Chain Theorems of Lines Circles and Planes

by

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Introduction

In this paper we will prove Clifford chain theorem for general lines in the plane by using real cross ratio lemma. We will then discuss the Clifford chain theorem for degenerate lines and will obtain 6 types of Clifford figures with examples. We will also find that the Clifford chain theorem fails for some cases and we will show some examples. At the end we will obtain three similar chain theorems for circles and planes by applying Clifford chain theorem for general lines in the plane under the transformations of stereographic projection and circle inversion. These three chain theorems are
(1): chain theorem for general circles on the sphere,
(2): chain theorem for general circles in the plane and
(3): chain theorem for general planes in the space.
We will also prove Miquel's pentagon theorem by applying Clifford chain theorem for four general lines.

In 1871, W. K. Clifford announced a series of theorems which we call Clifford line chains. Two straight lines determine a point, which is the intersection point. Three straight lines determine a circle, which is the circumcircle of the triangle formed by the three lines. In a 4 lines case, the four circumcircles, of the four triangles formed by the four lines taking three at each time, pass through a point, the Wallace or focal point of the 4 lines. In a 5 lines case, the five Wallace points obtained by omitting in turn each one of the five lines lie on a circle, which we call the Clifford circle of the 5 lines. In a 6 lines case, the six Clifford circles obtained by omitting in turn each one of the 6 lines pass through a point, which we call the Clifford point of the 6 lines. And so on, so that in a 2n lines case, the 2n Clifford circles obtained by omitting in turn each one of the 2n lines, pass through the Clifford point of the 2n lines; while in a 2n+1 lines case, the 2n+1 Clifford points obtained by omitting in turn each one of the 2n+1 lines, lie on the Clifford circle of the 2n+1 lines.

We will classify sets of lines in the Euclidean plane into two groups, general lines and degenerate lines. A set of lines are general, if no two lines are parallel and no three lines go through a same point. A set of lines are degenerate, if at least two lines are parallel or three lines go through a same point. First we are going to discuss the number of intersection points for a set of general lines in the chapter 1. We will give a very detailed proof of Clifford line-chain theorem inductively by using the real cross ratio lemma in chapter 2. In chapter 3, we are going to investigate the chain theorem for sets of degenerate lines and we will find that it holds for some cases and fails for some others. In chapter 4, we will find 6 types of Clifford figures with examples including some failures of the theorems. In chapter 5, we are going to prove the circle-chain theorem on the sphere by applying Clifford line-chain theorem under the stereographic projection. In chapter 6, we will obtain another two chain theorems and Miquel's pentagon theorem by using the previous two chain theorems under some transformations. The first chain theorem is about circles passing through a same point in the plane and the second chain theorem is about planes passing through a same point in the space.

The circles in the chain theorems as Clifford figures may have different forms. Consider the equation of a circle in the Euclidean plane as

$$ax^2 + ay^2 + bx + cy + d = 0 \quad \cdots\cdots\cdots \quad \boxed{1}$$

(1): If $a = 0$, $\boxed{1}$ becomes $bx + cy + d = 0$, which expresses a line.

(2): If $a \neq 0$, then we may write $\boxed{1}$ in the following form

$$\left(x + \frac{b}{2a}\right)^2 + \left(y + \frac{c}{2a}\right)^2 = \frac{b^2 + c^2 - 4ad}{4a^2} \quad \cdots\cdots\cdots \quad \boxed{2}$$

(1)′: If $b^2 + c^2 - 4ad = 0$, then $\boxed{2}$ expresses a point $\left(-\frac{b}{2a}, -\frac{c}{2a}\right)$.

(2)′: If $b^2 + c^2 - 4ad \neq 0$, then $\boxed{2}$ expresses an ordinary circle with centre at

$\left(-\frac{b}{2a}, -\frac{c}{2a}\right)$ and radius $\sqrt{\frac{b^2+c^2-4ad}{4a^2}}$.

Because of the reasons above, we may classify circles into three different kinds, line-circle denoted by $\boxed{L}$, point-circle denoted by $\boxed{P}$ (when it is a point at infinity, denoted by $\boxed{\infty}$) and ordinary circle by $\boxed{O}$. We may meet different circles when we deal with chain theorems in this paper. We will use these three symbols to refer to different forms of circles for simplicity.

Chapter 1

Intersection points of general lines in the plane

The intersection points of lines are playing important rules in the Clifford chain theorems. In order to investigate the chain theorem for general lines in the plane, we need to know how many intersection points the lines define. We may differentiate sets of lines in the plane into two kinds, general lines and degenerate lines. In this chapter we are going to make this more precise in terms of the number of intersection points.

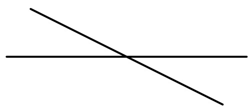
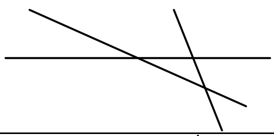
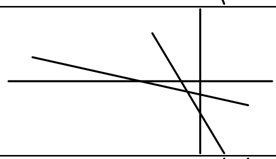
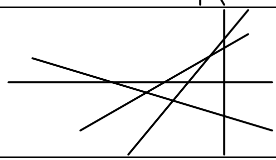
Lemma 1.1. Two general lines have an intersection point.

Lemma 1.2. Three general lines have three intersection points.

Proposition: Four general lines have six intersection points.

Proof:
Given four general lines L_1, L_2, L_3 and L_4. By Lemma 1.2., L_1, L_2, L_3 have 3 intersection points and L_4 meets each one of L_1, L_2, L_3 once. Therefore, four lines have $6 = 3 + 3$ intersection points.

In the same manner, we may formulate the number of intersection points for general lines in the following table. All the lines in the table are general lines.

number of lines	a line with the rest of the lines	sample graph	Number of intersection points
1	a line with 0 lines		0
2	a line with 1 line		$1 = 0 + 1$
3	a line with 2 lines		$3 = 1 + 2$
4	a line with 3 lines		$6 = 3 + 3$
5	a line with 4 lines		$10 = 6 + 4$
… . ..	… …	… …	… …
n	a line with n-1 lines		previous number of points $+ (\mathrm{n}-1)$ $= 1 + 2 + 3 + \dots + (\mathrm{n}-1)$ $= \mathrm{n}(\mathrm{n}-1)/2$

Theorem: n general lines have $\mathrm{n}(\mathrm{n}-1)/2$ intersection points.

Proof:

If $n = 2$, then $\mathrm{n(n-1)/2 = 1}$. It is true.

If $n = 3$, then $\mathrm{n(n-1)/2 = 3}$. It is true.

Assume that it is true for $n = m$, i.e., the number of intersection points is $\mathrm{m(m-1)/2}$. We need to show that it is true for $n = \mathrm{m+1}$. We take away one line from the set of $\mathrm{m+1}$ lines and consider the rest m lines in the plane. Then from the assumption, $\mathrm{m(m-1)/2}$ intersection points are defined by the m lines. Now, we put the line we took away into consideration, and get m more intersection points, one intersection point for each line of the m lines with the line. Hence, when $n = \mathrm{m+1}$, the number of intersection points is $\mathrm{m(m-1)/2 + m = (m+1)m/2}$, and it is true. Therefore, by the mathematics induction, the theorem is true for all $n \in N$.

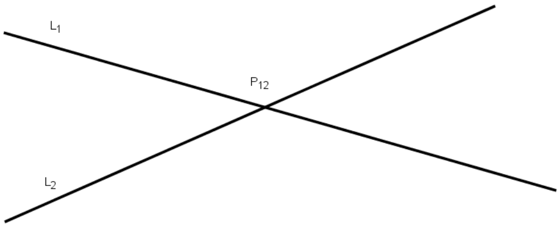
Remark: For n general lines in the plane, there are exactly $\mathrm{n(n-1)/2}$ intersection points. For n degenerate lines in the plane, there are fewer intersection points than $\mathrm{n(n-1)/2}$.

Chapter 2

Clifford chain theorem for general lines in the plane

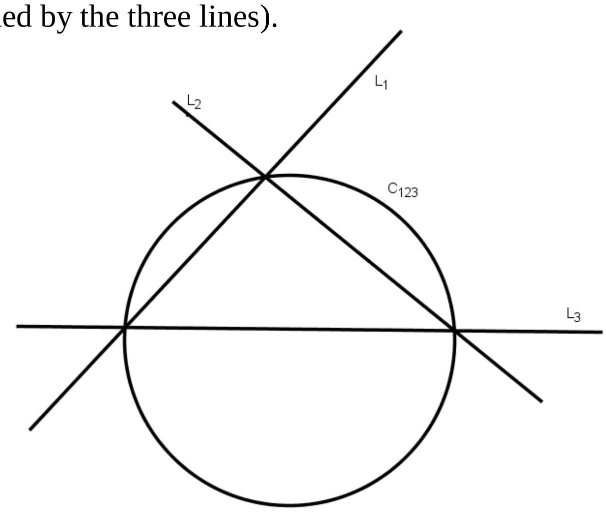
We are going to state the Clifford chain theorem for general lines in the plane and prove it in detail in this chapter. All the lines we are going to consider in this chapter are general lines. We will use real cross ratio lemma to prove it in the manner of mathematical induction. However, we may also use Simson's theorem to prove Clifford chain theorem for 4 general lines.

Lemma 2.1. Two general lines determine a point, which is the intersection point of the two lines.



P_{12} is the point defined by L_1 , L_2

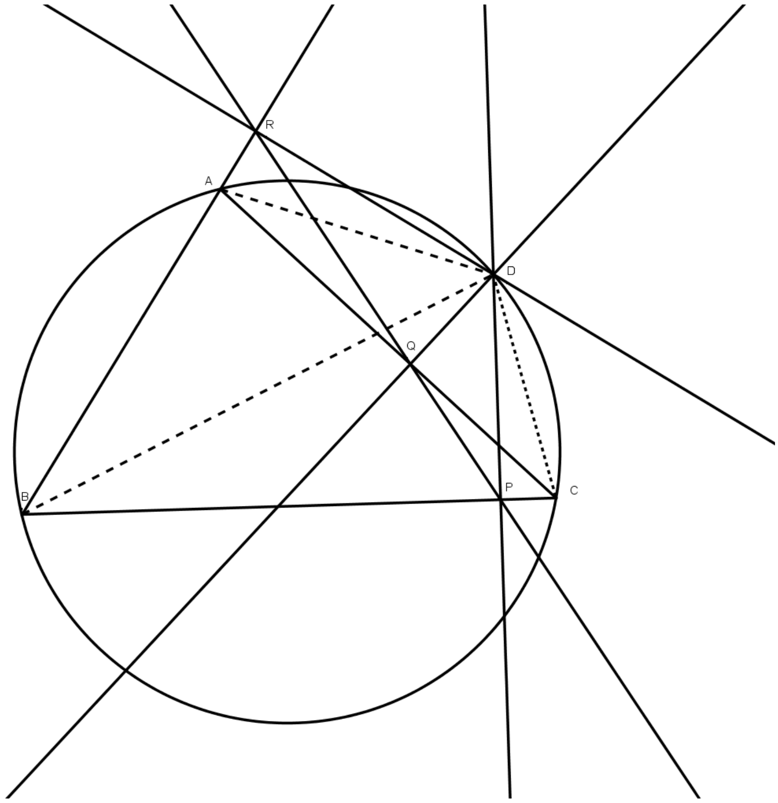
Lemma 2.2. Three general lines determine a circle, which is the circumcircle of the triangle formed by the three lines (which is the circle through the three intersection points defined by the three lines).



circle C_{123} is defined by the three lines

We are going to use the Simson's theorem to prove theorem 2.1..

Simson's Theorem[1]**:** Given ΔABC and a point D, let P, Q, R be the feet of the perpendiculars from the point D to (the extensions of) the sides BC, CA, AB, respectively. Then the points P, Q, R are collinear if and only if D is on the circumcircle of ΔABC.



D is on the circumcircle of ΔABC

Proof:

Assume that P, Q, R are collinear, need to prove that A, B, C, D are concyclic.

It is enough to show

$$\angle ADB = \angle ACB \quad \cdots \ \cdots \ (1).$$

Since P and Q are the feet on BC and AC, we have $\angle DQC = \angle DPC = 90°$ and so

[1] See theorem 2.5.1. in page 77, *Complex Numbers & Geometry*, by Liang-shin Han. The proof we give in this paper is different from the one in this book.

D, Q, P, C are concyclic and hence

$$\angle QDP = \angle QCP = \angle ACB \quad \cdots \ \cdots \ (2).$$

From (1) and (2) we know that we only need to show

$$\angle ADB = \angle QDP.$$

Since

$$\angle ADQ = \angle ADB + \angle BDQ$$

$$\angle BDP = \angle QDP + \angle BDQ,$$

We may only need to show $\angle ADQ = \angle BDP$.

Since D, Q, A, R are concyclic and R, P, Q are collinear, we obtain that

$$\angle ADQ = \angle ARQ = \angle BRP \ \cdots \ \cdots \ (3)\ .$$

Since P and R are the feet on BC and AB, we have $\angle DRB + \angle DPB = 180°$, and so $B, R, D, P,$ are concyclic and hence

$$\angle BRP = \angle BDP \quad \cdots \ \cdots \ (4).$$

From (3) and (4) we get $\angle ADQ = \angle BDP$, hence A, B, C, D are concyclic.

Suppose A, B, C, D are concyclic, we need to prove that P, Q, R are collinear.

Since $Q \in AC$ and only need to show

$$\angle PQC = \angle RQA \cdots \ \cdots \ (5).$$

Since R and Q are the feet on AB and AC, we have $\angle DRA = \angle DQA = 90°$ and so A, R, D, Q are concyclic and hence

$$\angle RQA = \angle RDA \quad \cdots \ \cdots \ (6).$$

Since P and Q are the feet on BC and AC, we have $\angle DQC = \angle DPC = 90°$ and so D, Q, P, C are concyclic and hence

$$\angle PQC = \angle PDC \quad \cdots \ \cdots \ (7).$$

From (5) , (6) and (7) we know that we only need to show

$$\angle RDA = \angle PDC \cdots \ \cdots \ (8).$$

Since $\angle RDB = 90° - \angle ABD,$

$\angle QDC = 90° - \angle QCD = 90° - \angle ACD,$

and $\angle ABD = \angle ACD$ (since A, B, C, D are concyclic),

we obtain $\angle RDB = \angle QDC.$

Where $\angle RDB = \angle RDA + \angle ADB,$

$\angle QDC = \angle QDP + \angle PDC$, i.e., $\angle RDA + \angle ADB = \angle QDP + \angle PDC \cdots \cdots$ (9).

From (8) and (9) we know that we only need to show

$$\angle ADB = \angle QDP.$$

$\angle ADB = \angle ACB = \angle QCP$, since A, B, C, D are concyclic and

$\angle QCP = \angle QDP$, since D, Q, P, C are concyclic.

So we obtain $\angle ADB = \angle QDP$, and get that P, Q, R are collinear.

■

Theorem 2.1. Four general lines determine a point, which is the common point of the four circles each defined by three lines.

Now we are going to prove theorem 2.1. by using Simson's Theorem.

Four lines determine a point, which means that each three of the lines determine a circle by lemma 2.2. and thus there are four such circles. These four circles intersect each other at two points, one of them is a common point of two lines and the other does not lie on any line, we will show that this point lies on all the four circles and thus is the required point.

