Optimization of replacement policy for a one-component system subject to Poisson shocks *†

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Abstract

In reliability engineering, system failures may occur due to intrinsic or extrinsic factors. For example, drinking water systems may fail due to ageing and deterioration (i.e., intrinsic factors) or flooding (i.e., extrinsic factors). An interesting question is: for such systems, how should preventive maintenance be scheduled? This paper investigates this question.

The paper develops a maintenance policy for repairable systems subject to extrinsic shocks. It assumes that a system may fail due to either intrinsic factors or extrinsic factors. Reliability indexes and the expected long run cost rate are then derived. A numerical example is given to illustrate the theoretical results.

Linear corrective maintenance; Hazard function; Geometric process; Repair-replacement policy; Poisson shock.

1 Introduction

In the reliability literature, most authors assume that system failures are due to system's intrinsic factors such as ageing and/or deterioration. In practice, failures can be due to intrinsic factors as well as extrinsic factors. For example, drinking water systems may fail due to flooding, which is an extrinsic factor. Reliability models that describe such shocks have been extensively investigated in the last two decades.

In the literature, shock models can be classified into the following three categories.

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- *Cumulative shock model* describes the scenario where a system fails if the accumulative damage due to shocks exceeds a threshold.
- *Extreme shock model* describes the scenario where a system fails if an individual shock exceeds a threshold.
- *δ-shock model* describes the scenario where a system fails if the interval of two consecutive shocks is shorter than a threshold δ.

In the literature, Sheu and Griffith (2002) investigate an extended block replacement policy under the assumption that shocks arrive according to a non-homogeneous Poisson process and that two types of failure may occur due to shocks. Type 1 failure can be rectified by a minimal repair while type 2 failure can only be removed by a major maintenance. Lam and Zhang (2004), and Wang and Zhang (2005) investigate maintenance policies for repairable systems subject to δ-shocks. They assume that shocks arrive according to a Poisson process and that repair time follows the geometric process (GP), which is a stochastic process proposed by Lam (1988). Tang and Lam (2006) propose a δ-shock maintenance model for a deteriorating system, assuming the shocks arrive according to a renewal process and that repair times follow the geometric process. Li and Zhao (2007) study the reliability of systems consisting of components subject to δ-shock shocks. Chen and Li (2008) derive a maintenance policy for a deteriorating system, assuming that system failures may be caused by either intrinsic or extrinsic factors. In their work, they assume that, with the number of repairs, the magnitude of shock damage the system can bear is decreasing and consecutive repair time is increasing. Cha and Lee (2010) study an extended stochastic failure model for a system subject to random shocks, assuming that a fatal shock causes systems to failure and a non-fatal shock may shorten system's working time. Wu (2012) considers a model under the assumption that the component may fail due to intrinsic or extrinsic factors.

On the other hand, optimization of maintenance policy based on the geometric process has attracted considerable attention in the reliability literature (Lam and Zhang, 2004; Lam, 1988; Chen and Li, 2008; Wang and Pham, 1996; Wu and Clements-Croome, 2006). The geometric process (Lam, 1988) defines an alternative to the non-homogeneous Poisson process: a sequence of random variables $\{X_k, k = 1, 2, \cdots\}$ forms a geometric process if the cumulative distribution function of X_k is given by $\tilde{F}(a^{k-1}t)$ for $k = 1, 2, \cdots$, where a is a positive constant and $\tilde{F}(t)$ is an arbitrary distribution function. Wang and Pham (1996) later refer to the geometric process as a quasi-renewal process. Wu and Clements-Croome (2006) extend the geometric process by replacing its parameter a^{k-1} with $\alpha a^{k-1} + \beta b^{k-1}$, where $a \geq 1$ and $0 < b \leq 1$. The geometric process has been applied to reliability analysis and maintenance policy optimization for various systems, see (Wang and Pham, 1996; Liang et al., 2012; Cheng and Li, 2012; Yu et al., 2013; Wang and Zhang, 2014), for example. In order to model the effectiveness of preventive maintenance (PM), Wu and Zuo (2010) review existing PM models, explore their interrelationships, and extend them to three new models: linear, nonlinear and their hybrid. A PM model is

- *linear* if the system has hazard rate $h_k(t)(k = 1, 2, \cdots)$ immediately after the kth PM with $h_k(t) = ah_{k-1}(t) + b$,

- *nonlinear* if $h_k(t) = h_{k-1}(\alpha t + \beta)$, and

- *hybrid* if $h_k(t) = ah_{k-1}(\alpha t + \beta) + b$, where a, b, α, β are non-negative parameters and $t > 0$.

Motivated by the above discussions, this paper investigates a linear corrective maintenance (CM) model for a single-component system under Poisson shocks. A maintenance policy N is considered under the following assumptions:

- repair on failed items is imperfect;

- after the kth repair, the intrinsic hazard rate function of the component becomes $h_k(t) = \alpha h_{k-1}(t) + \beta$, where $h_{k-1}(t)$ is the hazard rate function between the $(k-1)$th and the kth repair and $\alpha, \beta (k = 1, 2, \cdots)$ are parameters; and

- Repair time follows the geometric process.

To investigate maintenance policies for systems under the above assumptions is important because shocks occur from time to time in the real world and maintenance effect is not always perfect. As such, this paper derives the expected long run cost rate and optimal policy N^*.

The following assumptions are used in this paper.

1. Component failure may be due to either intrinsic factors or the extrinsic shocks;

2. Repair time follows the geometric process;

3. Repair effect on failed components is linear CM (corrective maintenance);

4. The system will fail if the threshold of a shock is greater than a threshold, which is a non-negative random variable with a general distribution function.

To the best of our knowledge, neither Assumption (3) nor Assumption (4) has been considered in the literature, which generates novelty.

The remainder of this paper is organized as follows. Section 2 recalls the geometric process defined by Lam (1988), and the linear preventive maintenance defined by Wu and Zuo (2010). Section 3 takes a further development on the proposed model. Section 4 derives explicit expressions for reliability indices. Section 5 studies a replacement policy and derives the average cost rate under policy N. Section 6 discusses two special cases. Section 7 presents two numerical examples. Section 8 closes the paper.

2 Definitions and assumptions

We first recall some definitions.

2.1 Definitions

Definition 1 (Ross, 1996) Assume that ξ and η are two random variables. If for every real number a, the inequality

$P(\xi \geq a) > P(\eta \geq a)$

holds, then ξ is stochastically greater than η, or η is stochastically less than ξ.

Definition 2 (Lam, 1988) Assume that $\{M(t), t \geq 0\}$ is a counting process, $\{X_k, k = 1, 2, \cdots\}$ is a sequence of independent non-negative random variables, and the distribution function of X_k is $\tilde{F}(a^{k-1}t), k = 1, 2, \cdots$, where $a > 0$ is a positive constant and $\tilde{F}(t)$ is an arbitrary distribution function.

If $S_n = \sum_{i=1}^{n} X_i, n \geq 1$ and

$$\{M(t) \geq n\} = \{S_n \leq t\}, \quad t \geq 0,\ n = 1, 2, \cdots,$$

then the counting process $\{M(t), t \geq 0\}$ is a geometric process (GP).

Obviously,

if $a > 1$, then $\{X_k, k = 1, 2, \cdots\}$ is stochastically decreasing,

if $a < 1$, then $\{X_k, k = 1, 2, \cdots\}$ is stochastically increasing, and

if $a = 1$, then $\{X_k, k = 1, 2, \cdots\}$ is renewal process.

Definition 3 (Wu and Zuo, 2010) The kth PM is linear PM if

$$h_k(t) = \alpha_k h_{k-1}(t) + \beta_k, \quad (k = 1, 2, \cdots) \tag{1}$$

where $h_{k-1}(t)$ and $h_k(t)$ are the hazard rate functions pre- and post- kth PM, respectively, α_k and β_k are positive parameters.

The parameters in (1) have their physical meanings. Parameter α_k indicates a degree of deterioration after PM. It is called an ageing alteration parameter of the linear PM model. Parameter β_k indicates the starting value of the hazard rate immediately after a PM. Therefore, β_k is called a location parameter.

Remark 2.1 From Definition 3, it follows that

(1) if $\alpha_k > 1$, then the system deteriorates faster than before;

(2) if $\alpha_k < 1$, then the system deteriorates slower than before;

(3) if $\alpha_k = 1$ and $\beta_k \neq 0$, then the system keeps the same shape but has a different location of the hazard rate as before;

(4) if $\beta_k > h_{k-1}(t_{k-1})$, then PM is a worse maintenance; and

(5) if $\beta_k < h_{k-1}(t_{k-1})$, then PM is a better maintenance,

where t_k is the time interval length between $(k-1)$th PM and kth PM.

