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# POLYNOMIAL POISSON ALGEBRAS: GEL'FAND-KIRILLOV PROBLEM AND POISSON SPECTRA

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## Abstract


We study the fields of fractions and the Poisson spectra of polynomial Poisson algebras.

First we investigate a Poisson birational equivalence problem for polynomial Poisson algebras over a field of arbitrary characteristic. Namely, the quadratic Poisson Gel'fand-Kirillov problem asks whether the field of fractions of a Poisson algebra is isomorphic to the field of fractions of a Poisson affine space, i.e. a polynomial algebra such that the Poisson bracket of two generators is equal to their product (up to a scalar). We answer positively the quadratic Poisson Gel'fand-Kirillov problem for a large class of Poisson algebras arising as semiclassical limits of quantised coordinate rings, as well as for their quotients by Poisson prime ideals that are invariant under the action of a torus. In particular, we show that coordinate rings of determinantal Poisson varieties satisfy the quadratic Poisson Gel'fand-Kirillov problem. Our proof relies on the so-called characteristic-free Poisson deleting derivation homomorphism. Essentially this homomorphism allows us to simplify Poisson brackets of a given polynomial Poisson algebra by localising at a generator.

Next we develop a method, the characteristic-free Poisson deleting derivations algorithm, to study the Poisson spectrum of a polynomial Poisson algebra. It is a Poisson version of the deleting derivations algorithm introduced by Cauchon [8] in order to study spectra of some noncommutative noetherian algebras. This algorithm allows us to define a partition of the Poisson spectrum of certain polynomial Poisson algebras, and to prove the Poisson Dixmier-Moeglin equivalence for those Poisson algebras when the base field is of characteristic zero. Finally, using both Cauchon's and our algorithm, we compare combinatorially spectra and Poisson spectra in the framework of (algebraic) deformation theory. In particular we compare spectra of quantum matrices with Poisson spectra of matrix Poisson varieties.

*to Alizée*

# Contents

# Introduction

Poisson algebras have been intensively and widely studied since their first appearance, both on their own and in connection with other areas of mathematics. For instance, we refer to [29] where Poisson structures are studied from the differential geometry point of view, [13] where links with number theory are made or [16] for the connection with noncommutative algebra, but this literature is of course non exhaustive. Our approach to Poisson algebras is intimately related to the study of their (algebraic) deformations. In fact our inspiration often comes from noncommutative algebra and we always try to see both worlds together in the same picture. In this thesis we study polynomial Poisson algebras, i.e. polynomial algebras in several variables endowed with Poisson structures. Our investigation focuses on two main aspects. First, we investigate the structure of their fields of fractions; second, we study their Poisson prime spectra, both on their own and in connection with the spectra of their deformations. For this purpose, we develop the so-called *characteristic free Poisson deleting derivation homomorphism* which helps us to understand both situations.

Let $\mathbb{K}$ be a field. Recall that a Poisson $\mathbb{K}$-algebra is a commutative $\mathbb{K}$-algebra endowed with a Poisson bracket, i.e. a skew-symmetric $\mathbb{K}$-bilinear map from $A \times A$ to $A$ satisfying the Jacobi identity and the Leibniz rule. Assuming that $A$ is a domain, we can uniquely extend the Poisson bracket to the field of fractions $\operatorname{Frac} A$ of $A$.

## Gel'fand-Kirillov problems

The first part of this thesis (Chapters 2, 3 and 4) is concerned with the Poisson structure of the field of fractions of polynomial Poisson algebras. Examples of polynomial Poisson algebras include the so-called *Poisson-Weyl algebras*. Recall that the *Poisson-Weyl algebra* of dimension $2r$ (or the $r$-th *Poisson-Weyl algebra*) is the polynomial algebra in $2r$ generators $X_1, \dots, X_r, Y_1, \dots, Y_r$ endowed with the Poisson bracket defined on the generators by $\{X_i, X_j\} = \{Y_i, Y_j\} = 0$ and $\{X_i, Y_j\} = \delta_{ij}$ for all $i, j$. Note that this is an algebraic version of the bracket given by Poisson for smooth functions on $\mathbb{R}^{2r}$. Poisson's bracket plays a crucial rôle in the context of Hamiltonian mechanics and is central in the study of Poisson manifolds. For instance, Darboux's theorem asserts that in a Poisson manifold the Poisson bracket takes, locally around each point where the Poisson matrix is locally constant, the same values (on a set of local coordinates) as the Poisson bracket described above, (up to the addition of some Casimir coordinates). Back to the algebraic setting, the field of fractions of the $r$-th Poisson-Weyl algebra is referred to as the *$r$-th Poisson-Weyl field*. It is a central object in the theory, and often, for a given polynomial Poisson algebra, one tries to decide whether it is Poisson birationally equivalent to a Poisson-Weyl algebra, that is, we would like to know whether there exists a Poisson algebras isomorphism between the field of fractions of the given polynomial Poisson algebra and a Poisson-Weyl field of appropriate dimension (possibly over a purely transcendental extension of the ground field).

This problem was first raised by Vergne in [37], where the author studied the case of the symmetric algebra $S(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ in characteristic 0, the polynomial algebra $S(\mathfrak{g})$ being endowed with the so-called *Kirillov-Kostant-Souriau Poisson structure*: for a basis $U_1, \dots, U_n$ of $\mathfrak{g}$, the Poisson bracket on $S(\mathfrak{g})$ is given by $\{U_i, U_j\} = [U_i, U_j]_{\mathfrak{g}}$ for all $i, j$. When $\mathfrak{g}$ is nilpotent, Vergne showed that the field of fractions of $S(\mathfrak{g})$ is Poisson isomorphic to the field of fractions of a Poisson-Weyl algebra over a purely transcendental extension of the ground field. In [35], this result was extended to the solvable case by Tauvel and Yu. Moreover, still assuming $\mathfrak{g}$ is solvable, they proved that this result also holds for any quotient of $S(\mathfrak{g})$ by a Poisson prime ideal.

The problem raised by Vergne takes its roots in the celebrated Gel'fand-Kirillov Conjecture [14] which is a problem of birational equivalence between enveloping algebras of Lie algebras and Weyl skewfields. More precisely, the Gel'fand-Kirillov Conjecture says that: "the skewfield of fractions of the enveloping algebra of any finite-dimensional complex algebraic Lie algebra is isomorphic to a Weyl skewfield". This conjecture was first proved for the Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ and for nilpotent Lie algebras in 1966 by Gel'fand

and Kirillov [14]. The solvable case was proved independently by Borho, Joseph and McConnell in 1973. However, the conjecture is not true in general: a class of counterexamples is found in 1996 by Alev, Ooms and Van Den Bergh [2]. More recently, Premet showed that the conjecture also fails for simple Lie algebras of the types $B_n$ ($n \geq 3$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$ and $F_4$, see [34]. Note that the algebras involved in the statement of the Gel'fand-Kirillov Conjecture are considered over algebraically closed fields of characteristic zero. However, the conjecture also makes sense in positive characteristic, see for instance [5]. In [34], the author refutes the Gel'fand-Kirillov Conjecture for the enveloping algebra of simple Lie algebras of certain types by actually refuting a modular version of the conjecture. This certainly shows that we should not restrict our attention only to the case where the characteristic is 0, but also study the modular case. This motivated us to study the Poisson structure of fields of fractions of polynomial Poisson algebras over a field of arbitrary characteristic.

With the appearance of quantum groups in the eighties, new skewfields of reference were needed, and a quantum version of the Gel'fand-Kirillov Conjecture was suggested by Alev and Dumas [1], and studied by numerous authors. In this context, the skewfields of reference are the skewfields of fractions of quantised Weyl algebras, or equivalently, the skewfields of fractions of quantum affine spaces. The latter having simpler defining relations, we will take them as skewfields of reference. Let $\boldsymbol{q} \in M_n(\mathbb{K}^\times)$ be a multiplicatively skew-symmetric matrix, i.e. $q_{ij}^{-1} = q_{ji}$ for all $i, j$. Then, the *quantum affine space* $\mathcal{O}_{\boldsymbol{q}}(\mathbb{K}^n)$ is the algebra given by $n$ generators $x_1, \dots, x_n$ and relations $x_i x_j = q_{ij} x_j x_i$ for all $i, j$. Quantum affine spaces are noetherian domains and therefore admit skewfields of fractions which we will refer to as *quantum affine skewfields*. It is shown in [1] that quantum affine skewfields are never isomorphic to Weyl skewfields. The quantum Gel'fand-Kirillov problem asks, given a "quantum algebra", if its skewfield of fractions is isomorphic to the skewfield of fractions of a quantum affine space $\mathcal{O}_{\boldsymbol{q}}(\mathbb{L}^n)$ where $\mathbb{L}$ is a purely transcendental extension of $\mathbb{K}$. Of course, the class of quantum algebras is not clearly defined but it should include quantised enveloping algebras and quantised coordinate rings for instance. The quantum Gel'fand-Kirillov problem has been successfully investigated for several families of algebras of these types. In particular, the deleting derivations algorithm of Cauchon [8] gives a positive answer for a large class of iterated Ore extensions and their prime quotients. We refer to [6, I.2.11 and II.10.4] for further information about this quantum version of the Gel'fand-Kirillov Conjecture.

If we return to the Poisson setting, it is easy to find polynomial Poisson algebras whose fields of fractions are not Poisson isomorphic to Poisson-Weyl algebras. Thus, as in the quantum case, we need to introduce other Poisson fields of reference as follows. A *Poisson affine field* is the field of fractions of a Poisson affine space, i.e. the field of fractions of a polynomial algebra in $n$ indeterminates $X_1, \ldots, X_n$, with Poisson bracket given by $\{X_i, X_j\} = \lambda_{ij} X_i X_j$ for some skew-symmetric matrix $(\lambda_{ij}) \in M_n(\mathbb{K})$. It was proved in [18] that Poisson-Weyl fields and Poisson affine fields are not isomorphic, so that Poisson affine fields were used in [18] as fields of reference for a Poisson version of the quantum Gel'fand-Kirillov problem. Namely, the quadratic Poisson Gel'fand-Kirillov problem asks whether a given polynomial Poisson algebra is Poisson birationally equivalent to a Poisson affine space. In [18], it was shown that the fields of fractions of a large class of Poisson algebras are Poisson isomorphic to Poisson affine fields (over purely transcendental extensions of the base field). The method used to prove these Poisson isomorphisms is based on a Poisson version of the deleting derivation homomorphism introduced by Cauchon in [8]. We note that, while Cauchon's deleting derivation homomorphism cannot be defined when the quantum parameter involved is a root of unity, Haynal [23] generalised Cauchon's construction to the root of unity case by using the notion of *higher derivation.*

The main aim of Chapters 2 and 3 is to establish the quadratic Poisson Gel'fand-Kirillov problem for a large class of polynomial Poisson algebras (and their quotients) over a field of arbitrary characteristic. In characteristic zero, the main tool used in [18] for the same purpose is the so-called Poisson deleting derivation homomorphism. This homomorphism is a Poisson algebra isomorphism between localisations of two Poisson-Ore extensions:

$$\begin{aligned} F : A[Y^{\pm 1}; \alpha]_P &\xrightarrow{\cong} A[X^{\pm 1}; \alpha, \delta]_P \\ A \ni a \quad &\longmapsto \sum_{i \geq 0} \left(\frac{-1}{s}\right)^i \frac{\delta^i(a)}{i!} X^{-i}, \\ Y \quad &\longmapsto X. \end{aligned}$$

under the assumptions that the derivation $\delta$ is locally nilpotent and $\alpha\delta = \delta(\alpha + s)$ for some $s \in \mathbb{K}^{\times}$.

Obviously, the above formula defining the Poisson deleting derivation homomorphism does not make sense in positive characteristic due to the division by $i!$. To overcome this problem and define a characteristic-free Poisson deleting derivation homomorphism, we observe that the sequence of linear maps $\left(\frac{\delta^i}{i!}\right)$ is a so-called iterative higher derivation which extends the derivation $\delta$ (that is, whose first terms are id and $\delta$). In Chapter 2, we construct a characteristic-free Poisson deleting derivation homomorphism in the case where the derivation $\delta$ extends to a so-called iterative *higher Poisson derivation*, i.e. an iterative higher derivation compatible with the Poisson structure.

In Chapter 3, we use the characteristic-free Poisson deleting derivation homomorphism repeatedly to prove that the quadratic Poisson Gel'fand-Kirillov problem holds for a large class of iterated Poisson-Ore extensions. We actually prove a stronger result by also considering Poisson prime quotients. More precisely, we show that if $P$ is a Poisson prime ideal of a polynomial Poisson algebra $A$ to which our construction applies, then there exists a Poisson prime ideal $Q$ in a Poisson affine space $B$ such that $\mathrm{Frac}\,(A/P) \cong \mathrm{Frac}\,(B/Q)$ as Poisson algebras (this proves the quadratic Poisson Gel'fand-Kirillov problem for $A$ since $Q = 0$ when $P = 0$). Additionally, if a torus $H$ is acting rationally by Poisson automorphisms on $A$ and if $P$ is invariant under this action, we show, modulo some technical assumptions, that the ideal $Q$ of the Poisson affine space $B$ is also invariant under the induced torus action on $B$. Under certain mild assumptions on the ground field, we prove that $B$ has only finitely many $H$-invariant Poisson prime ideals and that they are all generated by some of its generators. As a consequence, when $P$ is $H$-invariant, the quotient $B/Q$ is a Poisson affine space, so that the quotient $A/P$ also satisfies the quadratic Poisson Gel'fand-Kirillov problem.

Contrary to the characteristic zero case, there is one hypothesis in our result that is difficult to check: the existence of iterative higher Poisson derivations extending given derivations. In characteristic zero, the only iterative higher Poisson derivation extending a derivation $\delta$ is actually the *canonical higher derivation* $(\frac{\delta^i}{i!})$. In prime characteristic, the existence of an iterative higher Poisson derivation extending a given derivation is a harder problem. In Chapter 4 we tackle this problem using the so-called *semiclassical limit* process. More precisely, we show that the existence of a quantum version of the canonical higher derivation in a "quantum algebra" $R$ ensures (under mild hypotheses) the existence of a higher Poisson derivation in the semiclassical limit of $R$ (see Theorem 4.1.3). At the noncommutative level, the characteristic of the ground field does not influence the existence of a quantum version of the canonical higher derivation. The existence only depends on the genericity of the deformation parameter. However, in our case, the deformation parameter is always transcendental (to allow for the semiclassical limit process), thus ensuring the existence of quantum canonical higher derivations. As a consequence, we obtain many examples of Poisson algebras to which our result applies in Section 4.2. For instance, we obtain that the coordinate rings of Poisson matrix varieties and their $H$-invariant Poisson prime quotients, such as the coordinate rings of determinantal varieties, satisfy the quadratic Poisson Gel'fand-Kirillov problem (over a field of characteristic different of 2).

## Poisson spectrum

In the second part of this thesis (Chapters 5, 6 and 7) we turn our attention to the study of Poisson spectra of polynomial Poisson algebras. Different aspects of this topic have been investigated previously. For instance, the Poisson Dixmier-Moeglin equivalence is studied in [3], [15], [18] and [32], links between Poisson spectra and their quantum analogues are investigated in [22], [24] and [32] and Poisson spectra of Jacobian Poisson structures and generalisations in higher dimensions are studied in [26] and [25].

Inspired by [8], we develop a method to study the algebras of a class $\mathcal{P}$ of iterated Poisson-Ore extensions over a field of arbitrary characteristic. More precisely for $A \in \mathcal{P}$, the (characteristic-free) *Poisson deleting derivations algorithm* consists of performing several explicit changes of variables inside the field of fractions $\operatorname{Frac} A$ of $A$. At each step of the algorithm we obtain a sequence of $n$ algebraically independent elements of $\operatorname{Frac} A$, where the integer $n$ corresponds to the number of indeterminates in $A$. The subalgebra of $\operatorname{Frac} A$ generated by these elements is a Poisson algebra with a "simpler" Poisson bracket than the one obtained at the previous step. Moreover the Poisson algebras corresponding to two consecutive steps, say $C_{j+1}$ and $C_j$, satisfy:

$$C_{j+1}S_j^{-1} = C_j S_j^{-1}$$

for a given multiplicatively closed set $S_j$. After the last step, we get algebraically independent elements $T_1, \dots, T_n$ of $\operatorname{Frac} A$ such that the algebra $\overline{A}$ generated by the $T_i$s is a Poisson affine space. In particular, the algorithm shows that $\operatorname{Frac} A = \operatorname{Frac} \overline{A}$ as Poisson algebras. Therefore we retrieve the results of Poisson birational equivalence obtained in Chapter 3, that is the Poisson algebras of the class $\mathcal{P}$ satisfy the quadratic Poisson Gel'fand-Kirillov problem. Moreover, the algorithm returns explicit generators of $\operatorname{Frac} A$ such that $\operatorname{Frac} A$ is a Poisson affine field in these generators.

For a Poisson algebra $A$ we denote by $\text{P.Spec}\,(A)$ its *Poisson spectrum*, i.e. the set of prime ideals of $A$ which are also Poisson ideals. The set $\text{P.Spec}\,(A)$ is equipped with the induced Zariski topology from $\text{Spec}\,(A)$ the spectrum of $A$. When $A \in \mathcal{P}$, our algorithm allows us to define an embedding $\varphi$ from $\text{P.Spec}\,(A)$ to $\text{P.Spec}\,(\overline{A})$ called the *canonical embedding*. This embedding will be our main tool for studying Poisson spectra. One of its important properties is that for $P \in \text{P.Spec}\,(A)$ we have a Poisson algebra isomorphism

$$\operatorname{Frac}\Big(\frac{A}{P}\Big) \cong \operatorname{Frac}\Big(\frac{\overline{A}}{\varphi(P)}\Big).$$

Note that this isomorphism reduces the quadratic Poisson Gel'fand-Kirillov problem for the Poisson prime quotients of $A$ to the quadratic Poisson Gel'fand-Kirillov problem for the Poisson prime quotients of a Poisson affine space. The canonical embedding leads to a partition of $\text{P.Spec}\,(A)$ indexed by a subset $W'_P$ of $W := \mathscr{P}([\![1,n]\!])$, the powerset of $[\![1,n]\!] := \{1,\dots,n\}$. More precisely, for $w \in W$, we set:

$$\text{P.Spec}_w(\overline{A}) := \big\{P \in \text{P.Spec}\,(\overline{A}) \mid P \cap \{T_1,\dots,T_n\} = \{T_i \mid i \in w\}\big\},$$

where we recall that the $T_i$s are the generators of the Poisson affine space $\overline{A}$. These sets form a partition of $\text{P.Spec}\,(\overline{A})$ which induces a partition on $\text{P.Spec}\,(A)$ as follows:

$$\text{P.Spec}\,(A) = \bigsqcup_{w \in W'_P} \varphi^{-1}\big(\text{P.Spec}_w(\overline{A})\big), \quad \text{where:}$$

$$W'_P := \{w \in W \mid \varphi^{-1}\big(\text{P.Spec}_w(\overline{A})\big) \neq \emptyset\}.$$

This partition of $\text{P.Spec}\,(A)$ is called the *canonical partition*, and the elements of $W'_P$ will be called the *Cauchon diagrams associated to* $A$, or Cauchon diagrams for short. For $w \in W'_P$, the set $\varphi^{-1}\big(\text{P.Spec}_w(\overline{A})\big)$ is called the *stratum* associated to $w$. We study the topological and algebraic properties of those strata in Section 5.4. In particular for $w \in W'_P$ the image of the stratum associated to $w$ is a closed subset of $\text{P.Spec}_w(\overline{A})$ and $\varphi$ induces an homeomorphism from this stratum to its image. We show in Section 5.5

that this inclusion is actually an equality when we suppose that a torus acts rationally on $A$ by Poisson automorphisms. As shown in [15] such a torus action leads to another partition of P.Spec $(A)$ (under the assumption that char $\mathbb{K} = 0$). This partition, called the $H$-stratification, provides a great deal of information on P.Spec $(A)$, see for instance [15, Theorem 4.2]. In Section 5.5 we show that when both partitions can be considered they actually coincide.

A direct application of the results of Chapter 5 is given in Chapter 6. A subset of importance of P.Spec $(A)$ is the set consisting of Poisson primitive ideals. Recall that a (left) primitive ideal in a ring $R$ is the annihilator of a simple (left) $R$-module. It is usually not so easy to distinguish primitive ideals within prime ideals using their definition. Therefore other characterisations of primitive ideals have been investigated. For primitive ideals of enveloping algebras Dixmier and Moeglin suggested two characterisations, an algebraic one and a topological one. More precisely, let $R$ be a ring and $I$ a prime ideal in $R$. We say that $I$ is locally closed if $\{I\}$ is a locally closed point of Spec $(R)$ and that $I$ is rational if $Z(\mathrm{Frac}\, R/I)$ is an algebraic extension of the ground field. Dixmier [10] and Moeglin [30] showed that, for enveloping algebras, a prime ideal is primitive if and only if it is locally closed if and only if it is rational. More generally we say that the Dixmier-Moeglin equivalence holds for a given algebra (or a class of algebras) when the sets of primitive, locally closed and rational ideals coincide. Similarly, we say that the Poisson Dixmier-Moeglin equivalence holds for a Poisson algebra if the sets of Poisson primitive, Poisson locally closed and Poisson rational ideals coincide (see Section 6.1 for a precise definition of these notions). In Chapter 6 we prove that the Poisson Dixmier-Moeglin equivalence holds for all the algebras of the class $\mathcal{P}$ when char $\mathbb{K} = 0$. The Poisson deleting derivations algorithm allows us to reduce the proof to the case of Poisson affine spaces, which is solved by [15, Example 4.6] for instance. Moreover, the canonical partition gives us another characterisation for primitive ideals, namely, that they are the maximal ideals whithin their strata. To conclude Chapter 6 we prove a transfer result for Poisson-Ore extensions. More precisely we show that if the Poisson-Ore extension $A[X; \alpha]_P$ satisfies the Poisson Dixmier-Moeglin equivalence, then so does the Poisson-Ore extension $A[X; \alpha, \delta]_P$ (under some assumptions on the map $\delta$). For this purpose, we need to generalise some results of Chapter 5 essentially by constructing and studying a canonical embedding for the Poisson-Ore extension $A[X; \alpha, \delta]_P$ (which does not necessarily belong to $\mathcal{P}$).

In Chapter 7 we compare the Poisson spectrum of a Poisson algebra with the spectrum of (one of its) deformation. More precisely we define a class $\mathcal{R}$ of iterated Ore extensions over $\mathbb{K}[t^{\pm 1}]$ such that for an element $R_t$ of $\mathcal{R}$ the quotient algebra $R_q := R_t/(t-q)R_t$ (for a non root of unity $q \in \mathbb{K}^\times$) is a deformation of the Poisson algebra $A := R_t/(t-1)R_t$. Moreover, the class $\mathcal{R}$ is defined in such a way that Cauchon's deleting derivations algorithm can be applied to $R_q$ and our Poisson deleting derivations algorithm can be applied to $A$. Cauchon's algorithm leads to a partition of $\text{Spec}\,(R_q)$ indexed by the elements of a subset $W'$ of $W = \mathscr{P}([\![1,n]\!])$ for some integer $n \geq 1$. As explained previously the Poisson spectrum $\text{P.Spec}\,(A)$ is also partitioned into strata indexed by the elements of a subset $W'_P$ of $W$. The main goal of Chapter 7 is to compare the sets of Cauchon diagrams $W'$ and $W'_P$. More precisely we ask in Question 7.1.1 if these sets are equal (when $\text{char}\,\mathbb{K} = 0$). We answer positively Question 7.1.1 for three examples in small dimensions in Section 7.2 and in Appendix B. Finally, we investigate the case of quantum/Poisson matrices in Section 7.3. For this purpose, we construct a bijection between the set of Cauchon diagrams $W'_P$ and a set of combinatorial objects: rectangular grids whose boxes are coloured in black or white with a condition on the black boxes (see Definition 7.3.1). Cauchon showed in [9] that the set of Cauchon diagrams $W'$ is also in bijection with the same set of combinatorial objects. This answers positively Question 7.1.1 for these algebras. Unfortunately these positive results do not help to understand the general situation since they mostly rely on explicit computation of the sets of Cauchon diagrams.

Parts of the material of Chapters 2, 3 and 4 can be found in [28]. Some of the material in Chapters 5, 6 and 7 comes from [27]. Both [28] and [27] are joint work with my PhD supervisor Stéphane Launois.

# Chapter 1

# Polynomial Poisson algebras and their deformations

In this chapter we introduce the main objects studied in this thesis: Poisson algebras. Our study of Poisson algebras being closely related to the study of their (algebraic) deformations, we also recall the process of semiclassical limit which links the noncommutative algebra world with the Poisson world. We denote by $\mathbb{K}$ an arbitrary field.

## 1.1 Poisson algebra

Unless otherwise stated, by an algebra we mean an associative $\mathbb{K}$-algebra.

### 1.1.1 Definitions and first properties

**Definition 1.1.1.** A Poisson $\mathbb{K}$-algebra $A$ is a commutative $\mathbb{K}$-algebra endowed with a Poisson bracket, i.e. a skew-symmetric $\mathbb{K}$-bilinear map from $A \times A$ to $A$ satisfying:

- the Jacobi identity: $\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0$ for all $a,b,c \in A$,
- the Leibniz rule: $\{ab,c\} = a\{b,c\} + \{a,c\}b$ for all $a,b,c \in A$.

In particular, the Jacobi identity tells us that a Poisson algebra is also a Lie algebra. The Leibniz rule can be seen as a compatibility condition between the associative algebra and Lie algebra structures, and can be rephrased by saying that for all $a \in A$ the map $\{a, -\}$ is an (associative) $\mathbb{K}$-derivation of $A$. Such derivations are called *Hamiltonian derivations* and they forms a Lie subalgebra of $\mathrm{Der}(A)$, the Lie algebra of derivations on $A$.

*Remark* 1.1.2. (1) Let $A$ be a Poisson algebra generated by elements $X_1, \ldots, X_n$. Then the Poisson bracket is uniquely determined by the values $\{X_i, X_j\}$ for $1 \leq j < i \leq n$. Moreover if we have:

$$\{\{X_i, X_j\}, X_k\} + \{\{X_j, X_k\}, X_i\} + \{\{X_k, X_i\}, X_j\} = 0 \tag{1.1}$$

for all $1 \leq i, j, k \leq n$, then the Jacobi identity holds in $A$. Therefore to define a Poisson bracket on $A$ it is enough to give the values $\{X_i, X_j\}$ for $1 \leq j < i \leq n$ and then to check equation (1.1) for all $1 \leq i, j, k \leq n$.

(2) Suppose that $A = \mathbb{K}[X_1, \ldots, X_n]$. For $F \in A$ and all $1 \leq i \leq n$ we denote by $\frac{\partial F}{\partial X_i}$ the formal derivative of $F$ with respect to the indeterminate $X_i$. For all $F, G \in A$ we have:

$$\{F, G\} = \sum_{i,j=1}^{n} \frac{\partial F}{\partial X_i} \frac{\partial G}{\partial X_j} \{X_i, X_j\} = \sum_{1 \leq j < i \leq n} \Big( \frac{\partial F}{\partial X_i} \frac{\partial G}{\partial X_j} - \frac{\partial F}{\partial X_j} \frac{\partial G}{\partial X_i} \Big) \{X_i, X_j\}. \tag{1.2}$$

A proof of the first equality can be found in [29, Proposition 1.6]. The second equality arises by skew-symmetry.

*Example* 1.1.3. (1) Let $A$ be a commutative algebra. Then $A$ becomes a Poisson algebra by setting $\{a, b\} = 0$ for all $a, b \in A$. The algebra $A$ endowed with this Poisson bracket is referred to as an *abelian* Poisson algebra.

(2) Let $A = \mathbb{K}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$. We denote by $\delta_{ij}$ the Kronecker delta symbol. We define a Poisson bracket on $A$ by setting:

$$\{X_i, X_j\} = \{Y_i, Y_j\} = 0 \text{ and } \{X_i, Y_j\} = \delta_{ij} \text{ for all } i, j,$$

or equivalently by setting:

$$\{F, G\} = \sum_{1 \leq i \leq n} \Big( \frac{\partial F}{\partial X_i} \frac{\partial G}{\partial Y_i} - \frac{\partial F}{\partial Y_i} \frac{\partial G}{\partial X_i} \Big)$$

for all $F, G \in A$. This Poisson algebra is called the *n-th Poisson-Weyl algebra* and is denoted by $A_n^P(\mathbb{K})$.

(3) Let $A = \mathbb{K}[X_1, \ldots, X_n]$ and let $\boldsymbol{\lambda} = (\lambda_{ij}) \in M_n(\mathbb{K})$ be a skew-symmetric matrix. Then we define a Poisson bracket on $A$ by setting:

$$\{X_i, X_j\} = \lambda_{ij} X_i X_j,$$

for all $1 \leq i, j \leq n$. This Poisson algebra is called a *Poisson affine n-space* (associated to the matrix $\boldsymbol{\lambda}$) and is denoted by $\mathbb{K}_{\boldsymbol{\lambda}}[X_1, \ldots, X_n]$. We will refer to such a Poisson structure as a *quadratic* Poisson structure. When $n = 2$ we talk about Poisson affine *planes*.

(4) Let $\mathfrak{g}$ be a finite dimensional Lie algebra with basis $\{x_1, \ldots, x_n\}$. Then the symmetric algebra $S(\mathfrak{g}) = \mathbb{K}[x_1, \ldots, x_n]$ is a Poisson algebra for the Poisson bracket given by:

$$\{x_i, x_j\} = [x_i, x_j]_{\mathfrak{g}}$$

for all $1 \leq i, j \leq n$. This Poisson structure on $S(\mathfrak{g})$ is often called the *Kirillov-Kostant-Souriau* Poisson structure.

(5) Let $A := \mathcal{O}\big(M_2(\mathbb{K})\big) = \mathbb{K}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$. We define a Poisson structure on $A$ by setting:

$$\begin{aligned} &\{X_{11}, X_{12}\} = X_{11}X_{12}, \quad &&\{X_{11}, X_{22}\} = 2X_{12}X_{21}, \quad &&\{X_{12}, X_{22}\} = X_{12}X_{22}, \\ &\{X_{11}, X_{21}\} = X_{11}X_{21}, \quad &&\{X_{12}, X_{21}\} = 0, \quad &&\{X_{21}, X_{22}\} = X_{21}X_{22}. \end{aligned} \tag{1.3}$$

This Poisson algebra is called the algebra of $2 \times 2$ *Poisson matrix variety.* More generally one can endow the coordinate ring of the $m \times p$ matrix variety with a Poisson bracket defined using the Poisson structure (1.3) on $\mathcal{O}\big(M_2(\mathbb{K})\big)$ (see Section 4.2.1).

**Definition 1.1.4.** Let $A$ and $B$ be two Poisson algebras with Poisson brackets respectively denoted by $\{-,-\}_A$ and $\{-,-\}_B$. A *Poisson algebra homomorphism* is an algebra homomorphism $f : A \to B$ such that $f(\{a, b\}_A) = \{f(a), f(b)\}_B$ for all $a, b \in A$. A *Poisson algebra isomorphism* is a bijective Poisson algebra homomorphism.

In the rest of this thesis we will always omit subscripts indexing Poisson brackets when more than one is involved. Which Poisson bracket is intended will always be clear in context.

**Definition 1.1.5.** Let $A$ be a Poisson algebra.

(1) A *Poisson subalgebra* of $A$ is a subalgebra $B$ of $A$ such that $\{a, b\} \in B$ for all $a, b \in B$.

(2) A *Poisson ideal* of $A$ is an ideal $I$ of $A$ such that $\{a, x\} \in I$ for all $a \in A$ and $x \in I$.

(3) $A$ is *Poisson simple* if its only Poisson ideals are $\{0\}$ and $A$.

(4) The *Poisson centre* of $A$ is the Poisson subalgebra: $Z_P(A) := \{z \in A \mid \{z, -\} \equiv 0\}$.

*Remark* 1.1.6. Let $A$ be a Poisson algebra. The Poisson centre $Z_P(A)$ is a Lie ideal of $A$ but not always an associative ideal of $A$. We always have $\mathbb{K} \subseteq Z_P(A)$, and in positive characteristic (say $\operatorname{char} \mathbb{K} = p > 0$) we have $a^p \in Z_P(A)$ for all $a \in A$ since for all $a, b \in A$ we have:

$$\{a^p, b\} = pa^{p-1}\{a, b\} = 0.$$

*Remark* 1.1.7. Let $A$ be a Poisson algebra.

(1) If $I$ is a Poisson ideal of $A$ then there is a well-defined Poisson bracket on the quotient algebra $A/I$ defined by $\{\overline{a}, \overline{b}\} = \overline{\{a, b\}}$ for all $a, b \in A$, where $\overline{a}$ denote the image in $A/I$ of $a \in A$.

(2) Let $S$ be a multiplicatively closed subset of $A$. Then the Poisson bracket of $A$ extends uniquely to the localisation $AS^{-1}$ by setting:

$$\{as^{-1}, bt^{-1}\} = \{a, b\}s^{-1}t^{-1} - \{a, t\}bs^{-1}t^{-2} - \{s, b\}as^{-2}t^{-1} + \{s, t\}abs^{-2}t^{-2}$$

for all $a, b \in A$ and all $s, t \in S$. In particular if $A$ is a domain its Poisson structure uniquely extends to its field of fractions.

*Example* 1.1.8. Since the Laurent polynomial algebra $\mathbb{K}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and the field of rational functions $\mathbb{K}(X_1, \ldots, X_n)$ are both localisations of the polynomial algebra in $n$ indeterminates $\mathbb{K}[X_1, \ldots, X_n]$, a quadratic Poisson structure on $\mathbb{K}[X_1, \ldots, X_n]$ given by a matrix $\boldsymbol{\lambda} \in M_n(\mathbb{K})$ uniquely extends to $\mathbb{K}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and $\mathbb{K}(X_1, \ldots, X_n)$. These localisations endowed with those uniquely extended Poisson structures are respectively called the *Poisson torus* and the *Poisson affine field*, and respectively denoted by $\mathbb{K}_{\boldsymbol{\lambda}}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$

and $\mathbb{K}_{\boldsymbol{\lambda}}(X_1, \dots, X_n)$. One can easily check that in these localisations we have:

$$\{X_i^{-1}, X_j\} = -\lambda_{ij} X_i^{-1} X_j \quad \text{and} \quad \{X_i^{-1}, X_j^{-1}\} = \lambda_{ij} X_i^{-1} X_j^{-1}$$

for all $i, j$.

### 1.1.2 Poisson structure on the polynomial algebra in three variables

The choice of a polynomial $F \in \mathbb{K}[X, Y]$ always allows us to define a Poisson structure on $\mathbb{K}[X, Y]$ by setting $\{X, Y\} = F$. On a polynomial algebra in three variables the Jacobi identity imposes some restrictions. Let $A := \mathbb{K}[X_1, X_2, X_3]$, and $U, V, W \in A$. It is a straightforward verification to see that we define a Poisson bracket on $A$ by setting:

$$\{X_1, X_2\} = W, \quad \{X_2, X_3\} = U \quad \text{and} \quad \{X_3, X_1\} = V,$$

if and only if:

$$\operatorname{curl}(U, V, W) \cdot (U, V, W) = 0, \tag{1.4}$$

where:

$$\operatorname{curl}(U, V, W) := \left( \frac{\partial W}{\partial X_2} - \frac{\partial V}{\partial X_3}, \frac{\partial U}{\partial X_3} - \frac{\partial W}{\partial X_1}, \frac{\partial V}{\partial X_1} - \frac{\partial U}{\partial X_2} \right).$$

An important class of examples is as follows. Fix a polynomial $F \in A$ and set:

$$\{X_1, X_2\} = \frac{\partial F}{\partial X_3}, \quad \{X_2, X_3\} = \frac{\partial F}{\partial X_1} \quad \text{and} \quad \{X_3, X_1\} = \frac{\partial F}{\partial X_2}.$$

Since $\frac{\partial}{\partial X_j} \frac{\partial F}{\partial X_i} = \frac{\partial}{\partial X_i} \frac{\partial F}{\partial X_j}$ for all $1 \leq i, j \leq 3$, equation (1.4) is satisfied in that case. Such a Poisson structure is called a *Jacobian Poisson structure with potential $F$*. It is easy to see that $F \in Z_P(A)$, so that $\mathbb{K}[F] \subseteq Z_P(A)$ (with equality when $F$ is indecomposable, see [13, Lemma 1]).

By example, taking $F = \lambda X_1 X_2 X_3$ for $\lambda \in \mathbb{K}$ we obtain the Poisson affine space $\mathbb{K}_{\boldsymbol{\lambda}}[X_1, X_2, X_3]$ with $\lambda_{12} = -\lambda_{13} = \lambda_{23} = \lambda$.

We also refer to [25] where the authors study the Poisson prime ideals of Poisson algebras endowed with Jacobian Poisson structures.

### 1.1.3 Poisson spectrum

The *spectrum* of a (non-necessarily commutative) algebra $A$, denoted by $\text{Spec}\,(A)$, is the set of prime ideals of $A$. Recall that an ideal $P$ of $A$ is prime if whenever $IJ \subseteq P$ for some ideals $I, J$ of $A$, then either $I \subseteq P$ or $J \subseteq P$. Recall that when $A$ is commutative $P$ is prime if and only if the quotient $A/P$ algebra is a domain. It is easy to check that by setting:

$$V(I) := \{P \in \text{Spec}\,(A) \mid P \supseteq I\}$$

for all ideals $I$ of $A$ we define the closed sets of a topology on $\text{Spec}\,(A)$. This topology is referred to as the *Zariski topology*, and when speaking of the prime spectrum of a ring we will always think of it endowed with this topology.

If $A$ is a Poisson algebra, then the *Poisson spectrum* of $A$, denoted by $\text{P.Spec}\,(A)$, is the subset of $\text{Spec}\,(A)$ consisting of Poisson ideals. Equivalently $\text{P.Spec}\,(A)$ is the set of ideals which are both prime ideals and Poisson ideals.

*Remark* 1.1.9. Usually the Poisson spectrum of a Poisson algebra is defined in more generality. A Poisson-prime ideal $P$ is a Poisson ideal such that if whenever $IJ \subseteq P$ for some Poisson ideals $I, J$ of $A$, then either $I \subseteq P$ or $J \subseteq P$. It is clear that a Poisson and prime ideal is a Poisson-prime ideal. If $A$ is noetherian and the characteristic of the base field is zero then the converse is true thanks to [11, Lemma 3.3.2]. When working in positive characteristic latter on, we will restrict our attention on the study of Poisson and prime ideals.

The Poisson spectrum $\text{P.Spec}\,(A)$ can be endowed with the induced topology from $\text{Spec}\,(A)$. The closed sets are the sets:

$$V_P(I) := \{P \in \text{P.Spec}\,(A) \mid P \supseteq I\}$$

for all ideals $I$ of $A$. We remark that we can replace $I$ by the smallest Poisson ideal containing $I$ without changing these sets, so the closed sets consist of the $V_P(I)$ for all Poisson ideals $I$.

*Remark* 1.1.10. Later on we will be dealing with homeomorphisms between Poisson spectra. We remark that, in particular, such an homeomorphism and its inverse preserve inclusions. More precisely for two Poisson algebras $A$ and $B$ and an homeomorphism $\varphi$

from P.Spec $(A)$ and P.Spec $(B)$ we have:

$$P \subseteq Q \implies \varphi(P) \subseteq \varphi(Q), \text{ and}$$
$$I \subseteq J \implies \varphi^{-1}(I) \subseteq \varphi^{-1}(J),$$

for all $P, Q \in$ P.Spec $(A)$ and all $I, J \in$ P.Spec $(B)$. See [16, Assertion (a) of Lemma 9.4].

### 1.1.4 Ore extensions and Poisson-Ore extensions

In this section we present two classes of algebras, the so-called *iterated Ore extensions* or *skew polynomial rings* from noncommutative ring theory, and their Poisson analogues: *iterated Poisson-Ore extensions.*

#### 1.1.4.1 Ore extensions

Good references for Ore extensions are [17] and [6]. Let $\mathcal{R}$ be a (not necessarily commutative) algebra and $\sigma$ be an automorphism of $\mathcal{R}$. A *$\sigma$-derivation* of $\mathcal{R}$ is a linear map $\Delta$ from $\mathcal{R}$ to $\mathcal{R}$ such that for all $r, s \in \mathcal{R}$ we have:

$$\Delta(rs) = \sigma(r)\Delta(s) + \Delta(r)s.$$

In particular one can easily check that $\Delta(1) = 0$ and that an id-derivation is just a derivation in the associative sense. Following [6, Section I.1.11], the notation $\mathcal{T} := \mathcal{R}[x; \sigma, \Delta]$ means that:

(1) $\mathcal{T}$ is a free left $\mathcal{R}$-module with basis $\{x^i \mid i \geq 0\}$,

(2) $\mathcal{T}$ contains $\mathcal{R}$ as a subring and $x \in \mathcal{T}$,

(3) $xr = \sigma(r)x + \Delta(r)$ for all $r \in \mathcal{R}$,

(4) $\sigma$ is an automorphism of $\mathcal{R}$ and $\Delta$ is a $\sigma$-derivation of $\mathcal{R}$.

$\mathcal{T}$ is then called an *Ore extension* or a *skew polynomial ring* over $\mathcal{R}$. Existence of such objects can be proved, see [17, Chapter 1]. We have the following classical notations $\mathcal{R}[x; \Delta] := \mathcal{R}[x; \text{id}, \Delta]$ when $\sigma = \text{id}$, and $\mathcal{R}[x; \sigma] := \mathcal{R}[x; \sigma, 0]$ when $\Delta = 0$. To avoid confusion when using the notation $\mathcal{R}[x; f]$ we will always state whether $f$ is an automorphism or a derivation of $\mathcal{R}$. There is a noncommutative version of the Hilbert Basis Theorem for Ore extensions (see [6, Theorem I.1.13] for instance).

**Theorem 1.1.11.** *Let* $\mathcal{T} := \mathcal{R}[x;\sigma,\Delta]$ *be an Ore extension and suppose that* $\mathcal{R}$ *is a noetherian domain. Then* $\mathcal{T}$ *is a noetherian domain.*

In particular, if $\mathcal{R}$ is a noetherian domain, then $\mathcal{T}$ satisfies the (left and right) Ore conditions and admits a skewfield of fractions which we denote by $\operatorname{Frac}\mathcal{T}$.

Iterated Ore extensions can be constructed inductively. Starting from an Ore extension $\mathcal{T}_1 = \mathcal{R}[x_1;\sigma_1,\Delta_1]$, an automorphism $\sigma_2$ of $\mathcal{T}_1$ and a $\sigma_2$-derivation $\Delta_2$ of $\mathcal{T}_1$, we construct the Ore extension $\mathcal{T}_2 = \mathcal{T}_1[x_2;\sigma_2,\Delta_2]$. We can repeat this process. Therefore we say that an algebra $\mathcal{T}$ is an *iterated Ore extension over a ring* $\mathcal{R}$ if it is of the form:

$$\mathcal{T} := \mathcal{T}_n = \mathcal{R}[x_1;\sigma_1,\Delta_1]\cdots[x_n;\sigma_n,\Delta_n],$$

where $\sigma_i \in \operatorname{Aut}(\mathcal{T}_{i-1})$ and $\Delta_i$ is $\sigma_i$-derivation of $\mathcal{T}_{i-1}$ for all $1 \leq i \leq n$ (with the convention that $\mathcal{T}_0 := \mathcal{R}$).

*Example* 1.1.12. (1) Recall that the *n-th Weyl algebra* $A_n(\mathbb{K})$ is the algebra given by $2n$ generators $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ and relations:

$$[x_i,x_j] = [y_i,y_j] = 0 \text{ and } [x_i,y_j] = \delta_{ij}$$

for all $1 \leq i,j \leq n$, where $[x,y] := xy - yx$. $A_n(\mathbb{K})$ is an iterated Ore extension over $\mathbb{K}[y_1,\ldots,y_n]$:

$$A_n(\mathbb{K}) = \mathbb{K}[y_1,\ldots,y_n][x_1;\partial_{y_1}][x_2;\partial_{y_2}]\cdots[x_n;\partial_{y_n}]$$

where $\partial_{y_i}$ is the usual partial derivative with respect to $y_i$.

(2) Let $\boldsymbol{q} = (q_{ij}) \in M_n(\mathbb{K}^\times)$ be a multiplicatively skew-symmetric matrix, i.e. such that $q_{ji} = q_{ij}^{-1}$ for all $i,j$. The *quantum affine space* $\mathcal{O}_{\boldsymbol{q}}(\mathbb{K}^n)$ is the algebra given by $n$ generators $x_1,\ldots,x_n$ and relations:

$$x_ix_j = q_{ij}x_jx_i$$

for all $1 \leq i, j \leq n$. $\mathcal{O}_{\boldsymbol{q}}(\mathbb{K}^n)$ can be expressed as an iterated Ore extension as follows:

$$\mathcal{O}_{\boldsymbol{q}}(\mathbb{K}^n) = \mathbb{K}[x_1][x_2;\sigma_2]\cdots[x_n;\sigma_n]$$

where $\sigma_i$ is the automorphism of $\mathbb{K}[x_1][x_2;\sigma_2]\cdots[x_{i-1};\sigma_{i-1}]$ defined by $\sigma_i(x_j) = q_{ij}x_j$ for all $1 \leq j < i \leq n$.

(3) Let $q \in \mathbb{K}^{\times}$. The algebra of $2\times 2$ quantum matrices $R := \mathcal{O}_q\big(M_2(\mathbb{K})\big)$ is the algebra given by generators $x_{11}, x_{12}, x_{21}, x_{22}$ and relations:

$$\begin{aligned} &x_{11}x_{12} = qx_{12}x_{11}, \quad x_{11}x_{22} - x_{22}x_{11} = (q-q^{-1})x_{12}x_{21}, \quad x_{12}x_{22} = qx_{22}x_{12}, \\ &x_{11}x_{21} = qx_{21}x_{11}, \quad x_{12}x_{21} = x_{21}x_{12}, \qquad\qquad\qquad\qquad\quad x_{21}x_{22} = qx_{22}x_{21}. \end{aligned} \tag{1.5}$$

$R$ can be expressed as an iterated Ore extension over $\mathbb{K}[x_{11}]$ as follows:

$$R = \mathbb{K}[x_{11}][x_{12};\sigma_{12}][x_{21};\sigma_{21}][x_{22};\sigma_{22},\Delta_{22}], \text{ where:}$$

- $\sigma_{12}$ is the automorphism of $\mathbb{K}[x_{11}]$ such that $\sigma_{12}(x_{11}) = q^{-1}x_{11}$,
- $\sigma_{21}$ is the automorphism of $\mathbb{K}[x_{11}][x_{12};\sigma_{12}]$ such that $\sigma_{21}(x_{11}) = q^{-1}x_{11}$ and $\sigma_{21}(x_{12}) = x_{12}$,
- $\sigma_{22}$ is the automorphism of $\mathbb{K}[x_{11}][x_{12};\sigma_{12}][x_{21};\sigma_{21}]$ such that $\sigma_{22}(x_{11}) = x_{11}$, $\sigma_{22}(x_{12}) = q^{-1}x_{12}$ and $\sigma_{22}(x_{21}) = q^{-1}x_{21}$,
- $\Delta_{22}$ is the $\sigma_{22}$-derivation of the $\mathbb{K}[x_{11}][x_{12};\sigma_{12}][x_{21};\sigma_{21}]$ such that: $\Delta_{22}(x_{11}) = (q^{-1}-q)x_{12}x_{21}$ and $\Delta_{22}(x_{12}) = \Delta_{22}(x_{21}) = 0$.

#### 1.1.4.2 Poisson-Ore extensions

The set of derivations of an associative algebra $A$ is denoted by $\text{Der}(A)$. This is a Lie algebra for the commutator bracket.