Proof:

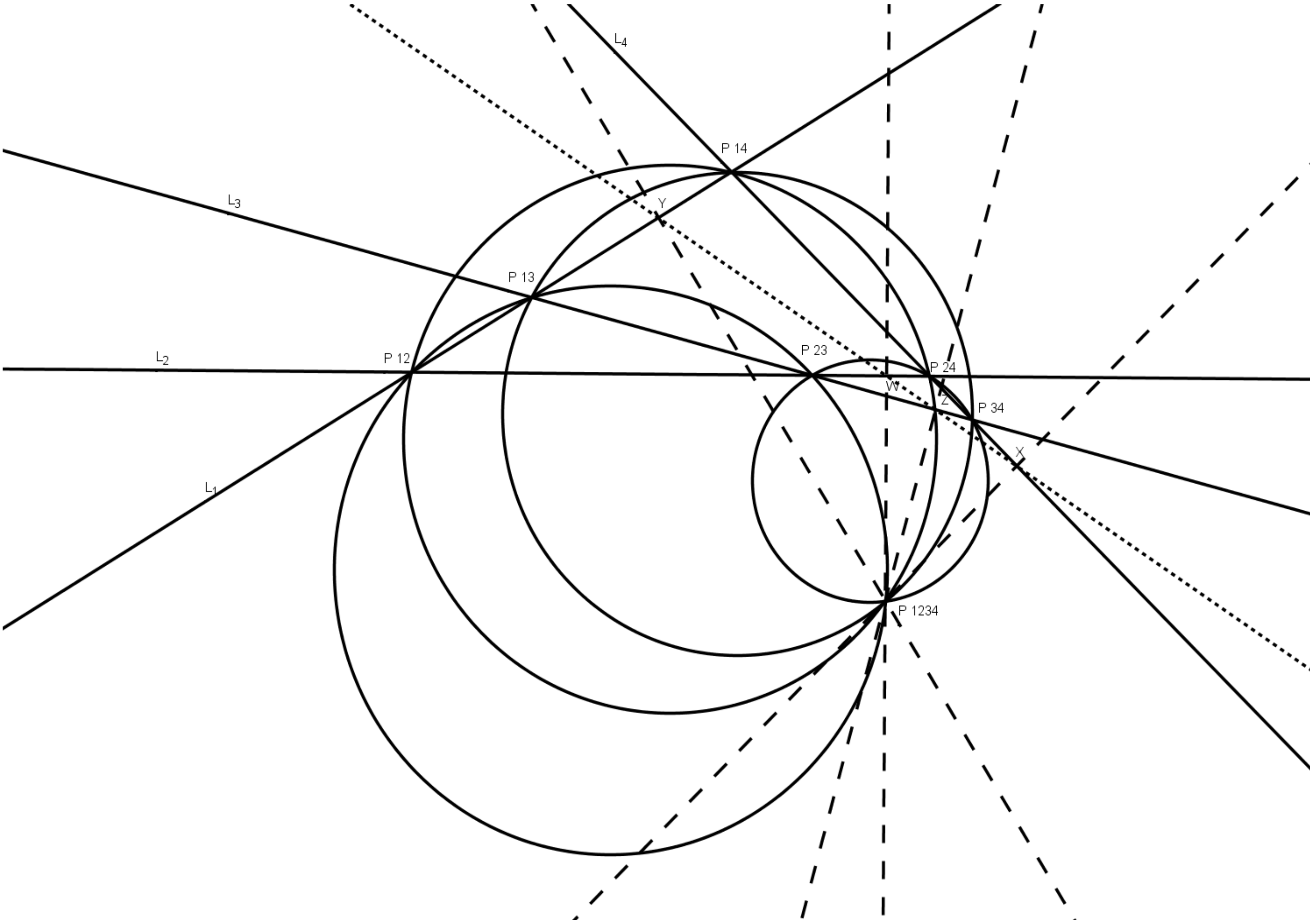
Let L_1, L_2, L_3, L_4 be the four general lines. Let $P_{ij} = L_i \cap L_j$ and C_{ijk} the circle through P_{ij} , P_{jk} , P_{ik} . Let $A = P_{12}$, $B = P_{14}$, $C = P_{24}$, $D = P_{34}$, $E = P_{23}$, $F = P_{13}$.

We consider two circles C_{134} and C_{124}. They have two common points and one of them is P_{14} and we denote the other by P_{1234}.

Let X, Y, Z be the feet of perpendiculars from P to the sides BD, BF and DF. Since $P_{1234} \in C_{134}$, we get X, Y, and Z are collinear by Simson's Theorem.

Let W be the perpendicular from P_{1234} to the side AC. Since X, Y are the feet of perpendiculars form P_{1234} to the sides CB and AB of the triangle ΔABC, and $P_{1234} \in C_{124}$. We get X, Y and W are collinear by Simson's Theorem. Hence, X, Y, Z and W are collinear.

Consider the triangle ΔAEF, since the feet W, Z, Y are collinear, we get $P_{1234} \in C_{123}$, by Simson's Theorem. Similarly, in the triangle ΔCDE, the feet X, Z, W are collinear, we obtain that $P_{1234} \in C_{234}$ by Simson's Theorem.



P_{1234} is the point determined by the four general lines

Hence, P_{1234} is the intersection point of the four circles, determined by the four triangles each formed by three lines of the four lines. Therefore, P_{1234} is the point determined by the four general lines. We may state this as

$$P_{1234} \in C_{123} \cap C_{124} \cap C_{134} \cap C_{234}.$$

■

Cross ratio[1]

Let A, B, C be three complex numbers. Then the complex number, in polar form $re^{i\theta}$ and defined as the quotient

$$re^{i\theta} = \frac{A-C}{B-C}$$

has modulus $r = A - C \; / \; B - C$ and argument $\theta = arg(A - C) - arg(B - C)$.

Let A, B, C, D be four complex numbers, no three of which are equal. Then their cross ratio is defined to be the quotient

$$(AB, CD) = \frac{A-C}{B-C} \Big/ \frac{A-D}{B-D} = \frac{(A-C)(B-D)}{(A-D)(B-C)}.$$

The argument of the cross ratio (AB, CD) is

$$\begin{aligned} arg(AB, CD) &= arg\big((A-C)(B-D)\big) - arg\big((A-D)(B-C)\big) \\ &= arg(A-C) + arg(B-D) - arg(A-D) - arg(B-C) \\ &= \{arg(A-C) - arg(B-C)\} - \{arg(A-D) - arg(B-D)\} \end{aligned}$$

Real cross ratio theorem[2]**:** The cross ratio (AB, CD) is a real number if and only if the four points A, B, C, D are concyclic or collinear.

Proof:

Let $u = |\angle ACB|$, $v = |\angle ADB|$, we have two cases when the points are not collinear.

The first case is that where D, C lie on the same side of AB and we have

$$\begin{aligned} arg(AB, CD) &= \{arg(A-C) - arg(B-C)\} - \{arg(A-D) - arg(B-D)\} \\ &= -u - (-v) \\ &= v - u \end{aligned}$$

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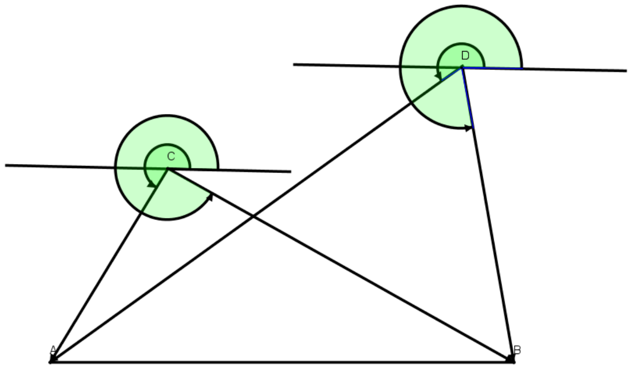
[1] See 4.9 Cross Ratios, page 120, *Geometry*, Roger Fenn.

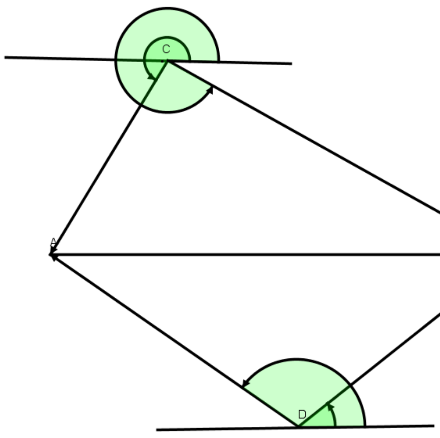
[2] See The real cross ratio theorem, page 121, *Geometry*, Roger Fenn. This proof is different from the book, where might be some problem.



D, C lie on the same side of AB

In the second case where D, C do not lie on the same side of AB and we have

$$\begin{aligned} arg(AB, CD) &= \{arg(A - C) - arg(B - C)\} - \{arg(A - D) - arg(B - D)\} \\ &= u - (-v) \\ &= u + v \end{aligned}$$



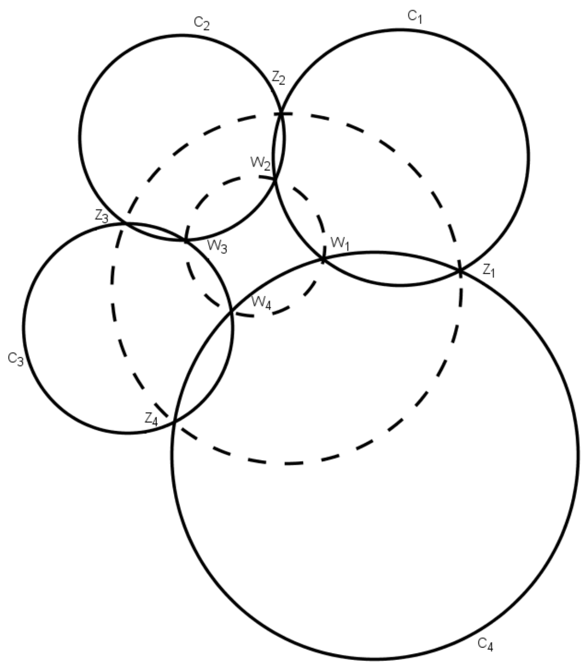
D, C do not lie on the same side of AB

The cross ratio (AB, CD) is a real number if and only if $arg(AB, CD)$ is zero or π.
That is either $v - u = 0$, i.e. $v = u$ or $u + v = \pi$ when the points are not collinear and this is precisely the condition that they lie on a circle.

■

Real Cross ratio Lemma[1]: Consider four circles, C_1, C_2, C_3, C_4. Let C_1, C_2 meet in the points Z_2, W_2; let C_2, C_3 meet in Z_3, W_3; let C_3, C_4 meet in Z_4, W_4 and let C_4, C_1 meet in Z_1, W_1. Then the points Z_1, Z_2, Z_3, Z_4 are concyclic[2] if and only if the points W_1, W_2, W_3, W_4 are concyclic or collinear.

Proof:



W_1, W_2, W_3, W_4 are concyclic

Consider the cross ratios

$$A_1 = (Z_1W_2, Z_2W_1),\ A_2 = (Z_2W_3, Z_3W_2)$$
$$A_3 = (Z_3W_4, Z_4W_3),\ A_4 = (Z_4W_1, Z_1W_4).$$

[1] See the application of the real cross ratio theorem at the bottom of the page 121, *Geometry*, Roger Fenn.
[2] In accordance with the introduction in the beginning of paper, by saying some points are concyclic we mean that they lie on an ordinary circle or a line, since a line can be considered as a circle.

Using real cross ratio theorem we know that all four are real because the corresponding points all lie on circles.

Z_1, W_2, Z_2, W_1 lie on C_1; Z_2, W_3, Z_3, W_2 lie on C_2;

Z_3, W_4, Z_4, W_3 lie on C_3; Z_4, W_1, Z_1, W_4 lie on C_4.

Let $Z = (Z_1Z_3, Z_2Z_4)$ and $W = (W_1W_3, W_2W_4)$.

The product

$$\frac{A_1A_3}{A_2A_4} = \frac{(Z_1W_2, Z_2W_1)(Z_3W_4, Z_4W_3)}{(Z_2W_3, Z_3W_2)(Z_4W_1, Z_1W_4)}$$

$$= \frac{\frac{(Z_1-Z_2)(W_2-W_1)}{(Z_1-W_1)(W_2-Z_2)}\frac{(Z_3-Z_4)(W_4-W_3)}{(Z_3-W_3)(W_4-Z_4)}}{\frac{(Z_2-Z_3)(W_3-W_2)}{(Z_2-W_2)(W_3-Z_3)}\frac{(Z_4-Z_1)(W_1-W_4)}{(Z_4-W_4)(W_1-Z_1)}}$$

$$= \frac{(Z_1-Z_2)(W_2-W_1)(Z_3-Z_4)(W_4-W_3)(Z_2-W_2)(W_3-Z_3)(Z_4-W_4)(W_1-Z_1)}{(Z_1-W_1)(W_2-Z_2)(Z_3-W_3)(W_4-Z_4)(Z_2-Z_3)(W_3-W_2)(Z_4-Z_1)(W_1-W_4)}$$

$$= \frac{(Z_1-Z_2)(Z_3-Z_4)(W_1-W_2)(W_3-W_4)(Z_2-W_2)(W_3-Z_3)(Z_4-W_4)(W_1-Z_1)}{(W_1-Z_1)(Z_2-W_2)(W_3-Z_3)(Z_4-W_4)(Z_3-Z_2)(W_3-W_2)(Z_1-Z_4)(W_1-W_4)}$$

$$= \frac{(Z_1-Z_2)(Z_3-Z_4)(W_1-W_2)(W_3-W_4)}{(Z_1-Z_4)(Z_3-Z_2)(W_1-W_4)(W_3-W_2)}$$

$$= (Z_1Z_3, Z_2Z_4)(W_1W_3, W_2W_4) = ZW$$

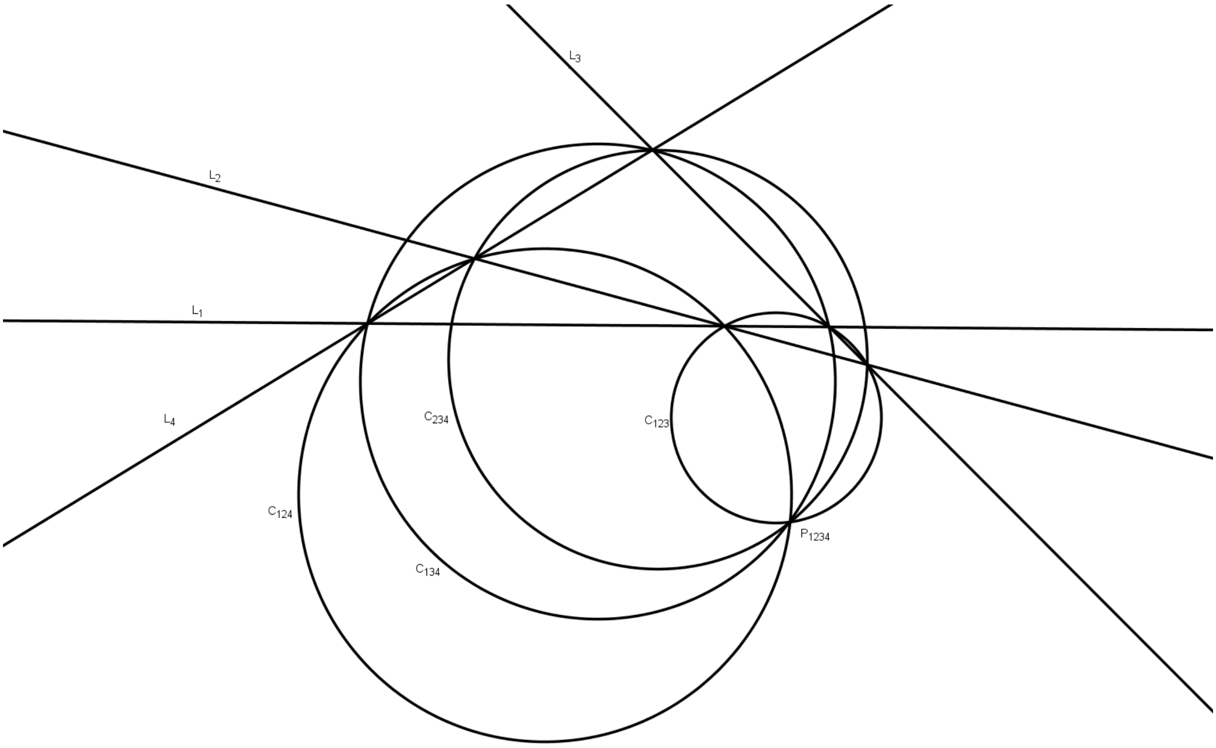
is therefore real. It follows that Z is real if and only if W is real. The result now follows by the real cross ratio theorem.

