the middle of a plateau.

The analogy to MCS theory starts with the hydrodynamical analogy of section 2.3.5; equations (2.65) and (2.67), and the CSGL equations (3.11) and (3.12). Written in terms of the total (real and statistical) fields $\boldsymbol{\mathcal{E}} = \mathbf{e} + \mathbf{E}^{\text{ext}}$ and $\mathcal{B} = b + B^{\text{ext}}$, the latter two equations become

$$\boldsymbol{\mathcal{E}} \times \hat{z} = -\frac{e}{\mu}\rho\mathbf{v} + \mathbf{E}^{\text{ext}} \times \hat{z}, \tag{5.1}$$

$$\mathcal{B} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}} = -\frac{e}{\mu}\rho + B^{\text{ext}}, \tag{5.2}$$

where $\hat{z}$ is a unit vector in the z-direction. B^{ext} and $\mathbf{E}^{\text{ext}}$ are the external fields. In addition to equation (3.21), we assume that there is no external electric field, i.e. $\mathbf{E}^{\text{ext}} = 0$, as before. The above two equations then imply that equations (2.65) and (2.67) and the Lorentz force $\mathbf{F}$ may be expressed with $\boldsymbol{\mathcal{E}}$ and $\mathcal{B}$ only.

As in section 3.3, we make a transformation to dimensionless quantities, effectively absorbing all parameters into the dimensionless constant μ. In these units, the external field plateau middle value is $B^{\text{ext}} = \frac{1}{\mu}$. The two equations above become

$$\boldsymbol{\mathcal{E}} \times \hat{z} = -\frac{1}{\mu}\rho\mathbf{v}, \tag{5.3}$$

$$\mathcal{B} = -\frac{1}{\mu}\rho + \frac{1}{\mu}, \tag{5.4}$$

and by using these we end up with the following expression for the Lorentz force $\mathbf{F}$:

$$\mathbf{F} = q(\boldsymbol{\mathcal{E}} + \mathcal{B}\mathbf{v} \times \hat{z}) = \frac{\boldsymbol{\mathcal{E}}}{1 - \mu\mathcal{B}}. \tag{5.5}$$

In the following part of this section, we will be considering small deviations from the ground state found in section 3.2.1, $\rho = 1 + \delta\rho$, $\mathbf{v} = \delta\mathbf{v}$ and assuming that the variations in the fields are slow (low frequency limit). All equations will therefore be linearized in $\delta\rho$ and $\delta\mathbf{v}$, and in derivatives. In the dimensionless CSGL theory, the potential in (2.67) is $V = \rho - 1$. Equation (2.67) becomes

$$\frac{\partial\delta\mathbf{v}}{\partial t} + \boldsymbol{\nabla}\delta\rho - \mathbf{F} = 0. \tag{5.6}$$

Combining this equation and equations (2.64) and (2.65) with equations (5.3) and (5.4) we find the following equations:

$$\dot{\mathcal{B}} + \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = 0, \tag{5.7}$$

$$\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\nabla} \times \hat{z}\mathcal{B} - \frac{1}{\mu}\boldsymbol{\mathcal{E}} \times \hat{z} = 0, \tag{5.8}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} + \frac{1}{\mu}\mathcal{B} = 0. \tag{5.9}$$

Equation (5.7) follows directly from (2.65), (5.8) follows from (5.6) by vector multiplication with $\hat{z}$, while (5.9) follows from (2.64) and (5.4) by observing that $\boldsymbol{\nabla} \times \mathbf{v} = -\boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}} = -\mathcal{B}$.

Equations (5.7)–(5.9) are equivalent to the field equations in Maxwell Chern Simons theory, equations (2.98)–(2.100), with a coupling constant $\mu \rightarrow -\frac{1}{\mu}$.

5.3 Lagrangian Approach

5.3.1 Introducing Vortices

In the hydrodynamical analogy of section 2.3.5, the assumption was made that the phase of the wave function is an analytical function of the coordinates, and therefore that there are no vortices in the system. Thus, we ended up with the source-less MCS equations (5.7)–(5.9). We will now use a less transparent, but more rigorous method to derive the approximate duality between CSGL and MCS theory, that also takes into account the possibility of vortices. Following Arovas and Freire[57], who have done a nice review of the corresponding duality between GL theory and regular Maxwell dynamics, we would like to transform the Lagrangian of CSGL theory (3.10) into the one for MCS theory (2.97). We start with the dimensionless CSGL Lagrangian

$$\mathcal{L} = \phi^*(i\partial_0 - A_0 - a_0)\phi + \frac{1}{2}\phi^*(\partial_i + iA_i + ia_i)^2\phi - \frac{1}{2}(\phi^*\phi - 1)^2 + \frac{\mu}{2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma. \quad (5.10)$$

The transformation starts by substituting $\phi = \sqrt{\rho}e^{iS}e^{i\chi}$. This is analogous to the hydrodynamical analogy of section 2.3.5, but since we will now also consider the possibilities of vortices in the system, we introduce the field $\chi(x)$ as the part of the wave function containing vortices. Therefore, this part need not be an analytical function. After making this substitution, we arrive at

$$\mathcal{L}' = -\rho(\partial_0 S + \partial_0\chi + A_0 + a_0) - \frac{\rho}{2}(\partial_i S + \partial_i\chi + A_i + a_i)^2 - \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho} - \frac{1}{2}(\rho - 1)^2 + \frac{\mu}{2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma, \quad (5.11)$$

where we have left out a total time derivative term $\frac{1}{2}\partial_0\rho$.

5.3.2 The Hubbard-Stratanovich Method

The Lagrangian (5.11) contains a squared term $-\frac{\rho}{2}(\partial_i S + \partial_i\chi + A_i + a_i)^2$, which we would like to simplify. This is possible with the introduction of a *Hubbard-Stratanovich field*. The idea behind the Hubbard-Stratanovich method is that we find a new Lagrangian such that integration over the new field gives us back the original Lagrangian. We achieve this as follows: Imagine a Lagrangian $\mathcal{L}_1 = \alpha\phi^2$, and that we would like the field ϕ to appear linearly only. The solution is to consider a Lagrangian of the form $\mathcal{L}_2 = -\beta\psi^2 + 2\gamma\psi\phi$, where ψ is a new dynamical field introduced to linearize the Lagrangian in ϕ. It is easy to see that integrating $\mathcal{L}_2$ over ψ gives us $\mathcal{L}_2' = \frac{\gamma^2}{\beta}\phi^2$, so that if $\frac{\gamma^2}{\beta} = \alpha$, $\mathcal{L}_2$ is essentially equivalent to $\mathcal{L}_1$.

Introducing a Hubbard–Stratanovich field $Q^i(x)$ into equation (5.11), we find

$$\mathcal{L}'' = -\rho(\partial_0 S + \partial_0\chi + A_0 + a_0) - Q^i(\partial_i S + \partial_i\chi + A_i + a_i) + \frac{1}{2\rho}\mathbf{Q}^2 - \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho} - \frac{1}{2}(\rho-1)^2 + \frac{\mu}{2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma. \quad (5.12)$$

Note that integrating over Q^i gets us right back to $\mathcal{L}'$. We may now integrate over S, which gives the constraint

$$\boldsymbol{\nabla}\cdot\mathbf{Q} + \partial_0\rho \equiv \partial_\mu Q^\mu = 0. \quad (5.13)$$

This means that we may write

$$\rho - 1 = \epsilon^{ij}\partial_i Z_j, \qquad Q^i = \epsilon^{ij}(\partial_j Z_0 - \partial_0 Z_j). \quad (5.14)$$

To keep track of some constant terms when making the above substitutions, we also introduce a non-dynamical gauge field z_μ, satisfying $\epsilon^{ij}\partial_i z_j = 1$ and $\partial_0 z_i = 0 = z_0$. We see that this makes us able to rewrite the coupling between the vortex field and the Hubbard-Stratanovich field as

$$Q^\mu\partial_\mu\chi = [\epsilon^{\mu\nu\sigma}\partial_\nu(Z_\sigma + z_\sigma)](\partial_\mu\chi). \quad (5.15)$$

5.3.3 Vortex current

The interaction term (5.15) may be rewritten, by leaving out a surface term, as $-(Z_\sigma + z_\sigma)\epsilon^{\mu\nu\sigma}\partial_\nu\partial_\sigma\chi$. We now claim that the field $J^\mu \equiv \epsilon^{\mu\nu\sigma}\partial_\nu\partial_\sigma\chi$ is in fact equal to the 3-current charge density of vortices. It is obvious that J^μ will be non-zero at a point x_0 only if χ is not analytical at this point. Imagine then that there is a vortex at x_0 with charge s, so that χ will have the property that it changes by $2\pi s$ when moving in a closed loop around x_0. Integrating J^0 over a neighborhood S of x_0 gives us by the theorem of Stokes,

$$\int_S J^0 dS = \oint_{\partial S} \boldsymbol{\nabla}\chi\cdot d\mathbf{l} = 2\pi s, \quad (5.16)$$

so that integrating J^0 over a larger area simply gives the total topological charge of the vortices in this area, multiplied by 2π. The analogous to the MCS charge of a CSGL vortex is thus $q \equiv 2\pi s$. Since the definition of J^μ also guarantees that $\partial_\mu J^\mu = 0$, we must then have that J^μ is the 3-current density of vortices. By introducing J^μ and Z^μ into (5.12), we end up with the Lagrangian

$$\mathcal{L}''' = -(Z_\mu + z_\mu)J^\mu - \epsilon^{\mu\nu\sigma}(A_\mu + a_\mu)\partial_\nu(Z_\sigma + z_\sigma) + \frac{\mathbf{E}^2}{2(1-B)} - \frac{1}{2}B^2 - \frac{(\boldsymbol{\nabla}B)^2}{8(1-B)} + \frac{\mu}{2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma. \quad (5.17)$$

Here we have introduced the fields $\mathbf{E} = -\boldsymbol{\nabla} Z^0 - \partial_0 \mathbf{Z}$ and $B = \boldsymbol{\nabla} \times \mathbf{Z}$. Note that these definitions are not equivalent to the previous definitions of $\mathbf{E}$ and B. Integrating over the original field a_μ, we finally arrive at

$$\mathcal{L}_{\text{new}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\epsilon^{\mu\nu\sigma} \partial_\nu A_\mu - J^\sigma)(Z_\sigma + z_\sigma) - \frac{1}{2\mu} \epsilon^{\mu\nu\sigma} (Z_\mu + z_\mu) \partial_\nu (Z_\sigma + z_\sigma) + \frac{1}{2} \frac{B\mathbf{E}^2}{1-B} - \frac{(\boldsymbol{\nabla} B)^2}{8(1-B)}. \quad (5.18)$$

So far, we have made no approximations, and this new Lagrangian describes our system just as well as the original (5.10). We see that linearizing (5.18) by leaving out the two last terms, gives us a Lagrangian similar to the Maxwell-Chern-Simons Lagrangian:

$$\mathcal{L}_{\text{lin}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\epsilon^{\mu\nu\sigma} \partial_\nu A_\mu - J^\sigma)(Z_\sigma + z_\sigma) - \frac{1}{2\mu} \epsilon^{\mu\nu\sigma} (Z_\mu + z_\mu) \partial_\nu (Z_\sigma + z_\sigma). \quad (5.19)$$

This Lagrangian gives the following equations of motion:

$$\boldsymbol{\nabla} \cdot \mathbf{E} + (\frac{1}{\mu} B + B^{\text{ext}} - \frac{1}{\mu}) = J^0, \quad (5.20)$$

$$\boldsymbol{\nabla} \times B\hat{z} - \dot{\mathbf{E}} - \hat{z} \times (\frac{1}{\mu} \mathbf{E} - \mathbf{E}^{\text{ext}}) = \mathbf{J}. \quad (5.21)$$

Here, B^{ext} and $\mathbf{E}^{\text{ext}}$ are the external fields related to A^μ. In addition, Faraday's equation (5.22) follows directly from the definitions of $\mathbf{E}$ and B,

$$\dot{B} + \boldsymbol{\nabla} \times \mathbf{E} = 0. \quad (5.22)$$

On the middle of the plateau of the FQHE, we have $\mathbf{E}^{\text{ext}} = 0$ and $B^{\text{ext}} = \frac{1}{\mu}$ in our dimensionless units, and this gives the familiar MCS equations;

$$\boldsymbol{\nabla} \cdot \mathbf{E} + \frac{1}{\mu} B = J^0, \quad (5.23)$$

$$\boldsymbol{\nabla} \times B\hat{z} - \dot{\mathbf{E}} - \frac{1}{\mu} \hat{z} \times \mathbf{E} = \mathbf{J}. \quad (5.24)$$

Note that the only change from equations (5.8) and (5.9) to equations (5.23) and (5.24) is the introduction of the vortex 3-current $(J^0, \mathbf{J})$. This current could also be introduced into (5.8) and (5.9) by realizing[25] that $\boldsymbol{\nabla} \partial_0 S = \partial_0 \boldsymbol{\nabla} S - \mathbf{J} \times \hat{z}$ when there are vortices present.

