

# Properties of range-based volatility estimators

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## Abstract

Volatility is not directly observable and must be estimated. Estimator based on daily close data is imprecise. Range-based volatility estimators provide significantly more precision, but still remain noisy volatility estimates, something that is sometimes forgotten when these estimators are used in further calculations.

First, we analyze properties of these estimators and find that the best estimator is the Garman-Klass (1980) estimator. Second, we correct some mistakes in existing literature. Third, the use of the Garman-Klass estimator allows us to obtain an interesting result: returns normalized by their standard deviations are approximately normally distributed. This result, which is in line with results obtained from high frequency data, but has never previously been recognized in low frequency (daily) data, is important for building simpler and more precise volatility models.

Key words: volatility, high, low, range

JEL Classification: C58, G17, G32

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## 1. Introduction

Asset volatility, a measure of risk, plays a crucial role in many areas of finance and economics. Therefore, volatility modelling and forecasting become one of the most developed parts of financial econometrics. However, since the volatility is not directly observable, the first problem which must be dealt with before modelling or forecasting is always a volatility measurement (or, more precisely, estimation).

Consider stock price over several days. From a statistician's point of view, daily relative changes of stock price (stock returns) are almost random. Moreover, even though daily stock returns are typically of a magnitude of 1% or 2%, they are approximately equally often positive and negative, making average daily return very close to zero. The most natural measure for how much stock price changes is the variance of the stock returns. Variance can be easily calculated and it is a natural measure of the volatility. However, this way we

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<sup>1</sup>I would like to thank to Jonas Andersson, Milan Bašta, Ove Rein Hetland, Lukáš Laffers, Michal Zdeněk and anonymous referees for helpful comments. Norwegian School of Economics, Department of Finance and Management Science, Helleveien 30, 5045 Bergen, Norway and Norwegian University of Science and Technology, Department of Industrial Economics and Technology Management, 7491 Trondheim, Norway, email: Peter.Molnar@iot.ntnu.no

can get only an average volatility over an investigated time period. This might not be sufficient, because volatility changes from one day to another. When we have daily closing prices and we need to estimate volatility on a daily basis, the only estimate we have is squared (demeaned) daily return. This estimate is very noisy, but since it is very often the only one we have, it is commonly used. In fact, we can look at most of the volatility models (e.g. GARCH class of models or stochastic volatility models) in such a way that daily volatility is first estimated as squared returns and consequently processed by applying time series techniques.

When not only daily closing prices, but intraday high frequency data are available too, we can estimate daily volatility more precisely. However, high frequency data are in many cases not available at all or available only over a shorter time horizon and costly to obtain and work with. Moreover, due to market microstructure effects the volatility estimation from high frequency data is rather a complex issue (see Dacorogna et al. 2001).

However, closing prices are not the only easily available daily data. For the most of financial assets, daily open, high and low prices are available too. Range, the difference between high and low prices is a natural candidate for the volatility estimation. The assumption that the stock return follows a Brownian motion with zero drift during the day allows Parkinson (1980) to formalize this intuition and derive a volatility estimator for the diffusion parameter of the Brownian motion. This estimator based on the range (the difference between high and low prices) is much less noisy than squared returns. Garman and Klass (1980) subsequently introduce estimator based on open, high, low and close prices, which is even less noisy. Even though these estimators have existed for more than 30 years, they have been rarely used in the past by both academics and practitioners. However, recently the literature using the range-based volatility estimators started to grow (e.g. Alizadeh, Brandt and Diebold (2002), Brandt and Diebold (2006), Brandt and Jones (2006), Chou (2005), Chou (2006), Chou, Liu (2010)). For an overview see Chou and Liu (2010).

Despite increased interest in the range-based estimators, their properties are sometimes somewhat imprecisely understood. One particular problem is that despite the increased accuracy of these estimators in comparison to squared returns, these estimators still only provide a noisy estimate of volatility. However, in some manipulations (e.g. division) people treat these estimators as if they were exact values of the volatility. This can in turns lead to flawed conclusions, as we show later in the paper. Therefore we study these properties.

Our contributions are the following. First, when the underlying assumptions of the range-based estimators hold, all of them are unbiased. However, taking the square root of these estimators leads to biased estimators of standard deviation. We study this bias. Second, for a given true variance, distribution of the estimated variance depends on the particular estimator. We study these distributions. Third, we show how the range-based volatility estimators should be modified in the presence of opening jumps (stock price at the beginning of the day typically differs from the closing stock price from the previous day).

Fourth, the property we focus on is the distribution of returns standardized

by standard deviations. A question of interest is how this is affected when the standard deviations are estimated from range-based volatility estimators. The question whether the returns divided by their standard deviations are normally distributed has important implications for many fields in finance. Normality of returns standardized by their standard deviations holds promise for simple-to-implement and yet precise models in financial risk management. Using volatility estimated from high frequency data, Andersen, Bollerslev, Diebold and Labys (2000), Andersen, Bollerslev, Diebold, Ebens (2001), Forsberg and Bollerslev (2002) and Thamakos and Wang (2003) show that standardized returns are indeed Gaussian. Contrary, returns scaled by standard deviations estimated from GARCH type of models (which are based on daily returns) are not Gaussian, they have heavy tails. This well-known fact is the reason why heavy-tailed distributions (e.g.  $t$ -distribution) were introduced into the GARCH models. We show that when properly used, range-based volatility estimators are precise enough to replicate basically the same results as those of Andersen et al. (2001) obtained from high frequency data. To our best knowledge, this has not been previously recognized in the daily data. Therefore volatility models built upon high and low data might provide accuracy similar to models based upon high frequency data and still keep the benefits of the models based on low frequency data (much smaller data requirements and simplicity).

The rest of the paper is organized in the following way. In Section 2, we describe existing range-based volatility estimators. In Section 3, we analyze properties of range-based volatility estimators, mention some caveats related to them and correct some mistakes in the existing literature. In Section 4 we empirically study the distribution of returns normalized by their standard deviations (estimated from range-based volatility estimators) on 30 stock, the components of the Dow Jones Industrial Average. Section 5 concludes.

## 2. Overview

Assume that price  $P$  follows a geometric Brownian motion such that log-price  $p = \ln(P)$  follows a Brownian motion with zero drift and diffusion  $\sigma$ .

$$dp_t = \sigma dB_t \tag{1}$$

Diffusion parameter  $\sigma$  is assumed to be constant during one particular day, but can change from one day to another. We use one day as a unit of time. This normalization means that the diffusion parameter in (1) coincides with the daily standard deviation of returns and we do not need to distinguish between these two quantities. Denote the price at the beginning of the day (i.e. at the time  $t = 0$ )  $O$  (open), the price in the end of the day (i.e. at the time  $t = 1$ )  $C$  (close), the highest price of the day  $H$ , and the lowest price of the day  $L$ . Then we can calculate open-to-close, open-to-high and open-to-low returns as

$$c = \ln(C) - \ln(O) \tag{2}$$

$$h = \ln(H) - \ln(O) \tag{3}$$

$$l = \ln(L) - \ln(O) \quad (4)$$

Daily return  $c$  is obviously a random variable drawn from a normal distribution with zero mean and variance (volatility)  $\sigma^2$

$$c \sim N(0, \sigma^2) \quad (5)$$

Our goal is to estimate (unobservable) volatility  $\sigma^2$  from observed variables  $c$ ,  $h$  and  $l$ . Since we know that  $c^2$  is an unbiased estimator of  $\sigma^2$ ,

$$E(c^2) = \sigma^2 \quad (6)$$

we have the first volatility estimator (subscript  $s$  stands for "simple")

$$\widehat{\sigma}_s^2 = c^2 \quad (7)$$

Since this simple estimator is very noisy, it is desirable to have a better one. It is intuitively clear that the difference between high and low prices tells us much more about volatility than close price. High and low prices provide additional information about volatility. The distribution of the range  $d \equiv h - l$  (the difference between the highest and the lowest value) of Brownian motion is known (Feller (1951)). Define  $P(x)$  to be the probability that  $d \leq x$  during the day. Then

