Mikhailov, Alexander V. and Papamikos, Georgios and Wang, Jing Ping (2016) Darboux transformation for the vector sine-Gordon equation and integrable equations on a sphere. Letters in Mathematical Physics, 106 (7). pp. 973-996. ISSN 0377-9017.

DOI

https://doi.org/10.1007/s11005-016-0855-5

Darboux transformation for the vector sine-Gordon equation and integrable equations on a sphere

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Abstract

We propose a method for construction of Darboux transformations, which is a new development of the dressing method for Lax operators invariant under a reduction group. We apply the method to the vector sine-Gordon equation and derive its Bäcklund transformations. We show that there is a new Lax operator canonically associated with our Darboux transformation resulting an evolutionary differential-difference system on a sphere. The latter is a generalised symmetry for the chain of Bäcklund transformations. Using the re-factorisation approach and the Bianchi permutability of the Darboux transformations we derive new vector Yang-Baxter map and integrable discrete vector sine-Gordon equation on a sphere.

Mathematics Subject Classification. 35Q51, 37K10, 37K35, 39A14
Keywords. The vector sine-Gordon equation, Lax representations, Darboux transformations, Bäcklund transformations, Yang-Baxter maps, integrable equations on a sphere

1 Introduction

Lax integrable partial differential equations have natural connections with integrable differential-difference and discrete equations via Darboux transformations. Namely, given a Lax representation for a partial differential equation, we can systematically construct Darboux transformations whose Bianchi permutability condition leads to an integrable difference equation, while the corresponding Bäcklund transformations are (often nonlocal) symmetries of these difference equations and are integrable differential–difference equations in their own right (see, for instance, [1, 2, 3, 4]).

In this paper we make steps towards the development of a systematic approach to construction of Darboux transformations, associated difference systems and integrable maps based on a natural extension of the dressing method. The motivation for this line of research is to cooperate the reduction groups [5, 6, 7] of Lax representations of integrable partial differential equations to integrable difference equations and corresponding Darboux transformations. The aim of our project is to describe Darboux transformations for Lax operators on Kac-Moody and automorphic Lie algebras and to extend them to corresponding Lax-Darboux schemes linking together partial differential, differential–difference and partial difference integrable systems sharing common symmetries and conservation laws. Recently, the authors of [8, 9] have completed a comprehensive study for the Lax operators of the nonlinear Schrödinger equation type and they derived some new discrete equations and new Yang-Baxter maps. In the paper [10] the method was applied to the case of Lax operators with the reduction group generated by inner and outer automorphisms of the Lax representation for two dimensional Volterra chain.

In this paper we study differential-difference and partial difference equations associated with the vector sine-Gordon equation

$$D_t\left(\frac{D_x\boldsymbol{\alpha}}{\beta}\right)=\boldsymbol{\alpha},\quad \beta^2+|\boldsymbol{\alpha}|^2=1, \tag{1}$$

where dependent variables $\boldsymbol{\alpha}=(\alpha^1,\cdots,\alpha^n)^T$ is n-dimensional real vector and $\beta\in\mathbb{R}$. Here and in what follows the upper index T denotes the transposition of a vector or a matrix and we denote $\boldsymbol{\alpha}^T\boldsymbol{\alpha}=|\boldsymbol{\alpha}|^2$. Sometimes it is convenient to write β and $\boldsymbol{\alpha}$ as an $n+1$-dimensional unit vector $\boldsymbol{v}^T=\begin{pmatrix}\beta, & \boldsymbol{\alpha}^T\end{pmatrix}$. Thus the vector sine-Gordon equation is an integrable partial differential system of equations with the vector dependent variable $\boldsymbol{v}$ is on a sphere $\boldsymbol{v}\in S^n=\{\boldsymbol{x}\in\mathbb{R}^{n+1}\,|\,\boldsymbol{x}^T\boldsymbol{x}=1\}$.

Equation (1) has a long history, it first appeared in [11] viewed as a reduction of the two-dimensional σ-model [12]. Its Lax representation was given in [13] and its Lagrangian formulation in [14]. Later, this equation reappeared in the study of moving frames in the Riemannian geometry [15]. The dressing method for construction of multi-soliton solutions of (1) and study of soliton interactions has been recently developed in [16]. There are several other generalisations of the scalar sine-Gordon equation in the literature. For example, the well known Budagov-Takhtajan system [17, 18, 19] is different from (1) if $n>1$. It admits soliton decays which are not possible for equation (1).

In this paper we use the Lax representation (10), (11) proposed in [15] and studied in [16]. For these Lax operators we derive a Darboux transformation (Theorem 1) with Darboux matrix

$$M_\nu(\boldsymbol{u})=I_{n+2}+\frac{i\nu}{\lambda-i\nu}Qa\rangle\langle a-\frac{i\nu}{\lambda+i\nu}a\rangle\langle aQ,\quad \langle a=(i,\boldsymbol{u}^T),\quad \boldsymbol{u}=\frac{\boldsymbol{v}_1+\boldsymbol{v}}{|\boldsymbol{v}_1+\boldsymbol{v}|}, \tag{2}$$

where I_{n+2} is the identity matrix of size $n+2$, $Q=\mathrm{diag}(-1,1,\ldots,1)$, and rigorously prove that

$$D_t\left(\frac{\boldsymbol{v}_1+\boldsymbol{v}}{|\boldsymbol{v}_1+\boldsymbol{v}|}\right)=-\frac{1}{2\nu}(\boldsymbol{v}_1-\boldsymbol{v}); \tag{3}$$

$$\frac{D_x\boldsymbol{\alpha}_1}{\beta_1}-\frac{D_x\boldsymbol{\alpha}}{\beta}=-\frac{2\nu}{|\boldsymbol{v}_1+\boldsymbol{v}|}\left(\boldsymbol{\alpha}_1+\boldsymbol{\alpha}\right), \tag{4}$$

is the Bäcklund transformation which depends on an arbitrary constant parameter $0\neq\nu\in\mathbb{R}$ and relates two solutions $\boldsymbol{v}$ and $\boldsymbol{v}_1$ of the vector sine-Gordon equation (1). Equations (3) and (4) can also be seen as two non-evolutionary differential–difference equations on a lattice with variables $\boldsymbol{v}_j\in S^n,\ j\in\mathbb{Z}$ and with a shift operator $\mathcal{S}_\nu:\boldsymbol{v}_j\mapsto\boldsymbol{v}_{j+1}$, assuming $\boldsymbol{v}_0=\boldsymbol{v}$.

For the Darboux matrix (2) there exists a new Lax operator

$$\mathcal{B}_\nu=D_\tau-i\nu\frac{Qa_{-1}\rangle\langle a-a\rangle\langle a_{-1}Q}{(\lambda-i\nu)\langle aQa_{-1}\rangle}+i\nu\frac{a_{-1}\rangle\langle aQ-Qa\rangle\langle a_{-1}}{(\lambda+i\nu)\langle aQa_{-1}\rangle},\qquad a_{-1}\rangle=\mathcal{S}_\nu^{-1}(a)\rangle \tag{5}$$

canonically associated with it. The existence of a Lax operator which is canonically associated with a Darboux matrix is quite remarkable. We first observed it in [10] where a local generalised symmetry for non-evolutionary differential–difference equations (Bäcklund transformations for two–dimensional Volterra chain) has been found. Our analysis of the variety of Darboux–Lax integrable differential–difference systems shows that local symmetries often correspond to such operators. They deserve more attention and we are planning to develop this direction of research in future publications.

The compatibility of the Darboux matrix (2) and Lax operator (5) also results in an evolutionary differential–difference equation

$$D_\tau(\boldsymbol{v}) = \frac{|\boldsymbol{v}_{-1}+\boldsymbol{v}|^2(\boldsymbol{v}_1+\boldsymbol{v}) - |\boldsymbol{v}_1+\boldsymbol{v}|^2(\boldsymbol{v}_{-1}+\boldsymbol{v})}{(\boldsymbol{v}_1^T+\boldsymbol{v}^T)(\boldsymbol{v}_{-1}+\boldsymbol{v}) + |\boldsymbol{v}_{-1}+\boldsymbol{v}||\boldsymbol{v}_1+\boldsymbol{v}|}, \qquad \boldsymbol{v}_k = \mathcal{S}_\nu^k(\boldsymbol{v}), \tag{6}$$

which is a local symmetry of the nonevolutionary integrable systems (3) and (4). System (6) is known, and it was found by Adler [20] in his classification of isotropic integrable Volterra-type lattices on the sphere with generalised symmetries. In this paper we equip this system with a Lax-Darboux representation and connect it to the vector sine-Gordon equation (1) and its Bäcklund chains (3) and (4).

The Bianchi permutability condition for two Darboux transformations with distinct parameters $\mu \neq \pm\nu$ resulting in two shift operators $\mathcal{S}_\nu,\ \mathcal{S}_\mu$ leads to the integrable discrete equation

$$\boldsymbol{v}_{1,0} = -\boldsymbol{v} + 2\mathbf{X}(\mathbf{X}^T\boldsymbol{v}) \tag{7}$$

where $\mathcal{S}_\nu^n\mathcal{S}_\mu^m\boldsymbol{v} = \boldsymbol{v}_{n,m}$ and

$$\mathbf{X} = \frac{(\nu^2-\mu^2)\mathbf{x} + 2\mu(\nu+\mu(\mathbf{x}^T\mathbf{y}))\mathbf{y}}{\nu^2+\mu^2+2\mu\nu(\mathbf{x}^T\mathbf{y})}, \qquad \mathbf{x} = \frac{\boldsymbol{v}_{1,1}+\boldsymbol{v}_{0,1}}{|\boldsymbol{v}_{1,1}+\boldsymbol{v}_{0,1}|}, \quad \mathbf{y} = \frac{\boldsymbol{v}_{0,1}+\boldsymbol{v}}{|\boldsymbol{v}_{0,1}+\boldsymbol{v}|}.$$

Equation (7) can also be uniquely resolved with respect to the variable $\boldsymbol{v}_{0,1}$. Bäcklund transformations (3), (4) and evolutionary equation (6) are its non-local and local symmetries.

Moreover, the re-factorisation of the product of two Darboux matrices $M_\nu(\mathbf{x})M_\mu(\mathbf{y}) = M_\mu(\mathbf{Y})M_\nu(\mathbf{X})$ leads to a new involutive Yang-Baxter map $R_{\nu,\mu} : S^n \times S^n \mapsto S^n \times S^n$

$$R_{\nu,\mu} : \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} \dfrac{(\nu^2-\mu^2)\mathbf{x} + 2\mu(\nu+\mu(\mathbf{x}^T\mathbf{y}))\mathbf{y}}{\nu^2+\mu^2+2\mu\nu(\mathbf{x}^T\mathbf{y})} \\ \\ \dfrac{(\mu^2-\nu^2)\mathbf{y} + 2\nu(\mu+\nu(\mathbf{x}^T\mathbf{y}))\mathbf{x}}{\nu^2+\mu^2+2\mu\nu(\mathbf{x}^T\mathbf{y})} \end{pmatrix}, \qquad \mathbf{x},\mathbf{y},\mathbf{X},\mathbf{Y} \in S^n. \tag{8}$$

The arrangement of this paper is as follows: In Section 2, we give basic definitions related to the Lax- Darboux scheme such as Darboux transformation, the shift operators and Bäcklund transformation. In Section 3, we derive the Bäcklund transformations for the vector sine-Gordon equation (1) together with the Darboux matrix. In Section 4, we construct a local generalised symmetry for these non-evolutionary equations using the Darboux matrix. Meanwhile, we provide the Lax representation for this symmetry flow, which is one of isotropic integrable Volterra-type lattices on the sphere [20]. In Section 5, we use the re-factorisation of two Darboux matrices to derive the corresponding Yang-Baxter map and integrable discrete vector sine-Gordon equation, which are new to the best of our knowledge.

