On the continued fraction expansion of certain Engel series

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Abstract

An Engel series is a sum of the reciprocals of an increasing sequence of positive integers, which is such that each term is divisible by the previous one. Here we consider a particular class of Engel series, for which each term of the sequence is divisible by the square of the preceding one, and find an explicit expression for the continued fraction expansion of the sum of a generic series of this kind. As a special case, this includes certain series whose continued fraction expansion was found by Shallit. A family of examples generated by nonlinear recurrences with the Laurent property is considered in detail, along with some associated transcendental numbers.

Keywords: continued fraction, nonlinear recurrence, transcendental number, Laurent property.

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1 Introduction

Given a sequence of positive integers (x_n), which is such that $x_n|x_{n+1}$ for all n, the sum of the reciprocals is the Engel series

$$\sum_{j=1}^{\infty} \frac{1}{x_j} = \sum_{j=1}^{\infty} \frac{1}{y_1 y_2 \cdots y_j}, \tag{1.1}$$

where $y_1 = x_1$ and $y_{n+1} = x_{n+1}/x_n$ for $n \geq 1$. (It should be assumed that (x_n) is eventually increasing, which guarantees the convergence of the sum. A brief introduction to Engel series can be found in [5].) In recent work [12], we considered some particular series of this kind that are generated by certain nonlinear recurrences of second order, and are such that the sequences (x_n) and (y_n) appear interlaced in the continued fraction expansion. Here we start from a sequence with the stronger property that $x_n^2|x_{n+1}$. Any initial 1s can be

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ignored, but in what follows it will be convenient to start with $x_1 = 1$ and take $x_2 \geq 2$, which implies $x_n \geq 2^{2^{n-2}}$ for $n \geq 2$, and we may write

$$x_1 = 1, \qquad x_n = \prod_{j=2}^{n} z_j^{2^{n-j}} \tag{1.2}$$

for some sequence of positive integers (z_n) with $z_2 \geq 2$. The corresponding Engel series is

$$S := \sum_{j=1}^{\infty} \frac{1}{x_j} = 1 + \sum_{j=2}^{\infty} \frac{1}{z_2^{2^{j-2}} z_3^{2^{j-3}} \cdots z_j}. \tag{1.3}$$

Our main result will be to show that in the generic case, when $z_2 \geq 3$ and $z_n \geq 2$ for $n \geq 3$, the continued fraction expansion of S has a universal structure, which we present explicitly.

There is a precedent for these results in the work of Shallit, who first found the continued fraction expansion of the sum

$$\sum_{k=0}^{\infty} \frac{1}{u^{2^k}} \tag{1.4}$$

in [16], for integer $u \geq 3$ (with $u = 2$ being a degenerate case), and went on [18] to obtain the continued fraction for the more general series

$$\sum_{k=0}^{\infty} \frac{1}{u^{c_k}}, \tag{1.5}$$

where (c_k) is a sequence of positive integers with some non-negative N such that $d_n := c_{n+1} - 2c_n \geq 0$ for all $n \geq N$. If we set $z_2 = u^{c_0}$ and $z_j = u^{d_{j-3}}$ for $j \geq 3$ in (1.3), and assume $d_n \geq 0$ for all $n \geq 0$, then $S - 1$ coincides with (1.5).

1.1 Outline of the paper

In the next section we prove the main result, namely the expression for the continued fraction expansion of a generic series of the form (1.3). Section 3 is devoted to an infinite family of examples of series of this type, which are generated by nonlinear recurrences with the Laurent property. For such nonlinear recurrence sequences, we show that the sum of the corresponding series (1.3) is a transcendental number. In the fourth section we consider the continued fractions obtained from certain degenerate cases, when either $z_2 = 2$ or $z_n = 1$ for $n \geq 3$, and some particular examples of these degenerate cases are examined in more detail. The final section contains some conclusions.

2 Continued fractions

We use the notation

$$[a_0; a_1, a_2, a_3, \ldots, a_j, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{a_j + \cdots}}}}$$

for continued fractions, and for the nth convergent of the continued fraction $[a_0; a_1, a_2, \ldots]$ we have

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n],$$

where the numerators p_n and denominators q_n are given in terms of the coefficients a_j according to the matrix identity

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}. \tag{2.1}$$

For what follows, it will also be convenient to note the identity obtained by taking the determinant of each side of (2.1), that is

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}. \tag{2.2}$$

For other basic results on continued fractions, the reader is referred to [3].