Another thing to note is that there is no dynamics in the field J^μ. All the dynamics of the vortices is hidden in the field Z^μ in the same manner as the vortices in the original model receive their dynamics from the fluid.

As a last note, if we are *not* on the middle of the plateau, we still have $B^{\text{ext}} = \frac{1}{\mu}\rho$ in the ground state, so that $\frac{1}{\mu} B + B^{\text{ext}} - \frac{1}{\mu} = 0$. Thus, if we rename $\frac{1}{\mu} B + B^{\text{ext}} - \frac{1}{\mu} \to \frac{1}{\mu} B$, then we will have exactly the same system as (5.23) and (5.24), where $B = 0$ and $\mathbf{E} = 0$ in the ground state.

5.3.4 Units and Dimensions

Returning for a moment to dimensional units, we find that we must introduce a new "speed of light" c to write the Lagrangian in a simple form such as (5.18). Defining

$$c \equiv \sqrt{\frac{\lambda\rho_0}{m}} \quad \text{and} \quad \partial_0 = \frac{1}{c}\partial_t \tag{5.25}$$

makes it possible to express equation (5.18) in dimensional units:

$$\begin{aligned}\mathcal{L}_{\text{new}} = &-\frac{m\rho_0}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m\rho_0}{c}(\frac{e}{m}\epsilon^{\mu\nu\sigma}\partial_\nu A_\mu - J^\sigma)(Z_\sigma + z_\sigma) \\ &- \frac{m\rho_0}{2\mu}\epsilon^{\mu\nu\sigma}(Z_\mu + z_\mu)\partial_\nu(Z_\sigma + z_\sigma) + \frac{m\rho_0}{2c}\frac{B\mathbf{E}^2}{1-\frac{B}{c}} - \frac{m\rho_0(\xi\boldsymbol{\nabla}B)^2}{8(1-\frac{B}{c})},\end{aligned} \tag{5.26}$$

where $\xi = \frac{1}{mc}$. In CSGL theory, c corresponds to the speed of sound, and ξ is in fact $\sqrt{2}$ times the coherence length defined earlier.

In dimensional units, Z^μ is defined by

$$\rho - \rho_0 = \frac{\rho_0}{c}\epsilon^{ij}\partial_i Z_j, \qquad Q^i = \rho_0\epsilon^{ij}(\partial_j Z_0 - \partial_0 Z_j). \tag{5.27}$$

5.4 Vortices in Motion

5.4.1 Lorentz Transformations

Before we go on to study the dynamics of vortices in detail, let us have a look at the fields from a moving vortex as described in the MCS theory.

The fields from a static vortex at the origin, $J^0 = 2\pi s\delta(\mathbf{r}), \mathbf{J} = 0$ have been given in equations (2.101) and (2.102):

$$B(\mathbf{r}) = -\frac{s}{\mu}K_0(\frac{r}{\mu}), \tag{5.28}$$

$$\mathbf{E}(\mathbf{r}) = \frac{s}{\mu}K_1(\frac{r}{\mu})\hat{\mathbf{e}}_r. \tag{5.29}$$

The Lagrangian (5.19) is Lorentz invariant, thus to find the fields for a moving vortex we only have to Lorentz transform the fields above. A Lorentz boost of the electromagnetic fields in two dimensions is given by

$$B' = \gamma B - \gamma\boldsymbol{\beta}\times\mathbf{E}, \tag{5.30}$$

$$\mathbf{E}' = \gamma\mathbf{E} - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) + \gamma\boldsymbol{\beta}\times B, \tag{5.31}$$

where $\boldsymbol{\beta}$ is the velocity of the moving frame (i.e. the vortex), and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. If β is small, we may linearize the second equation by leaving out the second term and approximate γ by $\gamma \approx 1$. Thus, the fields from a moving vortex, generated by boosting the above fields, become

$$B'(\mathbf{r}') = -\frac{s}{\mu}\left(K_0(\frac{r}{\mu}) + \beta_\theta K_1(\frac{r}{\mu})\right), \tag{5.32}$$

$$\mathbf{E}'(\mathbf{r}') = \frac{s}{\mu}\left(K_1(\frac{r}{\mu})\hat{\mathbf{e}}_r + \hat{z}\times\boldsymbol{\beta}K_0(\frac{r}{\mu})\right), \tag{5.33}$$

where β_θ is the θ-component of the velocity.

In section 3.7.1 we discussed an adiabatically moving vortex in CSGL theory. We found that the deviation in the charge density is given by equation (3.107). Furthermore, equation (3.67) gives the asymptotic form of $\omega = s - \alpha$. Using this equation, we see that equations (5.32) and (3.107) are in fact equivalent (up to a constant, which stems from the definitions of $\mathbf{E}$ in the duality to MCS theory) in the asymptotic limit.

5.4.2 Vortex Dynamics

The dynamics of vortices in CSGL theory is still not completely understood. Since the vortices represent quasiparticles, we would expect them to behave more or less like ordinary particles. In theory, it should be possible to find an effective Lagrangian for the vortices by integrating out the MCS fields Z^μ from equation (5.19).

In Coloumb gauge, equation (5.19) may be rewritten by leaving out surface terms as

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}Z_0\boldsymbol{\nabla}^2 Z_0 + [\epsilon^{ij}\partial_j(A_i + \frac{1}{\mu}Z_i + \frac{1}{\mu}z_i) - J^0][Z_0 + z_0] \\ + \frac{1}{2}(\partial_0 Z_i)^2 + \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{Z})^2 + [\epsilon^{ij}(\partial_0 A_j - \partial_j A_0) - J^i][Z_i + z_i] \\ - \frac{1}{2\mu}\epsilon^{ij}(Z_j + z_j)\partial_0(Z_i + z_i).\end{aligned} \tag{5.34}$$

In this Lagrangian we may integrate out the Z_0-field to produce

$$\begin{aligned}\mathcal{L}_{\text{eff}} = -\frac{1}{4\pi}\int d^2x'\tilde{J}^0(\mathbf{r},t)\ln(|\mathbf{r}-\mathbf{r}'|)\tilde{J}^0(\mathbf{r}',t) \\ + \frac{1}{2}(\partial_0 Z_i)^2 + \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{Z})^2 + [\epsilon^{ij}(\partial_0 A_j - \partial_j A_0) - J^i][Z_i + z_i] \\ - \frac{1}{2\mu}\epsilon^{ij}(Z_j + z_j)\partial_0(Z_i + z_i),\end{aligned} \tag{5.35}$$

where

$$\tilde{J}^0 = \epsilon^{ij}\partial_j(A_i + \frac{1}{\mu}Z_i + \frac{1}{\mu}z_i) - J^0. \tag{5.36}$$

But from equation (5.20) we see that to zeroth order, $\tilde{J}^0 = 0$, i.e.

$$\epsilon^{ij}\partial_j Z_i(x) = \mu J^0(x) = \mu \sum_k q_k \delta^{(2)}(\mathbf{r} - \mathbf{r}_k), \tag{5.37}$$

where we have inserted $\epsilon^{ij}\partial_i z_j = 1$ and used the (dimensionless) plateau middle value $B^{\text{ext}} = \frac{1}{\mu}$. This equation may be solved, yielding

$$Z_i(x) = \frac{\mu}{2\pi} \sum_k q_k \epsilon^{ij} \frac{x^j - x^j_k}{|\mathbf{r} - \mathbf{r}_k|^2}. \tag{5.38}$$

Inserting this and $J^i = \sum_k q_k \dot{x}^i_k \delta(\mathbf{r} - \mathbf{r}_k)$ gives the effective Lagrangian[37]

$$L_{\text{eff}} = \frac{1}{2} \sum_k q_k \epsilon^{ij} \dot{x}^i_k x^j_k + \mu \sum_k q_k A_0(\mathbf{r}_k) - \frac{\mu}{2\pi}\epsilon^{ij} \sum_{k,l} q_k q_l \dot{x}^i_k \frac{x^j_k - x^j_l}{|\mathbf{r}_k - \mathbf{r}_l|^2} + \frac{\mu^2}{2} \sum_{k,l} q_k q_l \delta^2(\mathbf{r}_k - \mathbf{r}_l) + \sum_k \Delta_k, \tag{5.39}$$

where the last term $\sum_k \Delta_k$ comes from the second and last term of equation (5.35) and contains the self-energy of the vortices[37].

For one single vortex, disregarding the last term of (5.39), we get the equation of motion

$$\epsilon^{ij} \frac{\partial x^j}{\partial t} = \mu \frac{\partial A_0(x)}{\partial x^i}. \tag{5.40}$$

This equation suggests that that the vortices are massless, since that would imply that the Lorentz force $q(\mathbf{E} + B\mathbf{v} \times \hat{z})$ vanishes, and this is exactly what the above equation is saying (remember that in our dimensionless units, $B = \frac{1}{\mu}$.) However, we must remember that we used a rather crude approximation to derive equation (5.38).

The third term in (5.39) describes the anyonic nature of the vortices. The contribution to the action from this term on a path where one vortex is interchanged with another is[37] $\Delta S = \frac{\pi}{2n+1}$.

The fourth term shows how the vortices "inherit" the delta function interaction between the electrons. If we had used another form of the interaction, we would have another term here.

5.5 Enhanced Vortex Dynamics

The effective Lagrangian for vortices in the previous section, equation (5.39) was derived using a crude approximation, and contains self-energy terms Δ_k which were not calculated. Using a more accurate method, we would like to improve on this situation.

As in the previous section, we would like to transform the MCS field equation (5.19) into an effective Lagrangian for the vortices, by treating these as point particles. We will do this by integrating out the MCS fields Z^μ using the MCS propagator, which has been found by Løvvik, and which is given in equation (2.105). This is an alternative to the approximation used to find equation (5.38), that hopefully will be more exact. Our goal will be to find a Lagrangian similar to the Lagrangian for a classical particle in an electromagnetic field,

$$L = \frac{1}{2}M\mathbf{v}^2 + Q\mathbf{v}\cdot\mathbf{A} + QA_0(\mathbf{r}), \tag{5.41}$$

where M the mass and Q the charge of the particle, $\mathbf{v}$ is the velocity of the particle and $\mathbf{A}$ and A_0 describe the (external) electromagnetic field. We will here follow D. P. Arovas and J. A. Freire[57], who have used a similar procedure for vortices in GL theory. In parallel to Arovas' and Freire's results, we will find that an expression like (5.41) may be found if we allow M and Q to be functions of the frequency, in a manner that will be defined below.

5.5.1 Self Interaction

As noted at the end of section 5.3.3, all the dynamics of the vortices is hidden within the MCS field Z^μ. The vortex field J^μ appears in the effective Lagrangian (5.19) only as an "external" current with no dynamics of its own. We would like to extract the parts of the MCS fields that give dynamics to the vortices so as to have explicit terms in the Lagrangian for this. We start with the parts of (5.19) that connect the MCS field to itself and to the vortex current:

$$\mathcal{L}_{\text{vortex}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\pi s J^\mu Z_\mu - \frac{1}{2\mu}\epsilon^{\mu\nu\sigma}Z_\mu\partial_v Z_\sigma. \tag{5.42}$$

We expect these terms to contain all the dynamics of the CSGL vortices, within the approximation used to obtain (5.19). From (5.42) we might integrate out the field Z^μ to produce

$$\mathcal{L}'_{\text{vortex}} = -J_\mu D^{\mu\nu}_{\text{MCS}} J_\nu, \tag{5.43}$$

where $D^{\mu\nu}_{\text{MCS}}$ is the MCS propagator.