In this paper, we will apply the linear maintenance concept to describe corrective maintenance, and refer it to as a linear CM. In addition, the following notions will be used.

- The time interval from the completion of the (n-1)th repair to the completion of the nth repair is called the nth cycle, where $n = 1, 2, \cdots$.

- The time interval from the time when a component is started to the time when the component fails due to intrinsic factors is referred to as the *intrinsic lifetime* of the component.

- The time interval from the time when a component is started to the time when the component fails due to extrinsic shocks is referred to as the *shock lifetime* of the component.

- The intrinsic hazard rate is the hazard rate of the component failure that is caused by intrinsic factor.

2.2 Notation

We summarize the notation in Table 1.

2.3 Assumptions

The following assumptions are made.

A1. A system consists of a component and a repairman. The component is new when it starts working at time $t = 0$. Once the component fails, a repairman will repair it immediately. As soon as a repair is completed, the component is put back into use again.

A2. The failure of the system may be due to intrinsic or extrinsic shocks. The effects of the intrinsic factors and the extrinsic shocks are statistically mutually independent **in the the same cycle**. The arrivals of shocks follow a Poisson process $\{N(t), t \geq 0\}$ with intensity $\lambda > 0$. The magnitudes of shocks $\{\widehat{X}_j^{(n)}, n = 1, 2, \cdots, \ j = 1, 2, \cdots\}$ are independent and identically distributed random variables with distribution function $F(t)$.

A3. When a shock arrives, it may affect the component. Each effect is independent of its history. The component will fail when the magnitude of a shock exceeds the threshold of the component at that time. The threshold under the jth shock in the nth cycle is an non-negative random variable $\varpi_j^{(n)}$ $(j = 1, 2, \cdots, \ n = 1, 2, \cdots)$. $\varpi_j^{(n)}$, $j = 1, 2, \cdots$ are i.i.d. with a distribution function $\Phi_n(x)$. The distribution of the intrinsic lifetime of the component in the first cycle is $D_1(t) = D(t) = 1 - \exp\{-\int_0^t h_0(x)\mathrm{d}x\}$, and the intrinsic hazard rate function is $h_0(t)$.

A4. The thresholds and the repair times upon failures follow two GPs, respectively. The repair is linear, i.e., $\{\varpi_1^{(n)}, n = 1, 2, \cdots\}$ forms a GP with ratio a, i.e., $\Phi_n(x) = \Phi(a^{n-1}x)$, and $\{Y_n, n = 1, 2, \cdots\}$ forms a GP with ratio b, i.e., $H_n(y) = H(b^{n-1}y)$. However, the intrinsic hazard rate of the system changes from $h_{n-1}(t)$ before the nth CM to $h_n(t) = \alpha_n h_{n-1}(t) + \beta_n$ after the nth CM, where $\alpha_n > 0, \beta_n > 0, \ n = 1, 2, \cdots$.

A5. All random variables and processes involved are statistically mutually independent.

Table 1: Notation

$\widehat{X}_j^{(n)}$:	magnitude of the jth shock in cycle n. $F(t) = P(\widehat{X}_j^{(n)} \leq t)$.
$\varpi_j^{(n)}$:	threshold of the component under the jth shock in cycle n. For $n \geq 1$, $\Phi_n(t) = P(\varpi_j^{(n)} \leq t) = \Phi(a^{n-1}t), \Phi(t) = P(\varpi_j^{(1)} \leq t)$.
ξ_n :	shock lifetime of the component in cycle n. For $n \geq 1$, $L_n(t) = P(\xi_n \leq t)$ $= 1 - e^{-r_n \lambda t}$, where $r_n = P(\widehat{X}_j^{(n)} > \varpi_j^{(n)}) = \int_0^\infty \Phi(a^{n-1}x)\mathrm{d}F(x)$.
η_n :	intrinsic lifetime of the component in cycle n. For $n \geq 1$, $D_n(t) = P(\eta_n \leq t)$ $= 1 - \exp(-\int_0^t h_{n-1}(x)\mathrm{d}x), D(t) = D_1(t)$, where $h_n(x) = \alpha_n h_{n-1}(x) + \beta_n$.
X_n :	lifetime of the component in cycle n. For $n \geq 1$, $G_n(t) = P(X_n \leq t)$ $= 1 - (1 - L_n(t))(1 - D_n(t)), \lambda_n = EX_n$, $g_n^*(s) = \int_0^\infty e^{-st}\mathrm{d}G_n(t)$, $\overline{G}_n^*(s) = \int_0^\infty (1 - G_n(t))e^{-st}\mathrm{d}t$.
Y_n :	failure repair time of the component in cycle n. $H_1(t) = H(t) = P(Y_1 \leq t)$ $= 1 - \exp\left(-\int_0^t \mu(y)\mathrm{d}y\right)$, $h^*(s) = \int_0^\infty e^{-st}\mathrm{d}H(t)$, $\overline{H}(t) = 1 - H(t)$. $H_n(t) = P(Y_n \leq t) = H(b^{n-1}t) = 1 - \exp\left(-\int_0^t b^{n-1}\mu(b^{n-1}y)\mathrm{d}y\right)$, $\mu = EY_1$.
Q :	the replacement time of the system. $M(t) = P(Q \leq t), \theta = EQ$.
a, b, α_k, β_k :	positive constants satisfying $a > 1, \alpha_k > 0, \beta_k > 0, (k = 2, 1, \cdots), 0 < b \leq 1$.
$S(t)$:	system state at time t, if the system is operating at time t, then $S(t) = 0$, if the system is being repaired at time t, then $S(t) = 1$.
$I(t)$:	number of cycles of system at time t.
$X(t)$:	elapsed working time of the component at time t.
$Y(t)$:	elapsed repair time of the system being repaired at time t.
$P_{0,i}(t, x)$:	state probability density function at time t when $S(t) = 0, I(t) = i, X(t) = x$. $P_{0,i}^*(s, x) = \int_0^\infty P_{0,i}(t, x)e^{-st}\mathrm{d}t$.
$P_{1,i}(t, y)$:	state probability density function at time t when $S(t) = 1, I(t) = i, Y(t) = y$. $P_{1,i}^*(s, y) = \int_0^\infty P_{1,i}(t, y)e^{-st}\mathrm{d}t$.
$A(t)$:	system availability at time t. $A^*(s) = \int_0^\infty A(t)e^{-st}\mathrm{d}t$
$m_f(t)$:	rate of occurrence of system failure at time t. $m_f^*(s) = \int_0^\infty m_f(t)e^{-st}\mathrm{d}t$
$R(t)$:	the system reliability at time t.
C_m :	repair cost per unit time of component.
C_w :	working reward per unit time.
C_r :	basic replacement cost of the system.
C_p :	cost proportional to the length of replacement time.
N :	number of failures of the component before replacement.
W :	length of a renewal cycle of the system under policy N.
C :	cost of a renewal cycle of the system under policy N.
$C(N)$:	long run average cost per unit time under policy N.

A possible realization of the system is shown in Fig. 1.

Figure 1

Fig. 1. A possible realization of the system.

3 Model development

Based on assumptions A3, A4 and A5, we can see that, the probability that one shock causes the component to fail in the nth cycle is

$$r_n = P(\widehat{X}_j^{(n)} > \varpi_j^{(n)}) = \int_0^\infty P(\varpi_j^{(n)} < x|\widehat{X}_j^{(n)} = x)\mathrm{d}P(\widehat{X}_j^{(n)} \le x) = \int_0^\infty \Phi(a^{n-1}x)\mathrm{d}F(x)$$

for $n = 1, 2, \cdots$ see (Wu and Wu, 2011; Wu, 2012).

Lemma 1(Wu, 2012) The distribution function of the shock lifetime of the component in the nth cycle is

$$L_n(t) = 1 - e^{-r_n\lambda t},\ t > 0,\ n = 1, 2, \cdots. \tag{2}$$

(i) If all of the first k CMs are linear, then from $A4$, we have

$$h_k(t) = A_k h_0(t) + B_k,$$

where $A_0 = 1, B_0 = 0, A_k = \prod_{i=1}^{k}\alpha_i, B_k = \sum_{i=1}^{k-1}\left(\beta_i \prod_{j=i+1}^{k}\alpha_j\right) + \beta_k$.

(ii) If all of the first k CMs are linear and all the parameters are identical, i.e., $\alpha_i = \alpha, \beta_i = \beta, i = 1, 2, \cdots, k$, then we have

$$h_k(t) = A'_k h_0(t) + B'_k,$$

where $A'_0 = 1, B'_0 = 0$, if $\alpha \neq 1$, then $A'_k = \alpha^k, B'_k = ((\alpha^k - 1)/(\alpha - 1))\beta$; **if** $\alpha = 1$**, then** $A'_k = 1, B'_k = k\beta$.