■

Another Proof of theorem 2.1.:

Proof:

Let L_1, L_2, L_3, L_4 be four general lines. Let P_{ij} be the intersection of the lines L_i and L_j. Let C_{ijk} be the circle through P_{ij}, P_{ik} and P_{jk}. Take three lines each time and we get four



P_{1234} is the point defined by the four general lines

circles $C_{123}, C_{124}, C_{134}$ and C_{234} by the lemma 2.2.. Only need to show that these four circles pass through a same point, P_{1234}.

P_{24}	$C_{234} \cap L_2$	P_{23}
P_{12}	$L_2 \cap L_1$	∞
P_{14}	$L_1 \cap C_{134}$	P_{13}
P_{1234}	$C_{134} \cap C_{234}$	P_{34}

Where P_{1234} is the intersection point (other than P_{34}) of circles C_{134} and C_{234}.
We have $\{P_{23}, \infty, P_{13}, P_{34}\} \in L_3$,
and thus $\{P_{24}, P_{12}, P_{14}, P_{1234}\} \in$ a circle by the real cross ratio lemma.
Since $\{P_{24}, P_{12}, P_{14}\} \in C_{124}$ we get $P_{1234} \in C_{124}$, i.e.,
$P_{1234} \in C_{234} \cap C_{134} \cap C_{124}$. We only need to show $P_{1234} \in C_{123}$.

P_{23}	C_{234}	$\cap$	L_2	P_{24}
P_{12}	L_2	$\cap$	L_1	∞
P_{13}	L_1	$\cap$	C_{134}	P_{14}
P_{1234}	C_{134}	$\cap$	C_{234}	P_{34}

We have $\{P_{24}, \infty, P_{14}, P_{34}\} \in L_4$,
and thus $\{P_{23}, P_{12}, P_{13}, P_{1234}\} \in$ a circle by the real cross ratio lemma.
Since $\{P_{23}, P_{12}, P_{13}\} \in C_{123}$, we get $P_{1234} \in C_{123}$.

■

Theorem 2.2. n general lines determine a circle if n is odd and a point if n is even. When n is odd the circle goes through the n points each defined by $n-1$ lines and when n is even the point is the common point of the n circels each defined by $n-1$ lines.

Proof:

We are going to use mathematical induction to prove this.
It is true for $n = 2, 3, 4$, which means that two lines define a point, three lines define a circle and four lines define a point, and on one hand the circle goes through the three points each defined by two lines and on the other hand it contains the point defined by the four lines.

Suppose as inductive hypothesis that for odd number smaller than n the lines define a circle and for even number smaller than n the lines define a point. Assume that any such circle, defined by $2m+1$ $(< n)$ lines, goes through $2m+1$ points each defined by $2m$ lines and contains the point defined by $2m+2$ $(< n)$ lines, the original $2m+1$ $(< n)$ lines with one added. Assume that any such point, defined by $2m$ $(< n)$ lines, is a common point of $2m$ circles each defined by $2m-1$ lines and lies on the circle defined by $2m+1$ $(< n)$ lines, the original $2m$ $(< n)$ lines with one added.

(1): We need to show that n general lines determine a circle when n is odd, and this circle passes through the n points each defined by $n-1$ general lines.

Suppose we are given n general lines $L_1, L_2, \cdots, L_n$, we take $n-1$ lines each time and we obtain n points by the hypothesis,

$$P_{12\cdots(n-1)}, P_{12\cdots(n-2)n}, \cdots, P_{23\cdots n}.$$

We claim that these n points are concyclic and we will finish the proof of (1) in $n-3$ steps by using the cross ratio lemma in each step. In the first step, we prove four points lie on a circle, $C_{12\cdots n}$, and then from the second step we prove each one of the remaining points lies on the same circle in each step. We may show $P_{23\cdots n}$ lies on $C_{12\cdots n}$ in the second step, $P_{13\cdots n}$ lies on $C_{12\cdots n}$ in the third step, $P_{124\cdots n}$ lies on $C_{12\cdots n}$ in the fourth step and so on $P_{12\cdots(n-5)(n-3)(n-2)(n-1)n}$ lies on $C_{12\cdots n}$ in the last step.

Step 1:

In order to use the lemma, we need to find four circles such that they intersect in pairs and one set of the intersections are from the given n points and the others are concyclic.
Choose four points related to a circle, say,

$$P_{12\cdots(n-1)}, P_{12\cdots(n-2)n}, P_{12\cdots(n-3)(n-1)n}, P_{12\cdots(n-4)(n-2)(n-1)n}$$

which are related to $C_{12\cdots(n-4)}$ (indices $12\cdots(n-4)$).

For $P_{12\cdots(n-1)}$, we have

$$P_{12\cdots(n-1)} \in C_{12\cdots(n-2)} \cap C_{12\cdots(n-3)(n-1)} \cap \cdots \cap C_{23\cdots(n-1)}$$

by the assumption. We choose any two circles related to $C_{12\cdots(n-4)}$, here we choose the first two circles $C_{12\cdots(n-2)}, C_{12\cdots(n-3)(n-1)}$ and from the hypothesis we get

$$P_{12\cdots(n-1)} \qquad C_{12\cdots(n-2)} \cap C_{12\cdots(n-3)(n-1)} \qquad P_{12\cdots(n-3)}$$

Where $P_{12\cdots(n-1)}$ and $P_{12\cdots(n-3)}$ are the two intersections of $C_{12\cdots(n-2)}$ and $C_{12\cdots(n-3)(n-1)}$. Take the second circle $C_{12\cdots(n-3)(n-1)}$, from the hypothesis there are only two points $P_{12\cdots(n-3)}$ and $P_{12\cdots(n-4)(n-1)}$ among those points the circle goes through and associated with indices $12\cdots(n-4)$, $P_{12\cdots(n-3)}$ has already chosen, so $P_{12\cdots(n-4)(n-1)}$ is the only choice.

Now find a point from the given four points

$P_{12\cdots(n-1)}, P_{12\cdots(n-2)n}, P_{12\cdots(n-3)(n-1)n}, P_{12\cdots(n-4)(n-2)(n-1)n}$, which lies on $C_{12\cdots(n-3)(n-1)}$ and associated with indices $12\cdots(n-4)$, $P_{12\cdots(n-3)(n-1)n}$ is the only choice, since $P_{12\cdots(n-1)}$ has already chosen. In the case of the next circle, the $n-2$ indices must come from $12\cdots(n-3)(n-1)n$ and $n-3$ of them must be $12\cdots(n-4)(n-1)$, so it must be $C_{12\cdots(n-4)(n-1)n}$, and we have

$$P_{12\cdots(n-3)(n-1)n} \qquad C_{12\cdots(n-3)(n-1)} \cap C_{12\cdots(n-4)(n-1)n} \qquad P_{12\cdots(n-4)(n-1)}$$

In the same manner we have

$$P_{12\cdots(n-4)(n-2)(n-1)n} \qquad C_{12\cdots(n-4)(n-1)n} \cap C_{12\cdots(n-4)(n-2)n} \qquad P_{12\cdots(n-4)n}$$

and

$$P_{12\cdots(n-3)(n-2)n} \qquad C_{12\cdots(n-4)(n-2)n} \cap C_{12\cdots(n-2)} \qquad P_{12\cdots(n-4)(n-2)}$$

Where

$$\{P_{12\cdots(n-3)}, P_{12\cdots(n-4)(n-1)}, P_{12\cdots(n-4)n}, P_{12\cdots(n-4)(n-2)}\} \in C_{12\cdots(n-4)}$$

from the hypothesis, hence by the real cross ratio lemma, we have

$$\{P_{12\cdots(n-1)}, P_{12\cdots(n-3)(n-1)n}, P_{12\cdots(n-4)(n-2)(n-1)n}, P_{12\cdots(n-3)(n-2)n}\} \in C_{12\cdots n}.$$

Step 2:

In order to show that $P_{2\cdots n}$ is on the defined circle $C_{12\cdots n}$ passing through

$P_{12\cdots(n-1)}, P_{12\cdots(n-3)(n-1)n}, P_{12\cdots(n-4)(n-2)(n-1)n}, P_{12\cdots(n-3)(n-2)n}$, we need to find four points from the given n points, which associated with indices differ from $12\cdots(n-4)$, say, $2\cdots(n-3)$, the only one choice is

$P_{12\cdots(n-1)}, P_{12\cdots(n-2)n}, P_{12\cdots(n-3)(n-1)n}, P_{2\cdots n}.$

From

$P_{12\cdots(n-1)} \in C_{12\cdots(n-2)} \cap C_{12\cdots(n-3)(n-1)} \cap \cdots \cap C_{2\cdots(n-1)}$,

we choose the first two circles associated with $2\cdots(n-3)$ and in the same way we get

$P_{12\cdots(n-1)}$	$C_{12\cdots(n-2)} \cap C_{12\cdots(n-3)(n-1)}$	$P_{12\cdots(n-3)}$
$P_{12\cdots(n-3)(n-1)n}$	$C_{12\cdots(n-3)(n-1)} \cap C_{2\cdots(n-3)(n-1)n}$	$P_{2\cdots(n-3)(n-1)}$
$P_{2\cdots n}$	$C_{2\cdots(n-3)(n-1)n} \cap C_{2\cdots(n-2)n}$	$P_{2\cdots(n-3)n}$
$P_{12\cdots(n-2)n}$	$C_{2\cdots(n-2)n} \cap C_{12\cdots(n-2)}$	$P_{2\cdots(n-3)(n-2)}$

Where

$\{P_{12\cdots(n-3)}, P_{2\cdots(n-3)(n-1)}, P_{2\cdots(n-3)n}, P_{2\cdots(n-3)(n-2)}\} \in C_{2\cdots(n-3)}$,

from the hypothesis, hence by the real cross ratio lemma, we have

$\{P_{12\cdots(n-1)}, P_{12\cdots(n-3)(n-1)n}, P_{2\cdots n}, P_{12\cdots(n-2)n}\} \in C_{12\cdots n}.$

Since $\{P_{12\cdots(n-1)}, P_{12\cdots(n-3)(n-1)n}, P_{12\cdots(n-2)n}\} \in C_{12\cdots n}$, so $P_{2\cdots n} \in C_{12\cdots n}$.

all the indices	related indices	the rest four indices			
<u>1 2 ...(n-4)</u> (n-3)(n-2)(n-1)n	1 2 ... (n-4)	(n-3)	(n-2)	(n-1)	n
1 <u>2 ... (n-3)</u>(n-2)(n-1)n	2 3 ... (n-3)	1	(n-2)	(n-1)	n
1 2 <u>3...(n-2)</u>(n-1)n	3...(n-2)	1	2	n-1	n
1 2 3 <u>4...(n-1)</u>n	4...(n-1)	1	2	3	n

1 2 3 4 <u>5...n</u>	5...n	1	2	3	4
... ...	... ...	...	...	...	...
step i <u>12...(i-5)</u>(i-4)(i-3)(i-2)(i-1)<u>i(i+1)...n</u>	12...(i-5)i(i+1)...n	(i-4)	(i-3)	(i-2)	(i-1)
... ...	... ...	...	...	...	...
step n-3 <u>1...(n-8)</u>(n-7)(n-6)(n-5)(n-4) <u>(n-3)...n</u>	1...(n-8) (n-3)...n	(n-7)	(n-6)	(n-5)	(n-4)

This table helps us to find the four circles and eight points in each step. For example, the second row is correspond to the second step, where the related indices are $2\cdots(n-3)$ and the rest are $1,(n-2),(n-1),n$, and we may get the four circles and eight points by substituting these two things into the first step with the correspondence between the two steps in the table and so on for the rest of the steps.

Step i:

$P_{1\cdots(i-2)i\cdots n}$ $\quad C_{1\cdots(i-3)i\cdots n} \cap C_{1\cdots(i-4)(i-2)i\cdots n}$ $\quad P_{1\cdots(i-4)i\cdots n}$

$P_{12\cdots(i-4)(i-2)\cdots n}$ $\quad C_{1\cdots(i-4)(i-2)i\cdots n} \cap C_{1\cdots(i-5)(i-2)\cdots n}$ $\quad P_{1\cdots(i-5)(i-2)i\cdots n}$

$P_{1\cdots(i-5)(i-3)\cdots n}$ $\quad C_{1\cdots(i-5)(i-2)\cdots n} \cap C_{1\cdots(i-5)(i-3)(i-1)\cdots n}$ $\quad P_{1\cdots(i-5)(i-1)\cdots n}$

$P_{12\cdots(i-3)(i-1)\cdots n}$ $\quad C_{1\cdots(i-5)(i-3)(i-1)\cdots n} \cap C_{1\cdots(i-3)i\cdots n}$ $\quad P_{1\cdots(i-5)(i-3)i\cdots n}$

The four points on the right side are on the circle $C_{1\cdots(i-5)i\cdots n}$, hence the four points on the left side are concyclic and we know that three of the points on the left come from the left side of the step above and so this circle is $C_{1\cdots n}$, and we have

$P_{1\cdots(i-2)i\cdots n} \in C_{1\cdots n}$.

The last step, step n-3

$P_{1\cdots(n-5)(n-3)\cdots n}$ $C_{1\cdots(n-6)(n-3)\cdots n} \cap C_{1\cdots(n-7)(n-5)(n-3)\cdots n}$ $P_{1\cdots(n-7)(n-3)\cdots n}$

$P_{1\cdots(n-7)(n-5)\cdots n}$ $C_{1\cdots(n-7)(n-5)(n-3)\cdots n} \cap C_{1\cdots(n-8)(n-5)\cdots n}$ $P_{1\cdots(n-8)(n-5)(n-3)\cdots n}$

$P_{1\cdots(n-8)(n-6)\cdots n}$ $C_{1\cdots(n-8)(n-5)\cdots n} \cap C_{1\cdots(n-8)(n-6)(n-4)\cdots n}$ $P_{1\cdots(n-8)(n-4)\cdots n}$

$P_{1\cdots(n-6)(n-4)\cdots n}$ $C_{1\cdots(n-8)(n-6)(n-4)\cdots n} \cap C_{1\cdots(n-6)(n-3)\cdots n}$ $P_{1\cdots(n-8)(n-6)(n-3)\cdots n}$

Where the four points on the right side are on the circle $C_{1\cdots(n-8)(n-3)\cdots n}$ by the hypothesis, the four points on the left are concyclic by the cross ratio lemma, hence $P_{1\cdots(n-5)(n-3)\cdots n} \in C_{1\cdots n}$ since the remaining three points on the left come from the left side of the step above and we know that they lie on so $C_{1\cdots n}$.