The MCS propagator is given in equation (2.105), but we will not need this expression. Instead we insert into (5.43) the current density for one single vortex moving in a path $\mathbf{R}(t)$,

$$J^\mu(x) = q\beta^\mu\delta^{(2)}(\mathbf{r} - \mathbf{R}(t)), \tag{5.44}$$

where $q = 2\pi s$ is the MCS charge. β^μ is the 3-velocity of the vortex, $\beta^0 = 1$, $\boldsymbol{\beta} = \dot{\mathbf{R}}(t)$. The expression $D^{\mu\nu}_{\text{MCS}}J_\nu$ is then simply the "retarded" MCS potential A^{ret}_μ, given in equation (2.106). In that equation μ was defined as $\mu \to -\frac{1}{\mu}$, so with our definition of μ, the potential is

$$A_\nu(\mathbf{r}) = \mathfrak{I}_\nu(\mu) - \mu\epsilon_{\mu\nu\lambda}\partial^\mu\left(\mathfrak{I}^\lambda(0) - \mathfrak{I}^\lambda(\mu)\right), \tag{5.45}$$

where

$$\mathfrak{I}^\lambda(\mu) = \int_{-\infty}^{\infty} \frac{\theta(t-t'-|\mathbf{r}-\mathbf{R}(t')|)\cos\left(\frac{1}{\mu}\sqrt{(t-t')^2-|\mathbf{r}-\mathbf{R}(t')|^2}\right)}{2\pi\sqrt{(t-t')^2-|\mathbf{r}-\mathbf{R}(t')|^2}} q\beta^\lambda(t')dt'. \tag{5.46}$$

Inserting this expression into (5.43), we find

$$\begin{aligned} S_{\text{vortex}} &= -\int d^3x J^\nu(x) A_\nu^{\text{ret}}(x) = -\int d^3x q\beta^\nu(t)\delta^{(2)}(\mathbf{r}-\mathbf{R}(t)) A_\nu^{\text{ret}}(x) \\ &= S_1 + \mu\left(S_2(0) - S_2(\mu)\right), \end{aligned} \tag{5.47}$$

where

$$S_1 = -q^2 \int d^3x \beta^\nu(t)\delta^{(2)}(\mathbf{r}-\mathbf{R}(t)) \times \int_{-\infty}^{\infty} \frac{\theta(t-t'-|\mathbf{r}-\mathbf{R}(t')|)\cos(\frac{\Delta x}{\mu})}{2\pi\Delta x} \beta_\nu(t')dt', \tag{5.48}$$

$$S_2(\mu) = -q^2 \int d^3x \beta^\nu(t)\delta^{(2)}(\mathbf{r}-\mathbf{R}(t)) \times \epsilon_{\mu\nu\lambda}\partial^\mu \int_{-\infty}^{\infty} \frac{\theta(t-t'-|\mathbf{r}-\mathbf{R}(t')|)\cos(\frac{\Delta x}{\mu})}{2\pi\Delta x} \beta^\lambda(t')dt', \tag{5.49}$$

and Δx is the Lorentz invariant interval between the events $(t, \mathbf{r})$ and $(t', \mathbf{R}(t'))$, $\Delta x = \sqrt{(t-t')^2 - |\mathbf{r}-\mathbf{R}(t')|^2}$.

5.5.2 Frequency Dependent Mass

In equation (5.48) we can integrate out the delta function to get

$$\begin{aligned} S_1 &= -q^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' (1 - \boldsymbol{\beta}(t)\boldsymbol{\beta}(t')) \frac{\theta(t-t'-|\mathbf{R}(t)-\mathbf{R}(t')|)\cos(\frac{\Delta x}{\mu})}{2\pi\Delta x} \\ &= -q^2 \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} ds (1 - \boldsymbol{\beta}(u)\boldsymbol{\beta}(u+s)) \frac{\theta(s-|\mathbf{R}(u+s)-\mathbf{R}(u)|)\cos(\frac{\Delta x}{\mu})}{2\pi\Delta x} \end{aligned} \tag{5.50}$$

where we have changed variables to $u = t'$ and $s = t - t'$, and where now $\Delta x = \sqrt{(t-t')^2 - |\mathbf{R}(t)-\mathbf{R}(t')|^2} = \sqrt{s^2 - |\mathbf{R}(u+s)-\mathbf{R}(s)|^2}$. The theta function is zero for $|\mathbf{R}(u+s)-\mathbf{R}(u)| > s$, which corresponds to $s < 0$ as long as the vortices move with sub-sonic speed. We therefore integrate s from 0 to ∞, where the theta function is always unity.

The Fourier transformation of the classical action $S = \int_{-\infty}^{\infty} \frac{1}{2} M v^2 dt$ is easily found to be $S = \int_{-\infty}^{\infty} \frac{1}{2} M\omega^2 |\mathbf{r}(\omega)|^2 d\omega$. With this in mind, we would like to express the self-interaction in the form

$$S_{\text{self}} = S_{\text{static}} + \int_{-\infty}^{\infty} d\omega \frac{1}{2} M(\omega)\omega^2 |\mathbf{R}(\omega)|^2 + \dots \tag{5.51}$$

i.e. as an approximation to a free particle, but with a "mass" $M(\omega)$ that may depend on the frequency ω. To achieve this form, we need to approximate the cosine in S_1;

$$\cos(\frac{\Delta x}{\mu}) = \cos(\frac{\sqrt{s^2 - s^2\overline{V}^2(u,s)}}{\mu}) \approx \cos(\frac{s}{\mu}) + \frac{s}{2\mu}\overline{V}^2(u,s)\sin(\frac{s}{\mu}), \tag{5.52}$$

where $\overline{V}(u,s) = \frac{|\mathbf{R}(u+s)-\mathbf{R}(u)|}{s}$, the average speed of the vortex between the times u and $u+s$, which is of order β. In this approximation we have assumed that $\frac{s}{\mu} \ll 1$ and $\overline{V}(u,s) \ll 1$. The former may be justified since the cosine will fluctuate quickly whenever this is not the case, and thus the contribution to the integral will vanish. The latter is just the assumption that the vortex moves slowly.

Using the 'slow-moving'-assumption, we also approximate the $\frac{1}{\Delta x}$ part as

$$\frac{1}{\Delta x} \approx \frac{1}{s}(1 + \frac{1}{2}\overline{V}^2(u,s)), \tag{5.53}$$

and end up with

$$\begin{aligned} S_1 = -q^2 \int_{-\infty}^{\infty} du \int_0^{\infty} & ds (1 - \boldsymbol{\beta}(u)\boldsymbol{\beta}(u+s)) \\ & \times \frac{1}{2\pi s}\left[\left(1 + \frac{1}{2}\overline{V}^2(u,s)\right)\cos(\frac{s}{\mu}) + \frac{s}{2\mu}\overline{V}^2(u,s)\sin(\frac{s}{\mu})\right]. \end{aligned} \tag{5.54}$$

We will now Fourier transform this expression with respect to u. This will give a contribution equal to that of a static vortex, which we will ignore for now, and a contribution proportional to $|\mathbf{R}(\omega)|^2$. We ignore higher powers of $\overline{V}(u,s)$ and β. Some practical properties of the Fourier transform,

$$\int_{-\infty}^{\infty} du \boldsymbol{\beta}(u)\boldsymbol{\beta}(u+s) = \int_{-\infty}^{\infty} d\omega e^{i\omega s}\boldsymbol{\beta}(\omega)\boldsymbol{\beta}(-\omega) = \int_{-\infty}^{\infty} d\omega \cos(\omega s)\omega^2 |\mathbf{R}(\omega)|^2, \tag{5.55}$$

$$\begin{aligned} \int_{-\infty}^{\infty} du |\mathbf{R}(u+s) - \mathbf{R}(u)|^2 &= \int_{-\infty}^{\infty} du \left(|\mathbf{R}(u+s)|^2 - 2\mathbf{R}(u+s)\cdot\mathbf{R}(u) + |\mathbf{R}(u)|^2\right) \\ &= 2\int_{-\infty}^{\infty} d\omega \left(1 - \cos(\omega s)\right)|\mathbf{R}(\omega)|^2, \end{aligned} \tag{5.56}$$

give us

$$S_1 = S_1^{\text{static}} + q^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{\delta}^{\infty} \frac{ds}{s} \Big\{ \cos(\omega s)\cos(\frac{s}{\mu}) - \frac{(1-\cos(\omega s))\cos(\frac{s}{\mu})}{s^2\omega^2} - \frac{(1-\cos(\omega s))\sin(\frac{s}{\mu})}{\mu s \omega^2} \Big\} \omega^2 |\mathbf{R}(\omega)|^2$$
$$\equiv S_1^{\text{static}} + q^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} M(\omega)\omega^2|\mathbf{R}(\omega)|^2, \quad (5.57)$$

where we have introduced an ultraviolet cutoff δ to accommodate for the fact that the vortices are not really point particles. Without this cutoff, the integral for $M(\omega)$ diverges. The cutoff is probably not necessary if we include terms with higher derivatives in the Lagrangian[57].

The quantity $M(\omega)$, which is interpreted as a "frequency dependent mass", can be computed using the cosine integral function $\text{Ci}(x) \equiv -\int_x^\infty \frac{\cos(\xi)}{\xi} d\xi$ (see appendix B for properties of the cosine integral):

$$M(\omega) = -\frac{\omega^2\mu^2+1}{4\mu^2\omega^2}\text{Ci}((\omega\mu+1)\frac{\delta}{\mu}) - \frac{\omega^2\mu^2+1}{4\mu^2\omega^2}\text{Ci}((\omega\mu-1)\frac{\delta}{\mu}) + \frac{\text{Ci}(\frac{\delta}{\mu})}{2\mu^2\omega^2} + \frac{\cos(\omega\delta)\sin(\frac{\delta}{\mu})}{2\mu\omega^2\delta} - \frac{\sin(\omega\delta)\cos(\frac{\delta}{\mu})}{2\omega\delta} + \frac{\cos(\omega\delta)\cos(\frac{\delta}{\mu})}{2\omega^2\delta^2} - \frac{\cos(\frac{\delta}{\mu})}{2\omega^2\delta^2} - \frac{\sin(\frac{\delta}{\mu})}{2\mu\omega^2\delta}. \quad (5.58)$$

Observe that if we let $\mu \to \infty$, we end up with the same expression as Arovas and Freire[57], as one would expect:

$$\lim_{\mu\to\infty} M(\omega) = -\frac{1}{2}\left\{ \text{Ci}(|\omega|\delta) + \frac{1-\cos(\omega\delta)}{\omega^2\delta^2} + \frac{sin(\omega\delta)}{\omega\delta} \right\}. \quad (5.59)$$

The interesting limit, however, is letting $\omega \to 0$, i.e. taking the low frequency limit. To first order in ω, assuming $\omega \ll \frac{1}{\mu}$, we get

$$M \underset{\omega\to 0}{\to} -\frac{1}{2}\left(\cos(\frac{\delta}{\mu}) + \text{Ci}(\frac{\delta}{\mu}) \right). \quad (5.60)$$

This means that the frequency dependent mass of a vortex in CSGL theory has a well defined low frequency limit, disregarding the cutoff parameter δ. This is in opposition to the similar case for a superfluid. To calculate the value of M in this limit, one must either find a value for the cutoff δ, or make a more thorough calculation retaining the term proportional to $(\boldsymbol{\nabla} B)^2$ in the Lagrangian (5.18). A natural value for δ would be the vortex radius, which in these units is

$$\delta = 1. \quad (5.61)$$

5.5.3 Frequency Dependent Charge

To compute $S_2(\mu)$, we need to differentiate the last integral in equation (5.49). One part of the result will contain a delta function, which will be nonzero at $\Delta x = 0$ only. It is evident that in this case $S_2(\mu) = S_2(0)$ and so the two contributions from S_2 to S will cancel. Differentiating the rest of the expression is straightforward, using

$$\partial^\mu \Delta x = \frac{x^\mu - x'^\mu}{\Delta x}, \tag{5.62}$$

$$\partial^\mu \cos(\frac{\Delta x}{\mu}) = -\sin(\frac{\Delta x}{\mu})\frac{x^\mu - x'^\mu}{\mu \Delta x}, \tag{5.63}$$

we arrive at

$$S_2(\mu) = -\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \epsilon_{\mu\nu\lambda}\beta^\nu(t)\beta^\lambda(t')\frac{\theta(t - t' - |\mathbf{R}(t) - \mathbf{R}(t')|)}{2\pi(\Delta x)^2} \cdot \left[\frac{1}{\mu}\sin(\frac{\Delta x}{\mu}) - \frac{\cos(\frac{\Delta x}{\mu})}{\Delta x}\right](x^\mu - x'^\mu). \tag{5.64}$$

To compute this expression, it is useful to note that

$$\epsilon_{\mu\nu\lambda}\beta^\nu(t)\beta^\lambda(t')(x^\mu - x'^\mu) = (t - t')\boldsymbol{\beta}(t) \times \boldsymbol{\beta}(t') + (\boldsymbol{\beta}(t') - \boldsymbol{\beta}(t)) \times (\mathbf{R}(t) - \mathbf{R}(t')). \tag{5.65}$$

Inserting this, and doing the same change of variables as in the previous section, we get

$$S_2(\mu) = -\int_{-\infty}^{\infty} du \int_0^{\infty} ds \frac{1}{2\pi(\Delta x)^2}\left[\frac{1}{\mu}\sin(\frac{\Delta x}{\mu}) - \frac{\cos(\frac{\Delta x}{\mu})}{\Delta x}\right] \cdot \left\{s\boldsymbol{\beta}(u+s) \times \boldsymbol{\beta}(u) + [\boldsymbol{\beta}(u) - \boldsymbol{\beta}(u+s)] \times [\mathbf{R}(u+s) - \mathbf{R}(u)]\right\}. \tag{5.66}$$

Now we approximate in the same fashion as in the previous section, using

$$\frac{\cos(\frac{\Delta x}{\mu})}{\Delta x} \approx \frac{1}{s}\cos(\mu s) + \frac{\overline{V}^2(u,s)}{2}[\frac{1}{s}\cos(\frac{s}{\mu}) + \frac{1}{\mu}\sin(\frac{s}{\mu})] \tag{5.67}$$

$$\frac{1}{\mu}\sin(\frac{\Delta x}{\mu}) \approx \frac{1}{\mu}\sin(\frac{s}{\mu}) + \frac{s}{2\mu^2}s\overline{V}^2(u,s)\cos(\frac{s}{\mu}), \tag{5.68}$$

and find

$$S_2(\mu) \approx -\frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_0^{\infty} ds \left[\frac{1}{\mu}\sin(\frac{s}{\mu}) - \frac{1}{s}\cos(\frac{s}{\mu})\right] \cdot \left\{\frac{1}{s}\boldsymbol{\beta}(u+s) \times \boldsymbol{\beta}(u) + \frac{1}{s^2}\left[\boldsymbol{\beta}(u) - \boldsymbol{\beta}(u+s)\right] \times \left[\mathbf{R}(u+s) - \mathbf{R}(u)\right]\right\}. \tag{5.69}$$