**Definition 1.1.13.** Let $A$ be a Poisson algebra.

(1) A *Poisson derivation* is a derivation $\alpha \in \text{Der}(A)$ such that:

$$\alpha(\{a,b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\} \text{ for all } a, b \in A.$$

The set of Poisson derivation of $A$ is denoted by $\text{Der}_P(A)$.

(2) Let $\alpha \in \mathrm{Der}_P(A)$. A *Poisson $\alpha$-derivation* is a derivation $\delta \in \mathrm{Der}(A)$ such that:

$$\delta(\{a,b\}) = \{\delta(a), b\} + \{a, \delta(b)\} + \alpha(a)\delta(b) - \delta(a)\alpha(b) \text{ for all } a, b \in A.$$

We remark that a Poisson 0-derivation is just a Poisson derivation. The definition of a Poisson-Ore extension is based on the following result of Oh [33, Theorem 1.1].

**Theorem 1.1.14.** *Let $\alpha$ and $\delta$ be $\mathbb{K}$-linear maps of a Poisson $\mathbb{K}$-algebra $A$. Then the polynomial algebra $B := A[X]$ is a Poisson algebra with Poisson bracket extending the Poisson bracket of $A$ and satisfying:*

$$\{X, a\} = \alpha(a)X + \delta(a) \text{ for all } a \in A,$$

*if and only if $\alpha \in \mathrm{Der}_P(A)$ and $\delta$ is a Poisson $\alpha$-derivation of $A$.*

**Definition 1.1.15.** Let $A$ be a Poisson algebra, $\alpha \in \mathrm{Der}_P(A)$ and $\delta$ be a Poisson $\alpha$-derivation of $A$. Set $B := A[X]$. The algebra $B$ endowed with the Poisson bracket from Theorem 1.1.14 is denoted by $B := A[X;\alpha,\delta]_P$ and called a *Poisson-Ore extension.* As usual we set $A[X;\alpha]_P := A[X;\alpha,0]_P$ when $\delta = 0$.

Poisson-Ore extensions satisfy the following universal property.

**Proposition 1.1.16.** *Let $B := A[X;\alpha,\delta]_P$ be a Poisson-Ore extension over a Poisson $\mathbb{K}$-algebra $A$ and $C$ be a Poisson $\mathbb{K}$-algebra. If $\psi : A \to C$ is a Poisson algebra homomorphism and if there exists $Y \in C$ such that:*

$$\{Y, \psi(a)\} = \psi(\alpha(a))Y + \psi(\delta(a))$$

*for all $a \in A$, then there exists a unique Poisson algebra homomorphism $\varphi : B \to C$ sending $X$ to $Y$ such that $\psi = \varphi \circ i$ where $i$ is the inclusion of $A$ in $B$.*

*Proof.* There is a well-defined $\mathbb{K}$-algebra homomorphism $\varphi : B \to C$ given by:

$$\varphi\Big(\sum_i a_i X^i\Big) := \sum_i \psi(a_i)Y^i,$$

where $a_i \in A$ for all $i$. Note that $\varphi(X) = Y$ and $\varphi(a) = \psi(a)$ for all $a \in A$. It is clear that this is the only possibility for $\varphi$, so it only remains to show that $\varphi$ is a Poisson algebra homomorphism. For all $a, b \in A$ we have $\varphi(\{a,b\}) = \psi(\{a,b\}) = \{\psi(a), \psi(b)\} = \{\varphi(a), \varphi(b)\}$, and:

$$\varphi(\{X,a\}) = \varphi(\alpha(a)X + \delta(a)) = \psi(\alpha(a))Y + \psi(\delta(a)) = \{Y, \psi(a)\} = \{\varphi(X), \varphi(a)\}.$$

$\square$

The construction of Poisson-Ore extensions can easily be iterated. We say that $R$ *is an iterated Poisson-Ore extension over* $A$ if:

$$R = A[X_1; \alpha_1, \delta_1]_P[X_2; \alpha_2, \delta_2]_P \cdots [X_n; \alpha_n, \delta_n]_P$$

where $A$ is a Poisson algebra, $\alpha_i$ is a Poisson derivation of the Poisson subalgebra $R_{i-1} := A[X_1; \alpha_1, \delta_1]_P \cdots [X_{i-1}; \alpha_{i-1}, \delta_{i-1}]_P$ of $R$, and $\delta_i$ is an $\alpha_i$-Poisson derivation of $R_{i-1}$ for all $1 \leq i \leq n$ (with the convention that $R_0 = A$).

*Example* 1.1.17. (1) The Poisson-Weyl algebra $A_n^P(\mathbb{K})$ is an iterated Poisson-Ore extension over the abelian Poisson algebra $\mathbb{K}[Y_1, \dots, Y_n]$:

$$A_n^P(\mathbb{K}) = \mathbb{K}[Y_1, \dots, Y_n][X_1; 0, \partial_{Y_1}]_P \cdots [X_n; 0, \partial_{Y_n}]_P,$$

where $\partial_{Y_i}$ denotes the usual partial derivative with respect to $Y_i$ for all $1 \leq i \leq n$. The following computations show that the maps $\partial_{Y_i}$ are Poisson 0-derivations. For all $1 \leq k, l \leq n$ and all $1 \leq p, q < i$ we have:

$$\begin{aligned}
&\partial_{Y_i}\big(\{Y_k, Y_l\}\big) = 0 = \{\partial_{Y_i}(Y_k), Y_l\} + \{Y_k, \partial_{Y_i}(Y_l)\},\\
&\partial_{Y_i}\big(\{X_p, X_q\}\big) = 0 = \{\partial_{Y_i}(X_p), X_q\} + \{X_p, \partial_{Y_i}(X_q)\},\\
&\partial_{Y_i}\big(\{X_p, Y_k\}\big) = \partial_{Y_i}(\delta_{kl}) = 0 = \{\partial_{Y_i}(X_p), Y_k\} + \{X_p, \partial_{Y_i}(Y_k)\}.
\end{aligned}$$

(2) A Poisson affine $n$-space $\mathbb{K}_{\boldsymbol{\lambda}}[X_1, \dots, X_n]$ is an iterated Poisson-Ore extension over $\mathbb{K}$:

$$\mathbb{K}[X_1][X_2; \alpha_2]_P \cdots [X_n; \alpha_n]_P,$$

where $\alpha_i$ is the Poisson derivation of the Poisson affine space $\mathbb{K}_{\boldsymbol{\lambda}_{i-1}}[X_1, \dots, X_{i-1}]$ such that $\alpha_i(X_j) = \lambda_{ij}X_j$ for all $1 \leq j < i \leq n$ and where $\boldsymbol{\lambda}_{i-1}$ is the $(i-1)\times(i-1)$ submatrix of $\boldsymbol{\lambda}$ obtained by deleting rows and columns indexed by $k \geq i$. The following computations show that the maps $\alpha_i$ are Poisson derivations. For all $1 \leq k, l < i \leq n$ we have:

$$\alpha_i(\{X_k, X_l\}) = \alpha_i(\lambda_{kl} X_k X_l) = \lambda_{kl}\alpha_i(X_k)X_l + \lambda_{kl} X_k \alpha_i(X_l) = \lambda_{kl}(\lambda_{ik} + \lambda_{il})X_k X_l,$$

and:

$$\{\alpha_i(X_k), X_l\} + \{X_k, \alpha_i(X_l)\} = (\lambda_{ik} + \lambda_{il})\{X_k, X_l\} = \lambda_{kl}(\lambda_{ik} + \lambda_{il})X_k X_l.$$

(3) The $2 \times 2$ Poisson matrix variety $A := \mathcal{O}\big(M_2(\mathbb{K})\big)$ can be expressed as an iterated Ore extension over $\mathbb{K}[X_{11}]$ as follows:

$$A = \mathbb{K}[X_{11}][X_{12}; \alpha_{12}]_P[X_{21}; \alpha_{21}]_P[X_{22}; \alpha_{22}, \delta_{22}]_P, \text{ where}$$

- $\alpha_{12}$ is the Poisson derivation of $\mathbb{K}[X_{11}]$ such that $\alpha_{12}(X_{11}) = -X_{11}$,
- $\alpha_{21}$ is the Poisson derivation of $\mathbb{K}[X_{11}][X_{12}; \alpha_{12}]_P$ such that $\alpha_{21}(X_{11}) = -X_{11}$ and $\alpha_{21}(X_{12}) = 0$,
- $\alpha_{22}$ is the Poisson derivation of $B := \mathbb{K}[X_{11}][X_{12}; \alpha_{12}]_P[X_{21}; \alpha_{21}]_P$ such that $\alpha_{22}(X_{11}) = 0$, $\alpha_{22}(X_{12}) = -X_{12}$ and $\alpha(X_{21}) = -X_{21}$,
- $\delta_{22}$ is the Poisson $\alpha_{22}$-derivation of $B$ such that $\delta_{22}(X_{12}) = \delta_{22}(X_{21}) = 0$ and $\delta_{22}(X_{11}) = -2X_{12}X_{21}$.

*Remark* 1.1.18. It is not necessarily obvious whether a given polynomial Poisson algebra $A$ can be expressed as a (iterated) Poisson-Ore extension. In all the examples we saw it was possible to use the "canonical generators" of a polynomial Poisson algebra to express it as an iterated Poisson-Ore extension. This is not always the case as the following example demonstrates. Let $A = \mathbb{C}[X, Y]$ with $\{X, Y\} = X^2 + Y^2$. The Poisson algebra $A$ cannot be expressed as a Poisson-Ore extension in the generators $X$ and $Y$ since we would need to have:

$$\{X, \mathbb{C}[Y]\} \subseteq \mathbb{C}[Y]X + \mathbb{C}[Y] \quad \text{or} \quad \{Y, \mathbb{C}[X]\} \subseteq \mathbb{C}[X]Y + \mathbb{C}[X].$$

But with the generators $X' := X + iY$ and $Y' := X - iY$ of $A$ it is easy to check that:

$$\{X', Y'\} = -2iX'Y'.$$

Thus $A$ can be expressed as a Poisson-Ore extension in the generators $X'$ and $Y'$. More precisely $A$ is a Poisson affine space and we have:

$$A = \mathbb{C}[X'][Y'; \alpha]_P,$$

where $\alpha$ is the Poisson derivation of $\mathbb{C}[X']$ such that $\alpha(X') = 2iX'$. Note that we only used a linear transformation here.

## 1.2 Semiclassical limit

Roughly speaking the semiclassical limit process is a way to obtain a Poisson algebra from a given noncommutative algebra. This process is explained in [12, Section 1.1.3] or [16, Section 2.1]. We recall here the *commutative fibre* version as described in [12, Section 1.1.3] since this is the version we will use later on.

Let $B$ be a principal ideal domain containing $\mathbb{K}$, and fix $h \in B$ such that the ideal $hB$ is maximal in $B$. Suppose that $\mathcal{R}$ is a not necessarily commutative torsion-free $B$-algebra such that the quotient $A := \mathcal{R}/h\mathcal{R}$ is commutative (note that $h$ is central in $\mathcal{R}$ by definition of $\mathcal{R}$). Then the algebra $A$ becomes a Poisson $\mathbb{K}$-algebra as follows. Since $A$ is commutative, for all $r, s \in \mathcal{R}$ the commutator $[r, s] := rs - sr$ belongs to the ideal $h\mathcal{R}$ and there exists a unique element $\gamma(r, s) \in \mathcal{R}$ such that $[r, s] = h\gamma(r, s)$. We set $\frac{[r,s]}{h} := \gamma(r, s)$. Finally, it is easy to see that for all $r, s \in \mathcal{R}$ the (well-defined) formula:

$$\{r + h\mathcal{R}, s + h\mathcal{R}\} := \frac{[r, s]}{h} + h\mathcal{R}$$

defines a Poisson bracket on $A$.

**Definition 1.2.1.** The Poisson algebra defined above is called the *semiclassical limit* at $h$ of the noncommutative algebra $\mathcal{R}$, and the algebra $\mathcal{R}$ is called a *quantisation* of the Poisson algebra $A$. For any $q \in \mathbb{K}$ such that the central element $h - q$ is not invertible in $\mathcal{R}$ the algebra $A_q := \mathcal{R}/(h - q)\mathcal{R}$ is called a *deformation* of the Poisson algebra $A$. The diagram of Figure 1.1 illustrates this situation.

Note that there exists a so-called *filtered/graded* version of the semiclassical process which is suitable in particular for enveloping algebras of Lie algebras, see [16, Section

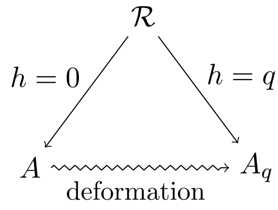


Figure 1.1

2.4]. The two constructions are linked in [16, Section 2.5]. We now give examples of the semiclassical limit process explained previously. More examples can be found in [12], [16] or [18].

*Example* 1.2.2. (1) Let $\mathcal{R} := A_1(\mathbb{K}[t])$ be the algebra over $\mathbb{K}[t]$ given by generators $x, y$ and relation $xy - yx = t$. We have $A := \mathcal{R}/t\mathcal{R} \cong \mathbb{K}[X, Y]$ where $X := x + t\mathcal{R}$ and $Y := y + t\mathcal{R}$. Therefore $A$ is a Poisson algebra with Poisson bracket given by:

$$\{X, Y\} = \frac{[x, y]}{t} + t\mathcal{R} = 1,$$

i.e. $A$ is the first Poisson-Weyl algebra $A_1^P(\mathbb{K})$. We remark that the algebra $A_1 := \mathcal{R}/(t-1)\mathcal{R}$ is isomorphic to the first Weyl algebra $A_1(\mathbb{K})$, so that $A_1(\mathbb{K})$ is a deformation of $A_1^P(\mathbb{K})$. This justifies the name Poisson-Weyl algebra given to $A_1^P(\mathbb{K})$.

(2) Let $\mathcal{R} = \mathcal{O}_t\big(M_2(\mathbb{K}[t^{\pm 1}])\big)$ be the $\mathbb{K}[t^{\pm 1}]$-algebra of $2 \times 2$ quantum matrices, where the relations are identical to the relations in (1.5), but where $q$ is replaced with $t$. We have $A = \mathcal{R}/(t-1)\mathcal{R} \cong \mathbb{K}[X_{11}, X_{12}, X_{21}, X_{22}]$ where $X_{ij} := x_{ij} + (t-1)\mathcal{R}$ for all $1 \le i, j \le 2$. The Poisson structure we obtain on $A$ is exactly the one given in (1.3). For example we compute:

$$\begin{aligned}
\{X_{11}, X_{22}\} &= \frac{x_{11}x_{22} - x_{22}x_{11}}{t-1} + (t-1)\mathcal{R} \\
&= \frac{(t - t^{-1})x_{12}x_{21}}{t-1} + (t-1)\mathcal{R} \\
&= \frac{t^{-1}(t-1)(t+1)x_{12}x_{21}}{t-1} + (t-1)\mathcal{R} \\
&= t^{-1}(t+1)x_{12}x_{21} + (t-1)\mathcal{R} \\
&= 2X_{12}X_{21}.
\end{aligned}$$

(3) Let $\mathfrak{g}$ be a finite dimensional Lie $\mathbb{K}$-algebra with basis $\{x_1, \dots, x_n\}$. Then the algebra

$U_t(\mathfrak{g})$ is the $\mathbb{K}$-algebra given by generators $x_1, \dots, x_n$ and $t$ with relations:

$$x_i x_j - x_j x_i = t[x_i, x_j]_{\mathfrak{g}} \text{ and } x_i t = t x_i,$$

for all $1 \leq i, j \leq n$. The algebra $U_0 := U_t(\mathfrak{g})/tU_t(\mathfrak{g})$ is isomorphic to the commutative polynomial algebra $S(\mathfrak{g}) \cong \mathbb{K}[X_1, \dots, X_n]$, where $X_i = x_i + tU_t(\mathfrak{g})$, and we obtain a Poisson bracket on $U_0$ given by:

$$\{X_i, X_j\} = [x_i, x_j]_{\mathfrak{g}} + tU_t(\mathfrak{g})$$

for all $1 \leq i, j \leq n$. We retrieve the Kirillov-Kostant-Souriau Poisson structure from (4) of Examples 1.1.3. We remark that the algebra $U_1 := U_t(\mathfrak{g})/(t-1)U_t(\mathfrak{g})$ is isomorphic to the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, so $U(\mathfrak{g})$ is a deformation of $S(\mathfrak{g})$.

(4) Let $\mathcal{R} := \mathcal{O}_{(t^{\lambda_{ij}})}\big((\mathbb{K}[t^{\pm 1}])^n\big)$ be a quantum affine space, where $(\lambda_{ij}) \in M_n(\mathbb{Z})$ is a skew-symmetric matrix. $\mathcal{R}$ is the algebra given by generators $x_1, \dots, x_n$ and relations $x_i x_j = t^{\lambda_{ij}} x_j x_i$. We remark that $\mathcal{R}$ is a domain and that $t-1$ is central in $\mathcal{R}$. Moreover the quotient algebra $R := \mathcal{R}/(t-1)\mathcal{R}$ is commutative and it is easy to see that $R = \mathbb{K}[X_1, \dots, X_n]$, where $X_i := x_i + (t-1)\mathcal{R}$ for all $1 \leq i \leq n$. Then, for all $1 \leq i, j \leq n$, we have:

$$\begin{aligned}
\{X_i, X_j\} &= \frac{t^{\lambda_{ij}} x_j x_i - x_j x_i}{t-1} + (t-1)\mathcal{R} \\
&= \frac{t^{\lambda_{ij}} - 1}{t-1} x_j x_i + (t-1)\mathcal{R} \\
&= \lambda_{ij} X_i X_j.
\end{aligned}$$

Thus $R$ is the Poisson affine space $\mathbb{K}_{(\lambda_{ij})}[X_1, \dots, X_n]$. For $q \in \mathbb{K}^{\times}$, we have:

$$\mathcal{R}/(t-q)\mathcal{R} \cong \mathcal{O}_{(q^{\lambda_{ij}})}\big(\mathbb{K}^n\big),$$

thus the quantum affine space $\mathcal{O}_{(q^{\lambda_{ij}})}\big(\mathbb{K}^n\big)$ is a deformation of the Poisson affine space $\mathbb{K}_{(\lambda_{ij})}[X_1, \dots, X_n]$. We remark that we only dealt with the so-called *uniparameter case*, i.e. when all the deformation parameters are all powers of a same element $q \in \mathbb{K}^{\times}$. To take into account the multiparameter case, some adaptations have to be made such as in [22], [16, Section 2.3] or [18, Section 2.2].

More examples will be given in Chapters 4 and 7.

## 1.3 Spectra of quantum tori and Poisson tori

Quantum (resp. Poisson) tori are helpful when studying quantum (resp. Poisson) affine spaces. In this section we recall classical results about quantum and Poisson tori. Let $(\lambda_{ij}) \in M_n(\mathbb{Z})$ be a skew-symmetric matrix and set $q_{ij} := q^{\lambda_{ij}}$ for all $i, j$ where $q \in \mathbb{K} \setminus \{0, 1\}$. Note that $(q_{ij})$ is a multiplicatively skew-symmetric matrix. The *quantum torus* associated to the matrix $(q_{ij})$ is the $\mathbb{K}$-algebra denoted by $\mathcal{T}$ given by generators $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ and relations $x_i x_j = q_{ij} x_j x_i$ for all $1 \leq i, j \leq n$. We denote by $T = \mathbb{K}_{(\lambda_{ij})}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ the Poisson torus associated to the matrix $(\lambda_{ij})$ as defined in Example 1.1.8.

The algebra $\mathcal{T}$ is a deformation of $T$ in the sense of Section 1.2. Indeed by setting $\mathcal{R}$ for the algebra generated over $\mathbb{K}[t^{\pm 1}]$ by generators $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ and relations $x_i x_j = t^{\lambda_{ij}} x_j x_i$ for all $1 \leq i, j \leq n$, we easily see that $\mathcal{T} \cong \mathcal{R}/(t-q)\mathcal{R}$ and that $T \cong \mathcal{R}/(t-1)\mathcal{R}$ as Poisson algebras.

For all $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ we set $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{T}$. Thus, as a vector space, $\mathcal{T}$ has a $\mathbb{K}$-basis of the form $\{x^\alpha \mid \alpha \in \mathbb{Z}^n\}$. The map $\sigma$ from $\mathbb{Z}^n \times \mathbb{Z}^n$ to $\mathbb{K}^\times$ defined by:

$$\sigma(\alpha, \beta) = \prod_{i,j=1}^{n} q^{\alpha_i \lambda_{ij} \beta_j}$$

for all $\alpha, \beta \in \mathbb{Z}^n$, allows us to express the commutation relations in $\mathcal{T}$ as follows:

$$x^\alpha x^\beta = \sigma(\alpha, \beta) x^\beta x^\alpha$$

for all $\alpha, \beta \in \mathbb{Z}^n$. The following result is due to Goodearl and Letzter [20].

**Lemma 1.3.1.** *We have:*

(1) *Every ideal of $\mathcal{T}$ is generated by its intersection with $Z(\mathcal{T})$.*

(2) *Set $\Sigma := \{\alpha \in \mathbb{Z}^n \mid \sigma(\alpha, -) \equiv 0\}$. Then:*

$$Z(\mathcal{T}) = \mathbb{K}[x^\alpha \mid \alpha \in \Sigma].$$

*Proof.* Assertion (1) is exactly assertion (1) of [20, Proposition 1.4] and assertion (2) is [20, Lemma 1.2]. □

On the Poisson side we have similar results. The assignment $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n} \in T$ for $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ allows us to identify the Laurent polynomial algebra $T$ with the group algebra $\mathbb{K}\mathbb{Z}^n$. The induced Poisson structure on $\mathbb{K}\mathbb{Z}^n$ is given by:

$$\{X^\alpha, X^\beta\} = b(\alpha, \beta) X^{\alpha+\beta}, \tag{1.6}$$

for all $\alpha, \beta \in \mathbb{Z}^n$, where $b$ is the skew-symmetric bilinear form from $\mathbb{Z}^n \times \mathbb{Z}^n$ to $\mathbb{K}$ defined by:

$$b(\alpha, \beta) = \sum_{i,j=1}^{n} \alpha_i \lambda_{ij} \beta_j,$$

for all $\alpha, \beta \in \mathbb{Z}^n$. The following result is due to Vancliff [36]. The author of [36] was working over $\mathbb{C}$ but the result is still true over more general base fields.

**Lemma 1.3.2.** *We have:*

(1) *Every Poisson ideal of $T$ is generated by its intersection with $Z_P(T)$.*

(2) *Set $S := \{\alpha \in \mathbb{Z}^n \mid b(\alpha, -) \equiv 0\}$. Then:*

$$Z_P(T) = \mathbb{K}[X^\alpha \mid \alpha \in S].$$

We give a proof of this result since it only appears in the literature over the field $\mathbb{C}$.

*Proof.* We start by proving assertion (2). It is clear from equation (1.6) that:

$$\mathbb{K}[X^\alpha \mid \alpha \in S] \subseteq Z_P(T).$$

Reciprocally, let $f \in Z_P(T)$ and write $f = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha X^\alpha$, where the scalars $a_\alpha$ are almost all zero. Then for all $\beta \in \mathbb{Z}^n$ we have:

$$0 = \{X^\beta, f\} = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha b(\beta, \alpha) X^{\alpha+\beta},$$

and since the monomials $X^{\alpha+\beta}$ are linearly independent we have $b(\beta, \alpha) = 0$ as long as

$a_\alpha \neq 0$. Therefore:

$$f = \sum_{\alpha \in S} a_\alpha X^\alpha \in \mathbb{K}[X^\alpha \mid \alpha \in S].$$

Thus we obtain $Z_P(T) = \mathbb{K}[X^\alpha \mid \alpha \in S]$ and assertion (2) is shown.

We now prove assertion (1). First notice that from assertion (2) we easily deduce that $T$ is a free $Z_P(T)$-module with basis $\{X^\gamma \mid \gamma \in \Gamma\}$ for a transversal $\Gamma$ of $S$ in $\mathbb{Z}^n$.

Let $P$ be a Poisson ideal of $T$. For $f \in P$ we can write $f = \sum_{i=1}^N f_i X^{\gamma_i}$, with $f_i \in Z_P(T)$ and $\gamma_i \in \Gamma$ for all $1 \leq i \leq N$ and $N$ minimal. We will show by induction on $N$ that if $\sum_{i=1}^N f_i X^{\gamma_i} \in P$, with $f_i \in Z_P(T) \setminus \{0\}$ and $\gamma_i \in \Gamma$ for all $1 \leq i \leq N$, then $f_i \in P$ for all $1 \leq i \leq N$.

For $N = 1$ the result is trivial since $X^{\gamma_1}$ is invertible. We now suppose that the result is true for $N-1$ and we will show that it is still true for $N$. Let $f = \sum_{i=1}^N f_i X^{\gamma_i} \in P$, with $f_i \in \mathbb{Z}_P(T) \setminus \{0\}$ and $\gamma_i \in \Gamma$ for all $1 \leq i \leq N$. Since $P$ is a Poisson ideal, for all $\alpha \in \mathbb{Z}^n$ we have:

$$\{X^\alpha, f\} X^{-\alpha} = \sum_{i=1}^N f_i b(\alpha, \gamma_i) X^{\gamma_i} \in P.$$

Thus for all $\alpha \in \mathbb{Z}^n$ we have $\{X^\alpha, f\}X^{-\alpha} - b(\alpha, \gamma_N) f \in P$, i.e.:

$$\sum_{i=1}^{N-1} f_i (b(\alpha, \gamma_i) - b(\alpha, \gamma_N)) X^{\gamma_i} \in P.$$

Since $\Gamma$ is a transversal, we have $\gamma_1 - \gamma_N \notin S$. Thus there exists $\alpha \in \mathbb{Z}^n$ such that $b(\alpha, \gamma_1) - b(\alpha, \gamma_N) \neq 0$. But then by the induction hypothesis we obtain that $f_1(b(\alpha, \gamma_i) - b(\alpha, \gamma_N)) \in P$, i.e. $f_1 \in P$. Therefore we have:

$$f - f_1 X^{\gamma_1} = \sum_{i=2}^N f_i X^{\gamma_i} \in P,$$

with $f_i \in Z_P(T) \setminus \{0\}$ and $\gamma_i \in \Gamma$ for all $2 \leq i \leq N$. Thus by induction hypothesis we obtain $f_i \in P$ for all $2 \leq i \leq N$. This conclude the induction.

We just have shown that $P \subseteq T(Z_P(T) \cap P)$. The reverse inclusion is trivial, and we conclude that $P$ is generated by its intersection with $Z_P(T)$. □

In particular these lemmas show that if $Z(\mathcal{T})$ (resp. $Z_P(T)$) is trivial, then $\mathcal{T}$ (resp.

$T$) is simple (resp. Poisson simple).

# Chapter 2

# The characteristic-free Poisson deleting derivation homomorphism

The aim of this chapter is to extend the Poisson deleting derivation homomorphism defined in characteristic 0 in [18] to the prime characteristic case. We first introduce the notion of higher Poisson derivation in Section 2.1. We use these higher Poisson derivations to overcome the characteristic problem, and thus define the characteristic-free Poisson deleting derivation homomorphism in Section 2.2. Section 2.3 is concerned with the compatibility of the characteristic-free Poisson deleting derivation homomorphism and the action of a torus acting rationally by Poisson automorphisms on the Poisson-Ore extension under consideration. This will be used later to prove the quadratic Poisson Gel'fand-Kirillov problem for torus-invariant prime factors of certain iterated Poisson-Ore extensions.

## 2.1 Higher Poisson derivation

The main tool to build Poisson birational isomorphisms between (certain) iterated Poisson-Ore extensions and Poisson affine spaces is the existence of higher derivations which are compatible with Poisson brackets. We define those higher derivations and give some of their properties. Higher derivations, also called *Hasse-Schmidt derivations*, have been previously studied in particular for their links with field extensions, see [38] or [39].

### 2.1.1 Definition and first properties

**Definition 2.1.1.** Let $A$ be a Poisson $\mathbb{K}$-algebra, $\alpha \in \mathrm{Der}_P(A)$ and $\eta \in \mathbb{K}$.

(1) A *higher derivation* on $A$ is a sequence of $\mathbb{K}$-linear maps $(D_i) := (D_i)_{i=0}^{\infty}$ such that:
$$D_0 = \mathrm{id}_A \text{ and } D_n(ab) = \sum_{i=0}^{n} D_i(a)D_{n-i}(b) \text{ for all } a, b \in A \text{ and all } n \geq 0. \tag{A1}$$
In particular $D_1$ is a derivation of $A$. Therefore we say that $\delta \in \mathrm{Der}(A)$ *extends to a higher derivation* if there exists a higher derivation $(D_i)$ on $A$ such that $D_1 = \delta$. A higher derivation is *iterative* if $D_i D_j = \binom{i+j}{i} D_{i+j}$ for all $i, j \geq 0$, and *locally nilpotent* if for all $a \in A$ there exists $n_a \geq 0$ such that $D_i(a) = 0$ for all $i \geq n_a$.

(2) A higher derivation $(D_i)$ is a *higher $\alpha$-skew Poisson derivation* if for all $a, b \in A$ and all $n \geq 0$:
$$D_n(\{a,b\}) = \sum_{i=0}^{n} \Big[\{D_i(a), D_{n-i}(b)\} + i\big(\alpha(D_{n-i}(a))D_i(b) - D_i(a)\alpha(D_{n-i}(b))\big)\Big]. \tag{A2}$$

(3) A higher $\alpha$-skew Poisson derivation is a *higher $(\eta, \alpha)$-skew Poisson derivation* if for all $i \geq 0$:
$$D_i\alpha = \alpha D_i + i\eta D_i. \tag{A3}$$

(4) We say that the derivation $\delta$ of a Poisson-Ore extension $A[X; \alpha, \delta]_P$ *extends to a higher $(\eta, \alpha)$-skew Poisson derivation* if there exists a higher $(\eta, \alpha)$-skew Poisson derivation $(D_i)$ on $A$ such that $D_1 = \delta$.

Note that if $A$ is finitely generated, say by $a_1, \dots, a_k$, a higher derivation $(D_i)$ is uniquely determined by the $D_i(a_j)$ for all $1 \leq j \leq k$ and all $i \geq 0$ thanks to Axiom (A1).

*Remark* 2.1.2. Let $A$ be a $\mathbb{K}$-algebra and $\delta \in \mathrm{Der}(A)$ which extends into an iterative higher derivation $(D_i)$. By induction we obtain $\delta^n = D_1^n = n! D_n$ for all $n \geq 0$. Therefore we have the following:

(1) In characteristic 0, the only iterative higher derivation $(D_i)$ on $A$ such that $D_1 = \delta$ is given by:
$$D_n = \frac{\delta^n}{n!}$$
for all $n \geq 0$. This iterative higher derivation is called the *canonical higher derivation associated to $\delta$*. Note that $(D_i)$ is locally nilpotent if $\delta$ is locally nilpotent.

(2) In characteristic $p > 0$ we have $D_1^p = 0$ and obtain no information on $D_p$ that way. One can show that the higher derivation $(D_i)$ is uniquely determined by the $D_{p^k}$ for $k \geq 0$. More precisely for $n = \sum_{k=0}^m n_k p^k$ the $p$-adic decomposition of $n$ we have:

$$D_n = \frac{D_1^{n_0} D_p^{n_1} \cdots D_{p^m}^{n_m}}{n_0! n_1! \cdots n_m!}.$$

See [38, Section III], the result for fields being trivially adapted for $\mathbb{K}$-algebras. In particular, we have $D_i = \frac{\delta^i}{i!}$ for all for $i < p$.

*Example* 2.1.3. Suppose $\mathbb{K}$ is of characteristic zero. Let $R = A[X; \alpha, \delta]_P$ be a Poisson-Ore extension where $A$ is a Poisson $\mathbb{K}$-algebra. If there exists $\eta \in \mathbb{K}^\times$ such that $\delta\alpha = \alpha\delta + \eta\delta$ then it follows from [18, Lemma 3.6] (with $s = -\eta$) that:

$$\delta^n(\{a, b\}) = \sum_{l+m=n} \binom{n}{l} \big(\{\delta^l(a), \delta^m(b)\} + m\delta^l(a)\alpha\delta^m(b) - l\delta^l(a)\delta^m\alpha(b)\big)$$

for all $a, b \in A$ and all $n \geq 0$. From this it is easily shown that the canonical higher derivation $\left(\frac{\delta^n}{n!}\right)$ is an iterative higher $(\eta, \alpha)$-skew Poisson derivation. The examples given in [18] provide a large family of $\alpha$-derivations $\delta$ satisfying $\delta\alpha = \alpha\delta + \eta\delta$ for some scalar $\eta \in \mathbb{K}^\times$, which extend to higher $(\eta, \alpha)$-skew Poisson derivations.

*Example* 2.1.4. Let $A := \mathbb{K}[X]$ and $\delta := \partial_X \in \mathrm{Der}(A)$. By setting:

$$D_i(X) := \begin{cases} X & i = 0, \\ 1 & i = 1, \\ 0 & i > 1, \end{cases}$$

we define an iterative higher derivation on $A$ extending $\delta$. For all $i, k \geq 0$ we can compute:

$$D_i(X^k) = \begin{cases} \binom{k}{i} X^{k-i} & k \geq i, \\ 0 & k < i. \end{cases} \tag{2.1}$$

Note that in characteristic zero the sequence $(D_i)$ and $(\frac{\delta^i}{i!})$ coincide. From (2.1) we see that $(D_i)$ is locally nilpotent. We now form the Poisson-Ore extension $B := A[Y; \alpha, \delta]_P$, where $\alpha := X\partial_X \in \mathrm{Der}_P(A)$ (note that $A$ is Poisson abelian). The higher derivation $(D_i)$ satisfies Axioms (A2) and (A3) of Definition 2.1.1 for $\eta = 1$ and $\alpha$ previously defined. Therefore $\delta$ extends to an iterative, locally niplotent higher $(1, \alpha)$-skew Poisson derivation on $A$.

We conclude this section by two technical lemmas.

**Lemma 2.1.5.** *Let $A$ be a Poisson $\mathbb{K}$-algebra, let $B \subseteq A$ be a Poisson subalgebra generated, as an algebra, by a finite set $\{b_1, \ldots, b_k\}$. Let $(D_i)$ be a higher derivation on $A$. If $D_i(b_j) \in B$ for all $i \geq 0$ and all $1 \leq j \leq k$, then $D_n(B) \subseteq B$ and $D_n(\{B, B\}) \subseteq B$ for all $n \geq 0$.*

*Proof.* We proceed by induction on the length of the monomials in the generators. For monomials of length one we have $D_n(b_j) \in B$ for all $n, j \geq 0$ by hypothesis. Let $T$ be a monomial of a given length and $b$ a generator. Suppose that $D_n(T) \in B$ for all $n \geq 0$. Then

$$D_n(Tb) = \sum_{i=0}^{n} D_i(T)D_{n-i}(b) \in B$$

by the induction hypothesis. This conclude the induction. Since $\{B, B\} \subseteq B$ we have $D_n(\{B, B\}) \subseteq B$. □

**Lemma 2.1.6.** *Let $A$ be a Poisson $\mathbb{K}$-algebra generated, as an algebra, by a finite set $\{a_1, \ldots, a_k\}$, and let $(D_i)$ be a higher derivation on $A$. If $(D_i)$ is locally nilpotent on $a_j$ for all $1 \leq j \leq k$, then $(D_i)$ is locally nilpotent on $A$.*

*Proof.* As in Lemma 2.1.5 we do an induction on the length of the monomials. If the length of a monomial is 1, the hypothesis gives the result. Suppose by induction that $(D_i)$ is locally nilpotent on a monomial $T$. Then there exists $p \geq 0$ such that $D_i(T) = 0$ for all $i \geq p$. Fix a generator $a_j$ of $A$. By hypothesis there exists $q \geq 0$ such that $D_i(a_j) = 0$ for all $i \geq q$. Thus we have:

$$D_n(Ta_j) = \sum_{i=0}^{n} D_i(T)D_{n-i}(a_j) = 0$$

for all $n \geq p + q$, i.e. $(D_i)$ is locally nilpotent on $Ta_j$. □

### 2.1.2 Higher derivation and localisation

The following proposition gives a criterion for a sequence of $\mathbb{K}$-linear maps to be a higher $(\eta, \alpha)$-skew Poisson derivation. This will be used later to extend a higher $(\eta, \alpha)$-skew Poisson derivation to certain localisations. For $\beta \in \mathrm{Der}_P(A)$, the Poisson bracket of $A$

uniquely extends to a Poisson bracket on the formal power series algebra $A[[X]]$ by setting $\{X, a\} = \beta(a)X$. This Poisson algebra is denoted by $A[[X;\beta]]_P$. The Poisson bracket of two elements of $A[[X;\beta]]_P$ is given by:

$$\{\sum_{i\geq 0} a_i X^i, \sum_{j\geq 0} b_j X^j\} = \sum_{n\geq 0}\Big(\sum_{i+j=n}\big(\{a_i, b_j\} + ia_i\beta(b_j) - j\beta(a_i)b_j\big)\Big)X^n,$$

where all the $a_i$s and the $b_j$s are in $A$. We remark that we have just extended by continuity the Poisson bracket of $A[X;\beta]_P$ to its completion $A[[X]]$. Note that the Poisson derivation $\beta$ of $A$ extends to a Poisson derivation of $A[[X;\beta]]_P$ by setting $\beta(X) = \eta X$ for any $\eta \in \mathbb{K}$ since:

$$\beta(\{X, a\}) = \big(\beta^2(a) + \eta\beta(a)\big)X = \{\beta(X), a\} + \{X, \beta(a)\}.$$

**Proposition 2.1.7.** *Let $(D_i)_{i=0}^{\infty}$ be a sequence of $\mathbb{K}$-linear maps on a Poisson $\mathbb{K}$-algebra $A$ with $D_0 = \mathrm{id}_A$, $\alpha \in \mathrm{Der}_P(A)$ and $\eta \in \mathbb{K}$.*

(a) *$(D_i)$ is a higher $\alpha$-skew Poisson derivation on $A$ if and only if the $\mathbb{K}$-linear map $\Psi : A \to A[[X;-\alpha]]_P$ given by $a \mapsto \sum_{i=0}^{\infty} D_i(a)X^i$ is a Poisson algebra homomorphism.*

(b) *Extend $\alpha$ to a Poisson derivation on $A[[X;-\alpha]]_P$ by setting $\alpha(X) = \eta X$. Assume that $(D_i)$ is a higher $\alpha$-skew Poisson derivation. Then $(D_i)$ is a higher $(\eta, \alpha)$-skew Poisson derivation if and only if the diagram of Figure 2.1 is commutative.*

$$\begin{array}{ccc} A[[X;-\alpha]]_P & \xrightarrow{\ \alpha\ } & A[[X;-\alpha]]_P \\ \Psi \big\uparrow & & \big\uparrow \Psi \\ A & \xrightarrow[\ \alpha\ ]{} & A \end{array}$$

Figure 2.1

*Proof.* (a) It is obvious that $\Psi$ is a $\mathbb{K}$-algebra homomorphism if and only if $(D_i)$ satisfies Axiom (A1). Let $a, b \in A$. We need to check that the equality $\Psi(\{a, b\}) = \{\Psi(a), \Psi(b)\}$

is equivalent to Axiom (A2):

$$\begin{aligned}\{\Psi(a),\Psi(b)\} &= \sum_{i,j}\{D_i(a)X^i, D_j(b)X^j\} \\ &= \sum_{i,j}\{D_i(a),D_j(b)\}X^{i+j} - iD_i(a)\alpha D_j(b)X^{i+j} + j\alpha D_i(a)D_j(b)X^{i+j} \\ &= \sum_{i,j}\Big(\{D_i(a),D_j(b)\} + j\alpha D_i(a)D_j(b) - iD_i(a)\alpha D_j(b)\Big)X^{i+j} \\ &= \sum_{n\geq 0}\sum_{i+j=n}\Big(\{D_i(a),D_j(b)\} + i\alpha D_j(a)D_i(b) - iD_i(a)\alpha D_j(b)\Big)X^n.\end{aligned}$$

Since $\Psi(\{a,b\}) = \sum_{n\geq 0} D_n(\{a,b\})X^n$ and $\{X^n \mid n \geq 0\}$ is a basis of $A[[X]]$, the equivalence is shown.

(b) We show that $\Psi\alpha = \alpha\Psi$ is equivalent to Axiom (A3). Let $a \in A$. Then we have:

$$\begin{aligned}\alpha\Psi(a) &= \sum_{i\geq 0}\alpha\big(D_i(a)X^i\big) \\ &= \sum_{i\geq 0}\alpha D_i(a)X^i + D_i(a)\alpha(X^i) \\ &= \sum_{i\geq 0}\big(\alpha D_i(a) + i\eta D_i(a)\big)X^i,\end{aligned}$$

On the other hand, we have:

$$\Psi\alpha(a) = \sum_{i\geq 0}\big(D_i\alpha(a)\big)X^i.$$

Hence $\Psi\alpha = \alpha\Psi$ if and only if $(D_i)$ satisfies Axiom (A3). □

**Proposition 2.1.8.** *Let $\alpha \in \mathrm{Der}_P(A)$, $\eta \in \mathbb{K}$ and $(D_i)$ a higher $(\eta,\alpha)$-skew Poisson derivation on a Poisson $\mathbb{K}$-algebra $A$. Let $S$ be a multiplicative set of regular elements of $A$. Then $(D_i)$ uniquely extends to a higher $(\eta,\alpha)$-skew Poisson derivation on $AS^{-1}$.*

*Proof.* A derivation $\beta$ of $A$ extends uniquely to $AS^{-1}$ by:

$$\beta(as^{-1}) = \beta(a)s^{-1} - as^{-2}\beta(s) \text{ for } a \in A \text{ and } s \in S. \tag{2.2}$$

So we can uniquely extend $\alpha$ and $D_1$ to $AS^{-1}$. Moreover if $\beta \in \mathrm{Der}_P(A)$ then after extension $\beta \in \mathrm{Der}_P(AS^{-1})$. Now suppose that $(D_i)$ extends to a higher $(\eta,\alpha)$-skew

Poisson derivation on $AS^{-1}$. For $a \in A$ and $s \in S$, we apply $D_n$ to the equation $a1^{-1} = (as^{-1})(s1^{-1})$ to get:

$$\begin{aligned} D_n(a)1^{-1} &= D_n\big((as^{-1})(s1^{-1})\big) \\ &= \sum_{i=0}^{n} D_i(as^{-1})D_{n-i}(s1^{-1}) \\ &= D_n(as^{-1})s1^{-1} + \sum_{i=0}^{n-1} D_i(as^{-1})D_{n-i}(s1^{-1}). \end{aligned}$$

This implies:

$$D_n(as^{-1}) = \Big(D_n(a) - \sum_{i=0}^{n-1} D_i(as^{-1})D_{n-i}(s)\Big)s^{-1},$$

thus proving uniqueness.

Let $\Psi : A \to A[[X; -\alpha]]_P$ be the $\mathbb{K}$-linear map defined in Proposition 2.1.7 and let:

$$\Phi : A[[X; -\alpha]]_P \to AS^{-1}[[X; -\alpha]]_P$$

be the canonical embedding. Consider the map $\Gamma := \Phi \circ \Psi$ from $A$ to $AS^{-1}[[X; -\alpha]]_P$ and note that $\Gamma$ is a $\mathbb{K}$-algebra Poisson algebra homomorphism by Proposition 2.1.7, since $(D_i)$ is a higher $\alpha$-skew Poisson derivation on $A$. For all $s \in S$, the constant term of $\Gamma(s)$ is a unit in $AS^{-1}$ and so $\Gamma(s)$ is a unit in $AS^{-1}[[X; -\alpha]]_P$. Hence $\Gamma$ extends to a $\mathbb{K}$-algebra homomorphism $\Gamma' : AS^{-1} \to AS^{-1}[[X; -\alpha]]_P$ such that $\Gamma'(as^{-1}) = \Gamma(a)\Gamma(s)^{-1}$. A straightforward computation shows that $\Gamma'$ is a Poisson algebra homomorphism.

We consider the diagram of Figure 2.2, where $\alpha$ has been extended to a Poisson deriva-

$$\begin{array}{ccc} AS^{-1}[[X; -\alpha]]_P & \xrightarrow{\ \alpha\ } & AS^{-1}[[X; -\alpha]]_P \\ \Gamma' \uparrow & & \uparrow \Gamma' \\ AS^{-1} & \xrightarrow[\ \alpha\ ]{} & AS^{-1} \end{array}$$

Figure 2.2

tion of $AS^{-1}[[X; -\alpha]]_P$ via (2.2) and $\alpha(X) = \eta X$. Since $\Gamma(a) = \sum_{i \geq 0}(D_i(a)1^{-1})X^i$, and

$(D_i)$ is a higher $(\eta, \alpha)$-skew Poisson derivation on $A$ for all $a \in A$ we have:

$$\begin{aligned}
\alpha\Gamma(a) &= \sum_{i\geq 0} \alpha\big((D_i(a)1^{-1})X^i\big) \\
&= \sum_{i\geq 0} \alpha(D_i(a)1^{-1})X^i + (D_i(a)1^{-1})\alpha(X^i) \\
&= \sum_{i\geq 0} (\alpha D_i(a)1^{-1} + i\eta D_i(a)1^{-1})X^i \\
&= \sum_{i\geq 0} (D_i\alpha(a)1^{-1})X^i = \Gamma\alpha(a).
\end{aligned}$$

Since $\Gamma$ is a $\mathbb{K}$-algebra homomorphism and $\alpha$ a $\mathbb{K}$-derivation we have:

$$\begin{aligned}
\alpha\Gamma'(as^{-1}) &= \alpha(\Gamma(a)\Gamma(s)^{-1}) \\
&= \alpha\Gamma(a)\Gamma(s)^{-1} - \Gamma(a)\Gamma(s)^{-2}\alpha\Gamma(s) \\
&= \Gamma\alpha(a)\Gamma(s)^{-1} - \Gamma(a)\Gamma(\alpha(s))\Gamma(s)^{-2} \\
&= \Gamma\alpha(a)\Gamma(s)^{-1} - \Gamma(a\alpha(s))\Gamma(s^2)^{-1} \\
&= \Gamma'(\alpha(a)s^{-1} - a\alpha(s)s^{-2}) \\
&= \Gamma'\alpha(as^{-1}).
\end{aligned}$$

Thus the diagram of Figure 2.2 is commutative, as desired.

Define a sequence $(D_i)$ on $AS^{-1}$ such that $D_i(as^{-1})$ is the coefficient of $X^i$ in $\Gamma'(as^{-1})$ for all $as^{-1} \in AS^{-1}$. Then, by Proposition 2.1.7, we conclude that this sequence is a higher $(\eta, \alpha)$-skew Poisson derivation on $AS^{-1}$ extending $(D_i)$ on $A$, as requested. □

## 2.2 Deleting derivation homomorphism

Let $A[X; \alpha, \delta]_P$ be a Poisson-Ore extension, where $A$ is a Poisson $\mathbb{K}$-algebra and set $S := \{X^n \mid n \geq 0\}$. The set $S$ is a multiplicative set (of regular elements) and we denote by $A[X^{\pm 1}; \alpha, \delta]_P$ the localisation $S^{-1}\big(A[X; \alpha, \delta]_P\big)$. Poisson brackets extend uniquely by localisation, so $A[X^{\pm 1}; \alpha, \delta]_P$ is also a Poisson algebra, called *Poisson-Ore Laurent algebra*. Suppose that the derivation $\delta$ extends to an iterative locally nilpotent higher $(\eta, \alpha)$-skew

Poisson derivation $(D_i)$ with $\eta \in \mathbb{K}^\times$. We define a map $\theta : A \to A[X^{\pm 1}; \alpha, \delta]_P$ by setting:

$$\theta(a) = \sum_{i \geq 0} \frac{1}{\eta^i} D_i(a) X^{-i} \qquad \text{for all } a \in A.$$

Note that this sum is finite since $(D_i)$ is locally nilpotent.