To begin with, it is convenient to treat the factors $z_2, z_3, \ldots$ of the sequence (x_n) as variables. For the first few partial sums we find the continued fraction expansions

$$S_1 = 1, \quad S_2 = \frac{z_2 + 1}{z_2} = [1; z_2], \quad S_3 = \frac{z_2^2 z_3 + z_2 z_3 + 1}{z_2^2 z_3} = [1; z_2 - 1, 1, z_3 - 1, z_2],$$

where the nth partial sum of (1.3) is denoted S_n. In general it is straightforward to write S_n as a fraction, that is

$$S_n = \frac{\sum_{j=1}^{n-1} \prod_{k=2}^{j} z_k^{2^{n-k} - 2^{j-k}} \prod_{\ell=j+1}^{n} z_\ell^{2^{n-\ell}} + 1}{z_2^{2^{n-2}} z_3^{2^{n-3}} \cdots z_n}, \tag{2.3}$$

where the denominator is x_n as given in (1.2); but the continued fraction expansion of the nth partial sum is best described recursively.

The basic pattern can be seen by looking at the continued fraction for the fourth partial sum, which is

$$S_4 = [1; z_2 - 1, 1, z_3 - 1, z_2, z_4 - 1, 1, z_2 - 1, z_3 - 1, 1, z_2 - 1].$$

Observe that the first five coefficients are the same as those of S_3, followed by $z_4 - 1$, 1, and then four more coefficients which almost coincide with the last four in S_3 in reverse order, except that there is $z_2 - 1$ in place of the final z_2 in S_3. This pattern persists, as described by the following

Proposition 2.1. *Given the initial set of coefficients*

$$[a_0^{(3)}; a_1^{(3)}, a_2^{(3)}, a_3^{(3)}, a_4^{(3)}] = [1; z_2 - 1, 1, z_3 - 1, z_2]$$

of the continued fraction expansion of S_3, of length $\ell_3 = 4$, define a sequence of sets of coefficients $(\{a_j^{(n)}\}_{j=0}^{\ell_n - 1})$ with $\ell_n = 3 \cdot 2^{n-2} - 1$ for $n = 3, 4, 5, \ldots$ recursively according to

$$a_j^{(n+1)} = a_j^{(n)} \quad \text{for} \quad j = 0, \ldots, \ell_n - 1,$$

$$a_{\ell_n}^{(n+1)} = z_{n+1} - 1, \qquad a_{\ell_n+1}^{(n+1)} = 1, \qquad a_{\ell_n+2}^{(n+1)} = a_{\ell_n-1}^{(n)} - 1,$$

and

$$a_j^{(n+1)} = a_{2\ell_n - j+1}^{(n)} \quad \text{for} \quad j = \ell_n + 3, \ldots, 2\ell_n.$$

Then the nth partial sum of the series (1.3) has the continued fraction expansion

$$S_n = [a_0^{(n)}; a_1^{(n)}, \ldots, a_{\ell_n-1}^{(n)}]. \tag{2.4}$$

Proof: This can be done similarly to the proof in [18], but we prefer to use matrix computations, in the same vein as [19]. The case $n = 3$ is easily verified directly. Proceeding by induction, suppose that the continued fraction expansion of S_n is given by (2.4), with coefficients $a_j^{(n)}$ for $j = 0, \ldots \ell_n - 1$ with $\ell_n = 3 \cdot 2^{n-2} - 1$ defined according the prescription above, and denote the numerators and denominators of the convergents by p_j and q_j respectively; so the final convergent gives

$$S_n = \frac{p_{\ell_n-1}}{q_{\ell_n-1}}, \qquad q_{\ell_n-1} = x_n.$$

Then for the next finite continued fraction defined by this recursive procedure there are a total of $\ell_{n+1} = 2\ell_n + 1 = 3 \cdot 2^{n-1} - 1$ coefficients, that is $a_j^{(n+1)}$ for $j = 0, \ldots \ell_{n+1} - 1$, and for the convergents we use $\tilde{p}_j$,$\tilde{q}_j$ to denote numerators/denominators, respectively. So by (2.1) we have