Proceeding to Fourier transform this expression, we arrive at

$$S_2(\mu) \approx -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\infty} ds\, [\omega s \sin(\omega s) + 2(\cos(\omega s) - 1)] \cdot \left[\frac{\sin(\frac{s}{\mu})}{\mu s^2} - \frac{\cos(\frac{s}{\mu})}{s^3}\right] i\omega \mathbf{R}(\omega) \times \mathbf{R}(\omega). \quad (5.70)$$

It remains to make an interpretation of this term. The interaction term for a classical particle in an electromagnetic field is $L_{\text{int}} = Q\mathbf{v} \cdot \mathbf{A}$, where $\mathbf{A}$ is the electromagnetic vector field. If this field is a constant magnetic field, we can use $\mathbf{A} = \frac{1}{2}B\hat{\mathbf{z}} \times \mathbf{r}$, and so the interaction term becomes

$$L_B = \frac{1}{2} QB\mathbf{r} \times \mathbf{v}. \quad (5.71)$$

If we now Fourier transform and let $\Lambda \equiv \frac{1}{2}QB$ be a function of the frequency, we find

$$L_B = i\omega\Lambda(\omega)\mathbf{R}(\omega) \times \mathbf{R}(-\omega). \quad (5.72)$$

The interpretation of the term $\mu[S_2(0) - S_2(\mu)]$ thus is that the particle (i.e. the vortex) has a frequency dependent charge, which is contained in the term $\Lambda(\omega)$. It should be noted that this term appears in the Lagrangian even if we in equation (5.42) left out the terms where the external field appears.

We now proceed to calculate $\Lambda(\omega)$:

$$\Lambda(\omega) = \mu[\lambda(\omega, \mu) - \lambda(\omega, 0))], \quad (5.73)$$

$$\begin{aligned}\lambda(\omega, \mu) &= \int_{\delta}^{\infty} ds\, [\omega s \sin(\omega s) + 2(\cos(\omega s) - 1)] \left[\frac{\sin(\frac{s}{\mu})}{\mu s^2} - \frac{\cos(\frac{s}{\mu})}{s^3}\right] \\ &= (\frac{2\omega}{\mu} - \frac{3}{2\mu^2})\text{Ci}(|\omega\mu - 1|\frac{\delta}{\mu}) - (\frac{2\omega}{\mu} + \frac{3}{2\mu^2})\text{Ci}(|\omega\mu + 1|\frac{\delta}{\mu}) \\ &\quad + \frac{3}{\mu^2}\text{Ci}(\frac{\delta}{\mu}) + \frac{3}{\mu\delta}\cos(\omega\delta)\sin(\frac{\delta}{\mu}) - \frac{3}{\mu\delta}\sin(\frac{\delta}{\mu}) \\ &\quad - \frac{1}{\delta^2}\cos(\omega\delta)\cos(\frac{\delta}{\mu}) + \frac{1}{\delta^2}\cos(\frac{\delta}{\mu}). \end{aligned} \quad (5.74)$$

The cutoff δ is not necessary when the full expression for $\Lambda(\omega)$ is calculated, because $\Lambda(\omega)$ is finite as $\delta \to 0$:

$$\lim_{\delta \to 0} \Lambda(\omega) = 2\omega \ln\left(\left|\frac{\omega\mu - 1}{\omega\mu + 1}\right|\right) - \frac{3}{2\mu}\ln|\omega^2\mu^2 - 1|. \quad (5.75)$$

In the limit $\mu \to \infty$, we find that Λ vanishes,

$$\Lambda(\omega) \approx -(4 + \frac{3}{2}\ln(\omega^2\mu^2))\frac{1}{\mu} + \mathcal{O}(\frac{1}{\mu^3}), \quad (5.76)$$

just as we would expect, since this term does not appear in the Lagrangian when we start with GL theory without the Chern–Simons term[57]. Expanding $\Lambda(\omega)$ for small ω, we further arrive at

$$\Lambda(\omega) = -\frac{5}{2}\mu\omega^2 + \mathcal{O}(\omega^3), \tag{5.77}$$

so the S_2 term of the Lagrangian does not contribute in the low frequency limit.

5.5.4 Full Lagrangian

To summarize, we write down the full Lagrangian for a free point vortex which we have found in this section:

$$L(\omega) = L_{\text{static}} + \frac{1}{2}M(\omega)\omega^2|\mathbf{R}(\omega)|^2 + \Lambda(\omega)i\omega\mathbf{R}(\omega)\times\mathbf{R}(-\omega), \tag{5.78}$$

$$\begin{aligned} M(\omega) = &-\frac{\omega^2\mu^2+1}{4\mu^2\omega^2}\mathrm{Ci}((\omega\mu+1)\frac{\delta}{\mu}) - \frac{\omega^2\mu^2+1}{4\mu^2\omega^2}\mathrm{Ci}((\omega\mu-1)\frac{\delta}{\mu}) + \frac{\mathrm{Ci}(\frac{\delta}{\mu})}{2\mu^2\omega^2} \\ &+ \frac{\cos(\omega\delta)\sin(\frac{\delta}{\mu})}{2\mu\omega^2\delta} - \frac{\sin(\omega\delta)\cos(\frac{\delta}{\mu})}{2\omega\delta} + \frac{\cos(\omega\delta)\cos(\frac{\delta}{\mu})}{2\omega^2\delta^2} \\ &- \frac{\cos(\frac{\delta}{\mu})}{2\omega^2\delta^2} - \frac{\sin(\frac{\delta}{\mu})}{2\mu\omega^2\delta}, \end{aligned} \tag{5.79}$$

$$\Lambda(\omega) = 2\omega\ln\left(\left|\frac{\omega\mu-1}{\omega\mu+1}\right|\right) - \frac{3}{2\mu}\ln|\omega^2\mu^2-1|. \tag{5.80}$$

The Lagrangian is written down as a function of the frequency ω rather than the time t and is thus the Fourier transform of the regular Lagrangian $L(t)$. δ is a cutoff of the order of the vortex core size, with $\delta = 1$ the natural choice in these units.

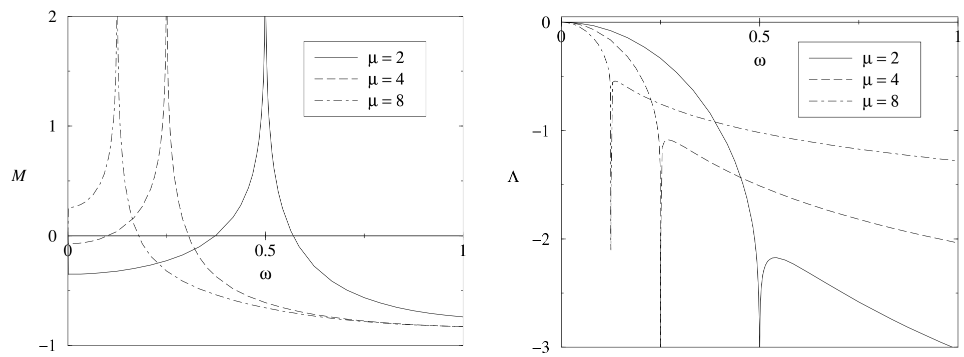
Both $M(\omega)$ and $\Lambda(\omega)$ have logarithmic singularities[1] at $\omega = \frac{1}{\mu}$. This behavior is seen in figure 5.1. In addition, we see in this figure that the mass is occasionally negative, and for certain values of μ it is even negative in the low frequency limit $\omega \to 0$. It is difficult to interpret this behavior physically, and it might signify that the approximations we have made is not valid, at least not for all ω.

5.6 Non-Lorentzian Corrections

The Lagrangian (5.18) contains a term proportional to $(\nabla B)^2$, which was left out in the linearized low frequency approximation (5.19). Keeping this term, we end up with the linearized Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu Z^\mu + \frac{1}{2\mu}\epsilon^{\mu\nu\sigma}Z_\mu\partial_\nu Z_\sigma - \frac{1}{8}(\nabla B)^2. \tag{5.81}$$

[1] note that $\mathrm{Ci}(x)$ is singular at $x = 0$, see appendix B.



(a) $M(\omega)$ (b) $\Lambda(\omega)$

Figure 5.1: Plots of $M(\omega)$ and $\Lambda(\omega)$.

In the Ginzburg-Landau case, Arovas and Freire[57] have found that adding this term makes the vortex propagator converge in the $s \to 0$ limit, so that the cutoff δ used in the previous section would not be needed. Furthermore, they found that the frequency dependent mass is strictly positive when including this term. With (5.81), the "photon" propagator becomes

$$\begin{aligned} D_{\mu\nu} = &- \frac{4-\mathbf{k}^2}{k^2(4-\mathbf{k}^2)-4M^2} g_{\mu\nu} + \frac{4M^2}{k^4[k^2(4-\mathbf{k}^2)-4M^2]} k_\mu k_\nu \\ &+ \frac{1}{k^2(4-\mathbf{k}^2)-4M^2} \epsilon_{\gamma\mu 0}\epsilon_{\lambda\nu 0} k^\gamma k^\lambda + \frac{4M}{k^2[k^2(4-\mathbf{k}^2)-4M^2]} \epsilon_{\mu\nu\sigma} i k^\sigma, \end{aligned} \tag{5.82}$$

where $M = \frac{1}{\mu}$. We have not been able to use this propagator to integrate the fields Z^μ out of the Lagrangian.

In dimensional variables, the Lagrangian (5.81) is[2]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} J_\mu Z^\mu + \frac{1}{2\mu} \epsilon^{\mu\nu\sigma} Z_\mu \partial_\nu Z_\sigma - \frac{1}{8} (\xi \boldsymbol{\nabla} B)^2, \tag{5.83}$$

where $\xi = \frac{1}{\sqrt{\lambda m \rho_0}} = \frac{1}{mc}$ and $c = \sqrt{\frac{\lambda \rho_0}{m}}$ is the speed of sound. The field equations become

$$\partial_i E^i - \frac{1}{\mu} B = \rho, \tag{5.84}$$

$$\epsilon_{ij}\partial_j B - \dot{E}^i - \frac{1}{\mu}\epsilon_{ij}E^j - \frac{\xi^2}{\mu}\epsilon_{ij}\partial_j \boldsymbol{\nabla}^2 B = \frac{1}{c} j^i. \tag{5.85}$$

[2] as can be seen by dividing equation (5.26) by $m\rho_0$

For a point vortex, we have $\rho = q\delta^{(2)}(\mathbf{r})$ and $\mathbf{j} = 0$, and by solving the last equation for $\mathbf{E}$ and inserting into the first, we arrive at

$$\mathbf{E}(\mathbf{r}) = \mu \boldsymbol{\nabla} B(\mathbf{r}) - \frac{\mu\xi^2}{4}\boldsymbol{\nabla}\boldsymbol{\nabla}^2 B(\mathbf{r}), \tag{5.86}$$

$$\mu^2\boldsymbol{\nabla}^2 B(\mathbf{r}) - \frac{\mu^2\xi^2}{4}(\boldsymbol{\nabla}^2)^2 B(\mathbf{r}) - B(\mathbf{r}) = \mu q\delta^{(2)}(\mathbf{r}). \tag{5.87}$$

Using rotational symmetry, the equation for $B(\mathbf{r}) = B(r)$ becomes

$$\mu^2\xi^2 r^3 B''''(r) + 2\mu^2\xi^2 r^2 B'''(r) - \mu^2(\xi^2 r + 4r^3)B''(r) \\ + \mu^2(\xi^2 - 4r^2)B'(r) + 4r^3 B(r) = -4\mu q r^3\delta^{(2)}(\mathbf{r}). \tag{5.88}$$

Unfortunately, it does not seem to be easy to solve this equation to find the fields $B(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$. Worse, even if we could solve it, the Lagrangian is no longer Lorentz invariant, so we could not find the fields for a moving vortex by Lorentz transforming it. We will therefore leave the subject of non-Lorentzian corrections to the MCS duality.

5.7 Plane Waves

We will again consider plane wave excitations of the FQHE system. This time we will look at excitations from a CSGL state where $\mathbf{j} = 0$ and $\rho = \rho_0$ (i.e. as in the ground state of the system), but where we will also assume that there is a constant density of vortices present in the system. In the dual representation, such excitations of the electrons correspond to excitations from the ground state $\mathbf{E} = 0$ and $B = 0$. The plane waves of CSGL theory are represented by electromagnetic waves in MCS theory, and the vortices by charged point particles, which act as sources for the fields. Ole Martin Løvvik[34] has studied this system and found a dispersion relation for it.