**Proposition 2.2.1.** *The $\mathbb{K}$-linear map $\theta : A \to A[X^{\pm 1}; \alpha, \delta]_P$ is a Poisson algebra homomorphism and satisfies the following identity:*

$$\{X, \theta(a)\} = \theta\big(\alpha(a)\big)X \qquad \textit{for all } a \in A.$$

*Proof.* $\theta$ is an algebra homomorphism because $(D_i)$ satisfies Axiom (A1). Let us show that $\theta$ respects the Poisson bracket using Axiom (A2) and the iterativity of $(D_i)$.

$$\begin{aligned}
\{\theta(a), \theta(b)\} &= \sum_{i,j \geq 0} \{\frac{1}{\eta^i} D_i(a) X^{-i}, \frac{1}{\eta^j} D_j(b) X^{-j}\} \\
&= \sum_{i,j \geq 0} \frac{1}{\eta^{i+j}} \Big( \{D_i(a), D_j(b)\} X^{-i-j} \\
&\qquad\qquad + D_j(b)\{D_i(a), X^{-j}\} X^{-i} + D_i(a)\{X^{-i}, D_j(b)\} X^{-j} \Big) \\
&= \sum_{i,j \geq 0} \frac{1}{\eta^{i+j}} \Big( \{D_i(a), D_j(b)\} X^{-i-j} + j D_j(b)\big(\alpha D_i(a) X + D_1 D_i(a)\big) X^{-i-j-1} \\
&\qquad\qquad\qquad - i D_i(a)\big(\alpha D_j(b) X + D_1 D_j(b)\big) X^{-i-j-1} \Big) \\
&= \sum_{i,j \geq 0} \frac{1}{\eta^{i+j}} \Big( \{D_i(a), D_j(b)\} + j\alpha D_i(a) D_j(b) - i D_i(a) \alpha D_j(b) \Big) X^{-i-j} \\
&\quad + \sum_{i,j \geq 0} \frac{1}{\eta^{i+j}} \Big( j D_j(b) D_1 D_i(a) - i D_i(a) D_1 D_j(b) \Big) X^{-i-j-1} \\
&= \sum_{i,j \geq 0} \frac{1}{\eta^{i+j}} \Big( \{D_i(a), D_j(b)\} + i\alpha D_j(a) D_i(b) - i D_i(a) \alpha D_j(b) \Big) X^{-i-j} \\
&\quad + \sum_{i,j \geq 0} \frac{1}{\eta^{i+j}} \Big( j(i+1) D_j(b) D_{i+1}(a) - i(j+1) D_i(a) D_{j+1}(b) \Big) X^{-i-j-1} \\
&= \sum_{t \geq 0} \frac{1}{\eta^t} \sum_{i+j=t} \Big( \{D_i(a), D_j(b)\} + i\big(\alpha D_j(a) D_i(b) - D_i(a) \alpha D_j(b)\big) \Big) X^{-t} \\
&\quad + \sum_{j,l \geq 1} \frac{jl}{\eta^{j+l-1}} D_j(b) D_l(a) X^{-j-l} - \sum_{i,k \geq 1} \frac{ik}{\eta^{i+k-1}} D_i(a) D_k(b) X^{-i-k} \\
&= \sum_{t \geq 0} \frac{1}{\eta^t} D_t(\{a, b\}) X^{-t} \\
&= \theta(\{a, b\}).
\end{aligned}$$

Finally we use Axiom (A3) and the iterativity of $(D_i)$ to show that $\{X, \theta(a)\} = \theta\big(\alpha(a)\big)X$. Indeed, we have:

$$\begin{aligned}
\{X, \theta(a)\} &= \sum_{i\geq 0} \frac{1}{\eta^i}\{X, D_i(a)\}X^{-i} \\
&= \sum_{i\geq 0} \frac{1}{\eta^i}\big(\alpha\big(D_i(a)\big)X + D_1D_i(a)\big)X^{-i} \\
&= \sum_{i\geq 0} \frac{1}{\eta^i}\alpha\big(D_i(a)\big)X^{-i+1} + \sum_{i\geq 0} \frac{1}{\eta^i}(i+1)D_{i+1}(a)X^{-i} \\
&= \sum_{i\geq 0} \frac{1}{\eta^i}\alpha\big(D_i(a)\big)X^{-i+1} + \sum_{i\geq 1} \frac{\eta}{\eta^i} iD_i(a)X^{-i+1} \\
&= \sum_{i\geq 0} \frac{1}{\eta^i}\big(\alpha\big(D_i(a)\big) + i\eta D_i(a)\big)X^{-i+1} \\
&= \sum_{i\geq 0} \frac{1}{\eta^i}D_i\big(\alpha(a)\big)X^{-i+1} \\
&= \theta\big(\alpha(a)\big)X.
\end{aligned}$$

□

We are now ready to state the main result of this section.

**Theorem 2.2.2.** *Let $A[X;\alpha,\delta]_P$ be a Poisson-Ore extension, where $A$ is a Poisson $\mathbb{K}$-algebra. Suppose that $\delta$ extends to an iterative, locally nilpotent higher $(\eta,\alpha)$-skew Poisson derivation $(D_i)$ on $A$ with $\eta \in \mathbb{K}^\times$. Then the algebra homomorphism $\theta : A \to A[X^{\pm 1}]$ defined by:*

$$\theta(a) = \sum_{i\geq 0} \frac{1}{\eta^i} D_i(a) X^{-i}$$

*uniquely extends to a Poisson $\mathbb{K}$-algebra isomorphism:*

$$\theta : A[Y^{\pm 1};\alpha]_P \xrightarrow{\cong} A[X^{\pm 1};\alpha,\delta]_P$$

*by setting $\theta(Y) = X$.*

*Proof.* Clearly $\theta$ extends uniquely to a $\mathbb{K}$-algebra homomorphism from $A[Y^{\pm 1}]$ to $A[X^{\pm 1}]$ by setting $\theta(Y) = X$. In view of Proposition 2.2.1 we know that $\theta(\{a,b\}) = \{\theta(a), \theta(b)\}$

for all $a, b \in A$. Moreover, for all $a \in A$ we have:

$$\theta(\{Y, a\}) = \theta(\alpha(a)Y) = \theta\big(\alpha(a)\big)\theta(Y) = \theta\big(\alpha(a)\big)X = \{X, \theta(a)\} = \{\theta(Y), \theta(a)\}.$$

Thus $\theta$ is a Poisson algebra homomorphism from $A[Y^{\pm 1}; \alpha]_P$ to $A[X^{\pm 1}; \alpha, \delta]_P$.

To conclude we show that $\theta$ is bijective. First, let $f \in A[Y^{\pm 1}]$ be a nonzero Laurent polynomial. We can write $f = \sum_{i=l}^{m} a_i Y^i$, where $l, m$ are two integers with $l \leq m$, and $a_i \in A$ for all $i \in \{l, \dots, m\}$ with $a_m \neq 0$. Observing that:

$$\theta(a_i Y^i) = \sum_{k \geq 0} \frac{1}{\eta^k} D_k(a_i) X^{i-k} = a_i X^i + \sum_{k \geq 1} \frac{1}{\eta^k} D_k(a_i) X^{i-k}$$

for all $i$, we can write $\theta(f) = a_m X^m + \sum_{i=j}^{m-1} b_i X^i$, for some $j < m$ and where $b_i \in A$ for all $j \leq i < m$. Thus $\theta(f) \neq 0$, and $F$ is injective.

For the surjectivity, we already have $\theta(Y^{\pm 1}) = X^{\pm 1}$, so we just need to check that $A \subset \mathrm{Im}(\theta)$. Let $a \in A$. Since $(D_i)$ is locally nilpotent, there exists $l \geq 0$ such that $D_l(a) = 0$. If $l \leq 1$, we have $\theta(a) = a$ and so $a \in \mathrm{Im}(\theta)$. Assume $l > 1$ and write $\theta(a) = a + \sum_{i=1}^{l-1} \frac{1}{\eta^i} D_i(a) X^{-i}$. Since $D_{l-i} D_i(a) = \binom{l}{i} D_l(a) = 0$ for $i = 1, \dots, l-1$, we have $D_i(a) \in \mathrm{Im}(\theta)$ for all $i = 1, \dots, l-1$ (we proceed by induction on $l$). Thus $\theta(a) - a$ is in the image of $\theta$ and so does $a$. Thus $\theta$ is surjective. □

We set $B := A[X; \alpha, \delta]_P$ and $S := \{X^i \mid i \geq 0\}$ so that we have $BS^{-1} = A[X^{\pm 1}; \alpha, \delta]_P$. We deduce immediately the following result.

**Corollary 2.2.3.** *$BS^{-1}$ contains a Poisson subalgebra $B'$ isomorphic to $A[Y; \alpha]_P$, and we have $B'S^{-1} = BS^{-1}$. In particular we have:*

$$\mathrm{Frac}\left(A[X; \alpha, \delta]_P\right) = \mathrm{Frac}\left(B'\right) \cong \mathrm{Frac}\left(A[Y; \alpha]_P\right).$$

*Proof.* Take $B' := \theta(A[Y; \alpha]_P)$. □

When $\mathrm{char}\, \mathbb{K} = 0$, Remark 2.1.2 and Example 2.1.3 show that Theorem 2.2.2 is [18, Theorem 3.7].

*Example* 2.2.4. Recall from Example 2.1.4 the Poisson-Ore extension $B := \mathbb{K}[X][Y; \alpha, \delta]_P$, where $\alpha := X\partial_X$ and $\delta = \partial_X$ so that we have $\{Y, X\} = XY + 1$. Recall that the derivation

$\delta$ extends to an iterative, locally nilpotent higher $(1, \alpha)$-skew Poisson derivation on $\mathbb{K}[X]$ given by:

$$D_i(X) := \begin{cases} X & i = 0 \\ 1 & i = 1 \\ 0 & i > 1. \end{cases}$$

Therefore we can apply Theorem 2.2.2 to $B$, and we have a Poisson algebra isomorphism:

$$\mathbb{K}[X][Z^{\pm 1}; \alpha]_P \cong \mathbb{K}[X][Y^{\pm 1}, \alpha, \delta]_P$$

sending $Z$ to $Y$. And so, by Corollary 2.2.3, there is inside $\mathbb{K}[X][Y^{\pm 1}, \alpha, \delta]_P$ a Poisson subalgebra isomorphic to the Poisson affine plane $\mathbb{K}[X][Z; \alpha]_P = \mathbb{K}_{\boldsymbol{\lambda}}[X, Z]$, where $\boldsymbol{\lambda} \in M_2(\mathbb{K})$ is the skew-symmetric matrix such that $\lambda_{12} = -1$. This is the algebra $\theta\big(\mathbb{K}[X, Z]\big) = \mathbb{K}[X', Y]$, where $X' := \theta(X) = X + Y^{-1}$. In this case it is easy to verify that:

$$\{Y, X'\} = \{Y, X\} = XY + 1 = Y(X + Y^{-1}) = YX'.$$

We conclude by saying that $\operatorname{Frac} B \cong \mathbb{K}(U, V)$ with $\{U, V\} = -UV$.

## 2.3 Case where a torus acts rationally: $H$-equivariance of the deleting derivation homomorphism

Let $A$ be a finitely generated Poisson $\mathbb{K}$-algebra. By an (algebraic) *torus* we mean a group of the form $(\mathbb{K}^{\times})^r$ for some $r > 0$. Suppose that a torus $H$ is acting by Poisson $\mathbb{K}$-algebra automorphisms on a Poisson-Ore extension $B := A[X; \alpha, \delta]_P$ such that $H(A) = A$. This means that there is a group homomorphism $\phi$ from $H$ to the group of Poisson automorphisms of $B$ (an automorphism $\sigma$ of $B$ is a Poisson automorphism is we have $\sigma(\{a, b\}) = \{\sigma(a), \sigma(b)\}$ for all $a, b \in B$). For $h \in H$ we set $h(a) := \phi(h)(a)$ for all $a \in B$. We suppose that the indeterminate $X$ is an $H$-eigenvector, that is for all $h \in H$ there exists $\mu \in \mathbb{K}$ such that $h(X) = \mu X$ (note that $\mu \neq 0$ since $h$ is an automorphism). Finally we assume that $H$ commutes with the derivation $\alpha$.

We show that under these assumptions the torus $H$ is also acting by Poisson automorphism on the Poisson-Ore extension $A[Y;\alpha]_P$. Let $h \in H$ and set $h(X) = \mu X$ for a scalar $\mu \in \mathbb{K}^\times$. Then $H$ is also acting by automorphisms on $A[Y;\alpha]_P$ via:

$$h\Big(\sum_{i=0}^{n} a_i Y^i\Big) = \sum_{i=0}^{n} h(a_i)\mu^i Y^i$$

for all $h \in H$. Note that $h(Y) = \mu Y$. Moreover this action respects the Poisson bracket of $A[Y;\alpha]_P$ since:

$$h(\{Y, a\}) = h(\alpha(a)Y) = h(\alpha(a))h(Y) = \mu\alpha(h(a))Y = \mu\{Y, h(a)\} = \{h(Y), h(a)\}.$$

These $H$-actions extend uniquely by localisation to $A[X^{\pm 1};\alpha,\delta]_P$ and $A[Y^{\pm 1};\alpha]_P$ since $X$ and $Y$ are $H$-eigenvectors. With a desire of clarity, we sometimes distinguish between the actions of $h \in H$ on $A[X^{\pm 1};\alpha,\delta]_P$ and $A[Y^{\pm 1};\alpha]_P$ by using subscripts: $h_X$ and $h_Y$. The following lemma gives conditions under which these actions commute with the deleting derivation homomorphism $\theta$ defined at the beginning of Section 2.2.

**Lemma 2.3.1.** *Suppose that $\delta$ extends to a higher $(\eta,\alpha)$-skew Poisson derivation $(D_i)$ on $A$ with $\eta \in \mathbb{K}^\times$. We denote by $\{a_1, \dots, a_l\}$ a set of generators of $A$. If for all $n \geq 0$ and all $1 \leq i \leq l$ we have:*

$$h\big(D_n(a_i)\big) = \mu^n D_n\big(h(a_i)\big)$$

*then $h_X\theta = \theta h_Y$, that is the diagram of Figure 2.3 is commutative.*

$$\begin{array}{ccc} A[X^{\pm 1};\alpha,\delta]_P & \xrightarrow{\;h_X\;} & A[X^{\pm 1};\alpha,\delta]_P \\ \theta \Big\uparrow & & \Big\uparrow \theta \\ A[Y^{\pm 1};\alpha]_P & \xrightarrow[\;h_Y\;]{} & A[Y^{\pm 1};\alpha]_P \end{array}$$

Figure 2.3

*Proof.* For all $1 \leq i \leq l$ we have:

$$\begin{aligned} h_X\big(\theta(a_i)\big) &= \sum_{k\geq 0} \frac{1}{\eta^k} h_X\big(D_k(a_i)\big) h_X(X^{-k}) \\ &= \sum_{k\geq 0} \frac{1}{\eta^k} \mu^k D_k\big(h_Y(a_i)\big)\mu^{-k} X^{-k} \end{aligned}$$

$$= \theta\big(h_Y(a_i)\big),$$

since $h_X(a) = h(a) = h_Y(a) \in A$ for all $a \in A$. We conclude by noting that:

$$h_X\big(\theta(Y)\big) = h_X(X) = \mu X = \mu\theta(Y) = \theta(\mu Y) = \theta\big(h_Y(Y)\big).$$

□

*Example* 2.3.2. Recall the Poisson-Ore extension $B := A[Y; \alpha, \delta]_P$, where $A := \mathbb{K}[X]$, $\alpha := X\partial_X$ and $\delta := \partial_X$ extends to an iterative, locally nilpotent higher $(1, \alpha)$-skew Poisson derivation $(D_i)$ on $A$ given by:

$$D_i(X) := \begin{cases} X & i = 0 \\ 1 & i = 1 \\ 0 & i > 1. \end{cases}$$

The torus $H = \mathbb{K}^\times$ acts rationally by Poisson algebra automorphisms on $B$ via $h(X) = hX$ and $h(Y) = h^{-1}Y$ for all $h \in H$. It is clear that $H$ also acts rationally by Poisson algebra automorphisms on $A[Y; \alpha]_P$ via the same rule. Fix $h \in H$. The corresponding eigenvalue for $Y$ is $\mu := h^{-1}$. Therefore the assumptions of Lemma 2.3.1 are satisfies since for all $i \geq 0$ we have:

$$h\big(D_i(X)\big) = \mu^i D_i\big(h(X)\big).$$

One can check that indeed $h$ and $\theta$ commute.

$$h\big(\theta(X)\big) = h(X + Y^{-1}) = hX + \big(h^{-1}Y\big)^{-1} = hX + hY^{-1} = \theta\big(h(X)\big).$$

In the next chapter we will see how the material developed in this chapter can be used to understand the structure of the field of fractions of certain iterated Poisson-Ore extensions.

# Chapter 3

# A quadratic Poisson Gel'fand-Kirillov problem

A Poisson algebra (which is a domain) satisfies the quadratic Poisson Gel'fand-Kirillov problem if its field of fractions is isomorphic to the field of fractions of a Poisson affine space, see Section 3.1 for more details. In this chapter we give a positive answer to the quadratic Poisson Gel'fand-Kirillov problem for Poisson algebras satisfying suitable conditions (see Section 3.3) and some of their quotients (see Section 3.4). This is achieved through repeated use of the characteristic-free Poisson deleting derivation homomorphism constructed in the previous chapter. In Section 3.2 we give some preliminary results which show that, after deleting the last derivation in an iterated Poisson-Ore extension, moving the last variable in first position does not affect the existence and properties of the needed higher Poisson derivations corresponding to the other variables. This is crucial as it allows for inductive use of the characteristic-free Poisson deleting derivation homomorphism in order to prove the main result of Section 3.2, namely Theorem 3.3.1. This theorem shows that, under suitable assumptions, there is a Poisson algebra isomorphism between the field of fractions of an iterated Poisson-Ore extension and a Poisson affine space, i.e. the iterated Poisson-Ore extension under consideration satisfies the quadratic Poisson Gel'fand-Kirillov problem.

Concerning Poisson prime factors of an iterated Poisson-Ore extension $A$, Theorem 3.3.1 tells us that they satisfy the quadratic Poisson Gel'fand-Kirillov problem if the corresponding Poisson prime factors of the Poisson affine space $B$ do (assertion (2) of Theorem

3.3.1). In characteristic zero, a Poisson prime factor of a Poisson affine space is always Poisson birationally equivalent to a Poisson affine space over a purely transcendental extension of the base field, see [18, Theorem 3.3]. However in prime characteristic this is not clear anymore, and we restrict ourselves to the Poisson prime ideals which are also invariant under the action of a torus $H$. In Section 3.4.2 we show that, under mild hypotheses, there are actually only finitely many $H$-invariant Poisson prime ideals in a Poisson affine space. Moreover, we explicitly describe all these ideals. As a consequence, the corresponding quotient algebras of $B$ satisfy the quadratic Poisson Gel'fand-Kirillov problem, and so we conclude from Theorem 3.3.1 that all $H$-invariant Poisson prime quotients of $A$ satisfy the quadratic Poisson Gel'fand-Kirillov problem.

## 3.1 The Quadratic Poisson Gel'fand-Kirillov problem

The quadratic Poisson Gel'fand-Kirillov problem is a problem of birational equivalence for polynomial Poisson algebras. It is a Poisson analogue of the quantum Gel'fand-Kirillov Conjecture.

*We say that a Poisson* $\mathbb{K}$*-algebra* $A$ *which is a domain satisfies the* quadratic Poisson Gel'fand-Kirillov problem *if there exists a Poisson* $\mathbb{K}$*-algebra isomorphism:*

$$\operatorname{Frac} A \cong \mathbb{K}_{\boldsymbol{\lambda}}(X_1, \dots, X_n),$$

*for an integer* $n \geq 1$ *and a skew-symmetric matrix* $\boldsymbol{\lambda} \in M_n(\mathbb{K})$*.*

Recall that $\mathbb{K}_{\boldsymbol{\lambda}}(X_1, \dots, X_n)$ denotes the field of fractions of the Poisson affine $n$-space $\mathbb{K}_{\boldsymbol{\lambda}}[X_1, \dots, X_n]$ (see (3) in Examples 1.1.3).

*Remark* 3.1.1. It is possible to relax the requirement in the problem set up above (as in [18]) by allowing the Poisson $\mathbb{K}$-algebra isomorphism to hold between $\operatorname{Frac} A$ and a Poisson affine field $\mathbb{L}_{\boldsymbol{\mu}}(X_1, \dots, X_t)$ for some field extension $\mathbb{K} \subseteq \mathbb{L}$ (an integer $t \geq 1$, and a skew-symmetric matrix $\boldsymbol{\mu} \in M_t(\mathbb{K})$). This version is used in [18] to include Poisson prime quotients of Poisson affine spaces (essentially to take into account that some indeterminates can become Poisson central in the quotient). It is also worthwhile noting that in this version the matrix $\boldsymbol{\mu}$ must take its coefficients in $\mathbb{K}$ and not in $\mathbb{L}$.

## 3.2 Preliminaries

In order to extend the results of the previous chapter to iterated Poisson-Ore extensions, we need to know the behaviour of a higher Poisson derivation when reordering the variables. This is the objective of the next two lemmas.

**Lemma 3.2.1.** *Let $A$ be a Poisson $\mathbb{K}$-algebra and $R = A[X;\alpha,\delta]_P[Y^{\pm 1};\beta]_P$ be an iterated Poisson-Ore extension, where $\beta(A) \subseteq A$ and $\beta(X) = \lambda X$ for $\lambda \in \mathbb{K}$.*

(1) *Then $R = A[Y^{\pm 1};\beta']_P[X;\alpha',\delta']_P$, where $\beta' = \beta|_A$, $\alpha'|_A = \alpha$, $\delta'|_A = \delta$, $\alpha'(Y) = -\lambda Y$ and $\delta'(Y) = 0$.*

(2) *If $\delta\alpha = \alpha\delta + \eta\delta$ in $A$, then $\delta'\alpha' = \alpha'\delta' + \eta\delta'$ in $A[Y^{\pm 1};\beta]_P$.*

(3) *Suppose further that $\delta$ extends to a higher $(\eta,\alpha)$-skew Poisson derivation $(D_i)$ on $A$ and that $\beta D_i = D_i\beta + i\lambda D_i$ for all $i \geq 0$. Then $\delta'$ extends to a higher $(\eta,\alpha')$-skew Poisson derivation $(D'_i)$ on $A[Y^{\pm 1};\beta]_P$ such that the restriction of $D'_i$ to $A$ coincides with $D_i$ for all $i \geq 0$, and $D'_i(Y) = 0$ for all $i > 0$.*

(4) *Keeping the assumptions of* (3) above, we have:

   (a) If $(D_i)$ is iterative, then $(D'_i)$ is iterative.

   (b) If $(D_i)$ is locally nilpotent, then $(D'_i)$ is locally nilpotent.

*Proof.* (1) Since $\beta(A) \subseteq A$ and $\{X, Y\} = -\lambda XY$ we can switch the variables $X$ and $Y$ in the expression of $R$ as a Poisson-Ore extension over $A$. The new maps we get are those described in (1).

(2) We only check the equality on a monomial $aY^i \in A[Y^{\pm 1}]$ since the derivations involved are $\mathbb{K}$-linear.

$$\begin{aligned}
\delta'\alpha'(aY^i) &= \delta'(\alpha'(a)Y^i + a\alpha'(Y^i)) \\
&= \delta'(\alpha'(a)Y^i) + \delta'(-i\lambda aY^i) \\
&= \delta'(\alpha'(a)Y^i) - i\lambda(\delta'(a)Y^i + \delta'(Y^i)a) \\
&= (\delta\alpha(a) - i\lambda\delta(a))Y^i \\
&= \big(\alpha\delta(a) + \eta\delta(a) - i\lambda\delta(a)\big)Y^i \\
&= (\alpha'\delta' + \eta\delta')(aY^i).
\end{aligned}$$

(3) Define a sequence of $\mathbb{K}$-linear maps $D'_i : A[Y^{\pm 1}; \beta']_P \to A[Y^{\pm 1}; \beta']_P$ for all $i \geq 0$ by

$$D'_i\Big(\sum_{j=-m}^{m} a_j Y^j\Big) = \sum_{j=-m}^{m} D_i(a_j) Y^j.$$

We check that $(D'_i)$ is a higher $(\eta, \alpha')$-skew Poisson derivation on $A[Y^{\pm 1}; \beta']_P$ satisfying all conditions of (3). First, it is clear that $D'_i(a) = D_i(a)$ for all $a \in A$. Moreover $D'_i(Y) = D_i(1)Y = 0$ for $i > 0$ and $D'_0 = \mathrm{id}$ on $A[Y^{\pm 1}; \beta']_P$. The following computation shows that $\delta'$ extends to $(D'_i)$:

$$D'_1\Big(\sum_{j=-m}^{m} a_j Y^j\Big) = \sum_{j=-m}^{m} D_1(a_j) Y^j = \sum_{j=-m}^{m} \delta(a_j) Y^j = \delta'\Big(\sum_{j=-m}^{m} a_j Y^j\Big).$$

It just remains to establish Axioms (A1), (A2) and (A3) of Definition 2.1.1 on monomials of $A[Y^{\pm 1}]$ (since the Poisson bracket is $\mathbb{K}$-bilinear and the $D'_i$ and the $D_i$ are $\mathbb{K}$-linear maps).

First, for all $a, b \in A$ and all $i, j \in \mathbb{Z}$:

$$\begin{aligned} D'_n\big((aY^i)(bY^j)\big) &= D_n(ab) Y^{i+j} \\ &= \sum_{k=0}^{n} D_k(a) D_{n-k}(b) Y^{i+j} \\ &= \sum_{k=0}^{n} D'_k(aY^i) D'_{n-k}(bY^j). \end{aligned}$$

Hence Axiom (A1) is proved. Next

$$\begin{aligned} D'_n(\{aY^i, bY^j\}) &= D'_n\big[\big(\{a,b\} + i\beta'(b)a - j\beta'(a)b\big)Y^{i+j}\big] \\ &= \big[D_n(\{a,b\}) + iD_n(\beta(b)a) - jD_n(\beta(a)b)\big]Y^{i+j} \\ &= \sum_{k=0}^{n} \Big[\{D_k(a), D_{n-k}(b)\} + k\big(\alpha D_{n-k}(a) D_k(b) - D_k(a) \alpha D_{n-k}(b)\big)\Big] Y^{i+j} \\ &\quad + i \sum_{k=0}^{n} D_{n-k}(a) D_k \beta(b) Y^{i+j} \\ &\quad - j \sum_{k=0}^{n} D_{n-k}(b) D_k \beta(a) Y^{i+j}, \end{aligned}$$

whereas

$$
\begin{aligned}
&\sum_{k=0}^{n}\{D'_k(aY^i), D'_{n-k}(bY^j)\} + k\big(\alpha' D'_{n-k}(aY^i)D'_k(bY^j) - D'_k(aY^i)\alpha' D'_{n-k}(bY^j)\big) \\
&= \sum_{k=0}^{n}\Big(\{D_k(a), D_{n-k}(b)\} + iD_k(a)\beta' D_{n-k}(b) - j\beta' D_k(a)D_{n-k}(b)\Big)Y^{i+j} \\
&\quad + \sum_{k=0}^{n} kD_k(b)\big(\alpha D_{n-k}(a)Y^i + D_{n-k}(a)\alpha'(Y^i)\big)Y^j \\
&\quad - \sum_{k=0}^{n} kD_k(a)\big(\alpha D_{n-k}(b)Y^j + D_{n-k}(b)\alpha'(Y^j)\big)Y^i \\
&= \sum_{k=0}^{n}\Big(\{D_k(a), D_{n-k}(b)\} + iD_k(a)\beta D_{n-k}(b) - j\beta D_k(a)D_{n-k}(b)\Big)Y^{i+j} \\
&\quad + \sum_{k=0}^{n} kD_k(b)\big(\alpha D_{n-k}(a) - i\lambda D_{n-k}(a)\big)Y^{i+j} \\
&\quad - \sum_{k=0}^{n} kD_k(a)\big(\alpha D_{n-k}(b) - j\lambda D_{n-k}(b)\big)Y^{i+j} \\
&= \sum_{k=0}^{n}\Big(\{D_k(a), D_{n-k}(b)\} + k\big(\alpha D_{n-k}(a)D_k(b) - D_k(a)\alpha D_{n-k}(b)\big)\Big)Y^{i+j} \\
&\quad + i\sum_{k=0}^{n} D_k(a)\beta D_{n-k}(b)Y^{i+j} - i\lambda\sum_{k=0}^{n} kD_{n-k}(a)D_k(b)Y^{i+j} \\
&\quad - j\sum_{k=0}^{n} D_{n-k}(b)\big(\beta D_k(a) - k\lambda D_k(a)\big)Y^{i+j} \\
&= \sum_{k=0}^{n}\Big(\{D_k(a), D_{n-k}(b)\} + k\big(\alpha D_{n-k}(a)D_k(b) - D_k(a)\alpha D_{n-k}(b)\big)\Big)Y^{i+j} \\
&\quad + i\sum_{k=0}^{n} D_{n-k}(a)\big(\beta D_k(b) - k\lambda D_k(b)\big)Y^{i+j} \\
&\quad - j\sum_{k=0}^{n} D_{n-k}(b)\big(\beta D_k(a) - k\lambda D_k(a)\big)Y^{i+j}.
\end{aligned}
$$

(In the last step of this computation we used a change of variable $k' = n-k$ in the second sum). Since $\beta D_k - \lambda k D_k = D_k\beta$ for all $k \geq 0$, Axiom (A2) is established. And finally, we get Axiom (A3) by computing:

$$
\begin{aligned}
(\alpha' D'_i + i\eta D'_i)(aY^l) &= \alpha'\big(D_i(a)Y^l\big) + i\eta D_i(a)Y^l \\
&= \big(\alpha D_i(a) - \lambda l D_i(a) + i\eta D_i(a)\big)Y^l \\
&= \big(D_i\alpha(a) - \lambda l D_i(a)\big)Y^l
\end{aligned}
$$

$$= D_i(\alpha(a) - \lambda la)Y^l$$
$$= D_i'\big((\alpha(a) - \lambda la)Y^l\big)$$
$$= D_i'\big(\alpha'(aY^l)\big),$$

for all $i \geq 0$ and $l \in \mathbb{Z}$.

(4a) If $(D_i)$ is iterative on $A$, then

$$D_i'D_j'(aY^l) = D_i'(D_j(a)Y^l) = D_iD_j(a)Y^l = \binom{i+j}{j} D_{i+j}(a)Y^l = \binom{i+j}{j} D_{i+j}'(aY^l)$$

for all $a \in A$, $l \in \mathbb{Z}$ and $i, j \geq 0$. Hence $(D_i')$ is iterative on $A[Y^{\pm 1}; \beta']_P$.
(4b) Suppose that $(D_i)$ is locally nilpotent on $A$. Using Lemma 2.1.6 we only need to check that $(D_i')$ is locally nilpotent on a set of generators of $A[Y^{\pm 1}]$. We take $A \cup \{Y^{\pm 1}\}$. For all $a \in A$ and $i \geq 0$ we have $D_i'(a) = D_i(a)$, so that $(D_i')^n(a) = 0$ for $n >> 0$. Moreover $D_i'(Y) = 0$ (which implies $D_i'(Y^{-1}) = 0$) for all $i > 0$. The result is shown. □

Lemma 3.2.1 can be generalised as follows.

**Lemma 3.2.2.** *Let $A$ be a Poisson $\mathbb{K}$-algebra and set:*

$$R := A[X_1; \alpha_1, \delta_1]_P \cdots [X_n; \alpha_n, \delta_n]_P[Y^{\pm 1}; \beta]_P,$$

*where $\beta(A) \subseteq A$ and $\beta(X_i) = \lambda_i X_i$ with $\lambda_i \in \mathbb{K}$ for all $1 \leq i \leq n$. We also set $R_0 := A$ and $R_j := A[X_1; \alpha_1, \delta_1]_P \cdots [X_j; \alpha_j, \delta_j]_P$ for $j = 1, \ldots, n$.*

(1) *Then $R = A[Y^{\pm 1}; \beta']_P[X_1; \alpha_1', \delta_1']_P \cdots [X_n; \alpha_n', \delta_n']_P$, where $\beta' = \beta|_A$, $\alpha_i'|_{R_j} = \alpha_i$, $\delta_i'|_{R_j} = \delta_i$, $\alpha_i'(Y) = -\lambda_i Y$ and $\delta_i'(Y) = 0$ for all $i = 1, \ldots, n$ and $j = 0, \ldots, i-1$.*

(2) *Set $R_j' := A[Y^{\pm 1}; \beta']_P[X_1; \alpha_1', \delta_1']_P \cdots [X_j; \alpha_j', \delta_j']_P$ and $R_0' = A[Y^{\pm 1}; \beta']_P$. For all $i$, if $\delta_i\alpha_i = \alpha_i\delta_i + \eta_i\delta_i$ on $R_{i-1}$, then $\delta_i'\alpha_i' = \alpha_i'\delta_i' + \eta_i\delta_i'$ on $R_{i-1}'$.*

(3) *Suppose that each $\delta_i$ extends to a higher $(\eta_i, \alpha_i)$-skew Poisson derivation $(D_{i,k})_{k=0}^{\infty}$, and that $\beta D_{i,k} = D_{i,k}\beta + k\lambda_i D_{i,k}$ on $R_{i-1}$ for all $i$ and $k$. Then each $\delta_i'$ extends to a higher $(\eta_i, \alpha_i')$-skew Poisson derivation $(D_{i,k}')_{k=0}^{\infty}$ on $R_{i-1}'$, where $D_{i,k}'$ coincides with $D_{i,k}$ on $R_j$, for $j < i$, and $D_{i,k}'(Y) = 0$ for $k > 0$.*

(4) *Keeping the assumptions of* (3) above, we have:

(a) If $(D_{i,k})_{k=0}^{\infty}$ is iterative, then $(D'_{i,k})_{k=0}^{\infty}$ is iterative.

(b) If $(D_{i,k})_{k=0}^{\infty}$ is locally nilpotent, then $(D'_{i,k})_{k=0}^{\infty}$ is locally nilpotent.

*Proof.* We prove all the results together using induction on $n$, based on Lemma 3.2.1. When $n = 1$ the result is exactly Lemma 3.2.1. Suppose that the assertions (1), (2), (3) and (4) are true for the rank $n-1$. We apply Lemma 3.2.1 to the iterated Poisson-Ore extension $R = R_{n-1}[X_n; \alpha_n, \delta_n]_P[Y^{\pm 1}; \beta]_P$. By assertion (1) of Lemma 3.2.1 we obtain that:

$$R = R_{n-1}[Y^{\pm 1}; \beta^*]_P[X_n; \alpha_n^*, \delta_n^*]_P,$$

where $\beta^* = \beta|_{R_{n-1}}$, $\alpha_n^*|_{R_{n-1}} = \alpha_n$, $\delta_n^*|_{R_{n-1}} = \delta_n$, $\alpha_n^*(Y) = -\lambda_n Y$ and $\delta_n^*(Y) = 0$. Moreover by assertion (2) of Lemma 3.2.1 we have $\delta_n^* \alpha_n^* = \alpha_n^* \delta_n^* + \eta_n \delta_n^*$ on $R_{n-1}[Y^{\pm 1}; \beta^*]_P$. Assertion (3) of Lemma 3.2.1 shows that the derivation $\delta_n^*$ extends to a higher $(\eta_n, \alpha_n^*)$-skew Poisson derivation $(D_{n,k}^*)_{k=0}^{\infty}$ on $R_{n-1}[Y^{\pm 1}; \beta^*]_P$, where $D_{n,k}^*$ coincides with $D_{n,k}$ on $R_{n-1}$, and $D_{n,k}^*(Y) = 0$ for all $k > 0$. Finally $(D_{n,k}^*)$ is iterative (resp. locally nilpotent) if $(D_{n,k})$ is iterative (resp. locally nilpotent) by assertion (4) of Lemma 3.2.1. The lemma then follows from the induction hypothesis. $\square$

## 3.3 A positive answer to Quadratic Poisson Gel'fand-Kirillov problem

The theorem below gives conditions under which (a quotient of) a suitable iterated Poisson-Ore extension is Poisson birationally equivalent to (a quotient of) a Poisson affine space. Recall that a *Poisson prime ideal* $P$ of a Poisson algebra $A$ is a prime ideal which is also a Poisson ideal, i.e. such that $\{a, u\} \in P$ for all $a \in A$ and $u \in P$. Suppose that $\mathbb{K}$ is infinite. Let $B$ be a Poisson $\mathbb{K}$-algebra supporting a torus $H$-action by Poisson automorphisms. An ideal $I$ of $B$ is said *$H$-invariant* if $H(I) = I$. The torus action is said *rational* if the action is semisimple ($B$ is the direct sum of its eigenspaces) and the corresponding characters are all rational, see [6, Theorem II.2.7] and Section 3.4.1.

**Theorem 3.3.1.** *Let $A = \mathbb{K}[X_1][X_2; \alpha_2, \delta_2]_P \cdots [X_n; \alpha_n, \delta_n]_P$ be an iterated Poisson-Ore extension such that each derivation $\delta_i$ extends to an iterative, locally nilpotent higher*

*$(\eta_i, \alpha_i)$-skew Poisson derivation $(D_{i,k})_{k=0}^{\infty}$ with $\eta_i \in \mathbb{K}^\times$ on:*

$$A_{i-1} = \mathbb{K}[X_1][X_2; \alpha_2, \delta_2]_P \cdots [X_{i-1}; \alpha_{i-1}, \delta_{i-1}]_P.$$

*Suppose furthermore that for all $1 \leq j < i \leq n$ there exists $\lambda_{ij} \in \mathbb{K}$ such that $\alpha_i(X_j) = \lambda_{ij} X_j$, and $\alpha_i D_{j,k} = D_{j,k}\alpha_i + k\lambda_{ij} D_{j,k}$ for all $k \geq 0$. Let $\boldsymbol{\lambda} = (\lambda_{ij})$ be the skew-symmetric matrix in $M_n(\mathbb{K})$ whose coefficients below the diagonal are the above scalars. Then:*

(1) *There exists a Poisson algebra isomorphism between* $\operatorname{Frac} A$ *and* $\mathbb{K}_{\boldsymbol{\lambda}}(Y_1, \ldots, Y_n)$.

(2) *For any Poisson prime ideal $P$ in $A$, there exists a Poisson prime ideal $Q$ in the Poisson affine space $B = \mathbb{K}_{\boldsymbol{\lambda}}[Y_1, \ldots, Y_n]$ such that the fields* $\operatorname{Frac} A/P$ *and* $\operatorname{Frac} B/Q$ *are isomorphic as Poisson algebras.*

(3) *Assume that the torus $H = (\mathbb{K}^\times)^r$ is acting rationally by Poisson automorphisms on $A$ such that each $X_i$ is an $H$-eigenvector, and $B$ is endowed with the induced $H$-action (for all $h \in H$ and all $1 \leq i \leq n$ there exists $\mu_i \in \mathbb{K}^\times$ such that $h(X_i) = \mu_i X_i$; then the action of $h$ on the generator $Y_i$ of $B$ is given by $h(Y_i) = \mu_i Y_i$). Moreover we suppose that $h\big(D_{i,k}(X_j)\big) = \mu_i^k D_{i,k}\big(h(X_j)\big)$ for all $1 \leq j < i \leq n$ and $k \geq 0$. Then, for any $H$-invariant Poisson prime ideal $P$ in $A$, there exists an $H$-invariant Poisson prime ideal $Q$ in $B = \mathbb{K}_{\boldsymbol{\lambda}}[Y_1, \ldots, Y_n]$ such that the fields* $\operatorname{Frac} A/P$ *and* $\operatorname{Frac} B/Q$ *are isomorphic as Poisson algebras.*

*Proof.* We prove these results all together by three inductions: first on $n$, second on the number $d$ of indices $i$ for which $\delta_i \neq 0$ and finally on the maximum index $t$ for which $\delta_t \neq 0$ (this last induction being downward). If $d = 0$ then set $t := n + 1$.

If $n = 1$ or $t = n + 1$ the result is shown. Indeed if $n = 1$, $\operatorname{Frac}(\mathbb{K}[X]) = \mathbb{K}(X)$ and if $t = n + 1$, then $d = 0$ and $A = \mathbb{K}[X_1][X_2; \alpha_2]_P \cdots [X_n; \alpha_n]_P = \mathbb{K}_{\boldsymbol{\lambda}}[X_1, \ldots, X_n] \cong B$. So we can assume that $n \geq 2$ and $t \leq n$.

Let $P$ be a Poisson prime ideal in $A$. Assertion (1) is satisfied when $P = Q = 0$ in (2). Assertions (2) and (3) are shown simultaneously. The proof splits in three cases: first if $X_n \in P$, next if $X_n \notin P$ and $t = n$, and finally if $X_n \notin P$ and $t < n$; each case will be solved by a different induction. Note that, for all $1 \leq i \leq n$, the $H$-actions on $A$ and $B$ induce, by restriction, $H$-actions on the subalgebras $A_i$ and $B_i := \mathbb{K}_{\boldsymbol{\lambda}_i}[X_1, \ldots, X_i]$, where $\boldsymbol{\lambda}_i$ is the upper left $i \times i$ submatrix of $\boldsymbol{\lambda}$. When $P$ is an $H$-invariant ideal of $A$ we also consider the induced action of $H$ on $A/P$. These actions are all rational actions by Poisson automorphisms, such that the generators of the algebras considered are $H$-eigenvectors.

**First case:** $X_n \in P$. Consider the Poisson algebra homomorphism $\Phi : A_{n-1} \to A/P$ defined by $\Phi(X_i) = \overline{X_i}$ for all $i < n$. Since $\Phi$ is surjective, there exists a Poisson prime ideal $P' = \ker(\Phi)$ in $A_{n-1}$ such that $A/P \cong A_{n-1}/P'$. Moreover it is clear that $P'$ is $H$-invariant if $P$ is $H$-invariant since the diagram of Figure 3.1 is commutative for all $h \in H$. By the first induction (on $n$), there exists an ($H$-invariant if $P$ is $H$-invariant) Poisson

$$\begin{array}{ccc} A_{n-1} & \xrightarrow{\quad\Phi\quad} & A/P \\ h \downarrow & & \downarrow h \\ A_{n-1} & \xrightarrow[\quad\Phi\quad]{} & A/P \end{array}$$

Figure 3.1

prime ideal $Q'$ in the algebra $B_{n-1}$ such that $\operatorname{Frac} A_{n-1}/P' \cong \operatorname{Frac} B_{n-1}/Q'$. Observe that $Q = Q' + BY_n$ is an ($H$-invariant if $P$ is $H$-invariant) Poisson prime ideal in $B$ such that $B_{n-1}/Q' \cong B/Q$. Thus $\operatorname{Frac} A/P \cong \operatorname{Frac} B/Q$.

**Second case:** $X_n \notin P$ and $t = n$. So $\delta_n \neq 0$. Set $A' = A_{n-1}[Y; \alpha_n]_P$. Since $\delta_n$ extends to an iterative, locally nilpotent higher $(\eta_n, \alpha_n)$-skew Poisson derivation $(D_{n,k})_{k=0}^{\infty}$ on $A_{n-1}$, it follows from Proposition 2.2.2 that $A_{n-1}[X_n^{\pm 1}; \alpha_n, \delta_n]_P \cong A_{n-1}[Y^{\pm 1}; \alpha_n]_P$ and so $A[X_n^{-1}] \cong A'[Y^{-1}]$. Thus there exists a Poisson prime ideal $P' = P[X_n^{-1}] \cap A'$ in $A'$ such that $\operatorname{Frac} A/P \cong \operatorname{Frac} A'/P'$, where $P' = 0$ if $P = 0$. As in Section 2.3, the action of $H$ on $A_{n-1}$ extends to $A_{n-1}[Y^{\pm 1}; \alpha_n]_P$ by setting $h(Y) = \mu_n Y$ (where $\mu_n \in \mathbb{K}^\times$ is defined by $h(X_n) = \mu_n X_n$). Then, if the ideal $P$ is $H$-invariant, the ideal $P'$ is $H$-invariant since the Poisson isomorphism $A_{n-1}[X_n^{\pm 1}; \alpha_n, \delta_n]_P \cong A_{n-1}[Y^{\pm 1}; \alpha_n]_P$ commutes with all $h \in H$ (choose $\{X_1, \dots, X_{n-1}\}$ for a generating set of $A_{n-1}$ and apply Lemma 2.3.1 with $A_{n-1}$ as coefficient ring). Finally, the number of nonzero maps among $\delta_2, \dots, \delta_{n-1}$ is $d - 1$, so the induction step (on $d$) gives the result for $\operatorname{Frac} A'/P'$ and so for $\operatorname{Frac} A/P$.

**Third case:** $X_n \notin P$ and $t < n$. Thus $\delta_n = 0$. By Lemma 3.2.2 we can write $A[X_n^{-1}]$ in the form:

$$A[X_n^{-1}] = \mathbb{K}[X_1][X_n^{\pm 1}; \alpha'_n]_P[X_2; \alpha'_2, \delta'_2]_P \cdots [X_{n-1}; \alpha'_{n-1}, \delta'_{n-1}]_P,$$

where $\alpha_i'(X_j) = \lambda_{ij} X_j$ for $j < i$ and $j = n$, and each $\delta_i'$ extends to an iterative, locally nilpotent higher $(\eta_i, \alpha_i')$-skew Poisson derivation $(D'_{i,k})_{k=0}^{\infty}$ on:

$$A'_{i-1} := \mathbb{K}[X_1][X_n^{\pm 1}; \alpha_n']_P[X_2; \alpha_2', \delta_2']_P \cdots [X_{i-1}; \alpha'_{i-1}, \delta'_{i-1}]_P.$$

It is clear that we have $\alpha_i' D'_{j,k} = D'_{j,k}\alpha_i' + k\lambda_{ij} D'_{j,k}$ and $h\big(D'_{i,k}(X_j)\big) = \mu_i^k D'_{i,k}\big(h(X_j)\big)$ for all $1 \le j < i \le n$, all $k \ge 0$ and all $h \in H$, since by Lemma 3.2.2 we have:

$$D'_{i,k}(X_j) = \begin{cases} D_{i,k}(X_j) & j < i \text{ and } k \ge 0, \\ X_n & j = n \text{ and } k = 0, \\ 0 & j = n \text{ and } k \ge 1. \end{cases}$$

We can now use the induction hypothesis since the derivation $\delta_t'$ is nonzero ($\delta_t'$ restricts to $\delta_t$) and occurs in position $t+1$ in the list $0, 0, \delta_2', \ldots, \delta'_{n-1}$. And thus the induction on $t$ gives our conclusion. □

By Example 2.1.3, when $\operatorname{char} \mathbb{K} = 0$, the hypotheses of [18, Theorem 3.9] imply those of our Theorem 3.3.1 (except Assertion (3)). Hence Assertions (1) and (2) of our Theorem 3.3.1 generalise [18, Theorem 3.9] to any characteristic.

## 3.4 Quadratic Poisson Gel'fand-Kirillov problem for quotients by $H$-invariant Poisson prime ideals

Assertions (2) and (3) of Theorem 3.3.1 tell us that $H$-invariant Poisson prime factors of the iterated Poisson-Ore extensions under consideration are Poisson birationally isomorphic to $H$-invariant Poisson prime factors of Poisson affine spaces. In this section, we go one step further and prove that these factor algebras satisfy the quadratic Poisson Gel'fand-Kirillov problem under some mild assumptions on the torus action (Hypothesis 3.4.1) and the base field $\mathbb{K}$.