$$\begin{pmatrix} \tilde{p}_{\ell_{n+1}-1} & \tilde{p}_{\ell_{n+1}-2} \\ \tilde{q}_{\ell_{n+1}-1} & \tilde{q}_{\ell_{n+1}-2} \end{pmatrix} = \mathbf{M}_n \begin{pmatrix} z_{n+1} - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{M}}_n^T, \tag{2.5}$$

where

$$\mathbf{M}_n = \begin{pmatrix} a_0^{(n)} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{\ell_n-1}^{(n)} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{\ell_n-1} & p_{\ell_n-2} \\ q_{\ell_n-1} & q_{\ell_n-2} \end{pmatrix}$$

and

$$\begin{aligned} \tilde{\mathbf{M}}_n^T &= \begin{pmatrix} a_{\ell_n-1}^{(n)} - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{\ell_n-2}^{(n)} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1^{(n)} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_{\ell_n-1}^{(n)} - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{\ell_n-1}^{(n)} & 1 \\ 1 & 0 \end{pmatrix}^{-1} \mathbf{M}_n^T \begin{pmatrix} a_0^{(n)} & 1 \\ 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} q_{\ell_n-1} - q_{\ell_n-2} & p_{\ell_n-1} - q_{\ell_n-1} + q_{\ell_n-2} - p_{\ell_n-2} \\ q_{\ell_n-2} & p_{\ell_n-2} - q_{\ell_n-2} \end{pmatrix}, \end{aligned}$$

with T denoting transpose. Thus the equation (2.5) simplifies to yield

$$\begin{pmatrix} \tilde{p}_{\ell_{n+1}-1} & \tilde{p}_{\ell_{n+1}-2} \\ \tilde{q}_{\ell_{n+1}-1} & \tilde{q}_{\ell_{n+1}-2} \end{pmatrix} = \begin{pmatrix} z_{n+1}q_{\ell_n-1}p_{\ell_n-1}+1 & z_{n+1}p_{\ell_n-1}\Delta_n - 1 \\ z_{n+1}q_{\ell_n-1}^2 & z_{n+1}q_{\ell_n-1}\Delta_n - 1 \end{pmatrix},$$

with $\Delta_n = p_{\ell_n-1} - q_{\ell_n-1}$, where we have used the fact that

$$\det \mathbf{M}_n = \begin{vmatrix} p_{\ell_n-1} & p_{\ell_n-2} \\ q_{\ell_n-1} & q_{\ell_n-2} \end{vmatrix} = -1$$

by (2.2), since ℓ_n is odd. Hence we have

$$\tilde{p}_{\ell_{n+1}-1} = z_{n+1}q_{\ell_n-1}p_{\ell_n-1}+1, \qquad \tilde{q}_{\ell_{n+1}-1} = z_{n+1}q_{\ell_n-1}^2 = x_{n+1},$$

so that

$$S_{n+1} = S_n + \frac{1}{x_{n+1}} = \frac{p_{\ell_n-1}}{q_{\ell_n-1}} + \frac{1}{z_{n+1}q_{\ell_n-1}^2} = \frac{\tilde{p}_{\ell_{n+1}-1}}{\tilde{q}_{\ell_{n+1}-1}}$$

which is the required result. □

Remark 2.2. *Note that, mutatis mutandis, both the recursive structure of the partial sums and the above inductive proof hold for the partial sums of an Engel series (1.1) if, for some positive integer n_0, the sequence (x_n) satisfies the weaker condition that $x_n^2|x_{n+1}$ for $n \geq n_0$ only. This is the analogue of the fact that for the series (1.5) in [18], $c_{n+1} - 2c_n \geq 0$ need only hold for $n \geq N$, for some N.*

The finite continued fraction expansions of the partial sums immediately yield the continued fraction for the full series (1.3), at least for a generic choice of factors $z_2, z_3, \ldots$ of the sequence (x_n).