We conclude from all the n-3 steps above that all the n points $\{P_{12\cdots(n-1)}, P_{12\cdots(n-2)n}, \cdots, P_{23\cdots n}\} \in C_{1\cdots n}$, i.e., it is true for odd number n, therefore, by mathematical induction the statement is true for all odd number.

(2): We need to show that n general lines determine a point when n is even, and this point is the common point of the n circles each defined by $n-1$ general lines.

Suppose we are given n lines $L_1, L_2, \cdots, L_n$, we take $n-1$ lines each time and we obtain n circles $C_{2\cdots n}, C_{13\cdots n},, C_{124\cdots n}, \cdots, C_{1\cdots(n-1)}$ by the hypothesis.
We claim that these n circles meet at a point and we need $n-2$ steps to finish the proof of (2) by using the cross ratio lemma in each step. In the first step, we prove the first three circles, $C_{2\cdots(n-1)}, C_{13\cdots(n-2)n},, C_{124\cdots(n-2)n}$ meet at a point, denote $P_{1\cdots n}$, and from the second step we prove each one of the remaining circles pass through $P_{1\cdots n}$ in each step. We may show $P_{1\cdots n} \in C_{1235\cdots n}$ in the second step, $P_{1\cdots n} \in C_{12346\cdots n}$ in the third step, $P_{1\cdots n} \in C_{123457\cdots n}$ in the fourth step and so on $P_{1\cdots n} \in C_{1\cdots(n-1)}$ in the last step. We could get four circles and eight points for each step from step above by using the correspondence in following table, however, the last step has a different pattern.

all the index	related indices	the rest four indices			
1 2 3 4 ... (n-1) n	4 ... (n-1)	1	2	3	n
1 2 3 4 5... (n-1) n	3 5 ... (n-1)	1	2	4	n
1 2 3 4 5 6 ... (n-1) n	3 4 6 ... (n-1)	1	2	5	n
... ...	... ...	...	...	...	...
step i 1 2 3 ...(i+1) (i+2) (i+3) ... (n-1)n	3 ...(i+1)(i+3) ... (n-1)	1	2	i+2	n
... ...	... ...	...	...	...	...
step (n-3) 1 2 3 ... (n-2) (n-1) n	3 ... (n-2)	1	2	(n-1)	n
step (n-2), the last step 1 2 3 ... (n-3) (n-2) (n-1) n	3 ... (n-3) (n-1)	1	2	(n-2)	n

Step1:

We have the following by the hypothesis

$P_{24\cdots n}$ $\quad C_{2\cdots n} \cap C_{24\cdots(n-1)}$ $\quad P_{2\cdots(n-1)}$

$P_{124\cdots(n-1)}$ $\quad C_{24\cdots(n-1)} \cap C_{14\cdots(n-1)}$ $\quad P_{4\cdots(n-1)}$

$P_{14\cdots n}$ $\quad C_{14\cdots(n-1)} \cap C_{13\cdots n}$ $\quad P_{13\cdots(n-1)}$

$P_{1\cdots n}$ $\quad C_{13\cdots n} \cap C_{2\cdots n}$ $\quad P_{3\cdots n}$

The four points on the right side are on the circle $C_{3\cdots(n-1)}$ from the hypothesis, hence the four points on the left side are concyclic by the cross ratio theorem.

Since $\quad P_{1\cdots n} \in C_{13\cdots n} \cap C_{2\cdots n}$ and

$$\{P_{24\cdots n}, P_{124\cdots(n-1)}, P_{14\cdots n}\} \subset C_{124\cdots n}$$

we get

$$P_{1\cdots n} \in C_{2\cdots n} \cap C_{13\cdots n} \cap C_{124\cdots n}.$$

Step 2:

We have the following by the hypothesis

$P_{235\cdots n}$	$C_{2\cdots n} \cap C_{235\cdots(n-1)}$	$P_{2\cdots(n-1)}$
$P_{1235\cdots(n-1)}$	$C_{235\cdots(n-1)} \cap C_{135\cdots(n-1)}$	$P_{35\cdots(n-1)}$
$P_{135\cdots n}$	$C_{135\cdots(n-1)} \cap C_{13\cdots n}$	$P_{13\cdots(n-1)}$
$P_{1\cdots n}$	$C_{13\cdots n} \cap C_{2\cdots n}$	$P_{3\cdots n}$

The four points on the right side are on the circle $C_{3\cdots(n-1)}$ from the hypothesis, hence the four points on the left side are concyclic by the cross ratio theorem.
Since

$$\{P_{235\cdots n}, P_{1235\cdots(n-1)}, P_{135\cdots n}\} \subset C_{1235\cdots n}$$

we get

$$P_{1\cdots n} \in C_{1235\cdots n}.$$

Step i:

We have the following by the hypothesis

$P_{2\cdots(i+1)(i+3)\cdots n}$	$C_{2\cdots n} \cap C_{23\cdots(i+1)(i+3)\cdots(n-1)}$	$P_{2\cdots(n-1)}$
$P_{1\cdots(i+1)(i+3)\cdots(n-1)}$	$C_{23\cdots(i+1)(i+3)\cdots(n-1)} \cap C_{13\cdots(i+1)(i+3)\cdots(n-1)}$	$P_{3\cdots(i+1)(i+3)\cdots(n-1)}$
$P_{13\cdots(i+1)(i+3)\cdots n}$	$C_{13\cdots(i+1)(i+3)\cdots(n-1)} \cap C_{13\cdots n}$	$P_{13\cdots(n-1)}$
$P_{1\cdots n}$	$C_{13\cdots n} \cap C_{2\cdots n}$	$P_{3\cdots n}$

The four points on the right side are on the circle $C_{3\cdots(n-1)}$ from the hypothesis, hence the four points on the left side are concyclic by the cross ratio theorem.

Since

$$\{P_{2\cdots(i+1)(i+3)\cdots n}, P_{1\cdots(i+1)(i+3)\cdots(n-1)}, P_{13\cdots(i+1)(i+3)\cdots n}\} \subset C_{1\cdots(i+1)(i+3)\cdots n}$$

we get

$$P_{1\cdots n} \in C_{1\cdots(i+1)(i+3)\cdots n}.$$

Step n-3:

We have the following by the hypothesis

$P_{2\cdots(n-2)n}$	$C_{2\cdots n} \cap C_{2\cdots(n-2)}$	$P_{2\cdots(n-1)}$
$P_{1\cdots(n-2)}$	$C_{2\cdots(n-2)} \cap C_{13\cdots(n-2)}$	$P_{3\cdots(n-2)}$
$P_{13\cdots(n-2)n}$	$C_{13\cdots(n-2)} \cap C_{13\cdots n}$	$P_{13\cdots(n-1)}$
$P_{1\cdots n}$	$C_{13\cdots n} \cap C_{2\cdots n}$	$P_{3\cdots n}$

The four points on the right side lie on the circle $C_{3\cdots(n-1)}$ from the hypothesis, hence the four points on the left side are concyclic by the cross ratio theorem.

Since

$$\{P_{2\cdots(n-2)n}, P_{1\cdots(n-2)}, P_{13\cdots(n-2)n}\} \subset C_{1\cdots(n-2)n}$$

we get

$$P_{1\cdots n} \in C_{1\cdots(n-2)n}.$$

The last step, step n-2:

$P_{2\cdots(n-1)}$	$C_{2\cdots n} \cap C_{2\cdots(n-3)(n-1)}$	$P_{2\cdots(n-3)(n-1)n}$
$P_{1\cdots(n-3)(n-1)}$	$C_{2\cdots(n-3)(n-1)} \cap C_{13\cdots(n-3)(n-1)}$	$P_{3\cdots(n-3)(n-1)}$
$P_{13\cdots(n-1)}$	$C_{13\cdots(n-3)(n-1)} \cap C_{13\cdots n}$	$P_{13\cdots(n-3)(n-1)n}$
$P_{1\cdots n}$	$C_{13\cdots n} \cap C_{2\cdots n}$	$P_{3\cdots n}$

The four points on the right side lie on the circle $C_{3\cdots(n-3)(n-1)n}$ from the hypothesis, hence the four points on the left side are concyclic by the cross ratio theorem.

Since

$$\left\{P_{2\cdots(n-1)}, P_{1\cdots(n-3)(n-1)}, P_{13\cdots(n-1)}\right\} \subset C_{1\cdots(n-1)}$$

we get

$$P_{1\cdots n} \in C_{1\cdots(n-1)}.$$

We conclude from all the n-2 steps above that all the n circles

$$P_{1\cdots n} \in C_{2\cdots n} \cap C_{13\cdots n} \cap C_{124\cdots n} \cap \cdots \cap C_{1\cdots(n-1)},$$

i.e., it is true for even number n, therefore, by mathematical induction the statement is true for all even numbers.

To summarize, this theorem is true for 2,3,4 general lines and we proved that it is true for odd number of general lines in (1) and even number of general lines in (2) based on the assumption, therefore, by mathematical induction, it is true for any number of general lines.

■

Chapter 3.

Clifford chain theorem for degenerate lines in the plane

In this chapter, we are going to investigate Clifford chain theorem in terms of degenerate sets of lines. In section 1, we will obtain the chain theorem for n lines where 2 lines are parallel and the rest are general. However, the theorem fails when more than 2 lines are parallel in a set of lines in section 2. In section 3, we will also find that the chain theorem holds for n lines where at least 3 lines are concurrent and the rest are general. In order to examine whether the n lines determine a point or circle, we put them in the real projective plane for simplicity. The model we choose here for the real projective plane is the extended Euclidean plane with a line at infinity, L_∞.

1. Two lines parallel and the rest in general position

In this section, we will obtain the chain theorem for n lines where 2 lines are parallel and the rest are general. We will also notice that the result remains the same if we just take the 2 parallel lines away from our consideration.

Now we are going to extend the real plane into the real projective plane by adding a point at infinity to each real line and all the points at infinity lie on the line at infinity L_∞. In doing so we may consider a line in the real projective plane as a close curve, for example, if L_i is a line in the real plane then $L_i \cup P_i$ is the corresponding line in projective plane, where P_i is the point at infinity on the direction of line L_i. Now we may say that parallel lines intersect at a point which lies on the line at infinity. The point at which the parallel lines intersect depends only on the slop of the lines, i.e. all parallel lines have a common point on L_∞.

There is no intersection for two parallel lines in the real plane; however, they have an intersection point at infinity in the real projective plane. Consider two lines, L_1, L_2 in general position then they have one intersection point P_{12} . If we pull P_{12} gradually into the distance in the plane, what we get is that when P_{12} goes to infinity the angle formed by L_1 , L_2 goes to zero and the lines tend to be parallel. We say that they have an intersection point $P_{12} \in L_\infty$. We may say that two parallel lines determine a point, which is the point at infinity on the direction of the two lines. In an other word, $L_1 \cup P_1$ and $L_2 \cup P_2$ are those two corresponding projective lines, where $P_1 = P_2$, since these two lines have the same slop and this point is the intersection point of these two projective lines and may be denoted by P_{12} . We say two parallel lines determine a point, which is the point at infinity on the direction of the two lines.

1.1. one general line and two parallel lines

Let $L_1 // L_2$ and L_3 be a general line. Now we take two lines each time and we get three intersection points in the real projective plane, P_{12} , P_{13} and P_{23} , where $P_{12} \in L_\infty$. We say that there is a degenerate circle, C_{123} , passing through these three points. C_{123} is a circle of the form $\boxed{L}$. In the projective plane, we may say C_{123} goes along L_3 and passes the infinity point P_3 on the direction of L_3 and P_{12} . We denote this circle $C_{123} = L_3 \cup L_\infty$.

Lemma 1.1. Three lines, two parallel and one general, define a circle, this circle is the union of the general line and the line at infinity.

1.2. Two general lines and two parallel lines

Let $L_1 // L_2$ and L_3 , L_4 be general. We take three lines each time and by doing so we can reduce the problem into situations we already discussed.

L_1 , L_2, L_3 define a circle form of $\boxed{L}$, $C_{123} = L_3 \cup L_\infty$, by the lemma 1.1.

L_1 , L_2, L_4 define a circle form of $\boxed{L}$, $C_{124} = L_4 \cup L_\infty$, by the lemma 1.1.

L_1 , L_3, L_4 define a circle form of $\boxed{O}$, C_{134} , by chapter 2

L_2 , L_3 , L_4 define a circle form of $\boxed{O}$, C_{234} , by chapter 2

and

$$C_{123} \cap C_{124} \cap C_{134} \cap C_{234} = P_{34}$$

this is the point defined by the four lines and we denote as $P_{1234} = P_{34}$.

Lemma 1.2. Four lines, two parallel and two general, define a point, this point is the intersection point of the two general lines, i.e., the point defined by the two general lines.

1.3. Three general lines and two parallel lines

Lemma 1.3. Five lines, two parallel and three general, define a circle which is the circle defined by the three general lines.

Proof:

Given $L_1 // L_2$ and L_3 , L_4 , L_5 general. We take four lines each time

L_1 , L_2 , L_3 , L_4 define a point $P_{1234} = P_{34}$, by the lemma 1.2.

L_1 , L_2 , L_3 , L_5 define a point $P_{1235} = P_{35}$, by the lemma 1.2.

L_1 , L_2 , L_4 , L_5 define a point $P_{1245} = P_{45}$, by the lemma 1.2.

L_1 , L_3 , L_4 , L_5 define a point P_{1345} , by chapter 2

L_2 , L_3 , L_4 , L_5 define a point P_{2345} , by chapter 2

Since P_{34} , P_{35} , P_{45} define a circle C_{345} and P_{1345} , P_{2345} lie on C_{345} by chapter 2. All the five points each determined by four lines lie on the circle C_{345} , which is the circle determined by the three general. We may consider this circle is determined by the five lines and denote $C_{345} = C_{12345}$.

■

1.4. Four general lines and two parallel lines

Lemma 1.4. Six lines, two parallel and four general, define a point which is the point determined by those four general lines.

Proof:

Given $L_1 \,//\, L_2$ and L_3, L_4, L_5, L_6 general. We take five lines each time and use the lemma 1.3. we get

L_1, L_2, L_3, L_4, L_5 define a circle $C_{12345} = C_{345}$

L_1, L_2, L_3, L_4, L_6 define a circle $C_{12346} = C_{346}$

L_1, L_2, L_3, L_5, L_6 define a circle $C_{12356} = C_{356}$

L_1, L_2, L_4, L_5, L_6 define a circle $C_{12456} = C_{456}$

And from chapter 2 we get

L_1, L_3, L_4, L_5, L_6 define a circle C_{13456}

L_2, L_3, L_4, L_5, L_6 define a circle C_{23456}

Need to show these six circles intersect at a point.
Since

$$C_{345} \cap C_{346} \cap C_{356} \cap C_{456} = P_{3456}$$

and $P_{3456} \in C_{13456} \cap C_{23456}$ by chapter 2, P_{3456} is the point we need and it is also the point defiend by the four general lines. We define $P_{123456} = P_{3456}$.