More precisely, set $[\![1, n]\!] := \{1, \ldots, n\}$ and $W := \mathscr{P}([\![1, n]\!])$, the set of subsets of $[\![1, n]\!]$. The key is to show that, under a suitable $H$-action, the only $H$-invariant Poisson prime ideals of a Poisson affine space $B = \mathbb{K}_{(\lambda_{ij})}[Y_1, \ldots, Y_n]$ are the ideals $J_w := \langle Y_i \mid i \in w \rangle$, where $w \in W$. This is achieved in Section 3.4.2. As a consequence the $H$-invariant

Poisson prime factors of $B$ are again Poisson affine spaces over $\mathbb{K}$, and therefore satisfy the quadratic Poisson Gel'fand-Kirillov problem. We conclude from Theorem 3.3.1 that $H$-invariant Poisson prime factors of the iterated Poisson-Ore extensions considered also satisfy the quadratic Poisson Gel'fand-Kirillov problem.

From now on, we require that the field $\mathbb{K}$ is infinite.

### 3.4.1 Assumptions on the $H$-action

In this section we recall some classical facts on rational torus action and present the hypotheses we need in the following section.

Let $r > 0$. Suppose that the torus $H = (\mathbb{K}^\times)^r$ is acting rationally by Poisson automorphisms on the iterated Poisson-Ore extension $A = \mathbb{K}[X_1][X_2; \alpha_2, \delta_2]_P \cdots [X_n; \alpha_n, \delta_n]_P$ such that each $X_i$ is an $H$-eigenvector, and suppose that there exist scalars $\lambda_{ij}$ for all $1 \leq j < i \leq n$ such that $\alpha_i(X_j) = \lambda_{ij} X_j$. The rational character group $X(H)$ of $H$ is identified with the group $\mathbb{Z}^r$ via the bijection:

$$\begin{array}{ccc} \mathbb{Z}^r & \longrightarrow & X(H) \\ \underline{x} = (x_1, \ldots, x_r) & \longmapsto & \Big( (h_1, \ldots, h_r) \longmapsto h_1^{x_1} \cdots h_r^{x_r} \Big). \end{array}$$

Since $H$ is a torus, the rationality of the action means that $A$ is the direct sum of its $H$-eigenspaces, and the corresponding eigenvalues are rational characters of $H$ (i.e. they are homomorphisms of algebraic varieties $(\mathbb{K}^\times)^r \to \mathbb{K}^\times$), see [6, Theorem II.2.7]. Fix $1 \leq i \leq n$. For all $h \in H$ we have $h(X_i) \in \mathbb{K}X_i$ since $X_i$ is an eigenvector. Thus we obtain a map $\underline{f}_i$ from $H$ to $\mathbb{K}^\times$ such that $h(X_i) = \underline{f}_i(h) X_i$. The map $\underline{f}_i$ is called the *character* or the *$H$-eigenvalue* associated to the eigenvector $X_i$. Since the $H$-action is rational the character $\underline{f}_i$ is rational, and $\underline{f}_i \in \mathbb{Z}^r$ under the correspondence previously described. For $\underline{\mu} = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r$ and $\underline{\nu} = (\nu_1, \ldots, \nu_r) \in \mathbb{Z}^r$, we set $(\underline{\mu} | \underline{\nu}) := \sum_{i=1}^r \mu_i \nu_i$.

In the following we restrict our attention on Poisson algebras satisfying Hypothesis 3.4.1. In Section 4 we will present many examples of such algebras.

**Hypothesis 3.4.1.** For all $1 \leq i \leq n$, there exists $\underline{\gamma}_i \in \mathbb{Z}^r$ such that:

- $\lambda_{ij} = (\underline{\gamma}_i | \underline{f}_j)$ for all $1 \leq j < i$;

- $\rho_i := (\underline{\gamma}_i | \underline{f}_i) \in \mathbb{K}^\times$.

Form the skew-symmetric matrix $\boldsymbol{\lambda} \in M_n(\mathbb{K})$ whose coefficients below the diagonal are the $\lambda_{ij}$ and, as in Assertion 3 of Theorem 3.3.1, endow $B = \mathbb{K}_{\boldsymbol{\lambda}}[Y_1, \ldots, Y_n]$ with the rational $H$-action by Poisson automorphisms induced by the $H$-action on $A$. Note that for all $1 \leq i \leq n$, the indeterminate $Y_i$ is an $H$-eigenvector with associated character $\underline{f}_i \in \mathbb{Z}^r$.

### 3.4.2 $H$-invariant ideals in Poisson affine spaces

For $w \in W$ we set $\overline{w} := [\![1, n]\!] \setminus w$. Assume $\overline{w} \neq \emptyset$. Recall that we set $J_w := \langle Y_i \mid i \in w \rangle$. We denote by $S_w$ the multiplicative set of $B/J_w$ generated by the $Y_i + J_w$ for $i \in \overline{w}$, and consider the algebra:

$$T = (B/J_w)S_w^{-1}.$$

We set $\overline{w} := \{l_1, \ldots, l_s\}$, where $1 \leq l_1 < \cdots < l_s \leq n$ and $s \in \{1, \ldots, n\}$. For all $i \in \{1, \ldots, s\}$, set $U_i := Y_{l_i} + J_w$ and for all $1 \leq j < i \leq n$, set $\lambda'_{ij} := \lambda_{l_i l_j}$. Then $T$ is the Poisson torus $T = \mathbb{K}_{(\lambda'_{ij})}[U_1^{\pm 1}, \ldots, U_s^{\pm 1}]$, where $(\lambda'_{ij})$ is the skew-symmetric matrix whose coefficients under the diagonal are the scalars $\lambda'_{ij}$ defined above.

Since the ideal $J_w$ and the multiplicative set $S_w$ are generated by $H$-eigenvectors, the torus $H$ is acting rationally by Poisson automorphisms on $T$ and for all $1 \leq i \leq s$ the indeterminate $U_i$ is an $H$-eigenvector with associated character $\underline{u}_i := \underline{f}_{l_i}$. Moreover for all $i \in \{1, \ldots, s\}$, we set $\underline{\gamma}'_i := \underline{\gamma}_{l_i}$ and $\rho'_i := \rho_{l_i}$. Thus we have $\lambda'_{ij} = (\underline{\gamma}'_i | \underline{u}_j)$ for all $1 \leq j < i \leq s$ and $\rho'_i = (\underline{\gamma}'_i | \underline{u}_i) \in \mathbb{K}^\times$ for all $1 \leq i \leq s$.

**Lemma 3.4.2.** *Let $(m_1, \ldots, m_s) \in \mathbb{Z}^s \setminus (0, \ldots, 0)$ and suppose that $U := U_1^{m_1} \cdots U_s^{m_s}$ is a Poisson central element in $T$. Then there exists $h \in H$ such that $h(U) = \varepsilon U$ with $\varepsilon \in \mathbb{K} \setminus \{0, 1\}$.*

*Proof.* We can assume that $m_s$ is nonzero. Otherwise replace $s$ by the largest $i$ such that $m_i \neq 0$ in the following. Start by noting that $U \in Z_p(T)$ implies that $0 = \{U, U_s\} = \big(\sum_{i<s} m_i \lambda'_{si}\big) U U_s$, i.e. $\sum_{i<s} m_i \lambda'_{si} = 0$.

Let $i < s$. Set $\underline{\gamma}'_s := (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r$. Thus we have $\lambda'_{si} = \sum_{j=1}^r \mu_j \nu_j$ with the notation $\underline{u}_i := (\nu_1, \ldots, \nu_r) \in \mathbb{Z}^r$. Let $q \in \mathbb{K}^\times$ and set $h_s := (q^{\mu_1}, \ldots, q^{\mu_r}) \in H$. Still identifying $X(H)$ with $\mathbb{Z}^r$, we have:

$$h_s(U_i) = \underline{u}_i(h_s)U_i = (q^{\mu_1})^{\nu_1} \cdots (q^{\mu_r})^{\nu_r} U_i = q^{\lambda'_{si}} U_i$$

for all $i < s$, and:

$$h_s(U_s) = \underline{u}_s(h_s)U_s = q^{(\underline{\gamma}'_s|\underline{u}_s)} U_s = q^{\rho'_s} U_s.$$

So $h_s(U) = q^{\sum_{i<s} m_i \lambda'_{si}} q^{\rho'_s m_s} U = q^{\rho'_s m_s} U$. By the assumptions on the ground field made at the beginning of Section 3.4, we can choose $q$ such that $q^{\rho'_s m_s} \neq 1$ (note that $\rho'_s m_s \neq 0$), and the result is shown. $\square$

The following proposition characterises the $H$-invariant Poisson prime ideals of $T$.

**Proposition 3.4.3.** *If $I$ is an $H$-invariant Poisson prime ideal of $T$, then $I = \langle 0 \rangle$.*

*Proof.* Suppose $I \neq \langle 0 \rangle$. By Lemma 1.3.2, there exists a nonzero Poisson central element $V \in I$. Write $V = \lambda_1 U^{m_1} + \cdots + \lambda_k U^{m_k}$ with $m_1, \ldots, m_k \in \mathbb{Z}^s$ pairwise distinct, $\lambda_1, \ldots, \lambda_k \in \mathbb{K}^\times$ and $k > 0$. Suppose that $V$ is chosen in such a way that $k$ is minimal. If $k = 1$, then $V$ is invertible and $I = T$, a contradiction, thus we suppose $k > 1$.

The monomials $U^{m_1}, \ldots, U^{m_k}$ are Poisson central, invertible and $U^{m_k}(U^{m_1})^{-1} = U^m$ with $m = m_k - m_1 \in \mathbb{Z}^s \setminus (0 \ldots, 0)$. Thus by Lemma 3.4.2 there exists $h \in H$ such that $h(U^{m_k}(U^{m_1})^{-1}) = \varepsilon U^{m_k}(U^{m_1})^{-1}$ with $\varepsilon \in \mathbb{K} \setminus \{0, 1\}$. Since $U_1, \ldots, U_s$ are $h$-eigenvectors, then so are $U^{m_1}, \ldots, U^{m_k}$ and we can write $h(U^{m_i}) = \nu_i U^{m_i}$ with $\nu_i \in \mathbb{K}^\times$ for all $1 \leq i \leq k$. Consider now the Poisson central element $W = V - \nu_1^{-1} h(V) \in I$. We have:

$$W = \sum_{i=1}^{k} \lambda_i (1 - \nu_i \nu_1^{-1}) U^{m_i} = \sum_{i=2}^{k} \lambda_i (1 - \nu_i \nu_1^{-1}) U^{m_i}.$$

Since $\varepsilon U^{m_k}(U^{m_1})^{-1} = h(U^{m_k}(U^{m_1})^{-1}) = \nu_k \nu_1^{-1} U^{m_k}(U^{m_1})^{-1}$ we have $\nu_k \nu_1^{-1} \neq 1$ and so $W \neq 0$. Thus $W$ is a nonzero Poisson central element of $I$ which can be written as a sum of at most $k-1$ monomials. This contradicts the choice of $k$. $\square$

Recall that the Poisson prime spectrum of $B$, denoted by $\text{P.Spec}\,(B)$, is the subset of Poisson ideals in $\text{Spec}\,(B)$. For all $w \in W$ we define a subset of $\text{P.Spec}\,(B)$ by setting:

$$\text{P.Spec}_w(B) := \Big\{ I \in \text{P.Spec}\,(B) \mid I \cap \{Y_1, \ldots, Y_n\} = \{Y_i \mid i \in w\} \Big\}.$$

These subsets form a partition of P.Spec$(B)$.

**Proposition 3.4.4.** *The only $H$-invariant Poisson prime ideals of $B$ are the ideals:*

$$J_w = \langle Y_i \mid i \in w \rangle$$

*for all $w \in W$.*

*Proof.* Let $I$ be an $H$-invariant Poisson prime ideal of $B$. There exists $w \in W$ such that $I \in \text{P.Spec}_w(B)$. If $w = \{1, \ldots, n\}$, then $J_w$ is a maximal ideal and thus $I = J_w$.

Suppose $w \neq \{1, \ldots, n\}$. Then $J_w \subset I$ and $I/J_w$ is a Poisson prime ideal of $B/J_w$ which does not intersect the multiplicative set $S_w$. Thus $P = (I/J_w)S_w^{-1}$ is a Poisson prime ideal of the Poisson torus $T = (B/J_w)S_w^{-1}$. Since $I$ is $H$-invariant and all elements of $S_w$ are $H$-eigenvectors, the ideal $P$ is $H$-invariant. Proposition 3.4.3 implies that $P = \langle 0 \rangle$ and so $I = J_w$, as desired. □

Combining Proposition 3.4.4 and Theorem 3.3.1 we obtain the main result of this section.

**Theorem 3.4.5.** *Let $A$ be an iterated Poisson-Ore extension satisfying all the hypotheses of Theorem 3.3.1. Assume that Hypothesis 3.4.1 is satisfied and that $\mathbb{K}$ is infinite. Then, for any $H$-invariant Poisson prime ideal $P$ of $A$, the field of fractions* Frac $A/P$ *is Poisson isomorphic to a Poisson affine field $\mathbb{K}_{\boldsymbol{\lambda}'}(Z_1, \ldots, Z_m)$, where $m \leq n$ and $\boldsymbol{\lambda}' \in M_m(\mathbb{K})$ is a skew-symmetric matrix.*

*Proof.* By Theorem 3.3.1 we have $\text{Frac}\, A/P \cong \text{Frac}\, B/Q$ where $B = \mathbb{K}_{\boldsymbol{\lambda}}[Y_1, \ldots, Y_n]$ and $Q$ is an $H$-invariant Poisson prime ideal of $B$. By Proposition 3.4.4 there exists $w \in W$ such that $Q = J_w$. Then $B/Q = \mathbb{K}_{\boldsymbol{\lambda}'}[\overline{Y_i} \mid i \notin w]$, where $\boldsymbol{\lambda}'$ is the skew-symmetric submatrix of $\boldsymbol{\lambda}$ obtained by deleting rows and columns indexed by $i \in w$. The result follows. □

Theorem 3.4.5 is new even in characteristic zero. In the following chapter, we prove a result that shows that the hypotheses of Theorem 3.4.5 are satisfied for large classes of polynomial Poisson algebras.

# Chapter 4

# Semiclassical limit and examples

In this chapter we give examples of Poisson $\mathbb{K}$-algebras satisfying the hypotheses of Theorem 3.4.5, so that they satisfy the quadratic Poisson Gel'fand-Kirillov problem described in Section 3.1. Most of our examples actually arise as semiclassical limits of quantum algebras described in [23, Section 5]. In order to prove a transfer result, one needs to address the existence of higher Poisson derivations on the Poisson algebras considered. Contrary to the characteristic zero case, higher derivations in prime characteristic seem not to be well understood. However, we can ensure their existence in arbitrary characteristic by the semiclassical limit process. This mainly relies on the fact that we can define a quantum analogue of a higher derivation independently of the characteristic of the base field, as long as the deformation parameter is transcendental over the base field (this is always the case in the setting of the semiclassical limit process). Our transfer result (Theorem 4.1.3) states, in particular, that this quantum analogue of a higher derivation induces a Poisson higher derivation on the semiclassical limit. More generally Theorem 4.1.3 gives conditions on a quantum algebra under which its semiclassical limit satisfies the quadratic Poisson Gel'fand-Kirillov problem. In Section 4.2 we illustrate our results with many examples including (coordinate rings of) matrix Poisson varieties or more generally (coordinate rings of) determinantal Poisson varieties.

We continue to assume that the ground field $\mathbb{K}$ is infinite.

## 4.1 Existence of higher Poisson derivation

We start this section by recalling the notion of $q$-integers and $q$-binomial coefficients, where $q$ is a nonzero non-root-of-unity element of $\mathbb{K}[t^{\pm 1}]$. Our conventions are as follows. For all $0 \leq k \leq i$ we set:

$$\begin{aligned} (i)_q &= q^{i-1} + q^{i-2} + \cdots + 1, \\ (i)!_q &= (i)_q (i-1)_q \cdots (1)_q, \\ \binom{i}{k}_q &= \frac{(i)!_q}{(i-k)!_q (k)!_q}. \end{aligned}$$

By convention $(0)!_q = 1$. In the following, we will use $q$-integers in the case where $q = t^\eta$ for $\eta \in \mathbb{Z}$.

The following proposition gives the existence of a higher $(\eta, \alpha)$-skew Poisson derivation on a Poisson-Ore extension which is the semiclassical limit of a suitable Ore extension. Let $\mathcal{A}$ be a $\mathbb{K}[t^{\pm 1}]$-algebra, $\sigma$ be a $\mathbb{K}[t^{\pm 1}]$-linear automorphism of $\mathcal{A}$ and $\Delta$ be a $\mathbb{K}[t^{\pm 1}]$-linear $\sigma$-derivation of $\mathcal{A}$. Recall that the multiplication in the Ore extension $\mathcal{R} := \mathcal{A}[x; \sigma, \Delta]$ is defined by:

$$xa = \sigma(a)x + \Delta(a)$$

for all $a \in \mathcal{A}$.

**Proposition 4.1.1.** *Let $\mathcal{A}$ be a torsion free $\mathbb{K}[t^{\pm 1}]$-algebra. Consider the Ore extension $\mathcal{R} = \mathcal{A}[x; \sigma, \Delta]$ and suppose that $R := \mathcal{R}/(t-1)\mathcal{R}$ is a commutative $\mathbb{K}$-algebra. Then:*

(1) *$R$ is a Poisson-Ore extension of the form $A[X; \alpha, \delta]_P$, where $A := \mathcal{A}/(t-1)\mathcal{A}$, $X := \overline{x}$, $\alpha \in \mathrm{Der}_P(A)$ and $\delta$ is a Poisson $\alpha$-derivation of $A$. More precisely, we have:*

$$\alpha := \frac{\sigma - \mathrm{id}}{t-1}\Big|_{t=1} \textit{ and } \delta := \frac{\Delta}{t-1}\Big|_{t=1},$$

*meaning that for all $a \in \mathcal{A}$ we have $\alpha(\overline{a}) = \frac{\sigma(a)-a}{t-1}|_{t=1}$ and $\delta(\overline{a}) = \frac{\Delta(a)}{t-1}|_{t=1}$.*

(2) *Suppose furthermore that $\Delta\sigma = t^\eta \sigma\Delta$ for some integer $\eta \in \mathbb{K}^\times$ and that:*

$$\Delta^i(\mathcal{A}) \subseteq (t-1)^i (i)!_{t^\eta} \mathcal{A}$$

*for all $i \geq 0$. Then $\delta$ extends to an iterative, higher $(\eta, \alpha)$-skew Poisson derivation*

*$(D_i)$ on $A$, which is locally nilpotent if $\Delta$ is locally nilpotent. More precisely, $D_i$ is defined by:*

$$D_i(\overline{a}) := \Big(\frac{\Delta^i(a)}{(t-1)^i(i)!_{t^\eta}}\Big)\Big|_{t=1}$$

*for all $a \in \mathcal{A}$.*

*Proof.* (1) First note that $(t-1)\mathcal{R} = (t-1)\mathcal{A}[x;\sigma,\Delta]$, where $(t-1)\mathcal{A}$ is a $(\sigma,\Delta)$-stable ideal of $\mathcal{A}$ (that is we have $\sigma\big((t-1)\mathcal{A}\big) = (t-1)\mathcal{A}$ and $\Delta\big((t-1)\mathcal{A}\big) \subseteq (t-1)\mathcal{A}$). So the corresponding quotient algebra is of the form:

$$R = \mathcal{R}/(t-1)\mathcal{R} = (\mathcal{A}/(t-1)\mathcal{A})\,[X] = A[X].$$

See for instance [6, Definition II.5.4]. We already know that $R$ is a Poisson algebra, so it just remains to prove that $R$ is a Poisson-Ore extension. Since $R$ is commutative, for all $a \in \mathcal{A}$ we have:

$$0 = \overline{xa} - \overline{ax} = \overline{(\sigma(a)-a)x + \Delta(a)} = \overline{(\sigma(a)-a)}X + \overline{\Delta(a)}.$$

So $(\sigma(a)-a) \in (t-1)\mathcal{A}$ and $\Delta(a) \in (t-1)\mathcal{A}$ for all $a \in \mathcal{A}$. The Poisson bracket between $\overline{a} \in A$ and $X$ is given by:

$$\{X,\overline{a}\} = \frac{\sigma(a)-a}{t-1}\Big|_{t=1} X + \frac{\Delta(a)}{t-1}\Big|_{t=1}.$$

We set

$$\alpha := \frac{\sigma - \mathrm{id}}{t-1}\Big|_{t=1} \text{ and } \delta := \frac{\Delta}{t-1}\Big|_{t=1}.$$

One can easily check that $\alpha$ and $\delta$ are well defined, that $\alpha \in \mathrm{Der}_P(A)$ and that $\delta$ is a Poisson $\alpha$-derivation on $A$. Thus:

$$\{X,\overline{a}\} = \alpha(\overline{a})X + \delta(\overline{a})$$

for all $a \in \mathcal{A}$, and the algebra $R$ is a Poisson-Ore extension of the form $A[X;\alpha,\delta]_P$.

(2) We claim that one defines an iterative, higher $(\eta,\alpha)$-skew Poisson derivation $(D_i)$ on $A$ by:

$$D_i(\overline{a}) := \Big(\frac{\Delta^i(a)}{(t-1)^i(i)!_{t^\eta}}\Big)\Big|_{t=1}$$

for all $i \geq 0$ and all $a \in \mathcal{A}$. First, since $\Delta^i(\mathcal{A}) \subseteq (t-1)^i(i)!_{t^\eta}\mathcal{A}$, it is straightforward to see that the map $D_i$ is well-defined for all $i \geq 0$. It remains to check that $(D_i)$ satisfies all the relevant axioms of Definition 2.1.1. Axiom (A1) follows from the fact that $\overline{\sigma(a)} = \overline{a}$ for all $a \in \mathcal{A}$. Set $d_i = \frac{\Delta^i}{(i)!_{t^\eta}}$ for all $i \geq 0$. Then (A3) follows easily from the identities:

$$d_i(\sigma - \mathrm{id}) = t^{i\eta}(\sigma - \mathrm{id})d_i + (t^{i\eta} - 1)d_i$$

for all $i \geq 0$. The higher derivation $(D_i)$ is iterative since $d_i d_j = \binom{i+j}{j}_{t^\eta} d_{i+j}$. Moreover, it is clear that $(D_i)$ is locally nilpotent if $\Delta$ is.

The verification of (A2) involves more computations, so the details are given here. Let $u, v \in \mathcal{A}$. Then one can easily check that:

$$d_n(uv) = \sum_{i=0}^{n} \sigma^{n-i} d_i(u) d_{n-i}(v),$$

so that for all $a, b \in \mathcal{A}$ we have:

$$d_n\Big(\frac{[a,b]}{t-1}\Big) = \frac{1}{(t-1)}\Big(\sum_{i=0}^{n-1} \sigma^{n-i} d_i(a) d_{n-i}(b) - \sum_{i=0}^{n-1} \sigma^{n-i} d_i(b) d_{n-i}(a) + d_n(a)b - d_n(b)a\Big).$$

Observe that for $i < n$:

$$\sigma^{n-i} d_i(a) d_{n-i}(b) = \sum_{j=1}^{n-i} \sigma^{n-i-j}(\sigma - \mathrm{id}) d_i(a) d_{n-i}(b) + d_i(a) d_{n-i}(b).$$

Thus:

$$\begin{aligned} d_n\Big(\frac{[a,b]}{t-1}\Big) &= \sum_{i=0}^{n} \frac{[d_i(a), d_{n-i}(b)]}{(t-1)} \\ &\quad + \frac{1}{(t-1)} \sum_{i=0}^{n-1} \Big( \sum_{j=1}^{n-i} \sigma^{n-i-j}(\sigma - \mathrm{id}) d_i(a) d_{n-i}(b) \\ &\qquad\qquad - \sum_{j=1}^{n-i} \sigma^{n-i-j}(\sigma - \mathrm{id}) d_i(b) d_{n-i}(a) \Big). \end{aligned}$$

Dividing by $(t-1)^n$, and then projecting onto $R$, we get:

$$D_n(\{\overline{a}, \overline{b}\}) = \sum_{i=0}^{n} \{D_i(\overline{a}), D_{n-i}(\overline{b})\} + \sum_{i=1}^{n} i\big(\alpha D_{n-i}(\overline{a}) D_i(\overline{b}) - \alpha D_{n-i}(\overline{b}) D_i(\overline{a})\big).$$

This proves (A2). $\square$

*Example* 4.1.2. Let $\mathcal{A} := \mathbb{K}[t^{\pm 1}, x]$ and let $\mathcal{R} := \mathcal{A}[y; \sigma, \Delta]$ be the Ore extension such that the automorphism $\sigma$ is defined by $\sigma(t) = t$ and $\sigma(x) = tx$, and such that the $\sigma$-derivation $\Delta$ is defined by $\Delta(t) = 0$ and $\Delta(x) = t - 1$. We obtain the following commutation rule in $\mathcal{R}$:

$$yx - txy = t - 1.$$

We have $R := \mathcal{R}/(t-1)\mathcal{R} \cong \mathbb{K}[X, Y]$, where $X := \overline{x}$ and $Y := \overline{y}$. Therefore we have:

$$R = \mathbb{K}[X][Y; \alpha, \delta]_P,$$

where $\alpha = X\partial_X$ and $\delta = \partial_X$. Indeed:

$$\alpha(X) = \Big(\frac{\sigma(x) - x}{t - 1}\Big)\Big|_{t=1} = X,$$

and:

$$\delta(X) = \Big(\frac{\Delta(x)}{t - 1}\Big)\Big|_{t=1} = 1.$$

Moreover we have:

$$\Delta^k(x) = \begin{cases} x & k = 0 \\ t - 1 & k = 1 \\ 0 & k > 1, \end{cases}$$

so that $\Delta^k(x) \in (t-1)^k (k)!_t \mathbb{K}[t^{\pm 1}, x]$ for all $k \geq 0$. We deduce that:

$$\Delta^k(\mathbb{K}[t^{\pm 1}, x]) \subseteq (t-1)^k (k)!_t \mathbb{K}[t^{\pm 1}, x] \quad \text{for all} \quad k \geq 0.$$

Since moreover $\Delta\sigma = t\sigma\Delta$, the assertion (2) of Proposition 4.1.1 shows that the derivation $\delta$ extends to an iterative, locally nilpotent higher $(1, \alpha)$-skew Poisson derivation on $\mathbb{K}[X]$. One may check that:

$$D_i(X) := \begin{cases} X & i = 0 \\ 1 & i = 1 \\ 0 & i > 1. \end{cases}$$

Note that the Poisson algebra obtained and this higher derivation are those presented in Example 2.1.4.

We can now state the main result of this section.

**Theorem 4.1.3.** *Let $\mathcal{R} = \mathbb{K}[t^{\pm 1}][x_1][x_2;\sigma_2,\Delta_2]\cdots[x_n;\sigma_n,\Delta_n]$ be an iterated Ore extension over $\mathbb{K}[t^{\pm 1}]$, and denote by $\mathcal{R}_j$ the subalgebra $\mathbb{K}[t^{\pm 1}][x_1][x_2;\sigma_2,\Delta_2]\cdots[x_j;\sigma_j,\Delta_j]$ for $1 \le j \le n$. We make the following assumptions:*

(H1) *The torus $H = (\mathbb{K}^\times)^r$ is acting rationally by $\mathbb{K}[t^{\pm 1}]$-algebra automorphisms on $\mathcal{R}$ such that for all $i \in \{1,\dots,n\}$:*

- *the indeterminate $x_i$ is an $H$-eigenvector with associated character $\underline{f}_i$;*

*and*

- *there exists $\underline{\gamma}_i \in \mathbb{Z}^r$ such that $\eta_i := -(\underline{\gamma}_i|\underline{f}_i) \in \mathbb{K}^\times$;*

(H2) *For all $2 \le i \le n$, we have $\Delta_i\sigma_i = t^{\eta_i}\sigma_i\Delta_i$;*

(H3) *For all $2 \le i \le n$ and $k \ge 0$, we have $\Delta_i^k(\mathcal{R}_{i-1}) \subseteq (t-1)^k(k)!_{t^{\eta_i}}\mathcal{R}_{i-1}$;*

(H4) *The automorphisms $\sigma_i$ satisfy $\sigma_i(x_j) = t^{\lambda_{ij}}x_j$ for $1 \le j < i \le n$, where $\lambda_{ij} := (\underline{\gamma}_i|\underline{f}_j)$.*

*Assume that $R := \mathcal{R}/(t-1)\mathcal{R}$ is commutative. Then, for any $H$-invariant Poisson prime ideal $P$ of $R$, the field* $\operatorname{Frac} R/P$ *is Poisson isomorphic to a Poisson affine field.*

*Proof.* We only need to check that $R$ satisfies all hypotheses of Theorem 3.4.5.

- First, we show that $R$ is an iterated Poisson-Ore extension of the form

$$R = \mathbb{K}[X_1][X_2;\alpha_2,\delta_2]_P\cdots[X_n;\alpha_n,\delta_n]_P,$$

where each $\delta_i$ extends to an iterative higher $(\eta_i,\alpha_i)$-skew Poisson derivation $(D_{i,k})_{k=0}^\infty$ on $R_{i-1} := \mathbb{K}[X_1][X_2;\alpha_2,\delta_2]_P\cdots[X_{i-1};\alpha_{i-1},\delta_{i-1}]_P$. This result is proved by induction on $n$ using Proposition 4.1.1. The case $n=1$ is trivial.

For $1 \le i \le n-1$, assume that $R_i = \mathbb{K}[X_1][X_2;\alpha_2,\delta_2]_P\cdots[X_i;\alpha_i,\delta_i]_P$. Then we have:

$$\begin{aligned} R_n = \frac{\mathcal{R}_n}{(t-1)\mathcal{R}_n} &= \frac{\mathcal{R}_{n-1}}{(t-1)\mathcal{R}_{n-1}}[X_n;\alpha_n,\delta_n]_P \\ &= \mathbb{K}[X_1][X_2;\alpha_2,\delta_2]_P\cdots[X_n;\alpha_n,\delta_n]_P, \end{aligned}$$

since $(t-1)\mathcal{R}_{n-1}$ is a $(\sigma_n, \Delta_n)$-stable ideal of $\mathcal{R}_{n-1}$. Note that

$$\alpha_n(X_j) = \frac{\sigma_n(x_j) - x_j}{t-1}\Big|_{t=1} = \frac{t^{\lambda_{nj}} - 1}{t-1} x_j\Big|_{t=1} = \lambda_{nj} X_j,$$

for all $1 \leq j \leq n$.

Hypotheses (H2) and (H3) ensure that Assertion 2 of Proposition 4.1.1 applies, so $\delta_n$ extends to an iterative higher $(\eta_n, \alpha_n)$-skew Poisson derivation $(D_{n,k})_{k=0}^{\infty}$ on $R_{n-1}$. It follows from Proposition 4.1.1 (and the induction hypothesis) that for $2 \leq j \leq n$ and $k \geq 0$ we have

$$D_{j,k} := \frac{\Delta_j^k}{(t-1)^k (k)!_{t^{\eta_j}}}\Big|_{t=1}.$$

• The next step is to show that for $2 \leq j < i \leq n$ and $k \geq 0$, we have the relations $\alpha_i D_{j,k} = D_{j,k}\alpha_i + k\lambda_{ij} D_{j,k}$.

First we show by induction (on $k$) the following identities:

$$\sigma_i \Delta_j^k = t^{k\lambda_{ij}} \Delta_j^k \sigma_i, \tag{4.1}$$

for $2 \leq j < i \leq n$. If $k = 1$ and $1 \leq l < j$, then we have

$$\sigma_i(x_j x_l) = \sigma_i\big(\sigma_j(x_l)x_j + \Delta_j(x_l)\big) = t^{\lambda_{ij}+\lambda_{il}+\lambda_{jl}} x_i x_j + \sigma_i \Delta_j(x_l),$$

and

$$\sigma_i(x_j)\sigma_i(x_l) = t^{\lambda_{ij}+\lambda_{il}}\big(\sigma_j(x_l) + \Delta_j(x_l)\big) = t^{\lambda_{ij}+\lambda_{il}+\lambda_{jl}} x_i x_j + t^{\lambda_{ij}} \Delta_j \sigma_i(x_l).$$

So $\sigma_i \Delta_j(x_l) = t^{\lambda_{ij}} \Delta_j \sigma_i(x_l)$ for all $1 \leq l < j < i \leq n$, as desired. Assume the result proved at rank $k$. Then we have

$$\sigma_i \Delta_j^{k+1} = (\sigma_i \Delta_j)\Delta_j^k = t^{\lambda_{ij}} \Delta_j \sigma_i \Delta_j^k = t^{(k+1)\lambda_{ij}} \Delta_j^{k+1} \sigma_i,$$

and (4.1) is proved.

Now it follows from (4.1) that:

$$(\sigma_i - \mathrm{id})\Delta_j^k = t^{k\lambda_{ij}} \Delta_j^k (\sigma_i - \mathrm{id}) + (t^{k\lambda_{ij}} - 1)\Delta_j^k.$$

Next, dividing both sides of this equation by $(t-1)^{k+1}(k)!_{t^{\eta_j}}$ , and then projecting on $R_{j-1}$, we obtain:

$$\alpha_i D_{j,k} = D_{j,k}\alpha_i + k\lambda_{ij} D_{j,k}.$$

• Then we show that the torus $H$ is acting rationally by Poisson automorphisms on $R$. Since $(t-1)\mathcal{R}$ is $H$-invariant, we can consider the induced action of $H$ on the quotient algebra $R$. This is a rational action by automorphisms. Moreover this action respects the Poisson bracket of $R$. Indeed for $f, g \in \mathcal{R}$, by setting $F = \overline{f}$ and $G = \overline{g}$, we have:

$$\begin{aligned} h(\{F,G\}) &= h\Big(\Big(\frac{[f,g]}{t-1}\Big)\Big|_{t=1}\Big) = \Big(h\Big(\frac{[f,g]}{t-1}\Big)\Big)\Big|_{t=1} \\ &= \Big(\frac{[h(f),h(g)]}{t-1}\Big)\Big|_{t=1} = \{h(F),h(G)\} \end{aligned}$$

for all $h \in H$.

• Fix $h \in H$ and set $h(x_j) = \mu_j x_j$, where $\mu_j \in \mathbb{K}^\times$ for all $1 \leq j \leq n$. We are now going to show that:

$$h\big(D_{i,k}(X_j)\big) = \mu_i^k D_{i,k}\big(h(X_j)\big)$$

for all $1 \leq j < i \leq n$ and all $k \geq 0$.

We start by observing that, for $k \geq 1$ and $1 \leq j < i \leq n$, we have:

$$x_i \Delta_i^{k-1}(x_j) = \sigma_i(\Delta_i^{k-1}(x_j))x_i + \Delta_i^k(x_j).$$

Thus:

$$\begin{aligned} \Delta_i^k(x_j) &= x_i\Delta_i^{k-1}(x_j) - \sigma_i(\Delta_i^{k-1}(x_j))x_i \\ &= x_i\Delta_i^{k-1}(x_j) - t^{\eta_i(1-k)+\lambda_{ij}}\Delta_i^{k-1}(x_j)x_i. \end{aligned}$$

Then it follows from an easy induction (on $k$) that for all $h \in H$ and $k \geq 0$ we have:

$$h(\Delta_i^k(x_j)) = \mu_j\mu_i^k\Delta_i^k(x_j). \tag{4.2}$$

Indeed, when $k = 1$, we have:

$$h(\Delta_i(x_j)) = h(x_ix_j - t^{\lambda_{ij}}x_jx_i) = \mu_i\mu_j\Delta_i(x_j).$$

Next, assuming the result proved at rank $(k-1)$ we get:

$$\begin{aligned} h(\Delta_i^k(x_j)) &= h(x_i\Delta_i^{k-1}(x_j) - t^{\eta_i(1-k)+\lambda_{ij}}\Delta_i^{k-1}(x_j)x_i) \\ &= \mu_i x_i \mu_j \mu_i^{k-1}\Delta_i^{k-1}(x_j) - t^{\eta_i(1-k)+\lambda_{ij}}\mu_j\mu_i^{k-1}\Delta_i^{k-1}(x_j)\mu_i x_i \\ &= \mu_j\mu_i^k\Delta_i^k(x_j), \end{aligned}$$

as desired. As $D_{j,k} := \left.\frac{\Delta_j^k}{(t-1)^k(k)!_{t^{\eta_j}}}\right|_{t=1}$, we deduce from (4.2) that:

$$h(D_{i,k}(X_j)) = \mu_i^k D_{i,k}\big(h(X_j)\big)$$

for all $k \geq 0$ and for all $1 \leq j < i \leq n$, as required.

- We conclude by noting that Hypothesis 3.4.1 is clearly satisfied with $\rho_i = -\eta_i = (\underline{\gamma}_i|\underline{f}_i)$ for all $1 \leq i \leq n$ since $X_i$ is an $H$-eigenvector with associated character $\underline{f}_i$ for all $1 \leq i \leq n$.

Hence all hypotheses of Theorem 3.4.5 are satisfied and so for any $H$-invariant Poisson prime ideal $P$ of $R$, the field $\mathrm{Frac}\, R/P$ is Poisson isomorphic to a Poisson affine field. □

When dealing with examples, the following lemma allows us to check Hypothesis (H3) of Theorem 4.1.3 only on the generators of the algebra under consideration.

**Lemma 4.1.4.** *Let $\mathcal{A}$ be a finitely generated $\mathbb{K}[t^{\pm 1}]$-algebra and form the Ore extension $\mathcal{R} = \mathcal{A}[x;\sigma,\Delta]$ with $\Delta\sigma = t^\eta\sigma\Delta$ for an integer $\eta \in \mathbb{K}^\times$. Let $\{a_1,\dots,a_n\}$ be a set of generators of $\mathcal{A}$. If the conditions $\Delta^i(a_k) \in (t-1)^i(i)!_{t^\eta}\mathcal{A}$ are satisfied for all $k \in \{1,\dots,n\}$ and $i \geq 0$, then:*

$$\Delta^i(\mathcal{A}) \subseteq (t-1)^i(i)!_{t^\eta}\mathcal{A}.$$

*Proof.* The result follows from an easy induction using the generalised quantum Leibniz formula:

$$\Delta^i(ab) = \sum_{k=0}^{i}\binom{i}{k}_{t^\eta}\sigma^{i-k}\Delta^k(a)\Delta^{i-k}(b)$$

for $a, b \in \mathcal{A}$. □

*Example* 4.1.5. We continue with the notation of Example 4.1.2. It is straightforward to see that the torus $H = \mathbb{K}^\times$ acts rationally on $\mathcal{R} = \mathbb{K}[t^{\pm 1}, x][y;\sigma,\Delta]$ by $\mathbb{K}[t^{\pm 1}]$-automorphisms via:

$$h(x) = hx \quad \text{and} \quad h(y) = h^{-1}y \quad \text{for all} \quad h \in H.$$

Thus $x$ is an $H$-eigenvector with associated character $f_x := 1$ and $y$ is an $H$-eigenvector with associated character $f_y := -1$. Moreover, for $\gamma := 1 \in \mathbb{Z}$ we have $1 = (\gamma \mid f_x)$ and $(\gamma \mid f_y) \in \mathbb{K}^\times$. Hence Hypothesis (H1) of Theorem 4.1.3 is satisfied. Hypotheses (H2), (H3) and (H4) are then easy to check with the computation of Example 4.1.2. Therefore Theorem 4.1.3 can be applied. For any $H$-invariant Poisson prime ideal $P$ of the Poisson-Ore extension:

$$R = \mathcal{R}/(t-1)\mathcal{R} = \mathbb{K}[X][Y; \alpha, \delta]_P,$$

the field $\mathrm{Frac}\,(R/P)$ is isomorphic to a Poisson affine field.

## 4.2 Examples

In this section we present several families of Poisson algebras satisfying the hypotheses of Theorem 4.1.3. Many iterated Ore extensions are described in [23, Section 5], and it is shown that lots of them actually satisfy the hypotheses of Theorem 4.1.3. As a consequence, their semiclassical limits and their quotients by $H$-invariant Poisson prime ideals satisfy the quadratic Poisson Gel'fand-Kirillov problem. This includes (but is not limited to) the semiclassical limits of:

- single parameter coordinate rings of odd-dimensional quantum Euclidean spaces;
- single parameter coordinate rings of quantum matrices;
- single parameter coordinate rings of even-dimensional quantum Euclidean spaces;
- single parameter coordinate rings of quantum symplectic spaces.

In this section we provide a detailed study of these examples. In particular, in the case of the coordinate rings of quantum matrices, we exhibit a family of $H$-invariant Poisson prime ideals: the so-called *determinantal ideals*.

### 4.2.1 Semiclassical limit of the coordinate ring of $m \times p$ quantum matrices

The single parameter coordinate ring of quantum matrices $\mathcal{A} := \mathcal{O}_t\big(M_{m,p}(\mathbb{K}[t^{\pm 1}])\big)$ is the $\mathbb{K}[t^{\pm 1}]$-algebra given by $mp$ generators $x_{11}, x_{12}, \ldots, x_{mp}$ and relations:

$$x_{ij}x_{kl} = \begin{cases} t^{-1}x_{kl}x_{ij} & i > k,\ j = l \\ t^{-1}x_{kl}x_{ij} & i = k,\ j > l \\ x_{kl}x_{ij} & i > k,\ j < l \\ x_{kl}x_{ij} - (t - t^{-1})x_{kj}x_{il} & i > k,\ j > l. \end{cases}$$

This algebra can be presented as an iterated Ore extension over $\mathbb{K}[t^{\pm 1}]$:

$$\mathcal{O}_t\big(M_{m,p}(\mathbb{K}[t^{\pm 1}])\big) = \mathbb{K}[t^{\pm 1}][x_{11}][x_{12}; \sigma_{12}, \Delta_{12}] \cdots [x_{mp}; \sigma_{mp}, \Delta_{mp}],$$

where the indeterminates are ordered using the lexicographic order, where $\sigma_{ij}$ is the $\mathbb{K}[t^{\pm 1}]$-automorphism of the appropriate subalgebra of $\mathcal{O}_t\big(M_{m,p}(\mathbb{K}[t^{\pm 1}])\big)$ defined by:

$$\sigma_{ij}(x_{kl}) = \begin{cases} t^{-1}x_{kl} & \text{if } i > k \text{ and } j = l \\ t^{-1}x_{kl} & \text{if } i = k \text{ and } j > l \\ x_{kl} & \text{if } i > k \text{ and } j \neq l, \end{cases}$$

for all $(k, l) <_{\text{lex}} (i, j)$, and where $\Delta_{ij}$ is the $\mathbb{K}[t^{\pm 1}]$-linear $\sigma_{ij}$-derivation such that:

$$\Delta_{ij}(x_{kl}) = \begin{cases} -(t - t^{-1})x_{kj}x_{il} & \text{if } i > k \text{ and } j > l \\ 0 & \text{otherwise} \end{cases}$$

for all $(k, l) <_{\text{lex}} (i, j)$.

Observe that the torus $H = (\mathbb{K}^{\times})^{m+p}$ acts rationally on $\mathcal{A}$ by automorphisms via:

$$h(t) = t \quad \text{and} \quad h(x_{ij}) = h_i h_{m+j} x_{ij}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq p$. So $x_{ij}$ is an $H$-eigenvector with associated character:

$$\underline{f}_{ij} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{m+p},$$

where the 1s occur in $i$-th and $(m + j)$-th positions. For $1 \leq i \leq m$ and $1 \leq j \leq p$, we define:

$$\underline{\gamma}_{ij} := (1, \ldots, 1, 0, -1, \ldots, -1, -2, -1, \ldots, -1) \in \mathbb{Z}^{m+p},$$

where the 0 occurs in $i$-th position and the $(-2)$ in $(m+j)$-th position. We have $(\underline{\gamma}_{ij}|\underline{f}_{ij}) = -2$ for all $1 \leq i \leq m$ and $1 \leq j \leq p$. To summarise, if $\operatorname{char}(\mathbb{K}) \neq 2$, Hypothesis (H1) of Theorem 4.1.3 is satisfied. For $(k,l) <_{\text{lex}} (i,j)$ we have:

$$(\underline{\gamma}_{ij}|\underline{f}_{kl}) = \begin{cases} -1 & \text{if } i > k \text{ and } j = l \\ -1 & \text{if } i = k \text{ and } j > l \\ 0 & \text{if } i > k \text{ and } j \neq l. \end{cases}$$

Note that for all $(k,l) <_{\text{lex}} (i,j)$ we have $\sigma_{ij}(x_{kl}) = t^{(\underline{\gamma}_{ij}|\underline{f}_{kl})} x_{kl}$. Thus Hypothesis (H4) of Theorem 4.1.3 is satisfied.

One can easily check that $\Delta_{ij}\sigma_{ij} = t^2\sigma_{ij}\Delta_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq p$. Thus, Hypothesis (H2) of Theorem 4.1.3 is satisfied. Let $\mathcal{A}_{ij}$ be the subalgebra of $\mathcal{A}$ generated over $\mathbb{K}[t^{\pm 1}]$ by $x_{11}, x_{12}, \ldots, x_{i,j-1}$. Note that $\Delta_{ij}^n(x_{kl}) = 0$ for all $n \geq 2$ and:

$$\Delta_{ij}(x_{kl}) = \begin{cases} -(t-1)(t^{-1}+1)x_{kj}x_{il} & \text{if } i > k \text{ and } j > l \\ 0 & \text{otherwise.} \end{cases}$$

So we have $\Delta_{ij}^n(x_{kl}) \in (t-1)^n (n)!_{t^2} \mathcal{A}_{ij}$ for all $(k,l) <_{\text{lex}} (i,j)$ and all $n \geq 0$, and Hypothesis (H3) of Theorem 4.1.3 is satisfied thanks to Lemma 4.1.4. So, if $\operatorname{char}\mathbb{K} \neq 2$, then we can apply Theorem 4.1.3 to $\mathcal{A}$.

Let $A = \mathcal{O}\big(M_{m,p}(\mathbb{K})\big) = \mathcal{A}/(t-1)\mathcal{A} = \mathbb{K}[X_{11}, \ldots, X_{mp}]$ be the semiclassical limit of $\mathcal{A}$, where $X_{ij} = x_{ij} + (t-1)\mathcal{A}$. For $(k,l) <_{\text{lex}} (i,j)$, the Poisson bracket on $A$ is given by:

$$\{X_{ij}, X_{kl}\} = \begin{cases} -X_{ij}X_{kl} & \text{if } i > k \text{ and } j = l \\ -X_{ij}X_{kl} & \text{if } i = k \text{ and } j > l \\ 0 & \text{if } i > k \text{ and } j < m \\ -2X_{kj}X_{il} & \text{if } i > k \text{ and } j > l. \end{cases}$$

We deduce from the above discussion the following result.

**Theorem 4.2.1.** *Assume that* $\operatorname{char}\mathbb{K} \neq 2$. *Let $P$ be an $H$-invariant Poisson prime ideal of $A = \mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$. The field of fractions of $A/P$ is Poisson isomorphic to a Poisson* *affine field* $\mathbb{K}_{\boldsymbol{\mu}}(Y_1, \ldots, Y_u)$*, where* $u \leq mp$ *and* $\boldsymbol{\mu} \in M_u(\mathbb{K})$ *is a skew-symmetric matrix.*

Note that when $\operatorname{char}\mathbb{K} = 2$, our methods do not apply to $A$. However in this case $A$ is already a Poisson affine space and the quadratic Poisson Gel'fand-Kirillov problem is trivial.

### 4.2.2 Quotients by Determinantal ideals

Assume that $\operatorname{char}\mathbb{K} \neq 2$. Determinantal ideals are ideals of $A = \mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$ generated by minors of a given size. More precisely set $n := \min(m,p)$ and let $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, p\}$ with $|I| = |J| \leq n$. We denote by $[I|J]$ the determinant:

$$[I|J] := \det\Big((X_{ij})_{(i,j)\in I\times J}\Big).$$

Such a determinant is called a *minor of size* $|I|$. For all $k \in \{0, \ldots, n-1\}$, the *determinantal ideal* $\mathcal{P}_k$ is the ideal generated by all $(k+1)\times(k+1)$ minors of $A$. Note that $\mathcal{P}_k$ contains all minors of size bigger than $k+1$ by Laplace Expansion.