$$P(x) = \sum_{n=1}^{\infty} (-1)^{n+1} n \left\{ \operatorname{Erfc} \left( \frac{(n+1)x}{\sqrt{2}\sigma} \right) - 2\operatorname{Erfc} \left( \frac{nx}{\sqrt{2}\sigma} \right) + \operatorname{Erfc} \left( \frac{(n-1)x}{\sqrt{2}\sigma} \right) \right\} \quad (8)$$

where

$$\operatorname{Erfc}(x) = 1 - \operatorname{Erf}(x) \quad (9)$$

and  $\operatorname{Erf}(x)$  is the error function. Using this distribution Parkinson (1980) calculates (for  $p \geq 1$ )

$$E(d^p) = \frac{4}{\sqrt{\pi}} \Gamma \left( \frac{p+1}{2} \right) \left( 1 - \frac{4}{2^p} \right) \zeta(p-1) (2\sigma^2) \quad (10)$$

where  $\Gamma(x)$  is the gamma function and  $\zeta(x)$  is the Riemann zeta function. Particularly for  $p = 1$

$$E(d) = \sqrt{8\pi}\sigma \quad (11)$$

and for  $p = 2$

$$E(d^2) = 4 \ln(2) \sigma^2 \quad (12)$$

Based on formula (12), he proposes a new volatility estimator:

$$\widehat{\sigma}_P^2 = \frac{(h-l)^2}{4 \ln 2} \quad (13)$$

Garman and Klass (1980) realize that this estimator is based solely on quantity  $h-l$  and therefore an estimator which utilizes all the available information  $c$ ,

$h$  and  $l$  will be necessarily more precise. Since search for the minimum variance estimator based on  $c$ ,  $h$  and  $l$  is an infinite dimensional problem, they restrict this problem to analytical estimators, i.e. estimators which can be expressed as an analytical function of  $c$ ,  $h$  and  $l$ . They find that the minimum variance analytical estimator is given by the formula

$$\widehat{\sigma_{GKprecise}^2} = 0.511 (h - l)^2 - 0.019 (c(h + l) - 2hl) - 0.383c^2 \quad (14)$$

The second term (cross-products) is very small and therefore they recommend neglecting it and using more practical estimator:

$$\widehat{\sigma_{GK}^2} = 0.5 (h - l)^2 - (2 \ln 2 - 1) c^2 \quad (15)$$

We follow their advice and further on when we talk about Garman-Klass volatility estimator (GK), we refer to (15). This estimator has additional advantage over (14) - it can be simply explained as an optimal (smallest variance) combination of simple and Parkinson volatility estimator.

Meilijson (2009) derives another estimator, outside the class of analytical estimators, which has even smaller variance than GK. This estimator is constructed as follows.

$$\widehat{\sigma_M^2} = 0.274\sigma_1^2 + 0.16\sigma_s^2 + 0.365\sigma_3^2 + 0.2\sigma_4^2 \quad (16)$$

where

$$\sigma_1^2 = 2 [(h' - c')^2 + l'] \quad (17)$$

$$\sigma_3^2 = 2 (h' - c' - l') c' \quad (18)$$

$$\sigma_4^2 = -\frac{(h' - c') l'}{2 \ln 2 - 5/4} \quad (19)$$

where  $c' = c, h' = h, l' = l$  if  $c > 0$  and  $c' = -c, h' = -l, l' = -h$  if  $c < 0$ .<sup>2</sup>

Rogers and Satchell (1991) derive an estimator which allows for arbitrary drift.

$$\widehat{\sigma_{RS}^2} = h(h - c) + l(l - c) \quad (20)$$

There are two other estimators which we should mention. Kunitomo (1992) derives a drift-independent estimator, which is more precise than all the previously mentioned estimators. However "high" and "low" prices used in his estimator are not the highest and lowest price of the day. The "high" and "low" used in this estimator are the highest and the lowest price relative to the trend line given by open and high prices. These "high" and "low" prices are unknown unless we have tick-by-tick data and therefore the use of this estimator is very limited.

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<sup>2</sup>This estimator is not analytical, because it uses different formula for days when  $c > 0$  than for days when  $c < 0$ .

Yang and Zhang (2000) derive another drift-independent estimator. However, their estimator can be used only for estimation of average volatility over multiple days and therefore we do not study it in our paper.

Efficiency of a volatility estimator  $\hat{\sigma}^2$  is defined as

$$Eff(\hat{\sigma}^2) \equiv \frac{var(\sigma_s^2)}{var(\hat{\sigma}^2)} \quad (21)$$

Simple volatility estimator has by definition efficiency 1, Parkinson volatility estimator has efficiency 4.9, Garman-Klass 7.4 and Meilijson 7.7. Rogers, Satchell has efficiency 6.0 for the zero drift and larger than 2 for any drift.

Remember that all of the studied estimators except for Rogers, Satchell are derived under the assumption of zero drift. However, for most of the financial assets, mean daily return is much smaller than its standard deviation and can therefore be neglected. Obviously, this is not true for longer time horizons (e.g. when we use yearly data), but this is a very good approximation for daily data in basically any practical application.

Further assumptions behind these estimators are continuous sampling, no bid-ask spread and constant volatility. If prices are observed only infrequently, then the observed high will be below the true high and observed low will be above the true low, as was recognized already by Garman and Klass (1980). Bid-ask spread has the opposite effect: observed high price is likely to happen at ask, observed low price is likely to happen at the low price and therefore the difference between high and low contains in addition bid-ask spread. These effects work in the opposite direction and therefore they will at least partially cancel out. More importantly, for liquid stocks both these effects are very small. In this paper we maintain the assumption of constant volatility within the day. This approach is common even in stochastic volatility literature (e.g. Alizadeh, Brandt and Diebold 2002) and assessing the effect of departing from this assumption is beyond the scope of this paper. However, this is an interesting avenue for further research.

### 3. Properties of range-based volatility estimators

The previous section provided an overview of range-based volatility estimators including their efficiency. Here we study their other properties. Our main focus is not their empirical performance, as this question has been studied before (e.g. Bali and Weinbaum (2005)). We study the performance of these estimators when all the assumptions of these estimators hold perfectly. This is more important than it seems to be, because this allows us to distinguish between the case when these estimators do not work (assumptions behind them do not hold) and the case when these estimators work, but we are misinterpreting the results. This point can be illustrated in the following example. Imagine that we want to study the distribution of returns standardized by their standard deviations. We estimate these standard deviations as a square root of the Parkinson volatility

estimator (13) and find that standardized returns are not normally distributed. Should we conclude that true standardized returns are not normally distributed or should we conclude that the Parkinson volatility estimator is not appropriate for this purpose? We answer this and other related questions.

To do so, we ran 500000 simulations, one simulation representing one trading day. During every trading day log-price  $p$  follows a Brownian motion with zero drift and daily diffusion  $\sigma = 1$ . We approximate continuous Brownian motion by  $n = 100000$  discrete intraday returns, each drawn from  $N(0, 1/\sqrt{n})$ .<sup>3</sup> We save high, low and close log-prices  $h, l, c$  for every trading day<sup>4</sup>.

### 3.1. Bias in $\sigma$

All the previously mentioned estimators are unbiased estimators of  $\sigma^2$ . Therefore, square root of any of these estimators will be a biased estimator of  $\sigma$ . This is direct consequence of well known fact that for a random variable  $x$  the quantities  $E(x^2)$  and  $E(x)^2$  are generally different. However, as I document later, using  $\sqrt{\widehat{\sigma^2}}$  as  $\widehat{\sigma}$ , as an estimator of  $\sigma$ , is not uncommon. Moreover, in many cases the objects of our interests are standard deviations, not variances. Therefore, it is important to understand the size of the error introduced by using  $\sqrt{\widehat{\sigma^2}}$  instead of  $\widehat{\sigma}$  and potentially correct for this bias. Size of this bias depends on the particular estimator.