2 The Lax-Darboux scheme

In this section, we recall the Lax representation of the vector sine-Gordon equation and introduce some basic definitions such as Darboux transformation and Bäcklund transformation.

The vector sine-Gordon equation (1) is equivalent to the compatibility condition $[\mathcal{L},\mathcal{A}] = 0$ for two linear problems [15], [16]

$$\mathcal{L}\Psi = 0, \qquad \mathcal{A}\Psi = 0, \tag{9}$$

where

$$\mathcal{L} = D_x - \lambda J - U \quad \text{and} \quad \mathcal{A} = D_t + \lambda^{-1} V, \tag{10}$$

and

$$J = \begin{pmatrix} 0 & 1 & \mathbf{0}^T \\ -1 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & 0_n \end{pmatrix}, \; U = \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & -\boldsymbol{\alpha}_x^T/\beta \\ \mathbf{0} & \boldsymbol{\alpha}_x/\beta & 0_n \end{pmatrix}, \; V = \begin{pmatrix} 0 & \boldsymbol{v}^T \\ -\boldsymbol{v} & 0_{n+1} \end{pmatrix}, \tag{11}$$

where $\mathbf{0}$ is n-dimensional zero column vector and 0_k denotes the $k \times k$ zero matrix. Without causing confusion, we sometimes simply write 0 instead for either of them.

A Darboux transformation is a linear map acting on a fundamental solution

$$\Psi \mapsto \overline{\Psi} = M\Psi, \qquad \det M \neq 0 \tag{12}$$

such that the matrix function $\overline{\Psi}$ is a fundamental solution of the linear problems

$$\overline{\mathcal{L}}\,\overline{\Psi} = 0, \qquad \overline{\mathcal{A}}\,\overline{\Psi} = 0 \tag{13}$$

with new "potentials" $\overline{\boldsymbol{\alpha}}$ and $\overline{\beta}$. The matrix M is often called the Darboux matrix. From the compatibility of (12) and (13) it follows that

$$D_x(M) = (\lambda J + \overline{U})M - M(\lambda J + U) = \lambda[J, M] + \overline{U}M - MU; \tag{14}$$

$$D_t(M) = (-\lambda^{-1}\overline{V})M - M(-\lambda^{-1}V) = -\lambda^{-1}(\overline{V}M - MV), \tag{15}$$

Equations lead to *Bäcklund transformations* for the vector sine-Gordon equation, which relate two solutions $\boldsymbol{\alpha}, \beta$ and $\overline{\boldsymbol{\alpha}}, \overline{\beta}$ of (1).

A Darboux transformation maps one compatible system (9) into another one (13). It defines a map $\mathcal{S} : \boldsymbol{v} \mapsto \overline{\boldsymbol{v}}$. The map (12) is invertible ($\det M \neq 0$) and it can be iterated

$$\cdots \underline{\Psi} \xrightarrow{\mathcal{S}} \Psi \xrightarrow{\mathcal{S}} \overline{\Psi} \xrightarrow{\mathcal{S}} \overline{\overline{\Psi}} \xrightarrow{\mathcal{S}} \cdots .$$

We introduce notations

$$\ldots \Psi_{-1} = \underline{\Psi},\; \Psi_0 = \Psi,\; \Psi_1 = \overline{\Psi},\; \Psi_2 = \overline{\overline{\Psi}}, \ldots,$$

$$\ldots \boldsymbol{v}_{-1} = \underline{\boldsymbol{v}},\; \boldsymbol{v}_0 = \boldsymbol{v},\; \boldsymbol{v}_1 = \overline{\boldsymbol{v}},\; \boldsymbol{v}_2 = \overline{\overline{\boldsymbol{v}}}, \ldots .$$

In these notations the maps $\mathcal{S}$ and $\mathcal{S}^{-1}$ increase and decrease the subscript index by one, and therefore we shall call it a $\mathcal{S}$–shift, or shift operator $\mathcal{S}$. The resulting Bäcklund transformations from the Lax-Darboux representations (14) and (15) are integrable differential difference equations.

A Darboux transformation with a parameter μ denoted by M_μ results in the $\mathcal{S}_\mu$ shift. If we also consider a Darboux transformation with a different choice of the parameter ν, then the corresponding shift we denote $\mathcal{S}_\nu$. Commuting shifts act on $\mathbb{Z}^2$ lattice where with the vertex (n, m) we associate the variable $\boldsymbol{v}_{n,m} = \mathcal{S}_\nu^n \mathcal{S}_\mu^m \boldsymbol{v}$. Commutativity of the shifts is (Bianchi permutability)

$$\mathcal{S}_\nu(M_\mu)M_\nu - \mathcal{S}_\mu(M_\nu)M_\mu = 0 \tag{16}$$

is equivalent to a system of partial-difference equations and we call it Darboux representation for this system of partial-difference equations. In literature such representation and Darboux matrices sometimes referred as discrete Lax representation and discrete Lax operators respectively. Differential difference equations (14) and (15) are the symmetries of this partial-difference equation.

We can also consider the re-factorisation of a product of two Darboux matrices, which leads to the Yang-Baxter map. In next sections, we construct the Darboux matrices for the vector sine-Gordon equation (1), and further produce its Bäcklund transformations, the associated Yang-Baxter map and the integrable difference equation.

3 Invariant Darboux matrix under the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

The Lax operators $\mathcal{L}$ and $\mathcal{A}$ are invariant under the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by three transformations ι r and s satisfying $\iota^2 = r^2 = s^2 = \mathrm{id}$. In particular, for operator $\mathcal{L}$ these transformations are:

$$\iota : \mathcal{L}(\lambda) \to -\mathcal{L}^\dagger(\lambda) = \mathcal{L}(\lambda), \tag{17}$$

$$r : \mathcal{L}(\lambda) \to \mathcal{L}^*(\lambda^*) = \mathcal{L}(\lambda), \tag{18}$$

$$s : \mathcal{L}(\lambda) \to Q\mathcal{L}(-\lambda)Q = \mathcal{L}(\lambda), \tag{19}$$

where $\mathcal{L}^\dagger(\lambda)$ is the formally adjoint operator defined by $\mathcal{L}^\dagger(\lambda) = -D_x - \lambda J^T - U^T$, matrix $Q = \mathrm{diag}(-1, 1, \ldots, 1)$ and * denotes the complex conjugation.

We assume that Darboux matrix $M(\lambda)$ is a rational function of the spectral parameter λ and is invariant with respect to the reduction group (17)–(19) action, namely,

$$\iota : M(\lambda) \mapsto (M(\lambda)^{-1})^T = M(\lambda), \tag{20}$$

$$r : M(\lambda) \mapsto M^*(\lambda^*) = M(\lambda), \tag{21}$$

$$s : M(\lambda) \mapsto QM(-\lambda)Q = M(\lambda). \tag{22}$$

Notice that the action of the automorphism ι (21) is different from (17) since M is an element of a Lie group rather than Lie algebra.

It is easy to show (see Proposition 1 in [16]) that a λ–independent Darboux matrix results in a constant linear map.

Proposition 1. [16] *Assume Darboux matrix M is invariant under the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and it is independent of spectral parameter λ. Then M is a constant matrix and of the form*

$$M = \pm \begin{pmatrix} 1 & 0 & \mathbf{0}^T \\ 0 & 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \Omega \end{pmatrix}, \tag{23}$$

where $\mathbf{0}$ is n-dimensional zero column vector and a constant (x, t - independent) matrix $\Omega \in O(n, \mathbb{R})$.

This implies that the constant Darboux matrix defines the transformation

$$\overline{\beta} = \beta \quad \text{and} \quad \overline{\boldsymbol{\alpha}} = \Omega\boldsymbol{\alpha}. \tag{24}$$

corresponding to $O(n, \mathbb{R})$ point symmetry of the vector sine-Gordon equation (1).

A rational in λ and reduction group invariant (20)–(22) matrices M must have poles on the orbits of the reduction group. They can be represented as a product of matrices which have poles in a single orbit. Here we shall study reduction group invariant Darboux matrix M_ν with two simple poles at $\lambda = \pm i\nu,\ \nu \neq 0,\ \nu \in \mathbb{R}$. It is the minimal possible number of simple poles for a reduction group invariant Darboux matrix and it corresponds to the dressing matrix for a single kink solution [16]. It is of the following form (cf. Proposition 2 in [16]):

$$M_\nu = I_{n+2} + \frac{i\nu}{\lambda - i\nu} A - \frac{i\nu}{\lambda + i\nu} QAQ, \qquad A \neq 0. \tag{25}$$

Proposition 2. *If matrix (25) is invariant under reduction group generated by ι, r and s, it can be represent as*

$$M_\nu(\boldsymbol{u}) = I_{n+2} + \frac{i\nu}{\lambda - i\nu} Qa\rangle\langle a - \frac{i\nu}{\lambda + i\nu} a\rangle\langle aQ, \qquad \langle a = (i, \boldsymbol{u}^T), \tag{26}$$

where $\boldsymbol{u}$ is a unit length real vector $\boldsymbol{u} \in S^n$.

Proof. Note that the given matrix is invariant under the action of s. It immediately follows from (20) that $M_\nu M_\nu^T = I_{n+2}$, which implies that

$$AA^T = A^T A = 0; \tag{27}$$

$$(I_{n+2} - \frac{1}{2} QAQ)A^T + A(I_{n+2} - \frac{1}{2} QA^T Q) = 0. \tag{28}$$

These identities correspond to the vanishing of the second and first order poles at $\lambda = \pm i\nu$, respectively. The invariance under the action of r implies

$$A^* = QAQ, \tag{29}$$

So the matrix A satisfies the conditions of Lemma 1 in Appendix. Hence, it is of the form

$$A = h \begin{pmatrix} -i \\ \mathbf{m} \end{pmatrix} \begin{pmatrix} i & \mathbf{u} \end{pmatrix}, \qquad 0 \neq h \in \mathbb{R}, \quad \mathbf{m}, \mathbf{u} \in \mathbb{R}^{n+1}, \quad |\mathbf{m}| = |\mathbf{u}| = 1.$$

Substituting it into (28), we obtain

$$h = 1, \qquad \mathbf{m} = \mathbf{u}.$$

Thus we have $A = Qa\rangle\langle a$, from which the result in the statement immediately follows. □

This proposition confirms that the kink solutions of rank 1 for the vector sine-Gordon equation (1) obtained in [16] are indeed generic and there are no kink solutions of higher ranks.