Lemma 2 (Wu and Zuo, 2010) If all of the first $(k-1)$ CMs are linear, then the distribution of the intrinsic lifetime of the component in the kth cycle is

$$D_k(t) = 1 - e^{B_{k-1}t}(1 - D(t))^{A_{k-1}}. \tag{3}$$

Lemma 3 (Wu and Zuo, 2010) If all of the first $(k-1)$ CMs are linear and all the parameters are identical, then the distribution of the intrinsic lifetime of the component in the kth cycle is

$$D_k(t) = 1 - e^{B'_{k-1}t}(1 - D(t))^{A'_{k-1}}. \tag{4}$$

Theorem 1 If all of the first $(n-1)$ CMs are linear, then the distribution of the lifetime of the component in the nth cycle is

$$G_n(t) = 1 - e^{-(r_n\lambda + B_{n-1})t}\left[1 - D(t)\right]^{A_{n-1}}, \ n = 1, 2, \cdots . \tag{5}$$

Proof. According to Assumptions $A2-A5$, ξ_n and η_n $(n = 1, 2, \cdots)$ are mutually independent and $X_n = \min\{\xi_n, \eta_n\}$. From Lemma 1 and 2, we have

$$\begin{aligned} G_n(t) &= P(X_n \leq t) = 1 - P(X_n \geq t) = 1 - P(\min\{\xi_n, \eta_n\} \geq t) = 1 - P(\xi_n \geq t, \eta_n \geq t) \\ &= 1 - P(\xi_n \geq t)P(\eta_n \geq t) = 1 - e^{-r_n\lambda t}e^{-B_{n-1}t}\left[1 - D(t)\right]^{A_{n-1}} \\ &= 1 - e^{-(r_n\lambda + B_{n-1})t}\left[1 - D(t)\right]^{A_{n-1}}, \ n = 1, 2, \cdots . \end{aligned}$$

□

The state space is $\Omega = \{0, 1\}$, where the operating state set is $W = \{0\}$ and the failure state set is $F = \{1\}$. According to the assumptions, $\{S(t), t \geq 0\}$ is not a Markov process. However, if we introduce the following supplementary variables $I(t)$, $X(t)$ and $Y(t)$, then $\{(S(t), I(t), X(t), Y(t)), t \geq 0\}$ is a continuous vector Markov process with the state space $\Omega^* = \{[0, i, x], [1, i, y]\}$, where i, x and y are the values taken by $I(t)$, $X(t)$ and $Y(t)$, respectively,see (Wu and Wu, 2011; Wu, 2012).

Remark 3.1

(1) When $S(t) = 0$, the $Y(t)$ vanishes and we ignore $Y(t)$ in the description of the states for simplicity;

(2) When $S(t) = 1$, the $X(t)$ vanishes and we ignore $X(t)$ in the description of the states for simplicity.

At time t, the state probabilities of the system are defined by

$$P_{0,i}(t, x)\mathrm{d}x = P(S(t) = 0, I(t) = i, x \leq X(t) \leq x + \mathrm{d}x), \ i = 1, 2, \cdots ,$$

and

$$P_{1,i}(t, y)\mathrm{d}y = P(S(t) = 1, I(t) = i, y \leq Y(t) \leq y + \mathrm{d}y), \ i = 1, 2, \cdots .$$

Then the following relations are valid:

$$\begin{aligned} p_{0,i}(t) = P(S(t) = 0, I(t) = i) = \textstyle\int_0^\infty P_{0,i}(t, x)\mathrm{d}x, \ i = 1, 2, \cdots , \\ p_{1,i}(t) = P(S(t) = 1, I(t) = i) = \textstyle\int_0^\infty P_{1,i}(t, y)\mathrm{d}y, \ i = 1, 2, \cdots . \end{aligned}$$

Since the process $\{(S(t), I(t), X(t), Y(t)), t \geq 0\}$ is a continuous vector Markov process, we can express it in a way considering the transitions occurring in t and $t + \Delta t$. Relating the state of the system at t and $t + \Delta t$, we can set up the following integro-differential equations for the system. **For example, for all $k \geq 1$, $x > 0$, if the system is in state $\{S(t + \Delta t) = 0, I(t + \Delta t) = k, X(t + \Delta t) = x + \Delta t\}$ at the moment $t + \Delta t$, the system must be in state $\{S(t) = 0, I(t) = k, X(t) = x\}$ at time t and it does not fail within**

the time interval $(t, t+\Delta t]$**. Then**

$$\begin{aligned}
P_{0,k}(t+\Delta t, x+\Delta t) &= P\{S(t+\Delta t)=0, I(t+\Delta t)=k, X(t+\Delta t)=x+\Delta t\} \\
&= P\{S(t)=0, I(t)=k, X(t)=x\} \times P\{S(t+\Delta t)=0, I(t+\Delta t)=k, \\
&\quad X(t+\Delta t)=x+\Delta t | S(t)=0, I(t)=k, X(t)=x\} + o(\Delta t) \\
&= P_{0,k}(t,x)\{1-[r_k\lambda + B_{k-1} + A_{k-1}h_0(x)]\,\Delta t\} + o(\Delta t).
\end{aligned}$$

Let Δt **tend to zero, then we have**

$$\left[\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}} + r_k\lambda + B_{k-1} + A_{k-1}h_0(x)\right] P_{0,k}(t,x) = 0, \;\; k \geq 1. \tag{6}$$

Similarly, for all $k \geq 1$**,** $y > 0$**, if the system is in state** $\{S(t+\Delta t)=1, I(t+\Delta t)=k, Y(t+\Delta t)=y+\Delta t\}$ **at time** $t+\Delta t$**, the system must be in state** $\{S(t)=1, I(t)=k, Y(t)=y\}$ **at time** t **and the failed component is not completed within time interval** $(t, t+\Delta t]$**, then**

$$\begin{aligned}
P_{1,k}(t+\Delta t, y+\Delta t) &= P\{S(t+\Delta t)=1, I(t+\Delta t)=k, Y(t+\Delta t)=y+\Delta t\} \\
&= P\{S(t)=1, I(t)=k, Y(t)=y\} \times P\{S(t+\Delta t)=1, I(t+\Delta t)=k, \\
&\quad Y(t+\Delta t)=y+\Delta t | S(t)=1, I(t)=k, Y(t)=y\} + o(\Delta t) \\
&= P_{1,k}(t,y)\left\{1-b^{k-1}\mu(b^{k-1}y)\Delta t\right\} + o(\Delta t).
\end{aligned}$$

Letting Δt **tend to zero, we have**

$$\left[\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{y}} + b^{k-1}\mu(b^{k-1}y)\right] P_{1,k}(t,y) = 0, \;\; k \geq 1. \tag{7}$$

For all $k \geq 2$**, if the system is in state** $\{S(t+\Delta t)=0, I(t+\Delta t)=k, 0 < X(t+\Delta t) \leq \Delta t\}$ **at time** $t+\Delta t$**, the system must be in state** $\{S(t)=1, I(t)=k, Y(t)=y\}$ **at time** t **(for** $y > 0$**) and the failed component has been completed within time interval** $(t, t+\Delta t]$**. Thus we have**

$$\begin{aligned}
P_{0,k}(t,0)\Delta t &= \int_0^{\Delta t} P_{0,k}(t+\Delta t, y)\mathrm{d}y \\
&= \int_0^{\infty} b^{k-2}\mu(b^{k-2}y)P_{1,k-1}(t,y)\Delta t\mathrm{d}y + o(\Delta t).
\end{aligned}$$

Letting Δt **tend to zero, we have the boundary condition:**

$$P_{0,k}(t,0) = \textstyle\int_0^{\infty} b^{k-2}\mu(b^{k-2}y)P_{1,k-1}(t,y)\mathrm{d}y, \;\; k \geq 2. \tag{8}$$

Similarly, for all $k \geq 1$**, in order that the system is in state** $\{S(t+\Delta t)=1, I(t+\Delta t)=k, 0 < Y(t+\Delta t) \leq \Delta t\}$ **at the moment** $t+\Delta t$**, the system must be in state** $\{S(t)=0, I(t)=k, X(t)=x\}$ **at time** t**, where** $x > 0$ **and the system has been failed**

in the time interval $(t, t+\Delta t]$. Then we have

$$\begin{aligned} P_{1,k}(t,0)\Delta t &= \int_0^{\Delta t} P_{1,k}(t+\Delta t, x)\mathrm{d}x \\ &= \int_0^{\infty} (r_k\lambda + B_{k-1} + A_{k-1}h_0(x))P_{0,k-1}(t,x)\Delta t\mathrm{d}x + o(\Delta t). \end{aligned}$$