Theorem 2.3. *For integer factors $z_2 \geq 3$ and $z_n \geq 2$ for all $n \geq 3$, the Engel series (1.3) has the continued fraction expansion*

$$S = [a_0; a_1, a_2, \ldots, a_j, \ldots] = [1; z_2 - 1, 1, z_3 - 1, z_2, z_4 - 1, 1, \ldots] \qquad (2.6)$$

where the coefficients are given by $a_j = \lim_{n\to\infty} a_j^{(n)}$.

Proof: The result follows from taking the limit $n \to \infty$ in (2.4), provided that none of the coefficients in the finite continued fractions vanish. To see that this is so, note that only $1, z_2 - 1, z_2$ and $z_3 - 1$ appear as coefficients in the continued fraction for S_3, and all of these are non-zero with the above conditions on the z_j. At each step of the recursion in Proposition 2.1 only $z_{n+1} - 1 \geq 1$ and $a_{\ell_n+2}^{(n+1)} = a_{\ell_n-1}^{(n)} - 1$ are potentially new coefficients, so we must check that $a_{\ell_n-1}^{(n)} - 1$ cannot vanish. For $n = 3$ the last coefficient is $a_4^{(3)} = z_2$, so in S_4 this gives $a_7^{(4)} = z_2 - 1$, while for $n \geq 4$ we have $a_{\ell_n-1}^{(n)} = a_1^{(n-1)} = z_2 - 1$, so $a_{\ell_n+2}^{(n+1)} = z_2 - 2$, which is why we require $z_2 \geq 3$. Thus the only numbers that appear as coefficients in the continued fraction expansion of S are $1, z_2, z_2 - 2$ and $z_j - 1$ for $j \geq 2$, and none of these are zero. □

3 Nonlinear recurrence sequences

Among nonlinear recurrences of the form

$$x_{n+N}\, x_n = f(x_{n+1}, \ldots, x_{n+N-1}), \tag{3.1}$$

where f is a polynomial in $N-1$ variables, there is a multitude of examples which surprisingly generate integer sequences. In a wide variety of cases, the recurrence (3.1) has the Laurent property: for certain special choices of f, all of the iterates belong to the ring $\mathbb{Z}[x_0^{\pm 1}, \ldots, x_{N-1}^{\pm 1}]$; as a consequence, if all the initial values are 1 (or ± 1), then each term of the sequence is an integer. Such sequences were popularized by Gale [8, 9], and subsequently Fomin and Zelevinsky found a useful technique - the Caterpillar Lemma [7] - which can be used to prove the Laurent property in many cases, i.e. for recurrences coming from cluster algebras [6] or in the more general setting of Laurent Phenomenon (LP) algebras [14].

In [10, 11] we classified recurrences of second order, of the form

$$x_{n+2}\, x_n = f(x_{n+1}). \tag{3.2}$$

For the Laurent property to hold, the recurrence (3.2) must belong to one of three classes, depending on the form of f: (i) $f(0) \neq 0$, in which case one can apply the framework of cluster algebras (when f is a binomial) or LP algebras (when it is not); (ii) $f(0) = 0$, $f'(0) \neq 0$; (iii) $f(0) = f'(0) = 0$. In the first two classes there are additional requirements on f, but in the third class one can take $f(x) = x^2 F(x)$ with arbitrary $F \in \mathbb{Z}[x]$.

In [12] we considered the case that $f(x) = x^2 F(x)$, where F has positive integer coefficients with $F(0) = 1$, and obtained the continued fraction expansion of the sum $\sum_{j=1}^{\infty} \frac{1}{x_j}$. In order to obtain an Engel series of the form (1.3), we should instead choose F so that (3.2) becomes

$$x_{n+2}\, x_n = x_{n+1}^{d_1} G(x_{n+1}), \tag{3.3}$$

where

$$d_1 \geq 3, \quad G(x) \in \mathbb{Z}_{\geq 0}[x], \quad \deg G = d_2 \geq 0, \quad G(0) \neq 0, \quad G(1) \geq 3. \tag{3.4}$$

From $x_{n+2}/x_{n+1}^2 = \frac{x_{n+1}}{x_n} \cdot x_{n+1}^{d_1-3} G(x_{n+1})$, we see by induction that, starting with the initial values $x_0 = x_1 = 1$, (x_n) is a sequence of positive integers such that $x_n^2 | x_{n+1}$ with $x_2 = G(1) \geq 3$; hence also $z_n = x_n / x_{n-1}^2 \geq 3$ for $n \geq 2$.