Proposition 3. *Matrix $M_\nu(\boldsymbol{u})$ (26) satisfies the compatibility condition (15) if and only if*

$$\overline{V} M_\nu^0(\boldsymbol{u}) = M_\nu^0(\boldsymbol{u}) V, \tag{30}$$

$$D_t \left(Qa\rangle\langle a\right) + \frac{1}{i\nu} \overline{V} Qa\rangle\langle a - \frac{1}{i\nu} Qa\rangle\langle aV = 0, \tag{31}$$

where

$$M_\nu^0(\boldsymbol{u}) = \lim_{\lambda \to 0} M_\nu(\boldsymbol{u}).$$

Proof. Taking the residue at $\lambda = 0$ on both sides of (15), we obtain (30). Taking the residue at $\lambda = i\nu$ on both sides of (15), we obtain (31). The residue at $\lambda = -i\nu$ will also vanishes due to the reduction group. □

We now convert the above conditions to the conditions for the components of vector $\langle a = (i, \boldsymbol{u}^T)$.

Proposition 4. *Matrix $M_\nu(\boldsymbol{u})$ given by (26) satisfies (30) if*

$$\boldsymbol{u} = \pm \frac{\overline{\boldsymbol{v}} + \boldsymbol{v}}{|\overline{\boldsymbol{v}} + \boldsymbol{v}|}. \tag{32}$$

Proof. It follows from (26) that

$$M_\nu^0(\boldsymbol{u}) = I_{n+2} - Qa\rangle\langle a - a\rangle\langle aQ = (M_\nu^0(\boldsymbol{u}))^T = (M_\nu^0(\boldsymbol{u}))^{-1}. \tag{33}$$

Thus we have $\overline{V} = M_\nu^0(\boldsymbol{u}) V M_\nu^0(\boldsymbol{u})$. Using (11), we get

$$\overline{\boldsymbol{v}} + \boldsymbol{v} = 2(\boldsymbol{u}^T \boldsymbol{v})\boldsymbol{u}.$$

It follows that $(\boldsymbol{u}^T\boldsymbol{v})^2 = \frac{1}{2}(\overline{\boldsymbol{v}}^T\boldsymbol{v} + 1) = \frac{1}{4}|\overline{\boldsymbol{v}} + \boldsymbol{v}|^2$. Thus we have $\boldsymbol{u} = \pm\frac{\overline{\boldsymbol{v}}+\boldsymbol{v}}{|\overline{\boldsymbol{v}}+\boldsymbol{v}|}$. □

In the rest of the paper we assume that $\overline{\boldsymbol{v}} + \boldsymbol{v} \neq \boldsymbol{0}$ and choose the positive sign in (32). We now compute the Bäcklund transformation and summarise the result for the Darboux matrix.

Theorem 1. *The vector sine-Gordon equation (1) possesses a Bäcklund transformation*

$$D_t\left(\frac{\overline{\boldsymbol{v}} + \boldsymbol{v}}{|\overline{\boldsymbol{v}} + \boldsymbol{v}|}\right) = -\frac{1}{2\nu}(\overline{\boldsymbol{v}} - \boldsymbol{v}); \quad \boldsymbol{v} = \begin{pmatrix} \beta \\ \boldsymbol{\alpha} \end{pmatrix}, \tag{34}$$

where $\nu \in \mathbb{R}$ is constant. The corresponding Darboux matrix is

$$M_\nu(\boldsymbol{u}) = I_{n+2} + \frac{i\nu}{\lambda - i\nu} Qa\rangle\langle a - \frac{i\nu}{\lambda + i\nu} a\rangle\langle aQ, \quad \langle a = (i, \boldsymbol{u}^T), \quad \boldsymbol{u} = \frac{\overline{\boldsymbol{v}} + \boldsymbol{v}}{|\overline{\boldsymbol{v}} + \boldsymbol{v}|}. \tag{35}$$

Proof. We derive the Bäcklund transformation from (31), which is equivalent to

$$Q\left(a_t\rangle + \frac{1}{i\nu} Q^{-1}\overline{V} Qa\rangle\right)\langle a + Qa\rangle\left(\langle a_t - \frac{1}{i\nu}\langle aV\right) = 0. \tag{36}$$

Therefore, there exists a scalar γ such that

$$a_t\rangle + \frac{1}{i\nu} Q^{-1}\overline{V} Qa\rangle = a\rangle\gamma; \qquad a_t\rangle + \frac{1}{i\nu} Va\rangle = -a\rangle\gamma. \tag{37}$$

Taking the sum of these two equations we get

$$a_t\rangle + \frac{1}{2i\nu}\left(Q^{-1}\overline{V}Q + V\right) a\rangle = 0.$$

that is,

$$(\boldsymbol{v}^T - \overline{\boldsymbol{v}}^T)\boldsymbol{u} = 0; \qquad D_t\boldsymbol{u} = -\frac{1}{2\nu}(\overline{\boldsymbol{v}} - \boldsymbol{v}).$$

The first equation is satisfied due to (32) and $|\boldsymbol{v}| = |\boldsymbol{v}_1| = 1$. The second one is the required equation (34). It is easy to check that equations (36) and (37) (with $\gamma = \frac{1}{\nu}\boldsymbol{v}^T\boldsymbol{u}$) are satisfied due to (34). □

Using the compatibility condition (14) between the Darboux matrix (35) and matrix U, we get the following Bäcklund transformation with respect to x:

Corollary 1. *For the Darboux matrix $M_\nu(\boldsymbol{u})$ given in Theorem 1, the compatibility condition (14) leads to the following Bäcklund transformation (with respect to x):*

$$\frac{D_x(\overline{\boldsymbol{\alpha}})}{\overline{\beta}} - \frac{D_x(\boldsymbol{\alpha})}{\beta} = -\frac{2\nu}{|\overline{\boldsymbol{v}} + \boldsymbol{v}|}\left(\overline{\boldsymbol{\alpha}} + \boldsymbol{\alpha}\right). \tag{38}$$

Proof. Substituting the Darboux matrix $M_\nu(\boldsymbol{u})$ into (14) and comparing the residues of all poles from both sides, we have

$$\overline{U} = U + i\nu\,[Qa\rangle\langle a - a\rangle\langle aQ, J]\,; \tag{39}$$

$$D_x\,(Qa\rangle\langle a) = \overline{U}Qa\rangle\langle a - Qa\rangle\langle aU + i\nu\big[J, Qa\rangle\langle a\big]. \tag{40}$$

By direct calculation, we obtain the identity (38) in the statement from (39).

We now check that (40) satisfies under the condition of (38). We rewrite it as

$$Q\left(D_x(a)\rangle - \overline{U}a\rangle + i\nu Ja\rangle\right)\langle a + Qa\rangle\left(\langle D_x(a) + \langle aU + i\nu\langle aJ\right) = 0. \tag{41}$$

Differentiating $|\overline{\boldsymbol{v}} + \boldsymbol{v}|^2$ with respect to x and using $\beta_x = -\frac{1}{\beta}\boldsymbol{\alpha}^T\boldsymbol{\alpha}_x$, we have

$$\begin{aligned} D_x(|\overline{\boldsymbol{v}} + \boldsymbol{v}|) &= \frac{1}{|\overline{\boldsymbol{v}} + \boldsymbol{v}|}(\overline{\boldsymbol{v}}^T + \boldsymbol{v}^T)D_x(\overline{\boldsymbol{v}} + \boldsymbol{v}) = \frac{1}{|\overline{\boldsymbol{v}} + \boldsymbol{v}|}(\beta\overline{\boldsymbol{\alpha}}^T - \overline{\beta}\boldsymbol{\alpha}^T)\left(\frac{\boldsymbol{\alpha}_x}{\beta} - \frac{\overline{\boldsymbol{\alpha}}_x}{\overline{\beta}}\right) \\ &= \frac{2\nu}{|\overline{\boldsymbol{v}} + \boldsymbol{v}|^2}(\beta\overline{\boldsymbol{\alpha}}^T - \overline{\beta}\boldsymbol{\alpha}^T)\,(\overline{\boldsymbol{\alpha}} + \boldsymbol{\alpha}) = \nu(\beta - \overline{\beta}). \end{aligned}$$

Using it and (38), by direct calculation we obtain

$$\begin{aligned} \langle D_x(a) + \langle aU + i\nu\langle aJ &= -\frac{\nu(\overline{\beta} + \beta)}{|\overline{\boldsymbol{v}} + \boldsymbol{v}|}\langle a; \\ D_x(a)\rangle - \overline{U}a\rangle + i\nu Ja\rangle &= \frac{\nu(\overline{\beta} + \beta)}{|\overline{\boldsymbol{v}} + \boldsymbol{v}|}a\rangle. \end{aligned}$$

Substituting it into (41), we see the identity is valid and thus we proved the statement. □

When $n = 1$, we take $\alpha = \sin\theta$ and $\beta = \cos\theta$. The vector sine-Gordon equation (1) reduces to the scalar sine-Gordon equation

$$\theta_{xt} = \sin\theta \tag{42}$$

We show that (34) and (38) give us its the well-known Bäcklund transformation.

In this case, we have $\frac{\overline{\boldsymbol{v}}+\boldsymbol{v}}{|\overline{\boldsymbol{v}}+\boldsymbol{v}|} = \begin{pmatrix} \cos\frac{\overline{\theta}+\theta}{2} \\ \sin\frac{\overline{\theta}+\theta}{2} \end{pmatrix}$. Substituting it into (34) and using trigonometric identities, we obtain

$$\overline{\theta}_t + \theta_t = \frac{2}{\nu}\sin\tfrac{\theta - \overline{\theta}}{2}. \tag{43}$$

Substituting the above α and β into the Bäcklund transformation (38), we get

$$\overline{\theta}_x - \theta_x = -\frac{2\nu}{2\cos\frac{\overline{\theta}-\theta}{2}}(\sin\overline{\theta} + \sin\theta) = -2\nu\sin\tfrac{\overline{\theta}+\theta}{2}. \tag{44}$$

Bäcklund transformations (38), (44) provide us with integrable discretisations of the vector (1) and scalar (42) sine-Gordon equations. There exists another integrable discretisation of these equations, recently published in [21], which has a different nature and is not directly related to the Bäcklund transformations of the vector or scalar sine-Gordon equations.

4 A local symmetry of the Bäcklund transformation

As discussed in Section 2 the Bäcklund transformations (34) and (38) can be viewed as integrable differential–difference equations. In this section, we show how to construct one of their generalised symmetries using the Darboux matrix (35). This symmetry is an integrable evolutionary equation, which belongs to the list of integrable Volterra-type lattice systems on a sphere satisfying the integrability conditions following from the existence of a formal recursion operator [20]. We construct a Lax operator, canonically associated with the Darboux matrix (35) which forms Lax-Darboux representation for this local symmetry.