To study this system we will need an equation for the influence of the electromagnetic field on the vortices, in addition to the MCS equations (5.20)–(5.22). We will here make use of the frequency dependent mass and the frequency dependent charge found in section 5.5. We will initially assume that apart from the interactions with the MCS fields, the vortices are free. The Lagrangian (5.78) is the Lagrangian for free vortices. In addition, we may extract the interaction with the MCS fields from the Lagrangian (5.19). Disregarding for a moment the frequency dependence of M and Λ, the total Lagrangian becomes

$$L = \frac{1}{2}M\mathbf{V}^2 + \Lambda\mathbf{r}\times\mathbf{v} + q\mathbf{V}\cdot[\mathbf{Z}(\mathbf{R}) + \mathbf{z}(\mathbf{R})] - qZ_0(\mathbf{R}). \tag{5.89}$$

The total set of equations of motion for the system may then be written

$$\partial_0 B + \epsilon^{ij}\partial_i E^j = 0, \tag{5.90}$$

$$\partial_i E^i + \frac{1}{\mu}(B + \mu B^{\text{ext}} - 1) = J^0, \tag{5.91}$$

$$\epsilon^{ij}\partial_j B - \partial_0 E^i + \frac{1}{\mu}\epsilon^{ij}E^j = J^i, \tag{5.92}$$

$$q[E^i + \epsilon^{ij}V^j(B-1)] + 2\epsilon^{ij}V^j\Lambda = M\partial_0 V^i. \tag{5.93}$$

By looking at equation (5.91), we see that the ground state of the system must be such that $B + \mu B^{\text{ext}} - 1 = \mu J^0$. We know that on the middle of a plateau in the FQHE, $B^{\text{ext}} = \frac{1}{\mu}$ and $B = 0$ in these units, and there are no vortices. This corresponds to the relation

$$B^{\text{ext}} = \frac{e}{\mu}\rho \tag{5.94}$$

for the original CSGL field. As B^{ext} is increased, we expect the latter relation (5.94) to continue to hold, so that the change in B^{ext} is absorbed into B and there will still not be any (free) vortices. However, at some point we expect B to reach a maximum, and vortices will start to appear. We therefore expect B to be a small constant B_C in the ground state.

We will now assume that $\mathbf{E}$ and B oscillates around the ground state, analogous to the treatment in section 3.2.2:

$$\mathbf{E} = \mathbf{E}_0 e^{i(kx-\omega t)}, \qquad B = B_0 e^{i(kx-\omega t)} + B_C. \tag{5.95}$$

We also assume that the vortices in the system are brought into collective oscillations:

$$\mathbf{J} = q\mathbf{J}_0 e^{i(kx-\omega t)}, \qquad J^0 = qN_0 e^{i(kx-\omega t)} + N_C, \tag{5.96}$$

where N_C is the constant background density of vortices with charge $q = 2\pi s$. We define longitudinal and transverse components of $\mathbf{E_0}$ and $\mathbf{J_0}$ as:

$$\mathbf{k}\cdot\mathbf{E}_0 \equiv kE_{0\parallel}, \qquad \mathbf{k}\cdot\mathbf{J}_0 \equiv kJ_{0\parallel}, \tag{5.97}$$

$$\mathbf{k}\times\mathbf{E}_0 \equiv kE_{0\perp}, \qquad \mathbf{k}\times\mathbf{J}_0 \equiv kJ_{0\perp}. \tag{5.98}$$

Inserting equation (5.90) into the oscillating part of equation (5.91) gives

$$i\omega E_{0\parallel} + \frac{1}{\mu}E_{0\perp} = \frac{\omega}{k}qN_0, \tag{5.99}$$

while the longitudinal component of equation (5.92) is

$$i\omega E_{0\parallel} + \frac{1}{\mu}E_{0\perp} = qJ_{0\parallel}. \tag{5.100}$$

This shows that

$$\omega N_0 = k J_{0\|}. \tag{5.101}$$

Substituting B_0 from equation (5.90) into the transverse component of equation (5.92), we find

$$q J_{0\perp} = \frac{i}{\omega}(\omega^2 - k^2) E_{0\perp} - \frac{1}{\mu} E_{0\|}. \tag{5.102}$$

Since the only charges in this system are the vortices, we have $\mathbf{J} = N_C \mathbf{V}$. We insert this into equation (5.93) while assuming that the oscillatory part of B in this equation may be omitted. This is a valid assumption provided that the following holds[34]:

$$|\rho_0| \ll qN, \tag{5.103}$$

$$\left| \frac{k E_{0\perp}^2}{qN\omega^2} \left(k^2 + \frac{1}{\mu^2} - \omega^2 \right) + \frac{\rho_0 E_{0\perp}}{qN\omega\mu} \right| \ll \left| \frac{1}{\omega\mu} E_{0\perp} + \frac{\rho_0}{k} \right|. \tag{5.104}$$

We find the following for the longitudinal and transverse component, respectively:

$$-i\omega M(\omega) J_{0\|} + b J_{0\|} = q N_C E_{0\|} + q J_{0\perp}(B_c - 1) + 2\Lambda(\omega) J_{0\perp}, \tag{5.105}$$

$$-i\omega M(\omega) J_{0\perp} + b J_{0\perp} = q N_C E_{0\perp} - q J_{0\|}(B_c - 1) - 2\Lambda(\omega) J_{0\|}, \tag{5.106}$$

where we have now included the ω-dependence of M and Λ explicitly. We have also included a damping factor $bV(\omega)$ on the left hand side of (5.93). Substituting $J_{0\|}$ and $J_{0\perp}$ from equations (5.100) and (5.102) then gives a homogeneous set of linear equations for $E_{0\|}$ and $E_{0\perp}$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E_{0\|} \\ E_{0\perp} \end{pmatrix} = 0, \tag{5.107}$$

$$A = \omega^2 M + ib\omega - q^2 N_C + \frac{q}{\mu} B_{\text{Eff}}, \tag{5.108}$$

$$B = -\frac{i}{\mu}(\omega M + ib) - iq\frac{\omega^2 - k^2}{\omega} B_{\text{Eff}}, \tag{5.109}$$

$$C = \frac{i}{\mu}(\omega M + ib) + iq\omega B_{\text{Eff}}, \tag{5.110}$$

$$D = M(\omega^2 - k^2) + ib\frac{\omega^2 - k^2}{\omega} - q^2 N_C + \frac{q}{\mu} B_{\text{Eff}}, \tag{5.111}$$

where $B_{\text{Eff}} = B_C - 1 + \frac{2}{q}\Lambda(\omega)$.

which can only be solved if the determinant $AD - BC = 0$. This gives the wanted dispersion relation:

$$k^2 = \omega^2 - \frac{1}{\mu^2} + \frac{q^2}{\mu^2}\frac{\mu^2 N_C(M\omega^2 + ib\omega)\omega^2 + N_C(M\omega^2 + ib\omega) - \mu^2 q^2 N_C^2\omega^2 + 2\mu q N_C B_{\text{Eff}}\omega^2}{-(M\omega^2 + ib\omega)^2 + q^2 N_C(M\omega^2 + ib\omega) + q^2 B_{\text{Eff}}^2\omega^2}. \tag{5.112}$$

We note about this relation is that if no vortices are present, we end up with the simple dispersion relation

$$k^2 = \omega^2 - \frac{1}{\mu^2}, \tag{5.113}$$

showing that the excitations are massive, with a mass $\frac{1}{\mu}$. This is the well known dispersion relation we have found before, but without the k^4-term. This term is missing since we have made a long wavelength approximation. This “vacuum” dispersion relation is independent of the external magnetic field.

The dispersion relation is plotted in figure 5.2, where the density N_C of vortices is gradually increased. At some values of ω, the value of k is seen to grow or decay very rapidly. These points correspond to divergences in the dispersion relation of the damping factor b is set to zero. Near these points, there are regions where k diminishes as ω grows. This phenomenon is called anomalous dispersion.

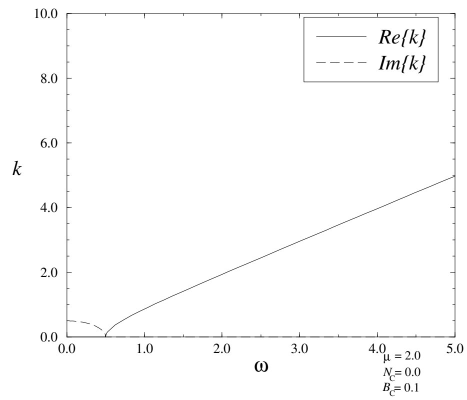
The points where the undamped system has a divergent dispersion relation may be regarded as resonance frequencies for the system. From the dispersion relation (5.112) we can find these points by setting $b = 0$. The divergent points are then found where the denominator in (5.112) is zero, i.e. at $\omega = 0$ and

$$\omega = \frac{q^2(N_C M + B_{\text{Eff}}^2)}{M^2}. \tag{5.114}$$

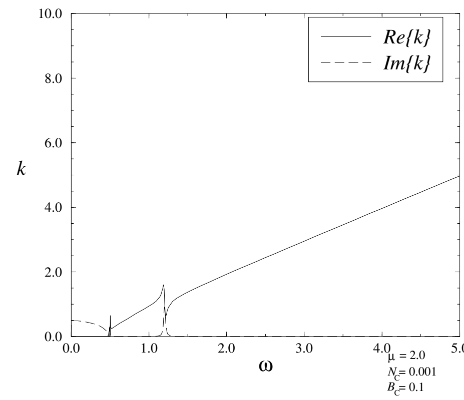
At this point, Løvvik found that the MCS theory has a narrow window of propagation. However, since the mass M and effective magnetic field B_{Eff} are dependent on the frequency, equation (5.114) may be fulfilled at several points. This is what we see in figure 5.2, where there are two resonance points with an area where plane waves may propagate between them.

The two separated areas where $\text{Re}\{k\} > 0$ in figure 5.2c,d may be interpreted as two different kinds of excitations. One can see that the single type of excitation for the “vacuum” system with no vortices, splits up into two modes, with the lower energy one having a higher value of $\frac{k}{\omega}$ than the “regular” mode, which has a higher energy.

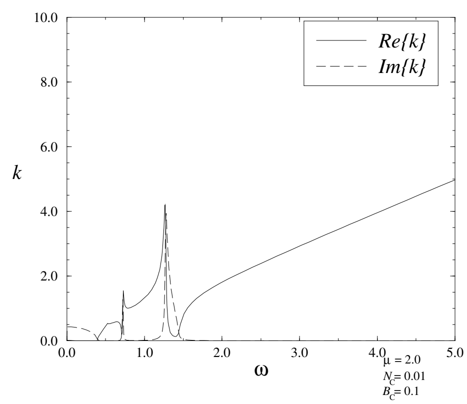
In figure 5.3, we have plotted the dispersion relation for the same parameters as in figure 5.2, except that $\mu = 25$. We see that qualitatively the results are the same, but one difference is evident: The mass of the lowest lying excitation increases when the density of vortices is increased before the new type of excitation appears.



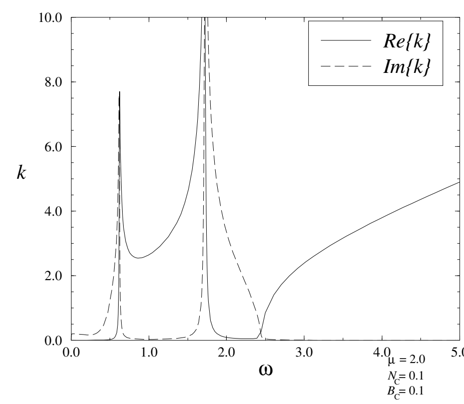
(a) $N_C = 0$



(b) $N_C = 0.001$



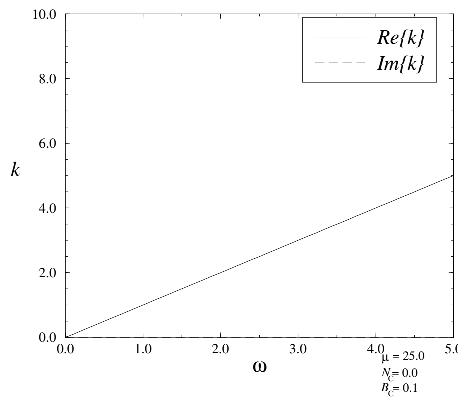
(c) $N_C = 0.01$



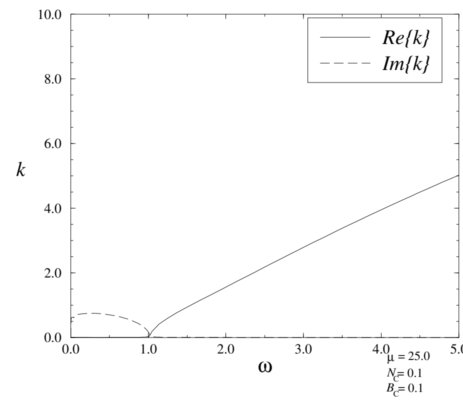
(d) $N_C = 0.1$

Figure 5.2: Plots of dispersion relation ($k(\omega)$) for $\mu = 2$, $b = 0.02$ and $B_C = 0.1$, and a varying density of vortices.