■

In the same manner, we will get the same result for n lines where 2 lines are parallel.

Theorem 3.1. n lines, where 2 lines are parallel and n-2 lines are general, determine a point when n is even and a circle when n is odd. This point is the point defined by those general lines and this circle is the circle defined by those general lines.

Proof:

We are going to prove this by using the mathematical induction.
It is true for n=3,4,5,6 by previous lemmas.
We assume that it is true for n=2m and n=2m+1, i.e., n=2m lines with 2 parallel lines and the rest are general define a point, which is the point defined by those general lines and n=2m+1 lines with 2 parallel lines the rest general define a circle, which is the circle defined by those general lines.

(1) Need to prove that it is true when n = 2(m+1) = 2m+2.

Given $L_1 // L_2$ and $L_3, \cdots, L_{2(m+1)}$ general. We take 2(m+1)-1=2m+1 lines each time and use the assumption and chapter 2, we obtain 2(m+1) circles

$$C_{1\cdots(2m+1)} = C_{3\cdots(2m+1)},$$
$$C_{1\cdots 2m2(m+1)} = C_{3\cdots 2m2(m+1)},$$
$$\cdots \qquad \cdots$$
$$C_{124\cdots 2(m+1)} = C_{4\cdots 2(m+1)},$$

and

$$C_{13\cdots 2(m+1)}, C_{23\cdots 2(m+1)}$$

It is enough to show these 2(m+1) circles meet at a point.
Since

$$C_{3\cdots(2m+1)} \cap C_{3\cdots 2m2(m+1)} \cap \cdots \cap C_{4\cdots 2(m+1)} = P_{3\cdots 2(m+1)}$$

and $P_{3\cdots 2(m+1)} \in C_{13\cdots 2(m+1)} \cap C_{23\cdots 2(m+1)}$, by chapter 2,
$P_{3\cdots 2(m+1)}$ is the point we need and it is also the point defined by the 2m general lines. We define $P_{1\cdots 2(m+1)} = P_{3\cdots 2(m+1)}$.

(2) Need to prove that it is true when n=2(m+1)+1=2m+3.

Given $L_1 \,//\, L_2$ and $L_3, \cdots, L_{2m+3}$ general. We take 2m+2 lines each time and use the assumption and chapter 2, we obtain 2m+3 points

$$P_{1\cdots 2(m+1)} = P_{3\cdots 2(m+1)},$$

$$P_{1\cdots(2m+1)(2m+3)} = P_{3\cdots(2m+1)(2m+3)},$$

$$\dots \qquad \dots$$

$$P_{124\cdots(2m+3)} = P_{4\cdots(2m+3)},$$

and

$$P_{13\cdots(2m+3)}, P_{23\cdots(2m+3)}$$

It is enough to show these 2m+3 points lie on a same circle.
Since

$$\{P_{3\cdots 2(m+1)}, P_{3\cdots(2m+1)(2m+3)}, \cdots, P_{4\cdots(2m+3)}\} \subset C_{3\cdots(2m+3)}$$

and $\{P_{13\cdots(2m+3)}, P_{23\cdots(2m+3)}\} \subset C_{3\cdots(2m+3)}$ by chapter 2,

$C_{3\cdots(2m+3)}$ is the circle we need and it is also the circle defined by the 2m+1 general lines.

We define $C_{1\cdots(2m+3)} = C_{3\cdots(2m+3)}$.

By the mathematics induction, the statement is true.

■

2. Three lines parallel and the rest in general position

In this section, we will obtain that the chain theorem fails for any set of lines whith more than five lines where three lines are parallel and the rest are general.

There is no intersection for two parallel lines in the real plane; however, they have an intersection point at infinity in the projective plane.

Let $L_1 // L_2 // L_3$, from the two-parallel-line case in section 1 we get three intersection points

$$P_{12} = P_{13} = P_{23} (= P_1 = P_2 = P_3) \in L_\infty.$$

Hence, L_1, L_2, L_3 define a point P_{12} at infinity, or may say that they defined a circle of the form $\boxed{P}$ and write $C_{123} = P_{12}$.

2.1. One general line and three parallel lines

Given $L_1 // L_2 // L_3$ and L_4 general. From the two-parallele-line case in section 1 we get

$$C_{124} = L_4 \cup L_\infty, C_{134} = L_4 \cup L_\infty, C_{234} = L_4 \cup L_\infty,$$

and $C_{123} = P_1$, from the above discussion.

These four circles intersect at P_1 , which may be considered as the point determined by the four lines and we denote $P_{1234} = P_1$.

Lemma 2.1. Four lines, three parallel and one general, determine a point which is the point at infinity on the direction of those parallel lines.

2.2. Two general lines and three parallel lines

Given $L_1 // L_2 // L_3$ and two general lines L_4, L_5. We take four lines each time

L_1, L_2, L_3, L_4 define a point $P_{1234} = P_1$, by the lemma 2.1.

L_1, L_2, L_3, L_5 define a point $P_{1235} = P_1$, by the lemma 2.1.

L_1, L_2, L_4, L_5 define a point $P_{1245} = P_{45}$, by the lemma 1.2.

L_1, L_3, L_4, L_5 define a point $P_{1345} = P_{45}$, by the lemma 1.2.

L_2, L_3, L_4, L_5 define a point $P_{2345} = P_{45}$, by the lemma 1.2.

Where P_{45} is a point in the plane and P_1 is the point at infinity on the direction of L_1 . We need these five lines define a circle of some form in order to hold the chain theorem. Now we have two points, P_{45} , P_1 , defined by these five lines. There are two ways to define a circle passing through P_{45} , P_1 , and these two circles are the form of $\boxed{L}$ in the plane. The first one is the line passing through P_{45} and parallel to L_1 , and the second one could be the union of L_∞ and any one of the three parallel lines. However, by doing so there is no point is defined by a set of six lines where three lines are parallel. For such six lines, we may have three line-circles and one ordinary circle and they do not meet at a point. Therefore, the chain theorem fails for any set of lines with more than five lines and three lines parallel.

3. m lines concurrent and the rest in general position

In this section, we will obtain the chain theorem for n lines where m ($3 \le m \le n$) lines are concurrent and the rest are general.

3.1. Three concurrent lines

We know three general lines L_1 , L_2 , L_3 define a circle C_{123} . If we move L_1 towards P_{23} , we find that C_{123} shrinks into a point $P_{12} = P_{13} = P_{23}$, which may be considered as the point defined by the three concurrent lines. We may say that these three lines also determine a circle of the form $\boxed{P}$. In this section we are going to analyse the situation where there are m lines go through a same point P and the others are in general position.

Lemma3.1.1. Three concurrent lines define a circle, which is the concurrency.

3.2. Three concurrent lines and one general line

Let L_1 , L_2 , L_3 be concurrent, P is the concurrency and let L_4 be general. We take three out of the four lines each time and get four circles

$C_{123} = P$, the concurrency of the three lines,

C_{124} , C_{134} and C_{234} .

We have

$$P = P_{12} = P_{13} = P_{23} \in C_{123} \cap C_{124} \cap C_{134} \cap C_{234} .$$

Define $P_{1234} = P$, and we say that these four lines determine a point, which is the point of the concurrency of the three lines.

Lemma 3.2. Four lines, where three lines are concurrent and one line is general, define a point which is the concurrency.

Lemma 3.3. Five lines, where three lines are concurrent and two lines are general, define a circle.

Proof:

Suppose we are given five lines, L_1, L_2, L_3 concurrent and L_4, L_5 general. We take four lines each time and obtain five points

$P_{1234}, P_{1235}, P_{1245}, P_{1345}, P_{2345}$, where $P_{1234} = P_{1235} = P$ by lemma 3.2.

P_{1245}	$C_{124} \cap C_{145}$	P_{14}
P_{1345}	$C_{145} \cap C_{345}$	P_{45}
P_{2345}	$C_{345} \cap C_{234}$	P_{34}
P_{1234}	$C_{234} \cap C_{124}$	P_{24}

Where $\{P_{14}, P_{45}, P_{34}, P_{24}\} \in L_4$, then $\{P_{1245}, P_{1345}, P_{2345}, P_{1234}\} \in$ a circle, by the real cross ratio lemma. We define this circle is C_{12345}.

■

3.3. m lines concurrent and the rest in general position

Remark: In terms of the proofs of lemma 3.2. and 3.3., we could still use the same proofs as In chapter 2, where we just need to apply $P = P_{12} = P_{13} = P_{23} = C_{123}$ for the three concurrent lines. In the same manner, if m lines $L_1, \cdots, L_m$ are concurrent and the rest are general, we may still use the same proof as in chapter 2, where we only need to apply $P = P_{t_1 t_2} = C_{l_1 l_2 l_3} = \cdots = P_{j_1 \cdots j_{m-2}} = C_{i_1 \cdots i_{m-1}} = P_{1 \cdots m}$, if m is even and $P = P_{t_1 t_2} = C_{l_1 l_2 l_3} = \cdots = C_{j_1 \cdots j_{m-2}} = P_{i_1 \cdots i_{m-1}} = C_{1 \cdots m}$, if m is odd. Where $\{t_1, t_2\}, \{l_1, l_2, l_3\}, \cdots, \{j_1, \cdots, j_{m-2}\}, \{i_1, \cdots, i_{m-1}\}$ are subsets of $\{1, \cdots, m\}$.

Here we just state the theorem for n lines with m lines concurrent and the rest general without giving the proof.

Theorem 3.2. n lines, where m $(3 \leq m \leq n)$ lines are concurrent and $n - m$ lines are general, define a point if n is even and a circle if n is odd.

Chapter 4

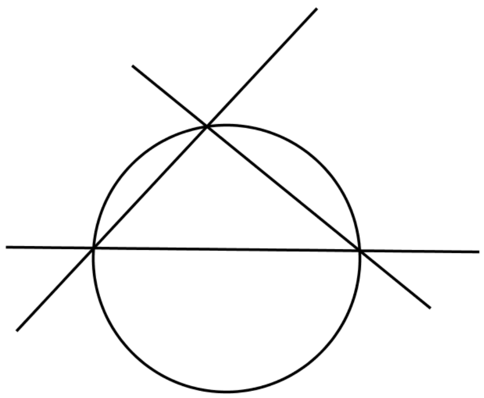
Different types of Clifford figures and the failure of Clifford theorem

In this chapter, we will present you 6 types of Clifford figures and the failure of the theorem with examples. However, it is difficult to find the certain condition for a set of lines such that we could know when a set of lines holds the theorem and when it fails. It is also difficult to know exactly what type of Clifford figures does a set of lines define when it holds the theorem.

We already discussed the Clifford chain theorem for any number n of general lines in chapter 2 and get that there is a Clifford point if n is even and a Clifford circle if n is odd. We also observed that for some special cases the Clifford figure has different forms and the theorem fails for some other cases in chapter 3. There are two main kinds of Clifford figures, namely, points and circles. However, we may have two kinds of points, ordinary points and points at infinity (nothing in the plane); three kinds of circles, ordinary circles, lines in the plane and points as we mentioned in the introduction for this paper.

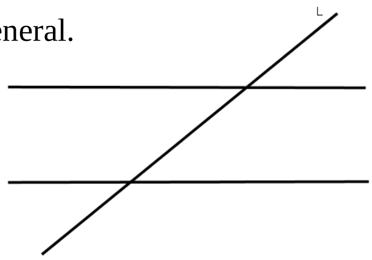
Type of lines	Number of lines	Clifford figure
(1)	odd	circle
(2)	odd	line
(3)	odd	point
(4)	odd	empty
(5)	even	point
(6)	even	empty

Example of type (1)**:** Three general lines define a circle in the plane.



Clifford circle of three general lines

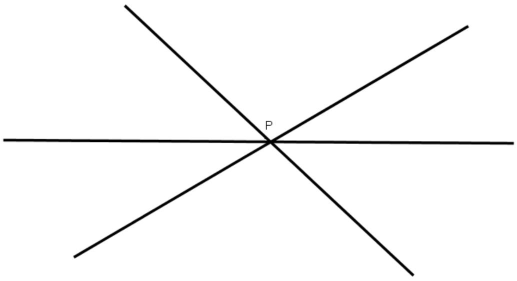
Example of type (2)**:** Three lines define a line in the plane, if two of them are parallel and one is general.



L is the Clifford figure of the three general lines

We know from chapter 2 that 3 general lines have a circle as their Clifford figure. Now we move some lines of the set of 3 lines such that the set finally approaches to a set of 3 lines with two of them parallel and one general. By doing so we obtained that the ordinary circle approaches to its limit, a line, the general line of the 3 lines.

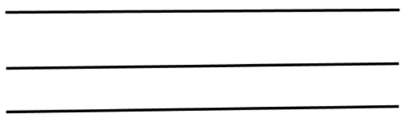
Example of type (3)**:** Three lines define a point in the plane, if they are concurrent.



P is the Clifford point of the three lines

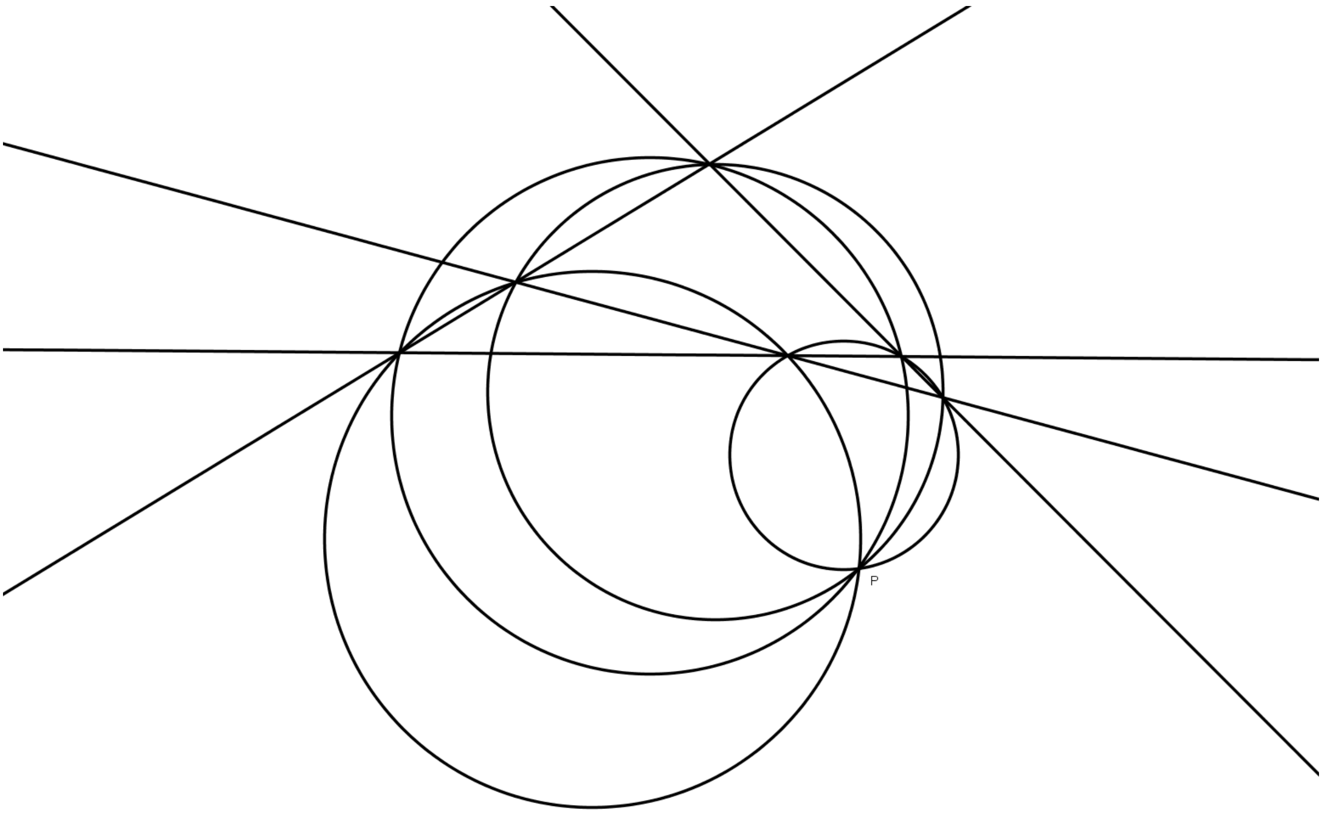
Three general lines define a Clifford circle, which is the circumcircle of the triangle formed by the lines. If we move some lines of the set of 3 lines such that the set approaches to a set of 3 concurrent lines. By doing so the Clifford circle approaches to its limit, a point, the concurrency.