The Lax operators $\mathcal{A}, \mathcal{L}$ (10) have simple poles at points of the degenerate orbits $\lambda = 0, \infty$ and are invariant with respect to the reduction group generated by the transformations (17)-(19). Starting from the Darboux matrix $M_\nu(\boldsymbol{u})$ (35) we construct a new Lax operator

$$\mathcal{B}_\nu = D_\tau - \mathcal{U}_\nu, \tag{45}$$

which is invariant with respect to the reduction group and matrix U_ν has the same set of poles as the logarithmic Fréchet derivative of the Darboux matrix. Thus, in the case of the Darboux matrix $M_\nu(\boldsymbol{u})$ (35) matrix $\mathcal{U}_\nu$ has the form

$$\mathcal{U}_\nu(\lambda) = \frac{i\nu B}{\lambda - i\nu} - \frac{i\nu QBQ}{\lambda + i\nu} \tag{46}$$

and satisfies

$$\mathcal{U}_\nu(\lambda) = -\mathcal{U}_\nu^T(\lambda), \qquad \mathcal{U}_\nu^*(\lambda^*) = \mathcal{U}_\nu(\lambda), \qquad Q\mathcal{U}_\nu(-\lambda)Q^{-1} = \mathcal{U}_\nu(\lambda). \tag{47}$$

For (46) conditions (47) are equivalent to

$$B = -B^T, \qquad B = QB^*Q. \tag{48}$$

We recall that (see (35))

$$M_\nu(\boldsymbol{u}) = I_{n+2} + \frac{i\nu}{\lambda - i\nu} A - \frac{i\nu}{\lambda + i\nu} QAQ, \qquad A = Qa\rangle\langle a, \quad \langle a = (i, \boldsymbol{u}^T), \quad |\boldsymbol{u}| = 1. \tag{49}$$

The compatibility condition

$$D_\tau M_\nu(\boldsymbol{u}) - \mathcal{S}_\nu(U_\nu)\ M_\nu(\boldsymbol{u}) + M_\nu(\boldsymbol{u})\ U_\nu = 0 \tag{50}$$

for the Lax-Darboux pair

$$D_\tau \Psi = U_\nu \Psi, \qquad \mathcal{S}_\nu \Psi = M_\nu(\boldsymbol{u})\Psi \tag{51}$$

is equivalent to the following two equations

$$B_1 A = AB, \tag{52}$$

$$A_\tau = B_1 - B - \frac{1}{2}\left(B_1 QAQ + QB_1 QA - QAQB - AQBQ\right), \tag{53}$$

where $B_1 = \mathcal{S}_\nu(B)$. Indeed, the left hand side of equation (50) is a rational matrix function of λ. This function is vanishing at $\lambda = \infty$ and conditions (52) and (53) are equivalent to the vanishing of second and first order poles at $\lambda = \pm i\nu$.

It is easy to check that

$$B = \frac{\gamma}{\langle aQa_{-1}\rangle}\left(Qa_{-1}\rangle\langle a - a\rangle\langle a_{-1}Q\right), \qquad a_{-1}\rangle = \mathcal{S}_\nu^{-1}(a)\rangle$$

is a solution of (52) satisfying the conditions (48) if γ is any real function of τ. Without any loss of generality we can set $\gamma = 1$ (by a point transformation). Substituting the expressions of A and B into (53), we get

$$\begin{aligned} &Q\left(a_\tau + \frac{a_{-1}\rangle}{\langle aQa_{-1}\rangle} + \frac{\langle a_1 a\rangle}{2\langle a_1Qa\rangle}Qa\rangle - \frac{a_1\rangle}{\langle a_1Qa\rangle} - \frac{\langle aa_{-1}\rangle}{2\langle aQa_{-1}\rangle}Qa\rangle\right)\langle a \\ &+Qa\rangle\left(\langle a_\tau - \frac{\langle a_1}{\langle a_1Qa\rangle} + \frac{\langle a_1a\rangle}{2\langle a_1Qa\rangle}\langle aQ + \frac{\langle a_{-1}}{\langle aQa_{-1}\rangle} - \frac{\langle aa_{-1}\rangle}{2\langle aQa_{-1}\rangle}\langle aQ\right) = 0. \end{aligned}$$

Hence, there exists a scalar ρ such that

$$\begin{aligned} &a_\tau\rangle + \frac{a_{-1}\rangle}{\langle aQa_{-1}\rangle} + \frac{\langle a_1a\rangle}{2\langle a_1Qa\rangle}Qa\rangle - \frac{a_1\rangle}{\langle a_1Qa\rangle} - \frac{\langle aa_{-1}\rangle}{2\langle aQa_{-1}\rangle}Qa\rangle = \rho a\rangle; \\ &\langle a_\tau - \frac{\langle a_1}{\langle a_1Qa\rangle} + \frac{\langle a_1a\rangle}{2\langle a_1Qa\rangle}\langle aQ + \frac{\langle a_{-1}}{\langle aQa_{-1}\rangle} - \frac{\langle aa_{-1}\rangle}{2\langle aQa_{-1}\rangle}\langle aQ = -\rho\langle a. \end{aligned}$$

From the above two identities it follows that $\rho = 0$ and

$$a_\tau\rangle + \frac{a_{-1}\rangle}{\langle aQa_{-1}\rangle} + \frac{\langle a_1a\rangle}{2\langle a_1Qa\rangle}Qa\rangle - \frac{a_1\rangle}{\langle a_1Qa\rangle} - \frac{\langle aa_{-1}\rangle}{2\langle aQa_{-1}\rangle}Qa\rangle = 0.$$

Due to $\langle a = (i, \boldsymbol{u}^T)$, the above equation becomes into

$$\boldsymbol{u}_\tau + \frac{\boldsymbol{u}_{-1}}{\boldsymbol{u}^T\boldsymbol{u}_{-1} + 1} + \frac{\boldsymbol{u}_1^T\boldsymbol{u} - 1}{2(\boldsymbol{u}_1^T\boldsymbol{u} + 1)}\boldsymbol{u} - \frac{\boldsymbol{u}_1}{\boldsymbol{u}_1^T\boldsymbol{u} + 1} - \frac{\boldsymbol{u}^T\boldsymbol{u}_{-1} - 1}{2(\boldsymbol{u}^T\boldsymbol{u}_{-1} + 1)}\boldsymbol{u} = 0.$$

So the unit vector $\boldsymbol{u}$ satisfies the equation

$$\boldsymbol{u}_\tau = 2(\mathcal{S}_\nu - 1)\frac{\boldsymbol{u}_{-1} + \boldsymbol{u}}{|\boldsymbol{u}^T + \boldsymbol{u}_{-1}|^2}. \tag{54}$$

We sum up the result in the following theorem:

Theorem 2. *The evolutionary vector equation (54) possesses a Lax-Darboux representation (51) with*

$$\begin{aligned} &M_\nu(\boldsymbol{u}) = I_{n+2} + \frac{i\nu}{\lambda - i\nu}Qa\rangle\langle a - \frac{i\nu}{\lambda + i\nu}a\rangle\langle aQ, \quad \langle a = (i, \boldsymbol{u}^T),\ \boldsymbol{u} \in S^n; \\ &U_\nu = \frac{i\nu}{\lambda - i\nu}\frac{Qa_{-1}\rangle\langle a - a\rangle\langle a_{-1}Q}{\langle aQa_{-1}\rangle} - \frac{i\nu}{\lambda + i\nu}\frac{a_{-1}\rangle\langle aQ - Qa\rangle\langle a_{-1}}{\langle aQa_{-1}\rangle}, \end{aligned} \tag{55}$$

where I_{n+2} is the identity matrix of size $n+2$, $Q = diag(-1, 1, \ldots, 1)$ and $\nu \in \mathbb{R}$ is a constant.

Equation (54) is is known as the discrete Heisenberg spin chain [22], whose link with discrete geometry was studied in [23].

It follows from (35) that the unit vector $\boldsymbol{u}$ is related to the unit vector $\boldsymbol{v}$ by

$$\boldsymbol{u} = \frac{\boldsymbol{v}_1 + \boldsymbol{v}}{|\boldsymbol{v}_1 + \boldsymbol{v}|}, \qquad \boldsymbol{v} = \begin{pmatrix} \beta \\ \boldsymbol{\alpha} \end{pmatrix}. \tag{56}$$

In the following proposition, we give the evolutionary equation for the unit vector $\boldsymbol{v}$.

Proposition 5. *Equation (54) and equation*

$$\boldsymbol{v}_\tau = \frac{|\boldsymbol{v}_{-1}+\boldsymbol{v}|^2(\boldsymbol{v}_1+\boldsymbol{v}) - |\boldsymbol{v}_1+\boldsymbol{v}|^2(\boldsymbol{v}_{-1}+\boldsymbol{v})}{(\boldsymbol{v}_1^T+\boldsymbol{v}^T)(\boldsymbol{v}_{-1}+\boldsymbol{v}) + |\boldsymbol{v}_{-1}+\boldsymbol{v}||\boldsymbol{v}_1+\boldsymbol{v}|} \tag{57}$$

are related by the Miura transformation (56).

Proof. Using (56), we can rewrite (57) into

$$\boldsymbol{v}_\tau = \frac{|\boldsymbol{v}_{-1}+\boldsymbol{v}|^2|\boldsymbol{v}_1+\boldsymbol{v}|\boldsymbol{u} - |\boldsymbol{v}_1+\boldsymbol{v}|^2|\boldsymbol{v}_{-1}+\boldsymbol{v}|\boldsymbol{u}_{-1}}{(\boldsymbol{u}^T\boldsymbol{u}_{-1}+1)|\boldsymbol{v}_{-1}+\boldsymbol{v}||\boldsymbol{v}_1+\boldsymbol{v}|} = \frac{|\boldsymbol{v}_{-1}+\boldsymbol{v}|\boldsymbol{u} - |\boldsymbol{v}_1+\boldsymbol{v}|\boldsymbol{u}_{-1}}{\boldsymbol{u}^T\boldsymbol{u}_{-1}+1}.$$

This leads to

$$\frac{\boldsymbol{v}_{1\tau}+\boldsymbol{v}_\tau}{|\boldsymbol{v}_1+\boldsymbol{v}|} = \frac{\boldsymbol{u}_1}{\boldsymbol{u}^T\boldsymbol{u}_1+1} - \frac{|\boldsymbol{v}_2+\boldsymbol{v}_1|\boldsymbol{u}}{(\boldsymbol{u}^T\boldsymbol{u}_1+1)|\boldsymbol{v}_1+\boldsymbol{v}|} + \frac{|\boldsymbol{v}_{-1}+\boldsymbol{v}|\boldsymbol{u}}{(\boldsymbol{u}^T\boldsymbol{u}_{-1}+1)|\boldsymbol{v}_1+\boldsymbol{v}|} - \frac{\boldsymbol{u}_{-1}}{\boldsymbol{u}^T\boldsymbol{u}_{-1}+1}$$

Substituting this into $\boldsymbol{u}_\tau$ and using the fact that $\boldsymbol{u}$ is the unit vector, we have

$$\boldsymbol{u}_\tau = \frac{\boldsymbol{v}_{1\tau}+\boldsymbol{v}_\tau}{|\boldsymbol{v}_1+\boldsymbol{v}|} - \boldsymbol{u}^T\frac{\boldsymbol{v}_{1\tau}+\boldsymbol{v}_\tau}{|\boldsymbol{v}_1+\boldsymbol{v}|}\boldsymbol{u} = \frac{\boldsymbol{u}_1+\boldsymbol{u}}{\boldsymbol{u}^T\boldsymbol{u}_1+1} - \frac{\boldsymbol{u}+\boldsymbol{u}_{-1}}{\boldsymbol{u}^T\boldsymbol{u}_{-1}+1} = 2(\mathcal{S}_\nu - 1)\frac{\boldsymbol{u}_{-1}+\boldsymbol{u}}{|\boldsymbol{u}^T+\boldsymbol{u}_{-1}|^2}$$

and thus we proved the statement. □

Both equations (34) and (57) are obtained from the same Darboux matrix M_ν. Thus they share the same generalised symmetries and conserved densities derived from the zero curvature conditions [24, 25, 10]. Therefore, equation (57) can be viewed as a symmetry of the nonevolutionary equations (34) and (38).