Fix $0 \leq k \leq n-1$. We claim that the Poisson field $\operatorname{Frac}(A/\mathcal{P}_k)$ is Poisson isomorphic to a Poisson affine field. For this, we just need to show that $\mathcal{P}_k$ is an $H$-invariant Poisson prime ideal by Theorem 4.2.1. First, it is well known that $\mathcal{P}_k$ is a prime ideal, see for instance [7, Theorem 6.3]. Moreover, $\mathcal{P}_k$ is clearly $H$-invariant, so to apply Theorem 4.2.1 to $A/\mathcal{P}_k$, it only remains to prove that $\mathcal{P}_k$ is a Poisson ideal. It is probably well known, but we have not been able to find the statement in the literature. The following lemma (re-)establishes this result.

**Lemma 4.2.2.** *For all* $0 \leq k \leq n-1$*, the ideal* $\mathcal{P}_k$ *is a Poisson ideal of* $A$*.*

*Proof.* Note that any minor of $A$ is the coset of a so-called *quantum minor* of $\mathcal{A}$. See [19, Introduction] for more details about quantum minors. In [19, Lemma **5.1**] the authors give (in the square case) commutation relations between quantum minors and generators of $\mathcal{A}$ which easily lead (by semiclassical limit) to the following Poisson brackets between minors and generators of $A$ in the square case. Set $N = \max(m,p)$. We consider the Poisson algebra $\mathcal{B} := \mathcal{O}_t\big(M_N(\mathbb{K})\big)$. Let $r, c \in \{1, \ldots, N\}$ and $I, J \subseteq \{1, \ldots, N\}$ with $|I| = |J| \geq 1$.

For $1 \leq i < j \leq N$, we define $[i,j] := \{i, i+1, \ldots, j\}$. The semiclassical limit process gives us the following Poisson brackets in $\mathcal{B}/(t-1)\mathcal{B} = \mathcal{O}\big(M_N(\mathbb{K})\big)$.

- If $r \in I$ and $c \in J$, then $\big\{X_{rc}, [I|J]\big\} = 0$.
- If $r \in I$ and $c \notin J$, then:

$$\big\{X_{rc}, [I|J]\big\} = -[I|J]X_{rc} - 2 \sum_{j \in J, j > c} (-1)^{-|J \cap [c,j]|} [I|J \cup \{c\} \setminus \{j\}] X_{rj}.$$

- If $r \notin I$ and $c \in J$, then:

$$\big\{X_{rc}, [I|J]\big\} = [I|J]X_{rc} + 2 \sum_{i \in I, i < r} (-1)^{-|I \cap [i,r]|} [I \cup \{r\} \setminus \{i\}|J] X_{ic}.$$

- If $r \notin I$ and $c \notin J$, then:

$$\begin{aligned}\big\{X_{rc}, [I|J]\big\} = 2 &\sum_{i \in I, i < r} (-1)^{-|I \cap [i,r]|} [I \cup \{r\} \setminus \{i\}|J] X_{ic} \\ - 2 &\sum_{j \in J, j > c} (-1)^{-|J \cap [c,j]|} [I|J \cup \{c\} \setminus \{j\}] X_{rj}.\end{aligned}$$

Since we have $\mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$ is a Poisson subalgebra of $\mathcal{O}\big(M_N(\mathbb{K})\big)$, the above formulae show in particular that $\big\{X_{rc}, [I|J]\big\} \in \mathcal{P}_k$ for all $[I|J] \in \mathcal{P}_k$ all $1 \leq r \leq m$, all $1 \leq c \leq p$ and all $0 \leq k \leq n-1$, i.e. $\mathcal{P}_k$ is a Poisson ideal of $\mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$ for all $0 \leq k \leq n-1$. $\square$

We are ready to conclude by the following result.

**Theorem 4.2.3.** *Let $0 \leq k \leq n-1$. The field of fractions $\operatorname{Frac} A/\mathcal{P}_k$ is Poisson isomorphic to a Poisson affine field $\mathbb{K}_{\boldsymbol{\mu}}(Y_1, \ldots, Y_u)$, where $u \leq mp$ and $\boldsymbol{\mu} \in M_u(\mathbb{K})$ is a skew-symmetric matrix.*

### 4.2.3 Semiclassical limits of the coordinate rings of quantum odd dimensional Euclidean spaces

Assume that $\operatorname{char} \mathbb{K} \neq 2$. Let $t$ be an indeterminate. We denote by $t^{1/2}$ a fixed square root of $t$ inside an algebraic closure of $\mathbb{K}(t)$. The coordinate ring of quantum odd-dimensional

Euclidean space, is the $\mathbb{K}[t^{\pm 1/2}]$-algebra $\mathcal{R}$ given by generators $w, y_1, \ldots, y_n, x_1, \ldots, x_n$ and relations:

$$\begin{aligned}
wy_i &= ty_iw && \text{for all } i,\\
wx_i &= t^{-1}x_iw && \text{for all } i,\\
y_iy_j &= t^{-1}y_jy_i && i > j,\\
x_ix_j &= tx_jx_i && i > j,\\
y_ix_j &= t^{-1}x_jy_i && i > j,\\
x_iy_j &= ty_jx_i && i \neq j,\\
x_iy_j &= y_ix_i + (t^{1/2} - t^{3/2})w^2 + \sum_{l<i}(1-t^2)y_lx_l && \text{for all } i.
\end{aligned}$$

$\mathcal{R}$ can be presented as an iterated Ore extension as follows:

$$\mathcal{R} = \mathbb{K}[t^{\pm 1/2}][w][y_1; \sigma_1][x_1; \tau_1, \Delta_1] \cdots [y_n, \sigma_n][x_n; \tau_n, \Delta_n],$$

where for all $1 \leq j < i \leq n$:

$$\begin{aligned}
\tau_i(y_j) &= ty_j, & \sigma_i(y_j) &= t^{-1}y_j,\\
\tau_i(x_j) &= tx_j, & \sigma_i(x_j) &= t^{-1}x_j,\\
\tau_i(y_i) &= y_i, & \sigma_i(w) &= t^{-1}w,\\
\tau_i(w) &= tw,
\end{aligned} \tag{4.3}$$

$$\Delta_i(y_j) = \Delta_i(x_j) = \Delta_i(w) = 0, \quad \text{and} \quad \Delta_i(y_i) = (t^{1/2} - t^{3/2})w^2 + \sum_{l<i}(1-t^2)y_lx_l.$$

We now check that $\mathcal{R}$ satisfies the assumptions of Theorem 4.1.3. The torus $H = (\mathbb{K}^\times)^{n+1}$ acts rationally by $\mathbb{K}[t^{\pm 1/2}]$-automorphisms on $\mathcal{R}$ by setting for all $h \in H$ and all $1 \leq i \leq n$:

$$\begin{aligned}
h(w) &= h_{n+1}w,\\
h(x_i) &= h_ix_i,\\
h(y_i) &= h_{n+1}^2h_i^{-1}y_i.
\end{aligned}$$

For all $1 \leq i \leq n$, the indeterminate $x_i$ is an eigenvector with associated character:

$$\underline{f_i} := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{n+1},$$

where the 1 is in $i$-th position. Similarly for all $1 \leq i \leq n$ the indeterminate $y_i$ is an eigenvector with associated character:

$$\underline{g_i} := (0, \ldots, 0, -1, 0, \ldots, 0, 2) \in \mathbb{Z}^{n+1},$$

where the $-1$ is in $i$-th position. Finally $w$ is an eigenvector with associated character:

$$\underline{e} := (0, \ldots, 0, 1) \in \mathbb{Z}^{n+1}.$$

For $1 \leq i \leq n$ set:

$$\underline{\gamma_i} := (1, \ldots, 1, 2, 0, \ldots, 0, 1) \in \mathbb{Z}^{n+1} \quad \text{and} \quad \underline{\gamma} := (-1, \ldots, -1) \in \mathbb{Z}^{n+1},$$

where the 2 is in $i$-th position. For $1 \leq j < i \leq n$ we have:

$$(\underline{\gamma_i} \mid \underline{e}) = (\underline{\gamma_i} \mid \underline{f_j}) = (\underline{\gamma_i} \mid \underline{g_j}) = 1, \tag{4.4}$$

and for all $1 \leq i \leq n$:

$$\begin{aligned} &(\underline{\gamma} \mid \underline{e}) = (\underline{\gamma} \mid \underline{f_i}) = (\underline{\gamma} \mid \underline{g_i}) = -1, \\ &(\underline{\gamma_i} \mid \underline{g_i}) = 0, \qquad (\underline{\gamma_i} \mid \underline{f_i}) = 2. \end{aligned} \tag{4.5}$$

Therefore Hypothesis (H1) is satisfied. It is easy to check that $\Delta_i \tau_i = t^{-2} \tau_i \Delta_i$ for all $1 \leq i \leq n$, and Hypothesis (H2) is then satisfied. For all $1 \leq i \leq n$ we have:

$$\Delta_i(y_i) = (t-1)\big( - t^{1/2}(t^2+t+1)w^2 - \sum_{l<i}(1+t)y_l x_l\big) \quad \text{and} \quad \Delta_i^2(y_i) = 0,$$

and by Lemma 4.1.4 we obtain Hypothesis (H3). Hypothesis (H4) follows from (4.3), (4.4) and (4.5). By the semiclassical limit process we obtain the Poisson algebra:

$$R := \mathcal{R}/(t-1)\mathcal{R} = \mathbb{K}[W, Y_1, \ldots, Y_n, X_1, \ldots, X_n],$$

with Poisson bracket given by:

$$\begin{aligned}
\{W, Y_i\} &= WY_i && \text{for all } i,\\
\{W, X_i\} &= -WX_i && \text{for all } i,\\
\{Y_i, Y_j\} &= -Y_iY_j && i > j,\\
\{X_i, X_j\} &= X_iX_j && i > j,\\
\{Y_i, X_j\} &= -Y_iX_j && i > j,\\
\{X_i, Y_j\} &= X_iY_j && i \neq j,\\
\{X_i, Y_j\} &= -3W^2 - 2\sum_{l<i} X_lY_l && \text{for all } i.
\end{aligned}$$

By Theorem 4.1.3 the Poisson algebra $R$ and all its $H$-invariant Poisson prime quotients satisfy the quadratic Poisson Gel'fand-Kirillov problem.

### 4.2.4 Semiclassical limits of coordinate rings of even-dimensional quantum Euclidean spaces

Assume that $\operatorname{char}\mathbb{K} \neq 2$. Let $\mathcal{R}$ be the $\mathbb{K}[t^{\pm 1}]$-algebra given by $2n$ generators $x_i, y_i$ for $1 \leq i \leq n$ and relations:

$$\begin{aligned}
y_iy_j &= t^{-1}y_jy_i && i < j,\\
x_ix_j &= tx_jx_i && i < j,\\
x_iy_j &= t^{-1}y_jx_i && i \neq j,\\
x_iy_i &= y_ix_i + \sum_{l<i}(1-t^{-2})y_lx_l && \text{for all } i.
\end{aligned}$$

$\mathcal{R}$ is an iterated Ore extension as follows:

$$\mathcal{R} = \mathbb{K}[t^{\pm 1}][y_1][x_1;\tau_1][y_2,\sigma_2][x_2;\tau_2,\Delta_2]\cdots[y_n,\sigma_n][x_n;\tau_n,\Delta_n],$$

where for all $1 \leq j < i \leq n$:

$$\begin{aligned}
\tau_i(y_j) &= t^{-1}y_j, & \sigma_i(y_j) &= ty_j,\\
\tau_i(x_j) &= t^{-1}x_j, & \sigma_i(x_j) &= tx_j,\\
\tau_i(y_i) &= y_i,
\end{aligned} \tag{4.6}$$

$$\Delta_i(y_j) = \Delta_i(x_j) = 0, \quad \text{and} \quad \Delta_i(y_i) = \sum_{l<i} (1 - t^{-2}) y_l x_l.$$

We now check that $\mathcal{R}$ satisfies the assumptions of Theorem 4.1.3. The torus $H = (\mathbb{K}^\times)^{n+1}$ acts rationally on $\mathcal{R}$ by $\mathbb{K}[t^{\pm 1}]$-automorphism by setting for all $1 \leq i \leq n$ and all $h \in H$:

$$h(x_i) = h_i x_i, \quad \text{and} \quad h(y_i) = h_1 h_{n+1} h_i^{-1} y_i.$$

For $1 \leq i \leq n$, the indeterminate $x_i$ is an eigenvector with associated character:

$$\underline{f_i} := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^{n+1},$$

where the 1 is in $i$-th position. For $1 < i \leq n$, the indeterminate $y_i$ is an eigenvector with associated character:

$$\underline{g_i} := (1, 0, \dots, 0, -1, 0, \dots, 0, 1) \in \mathbb{Z}^{n+1},$$

where the $-1$ is in $i$-th position, and $y_1$ is an eigenvector with associated character:

$$\underline{g_1} := (0, \dots, 0, 1) \in \mathbb{Z}^{n+1}.$$

For $1 < i \leq n$ set:

$$\begin{aligned}
\underline{\gamma_i} &:= (-1, \dots, -1, -2, 0, \dots, 0, -1) \in \mathbb{Z}^{n+1}, \\
\underline{\gamma_1} &:= (-2, 0, \dots, 0) \in \mathbb{Z}^{n+1}, \\
\underline{\gamma} &:= (1, \dots, 1) \in \mathbb{Z}^{n+1},
\end{aligned}$$

where the $-2$ is in $i$-th position in $\underline{\gamma_i}$. For all $1 \leq i \leq n$, the element $\underline{\gamma_i}$ corresponds to the indeterminate $x_i$, and the element $\underline{\gamma}$ corresponds to the indeterminate $y_i$. For $1 \leq j < i \leq n$ we have:

$$\begin{aligned}
(\gamma_i \mid f_j) &= -1, & (\gamma_i \mid g_i) &= 0, \\
(\gamma_i \mid f_i) &= -2, & (\gamma \mid f_j) &= 1, \\
(\gamma_i \mid g_j) &= -1, & (\gamma \mid g_j) &= 1, \\
(\gamma_1 \mid f_1) &= -2, & (\gamma_1 \mid g_1) &= 0.
\end{aligned} \tag{4.7}$$

Therefore (H1) is satisfied since $(\gamma_i \mid f_i) = -2$ and $(\gamma \mid g_i) = 1$ for all $1 \leq i \leq n$. It is easy to check that $\Delta_i \tau_i = t^2 \tau_i \Delta_i$ for all $1 \leq i \leq n$, and (H2) is then satisfied (note that since the $\sigma_i$-derivation associated to $y_i$ is zero, the condition (H2) for $\sigma_i$ is trivially satisfied). For all $1 \leq i \leq n$ we have:

$$\Delta_i(y_i) = (t-1) \sum_{l<i} t^{-2}(t+1) y_l x_l \quad \text{and} \quad \Delta_i^2(y_i) = 0,$$

and by Lemma 4.1.4 we obtain Hypothesis (H3). Hypothesis (H4) follows from (4.6) and (4.7) since for all $1 \leq j < i \leq n$ we have:

$$\begin{aligned} \tau_i(y_j) &= t^{(\gamma_i \mid g_j)} y_j, & \sigma_i(y_j) &= t^{(\gamma \mid g_j)} y_j, \\ \tau_i(x_j) &= t^{(\gamma_i \mid f_j)} x_j, & \sigma_i(x_j) &= t^{(\gamma \mid f_j)} x_j, \\ \tau_i(y_i) &= t^{(\gamma_i \mid g_i)} y_i, & \tau_1(y_1) &= t^{(\gamma_1 \mid g_1)} y_1. \end{aligned}$$

By the semiclassical limit process we obtain the Poisson algebra:

$$R := \mathcal{R}/(t-1)\mathcal{R} = \mathbb{K}[Y_1, \ldots, Y_n, X_1, \ldots, X_n],$$

with Poisson bracket given by:

$$\begin{aligned} \{Y_i, Y_j\} &= -Y_j Y_i & & i < j, \\ \{X_i, X_j\} &= X_j X_i & & i < j, \\ \{X_i, Y_j\} &= -X_i Y_j & & i \neq j, \\ \{X_i, Y_i\} &= 2 \sum_{l<i} X_l Y_l & & \text{for all } i. \end{aligned}$$

By Theorem 4.1.3 the Poisson algebra $R$ and all its $H$-invariant Poisson prime quotients satisfy the quadratic Poisson Gel'fand-Kirillov problem.

### 4.2.5 Semiclassical limits of coordinate rings of quantum symplectic spaces

Assume that $\operatorname{char}\mathbb{K} \neq 2$. Let $\mathcal{R}$ be the $\mathbb{K}[t^{\pm 1}]$-algebra given by $2n$ generators $x_i, y_i$ for $1 \leq i \leq n$ and relations:

$$\begin{aligned}
y_i y_j &= t y_j y_i && i < j,\\
x_i x_j &= t^{-1} x_j x_i && i < j,\\
x_i y_j &= t^{-1} y_j x_i && i \neq j,\\
x_i y_i &= t^{-2} y_i x_i + \sum_{l<i} (t^{-2} - 1) y_l x_l && \text{for all } i.
\end{aligned}$$

$\mathcal{R}$ is an iterated Ore extension as follows:

$$\mathcal{R} = \mathbb{K}[t^{\pm 1}][y_1][x_1; \tau_1][y_2, \sigma_2][x_2; \tau_2, \Delta_2] \cdots [y_n, \sigma_n][x_n; \tau_n, \Delta_n],$$

where for all $1 \leq j < i \leq n$:

$$\begin{aligned}
\tau_i(y_j) &= t^{-1} y_j, & \sigma_i(y_j) &= t^{-1} y_j,\\
\tau_i(x_j) &= t x_j, & \sigma_i(x_j) &= t x_j,\\
\tau_i(y_i) &= t^{-2} y_i,
\end{aligned} \tag{4.8}$$

$$\Delta_i(y_j) = \Delta_i(x_j) = 0, \quad \text{and} \quad \Delta_i(y_i) = \sum_{l<i} (t^{-2} - 1) y_l x_l.$$

We now check that $\mathcal{R}$ satisfies the assumptions of Theorem 4.1.3. The torus $H = (\mathbb{K}^\times)^{n+1}$ acts rationally on $\mathcal{R}$ by $\mathbb{K}[t^{\pm 1}]$-automorphisms by setting for all $1 \leq i \leq n$ and all $h \in H$:

$$h(x_i) = h_i x_i, \quad \text{and} \quad h(y_i) = h_1 h_{n+1} h_i^{-1} y_i.$$

For $1 \leq i \leq n$, the indeterminate $x_i$ is an eigenvector with associated character:

$$\underline{f_i} := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{n+1},$$

where the 1 is in $i$-th position. For $1 < i \leq n$, the indeterminate $y_i$ is an eigenvector with associated character:

$$\underline{g_i} := (1, 0, \ldots, 0, -1, 0, \ldots, 0, 1) \in \mathbb{Z}^{n+1},$$

where the $-1$ is in $i$-th position, and $y_1$ is an eigenvector with associated character:

$$\underline{g_1} := (0, \ldots, 0, 1) \in \mathbb{Z}^{n+1}.$$

For $1 \leq i \leq n$ set:

$$\underline{\gamma_i} := (1, \ldots, 1, 2, 0, \ldots, 0, -1) \in \mathbb{Z}^{n+1} \quad \text{and} \quad \underline{\gamma} := (1, \ldots, 1, -1) \in \mathbb{Z}^{n+1},$$

where the 2 is in $i$-th position. For all $1 \leq i \leq n$, the element $\underline{\gamma_i}$ corresponds to the indeterminate $x_i$, and the element $\underline{\gamma}$ corresponds to the indeterminate $y_i$. For $1 \leq j < i \leq n$ we have:

$$\begin{aligned} (\gamma_i \mid f_j) &= 1, & (\gamma_i \mid g_i) &= -2, \\ (\gamma_i \mid f_i) &= 2, & (\gamma \mid f_j) &= 1, \\ (\gamma_i \mid g_j) &= -1, & (\gamma \mid g_j) &= -1. \end{aligned} \tag{4.9}$$

Therefore (H1) is satisfied. It is easy to check that $\Delta_i \tau_i = t^{-2} \tau_i \Delta_i$ for all $1 \leq i \leq n$, and (H2) is then satisfied. For all $1 \leq i \leq n$ we have:

$$\Delta_i(y_i) = (1-t) \sum_{l<i} t^{-2}(1+t) y_l x_l \quad \text{and} \quad \Delta_i^2(y_i) = 0,$$

and by Lemma 4.1.4 we obtain Hypothesis (H3). Hypothesis (H4) follows from (4.8) and (4.9). By the semiclassical limit process we obtain the Poisson algebra:

$$R := \mathcal{R}/(t-1)\mathcal{R} = \mathbb{K}[Y_1, \ldots, Y_n, X_1, \ldots, X_n],$$

with Poisson bracket given by:

$$\begin{aligned} \{Y_i, Y_j\} &= Y_j Y_i & & i < j, \\ \{X_i, X_j\} &= -X_j X_i & & i < j, \\ \{X_i, Y_j\} &= -X_i Y_j & & i \neq j, \\ \{X_i, Y_i\} &= -2X_i Y_i - 2 \sum_{l<i} X_l Y_l & & \text{for all } i. \end{aligned}$$

By Theorem 4.1.3 the Poisson algebra $R$ and all its $H$-invariant Poisson prime quotients satisfy the quadratic Poisson Gel'fand-Kirillov problem.

### 4.2.6 An example in dimension 5

Let $\mathcal{R}$ be the $\mathbb{K}[t^{\pm 1}]$-algebra given by generators $x_1, x_2, x_3, x_4, x_5$ and relations:

$$\begin{aligned}
x_2x_1 &= t^{-1}x_1x_2, & x_4x_2 &= x_2x_4 + (1-t)x_3,\\
x_3x_1 &= x_1x_3 + (1-t)x_2, & x_5x_2 &= tx_2x_5 + t(1-t),\\
x_4x_1 &= tx_1x_4 + t(1-t), & x_4x_3 &= t^{-1}x_3x_4,\\
x_5x_1 &= tx_1x_5, & x_5x_3 &= x_3x_5 + (1-t)x_4,\\
x_3x_2 &= t^{-1}x_2x_3 & x_5x_4 &= t^{-1}x_4x_5.
\end{aligned}$$

$\mathcal{R}$ can be expressed as an iterated Ore extension as follows:

$$\mathcal{R} = \mathbb{K}[t^{\pm 1}][x_1][x_2;\sigma_2,\Delta_2]\cdots[x_5;\sigma_5,\Delta_5],$$

where:

$$\begin{aligned}
\sigma_i(x_{i-1}) &= t^{-1}x_{i-1}, & \Delta_i(x_{i-1}) &= 0,\\
\sigma_i(x_{i-2}) &= x_{i-2}, & \Delta_i(x_{i-2}) &= (1-t)x_{i-1},\\
\sigma_i(x_{i-3}) &= tx_{i-3}, & \Delta_i(x_{i-3}) &= t(1-t),\\
\sigma_i(x_{i-4}) &= tx_{i-4}, & \Delta_i(x_{i-4}) &= 0,
\end{aligned} \tag{4.10}$$

with the convention that $\sigma_i$ and $\Delta_i$ are defined on $x_{i-j}$ only when $1 \leq j < i \leq 5$. We now check that $\mathcal{R}$ satisfies the assumptions of Theorem 4.1.3. The torus $H = (\mathbb{K}^\times)^2$ acts rationally on $\mathcal{R}$ by $\mathbb{K}[t^{\pm 1}]$-automorphisms by setting for all $h = (h_1, h_2) \in H$:

$$h(x_i) := \begin{cases} h_ix_i & i = 1, 2\\ h_2h_1^{-1}x_3 & i = 3\\ h_{i-3}^{-1}x_i & i = 4, 5. \end{cases}$$

For $1 \leq i \leq 5$, we denote by $\underline{f_i}$ the character associated to the eigenvector $x_i$. We have:

$$\underline{f_1} := (1,0), \quad \underline{f_2} := (0,1), \quad \underline{f_3} := (-1,1), \quad \underline{f_4} := (-1,0), \quad \text{and} \quad \underline{f_5} := (0,-1).$$

One can check that Hypothesis (H1) is satisfied with the elements of $\mathbb{Z}^2$ defined as follows:

$$\underline{\gamma_2} := (-1,0), \quad \underline{\gamma_3} := (0,-1), \quad \underline{\gamma_4} := (1,0), \quad \text{and} \quad \underline{\gamma_5} := (1,1).$$

It is easy to see that $\Delta_i \sigma_i = t\sigma_i \Delta_i$ for $i = 2,3,4,5$, so that Hypothesis (H2) is satisfied. From (4.10) and Lemma 4.1.4 we obtain Hypothesis (H3). Finally easy computation leads to Hypothesis (H4). By the semiclassical limit we obtain the Poisson algebra:

$$R := \mathcal{R}/(t-1)\mathcal{R} = \mathbb{K}[X_1, \ldots, X_5],$$

with Poisson bracket given by:

$$\begin{aligned}
\{X_1, X_2\} &= X_1X_2, & \{X_2, X_4\} &= X_3,\\
\{X_1, X_3\} &= X_2, & \{X_2, X_5\} &= 1 - X_2X_5,\\
\{X_1, X_4\} &= 1 - X_1X_4, & \{X_3, X_4\} &= X_3X_4,\\
\{X_1, X_5\} &= -X_1X_5, & \{X_3, X_5\} &= X_4,\\
\{X_2, X_3\} &= X_2X_3, & \{X_4, X_5\} &= X_4X_5,
\end{aligned}$$

By Theorem 4.1.3 the Poisson algebra $R$ and all its $H$-invariant Poisson prime quotients satisfy the quadratic Poisson Gel'fand-Kirillov problem. In particular:

$$\operatorname{Frac} R \cong \mathbb{K}_{\boldsymbol{\lambda}}(Y_1, \ldots, Y_5),$$

where:

$$\boldsymbol{\lambda} := \begin{pmatrix} 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

# Chapter 5

# Poisson deleting derivations algorithm and the canonical embedding

For the rest of this thesis we turn our attention to the study of the Poisson spectra of certain polynomial Poisson algebras. In this chapter we introduce the *Poisson deleting derivations algorithm* which consists of applying several times the Poisson deleting derivation homomorphism to a (certain) iterated Poisson-Ore extension, keeping track of the generators at each steps.

In Section 5.1 we introduce the class $\mathcal{P}$ of iterated Poisson-Ore extensions we will study. The Poisson deleting derivations algorithm is then defined in Section 5.2. For a Poisson algebra $A \in \mathcal{P}$ the algorithm returns generators for a Poisson affine space $\overline{A}$. Recall that the Poisson spectrum of a Poisson algebra is the set of prime ideals that are also Poisson ideals. In Section 5.4 we define and study an embedding, the so-called *canonical embedding*, from the Poisson spectrum of $A$ to the Poisson spectrum of the Poisson affine space $\overline{A}$. Poisson spectra of Poisson affine spaces are well understood, and this knowledge together with the canonical embedding will help us to understand the Poisson spectrum of the Poisson algebra $A$. In particular we study a partition of the Poisson spectrum of $A$ indexed by (some) subsets of $\{1, \dots, n\}$ for some integer $n$. Moreover, the canonical embedding behaves nicely with this partition. Indeed, if $S_w$ is a part of the partition

($w \subseteq \{1, \dots, n\}$), then the canonical embedding induces an homeomorphism from $S_w$ to its image.

In Section 5.5 we add the hypothesis that a torus $H = (\mathbb{K}^\times)^r$ acts rationally by Poisson automorphisms on the Poisson algebra $A \in \mathcal{P}$. Firstly we study the compatibility of the Poisson deleting derivations algorithm with the torus action. Then we recall that a torus action leads to another partition of the Poisson spectrum as defined in [15] (when the characteristic is zero). We show that in fact these two partitions coincide in Section 5.5.3.

## 5.1 A class of iterated Poisson-Ore extensions

In this section, we introduce the class of Poisson algebras that we will study in the remaining chapters of this thesis.

**Hypothesis 5.1.1.**

(1) $A = \mathbb{K}[X_1][X_2; \alpha_2, \delta_2]_P \cdots [X_n; \alpha_n, \delta_n]_P$ is an iterated Poisson-Ore extension over $\mathbb{K}$. We set $A_i := \mathbb{K}[X_1][X_2; \alpha_2, \delta_2]_P \cdots [X_i; \alpha_i, \delta_i]_P$ for all $1 \leq i \leq n$.

(2) Suppose that for all $1 \leq j < i \leq n$ there exists $\lambda_{ij} \in \mathbb{K}$ such that $\alpha_i(X_j) = \lambda_{ij} X_j$. We set $\lambda_{ji} := -\lambda_{ij}$ for all $1 \leq j < i \leq n$.

(3) For all $2 \leq i \leq n$, assume that the derivation $\delta_i$ extends to an iterative, locally nilpotent higher $(\eta_i, \alpha_i)$-skew Poisson derivation $(D_{i,k})_{k=0}^\infty$ on $A_{i-1}$, where $\eta_i \in \mathbb{K}^\times$.

(4) Assume that $\alpha_i D_{j,k} = D_{j,k}\alpha_i + k\lambda_{ij} D_{j,k}$ for all $2 \leq j < i \leq n$ and all $k \geq 0$.

*Notation.* We denote by $\mathcal{P}$ the class of iterated Poisson-Ore extensions which satisfy Hypothesis 5.1.1.

*Remark* 5.1.2.

- If $A = \mathbb{K}[X_1][X_2; \alpha_2, \delta_2]_P \cdots [X_n; \alpha_n, \delta_n]_P \in \mathcal{P}$, then the intermediate Poisson algebras $A_i$ also belong to $\mathcal{P}$.
- When $\operatorname{char} \mathbb{K} = 0$, Hypothesis 5.1.1 can be simplified. Suppose that for all $2 \leq i \leq n$ the derivation $\delta_i$ is locally nilpotent and that we have $[\delta_i, \alpha_i] = \eta_i \delta_i$ with $\eta_i \in \mathbb{K}^\times$. Then we claim that hypotheses (3) and (4) of Hypothesis 5.1.1 are satisfied. Indeed recall that by Example 2.1.3, for all $2 \leq i \leq n$ the derivation $\delta_i$ uniquely extends to

an iterative, locally nilpotent higher $(\eta_i, \alpha_i)$-skew Poisson derivation on $A_{i-1}$, namely the sequence of $\mathbb{K}$-linear maps:

$$(D_{i,k})_k := \left(\frac{\delta_i^k}{k!}\right)_k.$$

Hence hypothesis (3) is satisfied. Moreover one can deduce hypothesis (4) as follows. For all $1 \leq l < j < i \leq n$ we have:

$$\begin{aligned}
\alpha_i\big(\{X_j, X_l\}\big) &= \alpha_i\big(\alpha_j(X_l)X_j + \delta_j(X_l)\big) \\
&= \lambda_{jl}(\lambda_{il} + \lambda_{ij})X_lX_j + \alpha_i\delta_j(X_l), \quad \text{and:} \\
\alpha_i\big(\{X_j, X_l\}\big) &= \{\alpha_i(X_j), X_l\} + \{X_j, \alpha_i(X_l)\} \\
&= (\lambda_{ij} + \lambda_{il})(\alpha_j(X_l)X_j + \delta_j(X_l)) \\
&= \lambda_{jl}(\lambda_{il} + \lambda_{ij})X_lX_j + \lambda_{ij}\delta_j(X_l) + \delta_j(\alpha_i(X_l)).
\end{aligned}$$

Thus we obtain $\alpha_i\delta_j = \delta_j\alpha_i + \lambda_{ij}\delta_j$ for all $1 < j < i \leq n$. By induction using the iterativity of $\delta_j$ we obtain for all $k \geq 0$ and all $1 < j < i \leq n$ that:

$$\alpha_i\delta_j^k = \delta_j^k\alpha_i + k\lambda_{ij}\delta_j^k. \tag{5.1}$$

By dividing by $k!$ both side of (5.1) we obtain hypothesis (4) since $D_{j,k} = \frac{\delta_j^k}{k!}$. To summarise: in $\operatorname{char}\mathbb{K} = 0$ hypotheses (3) and (4) reduce to hypothesis (3'):

(3') Assume that for all $2 \leq i \leq n$ the derivation $\delta_i$ is locally nilpotent and that $[\delta_i, \alpha_i] = \eta_i\delta_i$ for some $\eta_i \in \mathbb{K}^\times$.

- In characteristic zero we can characterise Poisson algebras of the class $\mathcal{P}$ for $n = 2$. Let $A = \mathbb{K}[X][Y, \alpha, \delta]_P \in \mathcal{P}$. We have $\alpha(X) = \lambda X$ for some $\lambda \in \mathbb{K}$, and one can check that $\delta$ is locally nilpotent if and only if $\delta(X) = \mu \in \mathbb{K}$. If $\mu = 0$ then $A$ is the Poisson affine plane $A = \mathbb{K}_\lambda[Y, X]$. If $\mu \neq 0$ by setting $X' = \mu^{-1}X$ we obtain $\{X', Y\} = \lambda X'Y + 1$. Moreover the equality $[\delta, \alpha] = \eta\delta$ for some nonzero scalar $\eta$ implies that $\lambda = \eta$ is nonzero. Therefore either $A$ is a Poisson plane (possibly the abelian polynomial Poisson algebra in two variables), or $A$ is isomorphic to the Poisson-Ore extension $\mathbb{K}[U][V; \lambda U\partial_U, \partial_U]_P$ for some $\lambda \in \mathbb{K}^\times$ by an isomorphism sending $X$ to $\mu U$ and $Y$ to $V$.

In the next sections we will need to use inductive arguments to define and study the Poisson deleting derivations algorithm. In the induction step we will need to re-arrange the order of the indeterminates of an iterated Poisson-Ore extension in $\mathcal{P}$. The following lemma will ensure that the new Poisson algebra is still in $\mathcal{P}$, so that one can apply the deleting derivation homomorphism to this new algebra, and thus proceed with the induction.

The restriction of a linear map $f$ to a subspace $V$ of its domain will be denoted by $f|_V$.

**Lemma 5.1.3.** *Let $A \in \mathcal{P}$ with $\delta_{j+1} = \cdots = \delta_n = 0$. With the notation of Hypothesis 5.1.1, we have the following.*

(1) *We can write $A = A_{j-1}[X_{j+1}; \beta_{j+1}]_P \cdots [X_n; \beta_n]_P[X_j; \alpha'_j, \delta'_j]_P$ where:*

- $\beta_i|_{A_{j-1}} = \alpha_i|_{A_{j-1}}$ *for all* $j < i \leq n$ *and* $\beta_i(X_l) = \lambda_{il}X_l$ *for all* $j < l < i$,
- $\alpha'_j|_{A_{j-1}} = \alpha_j$ *and* $\alpha'_j(X_l) = \lambda_{jl}X_l$ *for all* $j < l \leq n$,
- $\delta'_j|_{A_{j-1}} = \delta_j$ *and* $\delta'_j(X_l) = 0$ *for all* $j < l \leq n$.

(2) *$\delta'_j$ extends to an iterative, locally nilpotent higher $(\eta_j, \alpha'_j)$-skew Poisson derivation $(D'_{j,k})_{k=0}^{\infty}$ on $A_{j-1}[X_{j+1}; \beta_{j+1}]_P \cdots [X_n; \beta_n]_P$ such that the restriction of $D'_{j,k}$ to $A_{j-1}$ coincides with $D_{j,k}$ for all $k \geq 0$, and $D'_{j,k}(X_l) = 0$ for all $k > 0$ and all $j < l \leq n$.*

(3) *$A = A_{j-1}[X_{j+1}; \beta_{j+1}]_P \cdots [X_n; \beta_n]_P[X_j; \alpha'_j, \delta'_j]_P$ also belongs to $\mathcal{P}$.*

*Proof.* (1) Since $\{X_l, X_j\} = \lambda_{lj}X_lX_j$ for all $j < l \leq n$, the order of the variables $X_j, \dots, X_n$ can be changed. The resulting Poisson ($\alpha_i$-)derivations are those described above.
(2) This is an easy induction using Lemma 3.2.1.
(3) This follows directly from (1) and (2). □

## 5.2 Poisson deleting derivations algorithm

Let $A = \mathbb{K}[X_1][X_2; \alpha_2, \delta_2]_P \cdots [X_n; \alpha_n, \delta_n]_P \in \mathcal{P}$. We continue using the notation of Hypothesis 5.1.1.

We are now ready to describe the Poisson deleting derivations algorithm. For $j$ running from $n+1$ to 2 we define, by a decreasing induction, a sequence $(X_{1,j}, \dots, X_{n,j})$ of (algebraically independent) elements of $\operatorname{Frac} A$ and we set $C_j := \mathbb{K}[X_{1,j}, \dots, X_{n,j}]$. For $j = n+1$ we set $(X_{1,j}, \dots, X_{n,j}) := (X_1, \dots, X_n)$ so that $C_{n+1} = A$. Fix $2 \leq j \leq n$. Suppose that the sequence $(U_1, \dots, U_n) := (X_{1,j+1}, \dots, X_{n,j+1})$ is defined and that the algebra $C_{j+1}$ satisfies the following hypothesis:

**Hypothesis 5.2.1.**

(1) *$C_{j+1}$ is isomorphic to an iterated Poisson-Ore extension of the form:*

$$\mathbb{K}[X_1] \cdots [X_j; \alpha_j, \delta_j]_P [X_{j+1}; \beta_{j+1}]_P \cdots [X_n; \beta_n]_P$$

*by a Poisson isomorphism sending $U_i$ to $X_i$ for $1 \leq i \leq n$.*

(2) *For all $l \in \{j+1, \dots, n\}$, the map $\beta_l$ is a Poisson derivation such that $\beta_l(X_i) = \lambda_{li} X_i$ for all $1 \leq i < l$ and we have $\beta_l D_{i,k} = D_{i,k}\beta_l + k\lambda_{li} D_{i,k}$ for all $1 < i \leq j$ and all $k \geq 0$.*

Note that (1) of Hypothesis 5.2.1 allows us to express $C_{j+1}$ as the iterated Poisson-Ore extension:

$$\mathbb{K}[U_1] \cdots [U_j; \alpha_j, \delta_j]_P [U_{j+1}; \beta_{j+1}]_P \cdots [U_n; \beta_n]_P,$$

where by abuse of notation we denote again by $\alpha_i, \beta_i$ and $\delta_i$ the maps induced by the isomorphism of (1) of Hypothesis 5.2.1. In particular, for all $1 < i \leq j$, the derivation $\delta_i$ extends to an iterative, locally nilpotent higher $(\eta_i, \alpha_i)$-skew Poisson derivation $(D_{i,k})_{k=0}^{\infty}$ on the Poisson subalgebra $\mathbb{K}[U_1, \dots, U_{i-1}]$. The sequence $(V_1, \dots, V_n) := (X_{1,j}, \dots, X_{n,j})$ is then defined as follows:

$$V_i = \begin{cases} U_i & i \geq j, \\ \sum\limits_{k \geq 0} \frac{1}{\eta_j^k} D_{j,k}(U_i) U_j^{-k} & i < j. \end{cases}$$

**Proposition 5.2.2.** *Under the assumptions made since the beginning of Section 5.2, we have:*

(1) *The algebra $C_j$ is isomorphic to an iterated Poisson-Ore extension of the form:*

$$\mathbb{K}[X_1] \cdots [X_{j-1}; \alpha_{j-1}, \delta_{j-1}]_P [X_j; \beta_j]_P \cdots [X_n; \beta_n]_P$$

*by a Poisson isomorphism sending $V_i$ to $X_i$ for $1 \leq i \leq n$.*

(2) *For all* $j \leq l \leq n$*, the map* $\beta_l$ *is a Poisson derivation such that* $\beta_l(X_i) = \lambda_{li} X_i$ *for all* $1 \leq i < l$ *and we have* $\beta_l D_{i,k} = D_{i,k}\beta_l + k\lambda_{li} D_{i,k}$ *for all* $1 < i < j$ *and all* $k \geq 0$*.*

(3) *Set* $S_j = \{U_j^n \mid n \geq 0\} = \{V_j^n \mid n \geq 0\}$*. We have* $C_j S_j^{-1} = C_{j+1} S_j^{-1}$*.*

*Proof.* By Hypothesis 5.2.1 and Lemma 5.1.3 we can write:

$$C_{j+1} = \mathbb{K}[U_1]\cdots[U_{j-1};\alpha_{j-1},\delta_{j-1}]_P[U_{j+1};\beta'_{j+1}]_P\cdots[U_n;\beta'_n]_P[U_j;\alpha'_j,\delta'_j]_P,$$

where $\beta'_l$ for all $j < l \leq n$ and $\alpha'_j$ and $\delta'_j$ are defined as in assertion (1) of Lemma 5.1.3. In particular $\delta'_j$ extends to an iterative, locally nilpotent higher $(\eta_j, \alpha'_j)$-skew Poisson derivation $(D'_{j,k})_{k=0}^{\infty}$ on the Poisson algebra:

$$\widehat{C_{j+1}} := \mathbb{K}[U_1]\cdots[U_{j-1};\alpha_{j-1},\delta_{j-1}]_P[U_{j+1};\beta'_{j+1}]_P\cdots[U_n;\beta'_n]_P.$$

Therefore by applying Theorem 2.2.2 to the Poisson algebra $\widehat{C_{j+1}}[U_j;\alpha'_j,\delta'_j]_P$ we get a Poisson algebra isomorphism:

$$\begin{aligned} \theta : \widehat{C_{j+1}}[U_j^{\pm 1};\alpha'_j]_P &\xrightarrow{\cong} \widehat{C_{j+1}}[U_j^{\pm 1};\alpha'_j,\delta'_j]_P \\ \widehat{C_{j+1}} \ni a &\longmapsto \theta(a), \\ U_j &\longmapsto U_j. \end{aligned}$$

In particular, for all $1 \leq i \leq n$ with $i \neq j$, we have $\theta(U_i) = V_i$ since:

$$\theta(U_i) = \sum_{l\geq 0} \frac{1}{\eta_j^l} D'_{j,l}(U_i)U_j^{-l} = \begin{cases} \sum_{l\geq 0} \frac{1}{\eta_j^l} D_{j,l}(U_i)U_j^{-l} & i < j, \\ U_i & i > j. \end{cases}$$

Moreover $U_j = V_j$, thus we have:

$$\theta\big(\widehat{C_{j+1}}[U_j;\alpha'_j]_P\big) = \mathbb{K}[V_1]\cdots[V_{j-1};\alpha_{j-1},\delta_{j-1}]_P[V_{j+1};\beta'_{j+1}]_P\cdots[V_n;\beta'_n]_P[V_j;\alpha'_j]_P = C_j,$$

and by Corollary 2.2.3 we get $C_j S_j^{-1} = C_{j+1} S_j^{-1}$. This proves assertion (3).

Since $\{V_l, V_j\} = \lambda_{lj} V_j V_l$ for all $j < l \leq n$ we can bring back $V_j$ in the $j$-th position:

$$C_j = \mathbb{K}[V_1]\cdots[V_{j-1};\alpha_{j-1},\delta_{j-1}]_P[V_j;\beta''_j]_P\cdots[V_n;\beta''_n]_P,$$

where for all $j \leq l \leq n$, the map $\beta_l''$ is a Poisson derivation such that $\beta_l''(V_i) = \lambda_{li} V_i$ for all $1 \leq i < l$. This proves assertion (1).

Finally, the fact that $\beta_l'' D_{m,k} = D_{m,k} \beta_l'' + k\lambda_{lm} D_{m,k}$ for all $1 < m < j \leq l \leq n$ and all $k \geq 0$, follows directly from the equalities:

- $\beta_l D_{m,k} = D_{m,k} \beta_l + k\lambda_{lm} D_{m,k}$ for all $1 < m \leq j < l \leq n$ and all $k \geq 0$,
- $\beta_l(U_i) = \lambda_{li} U_i$ for all $j < l \leq n$ and all $1 \leq i < l$,
- $\alpha_j(U_i) = \lambda_{ji} U_i$ for all $1 \leq i < j$,
- $\beta_l''(V_i) = \lambda_{li} V_i$ for all $j \leq l \leq n$ and all $1 \leq i < l$.

This proves assertion (2). □

**Corollary 5.2.3.** *The algebra $\overline{A} := C_2$ is a Poisson affine space. More precisely, by setting $T_i := X_{i,2}$ for all $1 \leq i \leq n$ we have:*

$$\overline{A} = \mathbb{K}_{\boldsymbol{\lambda}}[T_1, \ldots, T_n],$$

*where $\boldsymbol{\lambda}$ is the skew-symmetric matrix defined by $\boldsymbol{\lambda} := (\lambda_{ij}) \in M_n(\mathbb{K})$.*

We illustrate in details the Poisson deleting derivations algorithm on an example in Appendix A.1.

## 5.3 Fields of fractions of $A$ and $\overline{A}$

In this section we show that there exists a localisation of $A \in \mathcal{P}$ isomorphic to a Poisson torus. Set $\Sigma$ for the multiplicative set in $\overline{A}$ generated by the $T_1, \ldots, T_n$. For $2 \leq j \leq n$ we define sets $\Sigma_j$ as follows:

$$\Sigma_2 := \Sigma \quad \text{and} \quad \Sigma_{j+1} := C_{j+1} \cap \Sigma_j.$$

**Proposition 5.3.1.** *We have:*

(1) *$\Sigma_j$ is a multiplicative set of $C_j$ containing $\{X_{j-1,j}, \ldots, X_{n,j}\}$ for all $2 \leq j \leq n+1$.*

(2) *For $2 \leq j \leq n$ we have $C_j \Sigma_j^{-1} = C_{j+1} \Sigma_{j+1}^{-1}$ as Poisson subalgebras of* Frac $A$.

*Proof.* If $j = 2$ assertion (1) is trivial. Let $2 \leq j \leq n$ and assume that assertion (1) is satisfied for this $j$. Since $\Sigma_j$ contains $\{V_{j-1}, \dots, V_n\}$ (recall that $V_i = X_{i,j}$ for all $1 \leq i \leq n$) the multiplicative set $\Sigma_{j+1}$ contains $U_l = V_l$ for $j \leq l \leq n$. And assertion (1) is proved.

We now prove assertion (2). From assertion (1) we obtain in particular that $S_j = \{U_j^n \mid n \geq 0\} = \{V_j^n \mid n \geq 0\} \subseteq \Sigma_j \cap \Sigma_{j+1}$. Thus by Proposition 5.2.2 we have:

$$C_{j+1} \subseteq C_{j+1}S_j^{-1} = C_j S_j^{-1} \subseteq C_j \Sigma_j^{-1}.$$

Since $\Sigma_{j+1} \subseteq \Sigma_j$ we have $C_{j+1}\Sigma_{j+1}^{-1} \subseteq C_j\Sigma_j^{-1}$. Reciprocally let $a \in C_j\Sigma_j^{-1}$ and write $a = c_1\sigma_1^{-1}$ for some $c_1 \in C_j$ and $\sigma_1 \in \Sigma_j$. Since $\Sigma_j \subseteq C_j \subseteq C_jS_j^{-1} = C_{j+1}S_j^{-1}$ we can write $\sigma_1 = c_2s_2^{-1}$ and $c_1 = c_3s_3^{-1}$ for some $c_2, c_3 \in C_{j+1}$ and some $s_2, s_3 \in S_j$. So $a = c_3s_2(c_2s_3)^{-1}$ with $c_3s_2 \in C_{j+1}$ since $S_j \subseteq C_{j+1}$. Moreover $c_2s_3 = \sigma_1s_2s_3 \in \Sigma_j \cap C_{j+1} = \Sigma_{j+1}$ since $S_j \subseteq \Sigma_j$. We conclude that $a = c_3s_2(c_2s_3)^{-1} \in C_{j+1}\Sigma_{j+1}^{-1}$, i.e. $C_j\Sigma_j^{-1} \subseteq C_{j+1}\Sigma_{j+1}^{-1}$ and assertion (2) is proved. □

In particular we have proved the following theorem.