As can be easily proved, an unbiased estimator  $\widehat{\sigma}_s$  of the standard deviation  $\sigma$  based on  $\sqrt{\widehat{\sigma_s^2}}$  is

$$\widehat{\sigma}_s = \sqrt{\widehat{\sigma_s^2}} \times \sqrt{\frac{\pi}{2}} = |c| \times \sqrt{\pi/2} \quad (22)$$

Using the results (11) and (13) we can easily find that an estimator of standard deviation based on range is

$$\widehat{\sigma}_P = \frac{h-l}{2} \times \sqrt{\frac{\pi}{2}} = \sqrt{\widehat{\sigma_P^2}} \times \sqrt{\frac{\pi \ln 2}{2}} \quad (23)$$

Similarly, when we want to evaluate the bias introduced by using  $\sqrt{\widehat{\sigma^2}}$  instead of  $\widehat{\sigma}$  for the rest of volatility estimators, we want to find constants  $c_{GK}$ ,  $c_M$  and  $c_{RS}$  such that

$$\widehat{\sigma}_{GK} = \sqrt{\widehat{\sigma_{GK}^2}} \times c_{GK} \quad (24)$$

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<sup>3</sup>Such a high  $n$  allows us to have almost perfectly continuous Brownian motion and having so many trading days allow us to know the distributions of range based volatility estimators with very high precision. Simulating these data took one months on an ordinary computer (Intel Core 2 Duo P8600 2.4 GHz, 2 GB RAM).

Note that we do not derive analytical formulas for the distributions of range-based volatility estimators. Since these formulas would not bring additional insights into the questions we study, their derivation is behind the scope of this paper.

<sup>4</sup>Open log-price is normalized to zero.

$$\widehat{\sigma}_M = \sqrt{\widehat{\sigma}_M^2} \times c_M \quad (25)$$

$$\widehat{\sigma}_{RS} = \sqrt{\widehat{\sigma}_{RS}^2} \times c_{RS} \quad (26)$$

From simulated high, low and close log-prices  $h$ ,  $l$ ,  $c$  we estimate volatility according to (7), (13), (15), (16), (20) and calculate mean of the square root of these volatility estimates. We find that  $c_s = 1.253$ ,  $c_P = 1.043$  (what is in accordance with theoretical values  $\sqrt{\pi/2} = 1.253$  and  $\sqrt{\pi \ln 2/2} = 1.043$ ) and  $c_{GK} = 1.034$ ,  $c_M = 1.033$  and  $c_{RS} = 1.043$ . We see that the square root of the simple volatility estimator is a severely biased estimator of standard deviation (bias is 25%), whereas bias in the square root of range-based volatility estimators is rather small (3% - 4%).

Even though it seems obvious that  $\sqrt{\widehat{\sigma}^2}$  is not an unbiased estimator of  $\sigma$ , it is quite common even among researchers to use  $\sqrt{\widehat{\sigma}^2}$  as an estimator of  $\sigma$ . I document this in two examples.

Bali and Weinbaum (2005) empirically compare range-based volatility estimators. The criteria they use are: mean squared error

$$MSE(\sigma_{estimated}) = E[(\sigma_{estimated} - \sigma_{true})^2] \quad (27)$$

mean absolute deviation

$$MAD(\sigma_{estimated}) = E[|\sigma_{estimated} - \sigma_{true}|] \quad (28)$$

and proportional bias

$$Prop.Bias(\sigma_{estimated}) = E[(\sigma_{estimated} - \sigma_{true})/\sigma_{true}] \quad (29)$$

For daily returns they find:

”The traditional estimator [(7) in our paper] is significantly biased in all four data sets. [...] it was found that squared returns do not provide unbiased estimates of the ex post realized volatility. Of particular interest, across the four data sets, extreme-value volatility estimators are almost always significantly less biased than the traditional estimator.”

This conclusion sounds surprising only until we realize that in their calculations  $\sigma_{estimated} \equiv \sqrt{\widehat{\sigma}^2}$ , which, as just shown, is not an unbiased estimator of  $\sigma$ . Actually, it is severely biased for a simple volatility estimator. Generally, if our interest is unbiased estimate of the standard deviation, we should use formulas (22)-(26).

A similar problem is in Bollen, Inder (2002). In testing for the bias in the estimators of  $\sigma$ , they correctly adjust  $\sqrt{\widehat{\sigma}_s^2}$  using formula (22), but they do not adjust  $\sqrt{\widehat{\sigma}_P^2}$  and  $\sqrt{\widehat{\sigma}_{GK}^2}$  by constants  $c_P$  and  $c_{GK}$ .



### 3.2. Distributional properties of range-based estimators

Daily volatility estimates are typically further used in volatility models. Ease of the estimation of these models depends not only on the efficiency of the used volatility estimator, but on its distributional properties too (Broto, Ruiz (2004)). When the estimates of relevant volatility measure (whether it is  $\sigma^2$ ,  $\sigma$  or  $\ln \sigma^2$ ) have approximately normal distribution, the volatility models can be estimated more easily.<sup>5</sup> We study the distributions of  $\widehat{\sigma^2}$ ,  $\sqrt{\widehat{\sigma^2}}$  and  $\ln \widehat{\sigma^2}$ , because these are the quantities modelled by volatility models. Most of the GARCH models try to capture time evolution of  $\sigma^2$ , EGARCH and stochastic volatility models are based on time evolution of  $\ln \sigma^2$  and some GARCH models model time evolution of  $\sigma$ .

Under the assumption of Brownian motion, the distribution of absolute value of return and the distribution of range are known (Karatzas and Shreve (1991), Feller (1951)). Using their result, Alizadeh, Brandt, Diebold (2002) derive the distribution of log absolute return and log range. Distribution of  $\widehat{\sigma^2}$ ,  $\sqrt{\widehat{\sigma^2}}$  and  $\ln \widehat{\sigma^2}$  is unknown for the rest of the range-based volatility estimators. Therefore we study these distributions. To do this, we use numerical evaluation of  $h$ ,  $l$  and  $c$  data, which are simulated according to the process (1) (<sup>6</sup>).

First we study the distribution of  $\widehat{\sigma^2}$  for different estimators. These distributions are plotted in Figure 1. Since all these estimators are unbiased estimators of  $\sigma^2$ , all have the same mean (in our case one). Variance of these estimators is given by their efficiency. From the inspection of Figure 1, we can observe that the density function of  $\widehat{\sigma^2}$  is approximately lognormal for range-based estimators. On the other hand, distribution of squared returns, which is  $\chi^2$  distribution with one degree of freedom, is very dispersed and reaches maximum at zero. Therefore, for most of the purposes, distributional properties of range-based estimators are more appropriate for further use than the squared returns. For the range, this was already noted by Alizadeh, Brandt, Diebold (2002). However, this is true for all the range-based volatility estimators. The differences in distributions among different range-based estimators are actually rather small.

The distributions of  $\sqrt{\widehat{\sigma^2}}$  are plotted in Figure 2. These distributions have less weight on the tails than the distributions of  $\widehat{\sigma^2}$ . This is not surprising, since the square root function transforms small values (values smaller than one) into larger values (values closer towards one) and it transforms large values (values larger than one) into smaller values (values closer to one). Again, the distributions of  $\sqrt{\widehat{\sigma^2}}$  for range range-based estimators have better properties

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<sup>5</sup>E.g. Gaussian quasi-maximum likelihood estimation, which plays an important role in estimation of stochastic volatility models, depends crucially on the near-normality of log-volatility.

<sup>6</sup>The fact that we do not search for analytical formula is not limiting at all. The analytical form of density function for the simplest range-based volatility estimator, range itself, is so complicated (it is an infinite series) that in the end even skewness and kurtosis must be calculated numerically.