In 2008, Adler presented the classification of isotropic integrable Volterra-type lattices on the sphere [20]. The author gave a list including six integrable equations denoted by V1–V6. Equation (54) is the V6 when $\delta = 1$ in the list. Equation (57) after scaling of τ is the V5 when $\epsilon = 1$ and $k = 2$. Here we established the relation between these two equations.

As pointed out by Adler [20], equation (54) is integrable without the constraint vector $\boldsymbol{u}$ being the unit vector. Introducing the transformation

$$\boldsymbol{w} = \frac{\boldsymbol{u}_{-1}+\boldsymbol{u}}{|\boldsymbol{u}_{-1}+\boldsymbol{u}|^2},$$

we have

$$\boldsymbol{w}_\tau = 2(\boldsymbol{w}^T\boldsymbol{w})(\boldsymbol{w}_1-\boldsymbol{w}_{-1}) - 4\boldsymbol{w}^T(\boldsymbol{w}_1-\boldsymbol{w}_{-1})\boldsymbol{w}, \tag{58}$$

which is one of vector modified Volterra lattices recently discussed in [26].

Equation (58) is well studied in [27], as an example of integrable Jordan triple. It is a Hamiltonian system with the Hamiltonian operator is

$$\mathcal{H}_{\boldsymbol{w}} = (\boldsymbol{w}^T\boldsymbol{w} - 2\boldsymbol{w}\boldsymbol{w}^T)(\mathcal{S}-\mathcal{S}^{-1})(\boldsymbol{w}^T\boldsymbol{w} - 2\boldsymbol{w}\boldsymbol{w}^T).$$

and the Hamiltonian function is $\rho_{\boldsymbol{w}} = -\ln|\boldsymbol{w}|$. The authors also gave its master symmetry

$$\tau_{\boldsymbol{w}} = (n-1)\boldsymbol{w}_\tau + 2(\boldsymbol{w}^T\boldsymbol{w})\boldsymbol{w}_1 - 4(\boldsymbol{w}^T\boldsymbol{w}_1)\boldsymbol{w}$$

to generate infinitely many commuting generalised symmetries and conserved densities starting from $\boldsymbol{w}_\tau$ and $\rho_{\boldsymbol{w}}$.

Using the Miura transformations, we are able to write down the corresponding Hamiltonian operators [4], Hamiltonian functions and master symmetries for equations (54) and (57) as follows:

$$\begin{aligned}
&\mathcal{H}_{\boldsymbol{u}} = (\mathcal{S}-1)(\mathcal{S}+1)^{-1}, \quad \rho_{\boldsymbol{u}} = \ln(|\boldsymbol{u}_1+\boldsymbol{u}|);\\
&\tau_{\boldsymbol{u}} = n\boldsymbol{u}_\tau + \frac{\boldsymbol{u}_1+\boldsymbol{u}}{|\boldsymbol{u}_1+\boldsymbol{u}|^2};\\
&\tau_{\boldsymbol{v}} = n\boldsymbol{v}_\tau; \qquad \rho_{\boldsymbol{v}} = \ln\left(\frac{(\boldsymbol{v}_1^T+\boldsymbol{v}^T)(\boldsymbol{v}_{-1}+\boldsymbol{v})}{|\boldsymbol{v}_{-1}+\boldsymbol{v}||\boldsymbol{v}_1+\boldsymbol{v}|}+1\right),
\end{aligned}$$

where we use the lower-index $\boldsymbol{u}$ and $\boldsymbol{v}$ to indicate their correspondences to equations (54) and (57) respectively. Using the above master symmetry, we are able to compute higher order symmetries for (57) sharing with both (34) and (38). Recently a new method for construction of master symmetries of homogeneous integrable evolution equations (the $\mathcal{O}$-scheme) was proposed in [28]. It would be very useful to extend the $\mathcal{O}$-scheme to the classes of equations studied in this paper.

For the scalar sine-Gordon equation (42), the local flow (57) becomes

$$\theta_\tau = \frac{\cos\frac{\theta_{-1}-\theta}{2}\sin\frac{\theta_1+\theta}{2} - \cos\frac{\theta_1-\theta}{2}\sin\frac{\theta_{-1}+\theta}{2}}{\cos\theta\left(\cos\frac{\theta_1-\theta_{-1}}{2}+1\right)} = \tan\frac{\theta_1-\theta_{-1}}{4},$$

which appeared in [29] as a differential-difference version of the modified Korteweg-de Vries equation.

5 Yang-Baxter map, integrable partial difference systems and the problem of re-factorisation

It is well known that the problem of re-factorisation of a product of Darboux matrices can be associated with the construction of Yang-Baxter maps [30, 31, 9]. A Darboux (or discrete Lax) representation for integrable partial difference equations can also be seen as the same problem of re-factorisation.

5.1 The Yang-Baxter map associated with the vector sine-Gordon equation.

Let us consider a product of two Darboux matrices and the problem of re-factorisation

$$M_\nu(\mathbf{x})M_\mu(\mathbf{y}) = M_\mu(\mathbf{Y})M_\nu(\mathbf{X}), \qquad \mu\neq\pm\nu, \quad \mathbf{x},\mathbf{y},\mathbf{X},\mathbf{Y}\in S^n, \tag{59}$$

where $M_\nu(\boldsymbol{u})$ is a Darboux matrix (55) defined in Theorem 2. On the left hand side of (59) the first and second factors are a matrices with poles at $\pm\nu$ and $\pm\mu$ respectively. On the right hand side the order of the factors is opposite. If this problem of a re-factorisation has a unique solution, namely for given $\mathbf{x},\mathbf{y}$ one can find vectors $\mathbf{X},\mathbf{Y}$ uniquely, then it defines a map $Y_{\nu,\mu}: S^n\times S^n\mapsto S^n\times S^n$,

$$Y_{\nu,\mu}:(\mathbf{x},\mathbf{y})\mapsto(\mathbf{X}(\mathbf{x},\mathbf{y};\nu,\mu),\mathbf{Y}(\mathbf{x},\mathbf{y};\nu,\mu)), \qquad \mathbf{x},\mathbf{y}\in S^n. \tag{60}$$

Starting from the map $Y_{\nu,\mu}$ we define maps $Y^{ij}_{\nu,\mu} : S^n \times S^n \times S^n \mapsto S^n \times S^n \times S^n,\ i < j$ as follows:

$$\begin{aligned} Y^{12}_{\nu,\mu} &: (\mathbf{x},\mathbf{y},\mathbf{z}) &\mapsto& (\mathbf{X}(\mathbf{x},\mathbf{y};\nu,\mu),\mathbf{Y}(\mathbf{x},\mathbf{y};\nu,\mu),\mathbf{z}),\\ Y^{13}_{\nu,\kappa} &: (\mathbf{x},\mathbf{y},\mathbf{z}) &\mapsto& (\mathbf{X}(\mathbf{x},\mathbf{z};\nu,\kappa),\mathbf{y},\mathbf{Y}(\mathbf{x},\mathbf{z};\nu,\kappa)),\\ Y^{23}_{\mu,\kappa} &: (\mathbf{x},\mathbf{y},\mathbf{z}) &\mapsto& (\mathbf{x},\mathbf{X}(\mathbf{y},\mathbf{z};\mu,\kappa),\mathbf{Y}(\mathbf{y},\mathbf{z};\mu,\kappa)). \end{aligned}$$

Definition 1. *The map $Y_{\nu,\mu}$ is called a Yang-Baxter map, if it satisfies the Yang-Baxter relation*

$$Y^{12}_{\nu,\mu} \circ Y^{13}_{\nu,\kappa} \circ Y^{23}_{\mu,\kappa} = Y^{23}_{\mu,\kappa} \circ Y^{13}_{\nu,\kappa} \circ Y^{12}_{\nu,\mu}. \tag{61}$$

It is easy to show that the map (60) associated with the problem of re-factorisation (59) satisfies the Yang-Baxter relation if the factorisation of the triple product $M_\nu(\mathbf{x})M_\mu(\mathbf{y})M_\kappa(\mathbf{z})$ is unique [32, 31].

Theorem 3. *Suppose that*

(i) *the re-factorisation problem (59) defines a unique map $Y_{\nu,\mu}$;*

(ii) *for three different real numbers μ,ν,κ satisfying $\mu \neq -\nu$, $\mu \neq -\kappa$ and $\kappa \neq -\nu$ it follows from*

$$M_\nu(\mathbf{x})M_\mu(\mathbf{y})M_\kappa(\mathbf{z}) = M_\nu(\mathbf{X})M_\mu(\mathbf{Y})M_\kappa(\mathbf{Z})$$

that $\mathbf{x} = \mathbf{X}, \quad \mathbf{y} = \mathbf{Y}, \quad \mathbf{z} = \mathbf{Z}$, that is, the factorisation is unique.

Then the corresponding map $Y_{\nu,\mu}$ is Yang-Baxter.

Proof. For any $\mathbf{x},\mathbf{y},\mathbf{z} \in S^n$ it follows from (59) that

$$\begin{array}{ccccccc} M_\nu(\mathbf{x})M_\mu(\mathbf{y})M_\kappa(\mathbf{z}) & = & M_\mu(\mathbf{Y})M_\nu(\mathbf{X})M_\kappa(\mathbf{z}) & = & M_\mu(\mathbf{Y})M_\kappa(\mathbf{Z})M_\mu(\tilde{\mathbf{X}}) & = & M_\kappa(\tilde{\mathbf{Z}})M_\mu(\tilde{\mathbf{Y}})M_\nu(\tilde{\mathbf{X}}) \\ (\mathbf{x},\mathbf{y},\mathbf{z}) & \overset{Y^{12}_{\nu,\mu}}{\longmapsto} & (\mathbf{X},\mathbf{Y},\mathbf{z}) & \overset{Y^{13}_{\nu,\kappa}}{\longmapsto} & (\tilde{\mathbf{X}},\mathbf{Y},\mathbf{Z}) & \overset{Y^{23}_{\mu,\kappa}}{\longmapsto} & (\tilde{\mathbf{X}},\tilde{\mathbf{Y}},\tilde{\mathbf{Z}}) \end{array}$$

Swapping the matrices in a different order we get

$$\begin{array}{ccccccc} M_\nu(\mathbf{x})M_\mu(\mathbf{y})M_\kappa(\mathbf{z}) & = & M_\nu(\mathbf{x})M_\kappa(\mathbf{Z})M_\mu(\mathbf{Y}) & = & M_\kappa(\hat{\mathbf{Z}})M_\nu(\mathbf{X})M_\mu(\mathbf{Y}) & = & M_\kappa(\hat{\mathbf{Z}})M_\mu(\hat{\mathbf{Y}})M_\nu(\hat{\mathbf{X}}) \\ (\mathbf{x},\mathbf{y},\mathbf{z}) & \overset{Y^{23}_{\mu,\kappa}}{\longmapsto} & (\mathbf{x},\mathbf{Y},\mathbf{Z}) & \overset{Y^{13}_{\nu,\kappa}}{\longmapsto} & (\mathbf{X},\mathbf{Y},\hat{\mathbf{Z}}) & \overset{Y^{12}_{\nu,\mu}}{\longmapsto} & (\hat{\mathbf{X}},\hat{\mathbf{Y}},\hat{\mathbf{Z}}) \end{array}$$

Thus we have

$$Y^{23}_{\mu,\kappa} \circ Y^{13}_{\nu,\kappa} \circ Y^{12}_{\nu,\mu} : (\mathbf{x},\mathbf{y},\mathbf{z}) \mapsto (\tilde{\mathbf{X}},\tilde{\mathbf{Y}},\tilde{\mathbf{Z}}), \quad Y^{12}_{\nu,\mu} \circ Y^{13}_{\nu,\kappa} \circ Y^{23}_{\mu,\kappa} : (\mathbf{x},\mathbf{y},\mathbf{z}) \mapsto (\hat{\mathbf{X}},\hat{\mathbf{Y}},\hat{\mathbf{Z}}).$$

The uniqueness of the factorisation that $(\hat{\mathbf{X}},\hat{\mathbf{Y}},\hat{\mathbf{Z}}) = (\tilde{\mathbf{X}},\tilde{\mathbf{Y}},\tilde{\mathbf{Z}})$ leads to the required Yang-Baxter relation (61) and thus we complete the proof. □

Let us compute the Yang-Baxter map (60) corresponding to the Darboux matrix (55).