Example of type (4)**:** Three parallel lines define nothing in the plane. The chain theorem fails for a set of lines where 3 lines are parallel.



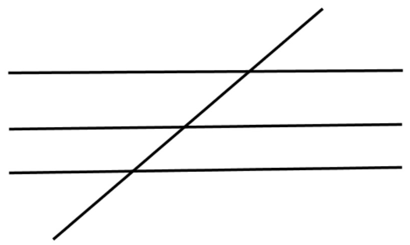
no Clifford figure defined

Example of type (5)**:** Four general lines define a point in the plane, we see this from chapter 2.



P is the Clifford point of the four lines

Example of type (6)**:** Four general lines define nothing in the plane, if three of them are parallel and one is general. Here, the three parallel lines as a subset of the lines is the type (4) and define nothing, hence, these four lines also define nothing. In another word, the chain theorem fails in this case.



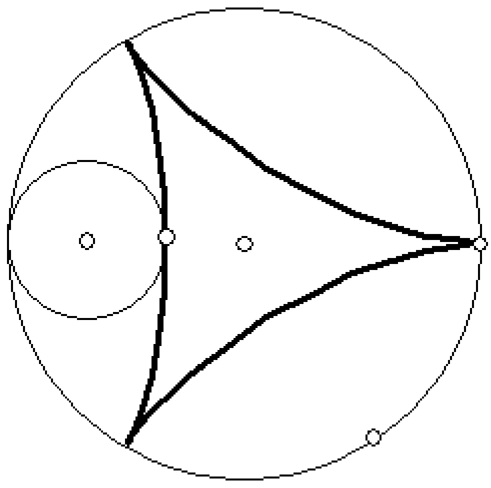
no Clifford figure defined

Remark: In theorem 2.2. in chapter 2, the circle, determined by odd number of general lines, may have different forms other than an ordinary circle. It is helpful to state the following example from the article, *The Failure of the Clifford Chain*, by Walter B. Carver.

Example: Five lines touching a deltoid define a line as the Clifford figure.

Definition of a deltoid: In geometry, a hypocycloid is a special plane curve generated by the trace of a fixed point on a smaller circle that rolls within a larger circle. When the radius of the smaller circle is r and the radius of the larger circle is $R = 3r$, this plane curve is a deltoid.

In this example, when all the five lines are in general position, we obtain a Clifford circle of the form $\boxed{L}$. However, in chapter 2 we just state that odd number of general lines define a circle without mentioning the forms of circles.



the black curve with three angles is a deltoid

Chapter 5

Chain theorem for general circles on the sphere

We are going to prove a chain theorem for general circles on the sphere. This chapter is an application of the Clifford chain theorem for general lines in the plane. We will prove this theorem by transforming Clifford chain theorem for general lines in the plane onto the sphere under stereographic projection. We will do this in detail in one chapter and will put some other applications together in chapter 6.

Observing all circles on the sphere passing through the north pole N, we may divide them into general circles and degenerate circles. Any two such circles have at least one intersection point, the north pole N. We say two such circles are parallel if they do not have a further intersection point other than N; we say two such circles are intersect if they have a further intersection point other than N; we say three such circles are concurrent if they pass a same point other than N; we say some such circles are of general position if no two of them parallel nor three of them concurrent.

The complex plane with ∞ added is called the extended complex plane and we write $C^+ = C \cup \{\infty\}$. Another name for C^+ is the Riemann sphere after B. Riemann. He was the first mathematician to identify the extended complex plane with a sphere under stereographic projection. We will fined that there is an one-to-one correspondence between the set of all general lines in the plane and the set of all general circles on the sphere under the stereographic projection. It is straight forward to consider the set of general circles on the sphere in terms of the chain theorem.

The sphere and planes in space

Suppose P is any plane in the space and $v = (a, b, c)$ is the unit normal vector to the plane P, i.e. $|v| = \sqrt{a^2 + b^2 + c^2} = 1$. Now we may express P as

$$ax + by + cz + e = 0$$

The distance between a point $B = (x_0, y_0, z_0)$ and the plane $ax + by + cz + e = 0$ is given by the following formula

$$D = \frac{|ax_0+by_0+cz_0+e|}{\sqrt{a^2+b^2+c^2}}.$$

The distance between the plane P and the origin $O = (0,0,0)$ is

$$D = \frac{|a\cdot 0+b\cdot 0+c\cdot 0+e|}{\sqrt{a^2+b^2+c^2}} = \frac{|e|}{\sqrt{c^2+d^2+e^2}} = |e|$$

Let S be the unit sphere centred at the origin. There are three possible relations between the plane P and the sphere S:

(1): If $|e| > 1$, then $P \cap S = \emptyset$.

(2): If $|e| = 1$, then $P \cap S = \{a\ point\}$, P is tangent to S at the point.

(3): If $|e| < 1$, then $P \cap S$ is a circle on S, in particular, is a big circle when $|e| = 0$.

Lemma: The section of a sphere made by any plane is a circle.

Proof:

For simplicity, we choose the sphere as the unit sphere S, centred at the origin O. Let C be the curve of the section of the sphere made by any plane P, then $C = S \cap P$.

Draw OO' perpendicular to the plane P; take any point $A \in C$ and join $OA, O'A$. Since OO' is perpendicular to the plane P, the angle $\angle OO'A$ is a right angle; therefore,

$$O'A = \sqrt{OA^2 - O'O^2}.$$

Now O and O' are fixed points, so OO' is constant; and $OA = 1$, being the radius of the sphere; hence $O'A$ is constant. Thus all the points in the curve C are equally distant from the fixed point O', O' lies on the same plane P as C does. Therefore, the section C is a circle of which O' is the centre. Here when the plane tangent to the sphere at O', then $O'A = 0$ and the section C is the tangent point; and when the plane pass through the origin O, then the section is a great circle, a circle on the sphere for which the centre is the origin.

This lemma tells us that any plane intersects the sphere defines a circle on the sphere which is the section of the sphere made by the plane. Since a circle lies in a plane and we may say that for any circle on the sphere, there is a plane for which the circle is the intersection circle.

Therefore, it is true that there is a one-to-one correspondence between the set of all the planes that intersect the sphere and the set of all the circles on the sphere.

Stereographic projection

Definition:

Let $P_0 = \{(x, y, z) \mid z = 0\}$ be the Euclidean plane,
$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ the unit sphere with the centre at the origin,
$N = (0,0,1)$ the northe pole on S. Stereographic projection is the projection of the unit sphere S from the North Pole N onto the plane P_0 through the equator.
The stereographic projection is a one-to-one correspondence between points in the set $S - \{N\}$ and the points in the set P_0. For any point $M \in S - \{N\}$, there is a unique line L through N and M, and this line intersects the plane P_0 in exactly one point M'. Define the stereographic projection of M to be this point M', and we denote this map by $\varphi(M) = M'$ and its inverse map by $\varphi^{-1}(M') = M$. We may also extend this map onto the whole sphere by assuming that the projection of N is the point at infinity in the plane P_0.

Formulas of stereographic projection

(1) Formula for stereographic projection from the sphere to the plane

Given any point $M = (m, n, l) \in S - \{N\}$, then there is a unique line L passing through N and M. We may choose the director vector for L as

$$\vec{v} = \overrightarrow{NM} = (m, n, l) - (0,0,1) = (m, n, l - 1) .$$

Let $A = (x, y, z)$ be any point on the line L, then the line L can be expressed by the following equation

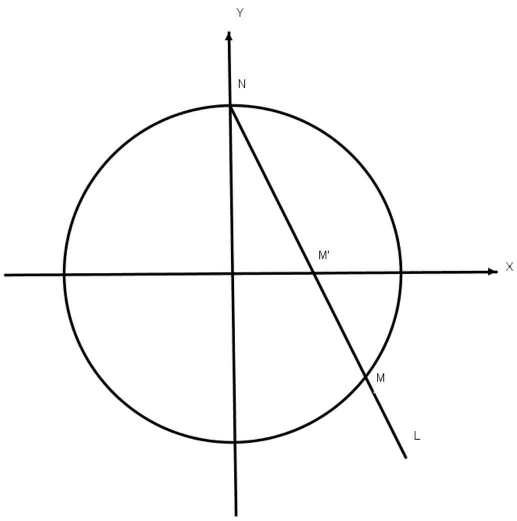
$$A = \vec{N} + t\vec{v},$$

$$(x, y, z) = (0,0,1) + t(m, n, l - 1)$$

$$= (tm, tn, 1 + t(l - 1))$$

or

$$\begin{cases} x = tm \\ y = tn \\ z = 1 + t(l-1) \end{cases}$$



stereographic projection of S from N onto $Z = 0$, shown in cross section

Since the line L intersects the plane $P_0 = \{(x, y, z) \mid z = 0\}$ in a unique point $\varphi(M) = M'$. We may fined M' by substituting $z = 0$ into the equation of L

$z = 1 + t(l-1) = 0 \Rightarrow t = \frac{1}{1-l}$, and yields

$$\begin{cases} x = \frac{m}{1-l} \\ y = \frac{n}{1-l} \\ z = 0 \end{cases}$$

So, $M' = (\frac{m}{1-l}, \frac{n}{1-l})$.

Therefore the formula for the map $\varphi: S - \{N\} \to P_0$ is

$$\varphi(m, n, l) = (\frac{m}{1-l}, \frac{n}{1-l})$$

(2) Formula for the inverse stereographic projection from the plane to the sphere

Given $M' = (a, b, 0)$ or (a, b)in the plane P_0 , then there is a unique line L passing through N and M'. We may choose the director vector for L as

$$\vec{v} = \overrightarrow{NM'} = (a, b, 0) - (0,0,1) = (a, b, -1) .$$

Let $A = (x, y, z)$ be any point on the line L, then the line L can be expressed by the following equation

$$A = \vec{N} + t\vec{v},$$

$$(x, y, z) = (0,0,1) + t(a, b, -1)$$

$$= (ta, tb, 1 - t)$$

or

$$\begin{cases} x = ta \\ y = tb \\ z = 1 - t \end{cases}$$

Since the line L intersects the sphere $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ in a unique point $\varphi^{-1}(M') = M$ other than N. We may fined M by combining the two equations of L and S.

$$(ta)^2 + (tb)^2 + (1 - t)^2 = 1,$$

$$t^2a^2 + t^2b^2 + 1 - 2t + t^2 = 1,$$

$$t^2a^2 + t^2b^2 - 2t + t^2 = 0,$$

$$t(ta^2 + tb^2 + t - 2) = 0,$$

$$t\{t(a^2 + b^2 + 1) - 2)\} = 0,$$

$$\begin{cases} t = 0 \text{ , } or \\ t = \dfrac{2}{a^2 + b^2 + 1} \end{cases}$$

If $t = 0$, we obtain $\begin{cases} x = 0 \\ y = 0 \\ z = 1 \end{cases}$ from the equation of the line and so we get one intersection of the sphere and the plane, which is the north pole, $N = (0,0,1)$.

If $t = \frac{2}{a^2+b^2+1}$, we obtain that

$$\begin{cases} x = \frac{2a}{a^2+b^2+1} \\ y = \frac{2b}{a^2+b^2+1} \\ z = \frac{a^2+b^2-1}{a^2+b^2+1} \end{cases}$$

And we get the other intersection of the sphere and the plane, which is

$$\varphi^{-1}(M') = M = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right).$$

Therefore the formula for the inverse map $\varphi^{-1}: P_0 \to S - \{N\}$ is

$$\varphi^{-1}(a,b) = (\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}).$$

Some properties of stereographic projection

(1) The image of a circle on the sphere is a circle(or a line) in the plane under the map

Proof:

Given a circle C on the sphere S, then there is one and only one plane P such that $C = P \cap S$. Let $\vec{v} = (c, d, e)$ be the normal vector to the plane P with $|\vec{v}| = \sqrt{c^2 + d^2 + e^2} = 1$.

We may express the plane P as

$$cx + dy + ez + f = 0 \quad \cdots\cdots\cdots \quad \boxed{1}$$

The distance between the plane P and the origin $O = (0,0,0)$ is

$$D = \frac{|c\cdot 0 + d\cdot 0 + e\cdot 0 + f|}{\sqrt{c^2+d^2+e^2}} = \frac{|f|}{\sqrt{c^2+d^2+e^2}} = |f|.$$

For any point $A \in C$, let $A' = (a, b)$ be the projection point of the point A, then

$$A = \varphi^{-1}(A') = \varphi^{-1}(a,b) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right)$$
$$= \left(\frac{2a}{Q+1}, \frac{2b}{Q+1}, \frac{Q-1}{Q+1}\right).$$

Where we temporarily denote $a^2 + b^2 = Q$, for simplicity. Since $A \in C = P \cap S$, substituting $A = (\frac{2a}{Q+1}, \frac{2b}{Q+1}, \frac{Q-1}{Q+1})$ into the equation $\boxed{1}$ of the plane, we get

$$\frac{2ac}{Q+1} + \frac{2bd}{Q+1} + \frac{eQ-e}{Q+1} + f = 0,$$
$$(e+f)Q + 2(ca+db) + (f-e) = 0,$$
$$(e+f)(a^2+b^2) + 2(ca+db) + (f-e) = 0 \quad \cdots\cdots\cdots \quad \boxed{2}$$

If $e = -f$, then equation $\boxed{2}$ becomes

$$2ca + 2db - 2e = 0,$$
$$ca + db - e = 0.$$

Which is the equation of a line in terms of (a,b).

At this time the equation $\boxed{1}$ of the plane P becomes

$$cx + dy + ez - e = 0$$

for which $N = (0,0,1)$ is satisfied and therefore, $N \in C = P \cap S$.

Therefore, we may say that for any circle C passing through N on the sphere, $\varphi(C)$ is a line in the plane.

If $e \neq -f$, the equation $\boxed{2}$ becomes

$$a^2 + b^2 + \frac{2c}{e+f}a + \frac{2d}{e+f}b = \frac{e-f}{e+f},$$
$$\left(a + \frac{c}{e+f}\right)^2 + \left(b + \frac{d}{e+f}\right)^2 = \frac{1-f^2}{(e+f)^2}$$

This expresses a circle with the centre at $\left(-\frac{c}{e+f}, -\frac{d}{e+f}\right)$ and radius $\frac{\sqrt{1-f^2}}{|e+f|}$.