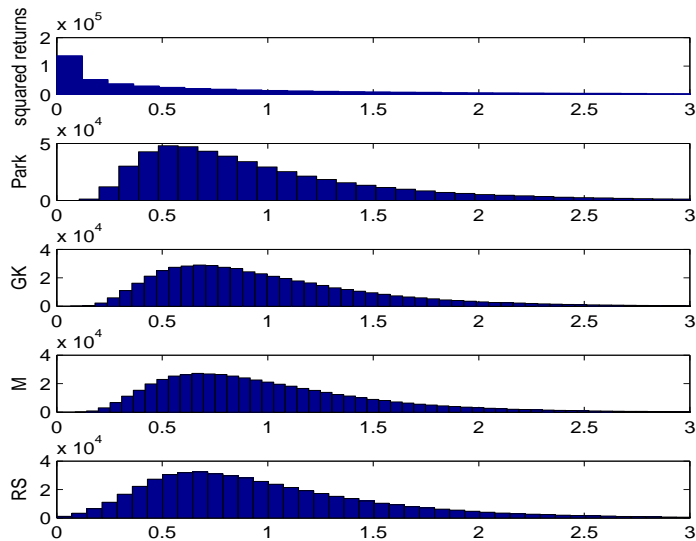


Figure 1: Distribution of variances estimated as squared returns and from Parkinson, Garman-Klass, Meilijson and Rogers-Satchell formulas.

than the distribution of the absolute returns. To distinguish the difference between different range-based volatility estimators, we calculate the summary statistics and present them in Table 1.

Table 1: The summary statistics for the square root of the volatility estimated as absolute returns and as a square root of the Parkinson, Garman-Klass, Meilijson and Rogers-Satchell formulas.

	mean	std	skewness	kurtosis
$ r $	0.80	0.60	1.00	3.87
$\sqrt{\widehat{\sigma}_P^2}$	0.96	0.29	0.97	4.24
$\sqrt{\widehat{\sigma}_{GK}^2}$	0.97	0.24	0.60	3.40
$\sqrt{\widehat{\sigma}_M^2}$	0.97	0.24	0.54	3.28
$\sqrt{\widehat{\sigma}_{RS}^2}$	0.96	0.28	0.46	3.44

No matter whether we rank these distributions according to their mean (which should be preferably close to 1) or according to their standard deviations (which should be the smallest possible), ranking is the same as in the previous case: the best is Meilijson volatility estimator, then Garman-Klass, next Roger-Satchell, next Parkinson and the last is the absolute returns.

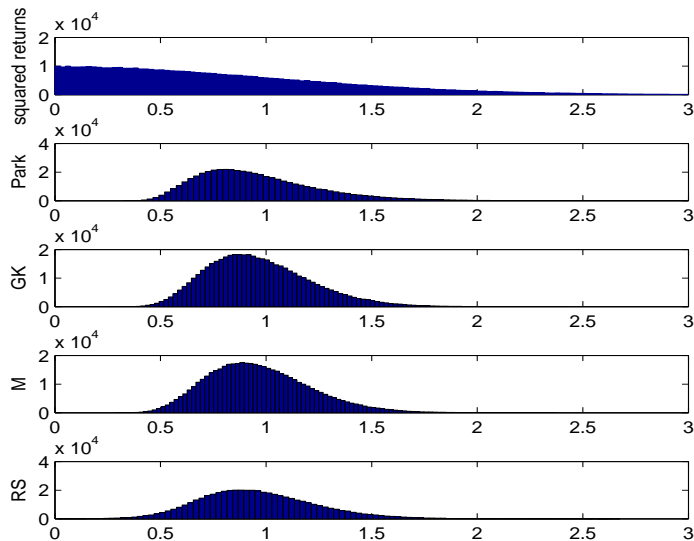


Figure 2: Distribution of square root of volatility estimated as squared returns and from Parkinson, Garman-Klass, Meilijson and Rogers-Satchell formulas.

In many practical applications, the mean squared error (MSE) of an estimator  $\hat{\theta}$

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] \quad (30)$$

is the most important criterion for the evaluation of the estimators, since MSE quantifies the difference between values implied by an estimator and the true values of the quantity being estimated. The MSE is equal to the sum of the variance and the squared bias of the estimator

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + (Bias(\hat{\theta}, \theta))^2 \quad (31)$$

and therefore in our case (when estimator with smallest variance has smallest bias) is the ranking according to MSE identical with the ranking according to bias or variance.

In the end, we investigate the distribution of  $\ln \hat{\sigma}^2$  (see Figure 3). As we can see, the logarithm of the squared returns is highly nonnormally distributed, but the logarithms of the range-based volatility estimators have distributions similar to the normal distribution. To see the difference among various range-based estimators, we again calculate their summary statistics (see Table 2).

Note that the true volatility is normalized to one. Normality of the estimator is desirable for practical reasons and therefore the ideal estimator should have

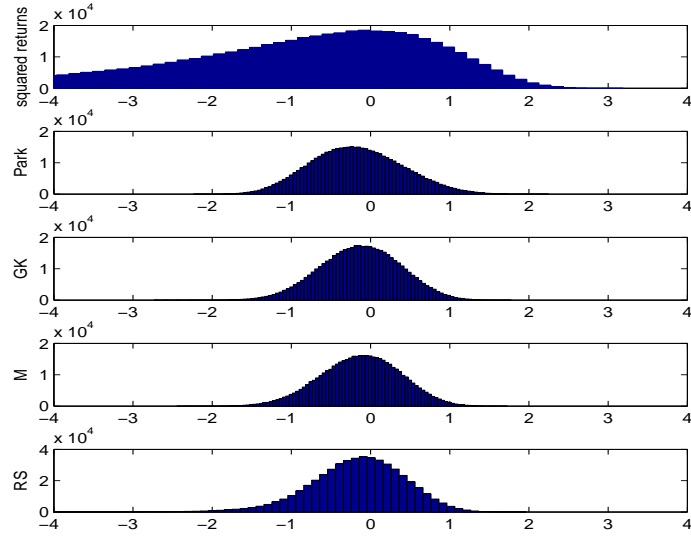


Figure 3: Distribution of the logarithm of volatility estimated as squared returns and from the Parkinson, Garman-Klass, Meilijson and Rogers-Satchell formulas.

Table 2: Summary statistics for logarithm of volatility estimated as a logarithm of squared returns and as a logarithm of Parkinson, Garman-Klass, Meilijson and Rogers-Satchell volatility estimators.

	mean	std	skewness	kurtosis
$\ln(r^2)$	-1.27	2.22	-1.53	6.98
$\ln(\widehat{\sigma}_P^2)$	-0.17	0.57	0.17	2.77
$\ln(\widehat{\sigma}_{GK}^2)$	-0.13	0.51	-0.09	2.86
$\ln(\widehat{\sigma}_M^2)$	-0.13	0.50	-0.14	2.86
$\ln(\widehat{\sigma}_{RS}^2)$	-0.17	0.61	-0.71	5.41

mean and skewness equal to zero, kurtosis close to three and standard deviation as small as possible. We see that from the five studied estimators the Garman-Klass and Meilijson volatility estimators, in addition to being most efficient, have best distributional properties.

### 3.3. Normality of normalized returns

As was empirically shown by Andersen, Bollerslev, Diebold, Labys (2000), Andersen, Bollerslev, Diebold, Ebens (2001), Forsberg and Bollerslev (2002) and Thamakos and Wang (2003) on different data sets, standardized returns (returns divided by their standard deviations) are approximately normally distributed. In other words, daily returns can be written as

$$r_i = \sigma_i z_i \tag{32}$$

where  $z_i \sim N(0,1)$ . This finding has important practical implications too. If returns (conditional on the true volatility) are indeed Gaussian and heavy tails in their distributions are caused simply by changing volatility, then what we need the most is a thorough understanding of the time evolution of volatility, possibly including the factors which influence it. Even though the volatility models are used primarily to capture time evolution of volatility, we can expect that the better our volatility models, the less heavy-tailed distribution will be needed for modelling of the stock returns. This insight can contribute to improved understanding of volatility models, which is in turn crucial for risk management, derivative pricing, portfolio management etc.