Theorem 4. *The Yang-Baxter map (60) corresponding to the Darboux matrix (55) is of the form*

$$\mathbf{X}(\mathbf{x},\mathbf{y};\nu,\mu) = \frac{(\nu^2-\mu^2)\mathbf{x} + 2\mu(\nu+\mu(\mathbf{x}^T\mathbf{y}))\mathbf{y}}{\nu^2+\mu^2+2\mu\nu(\mathbf{x}^T\mathbf{y})}, \tag{62}$$

$$\mathbf{Y}(\mathbf{x},\mathbf{y};\nu,\mu) = \frac{(\mu^2-\nu^2)\mathbf{y} + 2\nu(\mu+\nu(\mathbf{x}^T\mathbf{y}))\mathbf{x}}{\nu^2+\mu^2+2\mu\nu(\mathbf{x}^T\mathbf{y})}, \tag{63}$$

where $\mu \neq \pm\nu$ are constant and $\mathbf{x},\mathbf{y},\mathbf{X},\mathbf{Y} \in S^n$. The map is involutive $Y_{\nu,\mu} \circ Y_{\nu,\mu} = \mathrm{id}$ and it has the invariant $\mathbf{x}^T\mathbf{y} = \mathbf{X}^T\mathbf{Y}$.

Proof. We derive the map using the re-factorisation of two Darboux matrices (59) and adapting the notations

$$a\rangle = \begin{pmatrix} i \\ \mathbf{X} \end{pmatrix}, \quad b\rangle = \begin{pmatrix} i \\ \mathbf{Y} \end{pmatrix}, \quad \tilde{a}\rangle = \begin{pmatrix} i \\ \mathbf{x} \end{pmatrix}, \quad \tilde{b}\rangle = \begin{pmatrix} i \\ \mathbf{y} \end{pmatrix}.$$

At $\lambda = \infty$ equation (59) is satisfied. Taking the residue of (59) at $\lambda = i\nu$ we get

$$Q\tilde{a}\rangle\langle\tilde{a}\left(I + \frac{\mu}{\nu-\mu}Q\tilde{b}\rangle\langle\tilde{b} - \frac{\mu}{\nu+\mu}\tilde{b}\rangle\langle\tilde{b}Q\right) = \left(I + \frac{\mu}{\nu-\mu}Qb\rangle\langle b - \frac{\mu}{\nu+\mu}b\rangle\langle bQ\right)Qa\rangle\langle a, \tag{64}$$

which implies

$$Q\tilde{a}\rangle = \left(I + \frac{\mu}{\nu-\mu}Qb\rangle\langle b - \frac{\mu}{\nu+\mu}b\rangle\langle bQ\right)Qa\rangle\Gamma \tag{65}$$

$$\langle a = \Gamma\langle\tilde{a}\left(I + \frac{\mu}{\nu-\mu}Q\tilde{b}\rangle\langle\tilde{b} - \frac{\mu}{\nu+\mu}\tilde{b}\rangle\langle\tilde{b}Q\right) \tag{66}$$

for some scalar $\Gamma \neq 0$. Taking the first entry in (66) we find that

$$\Gamma = \frac{\nu^2 - \mu^2}{\nu^2 + \mu^2 + 2\mu\nu(\mathbf{x}^T\mathbf{y})}. \tag{67}$$

Substitution of (67) in (66) leads to (62). Similarly, the residue of (59) at $\lambda = i\mu$ yields (63).

By direct calculation, we can check that $\mathbf{x}^T\mathbf{y} = \mathbf{X}^T\mathbf{Y}$ using the fact $\mathbf{x}, \mathbf{y} \in S^n$. Indeed,

$$\mathbf{X}^T\mathbf{Y} = \frac{((\nu^2-\mu^2)^2 + 4\mu\nu(\nu + \mu(\mathbf{x}^T\mathbf{y}))(\mu + \nu(\mathbf{x}^T\mathbf{y})))(\mathbf{x}^T\mathbf{y})}{(\nu^2 + \mu^2 + 2\mu\nu(\mathbf{x}^T\mathbf{y}))^2} = \mathbf{x}^T\mathbf{y}.$$

To prove the involutivity of the map, we simply write the map into the matrix form

$$(\mathbf{X}, \mathbf{Y}) = (\mathbf{x}, \mathbf{y})\, P(\mathbf{x}, \mathbf{y}), \tag{68}$$

where

$$P(\mathbf{x}, \mathbf{y}) = \frac{1}{\nu^2 + \mu^2 + 2\mu\nu(\mathbf{x}^T\mathbf{y})}\begin{pmatrix} (\nu^2 - \mu^2) & 2\mu\nu + 2\nu^2(\mathbf{x}^T\mathbf{y}) \\ 2\nu\mu + 2\mu^2(\mathbf{x}^T\mathbf{y}) & \mu^2 - \nu^2 \end{pmatrix}$$

Notice that $P(\mathbf{x}, \mathbf{y}) = P(\mathbf{X}, \mathbf{Y})$ due to the invariance $\mathbf{x}^T\mathbf{y} = \mathbf{X}^T\mathbf{Y}$. It can be easily checked that $P^2(\mathbf{x}, \mathbf{y}) = I_2$, which immediately leads to the involutivity of the map.

It follows from (68) that

$$(\mathbf{x}, \mathbf{y}) = (\mathbf{X}, \mathbf{Y})\, P(\mathbf{X}, \mathbf{Y}), \tag{69}$$

which compatible to identities (65) and the one obtained from the residue of (59) at $\lambda = i\nu$ and $\lambda = i\mu$, respectively. Substitution of either (68) or (69) in (64) leads to the identity.

Finally, we show that the factorisation is unique. We know that the Darboux matrix is invertible and $(M_\nu(\mathbf{x}))^{-1} = M_\nu^T(\mathbf{x})$. Thus if $M_\nu(\mathbf{x})M_\mu(\mathbf{y})M_\kappa(\mathbf{z}) = M_\nu(\mathbf{X})M_\mu(\mathbf{Y})M_\kappa(\mathbf{Z})$, then

$$M_\mu(\mathbf{y})M_\kappa(\mathbf{z}) = M_\nu^T(\mathbf{x})M_\nu(\mathbf{X})M_\mu(\mathbf{Y})M_\kappa(\mathbf{Z}).$$

The left hand side is regular at $\lambda = i\nu$. The second order pole at $\lambda = i\nu$ in the right hand side vanishes only if $\mathbf{x}^T\mathbf{X} = 1$. Since both $\mathbf{x}$ and $\mathbf{X}$ are unit vectors, we have $\mathbf{x} = \mathbf{X}$. Then the right hand side of the equation is regular at $\lambda = i\nu$. In the same way, we can prove that $\mathbf{y} = \mathbf{Y}$ and $\mathbf{z} = \mathbf{Z}$. According to Theorem 3, the map defined by (62) and (63) is a Yang-Baxter map. □

Proposition 6. *Yang-Baxter map (62), (63) has a linear vector invariant*

$$\nu \mathbf{x} + \mu \mathbf{y} = \nu \mathbf{X} + \mu \mathbf{Y}. \tag{70}$$

Proof. The identity can be proved by a direct substitution of (62) and (63) in (70). Another way to prove this statement is to consider the Laurent expansion of equation (59) in λ at infinity. Then (70) follows immediately from the coefficients at λ^{-1}. □

5.2 Integrable partial difference system

Let us consider two Darboux transformations for the vector sine-Gordon system (1) corresponding to Darboux matrices $M_\nu(\mathbf{x})$ and $M_\mu(\mathbf{Y})$ with distinct positions of the poles $\mu \neq \pm\nu$. According to Theorem 1 the vectors $\mathbf{X}, \mathbf{y}$ can be expressed in terms of the original variables $\boldsymbol{v} = (\beta, \boldsymbol{\alpha}^T)^T$ as

$$\mathbf{X} = \frac{\mathcal{S}_\nu(\boldsymbol{v}) + \boldsymbol{v}}{|\mathcal{S}_\nu(\boldsymbol{v}) + \boldsymbol{v}|}, \qquad \mathbf{y} = \frac{\mathcal{S}_\mu(\boldsymbol{v}) + \boldsymbol{v}}{|\mathcal{S}_\mu(\boldsymbol{v}) + \boldsymbol{v}|}, \tag{71}$$

where $\mathcal{S}_\nu$ and $\mathcal{S}_\mu$ are the corresponding shift automorphisms. The Bianchi permutability condition for these two Darboux transformations (16) has the form

$$M_\nu(\mathcal{S}_\mu(\mathbf{X}))M_\mu(\mathbf{y}) = M_\mu(\mathcal{S}_\nu(\mathbf{y}))M_\nu(\mathbf{X}), \tag{72}$$

which coincides with (59) where

$$\mathbf{x} = \mathcal{S}_\mu(\mathbf{X}) = \frac{\boldsymbol{v}_{1,1} + \boldsymbol{v}_{0,1}}{|\boldsymbol{v}_{1,1} + \boldsymbol{v}_{0,1}|}, \quad \mathbf{Y} = \mathcal{S}_\nu(\mathbf{y}) = \frac{\boldsymbol{v}_{1,1} + \boldsymbol{v}_{1,0}}{|\boldsymbol{v}_{1,1} + \boldsymbol{v}_{1,0}|}. \tag{73}$$

Here we used the notation $\mathcal{S}_\nu^n \mathcal{S}_\mu^m \boldsymbol{v} = \boldsymbol{v}_{n,m}$ introduced in Section 2 and convention $\boldsymbol{v}_{0,0} = \boldsymbol{v}$. Having made identifications (71),(73) and using Theorem 4 we can show that the Bianchi permutability condition (72) is equivalent to a single quadrilateral equation for variables $\boldsymbol{v}_{i,j} \in S^n$ on the two dimensional lattice $\mathbb{Z}^2$. This equation can be written in a few equivalent forms.