At this time $N \notin P$, so $N \notin C$ and we may say that for any circle C on the sphere not passing through $N, \varphi(C)$ is a circle in the plane.

■

(2) The image of a circle and a line in the plane is a circle on the sphere under the inverse map

Proof:

(a): We are going to show that the inverse map send a circle in the plane P_0 to a circle on the sphere not passing through the north pole N.

Let C_0 be a circle in the plane P_0 and it has an equation

$$x^2 + y^2 + ax + by + c = 0 \quad \cdots\cdots\cdots \quad \boxed{1}$$

Then

$$\varphi^{-1}(C_0) = \{(u, v, w) \in S \mid \varphi(u, v, w) \in C_0\}$$

$$= \left\{(u, v, w) \in S \mid \left(\frac{u}{1-w}, \frac{v}{1-w}\right) \in C_0\right\}$$

Now we are going to investigate what kinds of elements we have in the set above.

Since $(u, v, w) \in S$, we have $u^2 + v^2 + w^2 = 1$.

Since $\left(\frac{u}{1-w}, \frac{v}{1-w}\right) \in C_0$, we have obtained the following equation by substituting it into the equation $\boxed{1}$

$$\left(\frac{u}{1-w}\right)^2 + \left(\frac{v}{1-w}\right)^2 + \frac{au}{1-w} + \frac{bv}{1-w} + c = 0$$

$$u^2 + v^2 + au(1-w) + bv(1-w) + c(1-w)^2 = 0$$

And $u^2 + v^2 = 1 - w^2$, we have

$$1 - w^2 + au(1-w) + bv(1-w) + c(1-w)^2 = 0$$

$$(1-w)[(1+w) + au + bv + c(1-w)] = 0$$

hence

$$1 - w = 0$$

or

$$(1+w) + au + bv + c(1-w) = 0$$

When $1 - w = 0$, $w = 1$, then $u = v = 0$ and therefore

$\varphi^{-1}(C_0) = \{(0,0,1)\} = \{N\}$ and so $C_0 = \{\infty\}$ by the definition and this is not an ordinary circle and so we ignore this.

Hance we have obtained that

$$\varphi^{-1}(C_0) = \{(u, v, w) \in S \mid (1 + w) + au + bv + c(1 - w) = 0\}$$
$$= \{(u, v, w) \in S \mid au + bv + (1 - c)w + (1 + c) = 0\}$$
$$= S \cap P$$

Where $au + bv + (1 - c)w + (1 + c) = 0$ is an equation of a plane *P* in the space which does not pass N. In order to prove $\varphi^{-1}(C_0) = S \cap P$ is a circle on the sphere, it is enough to show that the distance D between $O = (0,0,0)$ and *P* is less than 1.

The equation $\boxed{1}$ of the circle C_0 may be written as

$$\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \left(\frac{\sqrt{a^2 + b^2 - 4c}}{2}\right)^2$$

Where the radius $\left(\frac{\sqrt{a^2+b^2-4c}}{2}\right)^2 > 0$

$$a^2 + b^2 - 4c > 0$$

$$a^2 + b^2 > 4c$$

This yield

$$D = \frac{|a \times 0 + b \times o + (1 - c) \times 0 + (1 + c)|}{\sqrt{a^2 + b^2 + (1 - c)^2}}$$

$$= \frac{|1+c|}{\sqrt{a^2+b^2+c^2-2c+1}}$$

$$< \frac{|1+c|}{\sqrt{4c+c^2-2c+1}} = \frac{|1+c|}{\sqrt{c^2+2c+1}} = \frac{|1+c|}{\sqrt{(1+c)^2}} = 1.$$

(b): Now we are going to prove that the inverse map send a line in the plane P_0 to a circle on the sphere passing through the north pole N.

Given a line L in the plane P_0 and we may express L by the following euation

$$ax + by + c = 0, (a, b) \neq O$$

For any $A \in L$, let $A' = (u, v, w) = \varphi^{-1}(A) \in S$, then $A = \left(\frac{u}{1-w}, \frac{v}{1-w}\right)$ and by subsitituting it into the equation of *L* we get an equation for the point $A' = (u, v, w)$

$$\frac{au}{1-w} + \frac{bv}{1-w} + c = 0,$$

$$au + bv + c(1 - w) = 0 \ \cdots\cdots\cdots \ \boxed{1}$$

This is an equation of a plane in the space passing through N, which we denote by P, and we have

$$A' \in P \cap S, N \in P \ .$$

The distance between the plane *P* and the origin $O = (0{,}0{,}0)$ is

$$D = \frac{|a{\cdot}0 + b{\cdot}0 - c{\cdot}0 + c|}{\sqrt{a^2+b^2+c^2}} = \frac{|c|}{\sqrt{a^2+b^2+c^2}} < \frac{|c|}{|c|} = 1 \ .$$

That means the plane P and the sphere S intersect and the section is not a point, is an ordinary circle. Hance the projection of the line L in the plane P_0 is a circle on the sphere S. Let $C = \varphi^{-1}(L)$, we may write $C = P \cap S$.

From $N \in P$, we have

$N = (0{,}0{,}1) \in P \cap S = C$.

Therefore the inverse projection of a line in the plane P_0 is a circle on the sphere S passing through the north pole N.

In particular, when the line *L* passes through the origin, $c = 0$ and the distance between the plane P and the origin is $D = 0$. So $C = P \cap S$ is a big circle on the sphere passing through the two poles.

■

It is obvious that we may obtain the following two lemmas directly from the two properties of stereographic projection.

Lemma 5.1. Stereographic projection induces a bijection between the set of all the circles in the plane and the set of all the circles on the sphere which do not pass through north pole.

Lemma 5.2. Stereographic projection induces a bijection between the set of all the lines in the plane and the set of all the circles on the sphere which do pass through the north pole. In particular, stereographic projection induces a bijection between the set of all the general lines in the plane and the set of all the general circles on the sphere.

Real Cross ratio theorem on the sphere

Because of the properties (1) and (2) of the stereographic projection, the conditions and the results of the real cross ratio theorem remain unchanged under the map. Stereographic projection sends points on a circle or a line in the plane to points on a circle on the sphere and sends circles or lines through a same point in the plane to circles through a same point on the sphere. Therefore, by the two lemmas above, the real cross ratio lemma applies on the sphere. In other words, if some circles and lines have some incidence relations in the plane, then the corresponding circles have the exact same incidence relations. Therefore, it is enough to transform the chain theorem for lines in the plane into the chain theorem for circles on the sphere.

Real Cross ratio Lemma on the sphere:

Consider four circles on the sphere, C_1, C_2, C_3, C_4. Let C_1, C_2 meet in the points Z_2, W_2 ; let C_2, C_3 meet in Z_3, W_3; let C_3, C_4 meet in Z_4, W_4 and let C_4, C_1 meet in Z_1, W_1. Then the points Z_1, Z_2, Z_3, Z_4 are concyclic if and only if the points W_1, W_2, W_3, W_4 are concyclic.

For the set of all the general lines in the plane, we obtained a chain theorem in chapter 2 and we proved it in detail. By lemma 5.2., we know that stereographic projection induces a bijection between the set of all the general lines in the plane and the set of all the general circles on the sphere. And also we have the properties of stereographic projection including lemma 5.1., 5.2. and the real cross ration lemma on the sphere. Therefore, it is enough to state the chain theorem for the set of all the general circles on the sphere.

We then have a chain of theorems for general circles on the sphere:

(1) Through a point N on the sphere pass a number of general circles $C_1, C_2, C_3, \cdots$

(2) Each two general circles meet in one further point: P_{12} denotes the point of meeting of the circles C_1 and C_2.

(3) Three general circles C_1, C_2, C_3, give three such points $P_{12},\ P_{13},\ P_{23}$. The circle through

these points is called the circle C_{123}.

(4) Four general circles C_1, C_2, C_3, C_4 , give four circles like C_{123}. It is found that these four circles always meet in a point, called the point P_{1234}.

(5) Five general circles C_1, C_2, C_3, C_4, C_5, give five points like P_{1234}. It is found that these points always lie on a circle, called the circle C_{12345}.

And so on we may state this chain theorem as follows.

Theorem: n general circles on the sphere, determine a point on the sphere if n is even and a circle on the sphere if n is odd.

Chapter 6

Some applications of Clifford chain theorem

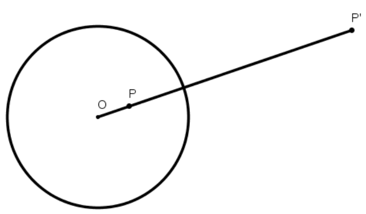
In this chapter, it is going to prove another two chain theorems for circles passing through a same point in the plane and planes passing through a same point in the space by using our line chain theorem in the plane and circle chain theorem on the sphere. We will also prove Miquel's pentagon theorem by applying Clifford line-chain theorem.

Application 1

Chain theorem for general circles in the plane

Circle Inversion

Definition: In the plane, the inverse of a point P in respect to a reference circle of centre O and radius R is a point P' such that P and P' are on the same ray going from O, and whose distance from O satisfies the equation $OP \times OP' = R^2$, we may say the inverse of the point P' is the point P. Assume that there is only one point at infinity in the plane and its inverse point is the centre O of the reference circle, then the inversion is a one-to-one transformation of the whole inversive plane.



P' is the inverse of P

Some Facts of Circle Inversion

(a) A line not passing through O is inverted into a circle passing through O, and vice versa; whereas a line passing through O is inverted into itself.

(b) A circle not passing through O is inverted into a circle not passing through O. The circle (or line) after inversion stays as before if and only if it is orthogonal to the reference circle at their points of intersection.

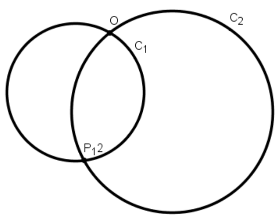
(c) Two circles have three types of relationship, intersecting, tangent and non-intersecting. A pair of circles of any one of these three types inverts into a pair of the same type (including, among pairs of "tangent circles", one circle and a tangent line, as well as two parallel lines).

Definition of general circles: Consider all circles passing through a common point O in the plane, we may divide them into general circles and degenerate circles. Any two such circles have at least one intersection point, O. We say two such circles are parallel if they do not have a further intersection point other than O; we say two such circles are intersect if they have a further intersection point other than O; we say three such circles are concurrent if they pass a same point other than O; we say some such circles are of general position if no two of them parallel nor three of them concurrent.

Circle chain theorem

We already proved the chain theorem for general lines in the plane. We may transform the situation of n general circles in the plane into the situation of n general lines in the plane by a circle inversion. We apply the chain theorem for general lines and finally inverts back to the original situation and get the corresponding chain theorem. Let the reference circle is a circle with radius R and centred at O.

Lemma 6.1. Two general circles in the plane determine a point.

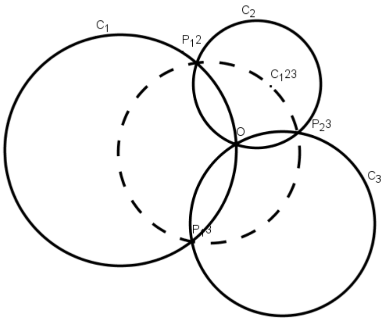


P_{12} is the point defined by C_1, C_2 through the point O

It is obvious that the intersection point, P_{12}, is the point determined by the two circles, C_1, C_2 passing through a same point O. However, we may also invert this situation into the situation of two general lines in the plane. Let L_1 , L_2 be the inverses of C_1 , C_2 and P'_{12} the inverse of P_{12} . Since P'_{12} is the point defined by L_1 and L_2, it is reasonable to take P_{12} as the point defined by C_1 and C_2.

■

Lemma 6.2. Three general circles in the plane determine a circle.



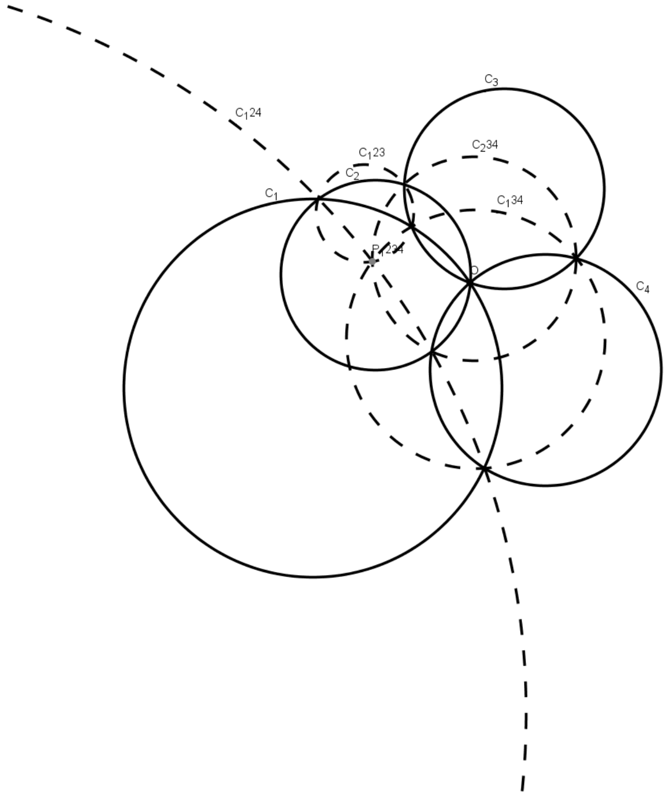
C_{123} is the circle defined by C_1, C_2, C_3 through the point O

Three general circles intersect in three points, P_{12}, P_{13}, P_{23}, and the circle, C_{123}, passing through these three points, is the circle we need. We may get the same result by inversion. Let L_1, L_2, L_3 be the inverses of C_1, C_2, C_3 and $P'_{12}, P'_{13}, P'_{23}$ the inverses of P_{12}, P_{13}, P_{23} respectivelly. Let C'_{123} is the circumcircle of the triangle formed by the three lines, which is the circle defined by the three lines. Hence, C_{123} the inverse of C'_{123}, passing through P_{12}, P_{13}, P_{23}, is the circle we need.

■

Theorem 6.1. Four general circles in the plane determine a point.

In order to understand the chain theorem for n general circles in the plane, we will describe theorem 6.1. in detail and also for getting a main idea about the proof of the theorem, we are going to give a very detailed proof for theorem 6.1. in particularly.



P_{1234} is the point defined by C_1, C_2, C_3, C_4 through the point O

Description of theorem 6.1.:

Given four general circles C_1, C_2, C_3, C_4, all passing through a same point O. Let C_1 and C_2 meet in P_{12}, C_1 and C_3 in P_{13}, C_1 and C_4 in P_{14}, C_2 and C_3 in P_{23}, C_2 and C_4 in P_{24}, C_3 and C_4 in P_{34}. Let C_{123} be the circle through P_{12}, P_{13}, P_{23}; C_{124} the circle through P_{12}, P_{14}, P_{24}; C_{134} the circle through P_{13}, P_{14}, P_{34}; C_{234} the circle through P_{23}, P_{24}, P_{34}. Then the four circles $C_{123}, C_{124}, C_{134}, C_{234}$ have a common point P_{1234}.