Intuitively, normality of the standardized returns follows from the Central Limit Theorem: since daily returns are just a sum of high-frequency returns, daily returns will be drawn from normal distribution.<sup>7</sup>

Since both this intuition and the empirical evidence of the normality of returns standardized by their standard deviations is convincing, it is appealing to require that one of the properties of a "good" volatility estimator should be that returns standardized by standard deviations obtained from this estimator will be normally distributed (see e.g. Bollen and Inder (2002)). However, this intuition is not correct. As I now show, returns standardized by some estimate of the true volatility do not need to, and generally will not, have the same properties as returns standardized by the true volatility. Therefore we need to understand whether the range-based volatility estimators are suitable for standardization of the returns. There are two problems associated with these volatility estimators: they are noisy and their estimates might be (and typically are) correlated with returns. These two problems might cause returns standardized by the estimated standard deviations not to be normal, even when the returns standardized by their true standard deviations are normally distributed.

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<sup>7</sup>given the limited time-dependence and some conditions on existence of moments.

### 3.3.1. Noise in volatility estimators

We want to know the effect of noise in volatility estimates  $\widehat{\sigma}_i$  on the distribution of returns normalized by these estimates ( $\widehat{z}_i = r_i/\widehat{\sigma}_i$ ) when true normalized returns  $z_i = r_i/\sigma_i$  are normally distributed. Without loss of generality, we set  $\sigma_i = 1$  and generate one million observations of  $r_i$ ,  $i \in \{1, \dots, 1000000\}$ , all of them are iid  $N(0,1)$ . Next we generate  $\widehat{\sigma}_{i,n}$  in such a way that  $\widehat{\sigma}$  is unbiased estimator of  $\sigma$ , i.e.  $E(\widehat{\sigma}_{i,n}) = 1$  and  $n$  represents the level of noise in  $\widehat{\sigma}_{i,n}$ . There is no noise for  $n = 0$  and therefore  $\widehat{\sigma}_{i,0} = \sigma_i = 1$ . To generate  $\widehat{\sigma}_{i,n}$  for  $i > 0$  we must decide upon distribution of  $\widehat{\sigma}_{i,n}$ . Since we know from the previous section that range-based volatility estimates are approximately lognormally distributed, we draw estimates of the standard deviations from lognormal distributions. We set the parameters  $\mu$  and  $s^2$  of lognormal distribution in such a way that  $E(\widehat{\sigma}_{i,n}) = 1$  and  $\text{Var}(\widehat{\sigma}_{i,n}) = n$ , particularly  $\mu = -\frac{1}{2} \ln(1+n)$ ,  $s^2 = \ln(1+n)$ . For every  $n$ , we generate one million observations of  $\widehat{\sigma}_{i,n}$ . Next we calculate normalized returns  $\widehat{z}_{i,n} = r_i/\widehat{\sigma}_{i,n}$ . Their summary statistics is in the Table 3.

Table 3: Summary statistics for a random variable obtained as ratio of normal random variable with zero mean and variance one and lognormal random variable with constant mean equal to one and variance increasing from 0 to 0.8.

$n = \text{Var}(\widehat{\sigma}_i)$	$\text{mean}(\widehat{z}_{i,n})$	$\text{std}(\widehat{z}_{i,n})$	$\text{skewness}(\widehat{z}_{i,n})$	$\text{kurtosis}(\widehat{z}_{i,n})$
0.0	0.0001	1.00	0.00	3.00
0.2	0.0003	1.32	0.02	6.22
0.4	0.0013	1.66	-0.01	11.80
0.6	-0.0007	2.03	0.03	19.76
0.8	0.0025	2.43	0.01	34.60

Obviously,  $\widehat{z}_{i,0}$ , which is by definition equal to  $r_i$ , has zero mean, standard deviation equal to 1, skewness equal to 0 and kurtosis equal to 3. We see that normalization by  $\widehat{\sigma}$ , a noisy estimate of  $\sigma$ , does not change  $E(\widehat{z})$  and skewness of  $\widehat{z}$ . This is natural, because  $r_i$  is distributed symmetrically around zero. On the other hand, adding noise increases standard deviation and kurtosis of  $\widehat{z}$ . When we divide normally distributed random variable  $r_i$  by random variable  $\widehat{\sigma}_i$ , we are effectively adding noise to  $r_i$ , making its distribution flatter and more dispersed with more extreme observations. Therefore, standard deviation increases. Since kurtosis is influenced mostly by extreme observations, it increases too.

### 3.3.2. Bias introduced by normalization of range-based volatility estimators

Previous analysis suggests that the more noisy volatility estimator we use for the normalization of the returns, the higher the kurtosis of the normalized returns will be. Therefore we could expect to find the highest kurtosis when using the Parkinson volatility estimator (13). As we will see later, this is not the case. Returns and estimated standard deviations were independent in the previous section, but this is not the case when we use range-based estimators.

Let us denote  $\sigma_{\text{PARK}} \equiv \sqrt{\widehat{\sigma_{\text{PARK}}^2}}$ ,  $\sigma_{\text{GK}} \equiv \sqrt{\widehat{\sigma_{\text{RS}}^2}}$ ,  $\sigma_M \equiv \sqrt{\widehat{\sigma_M^2}}$  and  $\sigma_{\text{RS},t} \equiv$

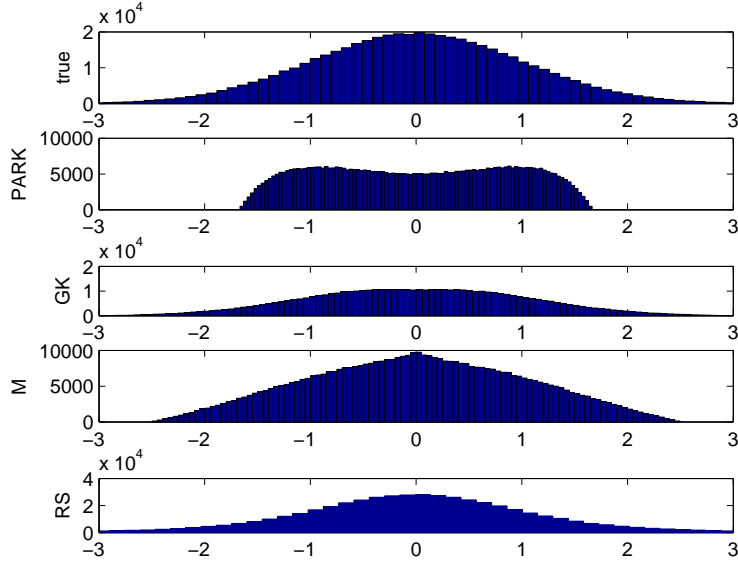


Figure 4: Distribution of normalized returns. "true" is the distribution of the stock returns normalized by the true standard deviations. This distribution is by assumption  $N(0,1)$ . PARK, GK, M and RS refer to distributions of the same returns after normalization by volatility estimated using the Parkinson, Garman-Klass, Meilijson and Rogers-Sanchell volatility estimators.

$\sqrt{\widehat{\sigma_{RS}^2}}$ . We study the distributions of  $\widehat{z}_{PARK,i} \equiv r_i/\sigma_{PARK,i}$ ,  $\widehat{z}_{GK,i} \equiv r_i/\sigma_{GK,i}$ ,  $\widehat{z}_{M,i} \equiv r_i/\sigma_{M,i}$ ,  $\widehat{z}_{RS,t} \equiv r_i/\sigma_{RS,i}$ . Histograms for these distributions are shown in Figure 4 and corresponding summary statistics are in Table 4.