Theorem 5. *Let $\boldsymbol{v} \in S^n$, $\nu \neq \mu \in \mathbb{R}$ and*

$$\mathbf{f} = \frac{(\nu^2 - \mu^2)\mathbf{x} + 2\mu(\nu + \mu(\mathbf{x}^T\mathbf{y}))\mathbf{y}}{\nu^2 + \mu^2 + 2\mu\nu(\mathbf{x}^T\mathbf{y})}, \quad \mathbf{F} = \frac{(\nu^2 - \mu^2)\mathbf{X} + 2\mu(\nu + \mu(\mathbf{X}^T\mathbf{Y}))\mathbf{Y}}{\nu^2 + \mu^2 + 2\mu\nu(\mathbf{X}^T\mathbf{Y})}$$

$$\mathbf{g} = \frac{(\mu^2 - \nu^2)\mathbf{y} + 2\nu(\mu + \nu(\mathbf{x}^T\mathbf{y}))\mathbf{x}}{\nu^2 + \mu^2 + 2\mu\nu(\mathbf{x}^T\mathbf{y})}, \quad \mathbf{G} = \frac{(\mu^2 - \nu^2)\mathbf{Y} + 2\nu(\mu + \nu(\mathbf{X}^T\mathbf{Y}))\mathbf{X}}{\nu^2 + \mu^2 + 2\mu\nu(\mathbf{X}^T\mathbf{Y})}$$

where

$$\mathbf{x} = \frac{\boldsymbol{v}_{1,1} + \boldsymbol{v}_{0,1}}{|\boldsymbol{v}_{1,1} + \boldsymbol{v}_{0,1}|}, \qquad \mathbf{y} = \frac{\boldsymbol{v}_{0,1} + \boldsymbol{v}}{|\boldsymbol{v}_{0,1} + \boldsymbol{v}|} \qquad \mathbf{X} = \frac{\boldsymbol{v}_{1,0} + \boldsymbol{v}}{|\boldsymbol{v}_{1,0} + \boldsymbol{v}|}, \qquad \mathbf{Y} = \frac{\boldsymbol{v}_{1,1} + \boldsymbol{v}_{1,0}}{|\boldsymbol{v}_{1,1} + \boldsymbol{v}_{1,0}|}. \tag{74}$$

Then the following equations are equivalent

$$\text{(a)} \quad \frac{\boldsymbol{v}_{1,0} + \boldsymbol{v}}{|\boldsymbol{v}_{1,0} + \boldsymbol{v}|} = \mathbf{f}, \qquad \text{(b)} \quad \frac{\boldsymbol{v}_{1,1} + \boldsymbol{v}_{1,0}}{|\boldsymbol{v}_{1,1} + \boldsymbol{v}_{1,0}|} = \mathbf{g},$$

$$\text{(c)} \quad \frac{\boldsymbol{v}_{0,1} + \boldsymbol{v}}{|\boldsymbol{v}_{0,1} + \boldsymbol{v}|} = \mathbf{G}, \qquad \text{(d)} \quad \frac{\boldsymbol{v}_{1,1} + \boldsymbol{v}_{0,1}}{|\boldsymbol{v}_{1,1} + \boldsymbol{v}_{0,1}|} = \mathbf{F},$$

$$\text{(e)} \quad \boldsymbol{v}_{1,0} = -\boldsymbol{v} + 2\mathbf{f}(\mathbf{f}^T\boldsymbol{v}), \qquad \text{(f)} \quad \boldsymbol{v}_{1,0} = -\boldsymbol{v}_{1,1} + 2\mathbf{g}(\mathbf{g}^T\boldsymbol{v}_{1,1}),$$

$$\text{(g)} \quad \boldsymbol{v}_{0,1} = -\boldsymbol{v} + 2\mathbf{G}(\mathbf{G}^T\boldsymbol{v}), \qquad \text{(h)} \quad \boldsymbol{v}_{0,1} = -\boldsymbol{v}_{1,1} + 2\mathbf{F}(\mathbf{F}^T\boldsymbol{v}_{1,1})$$

and each of them is equivalent to the Bianchi permutability condition (72).

Proof. Let us show that equation (a) is equivalent to (e): $\boldsymbol{v}_{1,0} = -\boldsymbol{v} + 2\mathbf{f}(\mathbf{f}^T\boldsymbol{v})$. Indeed, it follows from (a) that

$$2\mathbf{f}^T\boldsymbol{v} = 2\frac{\boldsymbol{v}_{1,0}^T + \boldsymbol{v}^T}{|\boldsymbol{v}_{1,0} + \boldsymbol{v}|}\boldsymbol{v} = 2\frac{\boldsymbol{v}_{1,0}^T\boldsymbol{v} + 1}{|\boldsymbol{v}_{1,0} + \boldsymbol{v}|} = |\boldsymbol{v}_{1,0} + \boldsymbol{v}|$$

and in particular $\mathbf{f}^T\boldsymbol{v} > 0$ and thus (a)⇒(e). Since $\mathbf{f}^T\boldsymbol{v} > 0$ we have (e)⇒(a):

$$\frac{\boldsymbol{v}_{1,0} + \boldsymbol{v}}{|\boldsymbol{v}_{1,0} + \boldsymbol{v}|} = \frac{2\mathbf{f}(\mathbf{f}^T\boldsymbol{v})}{|2\mathbf{f}(\mathbf{f}^T\boldsymbol{v})|} = \mathbf{f}.$$

In a similar way one can show that (b)⟺(f), (c)⟺(g) and (d)⟺(h). To demonstrate (e)⟺(f) we show that

$$-\boldsymbol{v} + 2\mathbf{f}(\mathbf{f}^T\boldsymbol{v}) = -\boldsymbol{v}_{1,1} + 2\mathbf{g}(\mathbf{g}^T\boldsymbol{v}_{1,1}) \tag{75}$$

by a direct computation. Let $|\boldsymbol{v}_{1,1} + \boldsymbol{v}_{0,1}| = x_s$ and $|\boldsymbol{v}_{0,1} + \boldsymbol{v}| = y_s$. Then

$$x_s^2 = 2(\boldsymbol{v}_{1,1}^T\boldsymbol{v}_{0,1} + 1), \quad y_s^2 = 2(\boldsymbol{v}_{0,1}^T\boldsymbol{v} + 1), \quad \mathbf{x}^T\mathbf{y} = x_s^{-1}y_s^{-1}(\boldsymbol{v}_{1,1}^T\boldsymbol{v}_{0,1} + \boldsymbol{v}_{0,1}^T\boldsymbol{v} + \boldsymbol{v}_{1,1}^T\boldsymbol{v} + 1), \tag{76}$$

which implies

$$(\boldsymbol{v}_{0,1}^T + \boldsymbol{v}^T)\boldsymbol{v}_{1,1} = x_s y_s \mathbf{x}^T\mathbf{y} - \frac{1}{2}y_s^2, \quad (\boldsymbol{v}_{1,1}^T + \boldsymbol{v}_{0,1}^T)\boldsymbol{v} = x_s y_s \mathbf{x}^T\mathbf{y} - \frac{1}{2}x_s^2. \tag{77}$$

Using (76), (77) we get

$$\begin{aligned}
&(\nu^2 + \mu^2 + 2\mu\nu(\mathbf{x}^T\mathbf{y}))^2(\mathbf{f}(\mathbf{f}^T\boldsymbol{v}) - \mathbf{g}(\mathbf{g}^T\boldsymbol{v}_{1,1}))\\
&\quad = \left((\nu^2 - \mu^2)(\boldsymbol{v}_{1,1}^T + \boldsymbol{v}_{0,1}^T)\boldsymbol{v}x_s^{-1} + \mu(\nu + \mu(\mathbf{x}^T\mathbf{y}))y_s\right)(\nu^2 - \mu^2)x_s^{-1}(\boldsymbol{v}_{1,1} + \boldsymbol{v}_{0,1})\\
&\quad -2\left((\mu^2 - \nu^2)(\boldsymbol{v}_{0,1}^T + \boldsymbol{v}^T)\boldsymbol{v}_{1,1}y_s^{-1} + \nu(\mu + \nu(\mathbf{x}^T\mathbf{y}))x_s\right)\nu(\mu + \nu(\mathbf{x}^T\mathbf{y}))x_s^{-1}(\boldsymbol{v}_{1,1} + \boldsymbol{v}_{0,1})\\
&\quad +2\left((\nu^2 - \mu^2)(\boldsymbol{v}_{1,1}^T + \boldsymbol{v}_{0,1}^T)\boldsymbol{v}x_s^{-1} + \mu(\nu + \mu(\mathbf{x}^T\mathbf{y}))y_s\right)\mu(\nu + \mu(\mathbf{x}^T\mathbf{y}))y_s^{-1}(\boldsymbol{v}_{0,1} + \boldsymbol{v})\\
&\quad -\left((\mu^2 - \nu^2)(\boldsymbol{v}_{0,1}^T + \boldsymbol{v}^T)\boldsymbol{v}_{1,1}y_s^{-1} + \nu(\mu + \nu(\mathbf{x}^T\mathbf{y}))x_s\right)\mu^2 - \nu^2)y_s^{-1}(\boldsymbol{v}_{0,1} + \boldsymbol{v})\\
&\quad = \frac{1}{2}(\nu^2 + \mu^2 + 2\mu\nu(\mathbf{x}^T\mathbf{y}))^2(\boldsymbol{v} - \boldsymbol{v}_{1,1}).
\end{aligned}$$

which leads to the required identity (75). Equation (59) is equivalent to (62), (63), which after the identification (71), (73) implies that the Bianchi permutability condition (72) is equivalent to equations (a) and (b). Thus (72) ⟺(a)⟺(b)⟺(e)⟺(f). In the same way we can show that equation (69) ⟺(c)⟺(d)⟺(g)⟺(h). Observation that (72)⟺(68)⟺(69) completes the proof. □

Thus, the condition (72) is equivalent to a quadrilateral equation (one take any equation from the list (a)-(h), depending on the problem) which is a new vector isotropic integrable system with dependent variable $\boldsymbol{v} \in S^n$ on a sphere. It is natural to refer this system as *discrete vector sine-Gordon* equation. Let us take equation (e)

$$\boldsymbol{v}_{1,0} = -\boldsymbol{v} + 2\mathbf{f}(\mathbf{f}^T\boldsymbol{v}) \tag{78}$$

as a representative. The Bianchi permutability condition (72) for two Darboux matrices play the role of a Darboux (or discrete Lax) representation for (78).