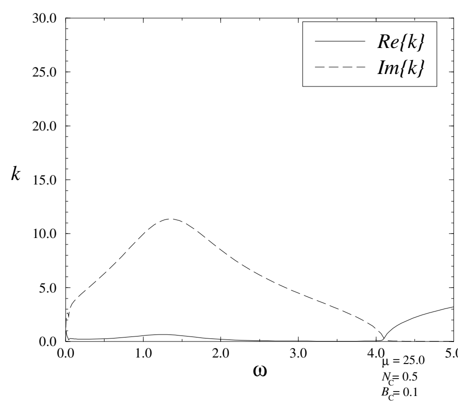
It would be nice if these phenomena could be related to phenomena in the fractional quantum Hall effect. However, there is a question of whether this model is a good description of the FQHE, and if so which values of μ would be physically relevant. There is also the mentioned problem with the frequency dependent mass $M(\omega)$ being negative. As there seems to be several problems in interpreting our model as a description of the FQHE, we have chosen to present the results without further discussion.



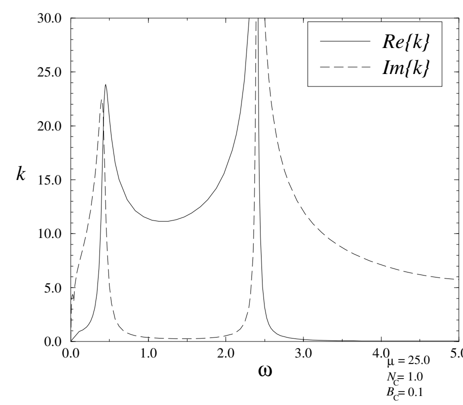
(a) $N_C = 0$



(b) $N_C = 0.1$



(c) $N_C = 0.5$



(d) $N_C = 1.0$

Figure 5.3: Plots of dispersion relation ($k(\omega)$) for $\mu = 25$, $b = 0.02$ and $B_C = 0.1$, and a varying density of vortices. Notice that the scale of the vertical axis is different on the two last plots than on the others.

Chapter 6

Conclusion

In this thesis we have studied some aspects of vortices in the Chern-Simons-Ginzburg-Landau (CSGL) theory for the fractional quantum Hall effect. The purpose has been to give a self-contained, comprehensive presentation of the properties of vortices in the CSGL theory, including results already obtained in the literature as well as our own results. We have provided some analytical results of CSGL theory, including how the charge and spin of vortices is given by their topological quantum number, and the asymptotic shape of the vortices far from the core. We have also reviewed the properties of the self-dual point of CSGL theory and its implication for the energy of vortices.

One major goal of the thesis has been to show how the Ginzburg-Landau (GL) theory for superconductors and superfluids and the Maxwell-Chern-Simons (MCS) theory for anyons in an electromagnetic field are connected to the Chern-Simons-Ginzburg-Landau theory. This has been studied both analytically and numerically. We have shown how the equations for the CSGL theory reduce to those of the GL theory in a certain limit, and how the MCS theory solution also may be obtained as a limit of the CSGL theory.

The main focus of the thesis has been the numerical results. We have solved the CSGL equations for a vortex numerically for a range of the dimensionless parameter, and shown how the size and energy of a vortex depends on this parameter. We have also studied the connection between the CSGL theory and the GL and MCS theories numerically, and found support for our analytical results.

In addition to the pure CSGL theory, we have studied some natural extensions of the CSGL theory. The first extension we have studied is the addition of a dynamical magnetic field. We have shown how the charge is no longer quantized when the magnetic field is made dynamical. The main study of the extensions has been numerical. We have shown how the inclusion of a dynamical magnetic field changes the size, energy and charge of a vortex, and we have found that the self-dual point of pure CSGL theory extends to a self-dual line.

The second extension we have studied is the extension of the CSGL wave function to a two-component spinor. We have shown how this extension allows another kind of vortex solutions, known as skyrmions, and studied some properties of these objects. Again, the main study has been numerical. We have shown how the size and spin of the skyrmions depend on the effective gyromagnetic ratio, and we have reproduced qualitatively results found by a different kind of study of a spin-dependent model for the fractional quantum Hall effect: that the lowest lying excitations for some parameter values is doubly charged skyrmions. Using our numerical results, we have obtained a phase diagram for the spin dependent CSGL theory.

The last part of this thesis has been devoted to the duality between the CSGL theory and the MCS theory. We have made a detailed derivation of the duality starting from the Lagrangian of CSGL theory. We have also attempted to use this duality to find a better description of the dynamics of vortices and a dispersion relation for a system with a gas of free vortices. Unfortunately, the results we have found have been difficult to interpret, so we have decided to present the results without attempting to give a full interpretation. We can only conclude that in this area there is still room for further study.

Appendix A

Notation

A.1 Abbreviations

2DEG Two-dimensional electron gas.

CSGL Chern-Simons-Ginzburg-Landau.

FQHE Fractional quantum Hall effect.

GL Ginzburg-Landau.

GPG Ginzburg-Pitaevskii-Gross.

IQHE Integer quantum Hall effect.

MCS Maxwell-Chern-Simons.

MCSGL Maxwell-Chern-Simons-Ginzburg-Landau.

QHE Quantum Hall effect.

A.2 Symbols

Greek indices ($\mu, \nu, \ldots$) can have values 0, 1, or 2. Latin indices ($i, j \ldots$) can have values 1 or 2. Components of 3-vector quantities are written as v^{μ} (contravariant) or v_{μ} (covariant), where $v_i = -v^i$. Components of 2-vectors are written as v^i, while the 2-vector it self is written in bold face: $\mathbf{v}$. Where indices are repeated, they are to be summed.

$A^{\mu}(x)$ Electromagnetic (gauge) field.

$a^{\mu}(x)$ Statistical gauge field.

$\mathcal{A}^{\mu}(x)$ Total gauge field, $\mathcal{A}^{\mu} = A^{\mu} + a^{\mu}$.

$B(x)$	Magnetic field (physical), $B = \epsilon^{ij}\partial_i A^j$.
$b(x)$	Statistical magnetic field, $b = \epsilon^{ij}\partial_i a^j$.
$\mathcal{B}(x)$	Total magnetic field, $\mathcal{B} = B + b$.
D^μ	Covariant derivative, $D_\mu = \partial_\mu + ie\mathcal{A}_\mu$.
E	Energy.
$\mathbf{E}(x)$	Electric field (physical), $E^i = -\partial_i A^0 - \partial_0 A^i$.
e	Electron charge ($e = -\|e\|$).
$\mathbf{e}(x)$	Statistical electric field, $e^i = -\partial_i a^0 - \partial_0 a^i$.
$\mathcal{E}(x)$	Energy density.
$\boldsymbol{\mathcal{E}}(x)$	Total electric field, $\boldsymbol{\mathcal{E}} = \mathbf{E} + \mathbf{e}$.
$F^{\mu\nu}$	Electromagnetic field tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
$f(\mathbf{r})$	Real part of wave function, $f(\mathbf{r}) = \|\phi(\mathbf{r})\|$.
$\mathbf{j}(x)$	Particle current density.
$j^\mu(x)$	3-current density.
L	Lagrangian functional.
$\mathcal{L}(x)$	Lagrangian density, $L = \int \mathcal{L} d^2x$.
m	Effective electron mass.
$\mathbf{p}$	Momentum.
q, Q	Charge.
$\mathbf{r}$	Coordinate vector.
r	Radial coordinate, $r = \|\mathbf{r}\|$.
t	Time coordinate.
$\mathbf{v}$	Velocity.
v^μ	3-velocity, $v^\mu = (1, \mathbf{v})$.
x^μ	Space-time coordinate, $x^\mu = (t, \mathbf{r})$.
$Z^\mu(x)$	MCS electromagnetic field (used in chapter 5).
z	Complex coordinate, $z = x + iy$.

$\alpha(r)$ — $\alpha(r) = r[A^\theta(r) + a^\theta(r)]$.

β^μ — 3-velocity, $\beta^\mu = v^\mu$.

γ — $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

$\delta^\mu{}_\nu$ — Kronecker-delta: $\delta^\mu{}_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$.

$\epsilon^{\mu\nu\sigma}$ — Totally antisymmetric tensor: $\epsilon^{012} = 1, \quad \epsilon^{\mu\nu\sigma} = \epsilon^{\nu\sigma\mu} = -\epsilon^{\nu\mu\sigma}$

ζ — Length scale of exponential decay of vortex, see equation (3.75).

θ — Polar coordinate.

κ — Dimensionless parameter of MCSGL theory, $\kappa^2 = \frac{2\lambda m^2}{\mu_0 e^2}$.

λ — Parameter of CSGL theory. In section 2.4.4, London penetration depth, $\lambda = \sqrt{\frac{m}{\mu_0 q^2 \rho}}$.

μ — Parameter of CSGL theory and MCS theory.

$\hat{\mu}$ — Dimensionless parameter of CSGL theory, $\hat{\mu} = \frac{m\lambda}{e^2}$.

ρ — Electron density, $\rho = \phi^*\phi$ (In section 4.2, $\rho = \phi^\dagger\phi$).

ρ^{ij} — Resistivity tensor, $j^i = \rho^{ij} E^j$.

ρ_H — Hall resistivity, $\rho_H = \rho^{xy}$.

ρ_L — Longitudinal resistivity, $\rho_L = \rho^{xx}$.

σ^{ij} — Conductivity tensor, $E^i = \sigma^{ij} j^j$.

σ_H — Hall conductivity, $\sigma_H = \sigma^{xy}$

σ_L — Longitudinal conductivity, $\sigma_H = \sigma^{xx}$

$\phi(\mathbf{r})$ — Wave function (bosonic).

$\chi(r)$ — Asymptotic form of $1 - f(r)$ as $r \to \infty$, see section 3.5.3.

$\psi(r)$ — Asymptotic form of $a_0(\mathbf{r})$, see section 3.5.3.

$\psi(\mathbf{r})$ — In section 2.3.4: Destruction operator for electrons

$\omega(r)$ — Asymptotic form of $s - \alpha(r)$, see section 3.5.3.

Appendix B

Special Functions

This appendix summarizes some properties of two kinds of special functions used in the thesis, based on Butkov[58].

B.1 Modified Bessel Functions

The modified Bessel functions are solutions of the equation

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \left(1 + \frac{m^2}{x^2}\right)\right] f(x) = 0, \tag{B.1}$$

where m in general may be and real number, but we will only consider the case where m is an integer. There are two kinds of modified bessel functions. The modified bessel functions of the first kind, $I_m(x)$, are given by the power series

$$I_m(x) = \sum_{k=0}^{\infty} \frac{x^{m+2k}}{k!(m+k)!2^{m+2k}}, \qquad m = 0, \pm 1, \pm 2, \dots. \tag{B.2}$$

The functions I_m obey the following equations:

$$I_{m-1} - I_{m+1} = \frac{2m}{x} I_m(x), \tag{B.3}$$

$$I_{m+1} = -\frac{m}{x} I_m(x) + \frac{dI_m(x)}{dx}, \tag{B.4}$$

$$I_{m-1} = \frac{m}{x} I_m(x) + \frac{dI_m(x)}{dx}, \tag{B.5}$$

$$\frac{d}{dx}[x^m I_m(x)] = x^m I_{m-1}(x), \tag{B.6}$$

$$\frac{d}{dx}[x^{-m} I_m(x)] = x^{-m} I_{m+1}(x). \tag{B.7}$$

The asymptotic form of $I_m(x)$ as $x \to \infty$ is given by

$$I_m(x) = \frac{e^x}{\sqrt{2\pi x}} + \mathcal{O}(x^{-\frac{3}{2}}). \tag{B.8}$$

The modified Bessel functions of the second kind, $K_m(x)$, are defined by

$$K_m(x) = \frac{(-1)^m}{2}\left[\frac{\partial I_{-\mu}(x)}{\partial\mu} - \frac{\partial I_{\mu}(x)}{\partial\mu}\right]_{\mu} = m. \tag{B.9}$$

These functions have a logarithmic divergence at the origin. They obey the following equations:

$$K_{m-1}(x) - K_{m+1}(x) = -\frac{2m}{x}K_m(x), \tag{B.10}$$

$$K_{m+1}(x) = \frac{m}{x}K_m(x) - \frac{dK_m(x)}{dx}, \tag{B.11}$$

$$K_{m-1}(x) = -\frac{m}{x}K_m(x) - \frac{dK_m(x)}{dx}, \tag{B.12}$$

$$\frac{d}{dx}[x^m K_m(x)] = -x^m K_{m-1}(x), \tag{B.13}$$

$$\frac{d}{dx}[x^{-m} K_m(x)] = -x^{-m} K_{m+1}(x). \tag{B.14}$$

The asymptotic form is

$$K_m(x) = \sqrt{\frac{\pi}{2x}}e^{-x} + \mathcal{O}(x^{-\frac{3}{2}}). \tag{B.15}$$

B.2 Sine and Cosine Integral Functions

The sine and cosine integral functions, $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ respectively, are defined in terms of integrals:

$$\mathrm{Si}(x) \equiv \int_0^{\infty} \frac{\sin(\xi)}{\xi}d\xi, \tag{B.16}$$

$$\mathrm{Ci}(x) \equiv -\int_x^{\infty} \frac{\cos(\xi)}{\xi}d\xi. \tag{B.17}$$

While $\mathrm{Si}(x)$ vanishes at the origin, the function $\mathrm{Ci}(x)$ has a logarithmic singularity[59]:

$$\mathrm{Ci}(x) = \gamma + \ln(x) + \mathcal{O}(x^2), \tag{B.18}$$

where γ is Euler's constant,

$$\gamma \equiv \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i} - \ln(n)\right) \approx 0.5772156649\ldots. \tag{B.19}$$

Appendix C

The Path Integral Method

This appendix provides a concise introduction to the path integral formulation of quantum mechanics. It is based mainly on Peskin and Schroeder[60] and Shankar[61].