Table 4: Summary statistics for returns normalized by different volatility estimates:  $\widehat{z}_{PARK,i} \equiv r_i/\sigma_{PARK,i}$ ,  $\widehat{z}_{GK,i} \equiv r_i/\sigma_{GK,i}$ ,  $\widehat{z}_{M,i} \equiv r_i/\sigma_{M,i}$ ,  $\widehat{z}_{RS,t} \equiv r_i/\sigma_{RS,i}$ .

	mean	std	skewness	kurtosis
$z_{true,i}$	0.00	1.00	0.00	3.00
$\widehat{z}_{P,i}$	0.00	0.88	-0.00	1.79
$\widehat{z}_{GK,i}$	0.00	1.01	0.00	2.61
$\widehat{z}_{M,i}$	0.00	1.02	0.00	2.36
$\widehat{z}_{RS,i}$	0.01	1.35	1.62	123.96

The true mean and skewness of these distributions are zero, because returns are symmetrically distributed around zero, triplets  $(h, l, c)$  and  $(-l, -h, -c)$  are equally likely and all the studied estimators are symmetric in the sense that they produce the same estimates for the log price following the Brownian motion  $B(t)$  and for the log price following Brownian motion  $-B(t)$ , par-

ticularly  $\widehat{\sigma}_{PARK}(h, l, c) = \widehat{\sigma}_{PARK}(-l, -h, c)$ ,  $\widehat{\sigma}_{GK}(h, l, c) = \widehat{\sigma}_{GK}(-l, -h, c)$ ,  $\widehat{\sigma}_M(h, l, c) = \widehat{\sigma}_M(-l, -h, c)$  and  $\widehat{\sigma}_{RD}(h, l, c) = \widehat{\sigma}_{RS}(-l, -h, c)$ .

However, it seems from Table 4 that distribution of  $\widehat{z}_{RS,i}$  is skewed. There is another surprising fact about  $\widehat{z}_{RS,i}$ . It has very heavy tails. The reason for this is that the formula (20) is derived without the assumption of zero drift. Therefore, when stock price performs one-way movement, this is attributed to the drift term and volatility is estimated to be zero. (If movement is mostly in one direction, estimated volatility will be nonzero, but very small). Moreover, this is exactly the situation when stock returns are unusually high. Dividing the largest returns by the smallest estimated standard deviations causes a lot of extreme observations and therefore very heavy tails. Due to these extreme observations the skewness of the simulated sample is different from the skewness of the population, which is zero. This illustrates that the generality (drift independence) of the Rogers and Satchell (1991) volatility estimator actually works against this estimator in cases when the drift is zero.

When we use the Parkinson volatility estimator for the standardization of the stock returns, we get exactly the opposite result. Kurtosis is now much smaller than for the normal distribution. This is in line with empirical finding of Bollen and Inder (2002). However, this result should not be interpreted that this estimator is not working properly. Remember that we got the result of the kurtosis being significantly smaller than 3 under ideal conditions, when the Parkinson estimator works perfectly (in the sense that it works exactly as it is supposed to work). Remember that this estimator is based on the range. Even though the range, which is based on high and low prices, seems to be independent of return, which is based on the open and close prices, the opposite is the case. Always when return is high, range will be relatively high too, because range is always at least as large as absolute value of the return.  $|r|/\sigma_{PARK}$  will never be larger than  $\sqrt{4 \ln 2}$ , because

$$\frac{|r|}{\sigma_{PARK}} = \frac{|r|}{\frac{h-l}{\sqrt{4 \ln 2}}} = \sqrt{4 \ln 2} \frac{|r|}{h-l} \leq \sqrt{4 \ln 2} \quad (33)$$

The correlation between  $|r|$  and  $\sigma_{PARK}$  is 0.79, what supports our argument. Another problem is that the distribution of  $\widehat{z}_{P,i}$  is bimodal.

As we can see from the histogram, distribution of  $\widehat{z}_{M,i}$  does not have any tails either. This is because the Meilijson volatility estimator suffers from the same type of problem as the Parkinson volatility estimator, just to a much smaller extent.

The Garman-Klass volatility estimator combines the Parkinson volatility estimator with simple squared return. Even though both, the Parkinson estimator and squared return are highly correlated with size of the return, the overall effect partially cancels out, because these two quantities are subtracted. Correlation between  $|r|$  and  $\sigma_{GK}$  is indeed only 0.36.  $\widehat{z}_{GK,i}$  has approximately normal distribution, as the effect of noise and the effect of correlation with returns to large extent cancels out.

We conclude this subsection with the appeal that we should be aware of the



assumptions behind the formulas we use. As range-based volatility estimators were derived to be as precise volatility estimators as possible, they work well for this purpose. However, there is no reason why all of these estimators should work properly when used for the standardization of the returns. We conclude that from the studied estimators the only estimator appropriate for standardization of returns is the Garman-Klass volatility estimator. We use this estimator later in the empirical part.

### 3.4. Jump component

So far in this paper, returns and volatilities were related to the trading day, i.e. the period from the open to the close of the market. However, most of the assets are not traded continuously for 24 hours a day. Therefore, opening price is not necessarily equal to the closing price from the previous day. We are interested in daily returns

$$r_i = \ln(C_i) - \ln(C_{i-1}) \quad (34)$$

simply because for the purposes of risk management we need to know the total risk over the whole day, not just the risk of the trading part of the day. If we do not adjust range-based estimators for the presence of opening jumps, they will of course underestimate the true volatility. The Parkinson volatility estimator adjusted for the presence of opening jumps is

$$\widehat{\sigma}_P^2 = \frac{(h-l)^2}{4 \ln 2} + j^2 \quad (35)$$

where  $j_i = \ln(O_i) - \ln(C_{i-1})$  is the opening jump. The jump-adjusted Garman-Klass volatility estimator is:

$$\widehat{\sigma}_{GK}^2 = 0.5 (h-l)^2 - (2 \ln 2 - 1) c^2 + j^2 \quad (36)$$

Other estimators should be adjusted in the same way. Unfortunately, including opening jump will increase variance of the estimator when opening jumps are significant part of daily returns.<sup>8</sup> However, this is the only way how to get unbiased estimator without imposing some additional assumptions. If we knew what part of the overall daily volatility opening jumps account for, we could find optimal weights for the jump volatility component and for the volatility within the trading day to minimize the overall variance of the composite estimator. This is done in Hansen and Lunde (2005), who study how to combine opening jump and realized volatility estimated from high frequency data into the most efficient estimator of the whole day volatility. However, the relation of opening jump and the trading day volatility can be obtained only from data. Moreover, there is no obvious reason why the relationship from the past should hold in the

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<sup>8</sup>Jump volatility is estimated with smaller precision than volatility within trading day.

future. Simply adding jump component makes range-based estimators unbiased without imposing any additional assumption.<sup>9</sup>

Adjustment for an opening jump is not as obvious as it seems to be and even researchers quite often make mistakes when dealing with this issue. The most common mistake is that the range-based volatility estimators are not adjusted for the presence of opening jumps at all (see e.g. Parkinson volatility estimator in Bollen, Inder (2002)). A less common mistake, but with worse consequences is an incorrect adjustment for the opening jumps. E.g. Bollen and Inder (2002) and Fiess and MacDonald (2002) refer to the following formula

$$\sigma_{GKwrong,i}^2 = 0.5 (\ln H_i - \ln L_i)^2 - (2 \ln 2 - 1) (\ln C_i - \ln C_{i-1})^2 \quad (37)$$

as Garman-Klass formula. This "Garman-Klass volatility estimator" will on average be even smaller than a Garman-Klass estimator not adjusted for jumps. Moreover, it sometimes produces negative estimates for volatility (variance  $\sigma^2$ ).

#### 4. Normalized returns - empirics

Andersen, Bollerslev, Diebold, and Ebens (2001) find that "although the unconditional daily return distributions are leptokurtic, the daily returns normalized by the realized standard deviations are close to normal." Their conclusion is based on standard deviations obtained these from high frequency data. We study whether (and to what extend) this result is obtainable when standard deviations are estimated from daily data only.