Using identification (71), (73) we can recast the vector invariant (70) of the Yang-Baxter map (Proposition 6) in a local conservation law for (78), that is,

$$(\mathcal{S}_\mu - 1)\nu\frac{\boldsymbol{v}_{1,0}+\boldsymbol{v}}{|\boldsymbol{v}_{1,0}+\boldsymbol{v}|} = (\mathcal{S}_\nu - 1)\mu\frac{\boldsymbol{v}_{0,1}+\boldsymbol{v}}{|\boldsymbol{v}_{0,1}+\boldsymbol{v}|}. \tag{79}$$

System (78) possess an infinite hierarchy of local conservation laws, which can be found using formal diagonalisation [25] of the Darboux matrices or using the master symmetry (59). Notice that for the equation (78) the unit vector $\boldsymbol{v}$ satisfies the identity

$$\frac{(\boldsymbol{v}_{1,1}^T+\boldsymbol{v}_{0,1}^T)(\boldsymbol{v}+\boldsymbol{v}_{0,1})}{|\boldsymbol{v}_{1,1}+\boldsymbol{v}_{0,1}||\boldsymbol{v}+\boldsymbol{v}_{0,1}|} = \frac{(\boldsymbol{v}_{1,1}^T+\boldsymbol{v}_{1,0}^T)(\boldsymbol{v}+\boldsymbol{v}_{1,0})}{|\boldsymbol{v}_{1,1}+\boldsymbol{v}_{1,0}||\boldsymbol{v}+\boldsymbol{v}_{1,0}|},$$

which follows from the invariant $\mathbf{x}^T\mathbf{y} = \mathbf{X}^T\mathbf{Y}$ (Theorem 4).

For the discrete vector sine-Gordon equation there is a well posed initial value problem with initial data given on the staircase

$$\{\boldsymbol{v}_{k,k}, \boldsymbol{v}_{k,k+1} \,|\, k\in\mathbb{Z},\ \boldsymbol{v}_{k,k}+\boldsymbol{v}_{k,k+1}\neq 0,\ \ |\boldsymbol{v}_{k,k}| = |\boldsymbol{v}_{k,k+1}| = 1\}.$$

To find the values $\boldsymbol{v}_{i,j}$ below the staircase ($i > j$) one can use equation (e) or (f), for values above the staircase ($j > i+1$) it is convenient to use equation (g) or (h) and their shifts (by $\mathcal{S}_\nu$ and $\mathcal{S}_\mu$). Thus the elimination map [33], which is a useful tool for study symmetries, conservation laws and other structures associated with this integrable system can be correctly defined.

The local symmetry (57) of the Bäcklund transformation (34) and (38) results in two symmetries

$$\begin{aligned} D_{\tau_\nu}\boldsymbol{v} &= \frac{|\boldsymbol{v}_{-1,0}+\boldsymbol{v}|^2(\boldsymbol{v}_{1,0}+\boldsymbol{v}) - |\boldsymbol{v}_{1,0}+\boldsymbol{v}|^2(\boldsymbol{v}_{-1,0}+\boldsymbol{v})}{(\boldsymbol{v}_{1,0}^T+\boldsymbol{v}^T)(\boldsymbol{v}_{-1,0}+\boldsymbol{v}) + |\boldsymbol{v}_{-1,0}+\boldsymbol{v}||\boldsymbol{v}_{1,0}+\boldsymbol{v}|}, \\ D_{\tau_\mu}\boldsymbol{v} &= \frac{|\boldsymbol{v}_{0,-1}+\boldsymbol{v}|^2(\boldsymbol{v}_{0,1}+\boldsymbol{v}) - |\boldsymbol{v}_{0,1}+\boldsymbol{v}|^2(\boldsymbol{v}_{0,-1}+\boldsymbol{v})}{(\boldsymbol{v}_{0,1}^T+\boldsymbol{v}^T)(\boldsymbol{v}_{0,-1}+\boldsymbol{v}) + |\boldsymbol{v}_{0,-1}+\boldsymbol{v}||\boldsymbol{v}_{0,1}+\boldsymbol{v}|} \end{aligned}$$

of the discrete vector sine-Gordon equation.

Bäcklund transformations (34) and (38) give the non-local symmetries of the discrete vector sine-Gordon equation as follows:

$$D_t\left(\frac{\boldsymbol{v}_{1,0}+\boldsymbol{v}}{|\boldsymbol{v}_{1,0}+\boldsymbol{v}|}\right) = \tfrac{1}{2\nu}(\boldsymbol{v}-\boldsymbol{v}_{1,0});\quad D_t\left(\frac{\boldsymbol{v}_{0,1}+\boldsymbol{v}}{|\boldsymbol{v}_{0,1}+\boldsymbol{v}|}\right) = \tfrac{1}{2\mu}(\boldsymbol{v}-\boldsymbol{v}_{0,1}); \tag{80}$$

$$(\mathcal{S}_\nu - 1)\frac{D_x\boldsymbol{\alpha}}{\beta} = -2\nu\frac{\boldsymbol{\alpha}_{1,0}+\boldsymbol{\alpha}}{|\boldsymbol{v}_{1,0}+\boldsymbol{v}|};\quad (\mathcal{S}_\mu - 1)\frac{D_x\boldsymbol{\alpha}}{\beta} = -2\mu\frac{\boldsymbol{\alpha}_{0,1}+\boldsymbol{\alpha}}{|\boldsymbol{v}_{0,1}+\boldsymbol{v}|}. \tag{81}$$

In the case $n = 1$, corresponding to the scalar sine-Gordon equation (42), we have

$$\frac{\boldsymbol{v}_{1,0}+\boldsymbol{v}}{|\boldsymbol{v}_{1,0}+\boldsymbol{v}|} = \begin{pmatrix} \cos\frac{\theta_{1,0}+\theta}{2} \\ \sin\frac{\theta_{1,0}+\theta}{2} \end{pmatrix};\qquad \mathbf{x} = \begin{pmatrix} \cos\frac{\theta_{1,1}+\theta_{0,1}}{2} \\ \sin\frac{\theta_{1,1}+\theta_{0,1}}{2} \end{pmatrix},\qquad \mathbf{y} = \begin{pmatrix} \cos\frac{\theta_{0,1}+\theta}{2} \\ \sin\frac{\theta_{0,1}+\theta}{2} \end{pmatrix}.$$

Substituting them into (78), we get

$$\left\{\begin{aligned} \sin\tfrac{\theta_{1,0}+\theta}{2} &= \frac{(\nu^2-\mu^2)\sin\frac{\theta_{1,1}+\theta_{0,1}}{2} + 2\mu(\nu+\mu\cos\frac{\theta_{1,1}-\theta}{2})\sin\frac{\theta_{0,1}+\theta}{2}}{\nu^2+\mu^2+2\mu\nu\cos\frac{\theta_{1,1}-\theta}{2}} \\ \cos\tfrac{\theta_{1,0}+\theta}{2} &= \frac{(\nu^2-\mu^2)\cos\frac{\theta_{1,1}+\theta_{0,1}}{2} + 2\mu(\nu+\mu\cos\frac{\theta_{1,1}-\theta}{2})\cos\frac{\theta_{0,1}+\theta}{2}}{\nu^2+\mu^2+2\mu\nu\cos\frac{\theta_{1,1}-\theta}{2}} \end{aligned}\right.$$

Using trigonometric identities, we can rewrite it as

$$\begin{cases}\Big(\nu\cos\frac{\theta_{1,1}+\theta_{0,1}+\theta_{1,0}+\theta}{4}+\mu\cos\frac{\theta_{0,1}+\theta_{1,0}+3\theta-\theta_{1,1}}{4}\Big)\Big(\nu\sin\frac{\theta_{1,1}+\theta_{0,1}-\theta_{1,0}-\theta}{4}+\mu\sin\frac{\theta_{0,1}+\theta-\theta_{1,1}-\theta_{1,0}}{4}\Big)=0;\\ \Big(\nu\sin\frac{\theta_{1,1}+\theta_{0,1}+\theta_{1,0}+\theta}{4}+\mu\sin\frac{\theta_{0,1}+\theta_{1,0}+3\theta-\theta_{1,1}}{4}\Big)\Big(\nu\sin\frac{\theta_{1,1}+\theta_{0,1}-\theta_{1,0}-\theta}{4}+\mu\sin\frac{\theta_{0,1}+\theta-\theta_{1,1}-\theta_{1,0}}{4}\Big)=0,\end{cases}$$

Since $\nu \neq \pm\mu$, it follows that

$$\nu\sin\frac{\theta_{1,1}+\theta_{0,1}-\theta_{1,0}-\theta}{4}=\mu\sin\frac{\theta_{1,1}+\theta_{1,0}-\theta_{0,1}-\theta}{4},$$

which is the classical discrete scalar sine-Gordon equation known as the Bianchi–Hirota equation.

Appendix

Here we give a proof of the lemma which we use in the proof of Proposition 2.

Lemma 1. *Let a matrix* $A \in \mathrm{Mat}_{n+2}(\mathbb{C})$ *satisfy*

$$AA^T = A^T A = 0; \tag{82}$$
$$A^* = QAQ, \tag{83}$$

where $Q = diag(-1, 1, \dots, 1)$*, then it can be represented as*

$$A = h\begin{pmatrix} -i \\ \mathbf{m}\end{pmatrix}\begin{pmatrix} i & \mathbf{n}^T\end{pmatrix}, \qquad 0 \neq h \in \mathbb{R}, \quad \mathbf{m},\mathbf{n} \in \mathbb{R}^{n+1}, \quad |\mathbf{m}| = |\mathbf{n}| = 1.$$

Proof. It follows from (83) that the form of matrix A is

$$A = \begin{pmatrix} h & -i\mathbf{e}^T \\ i\mathbf{f} & H\end{pmatrix}, \quad h \in \mathbb{R}, \quad \mathbf{e},\mathbf{f} \in \mathbb{R}^{n+1}, \quad H \in \mathrm{Mat}_{n+1}(\mathbb{R}). \tag{84}$$

It follows from (82) that

$$h^2 = \mathbf{e}^T\mathbf{e} = \mathbf{f}^T\mathbf{f}; \tag{85}$$
$$HH^T = \mathbf{f}\,\mathbf{f}^T; \quad H^T H = \mathbf{e}\,\mathbf{e}^T; \tag{86}$$
$$h\mathbf{f} - H\mathbf{e} = h\mathbf{e} - H^T\mathbf{f} = 0. \tag{87}$$

If $h = 0$ then $A = 0$. Now we assume that $h \neq 0$ and thus $\mathbf{e} \neq \mathbf{0}$, $\mathbf{f} \neq \mathbf{0}$. From (86), we obtain that $\mathrm{rank}(H) = \mathrm{rank}(HH^T) = \mathrm{rank}(\mathbf{f}\,\mathbf{f}^T) = 1$ and H can be represented as $H = \delta\mathbf{f}\mathbf{e}^T$, $\delta \in \mathbb{R}$. From (87) we deduce that $\delta = h^{-1}$. Thus

$$A = h\begin{pmatrix} 1 & -ih^{-1}\mathbf{e}^T \\ ih^{-1}\mathbf{f} & h^{-2}\mathbf{f}\mathbf{e}^T\end{pmatrix} = h\begin{pmatrix} -i \\ \mathbf{m}\end{pmatrix}\begin{pmatrix} i & \mathbf{n}^T\end{pmatrix}$$

where $\mathbf{n} = \mathbf{e}/h$, $\mathbf{m} = \mathbf{f}/h$ and $|\mathbf{n}| = |\mathbf{m}| = 1$ due to (85). □

Acknowledgements

The paper is supported by the Leverhulme Trust, EPSRC grant EP/I038675/1 and EPSRC grant EP/I038659/1. All authors gratefully acknowledge the financial support.

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