Example 3.1. *Taking $d_1 = 3$, and $G(x) = 3$ for all x (so $d_2 = 0$), the recurrence (3.3) becomes $x_{n+2} x_n = 3x_{n+1}^3$, which generates the sequence beginning* $1, 1, 3, 81, 531441, 5559060566555523, \ldots$. *In this case the recurrence can be solved explicitly to yield*

$$x_n = 3^{s_n}, \qquad s_n = \frac{5-\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^n - 1.$$

The sum of the reciprocals is

$$S = 1 + \frac{1}{3} + \frac{1}{3^4} + \frac{1}{3^{12}} + \frac{1}{3^{33}} + \cdots = [1; 2, 1, 8, 3, 80, 1, 2, 8, 1, 2, 19682, \ldots].$$

Remark 3.2. *In the above example, $S-1$ is a sum of the type (1.5) considered in [18], with $u = 3$ and $c_n = s_{n-2}$. More generally, any recurrence of the form $x_{n+2}x_n = ux_{n+1}^{d_1}$ with $d_1 \geq 3$ and $x_0 = x_1 = 1$ generates a sum of this type.*

For most choices of G it is not possible to give the general solution of the non-linear recurrence (3.3) in closed form. Nevertheless, one can adapt the methods of Aho and Sloane [2] to write a formula giving precise asymptotic information. By rewriting (3.3) in terms of logarithms we find that $\Lambda_n = \log x_n$ satisfies

$$\Lambda_{n+1} - (d_1 + d_2)\Lambda_n + \Lambda_{n-1} = \log c + \alpha_n, \quad \text{with} \quad \alpha_n = \log\left(\frac{G(x_n)}{cx_n^{d_2}}\right), \tag{3.5}$$

where $G(x) = cx^{d_2} +$ lower order. Since $\alpha_n = \log(1 + O(x_n^{-1})) = O(x_n^{-1})$ as $n \to \infty$, the leading order behaviour of Λ_n is determined by the linear expression on the left-hand side of (3.5), which has the characteristic equation $\lambda^2 - (d_1 + d_2)\lambda + 1 = 0$, with largest root

$$\lambda = \frac{d_1 + d_2 + \sqrt{(d_1 + d_2)^2 - 4}}{2} > 2. \tag{3.6}$$

The next two statements are equivalent to analogous formulae obtained for the sequences considered in [12].

Proposition 3.3. *For the initial conditions $x_0 = x_1 = 1$, the logarithm $\Lambda_n = \log x_n$ of each term of the sequence satisfying (3.3) is given by the formula*

$$\Lambda_n = \left(\frac{(1-\lambda^{-1})\lambda^n - (1-\lambda)\lambda^{-n}}{\lambda - \lambda^{-1}} - 1\right)\log c^{-\frac{1}{d_1+d_2-2}} + \sum_{k=1}^{n-1}\left(\frac{\lambda^{n-k} - \lambda^{k-n}}{\lambda - \lambda^{-1}}\right)\alpha_k, \tag{3.7}$$

where α_k is defined as in (3.5) and λ as in (3.6).

Corollary 3.4. *To leading order, the asymptotic approximation of the logarithm Λ_n is given by*

$$\Lambda_n \sim C\lambda^n, \tag{3.8}$$

where

$$C = \frac{1}{d_1 + d_2 - 2}\left(\frac{1-\lambda^{-1}}{\lambda - \lambda^{-1}}\right)\log c + \frac{1}{\lambda - \lambda^{-1}}\sum_{k=1}^{\infty}\lambda^{-k}\alpha_k,$$

and for the terms of the sequence $x_n \sim c^{-\frac{1}{d_1+d_2-2}}\exp(C\lambda^n)$.

The asymptotic behaviour of these nonlinear recurrence sequences is enough to show that the sum of the corresponding Engel series is transcendental.