We study stocks which were the components of the Dow Jow Industrial Average on January 1, 2009, namely AA, AXP, BA, BAC, C, CAT, CVX, DD, DIS, GE, GM, HD, HPQ, IBM, INTC, JNJ, JPM, CAG<sup>10</sup>, KO, MCD, MMM, MRK, SFT, PFE, PG, T, UTX, VZ and WMT. We use daily open, high, low and close prices. The data covers years 1992 to 2008. Stock prices are adjusted for stock splits and similar events. We have 4171 daily observations for every stock. These data were obtained from the CRSP database. We study DJI components to make our results as highly comparable as possible with the results of Andersen, Bollerslev, Diebold, and Ebens (2001).

For brevity, we study only two estimators: the Garman-Klass estimator (15) and the Parkinson estimator (13). We use the Garman-Klass volatility estimator because our previous analysis shows that it is the most appropriate one. We use the Parkinson volatility estimator to demonstrate that even though this estimator is the most commonly used range-based estimator, it should not be used for normalization of returns. Moreover, we study the effect of including or excluding a jump component into range-based volatility estimators.

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<sup>9</sup>These assumptions could be based on past data, but they would still be just assumptions.

<sup>10</sup>Since historical data for KFT (component of DJI) are not available for the complete period, we use its biggest competitor CAG instead.

First of all, we need to distinguish the daily returns and the trading day returns. By the daily returns we mean close-to-close returns, calculated according to formula (34). By the trading day returns we mean returns during the trading hours, i.e. open-to-close returns, calculated according to formula (2). We estimate volatilities accordingly: volatility of the trading day returns from (13) and (15) and the volatility of the daily returns using (35) and (36). Next we calculate standardized returns. We calculate standardized returns in three different ways: trading day returns standardized by trading day standard deviations (square root of trading day volatility), daily returns standardized by daily standard deviation and daily returns standardized by trading day standard deviation. Why do we investigate daily returns standardized by trading day standard deviations too? Theoretically, this does not make much sense because the return and the standard deviations are related to different time intervals. However, it is still quite common (see e.g. Andersen, Bollerslev, Diebold, and Ebens (2001)), because people are typically interested in daily returns, but the daily volatility cannot be estimated as precisely as trading day volatility. The volatility of the trading part of the day can be estimated very precisely from the high frequency data, whereas estimation of the daily volatility is always less precise because of the necessity of including the opening jump component. Therefore, trading day volatility is commonly used as a proxy for daily volatility. This approximation is satisfactory as long as the opening jump is small in comparison to trading day volatility, which is typically the case.

Now we calculate summary statistics for the different standardized returns as well as returns themselves. Results for the standard deviations are presented in Table 5 and results for the kurtosis are presented in Table 6. We do not put similar tables for mean and kurtosis into this paper, because these results are less interesting and can be summarized in one sentence: Mean returns are always very close to zero, independent of which standardization we used. Skewness is always very close to zero too.

The results for standard deviations and kurtosis are generally in line with the predictions from our simulations too. First let us discuss the standard deviations of the standardized returns. As Table 5 documents, normalization by standard deviations obtained from the Parkinson volatility estimator results in standard deviation smaller than one, approximately around 0.9 whereas normalization by standard deviation obtained from the Garman-Klass volatility estimator results in standard deviations larger than one, around 1.05. Normalization by standard deviations estimated from GARCH model is approximately 1.1. This is expected as well, because division by a noisy random variable increases the standard deviation.

Results for the kurtosis of standardized returns (see Table 6) are in line with the predictions from our simulations too. Return distributions have heavy tails (kurtosis significantly larger than 3). Second, the daily returns normalized by the standard deviations calculated from Garman-Klass formula are close to normal (kurtosis is close to 3). Third, the daily returns normalized by the standard deviations calculated from Parkinson formula have no tails (kurtosis is significantly smaller than 3). Fourth, normalization of daily returns by standard

Table 5: Standard deviations of the stock returns.  $r_{td}$  is an open-to-close return,  $r_d$  is a close-to-close return.  $\hat{\sigma}_{GK,td}$  ( $\hat{\sigma}_{P,td}$ ) is square root of Garman-Klass (Parkinson) volatility estimate without opening jump component.  $\hat{\sigma}_{GK,d}$  ( $\hat{\sigma}_{P,d}$ ) is square root of Garman-Klass (Parkinson) volatility estimate including opening jump component.  $\hat{\sigma}_{garch}$  is standard deviation estimated from GARCH(1,1) model based on daily returns.

	trading day returns			daily returns					
	$r_{td}$	$\frac{r_{td}}{\hat{\sigma}_{GK,td}}$	$\frac{r_{td}}{\hat{\sigma}_{P,td}}$	$r_d$	$\frac{r_d}{\hat{\sigma}_{GK,d}}$	$\frac{r_d}{\hat{\sigma}_{P,d}}$	$\frac{r_d}{\hat{\sigma}_{GK,td}}$	$\frac{r_d}{\hat{\sigma}_{P,td}}$	$\frac{r_d}{\hat{\sigma}_{garch}}$
AA	0.02	1.14	0.94	0.02	1.11	0.96	1.00	1.28	1.12
AXP	0.02	1.11	0.92	0.02	1.07	0.94	1.00	1.26	1.11
BA	0.02	1.04	0.89	0.02	1.02	0.92	1.00	1.20	1.10
BAC	0.02	1.12	0.93	0.02	1.08	0.94	1.00	1.26	1.12
C	0.02	1.11	0.91	0.03	1.05	0.92	1.01	1.26	1.12
CAT	0.02	1.10	0.92	0.02	1.08	0.95	1.00	1.28	1.13
CVX	0.01	1.11	0.92	0.02	1.08	0.95	1.00	1.25	1.09
DD	0.02	1.07	0.90	0.02	1.02	0.91	1.00	1.18	1.06
DIS	0.02	1.03	0.88	0.02	0.99	0.90	1.00	1.18	1.09
GE	0.02	1.07	0.91	0.02	1.03	0.93	1.00	1.20	1.09
GM	0.02	1.10	0.92	0.03	1.08	0.95	1.00	1.27	1.13
HD	0.02	1.06	0.90	0.02	1.02	0.92	1.00	1.20	1.10
HPQ	0.02	1.08	0.91	0.03	1.04	0.92	1.00	1.23	1.11
IBM	0.02	1.07	0.91	0.02	1.04	0.93	1.00	1.25	1.13
INTC	0.02	1.08	0.92	0.03	1.06	0.95	1.00	1.31	1.19
JNJ	0.01	1.06	0.89	0.02	1.00	0.90	1.00	1.17	1.06
JPM	0.02	1.06	0.90	0.02	1.03	0.92	1.00	1.22	1.10
CAG	0.01	1.09	0.89	0.02	0.98	0.87	1.00	1.15	1.01
KO	0.01	1.03	0.88	0.02	0.99	0.89	1.00	1.15	1.04
MCD	0.02	1.04	0.89	0.02	0.99	0.89	1.00	1.15	1.05
MMM	0.01	1.05	0.89	0.02	1.02	0.90	1.00	1.16	1.04
MRK	0.02	1.05	0.89	0.02	1.01	0.91	1.00	1.20	1.09
MSFT	0.02	1.04	0.90	0.02	1.03	0.93	1.00	1.24	1.14
PFE	0.02	1.08	0.91	0.02	1.04	0.92	1.00	1.22	1.10
PG	0.01	1.07	0.90	0.02	1.01	0.90	1.00	1.17	1.05
T	0.02	1.09	0.91	0.02	1.05	0.92	1.00	1.20	1.06
UTX	0.02	1.08	0.91	0.02	1.05	0.93	1.00	1.22	1.09
VZ	0.02	1.08	0.91	0.02	1.04	0.92	1.00	1.21	1.08
WMT	0.02	1.04	0.88	0.02	1.01	0.90	1.00	1.20	1.08
XOM	0.01	1.08	0.91	0.02	1.06	0.94	1.00	1.22	1.08
<b>mean</b>	<b>0.02</b>	<b>1.07</b>	<b>0.90</b>	<b>0.02</b>	<b>1.04</b>	<b>0.92</b>	<b>1.00</b>	<b>1.22</b>	<b>1.09</b>