Letting Δt tend to zero, we have

$$P_{1,k}(t,0) = \textstyle\int_0^{\infty}(r_k\lambda + B_{k-1} + A_{k-1}h_0(x))P_{0,k}(t,x)\mathrm{d}x, \ \ k \geq 1. \tag{9}$$

$$P_{0,1}(t,0) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0. \end{cases} \tag{10}$$

The initial conditions are

$$\begin{aligned} &P_{0,1}(0,x) = \delta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}, \\ &P_{0,k}(0,x) = 0, \ \ k = 2, 3, \cdots, \\ &P_{1,k}(0,y) = 0, \ \ k = 1, 2, \cdots. \end{aligned}$$

The Laplace transforms of the above differential equations are given by

$$\left[s + \frac{d}{dx} + r_k\lambda + B_{k-1} + A_{k-1}h_0(x)\right] P^*_{0,k}(s,x) = 0, \ \ k \geq 1, \tag{11}$$

$$\left[s + \frac{d}{dy} + b^{k-1}\mu(b^{k-1}y)\right] P^*_{1,k}(s,y) = 0, \ \ k \geq 1, \tag{12}$$

$$P^*_{0,1}(s,0) = 1, \tag{13}$$

$$P^*_{0,k}(s,0) = \textstyle\int_0^{\infty} b^{k-2}\mu(b^{k-2}y)P^*_{1,k-1}(s,y)\mathrm{d}y, \ \ k \geq 2, \tag{14}$$

$$P^*_{1,k}(s,0) = \textstyle\int_0^{\infty} \left[r_k\lambda + B_{k-1} + A_{k-1}h_0(x)\right] P^*_{0,k}(s,x)\mathrm{d}x, \ \ k \geq 1. \tag{15}$$

According to Eqs. (11) – (12), we have

$$P^*_{0,k}(s,x) = P^*_{0,k}(s,0)e^{-(s+r_k\lambda+B_{k-1})x - A_{k-1}\int_0^x h_0(u)\mathrm{d}u}, \ \ k \geq 1, \tag{16}$$

and

$$P^*_{1,k}(s,y) = P^*_{1,k}(s,0)e^{-sy}\overline{H}(b^{k-1}y), \ \ k \geq 1. \tag{17}$$

Substituting Eqs. (16) and (17) into (14) and (15), respectively, we have

$$P_{0,k}^*(s,0) = h^*(\frac{s}{b^{k-2}})P_{1,k-1}^*(s,0),\ k \geq 2, \tag{18}$$

and

$$P_{1,k}^*(s,0) = g_k^*(s)P_{0,k}^*(s,0),\ k \geq 1. \tag{19}$$

With Eqs. (13), (18) and (19), we obtain

$$P_{0,k}^*(s,0) = \prod_{i=1}^{k-1} g_i^*(s) \prod_{i=2}^{k} h^*(\frac{s}{b^{i-2}}) = \prod_{i=1}^{k-1} \left(g_i^*(s) h^*(\frac{s}{b^{i-1}}) \right),\ k \geq 1 \tag{20}$$

and

$$P_{1,k}^*(s,0) = \prod_{i=1}^{k} g_i^*(s) \prod_{i=2}^{k} h^*(\frac{s}{b^{i-2}}) = \prod_{i=1}^{k-1} \left(g_i^*(s) h^*(\frac{s}{b^{i-1}}) \right) g_k^*(s),\ k \geq 1 \tag{21}$$

Remark 3.2 Eqs. (20) and (21) are valid for $k = 1$ because an empty product is equal to 1 by convention.

Substituting Eqs. (13), (20) and (21) into (16) and (17), we have

$$P_{0,k}^*(s,x) = \left[\prod_{i=1}^{k-1} \left(g_i^*(s) h^*(\frac{s}{b^{i-1}}) \right) \right] e^{-(s+r_k\lambda+B_{k-1})x - A_{k-1}\int_0^x h_0(u)\mathrm{d}u},\ k \geq 1,$$

and

$$P_{1,k}^*(s,y) = \left[\prod_{i=1}^{k-1} \left(g_i^*(s) h^*(\frac{s}{b^{i-1}}) \right) g_k^*(s) \right] e^{-sy}\overline{H}(b^{k-1}y),\ k \geq 1.$$

With these explicit expressions, we can derive system reliability indices below.

4 Reliability indices

4.1 System availability

By definition, we have the system availability

$$A(t) = P(S(t) = 0) = \sum_{k=1}^{\infty} \int_0^{\infty} P_{0,k}(t,x)\mathrm{d}x.$$

The Laplace transform of $A(t)$ is given by

$A^*(s) = \sum\limits_{k=1}^{\infty} \int_0^{\infty} P_{0,k}^*(s,x)\mathrm{d}x$

$$=\int_0^\infty e^{-(s+r_1\lambda)x-\int_0^x h_0(u)\mathrm{d}u}\mathrm{d}x+\sum_{k=2}^{\infty}\left[\prod_{i=1}^{k-1}\left(g_i^*(s)h^*(\frac{s}{b^{i-1}})\right)\right]\int_0^\infty e^{-(s+r_k\lambda+B_{k-1})x-A_{k-1}\int_0^x h_0(u)\mathrm{d}u}\mathrm{d}x$$
$$=\sum_{k=1}^{\infty}\left[\prod_{i=1}^{k-1}\left(g_i^*(s)h^*(\frac{s}{b^{i-1}})\right)\right]\overline{G}_k^*(s).$$

If the component cannot be repaired as good as new, then for $s>0$, $0<\overline{G}_k^*(s)<\overline{G}_1^*(s)$**, and hence**

$$A^*(s)\leq\sum_{k=1}^{\infty}\left[\prod_{i=1}^{k-1}\left(g_i^*(s)h^*(\frac{s}{b^{i-1}})\right)\right]\overline{G}_1^*(s). \tag{22}$$

In this case, for $s>0$, $g_i^*(s)$ **is decreasing in** i **and** $0<g_i^*(s)<1$**. Since** $b<1$**, then** $h^*(\frac{s}{b^{i-1}})$ **is decreasing in** i **and** $0<h^*(\frac{s}{b^{i-1}})<1$**. Hence, we have** $\rho(s)=\lim_{k\to+\infty}\frac{\prod_{i=1}^{k}\left(g_i^*(s)h^*(\frac{s}{b^{i-1}})\right)}{\prod_{i=1}^{k-1}\left(g_i^*(s)h^*(\frac{s}{b^{i-1}})\right)}=\lim_{k\to+\infty}g_k^*(s)h^*(\frac{s}{b^{k-1}})<1$**. According to the ratio test, the series** $\sum_{k=1}^{\infty}\left[\prod_{i=1}^{k-1}\left(g_i^*(s)h^*(\frac{s}{b^{i-1}})\right)\right]$ **converges. Denote its sum by** $K(s)$**. Then,** $0\leq\lim_{s\to0}sA^*(s)\leq\lim_{s\to0}sK^*(s)\overline{G}_1^*(s)=0$**, which implies that** $\lim_{s\to0}sA^*(s)=0$**. According to the final value theorem of Laplace transform, the limiting availability of the system is given by**

$$A=\lim_{t\to+\infty}A(t)=\lim_{s\to0}sA^*(s)=0.$$

This agrees to our intuition. Since the component can not be repaired as good as new, the system availability will tend to be 0 as $t\to+\infty$.

4.2 System ROCOF

Let $M_f(t)$ be the expected number of failures in $(0,t]$, then its derivative $m_f(t)$ is the rate of occurrence of system failure (ROCOF) at time t, i.e., $m_f(t)$. Thus, $M_f(t)=\int_0^t m_f(u)\mathrm{d}u$.