Theorem 3.5. *Suppose that the sequence (x_n) with initial values $x_0 = x_1 = 1$ is generated by the recurrence (3.3) for some G satisfying the conditions (3.4). Then the sum S in (1.3) is a transcendental number.*

Proof: This is essentially identical to the proof of Theorem 4 in [12], so here we only sketch the argument. Recall that Roth's theorem says that if α is an irrational algebraic number then for an arbitrary fixed $\delta > 0$ there are only finitely many rational approximations p/q for which

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\delta}}. \tag{3.9}$$

The number S has an infinite continued fraction expansion, so it is irrational. From the asymptotics (3.8) it follows that for any $\epsilon > 0$ the growth condition

$$x_{n+1} > x_n^{\lambda-\epsilon} \tag{3.10}$$

holds for all sufficiently large n. By making a comparison with a geometric sum, this gives

$$\left|S - \frac{p_{\ell_n-1}}{q_{\ell_n-1}}\right| = \sum_{j=n+1}^{\infty} \frac{1}{x_j} < \frac{1}{x_n^{\lambda-\epsilon-\epsilon'}} = \frac{1}{q_{\ell_n-1}^{\lambda-\epsilon-\epsilon'}}$$

for any $\epsilon' > 0$ and n large enough. So if ϵ and ϵ' are chosen such that $\lambda-\epsilon-\epsilon' = 2+\delta > 2$, then $\alpha = S$ has infinitely many rational approximations satisfying (3.9), and hence must be transcendental. $\square$

Example 3.6. *Taking $d_1 = 3$, and $G(x) = 2x + 1$ for all x (so $d_2 = 1$), the recurrence (3.3) becomes*

$$x_{n+2}x_n = x_{n+1}^3(2x_{n+1} + 1), \tag{3.11}$$

which generates the sequence

$$1, 1, 3, 189, 852910317, 559991793772468776423807826163779 5, \ldots$$

with leading order asymptotics

$$x_n \sim \frac{e^{C(2+\sqrt{3})^n}}{\sqrt{2}}, \qquad C \approx 0.107812043.$$

The sum of the reciprocals is the transcendental number

$$S = [1; 2, 1, 20, 3, 23876, 1, 2, 20, 1, 2, 7697947188058154, \ldots] \approx 1.3386243.$$

Engel series of the form (1.3) can also be generated by nonlinear recurrences of higher order. For instance, one can take a recurrence of third order,

$$X_{n+3}X_n = X_{n+1}^{e_1}X_{n+2}^{e_2}H(X_{n+1}, X_{n+2}), \quad \text{with } e_1 \geq 1,\, e_2 \geq 2, \tag{3.12}$$

where the polynomial $H(X,Y) \in \mathbb{Z}_{\geq 0}[X,Y]$ is not divisible by either of its arguments. It is straightforward to show that the Laurent property holds for this recurrence, and from $X_{n+3}/X_{n+2}^2 = \frac{X_{n+1}}{X_n} \cdot X_{n+1}^{e_1-1} X_{n+2}^{e_2-2} H(X_{n+1}, X_{n+2})$, it is easy to see by induction that the initial values $X_0 = X_1 = X_2 = 1$ generate an integer sequence with $X_n^2 | X_{n+1}$ for all $n \geq 0$. Thus the sum of reciprocals starting from the index 2, that is

$$S' := \sum_{j=2}^{\infty} \frac{1}{X_j}, \tag{3.13}$$

is an Engel series of the form (1.3). Note that the condition $H(1,1) \geq 3$ should be imposed, in order for Theorem 2.3 to apply to this series.

A particular class of recurrences of the form (3.12) can be obtained by factorizing the terms of a sequence satisfying (3.3) as $x_n = X_n X_{n+1}$, which lifts the second order recurrence to

$$X_{n+3} X_n = (X_{n+1} X_{n+2})^{d_1 - 1} G(X_{n+1} X_{n+2}). \tag{3.14}$$

For a generic polynomial $H(X_{n+1}, X_{n+2})$ on the right-hand-side of (3.12), it is not immediately obvious which term will be dominant as $n \to \infty$, but in the special case (3.14) the same techniques as for the second order recurrence can be applied directly, to show that the leading order asymptotics is $\log X_n \sim C' \lambda^n$ for some $C' > 0$, where λ is given by (3.6). This means that Theorem 3.5 applies to the series (3.13) as well.