Table 6: Kurtosis of the stock returns.  $r_{td}$  is an open-to-close return,  $r_d$  is a close-to-close return.  $\hat{\sigma}_{GK,td}$  ( $\hat{\sigma}_{P,td}$ ) is square root of Garman-Klass (Parkinson) volatility estimate without opening jump component.  $\hat{\sigma}_{GK,d}$  ( $\hat{\sigma}_{P,d}$ ) is square root of Garman-Klass (Parkinson) volatility estimate including opening jump component.  $\hat{\sigma}_{garch}$  is standard deviation estimated from GARCH(1,1) model based on daily returns.

	trading day returns			daily returns					
	$r_{td}$	$\frac{r_{td}}{\hat{\sigma}_{GK,td}}$	$\frac{r_{td}}{\hat{\sigma}_{P,td}}$	$r_d$	$\frac{r_d}{\hat{\sigma}_{GK,d}}$	$\frac{r_d}{\hat{\sigma}_{P,d}}$	$\frac{r_d}{\hat{\sigma}_{GK,td}}$	$\frac{r_d}{\hat{\sigma}_{P,td}}$	$\frac{r_d}{\hat{\sigma}_{garch}}$
AA	9.63	2.84	1.76	11.63	2.73	1.87	3.48	2.56	4.62
AXP	8.46	3.03	1.81	9.62	2.84	1.91	4.10	2.70	5.00
BA	6.42	2.99	1.81	10.76	2.75	1.91	3.12	2.62	6.82
BAC	19.47	2.87	1.78	26.81	2.78	1.91	3.50	2.85	8.69
C	34.05	3.12	1.82	38.79	2.95	1.96	3.62	2.70	6.70
CAT	5.71	2.93	1.80	7.31	2.78	1.90	3.88	2.78	6.97
CVX	11.28	2.99	1.80	13.44	2.80	1.90	3.89	2.43	3.77
DD	7.07	2.98	1.81	7.53	2.84	1.95	3.54	2.63	5.23
DIS	6.75	2.93	1.81	11.04	2.76	1.94	4.23	3.78	9.88
GE	10.29	2.80	1.77	10.07	2.71	1.93	3.20	2.68	4.95
GM	43.27	2.93	1.82	26.30	2.74	1.89	3.73	2.80	7.41
HD	6.43	2.93	1.80	19.21	2.70	1.90	3.23	2.73	10.84
HPQ	7.63	2.92	1.80	9.29	2.77	1.93	3.30	2.81	9.73
IBM	6.87	2.82	1.78	9.44	2.75	1.91	3.74	3.90	8.17
INTC	6.45	2.62	1.76	8.59	2.57	1.87	3.89	4.56	6.86
JNJ	5.70	3.02	1.83	10.56	2.88	1.97	3.17	2.63	4.98
JPM	14.60	3.00	1.83	12.05	2.80	1.96	3.46	2.79	4.89
CAG	8.64	3.54	1.93	16.43	3.37	2.08	4.42	2.78	10.33
KO	7.81	3.12	1.86	8.56	2.94	1.98	3.41	2.57	6.66
MCD	8.56	3.05	1.84	7.48	2.84	1.95	3.14	2.48	5.26
MMM	6.86	3.08	1.84	7.60	3.01	1.99	3.59	2.68	8.72
MRK	6.64	2.96	1.82	24.22	2.78	1.92	4.35	4.83	42.92
MSFT	5.22	2.63	1.78	8.61	2.55	1.91	5.81	9.43	9.24
PFE	5.36	2.83	1.78	6.17	2.74	1.90	3.40	2.86	6.57
PG	8.22	2.96	1.83	75.61	2.89	1.97	3.46	3.11	17.85
T	6.23	3.00	1.81	7.40	2.90	1.96	3.32	2.42	4.29
UTX	9.11	3.01	1.79	32.55	2.81	1.91	3.83	3.01	28.66
VZ	6.89	2.98	1.79	7.80	2.88	1.93	4.67	2.61	4.52
WMT	6.59	3.19	1.86	5.98	2.99	1.97	3.72	3.19	4.41
XOM	11.30	2.91	1.77	12.62	2.81	1.91	3.21	2.44	4.11
<b>mean</b>	<b>10.25</b>	<b>2.97</b>	<b>1.81</b>	<b>15.45</b>	<b>2.82</b>	<b>1.93</b>	<b>3.71</b>	<b>3.15</b>	<b>8.97</b>

deviation estimated for trading day only, will cause upward bias in kurtosis. This is a consequence of the standardization by an incorrect standard deviation - sometimes (particularly in a situation when the opening jump is large), returns are divided by too small standard deviation, which will cause too many large observations for normalized returns.

The last column of Table 6 reports kurtosis of returns normalized by standard deviations estimated from GARCH(1,1) model with mean return fixed to zero. As we can see, these normalized returns are not Gaussian, they have fat tails. This is consistent with the fact that GARCH models with fat-tailed conditional distribution of returns fit data better than GARCH models with conditionally normally distributed returns. However, as is clear from this paper, this is the case simply because GARCH models always condition return distribution on the estimated volatility, which is only a noisy proxy of the true volatility. Therefore, even when distribution of returns conditional on the true volatility is Gaussian, distribution of returns conditional on estimated volatility will have heavy tails. This result has an important implication for volatility modelling: the more precisely we can estimate the volatility, the closer will be the conditional distribution of returns to the normal distribution.

## 5. Conclusion

Range-based volatility estimators provide significant increase in accuracy compared to simple squared returns. Even though efficiency of these estimators is known, there is some confusion about other properties of these estimators. We study these properties. Our main focus is the properties of returns standardized by their standard deviations.

First, we correct some mistakes in existing literature. Second, we study different properties of range-based volatility estimators and find that for most purposes, the best volatility estimator is the Garman-Klass volatility estimator. The Meilijson volatility estimator improves its efficiency slightly, but it is based on a significantly more complicated formula. However, performance of all the range-based volatility estimators is similar in most cases except for the case when we want to use them for standardization of the returns.

Returns standardized by their standard deviations are known to be normally distributed. This fact is important for the volatility modelling. This result was possible to obtain only when the standard deviations were estimated from the high frequency data. When the standard deviations were obtained from volatility models based on daily data, returns standardized by these standard deviations are not Gaussian anymore, they have heavy tails. Using simulations we show that even when returns themselves are normally distributed, returns standardized by (imprecisely) estimated volatility are not normally distributed; their distribution has heavy tails. In other words: the fact that standard volatility models show that even conditional distribution of returns has heavy tails does not mean that returns are not normally distributed. It means that these models cannot estimate volatility precisely enough and the noise in the volatility estimates causes the heavy tails.

It is not obvious whether range-based volatility estimators can be used for the standardization of the returns. Using simulations we find that for the purpose of returns standardization there are large differences between these estimators and we find that the Garman-Klass volatility estimator is the only one appropriate for this purpose. Putting all the results together, we rate the Garman-Klass volatility estimator as the best volatility estimator based on daily (open, high, low and close) data. We test this estimator empirically and we find that we can indeed obtain basically the same results from daily data as Andersen, Bollerslev, Diebold, and Ebens (2001) obtained from high-frequency (transaction) data. This is important, because the high-frequency data are very often not available or available only for a shorter time period and their processing is complicated. Since returns scaled by standard deviations estimated from GARCH type of models (based on daily returns) are not Gaussian (they have fat tails), our results show that the GARCH type of models cannot capture the volatility precisely enough. Therefore, in the absence of high-frequency data, further development of volatility models based on open, high, low and close prices is recommended.

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