Su and Shi (1995) has shown that

$$m_f(t)=\sum_{i\in W}\sum_{j\in F}\left[\sum_{k=1}^{\infty}\int_0^\infty\int_0^\infty\cdots[P_{i,k}(t,x_1,y_1,\cdots)a_{ijk}(x_1,y_1,\cdots)\mathrm{d}x_1\mathrm{d}y_1,\cdots]\right],$$

where the matrix $[a_{ijk}(x_1,y_1,\cdots)]$ **is an infinitesimal transition matrix of** $\{S(t),X_1(t),Y_1(t),\cdots\}$**, and** $P_{i,k}(t,x_1,y_1,\cdots)$ **is the probability that the system appears in the state** $(i,x_1,y_1,\cdots)$ **in** k**th cycle at time** t**. In view of the assumptions, we have** $W=\{0\}$, $F=\{1\}$, $a_{01k}(x)=r_k\lambda+B_{k-1}+A_{k-1}\int_0^x h_0(u)\mathrm{d}u$**, therefore,**

$$m_f(t)=\sum_{k=1}^{\infty}\int_0^\infty P_{0,k}(t,x)\left(r_k\lambda+B_{k-1}+A_{k-1}\int_0^x h_0(u)\mathrm{d}u\right)\mathrm{d}x.$$

The Laplace transform of $m_f(t)$ is

$$\begin{aligned} m_f^*(s) &= \sum_{k=1}^{\infty} \int_0^{\infty} P_{0,k}^*(s,x)\left(r_k\lambda + B_{k-1} + A_{k-1}\int_0^x h_0(u)\mathrm{d}u\right)\mathrm{d}x \\ &= \sum_{k=1}^{\infty}\left[\prod_{i=1}^{k-1}\left(g_i^*(s)h^*(\frac{s}{b^{i-1}})\right)\right]g_k^*(s). \end{aligned}$$

Likewise, **if the component can not be repaired as good as new, then** $m_f^*(s) \leq \sum_{k=1}^{\infty}\left[\prod_{i=1}^{k-1}\left(g_i^*(s)h^*(\frac{s}{b^{i-1}})\right)\right]g_1^*(s) = K(s)g_1^*(s)$, **we have** $0 \leq \lim_{s\to 0} sm_f^*(s) \leq \lim_{s\to 0} sK^*(s)g_1^*(s) = 0$**, which implies that** $\lim_{s\to 0} sm_f^*(s) = 0$**. According to the final value theorem for Laplace transform, the limiting ROCOF of the system is given by**

$$m_f = \lim_{t\to+\infty} m_f(t) = \lim_{s\to 0} sm_f^*(s) = 0.$$

4.3 System reliability and MTTFF

Since the system is consisted of one component, the reliability of the system is given by

$$\begin{aligned} R(t) &= P\{X_1 > t\} = P\{\min\{\xi_1,\eta_1\} > t\} = P\{\xi_1 > t, \eta_1 > t\} \\ &= P\{\xi_1 > t\}P\{\eta_1 > t\} = e^{-r_1\lambda t - \int_0^t h_0(x)dx}. \end{aligned}$$

Therefore, the mean time to the first failure (MTTFF) of the system is

$$\mathrm{MTTFF} = \int_0^{\infty} R(t)\mathrm{d}t = \int_0^{\infty} e^{-r_1\lambda t - \int_0^t h_0(x)\mathrm{d}x}\mathrm{d}t.$$

Remark 4.1 **If** $P(\varpi_j^{(n)} = \infty) = 1, b = 1, \mu(y) = \mu_0, \beta_n = \beta$ **and** $\alpha_n = \alpha = 0$**, i.e.,** $r_n = 0,\ j,n = 1,2,\cdots$**, shocks cause no harm on the component. Then the model reduces to the classical two-state Markov chain model. That is, the system is a one-component system where the component can be repaired as good as new, and the lifetime and the repair time of the component are exponential distribution with parameter** μ_0 **and** β**, respectively. Taking the limit as** α **tends to** 0**, we have** $\lim_{\alpha\to 0} A_k = \lim_{\alpha\to 0}\alpha^k = 0, \lim_{\alpha\to 0} B_k = \sum_{i=1}^{k-1}\beta(\lim_{\alpha\to 0}\alpha^{k-i}) + \beta = \beta$**. Then** $\lim_{\alpha\to 0} g_k^*(s) = \dfrac{\beta}{s+\beta}$**, and** $\lim_{\alpha\to 0}\overline{G}_k^*(s) = \dfrac{1}{s+\beta}$**,** $h^*(s) = \dfrac{\mu_0}{s+\mu_0}$**. The Laplace transform of the system availability**

is given by

$$
\begin{aligned}
A^*(s) &= \lim_{\alpha\to 0}\sum_{k=1}^{\infty}\left[\prod_{i=1}^{k-1}\left(g_i^*(s)h^*(s)\right)\right]\overline{G}_k^*(s) = \sum_{k=1}^{\infty}\left[\prod_{i=1}^{k-1}\left(\lim_{\alpha\to 0} g_i^*(s)h^*(s)\right)\right]\lim_{\alpha\to 0}\overline{G}_k^*(s) \\
&= \sum_{k=1}^{\infty}\left(\frac{\beta\mu_0}{(s+\beta)(s+\mu_0)}\right)^{k-1}\frac{1}{s+\beta} \\
&= \frac{s+\mu_0}{s(s+\beta+\mu_0)}
\end{aligned}
$$

In the same way, we can obtain the Laplace transform of the rate of occurrence of the system failure as fellow

$$m_f^*(s) = \frac{\beta(s+\mu_0)}{s(s+\beta+\mu_0)}$$

According to the final value theorem for Laplace transform, the limiting availability and the limiting ROCOF of the system are $A = \lim_{s\to 0} sA^*(s) = \dfrac{\mu_0}{\beta+\mu_0}$ **and** $m = \lim_{s\to 0} sm_f^*(s) = \dfrac{\beta\mu_0}{\beta+\mu_0}$**, respectively. This result agrees with the results of the classical two-states Markov chain model.**

5 Expected cost under policy N

In this section, we consider the repair-replacement policy N: a replacement is carried out if the number of failures reaches N. Our objective is to determine an optimal replacement policy N^* such that the expected long run average cost rate is minimized. Theoretical analysis will be carried out for this model. To this end, in addition to the assumptions in Section 2, we make one more assumption:

A6. A repair-replacement policy N is used, where N is the number of failures of the component. When a replacement is required, a new and identical component will be used, and the replacement time is a non-negative random variable Q.

The time interval between two replacements is referred to as a renewal cycle of the system. Let τ_1 be the time of the first replacement of the system. Denote the time between the (n-1)th replacement and the nth replacement of the system as $\tau_n,\ n = 2, 3, \cdots$.

It is obvious that $\{\tau_1, \tau_2, \cdots\}$ forms a renewal process. Let $C(N)$ be the expected long run cost rate of the system under the policy N, then we have

$$C(N) = \lim_{t\to\infty}\frac{\text{Expected cost within } [0,\text{t}]}{t}.$$

Since $\{\tau_1, \tau_2, \cdots\}$ is a renewal process, the time interval between two consecutive replacements is a renewal cycle. According to the renewal theorem Ross (1996), the long run average cost rate is given by

$$C(N) = \frac{\text{Expected cost incurred in a cycle}}{\text{Expected length of a cycle}} = \frac{E(C)}{E(W)}, \tag{23}$$

where W is the length of a renewal cycle of the system and C is the cost of a renewal cycle of the system under the policy N. In fact, it is observed that

$$W = \sum_{k=1}^{N} X_k + \sum_{k=1}^{N-1} Y_k + Q. \tag{24}$$

Then the expected length of a renewal cycle is

$$E(W) = \sum_{k=1}^{N} \int_0^{\infty} e^{-(r_k\lambda + B_{k-1})t - A_{k-1}\int_0^t h_0(x)\mathrm{d}x}\mathrm{d}t + \sum_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + \theta. \tag{25}$$

On the other hand,

$$C = C_r + C_m \sum_{k=1}^{N-1} Y_k + C_p Q - C_w \sum_{k=1}^{N} X_k, \tag{26}$$

and the expected cost within a cycle is given by

$$E(C) = C_r + C_m \sum_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + C_p\theta - C_w \sum_{k=1}^{N} \int_0^{\infty} e^{-(r_k\lambda + B_{k-1})t - A_{k-1}\int_0^t h_0(x)\mathrm{d}x}\mathrm{d}t. \tag{27}$$

Now we can give the main result of this section.