C.1 Path Integrals in Quantum Mechanics

In the 1940s, Richard Feynman invented a completely new formulation of quantum mechanics. The *path integral* formulation does not use a wave function or state vectors. Instead, it provides a way to directly calculate the amplitude for a process where a particle moves from one position to another, using the classical Lagrange function of the system. It produces the same results as and is equivalent to the more well-known Schrödinger and Heisenberg formulations.

The path integral formulation states that to find the amplitude for a process, calculate a phase factor $e^{iS[x(t)]}$ for each possible path $x(t)$ of the particle, where $S[x(t)]$ is the classical action for the path. Then sum all these phase factors to get the (total) amplitude. The amplitude for a process where a particle moves from x_i to x_f during a time T may then be written

$$\langle x_f| e^{-iHT} |x_i\rangle = \sum_{x(t)} e^{iS[x(t)]} \equiv \int \mathcal{D}x(t) e^{iS[x(t)]}, \tag{C.1}$$

where we have introduced the path integral measure $\mathcal{D}x(t)$.

To use the above expression, it is necessary to find a way to calculate the functional integral. The customary way is to discretize in time, reducing the continuous path $x(t)$ into a set of points $x(t_k)$, where $t_k = t_i + k\epsilon$, $k = 0, 1, 2, \ldots, N$. Letting $x_k \equiv x(t_k$, we then define the measure $\mathcal{D}x(t)$ as

$$\mathcal{D}x(t) = \frac{dx_1 dx_2 \cdots dx_{N-1}}{C^N} = \frac{1}{C} \prod_{k=1}^{N-1} \frac{dx_k}{C}. \tag{C.2}$$

The integral over all paths is then reduced to $N-1$ integrals over $\mathbb{R}$. The constant C appearing in (C.2) may in general depend on the "mesh spacing" $\epsilon = \frac{T}{N}$. We must also discretize the action. For a particle in a potential $V(x)$, the discretized action becomes

$$S = \int_{x_i}^{x_f} dt \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) \to \sum_{k=0}^{N-1} \left[\frac{m}{2}\frac{(x_{k+1}-x_k)^2}{\epsilon} - \epsilon V\left(\frac{x_{k+1}+x_k}{2}\right)\right]. \quad \text{(C.3)}$$

We may now calculate the value of the constant C by considering the contribution to the total amplitude from the last time slice. By (C.1) and (C.2), we have

$$\begin{aligned} &\langle x_f | e^{-iHT} | x_i \rangle \\ &= \int_{-\infty}^{\infty} \frac{dx_{N-1}}{C} \exp\left[im\frac{(x_f - x_{N-1})^2}{2\epsilon} - i\epsilon V\left(\frac{x_f + x_{N-1}}{2}\right)\right] \langle x_{N-1} | e^{-iH(T-\epsilon)} | x_i \rangle . \end{aligned} \quad \text{(C.4)}$$

We may then use the fact that the first term in this expression will tend to a delta function as $\epsilon \to 0$, so that we may expand in powers of $(x_{N-1} - x_f)$:

$$\begin{aligned} \langle x_f | e^{-iHT} | x_i \rangle &= \int_{-\infty}^{\infty} \frac{dx_{N-1}}{C} \exp\left[im\frac{(x_f - x_{N-1})^2}{2\epsilon}\right] [1 - i\epsilon V(x_f) + \cdots] \\ &\cdot \left[1 + (x_{N-1} - x_f)\frac{\partial}{\partial x_f} + \frac{1}{2}(x_{N-1}-x_f)^2 \frac{\partial^2}{\partial {x_f}^2} + \cdots\right] \langle x_f | e^{-iH(T-\epsilon)} | x_i \rangle . \end{aligned} \quad \text{(C.5)}$$

This integral may be easily evaluated, yielding

$$\langle x_f | e^{-iHT} | x_i \rangle = \left(\frac{1}{C}\sqrt{\frac{2\pi i\epsilon}{m}}\right)\left[1 - i\epsilon V(x_f) + \frac{i\epsilon}{2m}\frac{\partial^2}{\partial {x_f}^2} + \mathcal{O}(\epsilon^2)\right] \langle x_f | e^{-iH(T-\epsilon)} | x_i \rangle . \quad \text{(C.6)}$$

In the limit $\epsilon \to 0$, we must then have $C = \sqrt{\frac{2\pi i\epsilon}{m}}$. Comparing terms of order ϵ in the above expression, we then have

$$i\frac{\partial}{\partial T} \langle x_f | e^{-iHT} | x_i \rangle = \left[-\frac{1}{2m}\frac{\partial^2}{\partial {x_f}^2} + V(x_f)\right] \langle x_f | e^{-iHT} | x_i \rangle , \quad \text{(C.7)}$$

which is simply the Schrödinger equation. Thus we have justified the claim that the path integral formulation is equivalent to the Schrödinger formalism.

C.2 Path Integrals and the Aharonov-Bohm Effect

In section 2.2.4 we explained the Aharonov-Bohm effect by stating that particles pick up a phase factor when moving around a string of flux. This statement is easily understood using the path integral formalism. For a particle in an electromagnetic field, the classical Lagrangian is

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) - q\phi(\mathbf{r}), \qquad \text{(C.8)}$$

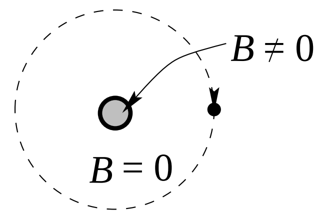


Figure 2.5: The Aharonov-Bohm Effect

where m is the mass and q is the charge of the particle. For a purely magnetic field, the vector field $\mathbf{A}$ gives the magnetic field B by $\boldsymbol{\nabla} \times \mathbf{A} = B$, and the potential ϕ vanishes. Thus, the contribution to the phase factor for a path $\mathbf{r}(t)$ going one complete turn around the solenoid is

$$e^{i\Delta S} = e^{i\int L(\mathbf{r}(t))dt} = e^{i\Delta S_0} e^{\left(iq\int_{t_i}^{t_f} \dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r})dt\right)} = e^{i\Delta S_0} e^{\left(iq\int \mathbf{A}\cdot d\mathbf{r}\right)}, \qquad \text{(C.9)}$$

where $e^{i\Delta S_0}$ is the contribution to the phase factor for the same path when there is no magnetic field. The last integral may be evaluated using the theorem of Stokes, giving

$$e^{i\Delta S} = e^{i\Delta S_0} e^{\left(iq\int_S d^2x \boldsymbol{\nabla}\times\mathbf{A}\right)} = e^{i\Delta S_0} e^{iq\Phi}, \qquad \text{(C.10)}$$

where S is the area enclosed by the path $\mathbf{r}(t)$, and Φ is the flux penetrating this area. For a setup like the one in figure 2.5, where all the flux is contained in a small area inaccessible to the particles, all paths circling the flux string once will thus gain the same additional phase factor $e^{iq\Phi}$ relative to the phase they would gain if the flux string was not there. This means that the total phase factor will have exactly the same addition.

Appendix D

Numerical Methods for Ordinary Differential Equations

This appendix will provide a description of the relaxation method used for numerical solution of the vortex equations in this thesis. The method used is based on the method in the book *Numerical Recipes in C* by William H. Press et.al.[62] and this introduction is based on that book.

D.1 Differential Equations and Boundary Values

Before we start the discussion about how to solve the differential equations (3.52), (3.55), and (3.56) numerically, we will review some basic features of systems of differential equations and their solutions. This discussion will be limited to ordinary differential equations (ODEs), e.g. equations where all the unknown functions are functions of the same variable, which we will call x.

First, we note that all systems of ODEs may be written on the form

$$\begin{aligned} F_1(y_1,\ldots,y_N;y'_1,\ldots,y'_N;y''_1,\ldots,y''_N;\ldots) &= 0 \\ F_2(y_1,\ldots,y_N;y'_1,\ldots,y'_N;y''_1,\ldots,y''_N;\ldots) &= 0 \\ &\vdots \\ F_M(y_1,\ldots,y_N;y'_1,\ldots,y'_N;y''_1,\ldots,y''_N;\ldots) &= 0, \end{aligned} \tag{D.1}$$

where $y_k \equiv y_k(x)$, $y'_k \equiv \frac{dy_k}{dx}$, etc., and $F_1 \ldots F_M$ are given functions. We also note that all such systems may be reduced to *first-order* systems only including a set $\tilde{y}_1,\ldots,\tilde{y}_{\tilde{N}}$ and their first derivatives $\tilde{y}'_1,\ldots,\tilde{y}'_{\tilde{N}}$, simply by defining new functions $\tilde{y}_i = y'_j$ where necessary. We will also *assume* that this set of equations may be brought onto the form

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2, \dots, y_N) \\ y_2' &= f_2(x, y_1, y_2, \dots, y_N) \\ &\vdots \\ y_N' &= f_N(x, y_1, y_2, \dots, y_N), \end{aligned} \tag{D.2}$$

which may be conveniently written on vector form

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}). \tag{D.3}$$

In addition to the set of equations (D.3), we will need a set of initial conditions

$$\mathbf{g}(\mathbf{y}(x^i)) = 0, \tag{D.4}$$

or a set of boundary conditions

$$\mathbf{g}^i(\mathbf{y}(x^i)) = 0, \qquad \mathbf{g}^f(\mathbf{y}(x^f)) = 0. \tag{D.5}$$

Note that the number of equations, the number of unknown functions and the number of initial conditions or the total number of boundary conditions must be the same. When the conditions on the solution is given as initial conditions (D.4), the problem consisting of (D.3) and (D.4) is called an *initial value problem.* The goal will then usually be to find a solution on the half-line $[x_0, \infty)$ or $(-\infty, x_0]$. When the problem is given as (D.3) and (D.5), it is called a *two-point boundary value problem*, and the goal will (usually) be to find a solution on the interval between points x_i and x_f.

There are three questions of particular interest when it comes to differential equations representing physical systems:

1) Is there a solution to the problem?
2) Is the solution unique?
3) Is the solution stable with respect to the initial conditions?

We will not go into detail on these questions, but we will mention a general theorem applicable to initial value problems:

Theorem D.1 *Let the functions* $\mathbf{f}$ *and* $\frac{\partial \mathbf{f}}{\partial y_1}, \frac{\partial \mathbf{f}}{\partial y_2}, \dots, \frac{\partial \mathbf{f}}{\partial y_N}$ *be continuous in a region* R *defined by* $x_i < x < x_f, y_1^i < y_1 < y_1^f, \dots, y_N^i < y_N < y_N^f$ *and let the point* $(x_0, \mathbf{y}(x_0))$ *be in* R*. Then there is a neighborhood of* x_0 *where there exists a unique solution* $\mathbf{y} = \boldsymbol{\phi}(x)$ *of the system (D.3) with initial conditions (D.4).*

Note that the neighborhood where the solution is valid might be very small. However, if we can find a solution which is continuous in some interval of x, the theorem guarantees that it is unique.

For two-point boundary value problems, the situation is more complicated, and it is not possible to say anything in general about the existence and uniqueness of solutions.

D.2 Numerical Methods

The basic principle behind all numerical methods for solving differential equations is to approximate the continuous interval $x = [a, b]$ on which one wants to solve the equations with a finite set of points (a *mesh*) $\{x_i\}_{i=0,1,2,\ldots,M}$, where $x_0 = a$ and $x_M = b$. The points x_i are in the simplest methods equally spaced, so that $x_i = a + i\Delta x$ where $\Delta x = \frac{b-a}{N-1}$.

The very simplest method for integrating an initial value problem is *Euler's method*. It involves approximating derivation with $\frac{d\mathbf{y}}{dx} \to \frac{\Delta \mathbf{y}}{\Delta x}$, i.e.

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y}) \quad \to \quad \mathbf{y}(x_{k+1}) = \mathbf{y}(x_k) + \Delta x \mathbf{f}(x_k) \tag{D.6}$$

This method illustrates the general theory of most numerical methods for initial value problems. One starts at one end of the interval, and then integrates towards the other end. Such a method is not directly applicable to a two point boundary value problem, where the solution depends on the boundary values in both ends of the interval.