**Theorem 5.3.2.** *There exists a multiplicative set $\mathcal{S}$ in $A$ such that:*

$$A\mathcal{S}^{-1} = \overline{A}\Sigma^{-1} = \mathbb{K}_{\boldsymbol{\lambda}}[T_1^{\pm 1}, \dots, T_n^{\pm 1}].$$

This theorem shows in particular that all the Poisson algebras of the class $\mathcal{P}$ satisfy the quadratic Poisson Gel'fand-Kirillov problem, retrieving assertion (1) of Theorem 3.3.1. Moreover the algorithm provides explicit generators for Frac $A$ such that Frac $A$ is a Poisson affine field in these generators.

## 5.4 The canonical embedding

Recall that for a Poisson algebra $B$ we denote by P.Spec$(B)$ its Poisson spectrum, i.e. the set of prime ideals of $B$ which are also Poisson ideals. P.Spec$(B)$ is endowed with the induced Zariski topology. In this section we focus on the behaviour of the Poisson spectrum of an iterated Poisson-Ore extension $A \in \mathcal{P}$ under the Poisson deleting derivations algorithm. We show that there is an embedding between P.Spec$(A)$ and P.Spec$(\overline{A})$. This is done by showing that, at each step of the algorithm there is an embedding between P.Spec $(C_{j+1})$ and P.Spec $(C_j)$.

Throughout this section, we use the notation of Hypothesis 5.1.1 and as previously we fix $2 \leq j \leq n$, and set $U_i := X_{i,j+1}$ and $V_i := X_{i,j}$ for all $1 \leq i \leq n$.

### 5.4.1 The embedding $\varphi_j : \text{P.Spec}\,(C_{j+1}) \to \text{P.Spec}\,(C_j)$

Recall that $U_j = V_j$ and set:

$$\Gamma_j^0(C_j) = \{P \in \text{P.Spec}\,(C_j) \mid V_j \notin P\}, \qquad \Gamma_j^1(C_j) = \{P \in \text{P.Spec}\,(C_j) \mid V_j \in P\},$$
$$\Gamma_j^0(C_{j+1}) = \{P \in \text{P.Spec}\,(C_{j+1}) \mid U_j \notin P\}, \quad \Gamma_j^1(C_{j+1}) = \{P \in \text{P.Spec}\,(C_{j+1}) \mid U_j \in P\}.$$

These sets partition P.Spec $(C_j)$ and P.Spec $(C_{j+1})$. Since we have $C_j S_j^{-1} = C_{j+1} S_j^{-1}$, contraction and extension of ideals provide bijections between $\Gamma_j^0(C_j)$ and $\Gamma_j^0(C_{j+1})$ (it is easy to show that the contraction or the extension of a Poisson ideal is again a Poisson ideal). More precisely we have the following result.

**Lemma 5.4.1.** *There is an homeomorphism $\varphi_j^0 : \Gamma_j^0(C_{j+1}) \to \Gamma_j^0(C_j)$ given by $\varphi_j^0(P) := PS_j^{-1} \cap C_j$ for $P \in \Gamma_j^0(C_{j+1})$. Its inverse is defined by $(\varphi_j^0)^{-1}(Q) := QS_j^{-1} \cap C_{j+1}$ for $Q \in \Gamma_j^0(C_j)$.*

We now want to compare $\Gamma_j^1(C_{j+1})$ and $\Gamma_j^1(C_j)$. For, we denote by $\langle U_j \rangle_P$ the smallest Poisson ideal in $C_{j+1}$ containing $U_j$ and for all $1 \leq i \leq n$, we denote by $\overline{U_i}$ the image of $U_i$ in the Poisson algebra $C_{j+1}/\langle U_j \rangle_P$.

**Lemma 5.4.2.** *The map $g_j : C_j \to C_{j+1}/\langle U_j \rangle_P$ given by $g_j(V_i) = \overline{U_i}$ for all $1 \leq i \leq n$ is a surjective Poisson algebra homomorphism.*

*Proof.* The map $g_j$ is the composition of the quotient map $\pi : C_{j+1} \to C_{j+1}/\langle U_j \rangle_P$ with the algebra isomorphism $\Psi : C_j \to C_{j+1}$ defined by $\Psi(V_i) = U_i$ for all $1 \leq i \leq n$. Thus clearly $g_j = \pi \circ \Psi$ is a surjective algebra homomorphism. Note that $\pi$ is a Poisson algebra homomorphism whereas $\Psi$ is not, so we cannot conclude directly. We show that $g_j(\{V_k, V_l\}) = \{g_j(V_k), g_j(V_l)\}$ for all $1 \leq l < k \leq n$. First if $k \geq j$ we have:

$$g_j(\{V_k, V_l\}) = g_j(\lambda_{kl} V_k V_l) = \lambda_{kl} \overline{U_k}\,\overline{U_l} = \{\overline{U_k}, \overline{U_l}\} = \{g_j(V_k), g_j(V_j)\}.$$

(Note that when $k = j$ we have $\overline{U_k} = 0$). If $k < j$ we have $\Psi(\delta_k(V_l)) = \delta_k(U_l)$ and:

$$\begin{aligned} g_j(\{V_k, V_l\}) = g_j\big(\lambda_{kl}V_kV_l + \delta_k(V_l)\big) &= \lambda_{kl}\overline{U_kU_l} + g_j\big(\delta_k(V_l)\big) \\ &= \lambda_{kl}\overline{U_kU_l} + \overline{\delta_k(U_l)} = \{\overline{U_k}, \overline{U_l}\} = \{g_j(V_k), g_j(V_l)\}. \end{aligned}$$

□

Set $N_j := \ker(g_j)$. Then since $C_j/N_j \cong C_{j+1}/\langle U_j\rangle_P$ there is an homeomorphism:

$$\varphi_j^1 : \Gamma_j^1(C_{j+1}) \to \{P \in \text{P.Spec}\,(C_j) \mid N_j \subseteq P\}$$

defined by $\varphi_j^1(P) := g_j^{-1}(P/\langle U_j\rangle_P)$ for $P \in \Gamma_j^1(C_{j+1})$. Since $V_j = U_j \in N_j$ we have $\{P \in \text{P.Spec}\,(C_j) \mid N_j \subseteq P\} \subseteq \Gamma_j^1(C_j)$ and:

**Lemma 5.4.3.** *There is an increasing and injective map $\varphi_j^1 : \Gamma_j^1(C_{j+1}) \to \Gamma_j^1(C_j)$ defined by $\varphi_j^1(P) = g_j^{-1}(P/\langle U_j\rangle_P)$ for $P \in \Gamma_j^1(C_{j+1})$, which induces an homeomorphism on its image.*

We can now define a map $\varphi_j : \text{P.Spec}\,(C_{j+1}) \to \text{P.Spec}\,(C_j)$ by setting:

$$\varphi_j(P) = \begin{cases} \varphi_j^0(P) \text{ if } P \in \Gamma_j^0(C_{j+1}), \\ \varphi_j^1(P) \text{ if } P \in \Gamma_j^1(C_{j+1}). \end{cases}$$

As a direct consequence of Lemmas 5.4.1 and 5.4.3 we get the following result.

**Proposition 5.4.4.** *The map $\varphi_j : \text{P.Spec}\,(C_{j+1}) \to \text{P.Spec}\,(C_j)$ is injective. For $\varepsilon \in \{0, 1\}$, the map $\varphi_j$ induces an homeomorphism from $\Gamma_j^\varepsilon(C_{j+1})$ to $\varphi_j\big(\Gamma_j^\varepsilon(C_{j+1})\big)$ which is a closed subset of $\Gamma_j^\varepsilon(C_j)$.*

### 5.4.2 The canonical partition of P.Spec $(A)$

**Definition 5.4.5.** We set $\varphi := \varphi_2 \circ \cdots \circ \varphi_n$. This is an injective map from $\text{P.Spec}\,(C_{n+1}) = \text{P.Spec}\,(A)$ to $\text{P.Spec}\,(C_2) = \text{P.Spec}\,(\overline{A})$ and we refer to it as the *canonical embedding.*

Let $W := \mathscr{P}([\![1, n]\!])$ denote the powerset of $[\![1, n]\!]$. For $w \in W$, we set:

$$\text{P.Spec}_w(\overline{A}) := \big\{P \in \text{P.Spec}\,(\overline{A}) \mid P \cap \{T_1, \ldots, T_n\} = \{T_i \mid i \in w\}\big\},$$

where we recall that the $T_i$s are the generators of the Poisson affine space $\overline{A}$. Note that these sets form a partition of P.Spec $(\overline{A})$. For all $w \in W$ we set:

$$\text{P.Spec}_w(A) := \varphi^{-1}\big(\text{P.Spec}_w(\overline{A})\big),$$

and $W'_P$ for the set of $w$ such that $\text{P.Spec}_w(A) \neq \emptyset$, i.e.:

$$W'_P := \{w \in W \mid \text{P.Spec}_w(A) \neq \emptyset\}.$$

Note that $W'_P$ is not empty since we always have $\varphi(\langle 0\rangle) = \langle 0\rangle \in \text{P.Spec}_\emptyset(\overline{A})$. We obtain a partition of P.Spec $(A)$:

$$\text{P.Spec}\,(A) = \bigsqcup_{w \in W'_P} \text{P.Spec}_w(A) \quad \text{and} \quad 1 \leq |W'_P| \leq |W| = 2^n.$$

**Definition 5.4.6.** This partition of P.Spec $(A)$ will be called the *canonical partition*, the elements of $W'_P$ will be called the *Cauchon diagrams associated to $A$*, or Cauchon diagrams for short. Finally, for $w \in W'_P$ the set $\text{P.Spec}_w(A)$ is called the *stratum* associated to $w$.

Note that the set $W'_P$ depends on the expression of $A$ as an iterated Poisson-Ore extension. We compute the sets of Cauchon diagrams for two examples in Appendices A.2 and B. See also Section 7.2.

### 5.4.3 A membership criterion for $\text{Im}(\varphi)$

The following results help us to understand whether a given Poisson prime ideal of $\overline{A}$ belongs to the image of the canonical embedding. This will be useful to understand better the canonical partition and when dealing with examples. We start this section with a membership criterion for $\text{Im}(\varphi_j)$. Recall that $N_j = \ker(g_j)$ was defined in Section 5.4.1.

**Lemma 5.4.7.** *Let $Q \in \text{P.Spec}\,(C_j)$. Then:*

$$Q \in \text{Im}(\varphi_j) \iff \big(\textit{either } U_j = V_j \notin Q, \textit{ or } N_j \subseteq Q\big).$$

*Proof.* This is clear since the map $\varphi_j^0$ is a bijection from $\Gamma_j^0(C_{j+1})$ to $\Gamma_j^0(C_j)$ and the map $\varphi_j^1$ is a bijection from $\Gamma_j^1(C_{j+1})$ to $\{Q \in \text{P.Spec}\,(C_j) \mid N_j \subseteq Q\}$. □

Set $f_1 := \mathrm{id}_{\mathrm{P.Spec}(\overline{A})}$. For all $2 \leq j \leq n$ we define a map $f_j : \mathrm{P.Spec}(C_{j+1}) \to \mathrm{P.Spec}(\overline{A})$ by setting $f_j := f_{j-1} \circ \varphi_j$. In particular we have $f_n = \varphi$. Note that the $f_j$s are injective maps. We deduce from Lemma 5.4.7 the following membership criterion for $\mathrm{Im}(\varphi)$.

**Proposition 5.4.8.** *Let $Q \in \mathrm{P.Spec}(\overline{A})$. The following are equivalent:*

- *$Q \in \mathrm{Im}(\varphi)$,*
- *for all $2 \leq j \leq n$ we have $Q \in \mathrm{Im}(f_{j-1})$ and either $X_{j,j} = X_{j,j+1} \notin f_{j-1}^{-1}(Q)$, or $N_j \subseteq f_{j-1}^{-1}(Q)$.*

*Remark* 5.4.9. To understand $N_j$ it is enough to understand $\langle U_j \rangle_P$ since $N_j = \Psi^{-1}(\langle U_j \rangle_P)$, where the algebra isomorphism $\Psi : C_j \to C_{j+1}$ is defined by $\Psi(V_i) = U_i$ for all $1 \leq i \leq n$ (see proof of Lemma 5.4.2). As $\{U_j, U_i\} = \lambda_{ji} U_j U_i + \delta_j(U_i)$ for all $i \in [\![1, j-1]\!]$, we deduce that:

$$\langle U_j, \delta_j(U_i) \mid i \in [\![1, j-1]\!] \rangle \subseteq \langle U_j \rangle_P.$$

By minimality of $\langle U_j \rangle_P$, the reverse inclusion will be satisfied if the left hand side is a Poisson ideal. However this is not always the case as the following example demonstrates. Let $A$ be the iterated Poisson-Ore extension $A := \mathbb{C}[X][Y; \beta, \Delta]_P[Z; \alpha, \delta]_P$, where $\beta := -X\partial_X$, $\alpha := X\partial_X - Y\partial_Y$, $\Delta := \partial_X$ and $\delta := Y^2\partial_X$, so that:

$$\{Y, X\} = -XY + 1,$$
$$\{Z, X\} = XZ + Y^2,$$
$$\{Z, Y\} = -YZ.$$

One can check that $A \in \mathcal{P}$, but that $\langle Z, Y^2 \rangle$ is not a Poisson ideal of $A$. This example will be studied in more detail in Section 7.2.2.

### 5.4.4 Topological and algebraic properties of the canonical embedding

In this section we investigate the topological and algebraic properties of the canonical embedding. We start with some useful results that will be used in this section as well as later on.

**Lemma 5.4.10.** *Let $P \in \mathrm{P.Spec}\,(C_{j+1})$ and $Q := \varphi_j(P) \in \mathrm{P.Spec}\,(C_j)$. For $j \leq l \leq n$ we have:*

$$U_l \in P \quad \Longleftrightarrow \quad V_l \in Q.$$

*Proof.* If $l = j$, then $(U_l \in P) \iff \big(P \in \Gamma_j^1(C_{j+1})\big)$ and $(V_l \in Q) \iff \big(Q \in \Gamma_j^1(C_j)\big)$, and the result is given by Proposition 5.4.4. We distinguish between two cases when $l > j$. First, if $P \in \Gamma_j^0(C_{j+1})$, then we have:

$$U_l \in P \quad \Rightarrow \quad U_l \in PS_j^{-1} \quad \Rightarrow \quad V_l = U_l \in C_j \cap PS_j^{-1} = Q,$$

and:

$$V_l \in Q \quad \Rightarrow \quad V_l \in QS_j^{-1} \quad \Rightarrow \quad U_l = V_l \in C_{j+1} \cap QS_j^{-1} = P.$$

Next, if $P \in \Gamma_j^1(C_{j+1})$, then we have:

$$U_l \in P \iff \overline{U_l} \in \frac{P}{\langle U_j \rangle_P} \iff g_j(V_l) \in \frac{P}{\langle U_j \rangle_P} \iff V_l \in g_j^{-1}\Big(\frac{P}{\langle U_j \rangle_P}\Big) = Q.$$

□

We deduce the following easy corollary. For $Q \in \mathrm{Im}(\varphi)$, we set $P_j := f_{j-1}^{-1}(Q) \in \mathrm{P.Spec}\,(C_j)$ for all $2 \leq j \leq n+1$. Note that in particular we have $Q = P_2$.

**Corollary 5.4.11.** *Let $Q \in \mathrm{Im}(\varphi)$ and fix $1 \leq l \leq n$. Then we have:*

$$T_l = X_{l,2} \in P_2 \iff X_{l,k} \in P_k,$$

*for all $2 \leq k \leq l+1$.*

*Proof.* We proceed by induction on $k$. For $k = 2$ the result is trivial. Assume that the result is shown for some $2 \leq k \leq l$. By Lemma 5.4.10 we have $X_{l,k+1} \in P_{k+1} \iff X_{l,k} \in P_k$ and the result follows. □

This corollary can be improved as follows. Let $1 \leq j \leq n$ and $w \in W$. Set $X_w := f_j^{-1}(\mathrm{P.Spec}_w(\overline{A})) \subseteq \mathrm{P.Spec}\,(C_{j+1})$. When $j \geq 2$, we also set $Y_w := f_{j-1}^{-1}(\mathrm{P.Spec}_w(\overline{A})) \subseteq \mathrm{P.Spec}\,(C_j)$, so that $X_w = \varphi_j^{-1}(Y_w)$ since $f_j = f_{j-1} \circ \varphi_j$. Note that the sets $X_w$ and $Y_w$ can be empty.

**Lemma 5.4.12.** *Let $P \in X_w$. For $j \leq l \leq n$ we have:*

$$l \in w \iff U_l \in P.$$

*Proof.* Note that since $l \geq j$ we have $U_l = X_{l,k} = T_l$ for all $2 \leq k \leq j+1$. If $j = 1$, we have $X_w = \text{P.Spec}_w(\overline{A})$ and the result comes from the definition of $\text{P.Spec}_w(\overline{A})$. Assume that $j \geq 2$ and the result shown for $j-1$. Since $l \geq j > j-1$ we obtain by Lemma 5.4.10 that $U_l \in P \iff V_l \in \varphi_j(P)$. Moreover $\varphi_j(P) \in Y_w$, thus the induction hypothesis shows that:

$$l \in w \iff V_l \in \varphi_j(P) \iff U_l \in P.$$

This concludes the induction. □

**Lemma 5.4.13.** *The set $f_j(X_w)$ is a closed subset of* $\text{P.Spec}_w(\overline{A})$*, and $f_j$ induces (by restriction) an homeomorphism from $X_w$ to $f_j(X_w)$.*

*Proof.* The result is trivial if $j = 1$. Assume that $j \geq 2$ and that the result is shown for $j-1$. By Lemma 5.4.12 (applied to $l = j$ for $j$ and $j-1$) we have:

- $(j \notin w) \quad \Rightarrow \quad (X_w \subseteq \Gamma_j^0(C_{j+1})$ and $Y_w \subseteq \Gamma_j^0(C_j))$,
- $(j \in w) \quad \Rightarrow \quad (X_w \subseteq \Gamma_j^1(C_{j+1})$ and $Y_w \subseteq \Gamma_j^1(C_j))$.

So in both cases we have $X_w \subseteq \Gamma_j^{\varepsilon}(C_{j+1})$ and $Y_w \subseteq \Gamma_j^{\varepsilon}(C_j)$ for $\varepsilon \in \{0,1\}$. Therefore we have $\varphi_j(X_w) = Y_w \cap Z$ where $Z = \varphi_j(\Gamma_j^{\varepsilon}(C_{j+1}))$ with $\varepsilon \in \{0,1\}$. By Proposition 5.4.4, $Y_w \cap Z$ is a closed subset of $Y_w$, and $\varphi_j$ induces an homeomorphism from $X_w$ to $Y_w \cap Z$. By the induction hypothesis $f_{j-1}$ induces an homeomorphism from $Y_w$ to $f_{j-1}(Y_w)$ which is a closed subset of $\text{P.Spec}_w(\overline{A})$. Thus $f_{j-1}(Y_w \cap Z)$ is a closed subset of $f_{j-1}(Y_w)$ (as the image of a closed subset by an homeomorphism), and so is a closed subset of $\text{P.Spec}_w(\overline{A})$. Since $f_j(X_w) = f_{j-1} \circ \varphi_j(X_w) = f_{j-1}(Y_w \cap Z)$, the first assertion is proved.

The map $f_j : X_w \to f_j(X_w) = f_{j-1}(Y_w \cap Z)$ is the composition of the two maps $\varphi_j : X_w \to Y_w \cap Z$ and $f_{j-1} : Y_w \cap Z \to f_{j-1}(Y_w \cap Z)$ which are both homeomorphisms. □

When $j = n$ we have $f_j = \varphi$ and $X_w = \text{P.Spec}_w(A)$, for all $w \in W$. We deduce the following result.

**Theorem 5.4.14.** *Let* $\varphi : \text{P.Spec}\,(A) \to \text{P.Spec}\,(\overline{A})$ *be the canonical embedding and* $w \in W'_P$*. Then* $\varphi\big(\text{P.Spec}_w(A)\big)$ *is a (non empty) closed subset of* $\text{P.Spec}_w(\overline{A})$*, and* $\varphi$ *induces (by restriction) an homeomorphism from* $\text{P.Spec}_w(A)$ *to* $\varphi\big(\text{P.Spec}_w(A)\big)$*.*

In a lot of examples (when the Poisson algebra considered is supporting a suitable torus action for instance, see Theorem 5.5.6) the inclusion of the previous theorem is actually an equality:

$$\varphi(\text{P.Spec}_w(A)) = \text{P.Spec}_w(\overline{A}).$$

However this is not true in general as the following example demonstrates.

*Example* 5.4.15. Assume that $\operatorname{char}\mathbb{K} = 0$. Let $B = \mathbb{K}_{\boldsymbol{\lambda}}[X_1, X_2, X_3]$ be the Poisson affine space where:

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Observe that $\alpha := -X_1\frac{\partial}{\partial X_1} - X_2\frac{\partial}{\partial X_2}$ is a Poisson derivation of $B$ and $\delta := (X_1+X_2)\frac{\partial}{\partial X_3}$ a Poisson $\alpha$-derivation of $B$. Thus we can form the Poisson-Ore extension $A = B[X_4; \alpha, \delta]_P$. Note that $\delta$ is locally nilpotent and that we have $\delta\alpha = \alpha\delta + \delta$. Thus by Remark 5.1.2 we have $A \in \mathcal{P}$, and we can apply the deleting derivations algorithm (note that there is only one step in the algorithm). The Poisson algebra $\overline{A}$ is the Poisson affine space $\mathbb{K}_{\boldsymbol{\lambda}'}[T_1, T_2, T_3, T_4]$ where:

$$\boldsymbol{\lambda}' = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix},$$

and where $T_1 = X_1$, $T_2 = X_2$, $T_3 = X_3 + (X_1 + X_2)X_4^{-1}$ and $T_4 = X_4$. The canonical embedding is the map $\varphi$ from $\text{P.Spec}\,(A)$ to $\text{P.Spec}\,(\overline{A})$ defined by:

$$P \longmapsto \begin{cases} PS^{-1} \cap \overline{A} & X_4 \notin P, \\ g^{-1}(P/\langle X_4\rangle_P) & X_4 \in P, \end{cases}$$

where $S$ is the multiplicative set of $A$ generated by $X_4$, and where:

$$g : \overline{A} \longrightarrow \frac{A}{\langle X_4\rangle_P},$$

$$T_i \quad \longmapsto X_i + \langle X_4 \rangle_P \quad \text{for} \quad i = 1, \ldots, 4.$$

Firstly we show that $\{4\} \in W'_P \subseteq W = \mathscr{P}([\![1,4]\!])$. Set $P := \langle X_4 \rangle_P = \langle X_4, X_1 + X_2 \rangle$. One can check that $P \in \text{P.Spec}\,(A)$. Since $X_4 \in P$, we can define a Poisson algebra isomorphism $A/P \cong \overline{A}/\varphi(P)$ by sending $X_i + P$ to $T_i + \varphi(P)$ for $1 \leq i \leq 4$ (this will be done in more generality in Lemma 5.4.18 later). Therefore we have $T_4 \in \varphi(P)$ and $T_1, T_2, T_3 \notin \varphi(P)$. Hence $\varphi(P) \in \text{P.Spec}_{\{4\}}(\overline{A})$ and $\{4\} \in W'_P$ (more generally the set of Cauchon diagrams of $A$ is computed in Appendix B.).

Secondly, since $\{4\} \in W'_P$, Theorem 5.4.14 tells us that the set $\varphi\big(\text{P.Spec}_{\{4\}}(A)\big)$ is a non-empty closed subset of $\text{P.Spec}_{\{4\}}(\overline{A})$. We will show that this inclusion is strict. For $Q \in \text{P.Spec}_{\{4\}}(A)$ we have $T_4 \in \varphi(Q) \in \text{P.Spec}_{\{4\}}(\overline{A})$, thus $\langle T_4, T_1 + T_2 \rangle \subseteq \varphi(Q)$ by Lemma 5.4.7. Hence we have the following inclusion:

$$\varphi\big(\text{P.Spec}_{\{4\}}(A)\big) \subseteq \{P \in \text{P.Spec}_{\{4\}}(\overline{A}) \mid T_4 \in P,\ T_1 + T_2 \in P\} \subseteq \text{P.Spec}_{\{4\}}(\overline{A}).$$

But it is clear that $\langle T_4 \rangle \in \text{P.Spec}_{\{4\}}(\overline{A})$. Thus:

$$\varphi\big(\text{P.Spec}_{\{4\}}(A)\big) \subsetneq \text{P.Spec}_{\{4\}}(\overline{A}).$$

In a similar fashion we can also show that $\{3,4\} \in W'_P$ and that:

$$\varphi\big(\text{P.Spec}_{\{3,4\}}(A)\big) \subsetneq \text{P.Spec}_{\{3,4\}}(\overline{A}).$$

To conclude this section we prove two results. We give a criterion for a Poisson prime ideal to belong to the image of the canonical embedding, and we exhibit a non empty subset of $W'_P$.

**Proposition 5.4.16.** *Let $w \in W'_P$, $P \in \text{P.Spec}_w(A)$ and $Q \in \text{P.Spec}_w(\overline{A})$ such that $\varphi(P) \subseteq Q$. Then $Q \in \text{Im}(\varphi)$.*

*Proof.* We prove by induction that $Q \in \text{Im}(f_j)$ for all $1 \leq j \leq n$. When $j = 1$ the result is trivial since $f_1$ is the identity on $\text{P.Spec}\,(\overline{A})$. Suppose that $Q \in \text{Im}(f_{j-1})$ for some $2 \leq j \leq n$. Since $f_j = f_{j-1} \circ \varphi_j$ it is enough to show that $f_{j-1}^{-1}(Q) \in \text{Im}(\varphi_j)$. First we remark that $\varphi(P) \subseteq Q$ implies that $f_{j-1}^{-1}(\varphi(P)) \subseteq f_{j-1}^{-1}(Q)$ by Lemma 5.4.13 (with $j$ replaced by $j-1$). We now distinguish between two cases.

Assume that $U_j \notin f_j^{-1}(\varphi(P))$. Then by Corollary 5.4.11 we have $T_j \notin \varphi(P)$ and so $j \notin w$. But then by Lemma 5.4.12 we have $U_j \notin f_{j-1}^{-1}(Q)$ and thus $f_{j-1}^{-1}(Q) \in \mathrm{Im}(\varphi_j)$ by Lemma 5.4.7.

Assume that $U_j \in f_j^{-1}(\varphi(P))$. Then:

$$N_j \subseteq \varphi_j\big(f_j^{-1}(\varphi(P))\big) = f_{j-1}^{-1}(\varphi(P)) \subseteq f_{j-1}^{-1}(Q),$$

and Lemma 5.4.7 shows that $f_{j-1}^{-1}(Q) \in \mathrm{Im}(\varphi_j)$.

This concludes the induction. The result follows by taking $j = n$. □

**Proposition 5.4.17.** *Set* $\overline{W} := \{w \in W \mid \delta_i = 0$ *for all* $i \in w\}$. *Then* $\overline{W} \subseteq W'_P$. *Moreover for* $w \in \overline{W}$ *we have* $\varphi\big(\mathrm{P.Spec}_w(A)\big) = \mathrm{P.Spec}_w(\overline{A})$.

*Proof.* Let $w \in \overline{W}$ and set $Q := \langle T_i \mid i \in w\rangle \in \mathrm{P.Spec}_w(\overline{A})$. We show that $Q \in \mathrm{Im}(\varphi)$. For, we will prove by induction that $Q \in \mathrm{Im}(f_{j-1})$ for all $2 \leq j \leq n+1$. For $j = 2$ the result is clear. Now suppose that $Q \in \mathrm{Im}(f_{j-1})$ for some $2 \leq j \leq n$. To show that $Q \in \mathrm{Im}(f_j)$ it is enough to show that $f_{j-1}^{-1}(Q) \in \mathrm{Im}(\varphi_j)$ since $f_j = f_{j-1} \circ \varphi_j$. As previously we set $P_j := f_{j-1}^{-1}(Q) \in \mathrm{P.Spec}(C_j)$. By Lemma 5.4.7 we have $P_j \in \mathrm{Im}(\varphi_j)$ if and only if either $X_{j,j+1} \notin P_j$, or $N_j \subseteq P_j$. We now distinguish between two cases. First suppose that $j \notin w$. Then $T_j \notin Q$ and by Lemma 5.4.11 we have $X_{j,j+1} \notin P_j$, i.e. $P_j \in \mathrm{Im}(\varphi_j)$. Now suppose that $j \in w$. Next $T_j \in Q$ and by Lemma 5.4.11 we get $X_{j,j+1} \in P_j$. But since $j \in w$ we have $\delta_j = 0$ and $N_j = \langle X_{j,j+1}\rangle$. Therefore $N_j \subseteq P_j$ and $P_j \in \mathrm{Im}(\varphi_j)$.

The second assertion follows from Proposition 5.4.16 since we just showed that for all $w \in \overline{W}$ we have $\langle T_i \mid i \in w\rangle \in \mathrm{Im}(\varphi)$. □

In particular this proposition shows that we always have $\{\emptyset, \{1\}\} \subseteq W'_P$ for $A \in \mathcal{P}$.

### 5.4.5 Poisson prime quotients of $A$ and $\overline{A}$

In this section we study the behaviour of the Poisson prime quotients of a Poisson algebra $A \in \mathcal{P}$ under the deleting derivations algorithm.

#### 5.4.5.1 Poisson prime quotients of $C_{j+1}$ and $C_j$

Fix $2 \leq j \leq n$, let $P \in \text{P.Spec}\,(C_{j+1})$ and set $Q := \varphi_j(P) \in \text{P.Spec}\,(C_j)$. As usual, to simplify notation we set $U_i := X_{i,j+1}$ and $V_i := X_{i,j}$ for all $i$. We also set $D := C_{j+1}/P$ and $E := C_j/Q$. Finally, we set $d_i := U_i + P$ and $e_i := V_i + Q$ for all $1 \leq i \leq n$.

**Lemma 5.4.18.** *If $d_j = 0$, then there is a Poisson algebra isomorphism between $E$ and $D$ sending $e_i$ to $d_i$ for all $1 \leq i \leq n$.*

*Proof.* $d_j = 0$ means that $P \in \Gamma_j^1(C_{j+1})$ and $Q = g_j^{-1}(P/\langle U_j \rangle_P)$. Thus we have a surjective Poisson algebra homomorphism:

$$C_j \longrightarrow \frac{C_{j+1}/\langle U_j \rangle_P}{P/\langle U_j \rangle_P} \cong C_{j+1}/P,$$

whose kernel is $Q$. □

**Lemma 5.4.19.** *Assume that $d_j \neq 0$ and set $\overline{S_j} := \{d_j^n \mid n \geq 0\}$. Then there is an injective Poisson algebra homomorphism $\Lambda : E \to D\overline{S_j}^{-1}$ defined by:*

$$\Lambda(e_i) := \begin{cases} d_i & i \geq j, \\ \sum\limits_{k \geq 0} \frac{1}{\eta_j^k} \overline{D_{j,k}(U_i)} d_j^{-k} & i < j, \end{cases}$$

*where $\overline{D_{j,k}(U_i)} := D_{j,k}(U_i) + P$.*

*Proof.* By assumption $P \in \Gamma_j^0(C_{j+1})$, so $QS_j^{-1} = PS_j^{-1}$ is an ideal in $C_jS_j^{-1} = C_{j+1}S_j^{-1}$ and we have the following identifications:

$$\frac{C_jS_j^{-1}}{QS_j^{-1}} = \frac{C_{j+1}S_j^{-1}}{PS_j^{-1}} \cong D\overline{S_j}^{-1}.$$

Thus the canonical embedding of $C_j$ in $C_jS_j^{-1}$ induces a well-defined injective Poisson algebra homomorphism $\Lambda$ from $E$ to $D\overline{S_j}^{-1}$ whose expression is clear from the equalities:

$$V_i = \begin{cases} U_i & i \geq j, \\ \sum\limits_{k \geq 0} \frac{1}{\eta_j^k} D_{j,k}(U_i) U_j^{-k} & i < j. \end{cases}$$

□

From Lemma 5.4.18 and Lemma 5.4.19, we can state:

**Corollary 5.4.20.** *$D$ and $E$ have the same Poisson field of fractions (if $U_j \notin P$, we identify $E$ with its image in $DS_j^{-1}$ by $\Lambda$ so that we have $D\overline{S_j}^{-1} = E\overline{S_j}^{-1}$).*

#### 5.4.5.2 Poisson prime quotients of $A$ and $\overline{A}$

From the previous section we deduce the following results about the Poisson prime quotients of $A$ and $\overline{A}$. Let $P$ be a Poisson prime ideal of $A$ and set $Q := \varphi(P)$. For all $2 \leq j \leq n+1$ we set:

- $P_j := \varphi_j \circ \cdots \circ \varphi_n(P) \in \text{P.Spec}\,(C_j)$, in particular $P_{n+1} = P$ and $P_2 = Q$;
- $D_j := C_j/P_j$, $D = D_{n+1} = A/P$ and $\overline{D} := D_2 = \overline{A}/\varphi(P)$;
- $d_{i,j} := X_{i,j} + P_j$ and $t_i = d_{i,2}$ for all $1 \leq i \leq n$.

Finally we set $G := \text{Frac}\,(D)$. From Corollary 5.4.20 we deduce that all the algebras $D_j$ have the same Poisson field of fractions $G$.

**Proposition 5.4.21.** *Let $2 \leq j \leq n+1$. There is a Poisson algebra homomorphism $\gamma_j : C_j \to G$ sending $X_{i,j}$ to $d_{i,j}$ for $1 \leq i \leq n$. Its image is $D_j$ and its kernel is $P_j$.*

Lemma 5.4.18 and Lemma 5.4.19 give us an algorithm to obtain the generators $d_{i,j}$ of $D_j$ from the generators $d_{i,j+1}$ of $D_{j+1}$.

**Proposition 5.4.22.** *Let $2 \leq j \leq n$.*

(1) *If $d_{j,j+1} = 0$ then $d_{i,j} = d_{i,j+1}$ for all $1 \leq i \leq n$.*

(2) *If $d_{j,j+1} \neq 0$, then $d_{j,j+1}$ is invertible in $G$ and we have:*

$$d_{i,j} = \begin{cases} d_{i,j+1} & i \geq j, \\ \sum_{k \geq 0} \frac{1}{\eta_j^k} \gamma_{j+1}\big(D_{j,k}(X_{i,j+1})\big) d_{j,j+1}^{-k} & i < j. \end{cases}$$

Let $w$ the element in $W'_P$ such that $P \in \text{P.Spec}_w(A)$. Then $Q \in \text{P.Spec}_w(\overline{A})$ and for all $1 \leq i \leq n$ we have:

$$t_i \neq 0 \iff i \notin w.$$

Set $\overline{w} = W \setminus w$ and let $\Pi$ be the multiplicative set of $\overline{D}$ generated by the $t_i$ for $i \in \overline{w}$. For $2 \leq j \leq n+1$ we define sets $\Pi_j$ as follows:

$$\Pi_2 := \Pi \quad \text{and} \quad \Pi_{j+1} := D_{j+1} \cap \Pi_j.$$

**Proposition 5.4.23.** *Then we have:*

(1) $\Pi_j$ *is a multiplicative set of* $D_j$ *which contains* $\{d_{i,j} \mid j-1 \leq i \leq n \text{ and } d_{i,j} \neq 0\}$, *for all* $2 \leq j \leq n+1$.

(2) *For all* $2 \leq j \leq n$ *we have* $D_j \Pi_j^{-1} = D_{j+1} \Pi_{j+1}^{-1}$ *(as Poisson subalgebras of* $G$*).*

*Proof.* (1) We proceed by induction. If $j = 2$ the result is clear. Assume that assertion (1) is true for some $2 \leq j \leq n$. We show that this assertion is still true for $j+1$. First note that it is clear that $\Pi_{j+1}$ is a multiplicative set of $D_{j+1}$. We now distinguish between two cases. If $d_{j,j+1} = 0$, we have $d_{i,j+1} = d_{i,j}$ for all $1 \leq i \leq n$, so:

$$\{d_{i,j+1} \mid j \leq i \leq n \text{ and } d_{i,j+1} \neq 0\} = \{d_{i,j} \mid j \leq i \leq n \text{ and } d_{i,j} \neq 0\} \subseteq \Pi_j \cap D_{j+1} = \Pi_{j+1}.$$

If $d_{j,j+1} \neq 0$ we have $d_{i,j+1} = d_{i,j} \in \Pi_j \cap C_{j+1}$ for all $j \leq i \leq n$ thus $\{d_{i,j+1} \mid j \leq i \leq n \text{ and } d_{i,j+1} \neq 0\} \subseteq \Pi_{j+1}$. Thus assertion (1) is shown.

(2) Let $2 \leq j \leq n$. If $d_{j,j+1} = 0$, we have $d_{i,j+1} = d_{i,j}$ for all $1 \leq i \leq n$ and the result follows. Assume that $d_{j,j+1} \neq 0$. Assertion (1) tells us, in particular, that $\overline{S_j} = \{d_{j,j+1}^n \mid n \geq 0\} \subseteq \Pi_{j+1}$. Since $d_{j,j+1} = d_{j,j}$ we also have $\overline{S_j} \subseteq \Pi_j$ and by Proposition 5.4.20 we obtain:

$$D_{j+1} \subseteq D_{j+1} \overline{S_j}^{-1} = D_j \overline{S_j}^{-1} \subseteq D_j \Pi_j^{-1}.$$

We can then conclude as in Proposition 5.3.1. □

We deduce the following theorem on the Poisson structure of the fields of fractions of the Poisson prime quotients of $A$.

**Theorem 5.4.24.** *There exists a multiplicative set* $\mathcal{S}'$ *in* $A/P$ *such that* $(A/P)\mathcal{S}'^{-1} = (\overline{A}/Q)\Pi^{-1}$ *and thus* $\mathrm{Frac}\,(A/P) = \mathrm{Frac}\,(\overline{A}/Q)$*.*

In particular this theorem says that in order to prove the quadratic Poisson Gel'fand-Kirillov problem for the Poisson prime quotients of $A$ it is enough to prove it for the Poisson prime quotients of the Poisson affine space $\overline{A}$. We retrieve Assertion (2) of Theorem 3.3.1 with the addition that the ideal $Q$ is now characterised by the canonical embedding.

### 5.4.6 Conditions under which $T_i$ belongs to $Q \in \mathrm{Im}(\varphi)$

As usual let $P \in \mathrm{P.Spec}\,(A)$ and set $Q := \varphi(P) \in \mathrm{P.Spec}\,(\overline{A})$. In this section we prove a result which gives conditions under which a generator $T_i$ of $\overline{A}$ belongs to $Q$. We now fix some notation. For $2 \leq j \leq n+1$ set $P_j := \varphi_j \circ \cdots \circ \varphi_n(P) \in \mathrm{P.Spec}\,(C_j)$ ($P = P_{n+1}$ and $Q = P_2$), as well as $D := C_{j+1}/P_{j+1}$ and $E := C_j/P_j$. For $2 \leq j \leq n+1$ we denote by $C_{j+1}^{<j}$ (resp. $C_j^{<j}$) the Poisson subalgebra of $C_{j+1}$ (resp. $C_j$) generated by $U_i$ (resp. $V_i$) for all $1 \leq i < j$. More precisely we have:

$$C_{j+1}^{<j} = \mathbb{K}[U_1][U_2; \alpha_2, \delta_2]_P \cdots [U_{j-1}; \alpha_{j-1}, \delta_{j-1}]_P,$$
$$C_j^{<j} = \mathbb{K}[V_1][V_2; \alpha_2, \delta_2]_P \cdots [V_{j-1}; \alpha_{j-1}, \delta_{j-1}]_P.$$

In particular by Proposition 5.2.2 there is a Poisson algebra isomorphism $\theta_j$ from $C_{j+1}^{<j}$ to $C_j^{<j}$ sending $U_i$ to $V_i$ for all $1 \leq i < j$.

Fix $2 \leq j \leq n$ and assume that $U_j \notin P_{j+1}$. We denote by $D^{<j}$ (resp. $E^{<j}$) the Poisson subalgebra of $D$ (resp. $E$) generated by $d_i := U_i + P_{j+1}$ (resp. $e_i := V_i + P_j$) for all $1 \leq i < j$. The following lemma shows that we can induce the homomorphism $\theta_j$ to the quotient under certain conditions.

**Lemma 5.4.25.** *Assume that $U_j \notin P_{j+1}$ and that $D_{j,k}(C_{j+1}^{<j} \cap P_{j+1}) \subseteq P_{j+1}$ for all $k \geq 0$. Then there exists a unique Poisson algebra homomorphism $\overline{\theta_j}$ from $D^{<j}$ to $E^{<j}$ sending $d_i$ to $e_i$ for all $1 \leq i < j$.*

*Proof.* Note that it is enough to show that $\theta_j(C_{j+1}^{<j} \cap P_{j+1}) \subseteq C_j^{<j} \cap P_j$. For $c \in C_{j+1}^{<j} \cap P_{j+1}$ we have:

$$\theta_j(c) = \sum_{k \geq 0} \frac{1}{\eta_j^k} D_{j,k}(c) U_j^{-k} \in C_j^{<j}.$$

Since $U_j \notin P_{j+1}$ we have $P_j = P_{j+1} S_j^{-1} \cap C_j$ where $S_j^{-1}$ is the multiplicative set generated by $U_j$. By assumption, for all $k \geq 0$ we have $D_{j,k}(c) \in P_{j+1}$, so for all $k \geq 0$:

$$\frac{1}{\eta_j^k} D_{j,k}(c) U_j^{-k} \in P_{j+1} S_j^{-1}.$$

Thus $\theta_j(c) \in C_j^{<j} \cap P_j$, and the proposition is proved. $\square$

We can now state the main result of this section.

**Proposition 5.4.26.** *Assume that* $D_{l,k}(C_{l+1}^{<l} \cap P_{l+1}) \subseteq P_{l+1}$ *for all* $k \geq 0$ *and all* $1 \leq l \leq j$. *If:*

$$X_{j,j+1} \in P_{j+1} \Rightarrow X_{i,j+1} \in P_{j+1}$$

*for some* $1 \leq i < j$, *then* $T_i \in Q$.

*Proof.* We first prove by a decreasing induction that $X_{i,l} \in P_l$ for all $i < l \leq j+1$. The base of the induction is precisely the hypothesis of the proposition. Assume that $X_{i,l+1} \in P_{l+1}$ for some $i < l \leq j$.

Case 1: If $T_l = X_{l,2} = X_{l,l+1} \in P_{l+1}$, then by Lemma 5.4.18 there is a Poisson algebra isomorphism between $C_l/P_l$ and $C_{l+1}/P_{l+1}$ sending $X_{i,l} + P_l$ to $X_{i,l+1} + P_{l+1}$ for all $1 \leq i \leq n$. Thus $X_{i,l+1} \in P_{l+1}$ implies that $X_{i,l} \in P_l$ and the result is shown in that case.

Case 2: If $T_l = X_{l,2} = X_{l,l+1} \notin P_{l+1}$, then by Lemma 5.4.25 there is a Poisson algebra homomorphism between $(C_{l+1}/P_{l+1})^{<l}$ and $(C_l/P_l)^{<l}$ sending $X_{i,l+1} + P_{l+1}$ to $X_{i,l} + P_l$ for all $1 \leq i < l$. Therefore $X_{i,l+1} \in P_{l+1}$ implies that $X_{i,l} \in P_l$ and the result is shown in that case.

In particular when $l = i+1$ we obtain $X_{i,i+1} \in P_{i+1}$. Then by Lemma 5.4.11 we conclude that $T_i \in Q$ and the proposition is shown. $\square$

## 5.5 Torus action and the Poisson deleting derivations algorithm

We keep notation from the previous sections. In particular $A$ is a Poisson algebra of the class $\mathcal{P}$. In Section 5.5.1 we introduce a rational torus action by Poisson automorphisms on $A$ and study its compatibility with the deleting derivations algorithm. In particular, this allows us to improve Theorem 5.4.14. In Section 5.5.2 we introduce a partition of P.Spec $(A)$, the so-called $H$-*stratification*, where $H$ is a torus acting rationally by Poisson automorphisms on $A$. The $H$-stratification is defined in [15, Section 4] and is a Poisson version of the *Goodearl-Letzter $H$-stratification* (see [6, II.2]), which is used to deal with spectra of certain noncommutative noetherian rings. Finally in Section 5.5.3 we compare the canonical partition of $\text{P.Spec}\,(A)$ with the $H$-stratification, and show that they coincide.

### 5.5.1 Compatibility of the torus action and the Poisson deleting derivations algorithm

Let $A \in \mathcal{P}$ and $r > 0$. Suppose that the torus $H = (\mathbb{K}^\times)^r$ is acting rationally by Poisson automorphisms on $A$ such that each $X_i$ is an $H$-eigenvector. We study the compatibility of the deleting derivations algorithm with this action. The action of $H$ on $A$ uniquely extends to $\operatorname{Frac} A$. The following lemma shows that for all $2 \leq j \leq n+1$ the torus $H$ is also acting rationally by Poisson automorphisms on the algebra $C_j$ such that each $X_{i,j}$ is an $H$-eigenvector.

**Lemma 5.5.1.** *For $h \in H$ set $h(X_i) = \mu_i X_i$, where $\mu_i \in \mathbb{K}^\times$ for all $1 \leq i \leq n$. Assume that $h\big(D_{j,k}(X_i)\big) = \mu_j^k D_{j,k}\big(h(X_i)\big)$ for all $1 \leq i < j \leq n$ and $k \geq 0$. Then $h(V_i) = \mu_i V_i$ for all $1 \leq i \leq n$.*

*Proof.* Recall that $V_i = X_{i,j}$ for all $1 \leq i \leq n$. Thus the result is trivial when $j = n+1$. Assume that the result is true for the rank $j+1$, i.e. that we have $h(U_i) = \mu_i U_i$ for all $1 \leq i \leq n$. If $i \geq j$ we have $h(V_i) = \mu_i V_i$ since $V_i = U_i$. If $i < j$ then

$$\begin{aligned} h(V_i) &= \sum_{k\geq 0} \frac{1}{\eta_j^k} h\big(D_{j,k}(U_i)\big) h(U_j^{-k}) \\ &= \sum_{k\geq 0} \frac{1}{\eta_j^k} \mu_j^k D_k\big(h(U_i)\big) \mu_j^{-k} U_j^{-k} \\ &= \sum_{k\geq 0} \frac{1}{\eta_j^k} D_k\big(\mu_i U_i\big) U_j^{-k} \\ &= \mu_i V_i. \end{aligned}$$

□

In particular $H$ acts rationally by Poisson automorphisms on the Poisson affine space $\overline{A}$, and for all $1 \leq i \leq n$ the indeterminate $T_i$ is an $H$-eigenvector.