Theorem 2 For this repair system, under replacement policy N the average cost is given by

$$C(N) = \frac{C_r + (C_m + C_w)\sum\limits_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + (C_p + C_w)\theta}{\sum\limits_{k=1}^{N} \lambda_k + \sum\limits_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + \theta} - C_w. \tag{28}$$

Proof. Substituting the numerator and denominator of Eq. (23) with (25) and (27), respectively, we can derive the expected long run average cost rate of the system under policy N. □

Let $q_1(N) = \sum\limits_{k=1}^{N} \lambda_k + \sum\limits_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + \theta$. Then

$$\begin{aligned} C(N+1) - C(N) &= \frac{C_r + (C_m + C_w)\sum\limits_{k=1}^{N} \frac{\mu}{b^{k-1}} + (C_p + C_w)\theta}{q_1(N+1)} - C_w \\ &\quad - \frac{C_r + (C_m + C_w)\sum\limits_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + (C_p + C_w)\theta}{q_1(N)} + C_w \\ &= \frac{(C_m + C_w)\mu\left(\sum\limits_{k=1}^{N} \lambda_k + \theta - \lambda_{N+1}\sum\limits_{k=1}^{N-1} b^{N-k}\right) - [C_r + (C_m + C_p)\theta]\left(\lambda_{N+1}b^{N-1} + \mu\right)}{b^{N-1} q_1(N+1) q_1(N)}. \end{aligned}$$

Since $b^{N-1}q_1(N+1)q_1(N)$ is always positive, it is clear that the sign of $C(N+1)-C(N)$ is the same as the sign of its numerator. Therefore, we can obtain following lemma:

Lemma 4 $C(N+1) \geq C(N) \Leftrightarrow B(N) \geq 1$, where

$$B(N) = \frac{(C_m + C_w)\mu\left(\sum\limits_{k=1}^{N}\lambda_k + \theta - \lambda_{N+1}\sum\limits_{k=1}^{N-1}b^{N-k}\right)}{\left[C_r + (C_m + C_p)\theta\right]\left(\lambda_{N+1}b^{N-1} + \mu\right)}. \tag{29}$$

Lemma 5 $B(N)$ is non-decreasing in N if $\lambda_{N+1} - b\lambda_{N+2} \geq 0$ for all integer N.

Proof. Now we consider the difference of $B(N+1)$ and $B(N)$ and obtain the following result.

$$\begin{aligned}
&B(N+1) - B(N)\\
=& \frac{(C_m + C_w)\mu\left(\sum\limits_{k=1}^{N+1}\lambda_k + \theta - \lambda_{N+2}\sum\limits_{k=1}^{N}b^{N+1-k}\right)}{\left[C_r + (C_m + C_p)\theta\right]\left(\lambda_{N+2}b^{N} + \mu\right)} - \frac{(C_m + C_w)\mu\left(\sum\limits_{k=1}^{N}\lambda_k + \theta - \lambda_{N+1}\sum\limits_{k=1}^{N-1}b^{N-k}\right)}{\left[C_r + (C_m + C_p)\theta\right]\left(\lambda_{N+1}b^{N-1} + \mu\right)}\\
=& \frac{(C_m + C_w)\mu}{\left[C_r + (C_m + C_p)\theta\right]\left(\lambda_{N+2}b^{N} + \mu\right)\left(\lambda_{N+1}b^{N-1} + \mu\right)}\left\{\left(\sum\limits_{k=1}^{N+1}\lambda_k + \theta - \lambda_{N+2}\sum\limits_{k=1}^{N}b^{N+1-k}\right)\times\right.\\
&\left.\left(\lambda_{N+1}b^{N-1} + \mu\right) - \left(\sum\limits_{k=1}^{N}\lambda_k + \theta - \lambda_{N+1}\sum\limits_{k=1}^{N-1}b^{N-k}\right)\left(\lambda_{N+2}b^{N} + \mu\right)\right\}.\\
=& \frac{(C_m + C_w)\mu}{\left[C_r + (C_m + C_p)\theta\right]\left(\lambda_{N+2}b^{N} + \mu\right)\left(\lambda_{N+1}b^{N-1} + \mu\right)}\left\{\sum\limits_{k=1}^{N+1}\lambda_k\lambda_{N+1}b^{N-1} + \theta\lambda_{N+1}b^{N-1}\right.\\
&-\lambda_{N+1}\lambda_{N+2}\sum\limits_{k=1}^{N}b^{2N-k} + \mu\sum\limits_{k=1}^{N+1}\lambda_k + \theta\mu - \mu\lambda_{N+2}\sum\limits_{k=1}^{N}b^{N+1-k} - \sum\limits_{k=1}^{N}\lambda_k\lambda_{N+2}b^{N}\\
&\left.- \theta\lambda_{N+2}b^{N} + \lambda_{N+1}\lambda_{N+2}\sum\limits_{k=1}^{N-1}b^{2N-k} - \mu\sum\limits_{k=1}^{N}\lambda_k - \theta\mu + \mu\lambda_{N+1}\sum\limits_{k=1}^{N-1}b^{N-k}\right\}\\
=& \frac{(C_m + C_w)\mu b^{N-1}(\lambda_{N+1} - b\lambda_{N+2})\left(\sum\limits_{k=1}^{N+1}\lambda_k + \theta + \sum\limits_{k=1}^{N}\frac{\mu}{b^{k-1}}\right)}{\left[C_r + (C_m + C_p)\theta\right]\left(\lambda_{N+2}b^{N} + \mu\right)\left(\lambda_{N+1}b^{N-1} + \mu\right)} \geq 0.
\end{aligned}$$

□

Since $a > 1$, then $r_k = P(\widehat{X}_j^{(k)} > \varpi_j^{(k)}) = \int_0^\infty \Phi(a^{k-1}x)\mathrm{d}F(x)$ is non-decreasing in k. If the *intrinsic lifetime* of the component becomes shorter and shorter after it has been repaired, then $-(r_k\lambda + B_{k-1})x - A_{k-1}\int_0^x h_0(u)\mathrm{d}u$ is non-increasing in k for all $x > 0$, and

$$\lambda_k = \int_0^\infty e^{-(r_k\lambda + B_{k-1})x - A_{k-1}\int_0^x h_0(u)\mathrm{d}u}\mathrm{d}x$$

is non-increasing. Therefore, $\lambda_{N+1} - b\lambda_{N+2} \geq 0$. Then we have the following lemma:

Lemma 6 $B(N)$ is non-decreasing in N if $\alpha_k \geq 1, (k = 1, 2, \cdots)$.

According to Lemmas 3 and 4, an analytical expression for an optimal policy for minimizing $C(N)$ can be obtained through analysis of $B(N)$. Thus, we have the following theorem.

Theorem 3 If $\lambda_{N+1} - b\lambda_{N+2} \geq 0$ for all integer N or $\alpha_k \geq 1, (k = 1, 2, \cdots)$, then the optimal replacement policy N^* can be determined by

$$N^* = \min\{N|B(N) \geq 1\}. \tag{30}$$

Furthermore, if $B(N^*) > 1$, then the optimal policy N^* is unique. Because $B(N)$ is non-decreasing in N, there exits an integer N^* such that

$$B(N) \geq 1 \Leftrightarrow N \geq N^*$$

Note that N^* is the minimum in (30), which is an optimal replacement policy. Furthermore, it is easy to verify that if $B(N^*) > 1$, then the optimal policy is existent and unique.

6 Special cases

In this section, we consider the following two special cases:

Case 1. If $P(\varpi_j^{(n)} = \infty) = 1$, i.e. $r_n = 0,\ j, n = 1, 2, \cdots$, then shocks cause no harm on the component. The system is a repairable system that the repair time of component is GP with linear corrective maintenance. Hence, the expected long run average cost rate of the system under policy N is given by

$$C(N) = \frac{C_r + C_m \sum_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + C_p\theta - C_w \sum_{k=1}^{N} \int_0^\infty e^{-B_{k-1}t - A_{k-1}\int_0^t h_0(x)\mathrm{d}x}\mathrm{d}t}{\sum_{k=1}^{N} \int_0^\infty e^{-B_{k-1}t - A_{k-1}\int_0^t h_0(x)\mathrm{d}x}\mathrm{d}t + \sum_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + \theta}. \tag{31}$$

Case 2. If $P(\eta_n = \infty) = 1,\ n = 1, 2, \cdots$, then the component failure is only caused by external shocks and the system is repairable. Hence, the expected long run average cost rate of the system under policy N can be obtained from (28) by taking $\alpha_k = \beta_k = 0$. It is given by

$$C(N) = \frac{C_r + (C_m + C_w)\sum_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + (C_p + C_w)\theta}{\sum_{k=1}^{N} \frac{1}{r_k\lambda} + \sum_{k=1}^{N-1} \frac{\mu}{b^{k-1}} + \theta} - C_w. \tag{32}$$

7 Numerical examples

To illustrate the above derivation, we carry out the following numerical experiments and assume the distribution function of the value of each shock, the threshold, the intrinsic lifetime and the repair time of a new component, and the replacement time of the system are

$$F(x) = \begin{cases} 1 - \exp\left(-\gamma x^2\right), & x > 0, \\ 0, & x \leq 0, \end{cases} \qquad \Phi(x) = \begin{cases} 1 - \exp\left(-\frac{\gamma}{2}x^2\right), & x > 0, \\ 0, & x \leq 0, \end{cases}$$

$$D(x)=\begin{cases}1-\exp\left(-\varepsilon x^2\right), & x>0,\\ 0, & x\le 0,\end{cases}\qquad H(x)=\begin{cases}1-\exp\left(-\nu x^2\right), & x>0,\\ 0, & x\le 0,\end{cases}$$

$$S(x)=\begin{cases}1-\exp\left(-\dfrac{x}{\theta}\right), & x>0,\\ 0, & x\le 0,\end{cases}$$

respectively, where $\gamma>0,\varepsilon>0,\nu>0,\theta>0$. Then, the probability that one shock causes the system to fail in the nth cycle is