One method for solving a two point boundary value problem is called the "shooting" method. The principle of this method is to use an iterative method like (D.6) anyway, guessing the unknown function values at the first boundary point. One may then compare the actual resulting values of the functions at the second boundary with the boundary conditions at this point, correct[1] the guessed boundary values at the first point if necessary and repeat this process until sufficient accuracy is reached.

For problems which are very sensitive to the boundary values, "shooting" is not practical. Such problems will require very accurate "guessing" of the unknown function values at the first boundary, and will at best result in very slow convergence towards the solution. The limited precision of computer systems might also have the result that a sufficiently accurate solution can not be found. We have found that for the vortex problem of CSGL theory, another method is superior by far, the so-called *relaxation* method.

To describe the idea of the relaxation method, we return to the problem of how to apply Euler's method (D.6) to a two point boundary value problem. If the set of differential equations is *linear*, this is actually a solvable problem. Defining $\mathbf{y}_k \equiv \mathbf{y}(x_k)$, (D.6) becomes

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \Delta x \mathbf{f}(x_k, \mathbf{y}_k). \tag{D.7}$$

Written out for the whole interval, this becomes

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{y}_0 + \Delta x \mathbf{f}(x_0, \mathbf{y}_0) \\
\mathbf{y}_2 &= \mathbf{y}_1 + \Delta x \mathbf{f}(x_1, \mathbf{y}_1) \\
&\vdots \\
\mathbf{y}_M &= \mathbf{y}_{M-1} + \Delta x \mathbf{f}(x_{M-1}, \mathbf{y}_{M-1}).
\end{aligned} \tag{D.8}$$

[1]In practice, a numerical method for finding the root of a function, such as Newton's method, would be used to find out how to correct the guessed values

The boundary conditions are now in $\mathbf{y}_0$ and $\mathbf{y}_M$. The above set of equations then consist of MN equations and MN unknowns. This system may be solved with a matrix method, although if the number of points M is large, we quickly get very large matrices! Unfortunately, for non-linear equations there is no general method to obtain an exact solution.

The relaxation method overcomes this problem by considering small deviations from a set of trial functions $\{\mathbf{y}_k\}$ only. If we assume that we have a good approximation to the solution, $\{\bar{\mathbf{y}}_k\}$, a better approximation $\mathbf{y}_k = \bar{\mathbf{y}}_k + \delta\mathbf{y}_k$ may be found by linearizing equation (D.7) in $\delta\mathbf{y}_k$ and solving it, which is now easy to do since the system is linear.

D.3 The Relaxation Method

The relaxation method used to compute the results in this thesis uses a variant of the Euler scheme (D.7). All function values are evaluated between grid points. This results in the following scheme:

$$\mathbf{E}_k \equiv \mathbf{y}_k - \mathbf{y}_{k-1} - (x_k - x_{k-1})\mathbf{f}\left(\frac{1}{2}(x_k + x_{k-1}), \frac{1}{2}(\mathbf{y}_k + \mathbf{y}_{k-1}\right) = 0, \qquad k = 1, 2, \ldots, M-1. \quad \text{(D.9)}$$

These $M-1$ equations together with the boundary conditions

$$\mathbf{E}_0 \equiv \mathbf{g}^i(\mathbf{y}_0) = 0 \quad \text{(D.10)}$$

$$\mathbf{E}_M \equiv \mathbf{g}^f(\mathbf{Y}_M) = 0 \quad \text{(D.11)}$$

defines our numerical problem. The vectors $\mathbf{E}_0$ and $\mathbf{E}_M$ have only n^i and n^f components not identical to zero, respectively, where n^i is the number of boundary conditions at the first point and n^f is the number of boundary conditions at the last point.

As mentioned, we will attempt to find a solution to this problem by considering small deviations from a trial function $\bar{\mathbf{y}}_k$. If we assume that $\bar{\mathbf{y}}_k$ is a good approximation to the solution of the problem, we may expand the functions $\mathbf{E}_k(\mathbf{y}_k, \mathbf{y}_{k-1})$ by

$$\begin{aligned}\mathbf{E}_k(\mathbf{y}_k, \mathbf{y}_{k-1}) &= \mathbf{E}_k(\bar{\mathbf{y}}_k + \Delta\mathbf{y}_k, \bar{\mathbf{y}}_{k-1} + \Delta\mathbf{y}_{k-1}) \\ &\approx \mathbf{E}_k(\bar{\mathbf{y}}_k, \bar{\mathbf{y}}_{k-1}) + \sum_{n=1}^{N} \frac{\partial \mathbf{E}_k}{\partial y_{n,k-1}} \Delta y_{n,k-1} + \sum_{n=1}^{N} \frac{\partial \mathbf{E}_k}{\partial y_{n,k}} \Delta y_{n,k}, \quad \text{(D.12)}\end{aligned}$$

where $\Delta\mathbf{y}_k \equiv \mathbf{y}_k - \bar{\mathbf{y}}_k$, and $\mathbf{y}_k$ is the true solution to the problem. Since $\mathbf{E}_k(\mathbf{y}_k, \mathbf{y}_{k-1}) = 0$, for each interior point x_k, $k = 1, \ldots, M$ we thus have the equation

$$\sum_{n=1}^{N} S_{j,n} \Delta y_{n,k-1} + \sum_{n=N+1}^{2N} S_{j,n} \Delta y_{n-N,k} = -E_{j,k}, j = 1, 2, \ldots, N, \quad \text{(D.13)}$$

where $(S_{j,n})$ is an $N \times 2N$ matrix for each k,

$$S_{j,n} \equiv \begin{cases} \frac{\partial E_{j,k}}{\partial y_{n,k-1}}, & n = 1, 2, \ldots, N \\ \frac{\partial E_{j,k}}{\partial y_{n-N,k-1}}, & n = N+1, N+2, \ldots, 2N \end{cases} . \tag{D.14}$$

At the boundary points, similar equations may be found. At the first point, we have

$$\sum_{n=1}^{N} S_{j,n} \Delta y_{n,0} = -E_{j,0}, \quad j = n^f + 1, n^f + 2, \ldots, N S_{j,n} \equiv \frac{\partial E_{j,0}}{\partial y_{n,0}}, \tag{D.15}$$

and at the last point we have

$$\sum_{n=1}^{N} S_{j,n} \Delta y_{n,M} = -E_{j,M}, \quad j = 1, 2, \ldots, n^f S_{j,n} \equiv \frac{\partial E_{j,M+1}}{\partial y_{n,M+1}}. \tag{D.16}$$

Note that for technical reasons, the matrix $(S_{j,n})$ at the first boundary is defined only for the n^i values of j from $n^f + 1$ to N (recall that the total number of boundary conditions, $n^i + n^f$ equals the number of equations N.)

The matrices $(S_{j,n})$ at each mesh point k may now be put together in one huge $NM \times NM$ matrix S which obeys the equation

$$\mathrm{S}\Delta\mathbf{y} = -\mathbf{E}, \tag{D.17}$$

and where the vectors $\mathbf{y}$ and $\mathbf{E}$ are given by

$$\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_M) \tag{D.18}$$

$$\mathbf{E} = (\mathbf{E}_0, \mathbf{E}_1, \ldots, \mathbf{E}_M). \tag{D.19}$$

Equation (D.17) may be solved by Gaussian elimination. Since the matrix has a special structure where only elements close to the diagonal are non-zero, an optimized Gaussian elimination procedure may be used, but we will not enter the technical details here.

The procedure of solving equation (D.17) for $\Delta\mathbf{y}$ is repeated until the accuracy of the solution is adequate.

Bibliography

[1] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[2] D. C. Tsui, H. L. Störmer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).

[3] J. M. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977).

[4] W. J. Duffin, *Electricity and Magnetism* (McGraw Hill, London, 1990).

[5] D. C. Tsui, Physica B **164**, 59 (1990).

[6] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).

[7] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 484 (1959).

[8] R. G. Chambers, Phys. Rev. Lett. **5**, 3 (1960).

[9] Ehrenberg and Siday, Proc. Phys. Soc. London B **62**, 8 (1949).

[10] *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).

[11] A. Karlhede, S. A. Kivelson, and S. L. Sondhi, in *Correlated electron systems*, edited by V. J. Emery (World Scientific, Singapore, 1993).

[12] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[13] B. I. Halperin, Phys. Rev. Lett **52**, 1583 (1984).

[14] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).

[15] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[16] D. R. Tilley and J. Tilley, *Superfluidity and Superconductivity*, 3rd ed. (Adam Hilger, Bristol, 1990).

[17] F. Ravndal, *Lecture Notes on Statistical Physics* (Institute of Physics, University of Oslo, 1995).

[18] V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950).

[19] V. L. Ginzburg and L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **34**, 1240 (1958).

[20] P. Nozières and D. Pines, *The Theory of Quantum Liquids* (Addison-Wesley, New York, 1990), Vol. 2.

[21] E. P. Gross, Nuovo Cimento **20**, 454 (1961).

[22] L. P. Pitaevskii, Sov. Phys. JETP **12**, 155 (1961).

[23] A. A. Abrikosov, Sov. Phys. JETP **5**, 1174 (1957).

[24] E. Madelung, Z. Phys **40**, 322 (1927).

[25] L. K. Myklebust, *Quantized Vortices*, Cand. scient. thesis, University of Oslo, 1996.

[26] L. Onsager, Nuovo Cimento **6**, Suppl. 2, 249 (1949).

[27] R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland, Amsterdam, 1955), Vol. 1, Chap. 2.

[28] W. F. Vinen, Nature **181**, 1524 (1958).

[29] W. F. Vinen, Proc. R. Soc. A **260**, 218 (1961).

[30] M. P. Kawatra and R. K. Pathria, Phys. Rev. **151**, 132 (1966).

[31] W. H. Kleiner, L. M. Roth, and S. H. Autler, Phys. Rev. A **133**, 1226 (1964).

[32] U. Essman and H. Träuble, Phys. Lett. A **24**, 526 (1967).

[33] S. S. Chern and J. Simons, Pro. Nat. Acad. Sci. **68**, 791 (1971).

[34] O. M. Løvvik, *On Maxwell–Chern–Simons Theory*, Cand. scient. thesis, University of Oslo, 1992.

[35] S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).

[36] N. Read, Phys. Rev. Lett. **62**, 86 (1989).

[37] S. C. Zhang, Int. J. Mod. Phys. B **6**, 25 (1992).

[38] M. Stone, Phys. Rev. B **42**, 212 (1990).

[39] S. Curnoe and N. Weiss, Int. J. Mod. Phys. A **11**, 329 (1996).

[40] J. M. Leinaas and S. Viefers, cond-mat/9712009 (unpublished).

[41] R. Tafelmayer, Nucl. Phys. B **396**, 386 (1993).

[42] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. **24**, 449 (1976).

[43] Z. F. Ezawa, M. Hotta, and A. Iwazaki, Phys. Lett. B **261**, 443 (1991).

[44] Z. F. Ezawa, M. Hotta, and A. Iwazaki, Phys. Rev. D **44**, 3906 (1991).

[45] R. Jackiw and S.-Y. Pi, Phys. Rev. Lett. **64**, 2969 (1990).

[46] J.-M. Duan, Phys. Rev. B **49**, 12381 (1994).

[47] B. I. Halperin, Helv. Phys. Acta **56**, 75 (1983).

[48] K. Moon *et al.*, Phys. Rev. B **51**, 5138 (1995).

[49] T. Chakraborty and P. Pietiläinen, *The Fractional Quantum Hall Effect* (Springer-Verlag, Berlin, 1988).

[50] D. Lilliehöök, *Generalized Quantum Hall Skyrmions*, Filosofie licentiatexamen, Stockholm University, 1998.

[51] S. L. Sondhi, A. Karlhede, S. A. Kivelson, and E. H. Rezayi, Phys. Rev. B **47**, 16419 (1993).

[52] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).

[53] D. Lilliehöök, K. Lejnell, A. Karlhede, and S. L. Sondhi, Phys. Rev. B **56**, 6805 (1997).

[54] M. P. A. Fisher and D. H. Lee, Phys. Rev. B **39**, 2756 (1989).

[55] D.-H. Lee and M. P. A. Fisher, Phys. Rev. Lett. **63**, 903 (1989).

[56] D.-H. Lee and S.-C. Zhang, Phys. Rev. Lett. **66**, 1220 (1991).

[57] D. P. Arovas and J. A. Freire, Phys. Rev. B **55**, 1068 (1997).

[58] E. Butkov, *Mathematical Physics* (Addison-Wesley, Reading, Massachusetts, 1968).

[59] *Handbook of Mathematical Functions*, edited by M. Abranowitz and I. E. Stegun (National Bureau of Standards, New York, 1970).

[60] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, Massachusetts, 1995).

[61] R. Shankar, *Principles of Quantum Mechanics* (Plenum Press, New York, 1994).

[62] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C*, 2nd ed. (Cambridge University Press, Cambridge, 1992).