*Remark* 5.5.2. If $\operatorname{char}\mathbb{K}=0$, the hypothesis of Lemma 5.5.1 can be simplified. We claim that, for a fixed $h\in H$, the assumption:

$$h\big(D_{j,k}(X_i)\big)=\mu_j^k D_{j,k}\big(h(X_i)\big)\quad\text{for all}\quad 1\leq i<j\leq n\quad\text{and}\quad k\geq 0,$$

is trivially satisfied. Indeed we show by induction that for all $k\geq 0$, all $h\in H$ and all $1\leq i<j\leq n$ we have:

$$h\big(\delta_j^k(X_i)\big)=\mu_j^k\delta_j^k\big(h(X_i)\big). \tag{5.2}$$

This will prove the claim since when $\operatorname{char}\mathbb{K}=0$, we have $D_{j,k}=\frac{\delta_j^k}{k!}$ for all $k\geq 0$ and all $2\leq j\leq n$. We now proceed with the induction. Fix $h\in H$ and set $h(X_i)=\mu_i X_i$, where $\mu_i\in\mathbb{K}^\times$ for all $1\leq i\leq n$. Suppose $k=1$. For $1\leq i<j\leq n$ we have:

$$\begin{aligned}h\big(\delta_j(X_i)\big)&=h(\{X_j,X_l\}-\alpha_j(X_l)X_j)\\&=\mu_j\mu_l(\{X_j,X_l\}-\lambda_{jl}X_lX_j)\\&=\mu_j\mu_l\delta_j(X_l)=\mu_j\delta_j\big(h(X_l)\big).\end{aligned}$$

Now suppose that equation (5.2) is satisfied for a rank $k$. We have:

$$\begin{aligned}h\big(\delta_j^{k+1}(X_i)\big)&=h\big(\{X_j,\delta_j^k(X_l)\}-\alpha_j\delta_j^k(X_l)X_j\big)\\&=\{h(X_j),h\delta_j^k(X_l)\}-h\big(\delta_j^k\alpha_j(X_l)-\eta_jk\delta_j(X_l)\big)h(X_j)\\&=\mu_j^{k+1}\{X_j,\delta_j^kh(X_l)\}-\mu_j\big(h\delta_j^k(\lambda_{jl}X_l)-\eta_jkh\delta_j^k(X_l)\big)X_j\\&=\mu_j^{k+1}\Big(\{X_j,\delta_j^kh(X_l)\}-\big(\lambda_{jl}\delta_j^kh(X_l)+\eta_jk\delta_j^kh(X_l)\big)X_j\Big)\\&=\mu_j^{k+1}\Big(\{X_j,\delta_j^kh(X_l)\}-\big(\delta_j^k\alpha_jh(X_l)+\eta_jk\delta_j^kh(X_l)\big)X_j\Big)\\&=\mu_j^{k+1}\big(\{X_j,\delta_j^kh(X_l)\}-\alpha_j\delta_j^kh(X_l)X_j\big)\\&=\mu_j^{k+1}\delta_j\delta_j^kh(X_l)=\mu_j^{k+1}\delta_j^{k+1}h(X_l).\end{aligned}$$

This concludes the induction.

Recall that an ideal $I$ of a given algebra endowed with a torus action is $h$-*invariant* (with $h\in H$) if $h(I)=I$. If $h(I)=I$ for all $h\in H$, then $I$ is said $H$-*invariant*. Let

$1 \leq j \leq n$. For an ideal $I$ in $C_{j+1}$ we set:

$$(H : I) := \bigcap_{h \in H} h(I).$$

Equivalently $(H : I)$ is the largest $H$-invariant ideal contained in $I$. Moreover if $I$ is a Poisson ideal then $(H : I)$ is a Poisson ideal. Indeed since each $h \in H$ acts by Poisson automorphisms on $C_{j+1}$ the ideal $h(I)$ is a Poisson ideal for all $h \in H$. However when $I$ is a prime ideal is it not clear whether $(H : I)$ is also prime. It is true under the assumptions that $\mathbb{K}$ is infinite and that the algebra we are working with is noetherian, see for instance [6, Proposition II.2.9]. For simplicity we assume that $\operatorname{char} \mathbb{K} = 0$ for the remainder of this chapter. In particular Remarks 5.1.2 and 5.5.2 apply. The subset of $\text{P.Spec}\,(C_{j+1})$ consisting of $H$-invariant ideals is denoted by $H\text{-P.Spec}\,(C_{j+1})$. By the previous discussion if $P \in \text{P.Spec}\,(C_{j+1})$ then $(H : P) \in H\text{-P.Spec}\,(C_{j+1})$.

Recall that $\langle U_j \rangle_P$ is the smallest Poisson ideal in $C_{j+1}$ containing $U_j$ (equivalently containing $\langle U_j \rangle$).

**Lemma 5.5.3.** *The ideal $\langle U_j \rangle_P$ is an $H$-invariant ideal of $C_{j+1}$.*

*Proof.* Set $I := (H : \langle U_j \rangle_P)$. We will show that $I = \langle U_j \rangle_P$. Note that $U_j \in I$ since $U_j$ is an $H$-eigenvector. Therefore we have:

$$\langle U_j \rangle \subseteq I \subseteq \langle U_j \rangle_P.$$

Since $\langle U_j \rangle_P$ is a Poisson ideal, the ideal $I$ is a Poisson ideal, and by minimality of $\langle U_j \rangle_P$ we conclude that $\langle U_j \rangle_P = I$. This shows that $\langle U_j \rangle_P$ is $H$-invariant. □

We can now prove that $h \in H$ commutes with the embedding $\varphi_j$.

**Lemma 5.5.4.** *Let $2 \leq j \leq n$. If $P \in \text{P.Spec}\,(C_{j+1})$ and $h \in H$ we have:*

$$\varphi_j\big(h(P)\big) = h\big(\varphi_j(P)\big).$$

*Proof.* Recall that $S_j = \{U_j^n \mid n \geq 0\}$. Assume first that $P \in \Gamma_j^0(C_{j+1})$, i.e. that $U_j \notin P$. Since $U_j$ is an $h$-eigenvector (Lemma 5.5.1) we have $h(P) \in \Gamma_j^0(C_{j+1})$. Thus

$\varphi_j(h(P)) = C_j \cap h(P)S_j^{-1} = C_j \cap h(PS_j^{-1})$ and:

$$\varphi_j\big(h(P)\big) = h(C_j) \cap h(PS_j^{-1}) = h\big(C_j \cap PS_j^{-1}\big) = h\big(\varphi_j(P)\big).$$

Assume now that $P \in \Gamma_j^1(C_{j+1})$. By Lemma 5.5.3 the ideal $\langle U_j \rangle_P$ is $H$-invariant, so $h \in H$ induces a Poisson automorphism $\overline{h}$ of $C_{j+1}/\langle U_j \rangle_P$ sending $U_i + \langle U_j \rangle_P$ to $h(U_i) + \langle U_j \rangle_P$. Recall from Lemma 5.4.2 the Poisson homomorphism $g_j : C_j \to C_{j+1}/\langle U_j \rangle_P$ sending $V_i$ to $U_i + \langle U_j \rangle_P$. In view of Lemma 5.5.1 the diagram of Figure 5.1 is commutative.

$$\begin{array}{ccc} C_j & \xrightarrow{\quad g_j \quad} & C_{j+1}/\langle U_j \rangle_P \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \overline{h}} \\ C_j & \xrightarrow[\quad g_j \quad]{} & C_{j+1}/\langle U_j \rangle_P \end{array}$$

Figure 5.1

Since $\langle U_j \rangle_P \subseteq P$ we have $\langle U_j \rangle_P \subseteq h(P)$ and $\overline{h}(P/\langle U_j \rangle_P) = h(P)/\langle U_j \rangle_P$. Thus, using the commutativity of the diagram of Figure 5.1 and the fact that $h$ and $\overline{h}$ are bijective maps, we have $g_j^{-1}(h(P)/\langle U_j \rangle_P) = h\big(g_j^{-1}(P/\langle U_j \rangle_P)\big)$. This means that $\varphi_j\big(h(P)\big) = h\big(\varphi_j(P)\big)$. □

We deduce that $h \in H$ commutes with the canonical embedding.

**Lemma 5.5.5.** *Let* $P \in \mathrm{P.Spec}\,(A)$ *and* $h \in H$. *We have:*

(1) $\varphi\big(h(P)\big) = h\big(\varphi(P)\big)$.

(2) *If* $P \in \mathrm{P.Spec}_w(A)$ *for some* $w \in W_P'$*, then* $h(P) \in \mathrm{P.Spec}_w(A)$.

*Proof.* Assertion (1) comes from Lemma 5.5.4 and the equality $\varphi = \varphi_2 \circ \cdots \circ \varphi_n$. If $P \in \mathrm{P.Spec}_w(A)$, then $\varphi(P) \in \mathrm{P.Spec}_w(\overline{A})$. Since $T_i$ is an $h$-eigenvector for all $1 \leq i \leq n$ we have $\varphi\big(h(P)\big) = h\big(\varphi(P)\big) \in \mathrm{P.Spec}_w(\overline{A})$, thus $h(P) \in \mathrm{P.Spec}_w(A)$. □

Recall that, by Proposition 3.4.4, the only $H$-invariant Poisson prime ideals of the Poisson affine space $\overline{A}$ are the ideals:

$$J_w = \langle T_i \mid i \in w \rangle,$$

for all $w \in W = \mathscr{P}(\llbracket 1, n \rrbracket)$, provided that Hypotheses 3.4.1 is satisfied (recall that we assume that $\operatorname{char} \mathbb{K} = 0$). We can now state the following improvement of Theorem 5.4.14.

**Theorem 5.5.6.** *Assume that Hypothesis 3.4.1 is satisfied and let $w \in W'_P$. Then:*

$$\varphi\big(\text{P.Spec}_w(A)\big) = \text{P.Spec}_w(\overline{A}).$$

*Proof.* By Theorem 5.4.14, $\varphi\big(\text{P.Spec}_w(A)\big)$ is a closed subset of $\text{P.Spec}_w(\overline{A})$ so there exists a proper ideal $I$ in $\overline{A}$ containing $J_w = \langle T_i \mid i \in w \rangle$ such that:

$$\varphi\big(\text{P.Spec}_w(A)\big) = \big\{Q \in \text{P.Spec}_w(\overline{A}) \mid Q \supseteq I\big\}.$$

Let $Q \in \varphi\big(\text{P.Spec}_w(A)\big)$, and write $Q = \varphi(P)$ for some $P \in \text{P.Spec}_w(A)$. For $h \in H$ we have $h(Q) = \varphi\big(h(P)\big) \in \varphi\big(\text{P.Spec}_w(A)\big)$ by Lemma 5.5.5 and thus $I \subseteq h(Q)$. By setting $J := (H : Q)$, we have $J_w \subseteq I \subseteq J \subseteq Q$, so:

$$\Big(\{T_1, \ldots, T_n\} \cap J_w\Big) \subseteq \Big(\{T_1, \ldots, T_n\} \cap J\Big) \subseteq \Big(\{T_1, \ldots, T_n\} \cap Q\Big).$$

But since $J \in H\text{-P.Spec}(\overline{A})$ we can write $J = J_{w'}$ for some $w' \in W$ by Proposition 3.4.4. Hence $w \subseteq w' \subseteq w$, so $J_{w'} = J_w$ and $I = J_w$. □

*Remark* 5.5.7. In particular this theorem shows that the Poisson algebra $A$ from example 5.4.15 cannot be endowed with a rational action satisfying Hypothesis 3.4.1 and such that the generators of $A$ are eigenvectors.

### 5.5.2 Stratification of P.Spec $(A)$

In this section we present another partition of $\text{P.Spec}(A)$ for $A \in \mathcal{P}$. This partition was introduced in [15, Section 4] for Poisson algebras endowed with torus actions. More precisely, let $r > 0$ and suppose that $B$ is a Poisson $\mathbb{K}$-algebra and that a torus $H = (\mathbb{K}^\times)^r$ acts on $B$ by Poisson automorphisms. For any $J \in H\text{-P.Spec}(B)$ of $B$ we set:

$$\text{P.Spec}_J(B) = \{P \in \text{P.Spec}(B) \mid (H : P) = J\},$$

and we obtain a partition:

$$\text{P.Spec}\,(B) = \bigsqcup_{J \in H\text{-P.Spec}\,(B)} \text{P.Spec}_J(B),$$

since for all $P \in \text{P.Spec}\,(B)$ we have $(H : P) \in H\text{-P.Spec}\,(B)$ (recall that $\operatorname{char} \mathbb{K} = 0$). This partition is called the *$H$-stratification* of $\text{P.Spec}\,(B)$ and each set $\text{P.Spec}_J(B)$ is called an *$H$-stratum.* For the remainder of this chapter we want to consider both the canonical partition and the $H$-stratification for a given algebra $A \in \mathcal{P}$. We gather the assumptions we need for this in the following list. For $A \in \mathcal{P}$ we suppose that:

- a torus $H = (\mathbb{K}^\times)^r$ is acting rationally by Poisson automorphisms on $A$ such that each $X_i$ is an $H$-eigenvector,
- $H$ satisfies Hypothesis 3.4.1,
- $\operatorname{char} \mathbb{K} = 0$.

Note that by Lemma 5.5.1 these assumptions imply that $H$ acts rationally by Poisson automorphisms on $C_j$ for all $2 \leq j \leq n+1$. Thus we can consider the $H$-strata $\text{P.Spec}_J(C_j)$ for all $J \in H\text{-P.Spec}\,(C_j)$.

### 5.5.3 Canonical partition and $H$-stratification

In this section we show that the canonical partition and the $H$-stratification are actually the same partition of $\text{P.Spec}\,(A)$.

**Lemma 5.5.8.** *Let $\varepsilon \in \{0, 1\}$, $2 \leq j \leq n$, and $J \in H\text{-P.Spec}\,(C_{j+1})$. If $J \in \Gamma_j^\varepsilon(C_{j+1})$, then:*

$$\text{P.Spec}_J(C_{j+1}) \subseteq \Gamma_j^\varepsilon(C_{j+1}).$$

*Similarly, if $J \in H\text{-P.Spec}\,(C_j) \cap \Gamma_j^\varepsilon(C_j)$, then:*

$$\text{P.Spec}_J(C_j) \subseteq \Gamma_j^\varepsilon(C_j).$$

*Proof.* Assume that $J \in H\text{-P.Spec}\,(C_{j+1})$ and recall that $U_j = X_{j,j+1}$. If $\varepsilon = 1$, we have $U_j \in J$ and if $P \in \text{P.Spec}_J(C_{j+1})$ we deduce that $U_j \in P$, since $J \subseteq P$. Assume that $\varepsilon = 0$, so that $U_j \notin J$, and suppose that there exists $P \in \text{P.Spec}_J(C_{j+1})$ such that $U_j \in P$. Then, since $U_j$ is an $H$-eigenvector we have $U_j \in (H : P) = J$, and we reach a contradiction. Thus $U_j \notin P$. We deal similarly with the second assertion of the lemma. $\square$

We obtain the following result on the image of a $H$-strata in $\mathrm{P.Spec}\,(C_{j+1})$ under the embedding $\varphi_j$.

**Lemma 5.5.9.** *Let $2 \leq j \leq n$ and $J \in H\text{-}\mathrm{P.Spec}\,(C_{j+1})$. Then:*

(1) $J' = \varphi_j(J) \in H\text{-}\mathrm{P.Spec}\,(C_j)$.

(2) $\varphi_j\big(\mathrm{P.Spec}_{\,J}(C_{j+1})\big) \subseteq \mathrm{P.Spec}_{\,J'}(C_j)$.

*Proof.* We obtain assertion (1) by Lemma 5.5.4 since for all $h \in H$ we have:

$$h(J') = h(\varphi_j(J)) = \varphi_j\big(h(J)\big) = \varphi_j(J) = J'.$$

Let $P \in \mathrm{P.Spec}_{\,J}(C_{j+1})$ and set $Q := \varphi_j(P)$. We have $(H : P) = J$ and we want to show that $(H : Q) = J'$. Let $\varepsilon \in \{0, 1\}$ such that $J \in \Gamma_j^{\varepsilon}(C_{j+1})$. Then we have $P \in \Gamma_j^{\varepsilon}(C_{j+1})$ by Lemma 5.5.8. So $Q$ and $J'$ belong to $Y_\varepsilon := \varphi_j\big(\Gamma_j^{\varepsilon}(C_{j+1})\big)$. Recall that $Y_\varepsilon$ is a closed subset of $\Gamma_j^{\varepsilon}(C_j)$ (Proposition 5.4.4), thus there exists an ideal $I \in C_j$ such that:

$$Y_\varepsilon = \{T \in \Gamma_j^{\varepsilon}(C_j) \mid T \supseteq I\}.$$

Since $J \subseteq P$ and $\varphi_j$ is increasing on $\Gamma_j^{\varepsilon}(C_{j+1})$, we have $J' \subseteq Q$. But $J'$ is $H$-invariant so we have $J' \subseteq (H : Q)$. We set $J'' := (H : Q)$. Since $Q \in \Gamma_j^{\varepsilon}(C_j)$, Lemma 5.5.8 shows that $J'' \in \Gamma_j^{\varepsilon}(C_j)$. Since $J' \in Y_\varepsilon$, we have $I \subseteq J' \subseteq J''$, so that $J'' \in Y_\varepsilon$. In particular $J'' \in \mathrm{Im}(\varphi_j)$ and we set $J''' := \varphi_j^{-1}(J'') \in \Gamma_j^{\varepsilon}(C_{j+1})$. For $h \in H$, we have $J' \subseteq J'' \subseteq h(Q)$ i.e.:

$$\varphi_j(J) \subseteq \varphi_j(J''') \subseteq \varphi_j(h(P)).$$

Recall that $J$ and $J'''$ belong to $\Gamma_j^{\varepsilon}(C_{j+1})$. Moreover $h(P) \in \Gamma_j^{\varepsilon}(C_{j+1})$ since $h(P) \in \mathrm{P.Spec}_{\,J}(C_{j+1})$ and $J \in \Gamma_j^{\varepsilon}(C_{j+1})$, see Lemma 5.5.8. By Proposition 5.4.4 we deduce that $J \subseteq J''' \subseteq h(P)$ for all $h \in H$. Therefore we have:

$$J \subseteq J''' \subseteq (H : P) = J,$$

so that $J''' = J$. Thus $\varphi_j(J''') = \varphi_j(J)$ and we obtain $(H : Q) = J'$ as desired. $\square$

The following lemma shows in particular that there are only finitely many ideals in $H$-P.Spec $(A)$.

**Lemma 5.5.10.** *Let $J \in H$-P.Spec $(A)$ and $w \in W'_P$ such that $J \in \text{P.Spec}_w(A)$. Then:*

(1) $\varphi(J) = J_w$.

(2) $\text{P.Spec}_J(A) \subseteq \text{P.Spec}_w(A)$.

*Proof.* By Lemma 5.5.5 we have:

$$\varphi(J) = \varphi\big(h(J)\big) = h\big(\varphi(J)\big)$$

for all $h \in H$, so $\varphi(J) \in H$-P.Spec $(\overline{A})$. Since moreover $\varphi(J) \in \text{P.Spec}_w(\overline{A})$ we obtain $\varphi(J) = J_w$ by Proposition 3.4.4. This proves assertion (1).

By Lemma 5.5.9 we have $\varphi(\text{P.Spec}_J(A)) \subseteq \text{P.Spec}_{\varphi(J)}(\overline{A}) = \text{P.Spec}_{J_w}(\overline{A})$, and assertion (2) will follow if we show that $\text{P.Spec}_{J_w}(\overline{A}) \subseteq \text{P.Spec}_w(\overline{A})$. Let $Q \in \text{P.Spec}_{J_w}(\overline{A})$. Then $J_w \subseteq Q$ and we have:

$$\{T_i \mid i \in w\} = J_w \cap \{T_1, \dots, T_n\} \subseteq Q \cap \{T_1, \dots, T_n\}.$$

Assume that there exists $i \in \overline{w} := [\![1, n]\!] \setminus w$ such that $T_i \in Q$. Then $T_i \in (H : Q) = J_w$ since $T_i$ is an $H$-eigenvector, a contradiction. Therefore $Q \in \text{P.Spec}_w(\overline{A})$ and assertion (2) is proved. □

We can thus conclude by the following theorem.

**Theorem 5.5.11.** *Let $A \in \mathcal{P}$ and suppose that a torus $H = (\mathbb{K}^\times)^r$ is acting rationally by Poisson automorphisms on $A$ such that each $X_i$ is an $H$-eigenvector. Moreover assume that $H$ satisfies Hypothesis 3.4.1 and that $\operatorname{char} \mathbb{K} = 0$. Then, the canonical partition:*

$$\text{P.Spec}\,(A) = \bigsqcup_{w \in W'_P} \text{P.Spec}_w(A)$$

*coincides with the $H$-stratification:*

$$\text{P.Spec}\,(A) = \bigsqcup_{J \in H\text{-P.Spec}\,(A)} \text{P.Spec}_J(A).$$

The canonical partition is independent of the choice of the torus $H$, and the $H$-stratification is independent of the expression of $A$ as an iterated Poisson-Ore extension, thus we obtain the following corollary.

**Corollary 5.5.12.** *With the hypotheses of Theorem 5.5.11, the $H$-stratification is independent of the choice of the torus $H$, and the canonical partition of the expression of $A$ as an iterated Poisson-Ore extension.*

# Chapter 6

# Poisson primitive spectrum

In this chapter we turn our attention to the Poisson primitive spectra of the algebras of the class $\mathcal{P}$. In particular the Poisson deleting derivations algorithm allows us to prove the Poisson Dixmier-Moeglin equivalence for these algebras when $\text{char}\, \mathbb{K} = 0$. We first recall what is the Poisson Dixmier-Moeglin equivalence in Section 6.1. Then in Section 6.2, we give the proof of this equivalence for the Poisson algebras of the class $\mathcal{P}$. Even better, we show that the Poisson primitive ideals are exactly the Poisson prime ideals that are maximal in their strata. Finally in Section 6.3 we prove a transfer result for the Poisson Dixmier-Moeglin equivalence of Poisson-Ore extensions. For, we first need to generalise some results of Chapter 5. In all this chapter we assume that $\text{char}\, \mathbb{K} = 0$.

## 6.1 Poisson Dixmier-Moeglin equivalence

In representation theory one often wants to classify the simple modules of a given algebra. It is usually a difficult problem and one can focus first on studying their annihilators, the so-called *primitive ideals*. Dixmier [10] and Moeglin [30] studied these ideals for enveloping algebras of finite dimensional Lie algebras. They gave algebraic and topological characterisations for primitive ideals in these algebras. Let $R$ be a noetherian algebra. A prime ideal $P$ of $R$ is said *rational* provided that the field $Z(\text{Frac}\, R/P)$ is algebraic over the ground field. The ideal $P$ is said to be locally closed if the point $\{P\}$ is locally closed in $\text{Spec}\,(R)$ (with respect to the Zariski topology). Dixmier and Moeglin showed that for all finite dimensional complex Lie algebras $\mathfrak{g}$, the set of primitive ideals, rational ideals and locally closed ideals of $U(\mathfrak{g})$ are equal. More generally, we say that the *Dixmier-Moeglin equivalence* holds for a given noetherian algebra if the sets of primitive ideals, locally closed ideals and rational ideals coincide. This equivalence has been proved for several families of algebras such as quantised coordinate rings, see [21] for instance, or twisted coordinate rings [4].

A Poisson version of the Dixmier-Moeglin equivalence for Poisson algebra is investigated in [32], [15] or [3] for instance. Let $A$ be a Poisson $\mathbb{K}$-algebra and $P \in \text{P.Spec}\,(A)$. The ideal $P$ is said *locally closed* if the point $\{P\}$ is a locally closed point of $\text{P.Spec}\,(A)$. Recall that the Poisson centre of a Poisson algebra $B$ is the Poisson subalgebra $Z_P(B) := \{a \in B \mid \{a, -\} \equiv 0\}$. The ideal $P$ is said *Poisson rational* provided the field $Z_P\big(\text{Frac}\,(A/P)\big)$ is algebraic over the ground field $\mathbb{K}$. For $J$ an ideal of $A$, there is a largest Poisson ideal contained in $J$ that is called the *Poisson core* of $J$. Poisson cores of maximal ideals of $A$ are called *Poisson primitive ideals.* As recalled in [16], for a solvable finite dimensional complex Lie algebra $\mathfrak{g}$, the set of primitive ideals of $U(\mathfrak{g})$ is homeomorphic to the set of Poisson primitive ideals in $S(\mathfrak{g})$, the symmetric ideal of $\mathfrak{g}$ endowed with the Kirillov-Kostant-Souriau Poisson bracket (see 4 of Examples 1.1.3). Moreover it is shown in [31, Corollary 8] that the annihilator of a simple Poisson module is a Poisson primitive ideal. We say that the Poisson Dixmier-Moeglin equivalence holds for the Poisson algebra $A$ if the following sets coincide:

(1) the set of Poisson primitive ideals;
(2) the set of locally closed Poisson ideals;
(3) the set of Poisson rational ideals.

It is shown in [32] that we have the inclusions $(2) \subseteq (1) \subseteq (3)$ for all finitely generated Poisson algebras over a base field of characteristic zero. However the inclusion $(3) \subseteq (2)$ is not always satisfied as there exist counter-examples in all Krull dimension $d \geq 4$, see [3]. The Poisson Dixmier-Moeglin equivalence holds for several families of Poisson algebras such as Poisson tori [32], Poisson algebras supporting torus action [15], [18] and Poisson algebras with generalised Jacobian Poisson structures [26] for instance. In the next section we show that the algebras of the the class $\mathcal{P}$ also satisfy the Poisson Dixmier-Moeglin equivalence.

## 6.2 Poisson Dixmier-Moeglin equivalence for the algebras of the class $\mathcal{P}$

It is known that Poisson affine spaces satisfy the Poisson Dixmier-Moeglin equivalence, see [15, Example 4.6] for instance. In this section this fact together with the canonical embedding will allow us to prove the Poisson Dixmier-Moeglin equivalence for all algebras of the class $\mathcal{P}$. All algebras of the class $\mathcal{P}$ are finitely generated, therefore it only remains to show the Poisson rational ideals of $A \in \mathcal{P}$ are also locally closed. We will freely continue to use the notation of Chapter 5. Recall that for an ideal $I$ of a Poisson algebra $A$ we set:

$$V_P(I) = \{Q \in \text{P.Spec}\,(A) \mid Q \supseteq I\} \quad \text{and} \quad W_P(I) = \{Q \in \text{P.Spec}\,(A) \mid Q \not\supseteq I\}.$$

The following lemma is a Poisson version of [6, Lemma II.7.7].

**Lemma 6.2.1.** *Let $A$ be a Poisson algebra and $P \in \text{P.Spec}\,(A)$. Then $P$ is locally closed if and only if the intersection of all the Poisson prime ideals properly containing $P$ is an ideal properly containing $P$.*

*Proof.* Let $\mathcal{I}$ be the intersection of all the Poisson prime ideals of $A$ properly containing $P$. If $P \subsetneq \mathcal{I}$, then $W(\mathcal{I}) \cap V(P) = \{P\}$, i.e. $\{P\}$ is a locally closed point of P.Spec $(A)$. Conversely, if $P$ is locally closed, then there are ideals $I$ and $L$ in $A$ such that $V(I) \cap W(L) = \{P\}$. Let $Q \in \text{P.Spec}\,(A)$ such that $P \subsetneq Q$. Then $Q \in V(I)$ and so $Q \notin W(L)$, i.e. $L \subseteq Q$ and $L \subseteq \mathcal{I}$. We conclude that $P \subsetneq L + P \subseteq \mathcal{I}$. $\square$

Hence $P$ is locally closed if and only if the intersection of all non trivial Poisson prime ideals in $A/P$ is non trivial.

**Proposition 6.2.2.** *Let $A \in \mathcal{P}$. Then Poisson rational ideals of $A$ are Poisson locally closed ideals.*

*Proof.* Recall that by applying the Poisson deleting derivations algorithm to the Poisson algebra $A$ we get a sequence of Poisson algebras $C_j$ where $j$ runs from $n+1$ to 2 such that $C_{n+1} = A$ and $C_2 = \overline{A}$ is a Poisson affine space. We will show by an increasing induction on $j$ that all Poisson rational ideals of $C_j$ are locally closed. When $j = 2$ the algebra $\overline{A}$ is a Poisson affine space and the result comes from [15, Example 4.6]. Assume that for some

$2 \leq j \leq n$ the Poisson rational ideals of $C_j$ are locally closed. Let $P \in \text{P.Spec}\,(C_{j+1})$ be a Poisson rational ideal. We distinguish between two cases: either $U_j \in P$, or $U_j \notin P$.

Case 1: If $U_j \in P$, then by Lemma 5.4.18 we get a Poisson algebra isomorphism between $C_{j+1}/P$ and $C_j/\varphi_j(P)$. Thus we get:

$$Z_P\Big(\text{Frac}\,\Big(\frac{C_{j+1}}{P}\Big)\Big) \cong Z_P\Big(\text{Frac}\,\Big(\frac{C_j}{\varphi_j(P)}\Big)\Big),$$

and the Poisson prime ideal $\varphi_j(P)$ is rational in $C_j$ since by assumption $P$ is rational in $C_{j+1}$. Therefore by the induction hypothesis $\varphi_j(P)$ is locally closed in $C_j$. By Lemma 6.2.1 we can write:

$$\bigcap_{Q \neq 0} Q \neq \{0\}, \qquad \text{where} \qquad Q \in \text{P.Spec}\,\big(C_j/\varphi_j(P)\big),$$

or equivalently using the isomorphism $C_{j+1}/P \cong C_j/\varphi_j(P)$:

$$\bigcap_{Q \neq 0} Q \neq \{0\}, \qquad \text{where} \qquad Q \in \text{P.Spec}\,\big(C_{j+1}/P\big),$$

which means that $P$ is locally closed in $C_{j+1}$.

Case 2: If $U_j \notin P$, then by Lemma 5.4.19 we get the equality:

$$C_j S_j^{-1}/Q S_j^{-1} = C_{j+1} S_j^{-1}/P S_j^{-1},$$

which leads to the isomorphism:

$$Z_P\Big(\text{Frac}\,\Big(\frac{C_{j+1}}{P}\Big)\Big) \cong Z_P\Big(\text{Frac}\,\Big(\frac{C_j}{\varphi_j(P)}\Big)\Big).$$

Therefore $\varphi_j(P) \in \text{P.Spec}\,(C_j)$ is Poisson rational, and so is locally closed. We now introduce a few pieces of notation:

$$\begin{aligned}
\mathcal{F}_j^0 &:= \{Q \in \text{P.Spec}\,(C_j) \mid \varphi_j(P) \subsetneq Q \text{ and } V_j \notin Q\},\\
\mathcal{F}_j^1 &:= \{Q \in \text{P.Spec}\,(C_j) \mid \varphi_j(P) \subsetneq Q \text{ and } V_j \in Q\},\\
\mathcal{F}_{j+1}^0 &:= \{Q \in \text{P.Spec}\,(C_{j+1}) \mid P \subsetneq Q \text{ and } U_j \notin Q\},\\
\mathcal{F}_{j+1}^1 &:= \{Q \in \text{P.Spec}\,(C_{j+1}) \mid P \subsetneq Q \text{ and } U_j \in Q\},
\end{aligned}$$

$$\mathcal{T}_j^0 := \bigcap_{Q \in \mathcal{F}_j^0} Q, \quad \mathcal{T}_j^1 := \bigcap_{Q \in \mathcal{F}_j^1} Q, \quad \mathcal{T}_{j+1}^0 := \bigcap_{Q \in \mathcal{F}_{j+1}^0} Q, \quad \text{and} \quad \mathcal{T}_{j+1}^1 := \bigcap_{Q \in \mathcal{F}_{j+1}^1} Q.$$

Let $\mathcal{I}$ be the intersection of all the Poisson prime ideals of $C_{j+1}$ properly containing $P$. We have:

$$\Big(P \text{ locally closed}\Big) \iff \Big(P \subsetneq \mathcal{I}\Big) \iff \Big(P \subsetneq (\mathcal{T}_{j+1}^0 \cap \mathcal{T}_{j+1}^1)\Big). \tag{6.1}$$

By the induction hypothesis we have:

$$\varphi_j(P) \subsetneq \left(\mathcal{T}_j^0 \cap \mathcal{T}_j^1\right) \quad \text{so that} \quad \varphi_j(P) = PS_j^{-1} \cap C_j \subsetneq \mathcal{T}_j^0.$$

Since the map $\varphi_j$ restricts to an homeomorphism from $\mathcal{F}_{j+1}^0$ to $\mathcal{F}_j^0$ we have:

$$\varphi_j(P) \subsetneq \mathcal{T}_j^0 \iff P \subsetneq \mathcal{T}_{j+1}^0.$$

Therefore there exists $a \in \left(\mathcal{T}_{j+1}^0 \setminus P\right)$. Moreover by definition we have $U_j \in \left(\mathcal{T}_{j+1}^1 \setminus P\right)$. Since $P$ is a prime ideal and $a, U_j \notin P$ it clear that:

$$aU_j \in \Big(\mathcal{T}_{j+1}^0 \cap \mathcal{T}_{j+1}^1 \setminus P\Big),$$

and by (6.1) we obtain that $P$ is locally closed. This concludes the induction. The case $j = n$ gives us the result for $C_{n+1} = A$. $\square$

We are now ready to state the main results of this section.

**Theorem 6.2.3.** *Let $A \in \mathcal{P}$. Then $A$ satisfies the Poisson Dixmier-Moeglin equivalence.*

We deduce the following corollary which links the Poisson primitive ideals of $A$ with the Poisson primitive ideals of $\overline{A}$.

**Corollary 6.2.4.** *Let $A \in \mathcal{P}$. Then for all $P \in \mathrm{P.Spec}\,(A)$ we have the following equivalence:*

$$P \textit{ is Poisson primitive in } A \iff \varphi(P) \textit{ is Poisson primitive in } \overline{A}.$$

We can also describe the primitive ideals of $A \in \mathcal{P}$ inside their strata, namely they are exactly the maximal ideals in their respective strata.

**Proposition 6.2.5.** *Let $A \in \mathcal{P}$. Suppose that $w \in W'_P$ and let $P \in \mathrm{P.Spec}_w(A)$. Then:*

$$P \text{ is Poisson primitive} \iff P \text{ is maximal in } \mathrm{P.Spec}_w(A).$$

*Proof.* First suppose that $P$ is a Poisson primitive ideal. Then $\varphi(P) \in \mathrm{P.Spec}_w(\overline{A})$ is Poisson primitive in $\overline{A}$ by Corollary 6.2.4. By [15, Theorem 4.3, Example 4.6], $\varphi(P)$ is maximal in $\mathrm{P.Spec}_w(\overline{A})$. Now let $P' \in \mathrm{P.Spec}_w(A)$ be such that $P \subseteq P'$. Since $\varphi$ induces an homeomorphism from $\mathrm{P.Spec}_w(A)$ to $\varphi(\mathrm{P.Spec}_w(A)) \subseteq \mathrm{P.Spec}_w(\overline{A})$, we have $\varphi(P) \subseteq \varphi(P')$ inside $\mathrm{P.Spec}_w(\overline{A})$. By maximality of $\varphi(P)$ we get $\varphi(P) = \varphi(P')$, i.e. $P = P'$, and $P$ is maximal in $\mathrm{P.Spec}_w(A)$.

Conversely, suppose that $P$ is maximal in $\mathrm{P.Spec}_w(A)$. Then $\varphi(P)$ is maximal in $\varphi\big(\mathrm{P.Spec}_w(A)\big)$ by Theorem 5.4.14. Recall that $\varphi\big(\mathrm{P.Spec}_w(A)\big) \subseteq \mathrm{P.Spec}_w(\overline{A})$ by Theorem 5.4.14, and let $Q \in \mathrm{P.Spec}_w(\overline{A})$ such that $\varphi(P) \subseteq Q$. By Proposition 5.4.16 we have $Q \in \mathrm{Im}(\varphi)$, i.e. $Q \in \varphi\big(\mathrm{P.Spec}_w(A)\big)$ and by maximality of $\varphi(P)$ in $\varphi\big(\mathrm{P.Spec}_w(A)\big)$ we have $Q = \varphi(P)$. Therefore $\varphi(P)$ is maximal in $\mathrm{P.Spec}_w(\overline{A})$. By [15, Theorem 4.3, Example 4.6] this shows that $\varphi(P)$ is Poisson primitive in $\overline{A}$. We conclude by Corollary 6.2.4 that $P$ is Poisson primitive in $A$. □

*Example* 6.2.6. The algebra $A$ of Example 5.4.15 satisfies the Poisson Dixmier-Moeglin equivalence. Note that this algebra is not covered by [15, Theorem 4.3].

## 6.3 A transfer result for Poisson-Ore extensions

In this section we prove a transfer result for Poisson-Ore extensions. More precisely, we show that, under certain assumptions on $\delta$, if the Poisson-Ore extension $A[X;\alpha]_P$ satisfies the Poisson Dixmier-Moeglin equivalence, then the Poisson-Ore extension $A[X;\alpha,\delta]_P$ satisfies the Poisson Dixmier-Moeglin equivalence. We first need to extend some results of Section 5.4. We will construct a canonical embedding for a Poisson-Ore extension $B := A[X;\alpha,\delta]_P$ which can be "deleted" (i.e. satisfying the assumptions of Theorem 2.2.2).

### 6.3.1 Poisson deleting derivation homomorphism and canonical embedding for $A[X;\alpha,\delta]_P$

We recall the Poisson deleting derivation homomorphism in characteristic zero.

**Theorem 6.3.1.** [18, Theorem 3.7], or see also [Theorem 2.2.2]. *Let $B := A[X;\alpha,\delta]_P$ be a Poisson-Ore extension, where $A$ is a Poisson $\mathbb{K}$-algebra. Suppose that $[\delta,\alpha]=\eta\delta$ for some nonzero scalar $\eta$. Then there is a Poisson algebra isomorphism from $A[Y^{\pm1};\alpha]_P$ to $A[X^{\pm1};\alpha,\delta]_P$ such that:*

$$\theta(a)=\sum_{i\geq 0}\frac{1}{\eta^i}\frac{\delta^i(a)}{i!}X^{-i},$$

*and $\theta(Y)=X$.*

Set $S := \{X^i \mid i \geq 0\}$, so that $BS^{-1} = A[X^{\pm1};\alpha,\delta]_P$. Note that we define a Poisson derivation of $A' := \theta(A)$ by setting $\alpha' := \theta\circ\alpha\circ\theta^{-1}$.

**Corollary 6.3.2.** *The Poisson subalgebra $\overline{B} := \theta(A[Y;\alpha]_P) = \theta(A)[X;\alpha']_P$ of $BS^{-1}$ is Poisson isomorphic to $A[Y;\alpha]_P$, and we have $\overline{B}S^{-1} = BS^{-1}$.*

We will now define an embedding from P.Spec $(B)$ to P.Spec $(\overline{B})$. First, we partition P.Spec $(\overline{B})$ and P.Spec $(B)$ by the following sets:

$$\Gamma^0(\overline{B}) = \{P \in \text{P.Spec}\,(\overline{B}) \mid X \notin P\}, \qquad \Gamma^1(\overline{B}) = \{P \in \text{P.Spec}\,(\overline{B}) \mid X \in P\},$$
$$\Gamma^0(B) = \{P \in \text{P.Spec}\,(B) \mid X \notin P\}, \qquad \Gamma^1(B) = \{P \in \text{P.Spec}\,(B) \mid X \in P\}.$$

Since $\overline{B}S^{-1} = BS^{-1}$, contraction and extension of ideals provide bijections between $\Gamma^0(\overline{B})$ and $\Gamma^0(B)$.

**Lemma 6.3.3.** *There is an homeomorphism $\varphi^0 : \Gamma^0(B) \to \Gamma^0(\overline{B})$ given by $\varphi^0(P) := PS^{-1}\cap\overline{B}$ for $P \in \Gamma^0(B)$. Its inverse is defined by $(\varphi^0)^{-1}(Q) := QS^{-1}\cap B$ for $Q \in \Gamma^0(\overline{B})$.*

We denote by $\langle X\rangle_P$ the smallest Poisson ideal of $B$ containing $X$ and by $\overline{a}$ the image of $a \in A$ in the Poisson algebra $B/\langle X\rangle_P$.

**Lemma 6.3.4.** *There is a surjective Poisson algebra homomorphism $g : \overline{B} \to B/\langle X\rangle_P$.*

*Proof.* First notice that there is a Poisson algebra homomorphism $f$ from $\theta(A)$ to $B/\langle X\rangle_P$ given by $f\big(\theta(a)\big) := \overline{a}$ for all $a \in A$. Now since:

$$\{\overline{X}, f\big(\theta(a)\big)\} = \{\overline{X}, \overline{a}\} = \overline{\alpha(a)X} = f\big(\alpha'(\theta(a))\big)\overline{X}$$

for all $a \in A$, the Poisson algebra homomorphism $f$ extends by universal property of Poisson-Ore extension (see Proposition 1.1.16) to a Poisson algebra homomorphism $g$ from $\overline{B} = \theta(A)[Y;\alpha']_P$ to $B/\langle X\rangle_P$, sending $Y$ to $\overline{X}$. The map $g$ is clearly surjective. □

Set $N := \ker(g)$. Then there is an homeomorphism:

$$\varphi^1 : \Gamma^1_j(B) \to \{P \in \text{P.Spec}\,(\overline{B}) \mid N \subseteq P\}$$

defined by $\varphi^1(P) := g^{-1}(P/\langle X\rangle_P)$ for $P \in \Gamma^1(B)$. Since $X \in N$ we have:

$$\{P \in \text{P.Spec}\,(\overline{B}) \mid N \subseteq P\} \subseteq \Gamma^1(\overline{B}).$$

Therefore we obtain the following lemma.

**Lemma 6.3.5.** *There is an increasing and injective map $\varphi^1 : \Gamma^1(B) \to \Gamma^1(\overline{B})$ defined by $\varphi^1(P) = g^{-1}(P/\langle X\rangle_P)$ for $P \in \Gamma^1(B)$, which induces an homeomorphism on its image.*

We define a map $\varphi : \text{P.Spec}\,(B) \to \text{P.Spec}\,(\overline{B})$ by setting:

$$\varphi(P) = \begin{cases} \varphi^0(P) \text{ if } X \notin P, \\ \varphi^1(P) \text{ if } X \in P. \end{cases}$$

We have:

**Proposition 6.3.6.** *The map $\varphi : \text{P.Spec}\,(B) \to \text{P.Spec}\,(\overline{B})$ is injective. For $\varepsilon \in \{0,1\}$, the map $\varphi$ induces an homeomorphism from $\Gamma^\varepsilon(B)$ to $\varphi\big(\Gamma^\varepsilon(B)\big)$ which is a closed subset of $\Gamma^\varepsilon(\overline{B})$.*

We now study the behaviour of the Poisson prime quotients of a Poisson-Ore extension $B = A[X;\alpha,\delta]_P$ under the deleting derivation homomorphism. Let $P \in \text{P.Spec}\,(B)$ and set $Q := \varphi(P) \in \text{P.Spec}\,(\overline{B})$.

**Lemma 6.3.7.** *If $X \in P$, then there is a Poisson algebra isomorphism between $\overline{B}/Q$ and $B/P$.*

*Proof.* Since $X \in P$, we have $P \in \Gamma^1(B)$ and $Q = g^{-1}(P/\langle X \rangle_P)$. Thus there is a surjective Poisson algebra homomorphism:

$$\overline{B} \longrightarrow \frac{B}{\langle X \rangle_P} \longrightarrow \frac{B/\langle X \rangle_P}{P/\langle X \rangle_P} \cong B/P,$$

whose kernel is $Q$. □

We set $\overline{S} := \{X^i + P \mid i \geq 0\}$ and $\overline{S'} := \{X^i + Q \mid i \geq 0\}$.

**Lemma 6.3.8.** *If $X \notin P$, then there is a Poisson algebra isomorphism $\big(\overline{B}/Q\big)\overline{S'}^{-1} \cong \big(B/P\big)\overline{S}^{-1}$.*

*Proof.* By assumption $P \in \Gamma^0(B)$, so $QS^{-1} = PS^{-1}$ is an ideal in $\overline{B}S^{-1} = BS^{-1}$. Thus:

$$\big(\overline{B}/Q\big)\overline{S}'^{-1} \cong \frac{\overline{B}S^{-1}}{QS^{-1}} = \frac{BS^{-1}}{PS^{-1}} \cong \big(B/P\big)\overline{S}^{-1}.$$

Both isomorphisms come from the fact that $P \cap S = Q \cap S = \emptyset$. □

### 6.3.2 Transfer result

We now prove our transfer result for Poisson-Ore extensions. The proof is essentially the same as the proof of Proposition 6.2.2.

**Theorem 6.3.9.** *Let $A$ be a finitely generated Poisson $\mathbb{K}$-algebra, $\alpha \in \mathrm{Der}_P(A)$ and $\delta$ be a locally nilpotent Poisson $\alpha$-derivation such that $\delta\alpha - \alpha\delta = \eta\delta$ for some nonzero scalar $\eta$. If the Poisson-Ore extension $A[X;\alpha]_P$ satisfies the Poisson Dixmier-Moeglin equivalence, then the Poisson-Ore extension $A[X;\alpha,\delta]_P$ satisfies the Poisson Dixmier-Moeglin equivalence.*

*Proof.* Since $A$ is finitely generated, $B =: A[X;\alpha,\delta]_P$ is finitely generated. Thus, by [32], locally closed Poisson ideals of $B$ are Poisson primitive ideals, and Poisson primitive ideals are Poisson rational ideals. Therefore it only remains to show that Poisson rational ideals of $B$ are locally closed Poisson ideals. Note that since the algebra $\overline{B} = \theta(A)[X;\alpha']_P$ is Poisson isomorphic to $A[X;\alpha]_P$, the assumption of the theorem tells us that $\overline{B}$ satisfies the Poisson Dixmier-Moeglin equivalence. Let $P \in \text{P.Spec}\,(B)$ be a Poisson rational ideal. Recall that $P$ is locally closed if and only if the intersection of all non trivial Poisson prime ideals in $B/P$ is non trivial. We distinguish between two cases: either $X \in P$, or $X \notin P$.

Case 1: If $X \in P$, then by Lemma 6.3.7 we get a Poisson algebra isomorphism between $B/P$ and $\overline{B}/\varphi(P)$, and the result follows.

Case 2: If $X \notin P$, then by Lemma 6.3.8 we obtain the isomorphism:

$$Z_P\Big(\text{Frac}\left(\frac{B}{P}\right)\Big) \cong Z_P\Big(\text{Frac}\left(\frac{\overline{B}}{\varphi(P)}\right)\Big).$$

Therefore $\varphi(P) \in \text{P.Spec}\,(\overline{B})$ is Poisson rational, and so is locally closed. We set:

$$\begin{aligned}
\mathcal{F}^0 &:= \{Q \in \text{P.Spec}\,(\overline{B}) \mid \varphi(P) \subsetneq Q \text{ and } X \notin Q\},\\
\mathcal{F}^1 &:= \{Q \in \text{P.Spec}\,(\overline{B}) \mid \varphi(P) \subsetneq Q \text{ and } X \in Q\},\\
\mathcal{H}^0 &:= \{Q \in \text{P.Spec}\,(B) \mid P \subsetneq Q \text{ and } X \notin Q\},\\
\mathcal{H}^1 &:= \{Q \in \text{P.Spec}\,(B) \mid P \subsetneq Q \text{ and } X \in Q\},
\end{aligned}$$

$$\mathcal{T}^0 := \bigcap_{Q\in\mathcal{F}^0} Q, \quad \mathcal{T}^1 := \bigcap_{Q\in\mathcal{F}^1} Q, \quad \mathcal{R}^0 := \bigcap_{Q\in\mathcal{H}^0} Q, \quad \text{and} \quad \mathcal{R}^1 := \bigcap_{Q\in\mathcal{H}^1} Q.$$

Let $\mathcal{I}$ be the intersection of all the Poisson prime ideals of $B$ properly containing $P$. We have:

$$\Big(P \text{ locally closed}\Big) \iff \Big(P \subsetneq \mathcal{I}\Big) \iff \Big(P \subsetneq (\mathcal{R}^0 \cap \mathcal{R}^1)\Big). \tag{6.2}$$

By the induction hypothesis we have $\varphi(P) \subsetneq \big(\mathcal{T}^0 \cap \mathcal{T}^1\big)$, so that $\varphi(P) \subsetneq \mathcal{T}^0$. Since the map $\varphi$ restricts to an homeomorphism from $\mathcal{H}^0$ to $\mathcal{F}^0$ we obtain:

$$\varphi(P) \subsetneq \mathcal{T}^0 \iff P \subsetneq \mathcal{R}^0.$$

Therefore there exists $a \in \big(\mathcal{R}^0 \setminus P\big)$. Moreover by definition we have $X \in \big(\mathcal{R}^1 \setminus P\big)$. Since $P$ is a prime ideal and $a, X \notin P$ it clear that:

$$aX \in \Big(\mathcal{R}^0 \cap \mathcal{R}^1 \setminus P\Big),$$

and by (6.2) we obtain that $P$ is locally closed. □

We also get for free the following corollary.