$r_n=\int_0^\infty \Phi(a^{n-1}x)\mathrm{d}F(x)=\dfrac{a^{2n-2}}{a^{2n-2}+2},\ n=1,2,\cdots.$

Following $H(x)$, we have

$$\mu=\int_0^\infty \overline{H}(x)\mathrm{d}x=\int_0^\infty e^{-\nu x^2}\mathrm{d}x=\frac{\sqrt{\pi}}{2\sqrt{\nu}}.$$

According to Lemma 1, we have the distribution of the shock lifetime of the component in the nth cycle ξ_n

$$L_n(x)=1-\exp\left(-\frac{a^{2n-2}\lambda}{a^{2n-2}+2}x\right),x\ge 0.$$

We also assume $\alpha_k=\alpha,\beta_k=\beta,k=1,2,\cdots$. Then the distribution function of X_n is given by

$$G_n(x)=1-\exp\left\{-\left(\frac{a^{2n-2}\lambda}{a^{2n-2}+2}x+\frac{(\alpha^{n-1}-1)\beta}{\alpha-1}x+\alpha^{n-1}\varepsilon x^2\right)\right\},x\ge 0.$$

Denote $v_k=\alpha^{k-1}\varepsilon$ and $b_k=\dfrac{a^{2k-2}\lambda}{a^{2k-2}+2}+\dfrac{(\alpha^{k-1}-1)\beta}{\alpha-1}$. Then,

$$\begin{aligned}\lambda_k &= \int_0^\infty e^{-b_kx-v_kx^2}dx=e^{\frac{b_k^2}{4v_k}}\int_{\frac{b_k}{2v_k}}^\infty e^{-v_kt^2}\mathrm{d}t=e^{\frac{b_k^2}{4v_k}}\left(\int_0^\infty e^{-v_kt^2}\mathrm{d}t-\int_0^{\frac{b_k}{2v_k}} e^{-v_kt^2}\mathrm{d}t\right)\\ &= e^{\frac{b_k^2}{4v_k}}\left(\frac{\sqrt{\pi}}{2\sqrt{v_k}}-\int_0^{\frac{b_k}{2v_k}} e^{-v_kt^2}\mathrm{d}t\right). \end{aligned}\tag{33}$$

Substituting μ and λ_k into (28) and (29), respectively, we obtain the expression for $C(N)$ and $B(N)$ as follows:

$$C(N)=\frac{C_r+(C_m+C_w)\sum_{k=1}^{N-1}\frac{\sqrt{\pi}}{2\sqrt{\nu}b^{k-1}}+(C_p+C_w)\theta}{\sum_{k=1}^{N}e^{\frac{b_k^2}{4v_k}}\left(\frac{\sqrt{\pi}}{2\sqrt{v_k}}-\int_0^{\frac{b_k}{2v_k}}e^{-v_kt^2}\mathrm{d}t\right)+\sum_{k=1}^{N-1}\frac{\sqrt{\pi}}{2\sqrt{\nu}b^{k-1}}+\theta}-C_w,\tag{34}$$

$$B(N)=\frac{\frac{(C_m+C_w)\sqrt{\pi}}{2\sqrt{\nu}}\left[\sum_{k=1}^{N}e^{\frac{b_k^2}{4v_k}}\left(\frac{\sqrt{\pi}}{2\sqrt{v_k}}-\int_0^{\frac{b_k}{2v_k}}e^{-v_k t^2}\mathrm{d}t\right)+\theta-e^{\frac{b_{N+1}^2}{4v_{N+1}}}\left(\frac{\sqrt{\pi}}{2\sqrt{v_{N+1}}}-\int_0^{\frac{b_{N+1}}{2v_{N+1}}}e^{-v_{N+1}t^2}\mathrm{d}t\right)\sum_{k=1}^{N-1}b^{N-k}\right]}{[C_r+(C_m+C_p)\theta]\left[e^{\frac{b_{N+1}^2}{4v_{N+1}}}\left(\frac{\sqrt{\pi}}{2\sqrt{v_{N+1}}}-\int_0^{\frac{b_{N+1}}{2v_{N+1}}}e^{-v_{N+1}t^2}\mathrm{d}t\right)b^{N-1}+\frac{\sqrt{\pi}}{2\sqrt{\nu}}\right]}. \tag{35}$$

Then we can determine the optimal repair-replacement policy N^* by numerical methods such that $C(N^*)$ is minimized. We will see in the two numerical examples of Subsection 7.1 that the optimal policy N^* for $C(N)$ is unique.

7.1 Optimal policy N^*

In this subsection, we consider two situations about the deterioration degree of the system: the system deteriorates faster than before, i.e $\alpha > 1$ and the system deteriorates slower than before, i.e., $\alpha < 1$, and then we provide numerical examples to show an approach to determining N^*.

Situation 1: The system deteriorates faster than before. The system is simulated with the parameters $a = 1.05, b = 0.95, \lambda = 0.002, \varepsilon = 0.0001, \nu = 0.004, \alpha = 1.01, \beta = 0.0006, \theta = 5, C_w = 30, C_m = 20, C_p = 10$, and $C_r = 4500$. By numerical calculation, the average cost rate of the system is shown in Table 2, which corresponds to Fig. 2. From Table 2 and Fig. 2, we can see that the value $C(N)$ decreases when the number of failures N increases from 1 to 11, and then the value $C(N)$ increases with N. This shows that there exists a unique optimal repair-replacement policy N^*. From Table 2, we can find that the optimal number of failures for replacement is $N^* = 11$, and the corresponding minimal long run average cost per unit time is $C(N^*) = -15.8967$.

The same conclusion can be drawn by calculating the values of $B(N)$ and the results are presented in Table 3 and Fig. 3. It follows from Table 3 that $B(11) = 1.0908$ and 11 is the first integer such that $B(N) \geq 1$ and $N^* = 11$ is the optimal replacement policy from Theorem 3. Moreover, the optimal replacement policy is unique since $B(N^*) = B(11) = 1.0908 > 1$.

Situation 2: The system deteriorates slower than before. The system is simulated with the same parameters as Situation 1 except $\alpha = 0.98$. By numerical calculation, the average cost rate of the system is shown in Table 2, which corresponds to Fig. 2. Similarly, from Table 2 and Fig. 2, we know that the value $C'(N)$ decreases when the number of failures N changes from 1 to 12, and then increases with N. The optimal repair-replacement policy N^* exists uniquely, the optimal number of failures for replacement is $N^* = 12$, and the corresponding minimal long run average cost per unit time is $C'(N^*) = -16.5555$.

The same conclusion can be arrived at by calculating the values of $B'(N)$, from the results in Table 3 and Fig. 3 , we obtain: $B'(12) = 1.0890 > 1$ and 12 is the first integer such that $B'(N) \geq 1$. According to Theorem 3, the optimal replacement policy $N^* = 12$ is unique.

Comparing $C(N)$ and $C'(N)$ in Fig. 2 and Table 2, we obtain

$$C(N^*) = C(11) = -15.8967 > C'(N^*) = C'(12) = -16.5555,$$

and for the same N,

$$C(N) > C'(N).$$

Thus, the maintenance policy on the model with $\alpha = 0.98 < 1$ costs less than the maintenance policy on the model with $\alpha = 1.01 > 1$.

Table 2. The expected long run cost per unit time versus the repair-replacement policy N

Table 2

Figure 2

Fig. 2. The average cost rate $C(N)$ versus the repair-replacement policy N.

Table 3. The values of auxiliary function $B(N)$ (or $B'(N)$) versus the values of N

Table 3

Figure 3

Fig. 3. The values of auxiliary function $B(N)$ (or $B'(N)$) versus the values of N.

7.2 Sensitivity analysis

If we keep the values of parameters in Situation 2 in Section 7.1 unchanged, we obtain the results as shown in Table 4.

Table 4 shows how much N^* and $C(N^*)$ change when the parameter b increases from 0.81 to 0.99. From Table 4, we have the following results. N^* is sensitive to a small change in parameter b when b is larger than 0.91. It becomes stable when b is smaller than 0.91; it changes from 6 to 8 when b increases from 0.81 to 0.89. However, the average cost rate $C(N^*)$ decreases in b.