Proof:

We will prove this by inverting the situation of the four general circles C_1, C_2, C_3, C_4 into the corresponding situation of four general lines L_1, L_2, L_3, L_4, and apply the line chain theorem for four lines from chapter 2 and get a point, and finally we obtain the corresponding point for the four circles by getting everything inverted back into the original situation.

Now we have obtained four general lines L_1, L_2, L_3, L_4 in the plane. Let L_1 and L_2 meet in P'_{12}, L_1 and L_3 in P'_{13}, L_1 and L_4 in P'_{14}, L_2 and L_3 in P'_{23}, L_2 and L_4 in P'_{24}, L_3 and L_4 in P'_{34}. Let C'_{123} be the circle through $P'_{12}, P'_{13}, P'_{23}$; C'_{124} the circle through $P'_{12}, P'_{14}, P'_{24}$; C'_{134} the circle through $P'_{13}, P'_{14}, P'_{34}$; C'_{234} the circle through $P'_{23}, P'_{24}, P'_{34}$. Then the four circles $C'_{123}, C'_{124}, C'_{134}, C'_{234}$ have a common point P'_{1234} by the line chain theorem form chapter 2.

Now we get the situation of lines above inverted back into the four circles case:
Applying the facts of circle inversion, we obtain that P'_{12} gets inverted back into P_{12}, the intersection point of C_1 and C_2; P'_{13} into P_{13}, the intersection point of C_1 and C_3; P'_{14} into P_{14}, the intersection point of C_1 and C_4; P'_{23} into P_{23}, the intersection point of C_2 and C_3; P'_{24} into P_{24}, the intersection point of C_2 and C_4; P'_{34} into P_{34}, the intersection point of C_3 and C_4. Hence, C'_{123} gets inverted back into C_{123}, C'_{124} into C_{124}, C'_{134} into C_{134}, C'_{234} into C_{234} and gets the common point P'_{1234} of $C'_{123}, C'_{124}, C'_{134}, C'_{234}$ into P_{1234}, the common point of $C_{123}, C_{124}, C_{134}, C_{234}$. Therefore, P_{1234} is the point determined by the four general circles.

■

Circle chain theorem: n general circles in the plane determine a point if n is even and a circle if n is odd.

Application 2

Chain theorem for general planes in the space

Planes through a point in the space

Let $L_{P_0} = \{all\ the\ lines\ in\ the\ plane\ P_0\}$,

$C_N = \{all\ the\ circles\ through\ N\ on\ the\ sphere\ S\}$,

$P_N = \{all\ the\ planes\ through\ N\ on\ the\ sphere\ S, except\ the\ the\ one\ tangents\ to\ S\}$.

Lemma 5.2. tells us that there is a one-to-one correspondence between L_{P_0} and C_N. We know that for each fair of line $L \in L_{P_0}$ and circle $C \in C_N$ under the correspondence, there is a unique plane $P \in P_N$ such that $L \in P, C = P \cap S$. Hence, there is a one-to-one correspondence between the three sets, L_{P_0}, C_N, P_N. This inspired us to consider all the planes through a point in the space by using S and some results in chapter 5 in terms of the chain theorem.

We may assume that N is the point that all the planes pass through in the space, where N is the north pole of the sphere S.

For any two planes $P_1, P_2 \in P_N$, there is an intersection line $l_{12} = P_1 \cap P_2$ through N, there are two possible relations for the two planes:

(a) l_{12} is tangent to the sphere, i.e. $l_{12} \cap S = \{N\}$, the two corresponding circles are parallele on S in this case.

For $P_1, P_2 \in P_N$, there are $C_1, C_2 \in C_N$ such that $C_1 = P_1 \cap S, C_2 = P_2 \cap S$.

Then
$$\begin{aligned}\{N\} &= l_{12} \cap S \\ &= (P_1 \cap P_2) \cap S \\ &= (P_1 \cap S) \cap (P_2 \cap S) \\ &= C_1 \cap C_2\end{aligned}$$

(b) l_{12} goes through the sphere and has a further intersection point P_{12} other then N with the sphere, i.e., $l_{12} \cap S = \{N, P_{12}\}$, the two corresponding circles intersect in P_{12} on S.

For $P_1, P_2 \in P_N$, there are $C_1, C_2 \in C_N$ such that $C_1 = P_1 \cap S, C_2 = P_2 \cap S$.

Then
$$\begin{aligned}\{N, P_{12}\} &= l_{12} \cap S \\ &= (P_1 \cap P_2) \cap S \\ &= (P_1 \cap S) \cap (P_2 \cap S) \\ &= C_1 \cap C_2\end{aligned}$$

For three planes such that the intersection line of any two of them goes through the sphere.

Let $P_1, P_2, P_3 \in P_N$;

$l_{12} = P_1 \cap P_2$, $l_{13} = P_1 \cap P_3$, $l_{23} = P_2 \cap P_3$;

$\{P_{12}, N\} = l_{12} \cap S$, $\{P_{13}, N\} = l_{13} \cap S$, $\{P_{23}, N\} = l_{23} \cap S$.

Then there are two possible relations for the three planes:

(c) $l_{12} = l_{13} = l_{23}$, and so $P_{12} = P_{13} = P_{23}$, the three corresponding circles are concurrent and the concurrency is P_{12} on S.

(d) l_{12}, l_{13}, l_{23} are different and so P_{12}, P_{13}, P_{23} are three different points on S, the three corresponding cirlces are not concurrent in this case.

Definition: We define G is the set of all the general planes in P_N such that (b) and (d) are satisfied, in another world, G includes all the planes in P_N such that no two planes satisfy (a) and no three planes satisfy (c). We call all the planes in G general planes.

Theorem: n general planes determine a point on the sphere if n is even and a circle on the sphere if n is odd.

Proof:

(1) $n = 2$.

For any two general planes $P_1, P_2 \in G \subseteq P_N$, there are two general circles $C_1, C_2 \in C_N$, such that,$C_1 = P_1 \cap S, C_2 = P_2 \cap S$. From chapter 5 we know that C_1, C_2 determine a point on the sphere, P_{12} and $\{P_{12}, N\} = l_{12} \cap S$, i.e. P_{12} is the only intersection point of the two planes which lies on the sphere but differ from N. Define P_{12} is the point which is determined by P_1, P_2.

(2) $n = 3$.

For any three general planes $P_1, P_2, P_3 \in G \subseteq P_N$, there are three general circles

$C_1, C_2, C_3 \in C_N$, such that,

$C_1 = P_1 \cap S,\ C_2 = P_2 \cap S,\ C_3 = P_3 \cap S.$

From chapter 5 we know that C_1, C_2, C_3 determine a circle on the sphere, C_{123}, the circle passing through the three intersection points P_{12}, P_{13}, P_{23} of the planes on the sphere differ from N. We may define that C_{123} is the circle on the sphere which is determined by the three planes.

(3) $n = m$.

Let $P_1, \cdots, P_m$ be m general planes and $C_1, \cdots, C_m$ the corresponding general circles on the sphere. Assume that it is true for $n < m$ general planes,i.e., they determine a point if n is even and a circle if n is odd. Only need to show that it remains true for m general planes.

If m is even, we take $m-1$ planes into consideration each time and get m circles, $C_{12\cdots(m-1)}$, $C_{12\cdots(m-2)m}$, $\cdots$, $C_{23\cdots m}$, each one is determined by the corresponding $m-1$ planes each time by the assumption. We know from chapter 5 that each one of these circles is determined by $m-1$ circles from $C_1, \cdots, C_m$ and $C_{12\cdots(m-1)}$, $C_{12\cdots(m-2)m}$, $\cdots$, $C_{23\cdots m}$ pass through the same point $P_{12\cdots m}$ on the sphere defined by $C_1, \cdots, C_m$. So, it is reasonble to define $P_{12\cdots m}$ is the point determind by the m general planes.

If m is odd, we take $m-1$ planes into consideration each time and get m points, $P_{12\cdots(m-1)}$, $P_{12\cdots(m-2)m}$, $\cdots$, $P_{23\cdots m}$, each one is determined by the corresponding $m-1$ planes each time by the assumption. We know from chapter 5 that each one of these points is determined by $m-1$ circles from $C_1, \cdots, C_m$ and $P_{12\cdots(m-1)}$, $P_{12\cdots(m-2)m}$, $\cdots$, $P_{23\cdots m}$ lie on the same circle $C_{12\cdots m}$ on the sphere defined by $C_1, \cdots, C_m$. So, it is reasonable to define $C_{12\cdots m}$ is the circle determind by the m general planes.

Therefore, by the mathematical induction, the theorem is true for any number of general planes.

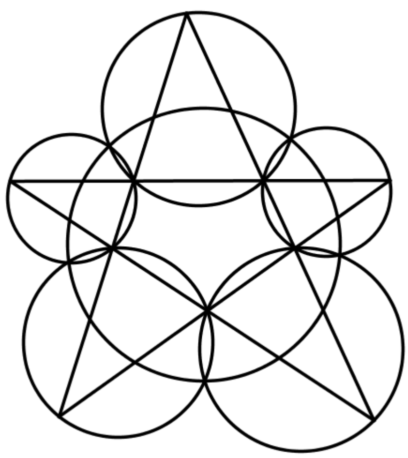
■

Remarks: Here we take the sphere as a reference and transforms the chain theorem for planes passing through a same point in the space under the correspondence between the circles through a same point on the sphere and the planes through the same point in the space. However, we may also take any plane not passing through the common point of the planes as a reference plane and use the correspondence between planes in the space and lines in the plane. For any plane passing through the common point in the space, there is a line in the reference plane such that the line is cut by the plane. The same chain theorem may be achieved by transforming the line chain theorem in the plane under the correspondence.

Application 3

Miquel's pentagon theorem

Miquel's pentagon theorem: If five general lines form a pentagon and the sides are extended to form a pentagram, the five lines intersect to form triangles on each side of the pentagon. Draw the circumcircles of each of these triangles. Then the five new points formed by the intersection of these five circles lie on another circle.



the biggest circle is defined by the five lines

Proof:

Let $L_1,\ L_2,\ L_3,\ L_4,\ L_5$ be the five lines which forms the five sides of the pentagon successively. Let Δ_{ijk} be the triangle formed by the three lines $L_i,\ L_j, L_k$ and C_{ijk} be the circumcircle of the triangle Δ_{ijk}. Then

$$P_{12} = L_1 \cap L_2, P_{23} = L_2 \cap L_3, P_{34} = L_3 \cap L_4, P_{45} = L_4 \cap L_5, P_{15} = L_1 \cap L_5$$

are the five vertices of the pentagon.

By applying the theorem 2.1., we get

$$C_{123} \cap C_{234} = \{P_{23}, P_{1234}\},\ \ C_{234} \cap C_{345} = \{P_{34}, P_{2345}\},$$
$$C_{345} \cap C_{145} = \{P_{45}, P_{1345}\},\quad C_{145} \cap C_{125} = \{P_{15}, P_{1245}\},$$
$$C_{125} \cap C_{123} = \{P_{12}, P_{1235}\}.$$

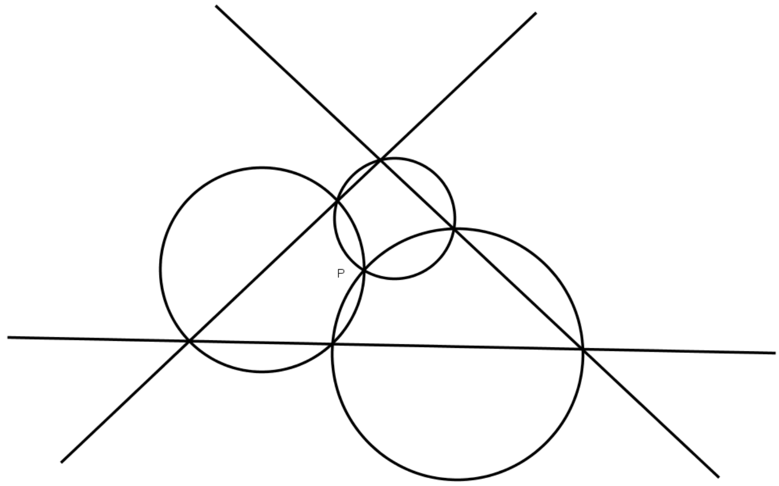
Where the second intersection point of each pair circles is the Clifford point determined by the four related lines, for example, P_{1234} is the Clifford point determined by L_1, L_2, L_3, L_4. Then it is clear that these five points P_{1234}, P_{2345}, P_{1345}, P_{1245}, P_{1235} lie on the circle, C_{12345}, by the theorem 2.2. and the proof is complete.

■

Remark: We are not going to prove Miquel's triangle theorem by applying Clifford Chain theorem. We will show that the chain theorem for four general lines in the plane may apply for one special case of the Miquel's triangle theorem.

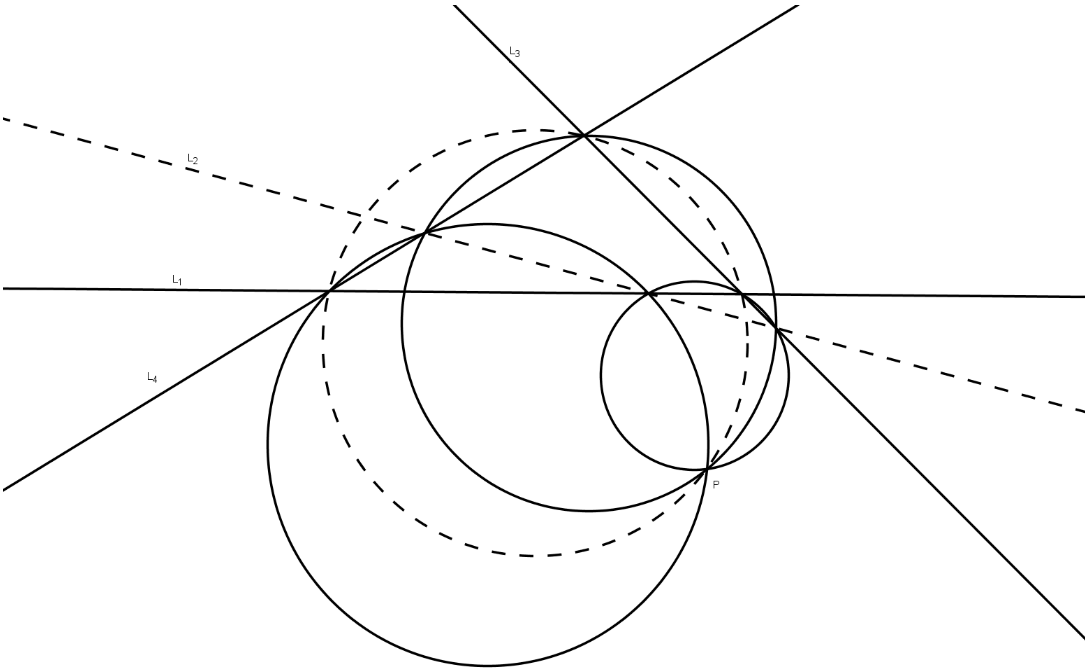
Miquel's triangle theorem: If a point is picked at random on each side of a triangle, then the three circles that are determined by each vertex and the two points on the adjacent sides are concurrent.

The point where the three circles intersect is called the Miquel point of the triangle.



P is the Miquel point of the triangle

When the three points lie on a line, the Miquel point of the original triangle coincides with the Clifford point of the 4 general lines, 3 lines of the triangle and the line the three points lie on.



P is the Clifford point of the four lines and also the Miquel point of the triangle formed by the three black lines

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