Example 3.7. *Setting $x_n = X_n X_{n+1}$ in (3.11) gives the recurrence*

$$X_{n+3} X_n = X_{n+1}^2 X_{n+2}^2 (2 X_{n+1} X_{n+2} + 1), \tag{3.15}$$

which generates the sequence beginning

$$1, 1, 1, 3, 63, 13538259, 4136364903142041945155635 05, \ldots.$$

To leading order, $\log X_n \sim C'(2+\sqrt{3})^n$, with $C' \approx 0.0227833$. The sum of the reciprocals in (3.13) is the transcendental number

$$S' = [1; 2, 1, 6, 3, 3410, 1, 2, 6, 1, 2, 2256800700104, \ldots] \approx 1.3492064.$$

For other examples of transcendental numbers whose complete continued fraction expansion is known, see [4] and references.

4 Degenerate cases

If either $z_2 = 2$ or $z_n = 1$ for some $n \geq 3$, then one of the coefficients in the continued fraction becomes zero, and Theorem 2.3 is no longer valid. To obtain a continued fraction with non-zero coefficients, one can use the replacement rule $[\ldots, a, 0, b, \ldots] \longrightarrow [\ldots, a+b, \ldots]$ (see Proposition 3 in [15]) to remove the zero. Each such replacement, decreases the length of a finite continued fraction by two, so in degenerate cases the length of the continued fraction expansion of S_n is typically shorter than the generic value $\ell_n = 3 \cdot 2^{n-2} - 1$. Here we present the expansion for two particular degenerate cases, omitting details of the proof.

4.1 The case $z_2 = 2$

For generic values of the factors z_j, the sequence of lengths ℓ_n of partial sums begins $1, 2, 5, 11, 23$ for $n = 1, 2, 3, 4, 5$. When $z_2 = 2$ and $z_j \geq 2$ for all $j \geq 3$, the first few continued fractions for the partial sums of S are $S_1 = 1$,

$$S_2 = [1; 2], \quad S_3 = [1; 1, 1, z_3 - 1, 2], \quad S_4 = [1; 1, 1, z_3 - 1, 2, z_4 - 1, 1, 1, z_3 - 1, 2],$$

which are of the same length as in the generic case, except for S_4 being of length 10, since at the end $[\ldots, 1, 1] \to [\ldots, 2]$. The first zero appears in S_5, which contains a single coefficient $z_2 - 2$, so removing this and making the final replacement $[\ldots, 1, 1] \to [\ldots, 2]$, results in the length being 20:

$$S_5 = [1; 1, 1, z_3 - 1, 2, z_4 - 1, 1, 1, z_3 - 1, 1, 1, z_5 - 1, 2, z_3 - 1, 1, 1, z_4 - 1, 2, z_3 - 1, 2].$$

Thereafter the pattern continues with the continued fraction doubling in length at each step, as described by the following

Theorem 4.1. *When $z_2 = 2$ and $z_j \neq 1$ for $j \geq 3$, the Engel series (1.3) has the continued fraction expansion*

$$S = [a_0; a_1, a_2, \ldots, a_j, \ldots] = [1; 1, 1, z_3 - 1, 2, z_4 - 1, 1, \ldots]$$

with coefficients given by $a_j = \lim_{n \to \infty} a_j^{(n)}$, where $(\{a_j^{(n)}\}_{j=0}^{\ell_n - 1})$, the sequence of sets of coefficients of the finite continued fractions for partial sums S_n, of length $\ell_n = 5 \cdot 2^{n-3}$ for $n = 4, 5, \ldots$, is defined by starting from

$$[a_0^{(4)}; a_1^{(4)}, \ldots, a_9^{(4)}] = [1; 1, 1, z_3 - 1, 2, z_4 - 1, 1, 1, z_3 - 1, 2],$$

and obtaining subsequent coefficients according to

$$a_j^{(n+1)} = a_j^{(n)} \quad \text{for} \quad j = 0, \ldots, \ell_n - 2,$$

$$a_{\ell_n - 1}^{(n+1)} = 1, \qquad a_{\ell_n}^{(n+1)} = 1, \qquad a_{\ell_n + 1}^{(n+1)} = z_{n+1} - 1,$$

$$a_j^{(n+1)} = a_{2\ell_n - j + 1}^{(n)} \quad \text{for} \quad j = \ell_n + 2, \ldots, 2\ell_n - 2, \qquad \text{and} \quad a_{2\ell_n - 1}^{(n+1)} = 2.$$