Similarly, Table 4 shows how N^* and $C(N^*)$ vary when the ageing alteration parameter α increases from 0.9 to 1.2. From Table 4, we have the following results. N^* is non-increasing in α. It is sensitive to a small change of parameter α when α is smaller than 1.03. N^* changes from 9 to 20 when parameter α decreases from 1.03 to 0.90 and it becomes stable when b is larger than 1.03 and it varies from 8 to 10 when α decreases from 1.2 to 1.07. The average cost rate $C(N^*)$ increases in α. It can also be seen that the optimal N^* is non-increasing in β, but the average cost rate $C(N^*)$ increases in β.

Table 4. Optimal N^* and $C(N^*)$ versus for different values of b, α and β

Table 4

8 Conclusions

This paper derives reliability indices and the expected cost rate for a repairable one-component system that may fail due to system's intrinsic and extrinsic factors. Maintenance policy is derived when both the thresholds of shocks and the repair times of the system follow geometric processes. A numerical example is given to illustrate the theoretical results of the model.

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9 Figure captions page, tables, and figures.

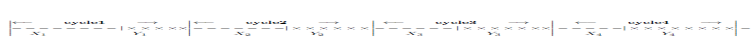

Figure 1: A possible realization of the system.

Table 2: The expected long run cost per unit time versus repair-replacement policy N

N	$C(N)$	$C'(N)$	N	$C(N)$	$C'(N)$	N	$C(N)$	$C'(N)$	N	$C(N)$	$C'(N)$
1	21.9995	21.9995	11	**-15.8967**	-16.6277	21	-13.8009	-15.2200	31	-9.7056	-11.9254
2	-1.0344	-1.2130	12	-15.8627	**-16.6555**	22	-13.4529	-14.9487	32	-9.2354	-11.5367
3	-8.0859	-8.3483	13	-15.7711	-16.6273	23	-13.0887	-14.6622	33	-8.7569	-11.9254
4	-11.3900	-11.7169	14	-15.6323	-16.5534	24	-12.7094	-14.3616	34	-8.2708	-10.7336
5	-13.2279	-13.6132	15	-15.4539	-16.4417	25	-12.3160	-14.0478	35	-7.7776	-10.3199
6	-14.3389	-14.7805	16	-15.2418	-16.2978	26	-11.9094	-13.7216	36	-7.2779	-9.8987
7	-15.0344	-15.5321	17	-15.0004	-16.1261	27	-11.4905	-13.3836	37	-6.7724	-9.4705
8	-15.4686	-16.0228	18	-14.7331	-15.9300	28	-11.0599	-13.0345	38	-6.2616	-9.0355
9	-15.7263	-16.3382	19	-14.4427	-15.7123	29	-10.6185	-12.6747	39	-5.7463	-8.5942
10	-15.8584	-16.5291	20	-14.1313	-15.4750	30	-10.1668	-12.3049	40	-5.2269	-8.1470

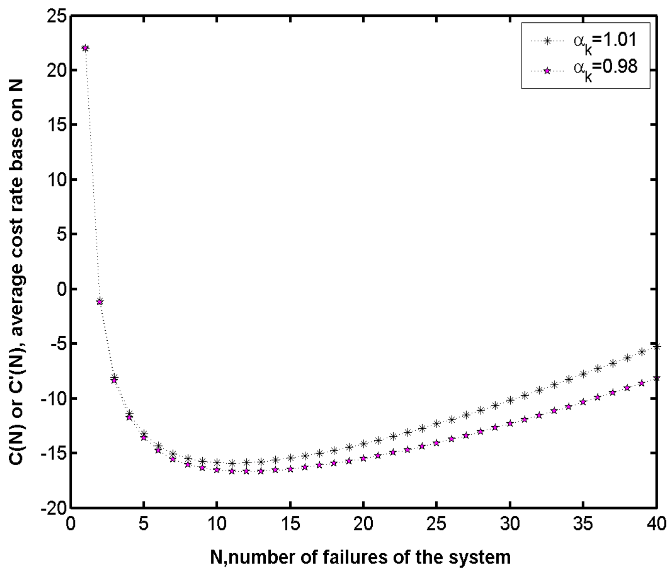

Figure 2: A plot of average cost rate $C(N)$ and $C'(N)$ for policy N.

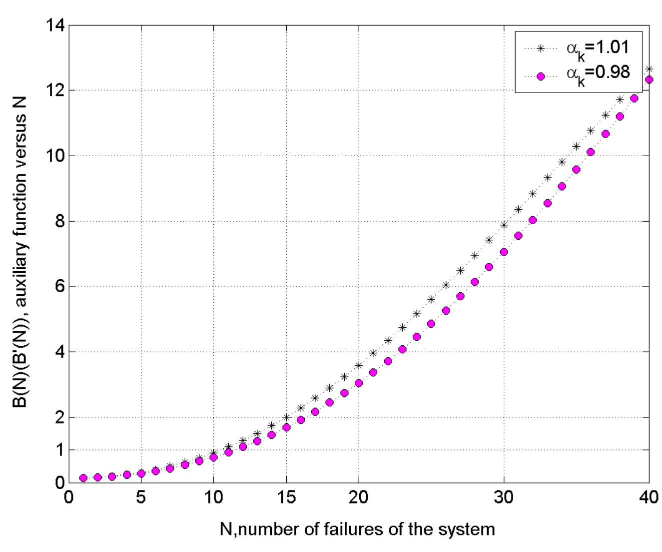

Figure 3: A plot of the auxiliary function, $B(N)$ and $B'(N)$ versus N.

Table 3: Results of $B(N)$ and $B'(N)$

N	$B(N)$	$B'(N)$	N	$B(N)$	$B'(N)$	N	$B(N)$	$B'(N)$	N	$B(N)$	$B'(N)$
1	0.1403	0.1386	11	**1.0908**	0.9256	21	3.9610	3.3732	31	8.36060	7.5452
2	0.1633	0.1579	12	1.2878	**1.0890**	22	4.3489	3.7180	32	8.8400	8.0394
3	0.1200	0.1885	13	1.5053	1.2699	23	4.7518	4.0803	33	9.3214	8.5447
4	0.2511	0.2311	14	1.7433	1.4687	24	5.1684	4.4596	34	9.8037	9.0602
5	0.3178	0.2864	15	2.0018	1.6856	25	5.5976	4.8555	35	10.2856	9.5852
6	0.4009	0.3553	16	2.2806	1.9209	26	6.0379	5.2674	36	10.7662	10.1188
7	0.5014	0.4383	17	2.5793	2.1746	27	6.4881	5.6948	37	11.2446	10.6602
8	0.6200	0.5363	18	2.8974	2.4468	28	6.9468	6.1369	38	11.7197	11.2087
9	0.7573	0.6497	19	3.2342	2.7373	29	7.4127	6.5932	39	12.1909	11.7633
10	0.9141	0.7793	20	3.5891	3.0462	30	7.8844	7.0629	40	12.6573	12.3234

Table 4: Optimal N^* and $C(N^*)$ versus for different values of b,α and β

d	$\alpha = 0.98, \beta = 0.0006$		α	$b = 0.95, \beta = 0.0006$		β	$b = 0.95, \alpha = 0.98$	
	N^*	$C(N^*)$		N^*	$C(N^*)$		N^*	$C(N^*)$
0.99	20	-18.5724	0.90	16	-19.0220	0.0004	13	-17.2181
0.988	19	-18.4476	0.91	15	-18.6955	**0.0006**	**12**	**-16.6555**
0.985	18	-18.2704	0.92	15	-18.3813	0.0008	11	-16.1277
0.98	17	-17.9940	0.93	14	-18.0722	0.0012	11	-15.1706
0.97	15	-17.4963	0.94	14	-17.7715	0.0014	10	-14.7413
0.96	13	-17.0569	0.95	13	-17.4806	0.0016	10	-14.3231
0.95	**12**	**-16.6555**	0.96	13	-17.1968	0.0018	10	-13.9157
0.94	11	-16.2849	0.97	12	-16.9199	0.0020	9	-13.5339
0.93	10	-15.9344	**0.98**	**12**	**-16.6555**	0.0030	8	-11.7962
0.92	10	-15.6106	**1.01**	**11**	**-15.8967**	0.0040	8	-10.3195
0.91	9	-15.3050	1.02	11	-15.6540	0.0050	7	-8.9785
0.89	8	-14.7252	1.03	10	-15.4144	0.0070	7	-6.7277
0.87	8	-14.2025	1.07	10	-14.5369	0.0090	7	-4.7930
0.86	7	-13.9425	1.08	9	-14.3274	0.0100	6	-3.9366
0.82	7	-13.0079	1.14	9	-13.1695	0.0130	6	-1.7113
0.81	6	-12.8184	1.20	8	-12.1596	0.0150	6	-0.4283