In order to obtain a sequence (x_n) of this degenerate type from a second order recurrence of the form (3.3), the conditions (3.4) should be modified so that $G(1) = 2$, which requires that $G(x) = x^{d_2} + 1$ for some non-negative integer d_2. So the recurrence becomes

$$x_{n+2}\, x_n = x_{n+1}^{d_1} (x_{n+1}^{d_2} + 1), \tag{4.1}$$

with $d_1 \geq 3$, $d_2 \geq 0$. The results of Proposition 3.3, Corollary 3.4 and Theorem 3.5 all apply without alteration to sequences obtained from (4.1).

Example 4.2. *Taking $d_1 = 3$, and $G(x) = x + 1$ for all x, the recurrence (3.3) becomes $x_{n+2} x_n = x_{n+1}^3 (x_{n+1} + 1)$, which generates the sequence*

$$1, 1, 2, 24, 172800, 37150633525248000000, \ldots$$

with asymptotics $x_n \sim e^{C(2+\sqrt{3})^n}$, $C \approx 0.06224548$. The sum of the reciprocals is the transcendental number

$$S = [1; 1, 1, 5, 2, 299, 1, 1, 5, 1, 1, 1244167199, 2, 5, 1, 1, 299, \ldots] \approx 1.54167245.$$

4.2 The case $z_j = 1$ for $j \geq 3$

If we set $z_2 = u$ and all other factors $z_j = 1$ then $x_n = u^{2^{n-2}}$ for $n \geq 2$; the expansion (2.6) is no longer valid because each coefficient $z_j - 1$ becomes zero for $j \geq 3$. In that case, the sum of the reciprocals is $S = 1 + \sum_{k=0}^{\infty} u^{-2^k}$, so that $S - 1$ coincides with (1.4). The continued fractions for the partial sums were first obtained in [16], and a nonrecursive description was given in [17]. The sequence of lengths begins 1,2,3,5,9,17, with $\ell_n = 2^{n-2} + 1$ for $n \geq 3$, and the full continued fraction is

$$S = [1; u-1, u+2, u, u, u-2, u, u+2, u, u-2, u+2, u, u-2, u, u, u+2, u, \ldots]$$

for $u \neq 2$. The only numbers that appear as coefficients in this continued fraction are $1, u-2, u-1, u, u+2$.

However, the case $u = 2$ is special, since some of the coefficients in the above expansion become zero. The sequence of lengths of partial sums starts with 1,2,3,5,7,11, and $\ell_n = 2^{n-3} + 3$ for $n \geq 4$. Only the numbers 1,2,4,6 appear as coefficients in the continued fraction for the series, which is

$$S = [1; 1, 4, 2, 4, 4, 6, 4, 2, 4, 6, 2, 4, 6, 4, 4, 2, 4, 6, \ldots].$$

The argument used to prove Theorem 3.5, based on Roth's theorem, does not apply to the partial sums of the series (1.4), since the sequence (x_n) does not grow fast enough: in contrast to (3.10), it satisfies the recurrence $x_{n+1} = x_n^2$. However, a direct proof of transcendence of (1.4), valid for all integers $u \geq 2$, was first given by Kempner [13]; various alternative proofs are collected in [1].

5 Conclusions

We have found the continued fraction expansion for an Engel series of the special type (1.3). In some cases, coming from nonlinear recurrence sequences, it has been shown that this produces transcendental numbers. We expect that the sum (1.3) should be transcendental for any choice of the factors $z_2 \geq 2$ and $z_j \geq 1$ for $j \geq 3$. However, we do not know of a simple way to prove this in general.

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