**Corollary 6.3.10.** *For all $P \in \mathrm{P.Spec}\,(B)$ we have the following equivalence:*

$$P \text{ is Poisson primitive in } B \iff \varphi(P) \text{ is Poisson primitive in } \overline{B}.$$

# Chapter 7

# Comparison of Spectra and Poisson spectra

In this chapter we study the link between the spectrum of a quantum algebra and the Poisson spectrum of its Poisson analogue. More precisely, Cauchon [8] uses his deleting derivations algorithm to obtain information on the spectra of the algebras of a class $\mathcal{Q}$ of iterated Ore extensions (i.e. the algebras satisfying the hypotheses of [8, Section 3.1]). Often the algebras of the class $\mathcal{Q}$ are deformations of Poisson algebras of the class $\mathcal{P}$. In the algebraic deformation context we can see both a given algebra $R$ of the class $\mathcal{Q}$ and its Poisson analogue $A$ as quotients of the same iterated Ore extension. The goal of this chapter is to compare the spectrum of $R$ with the Poisson spectrum of $A$ in such a situation. We now make precise the context we will work with in this chapter.

## 7.1 A class of iterated Ore extensions and a question

Suppose that $R_t$ is an iterated Ore extension over $\mathbb{K}[t^{\pm 1}]$:

$$R_t = \mathbb{K}[t^{\pm 1}][x_1][x_2; \sigma_2, \Delta_2] \cdots [x_n; \sigma_n, \Delta_n],$$

such that for all $2 \leq i \leq n$:

- $R_t^{<i}$ denotes the subalgebra of $R_t$ generated by $t^{\pm 1}, x_1, \dots, x_{i-1}$,

- $\sigma_i$ is the automorphism of $R_t^{<i}$ such that $\sigma_i(t) = t$ and $\sigma_i(x_j) = t^{\lambda_{ij}} x_j$ for all $1 \leq j < i$, where the $\lambda_{ij}$s are integers,
- $\Delta_i$ is a locally nilpotent $\sigma_i$-derivation of $R_t^{<i}$ such that $\Delta_i(t) = 0$,
- $\sigma_i \Delta_i = t^{\eta_i} \Delta_i \sigma_i$ for some integer $\eta_i \in \mathbb{K}^\times$,
- $\Delta_i^k(R_t^{<i}) \subseteq (t-1)^k (k)!_{t^{\eta_i}} R_t^{<i}$ for all $k \geq 0$,
- $A := R_t/(t-1)R_t$ is commutative.

*Notation.* We denote by $\mathcal{R}$ the class of algebra satisfying such hypotheses.

Let $R_t \in \mathcal{R}$. For a non root of unity $q \in \mathbb{K}^\times$ the element $t - q$ is central in $R_t$ and we denote by $R_q$ the quotient algebra $R_q := R_t/(t-q)R_t$. Recall from [8, Section 3.1] that an iterated Ore extension:

$$R = \mathbb{K}[x_1][x_2; \sigma_2, \Delta_2] \cdots [x_n; \sigma_n, \Delta_n],$$

belongs to the class $\mathcal{Q}$ if for all $1 < i \leq n$:

- $\sigma_i$ is an automorphism of the appropriate subalgebra such that $\sigma_i(x_j) = q_{ij} x_j$ with $q_{ij} \in \mathbb{K}^\times$ for all $1 \leq j < i$,
- $\Delta_i$ is a locally nilpotent $\sigma_i$-derivation of the appropriate subalgebra,
- $\sigma_i \Delta_i = l_i \Delta_i \sigma_i$ for a non root of unity $l_i \in \mathbb{K}^\times$.

Thus it is clear that $R_q \in \mathcal{Q}$. Moreover the algebra $A = R_t/(t-1)R_t$ is a Poisson algebra which belongs to the class $\mathcal{P}$ thanks to Theorem 4.1.3. In the language of Section 1.2 the algebra $R_q$ is a deformation of the Poisson algebra $A$, and $A$ is the semiclassical limit of the algebra $R_t$ at $t-1$.

Fix a non root of unity $q \in \mathbb{K}^\times$ and consider the algebra $R_q$. Since $R_q \in \mathcal{Q}$, Cauchon's deleting derivations algorithm can be applied. In particular we obtain a partition of the spectrum Spec$(R_q)$ of $R_q$ indexed by a subset $W'$ of $W = \mathscr{P}([\![1, n]\!])$ (see [8, Notation 4.4.1]). On the other hand the Poisson algebra $A$ belongs to the class $\mathcal{P}$ and, as shown in Section 5.4.2, we obtain a partition of the Poisson spectrum P.Spec$(A)$ of $A$ indexed by a subset $W'_P$ of $W$. Recall that the set $W'$ is the set of Cauchon diagrams of $R_q$ and $W'_P$ is the set of Cauchon diagrams of $A$. As it is often the case in deformation-quantisation theory that quantum objects and their Poisson analogues share similarities, it is natural to ask whether or not these sets coincide.

**Question 7.1.1.** Let $R_t \in \mathcal{R}$ and assume that $\operatorname{char} \mathbb{K} = 0$. Do the sets $W'$ and $W'_P$ coincide?

In Section 7.2 we answer positively Question 7.1.1 for two examples in small dimensions. The general strategy is the same: in both situations we describe explicitly the sets $W'$ and $W'_P$. Firstly, using Lemma 5.4.7 it is possible to exclude some elements of $W$. Then we exhibit ideals belonging to the remaining strata. The major difficulty, especially in the second example, is that at some point we have to check the primality of some given ideals. This is doable here because of the small dimensions we are working with, but this won't generalise in higher dimensions, especially when the algorithm involves several steps. A third example is given in Appendix B.

In Section 7.3 we answer positively Question 7.1.1 for the algebra of $m \times p$ quantum matrices over $\mathbb{K}[t^{\pm 1}]$, that we denote by $R_t := \mathcal{O}_t\big(M_{m,p}(\mathbb{K}[t^{\pm 1}])\big)$. It is well known that the algebra $R_t$ belongs to the class $\mathcal{R}$ (the details are given in Section 4.2.1). Cauchon gave a combinatorial description of the set $W'$ in [9, Théorème 3.2.1]. More precisely there is a bijection between $W'$ and the set $\mathcal{G}$ consisting of all $m \times p$ grids with black or white boxes satisfying the property that if a box is black, then every box strictly to its left is black or every box strictly above it is black. We will show that, when $\operatorname{char} \mathbb{K} \neq 2$, the set $W'_P$ of Cauchon diagrams for the matrix Poisson variety $A = \mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$ is also in bijection with the set $\mathcal{G}$ answering positively Question 7.1.1 for $\mathcal{O}_t\big(M_{m,p}(\mathbb{K}[t^{\pm 1}])\big)$.

When $\operatorname{char} \mathbb{K} = 2$ the Poisson algebra $A = \mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$ is a Poisson affine space. In that case, the Poisson deleting derivations algorithm is trivial and we have $W'_P = W$. However $W' \subsetneq W$ since the deformation parameter $q$ is not a root of unity. To generalise Question 7.1.1 to positive characteristic one would like to compare the Cauchon diagrams of $A$ with the Cauchon diagrams of $R_q$ when $q$ is a root of unity. For instance, in $\operatorname{char} \mathbb{K} = 2$, one would like to compare $W'_P$ with the set $W'$ obtained by taking $q$ to be a primitive second root of unity. However there is no primitive second root of unity in a field of characteristic 2. More generally, in a field of characteristic $p > 0$ there is no primitive root of unity of order divisible by $p$. This shows that Question 7.1.1 cannot be directly adapted to the positive characteristic case.

## 7.2 Examples

In this section we answer positively Question 7.1.1 for two examples. We suppose that $\operatorname{char}\mathbb{K} = 0$.

### 7.2.1 The quantum-Weyl algebra and its Poisson analogue

We recall from Example 4.1.2 the Ore extension $R_t = \mathbb{K}[t^{\pm 1}][x][y;\sigma,\Delta]$ where $\sigma$ is the automorphism of $\mathbb{K}[t^{\pm 1}][x]$ such that $\sigma(x) = tx$ and $\sigma(t) = t$, and $\Delta$ is the $\sigma$-derivation of $\mathbb{K}[t^{\pm 1}][x]$ such that $\Delta(x) = t - 1$ and $\Delta(t) = 0$. In particular we have:

- $yx = txy + t - 1$,
- $\Delta\sigma = t\sigma\Delta$,
- $\Delta^k(x) \in (t-1)^k (k)!_t \mathbb{K}[t^{\pm 1}, x]$.

Thus $R_t \in \mathcal{R}$. We first consider the Poisson algebra $A := R_t/(t-1)R_t \in \mathcal{P}$, then the algebra $R_q := R_t/(t-q)R_t \in \mathcal{Q}$ for a non root of unity $q \in \mathbb{K}^\times$. In this small example we will actually be able to describe explicitly all the elements of the non empty strata of $\operatorname{Spec}(R_q)$ and $\operatorname{P.Spec}(A)$. We require the field $\mathbb{K}$ to be algebraically closed.

#### 7.2.1.1 Poisson Spectrum of $A$

We have $A = \mathbb{K}[X][Y;\alpha,\delta]_P$ with $\alpha := X\partial_X$ and $\delta := \partial_X$. The Poisson deleting derivations algorithm returns (after one step) the Poisson affine space $\overline{A} = \mathbb{K}[X'][Y;\alpha]_P$ with $X' = X + Y^{-1}$. Since $\langle Y \rangle_P = \langle Y, 1 \rangle = A$, Lemma 5.4.7 gives us the following equivalence. For all $Q \in \operatorname{P.Spec}(\overline{A})$ we have:

$$\big(Q \in \operatorname{Im}(\varphi)\big) \iff \big(Y \notin Q\big). \tag{7.1}$$

Therefore for all $P \in \operatorname{P.Spec}(A)$ we have $\varphi(P) = PS^{-1} \cap \overline{A}$, where $S = \{Y^i \mid i \geq 0\}$. We want to find the set $W'_P \subseteq W = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ of Cauchon diagrams of $A$. Recall that by Proposition 5.4.17, we always have $\{\emptyset, \{1\}\} \subseteq W'_P$. From the equivalence (7.1) it is easy to see that:

$$w \in W'_P \iff 2 \notin w.$$

Thus $W'_P = \{\emptyset, \{1\}\}$. In the remainder of this section we describe the non empty strata of $A$.

**Lemma 7.2.1.** *We have* $\text{P.Spec}_{\emptyset}(\overline{A}) = \{\langle 0 \rangle\}$.

*Proof.* We denote by $T$ the localisation of $\overline{A}$ at the multiplicative set consisting of its (nonzero) monomials. Then $T = \mathbb{K}_1[X'^{\pm 1}, Y^{\pm 1}]$ is a Poisson torus and there is a $1-1$ correspondence:

$$\text{P.Spec}_{\emptyset}(\overline{A}) = \{P \in \text{P.Spec}\,(\overline{A}), X', Y \notin P\} \longleftrightarrow \{Q \in \text{P.Spec}\,(T)\}.$$

By Lemma 1.3.2 we have $Z_P(T) = \mathbb{K}$ and so $\text{P.Spec}\,(T) = \{\langle 0 \rangle\}$. Therefore we obtain $\text{P.Spec}_{\emptyset}(A) = \{\langle 0 \rangle\}$ since $\varphi\big(\text{P.Spec}_{\emptyset}(A)\big) \subseteq \text{P.Spec}_{\emptyset}(\overline{A})$. □

We now consider the stratum $\text{P.Spec}_{\{1\}}(A)$.

**Lemma 7.2.2.** *We have:*

$$\text{P.Spec}_{\{1\}}(A) = \{\langle XY + 1 \rangle, \langle XY + 1, Y - \lambda \rangle,\ \lambda \in \mathbb{K}^{\times}\}.$$

*Proof.* Since $\text{P.Spec}_{\emptyset}(\overline{A}) = \{\langle 0 \rangle\}$ we obtain $\varphi\big(\text{P.Spec}_{\{1\}}(A)\big) = \text{P.Spec}_{\{1\}}(\overline{A})$ by Lemma 5.4.1, and therefore to understand $\text{P.Spec}_{\{1\}}(A)$ it is enough to understand $\text{P.Spec}_{\{1\}}(\overline{A})$. There is a $1-1$ correspondence:

$$\{P \in \text{P.Spec}\,(\overline{A}) \mid \langle X' \rangle \subseteq P\} \longleftrightarrow \text{P.Spec}\,\big(\overline{A}/\langle X' \rangle\big),$$

and we have:

$$\text{P.Spec}\,\big(\overline{A}/\langle X' \rangle\big) \cong \text{P.Spec}\,(\mathbb{K}[Y]) = \text{Spec}\,(\mathbb{K}[Y]) = \{\langle Y - \lambda \rangle \mid \lambda \in \mathbb{K}\}.$$

So we conclude that:

$$\text{P.Spec}_{\{1\}}(\overline{A}) = \{\langle X' \rangle, \langle X', Y - \lambda \rangle \mid \lambda \in \mathbb{K}^{\times}\}.$$

To finish the proof we need to compute $\varphi^{-1}\big(\text{P.Spec}_{\{1\}}(\overline{A})\big)$. First we will show that

$\varphi(\langle XY+1\rangle) = \langle X'\rangle$. We have:

$$\{XY+1, X\} = X(XY+1),$$
$$\{XY+1, Y\} = -Y(XY+1).$$

So $\langle XY+1\rangle$ is a Poisson ideal. Moreover we have $A/\langle XY+1\rangle \cong \mathbb{K}[Z^{\pm 1}]$, so $\langle XY+1\rangle$ is a prime ideal. Therefore $\langle XY+1\rangle \in \text{P.Spec}\,(A)$. We now compute the image of $\langle XY+1\rangle$ by the canonical embedding.

$$\varphi(\langle XY+1\rangle) = \langle XY+1\rangle S^{-1} \cap \overline{A} = \{(XY+1)fY^{-i} \mid f \in A,\ i \geq 0\} \cap \overline{A}.$$

So $X' = X + Y^{-1} = (XY+1)Y^{-1} \in \varphi(\langle XY+1\rangle)$ and:

$$\langle X'\rangle \subseteq \varphi(\langle XY+1\rangle). \tag{7.2}$$

We now show that $\langle XY+1\rangle \subseteq \varphi^{-1}(\langle X'\rangle)$. It is clear that $\langle X'\rangle \in \text{Im}(\varphi)$, and we have:

$$\varphi^{-1}(\langle X'\rangle) = \langle X'\rangle S^{-1} \cap A = \{X' fY^{-i} \mid f \in \overline{A},\ i \geq 0\} \cap A.$$

Thus $XY+1 = X'Y \in \varphi^{-1}(\langle X'\rangle)$ so that:

$$\langle XY+1\rangle \subseteq \varphi^{-1}(\langle X'\rangle). \tag{7.3}$$

Notice that by definition we have $\varphi^{-1}(\langle X'\rangle) \in \text{P.Spec}_{\{1\}}(A)$. Moreover $\langle XY+1\rangle$ also belongs to $\text{P.Spec}_{\{1\}}(A)$. Indeed, we already showed that $X' \in \varphi(\langle XY+1\rangle)$, and if $Y \in \varphi(\langle XY+1\rangle)$, then $Y \in \langle XY+1\rangle$, a contradiction. Thus both $\varphi^{-1}(\langle X'\rangle)$ and $\langle XY+1\rangle$ belong to $\text{P.Spec}_{\{1\}}(A)$. Since $\varphi$ induces an homeomorphism from $\text{P.Spec}_{\{1\}}(A)$ to $\text{P.Spec}_{\{1\}}(\overline{A})$ we deduce from equations (7.2) and (7.3) that:

$$\varphi(\langle XY+1\rangle) \subseteq \varphi\big(\varphi^{-1}(\langle X'\rangle)\big) \subseteq \varphi(\langle XY+1\rangle),$$

i.e. $\varphi(\langle XY+1\rangle) = \langle X'\rangle$. In the same way we can show that for all $\lambda \in \mathbb{K}^{\times}$ we have:

$$\varphi(\langle XY+1, Y-\lambda\rangle) = \langle X', Y-\lambda\rangle.$$

Therefore we obtain:

$$\text{P.Spec}_{\{1\}}(A) = \{\langle XY+1\rangle, \langle XY+1, Y-\lambda\rangle \mid \lambda \in \mathbb{K}^\times\}.$$

□

We conclude that the Poisson spectrum of $A$ can be represented as follows:

$$\langle 0\rangle \subseteq \langle XY+1\rangle \subseteq \langle XY+1, Y-\lambda\rangle, \tag{7.4}$$

for all $\lambda \in \mathbb{K}^\times$. The Poisson primitive ideals are those which are maximal in their strata (Proposition 6.2.5), i.e. the ideals:

$$\langle 0\rangle \quad \text{and} \quad \langle XY+1, Y-\lambda\rangle \quad \text{for all} \quad \lambda \in \mathbb{K}^\times.$$

*Remark* 7.2.3. For all $\lambda \in \mathbb{K}^\times$ we have:

$$\langle XY+1, X-\lambda\rangle = \langle XY+1, Y+\lambda^{-1}\rangle,$$

since $X-\lambda = -\lambda\big((XY+1) - X(Y+\lambda^{-1})\big)$. This explains the apparent lack of symmetry in $X$ and $Y$ in (7.4).

#### 7.2.1.2 Spectrum of $R_q$

Let $q \in \mathbb{K}^\times$ be a non root of unity. $R_q$ is the Ore extension $R_q = \mathbb{K}[x][y;\sigma,\Delta]$ with $\sigma(x) = qx$ and $\Delta(x) = q-1$, so that:

$$yx = qxy + q - 1.$$

Note that this algebra is isomorphic to the first quantum-Weyl algebra $A_1^q(\mathbb{K})$ since $q \neq 1$. We will show that the set of Cauchon diagrams for $R_q$ is also $W' = \{\emptyset, \{1\}\}$. We denote by $\varphi$ the canonical embedding from $\text{Spec}\,(R_q)$ to $\text{Spec}\,(\overline{R_q})$ (see [8, Definition 4.4.1]), where the algebra $\overline{R_q}$ is a quantum affine plane in the generators $x' := x + y^{-1}$ and $y$.

Suppose that $2 \in w \in W'$ and let $P \in \text{Spec}_w(R_q)$. Then $y \in \varphi(P)$, i.e. $y \in P$ by [8, Lemme 4.3.3]. But then:

$$q - 1 = yx - qxy \in P,$$

a contradiction. This shows that $W' \subseteq \{\emptyset, \{1\}\}$.

• Secondly suppose that $2 \notin w \in W$. Then $y$ does not belong to any ideal in $\operatorname{Spec}_w(\overline{R_q})$. Thus $\operatorname{Spec}_w(\overline{R_q}) \subseteq \operatorname{Im}(\varphi)$ by [8, Lemme 4.3.1] and so $w \in W'_P$.

Therefore we have $W' = \{\emptyset, \{1\}\}$. In particular $W' = W'_P$ and Question 7.1.1 is positively answered for the algebra $R_t$. We will now compute the strata $\operatorname{Spec}_{\emptyset}(R_q)$ and $\operatorname{Spec}_{\{1\}}(R_q)$.

**Lemma 7.2.4.** *We have:*

$$\operatorname{Spec}_{\emptyset}(R_q) = \{\langle 0 \rangle\},$$
$$\operatorname{Spec}_{\{1\}}(R_q) = \{\langle xy + 1 \rangle, \langle xy + 1, y - \lambda \rangle \mid \lambda \in \mathbb{K}^{\times}\}.$$

*Proof.* Recall that $q$ is not a root of unity and that the field $\mathbb{K}$ is algebraically closed. Therefore, by [6, II.1.2] we obtain:

$$\operatorname{Spec}_{\emptyset}(\overline{R_q}) = \{\langle 0 \rangle\},$$
$$\operatorname{Spec}_{\{1\}}(\overline{R_q}) = \{\langle x' \rangle, \langle x', y - \lambda \rangle \mid \lambda \in \mathbb{K}^{\times}\}.$$

Since $\varphi\big(\operatorname{Spec}_{\emptyset}(R_q)\big) \subseteq \operatorname{Spec}_{\emptyset}(\overline{R_q})$ we obtain that $\operatorname{Spec}_{\emptyset}(R_q) = \{\langle 0 \rangle\}$. Notice that the element $xy + 1$ is normal in $R_q$ and that, since $R_q/\langle xy + 1 \rangle \cong \mathbb{K}[X^{\pm 1}]$, the ideal $\langle xy + 1 \rangle$ is prime in $R_q$. We now show that:

$$\varphi(\langle xy + 1 \rangle) = \langle x' \rangle. \tag{7.5}$$

Since:

$$\varphi(\langle xy + 1 \rangle) = \langle xy + 1 \rangle S^{-1} \cap \overline{R_q},$$

where $S = \{y^i \mid i \geq 0\}$, the equality $x' = (xy + 1)y^{-1}$ shows that:

$$\langle x' \rangle \subseteq \varphi(\langle xy + 1 \rangle). \tag{7.6}$$

Now note that $\langle x' \rangle \in \text{Spec}_{\{1\}}(\overline{R_q}) \subseteq \text{Im}(\varphi)$. So we have:

$$\varphi^{-1}(\langle x' \rangle) = \langle x' \rangle S^{-1} \cap R_q,$$

and the equality $xy + 1 = x'y$ gives us

$$\langle xy + 1 \rangle \subseteq \varphi^{-1}(\langle x' \rangle). \tag{7.7}$$

It is easy to see that both $\varphi^{-1}(\langle x' \rangle)$ and $\langle xy + 1 \rangle$ belong to $\text{Spec}_{\{1\}}(R_q)$. Since $\varphi$ induces an homeomorphism from $\text{Spec}_{\{1\}}(R_q)$ to $\text{Spec}_{\{1\}}(\overline{R_q})$ we deduce from equations (7.6) and (7.7) that:

$$\varphi(\langle xy + 1 \rangle) \subseteq \varphi\big(\varphi^{-1}(\langle x' \rangle)\big) \subseteq \varphi(\langle xy + 1 \rangle),$$

i.e. $\varphi(\langle xy + 1 \rangle) = \langle x' \rangle$. In a similar way we obtain that:

$$\varphi(\langle xy + 1, y - \lambda \rangle) = \langle x', y - \lambda \rangle,$$

for all $\lambda \in \mathbb{K}^{\times}$. Since $\varphi\big(\text{Spec}_{\{1\}}(R_q)\big) = \text{Spec}_{\{1\}}(\overline{R_q})$, this shows that:

$$\text{Spec}_{\{1\}}(R_q) = \{\langle xy + 1 \rangle, \langle xy + 1, y - \lambda \rangle \mid \lambda \in \mathbb{K}^{\times}\}.$$

□

We can now represent the spectrum of $R_q$ as follows:

$$\langle 0 \rangle \subseteq \langle xy + 1 \rangle \subseteq \langle xy + 1, y - \lambda \rangle,$$

for all $\lambda \in \mathbb{K}^{\times}$. Note that from our descriptions of $\text{Spec}(R_q)$ and $\text{P.Spec}(A)$ we can easily deduce that these two spectra are homeomorphic.

### 7.2.2 An example with a two step algorithm

Let $R_t$ be the iterated Ore extension over $\mathbb{K}[t^{\pm 1}]$ given by:

$$R_t := \mathbb{K}[t^{\pm 1}][x][y; \sigma_1, \Delta_1][z; \sigma_2, \Delta_2],$$

where:

- $\sigma_1$ is the automorphism of $\mathbb{K}[t^{\pm 1}][x]$ such that $\sigma_1(x) = t^{-1}x$ and $\sigma_1(t) = t$,
- $\Delta_1$ is the $\sigma_1$-derivation of $\mathbb{K}[t^{\pm 1}][x]$ such that $\Delta_1(x) = t - 1$ and $\Delta_1(t) = 0$,
- $\sigma_2$ is the automorphism of $\mathbb{K}[t^{\pm 1}][x][y; \sigma_1, \Delta_1]$ such that $\sigma_2(x) = tx$, $\sigma_2(y) = t^{-1}y$ and $\sigma_2(t) = t$,
- $\Delta_2$ is the $\sigma_2$-derivation of $\mathbb{K}[t^{\pm 1}][x][y; \sigma_1, \Delta_1]$ such that $\Delta_2(x) = (t-1)y^k$ for some integer $k \geq 1$ and $\Delta_2(y) = \Delta_2(t) = 0$.

Therefore we have $t \in Z(R_t)$ and the relations:

$$yx = t^{-1}xy + t - 1, \quad zx = txz + (t-1)y^k \quad \text{and} \quad zy = t^{-1}yz.$$

One can check that we have:

$$\Delta_1\sigma_1 = t^{-1}\sigma_1\Delta_1 \quad \text{and} \quad \Delta_2\sigma_2 = t^{1+k}\sigma_2\Delta_2.$$

Moreover for all $i \geq 0$ we have:

$$\Delta_1^i(x) \in (i)!_{t^{-1}}(t-1)^i\mathbb{K}[t^{\pm 1}][x] \quad \text{and} \quad \Delta_2^i(x) \in (i)!_{t^{k+1}}(t-1)^i\mathbb{K}[t^{\pm 1}][x][y; \sigma_1, \Delta_1].$$

Finally, it is clear that the quotient algebra $A := R_t/(t-1)R_t$ is commutative. Therefore $R_t \in \mathcal{R}$.

#### 7.2.2.1 Cauchon diagrams for $A$

$A$ is the iterated Poisson-Ore extension:

$$A = \mathbb{K}[X][Y; \alpha_1, \delta_1]_P[Z; \alpha_2, \delta_2]_P,$$

where $\alpha_1 = -X\partial_X$, $\alpha_2 = X\partial_X - Y\partial_Y$, $\delta_1 = \partial_X$ and $\delta_2 = Y^k\partial_X$. We have:

$$\{Y, X\} = -XY + 1,$$
$$\{Z, X\} = XZ + Y^k,$$
$$\{Z, Y\} = -YZ.$$

Note that this is actually a Jacobian Poisson structure whose potential is:

$$W := XYZ + \frac{1}{k+1}Y^{k+1} - Z.$$

In particular we have $\mathbb{K}[W] \subseteq Z_P(A)$. When $k = 2$ we retrieve the Poisson algebra given in Remark 5.4.9. Note that the derivation $\delta_1$ (resp. $\delta_2$) extends to a locally nilpotent iterative higher $(-1, \alpha_1)$-skew (resp. $(k+1, \alpha_2)$-skew) Poisson derivation on $\mathbb{K}[X]$ (resp. $\mathbb{K}[X][Y; \alpha_1, \delta_1]_P$).

We now compute the set $W'_P$ of Cauchon diagrams of $A$. The deleting derivations algorithm gives us new indeterminates:

$$\begin{aligned}
&X' = X + \frac{1}{k+1}Y^k Z^{-1},\\
&Y' = Y,\\
&X'' = X' - Y^{-1} = X + \frac{1}{k+1}Y^k Z^{-1} - Y^{-1},
\end{aligned}$$

with a sequence of iterated Poisson-Ore extensions:

$$\begin{aligned}
&A_4 := \mathbb{K}[X][Y; \alpha_1, \delta_1]_P[Z; \alpha_2, \delta_2]_P = A,\\
&A_3 := \mathbb{K}[X'][Y; \alpha_1, \delta_1]_P[Z; \alpha_2]_P,\\
&A_2 := \mathbb{K}[X''][Y; \alpha_1]_P[Z; \alpha_2]_P = \overline{A}.
\end{aligned}$$

The algebra $\overline{A}$ is the Poisson affine space $\mathbb{K}_{\boldsymbol{\lambda}}[X'', Y, Z]$, where $\lambda_{12} = -\lambda_{13} = \lambda_{23} = 1$. Note that we have $W = X''YZ$. The canonical embedding is the composite map $\varphi = \varphi_2 \circ \varphi_3$, where $\varphi_3 : \text{P.Spec}\,(A_4) \longrightarrow \text{P.Spec}\,(A_3)$ and $\varphi_2 : \text{P.Spec}\,(A_3) \longrightarrow \text{P.Spec}\,(A_2)$ are defined as in Section 5.4.1.

**Lemma 7.2.5.** *We have* $W'_P = \{\emptyset, \{1\}\}$.

*Proof.* Let $w \in W'_P$ and $P \in \text{P.Spec}_w(A)$. Suppose first that $2 \in w$. Then $Y \in \varphi(P)$ and by Corollary 5.4.11 we obtain $Y \in \varphi_3(P)$. We now distinguish between two cases: either $Z \in \varphi_3(P)$, or $Z \notin \varphi_3(P)$. Suppose that $Z \in \varphi_3(P)$, then the Poisson algebra isomorphism $A/P \cong A_3/\varphi_3(P)$ (Lemma 5.4.18) shows that $Y \in P$. Suppose now that $Z \notin \varphi_3(P)$. Then:

$$Y \in \varphi_3(P)[Z^{-1}] \cap A = P.$$

Therefore in both cases we have $Y \in P$. But then:

$$1 = \{Y, X\} + XY \in P,$$

a contradiction to the primality of $P$. Thus $2 \notin w$.

Suppose now that $3 \in w$. Then $Z \in \varphi(P)$, and $Z \in P$ by Corollary 5.4.11. But then:

$$Y^k = \{Z, X\} - XZ \in P,$$

and by primality of $P$ we have $Y \in P$. We conclude as previously that this is impossible and $3 \notin w$.

We proved that $W'_P \subseteq \{\emptyset, \{1\}\}$. The reverse inclusion can be obtained from Proposition 5.4.17. Thus we have $W'_P = \{\emptyset, \{1\}\}$. $\square$

In the remainder of this section we compute explicitly the preimage of the ideal $\langle X'' \rangle \in \text{P.Spec}_{\{1\}}(\overline{A})$. We claim that the ideal $\langle W \rangle$ of $A$ is a Poisson prime ideal such that $\varphi(\langle W \rangle) = \langle X'' \rangle$. We start by justifying that $\langle W \rangle \in \text{P.Spec}(A)$.

First we show that $W$ is irreducible in $A$. Suppose that $W = G_1 G_2$ with $G_1, G_2 \in \mathbb{K}[X, Y, Z]$. Since $\deg_X(W) = 1$ we can suppose that $G_1 = AX + B$ and $G_2 = C$, where $A, B, C \in \mathbb{K}[Y, Z]$. But then we have $AC = YZ$ and $BC = \frac{1}{k+1}Y^{k+1} - Z$. From the first equality we deduce that up to a nonzero scalar we have $C \in \{1, Y, Z, YZ\}$. Suppose that $C \neq 1$. Then from the second equality, either $Y$ or $Z$ must divides $\frac{1}{k+1}Y^{k+1} - Z$. This is a contradiction and $G_2 = C = 1$, i.e. $W$ is irreducible. We conclude that $\langle W \rangle$ is a prime ideal since $A$ is a unique factorisation domain. Moreover since $W \in Z_P(A)$ it is clear that $\langle W \rangle$ is a Poisson ideal.

Now consider the ideal $\langle X'Y - 1 \rangle$ of $A_3$. It is easy to check that $\langle X'Y - 1 \rangle \in \text{P.Spec}(A_3)$, and that $Y, Z \notin \langle X'Y - 1 \rangle$. In particular we have:

$$\varphi_2(\langle X'Y - 1 \rangle) = \langle X'Y - 1 \rangle [Y^{-1}] \cap \overline{A},$$

and since $X'' = (X'Y - 1)Y^{-1}$, it is clear that $\langle X'' \rangle \subseteq \varphi_2(\langle X'Y - 1 \rangle)$. Thus we have

$X'' \in \varphi_2(\langle X'Y-1\rangle)$ and $Y, Z \notin \varphi_2(\langle X'Y-1\rangle)$, so:

$$\langle X'Y-1\rangle \in \mathrm{P.Spec}_{\{1\}}(A_3) := \varphi_2^{-1}\big(\mathrm{P.Spec}_{\{1\}}(\overline{A})\big).$$

Note that $\langle X''\rangle \in \mathrm{Im}(\varphi_2)$ since $Y \notin \langle X''\rangle$, and so we have:

$$\varphi_2^{-1}(\langle X''\rangle) = \langle X''\rangle[Y^{-1}] \cap A_3.$$

Because $X''Y = X'Y - 1$ it is clear that $\langle X'Y-1\rangle \subseteq \varphi_2^{-1}(\langle X''\rangle)$. Since both the ideals $\langle X'Y-1\rangle$ and $\varphi_2^{-1}(\langle X''\rangle)$ belong to $\mathrm{P.Spec}_{\{1\}}(A_3)$ and since $\varphi_2$ induces an homeomorphism from $\mathrm{P.Spec}_{\{1\}}(A_3)$ to $\mathrm{P.Spec}_{\{1\}}(\overline{A})$ we obtain that:

$$\varphi_2(\langle X'Y-1\rangle) = \langle X''\rangle.$$

We have $Z \notin \langle X'Y-1\rangle$, so $\langle X'Y-1\rangle \in \mathrm{Im}(\varphi_2)$ and:

$$\varphi_3^{-1}(\langle X'Y-1\rangle) = \langle X'Y-1\rangle[Z^{-1}] \cap A.$$

Since $W = X''YZ = (X'Y-1)Z$ we have $\langle W\rangle \subseteq \varphi_3^{-1}(\langle X'Y-1\rangle)$. On the other hand by a degree argument one easily deduce that $Y, Z \notin \langle W\rangle$. Thus we have:

$$\varphi_3(\langle W\rangle) = \langle W\rangle[Z^{-1}] \cap A_3.$$

Therefore we get that $\langle X'Y-1\rangle \subseteq \varphi_3(\langle W\rangle)$ since $X'Y-1 = WZ^{-1}$. It is clear that $\varphi_3^{-1}(\langle X'Y-1\rangle)$ belongs to $\mathrm{P.Spec}_{\{1\}}(A)$. Moreover we deduce from $Y, Z \notin \langle W\rangle$ that $Y, Z \notin \varphi(\langle W\rangle)$, and that $X'' = WY^{-1}Z^{-1} \in \varphi(\langle W\rangle)$. Therefore $\varphi(\langle W\rangle) \in \mathrm{P.Spec}_{\{1\}}(\overline{A})$ and $\langle W\rangle \in \mathrm{P.Spec}_{\{1\}}(A)$. Since $\varphi_3$ induces an homeomorphism from $\mathrm{P.Spec}_{\{1\}}(A)$ to $\mathrm{P.Spec}_{\{1\}}(A_3)$ we finally obtain that:

$$\varphi_3(\langle W\rangle) = \langle X'Y-1\rangle.$$

We conclude that:

$$\varphi(\langle W\rangle) = \varphi_2 \circ \varphi_3(\langle W\rangle) = \varphi_2(\langle X'Y-1\rangle) = \langle X''\rangle.$$

#### 7.2.2.2 Cauchon diagrams for $R_q$

We now consider the algebra $R_q := R_t/(t-q)R_t$ for a non root of unity $q \in \mathbb{K}^\times$. We express $R_q$ as an iterated Ore extension as follows:

$$R_q = \mathbb{K}[x][y;\sigma_1,\Delta_1][z;\sigma_2,\Delta_2],$$

where the $\sigma_i$s and the $\Delta_i$s are defined analogously as at the beginning of the section, replacing $t$ by $q$. We have the relations:

$$\begin{aligned} yx &= q^{-1}xy + q - 1, \\ zx &= qxz + (q-1)y^k, \\ zy &= q^{-1}yz, \end{aligned}$$

where $k \geq 1$ is an integer. We can apply Cauchon's deleting derivations algorithm to $R_q$, and we will compute the set $W'$ of Cauchon diagrams of $R_q$. We can show that:

$$\{\emptyset\} \subseteq W' \subseteq \{\emptyset, \{1\}\}$$

similarly to the way we showed that $\{\emptyset\} \subseteq W'_P \subseteq \{\emptyset, \{1\}\}$ in the previous section. The tools needed are either in [8], or can be easily deduced from it. Only one detail requires explanation. Suppose that $y^k \in P$ for some prime ideal $P \in \mathrm{Spec}(R_q)$. To pass from $y^k \in P$ to $y \in P$, we need complete primeness of $P$. One can show that all prime ideals of $R_q$ are completely prime by [6, Theorem II.6.9]. Indeed there is an action of the torus $\mathbb{K}^\times$ on $R_q$ given by $\lambda(x) = \lambda x$, $\lambda(y) = \lambda^{-1}y$ and $\lambda(z) = \lambda^{-1-k}z$ for $\lambda \in \mathbb{K}^\times$ and satisfying the appropriate assumptions.

We will now show that $\{1\} \in W'$. For this, we need the following elements that we obtain by deleting the variables $z$ and $y$.

$$\begin{aligned} x' &:= x + q^k \frac{q-1}{q^{k+1}-1} y^k z^{-1}, \\ x'' &:= x' - qy^{-1} = x + q^k \frac{q-1}{q^{k+1}-1} y^k z^{-1} - qy^{-1}, \\ w &:= x''yz = xyz + q^{k+1} \frac{q-1}{q^{k+1}-1} y^{k+1} - qz. \end{aligned}$$

Note that $w \in Z(R_q)$. We denote by $\langle w \rangle$ the ideal generated by $w$ inside $R_q$ and by $\langle x'' \rangle$ the ideal generated by the normal element $x''$ inside $\overline{R_q}$. Since $\overline{R_q}$ is a quantum affine space it is clear that $\langle x'' \rangle \in \mathrm{Spec}\,(\overline{R_q})$. We will show that $\varphi(\langle w \rangle) = \langle x'' \rangle$. First we need to justify that $\langle w \rangle \in \mathrm{Spec}\,(R_q)$. We denote by $\Sigma$ the multiplicative set generated by $y$ and $z$ in $\overline{R_q}$. If we denote by $S$ the multiplicative set generated by $y$ and $z$ in $R_q$ one can show, adapting [8, Proposition 3.3.1], that both $\Sigma$ and $S$ satisfy the Ore conditions, that $\Sigma = S$ and that we have the equality:

$$\overline{R_q}\Sigma^{-1} = R_q S^{-1}.$$

The prime ideal $\langle x'' \rangle$ of $\overline{R_q}$ induces a prime ideal $J$ in $\overline{R_q}\Sigma^{-1}$. Since $w = x''yz$, the ideal $J$ is generated by $w$ (inside $\overline{R_q}\Sigma^{-1} = R_q S^{-1}$). Set $I := J \cap R_q \in \mathrm{Spec}\,(R_q)$. We will show that $I = \langle w \rangle$, proving that $\langle w \rangle$ is a prime ideal. Since $w \in J \cap R_q$ it is clear that $\langle w \rangle \subseteq I$. Reciprocally let $u \in I$. Since $u \in J$ we can write $u = wa$, where $a \in \overline{R_q}\Sigma^{-1} = R_q S^{-1}$. Therefore there exists $s \in S$ such that $as \in R_q$. All elements of $S$ are of the form $\beta y^i z^j$ for some scalar $\beta \in \mathbb{K}^\times$ and integers $i, j \geq 0$. Thus by rescaling $u$ if necessary we can assume that $\beta = 1$. Now choose $i, j$ minimal such that $uy^i z^j = wa'$ with the property that $a' := ay^i z^j$ belongs to $R_q$. We will show that $i = j = 0$. Suppose that $j \geq 1$. Then all monomials in $wa' = uy^i z^j$ must contain $z$. Write $a' = \sum r_l z^l \in R_q$ where the $r_i$s are polynomials in $x$ and $y$. Then:

$$wa' = \sum wr_l z^l = \sum r_l w z^l = \sum r_l (xy - q) z^{l+1} + \sum r_l \alpha z^l,$$

where $\alpha \in \mathbb{K}[y]$ is nonzero. But then, since $uy^i z^j = wa'$ we must have $r_0 \alpha = 0$, i.e. $r_0 = 0$. Therefore we can write $a' = a''z$ for some $a'' \in R_q$. But then we have:

$$uy^i z^{j-1} = wa'',$$

for some $a'' \in R_q$, contradicting the minimality of $j$. We conclude that $j = 0$. Suppose now that $i \geq 1$. Then all monomials in $uy^i = wa'$ must contains $y$. We now decompose $a'$ in the basis $\{x^{u_1} z^{u_2} y^{u_3} \mid \underline{u} = (u_1, u_2, u_3) \in \mathbb{N}^3\}$ of $R_q$ (this is indeed a basis since $y$ and $z$ $q$-commute). We express $a'$ in this basis:

$$a' = \sum_{\underline{u}} \lambda_{\underline{u}} x^{u_1} z^{u_2} y^{u_3} \in R_q,$$

where the scalars $\lambda_{\underline{u}}$ are almost all zero. Then by writing $w = qxzy + \mu y^{k+1} - qz$, where $\mu \in \mathbb{K}^\times$ we have:

$$\begin{aligned} wa' &= \sum_{\underline{u}} \lambda_{\underline{u}} x^{u_1} w z^{u_2} y^{u_3} \\ &= \sum_{\underline{u}} \lambda_{\underline{u}} q^{u_2+1} x^{u_1+1} z^{u_2+1} y^{u_3+1} \\ &\quad + \sum_{\underline{u}} \mu \lambda_{\underline{u}} q^{u_2(k+1)} x^{u_1} z^{u_2} y^{u_3+k+1} \\ &\quad - \sum_{\underline{u}} \lambda_{\underline{u}} q x^{u_1} z^{u_2+1} y^{u_3}. \end{aligned}$$

But then, since $uy^i = wa'$, all monomials of $wa'$ must contain $y$. Therefore we must have $\lambda_{\underline{u}} = 0$ when $u_3 = 0$, i.e. $a' = a''y$ for some $a'' \in R_q$. But then $uy^{i-1} = wa''$ with $a'' \in R_q$. This contradicts the minimality of $i$, so $i = 0$, and, finally, $u = wa$ for some $a \in R_q$, i.e. $u \in \langle w \rangle$. We have shown that $I = \langle w \rangle$, in particular $\langle w \rangle$ is a prime ideal in $R_q$. It is now easy to check that $\varphi(\langle w \rangle) = \langle x'' \rangle$ thanks to the equality $w = x''yz$.

We can finally conclude that $W' = \{\emptyset, \{1\}\} = W'_P$, answering positively Question 7.1.1 for the algebra $R_t$.

## 7.3 Cauchon diagrams for matrix Poisson varieties

In this section we give a combinatorial description of Cauchon diagrams for the matrix Poisson varieties. Assume that $\operatorname{char} \mathbb{K} \neq 2$. Recall from Section 4.2.1 that the coordinate ring of the variety of $m \times p$ Poisson matrices $A = \mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$ is the polynomial algebra $\mathbb{K}[X_{ij} \mid 1 \le i \le m,\ 1 \le j \le p\,]$ endowed with the Poisson bracket given by:

$$\{X_{ij}, X_{kl}\} = \begin{cases} X_{ij}X_{kl} & \text{if } i < k \text{ and } j = l, \\ X_{ij}X_{kl} & \text{if } i = k \text{ and } j < l, \\ 0 & \text{if } i < k \text{ and } j > l, \\ 2X_{il}X_{kj} & \text{if } i < k \text{ and } j < l. \end{cases}$$

We showed in Section 4.2.1 that $A$ is the semiclassical limit of an iterated Ore extension belonging to the class $\mathcal{R}$. Therefore the Poisson algebra $A$ belongs to the class $\mathcal{P}$. In Section 7.3.1 we will show how the Poisson deleting derivations algorithm applies to $A$.

As usual we will denote by $W'_P \subseteq W = \mathcal{P}\big([\![1,m]\!] \times [\![1,p]\!]\big)$ the set of Cauchon diagrams for $A$.

We denote by $G$ the set consisting of all $m \times p$ grids whose boxes are colored either in black or in white. There is a bijection:

$$\begin{aligned} \xi \ : \ W &\longrightarrow G \\ w \ &\longmapsto \mathcal{C}_w, \end{aligned}$$

where $\mathcal{C}_w$ is the $m \times p$ grid such that for all $(i,j) \in [\![1,m]\!] \times [\![1,p]\!]$ the box in position $(i,j)$ is black if and only if $(i,j) \in w$. For later purpose we set $w_{\mathcal{C}} := \xi^{-1}(\mathcal{C}) \in W$ for $\mathcal{C} \in G$. We want to understand the set of Cauchon diagrams $W'_P$ under this bijection. We denote by $\mathcal{G}$ the subset of $G$ defined as follows.

**Definition 7.3.1.** The set $\mathcal{G}$ consists of all $m \times p$ grids with black or white boxes satisfying the following property. If a box is black, then every box strictly to its left is black or every box strictly above it is black. We will refer to this property as the *Cauchon property*.

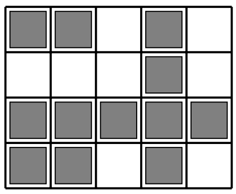

Figure 7.1: An example of an element $\mathcal{C} \in \mathcal{G}$ for $m = 4$ and $p = 5$.

For the element $\mathcal{C} \in \mathcal{G}$ of Figure 7.1 we have:

$$\begin{aligned} \xi^{-1}(\mathcal{C}) = \{&(1,1),(1,2),(1,4),(2,4),(3,1),(3,2), \\ &(3,3),(3,4),(3,5),(4,1),(4,2),(4,4)\} \in \mathcal{P}\big([\![1,4]\!] \times [\![1,5]\!]\big). \end{aligned}$$

In Section 7.3.3 we will prove that $\xi$ induces by restriction a bijection between $W'_P$ and $\mathcal{G}$.

### 7.3.1 Poisson deleting derivations algorithm and matrix Poisson varieties

From Section 4.2.1 we obtain the following:

- $A$ can be expressed as an iterated Poisson-Ore extension:

$$A = \mathbb{K}[X_{11}][X_{12}; \alpha_{12}, \delta_{12}]_P \cdots [X_{mp}; \alpha_{mp}, \delta_{mp}]_P$$

  for suitable maps $\alpha_{uv}$ and $\delta_{uv}$, where the indeterminates are ordered in the lexicographic order.

- For all $(1,2) \leq (u,v) \leq (m,p)$ the Poisson $\alpha_{uv}$-derivation $\delta_{uv}$ extends to an iterative, locally nilpotent higher $(2,\alpha_{uv})$-skew Poisson derivation $(D_{uv,k})$ on the appropriate Poisson subalgebra. A short computation (based on the formula of Proposition 4.1.1) leads to:

$$D_{uv,k}(X_{ij}) = \begin{cases} X_{ij} & k = 0 \\ -2X_{iv}X_{uj} & k = 1 \\ 0 & k > 1, \end{cases}$$

  for all $(1,1) \leq (i,j) < (u,v)$.

We now make explicit the Poisson deleting derivations algorithm in the context of matrix Poisson varieties. For $u \in \big(([\![1,m]\!] \times [\![1,p]\!]) \setminus \{(1,1),(m,p)\}\big)$, we will often denote by $u^-$ (respectively $u^+$) the largest (respectively smallest) element of $[\![1,m]\!] \times [\![1,p]\!]$ which is smaller (respectively larger) than $u$, with respect to the lexicographic order. By convention we set $(1,1)^+ := (1,2)$, $(m,p)^- := (m,p-1)$ and $(m,p)^+ := (m,p+1)$.

The Poisson deleting derivations algorithm (Section 5.2) returns for all $(1,2) \leq (u,v) \leq (m,p)^+$ a matrix $\big(X_{ij}^{(u,v)}\big)_{ij} \in M_{m,p}(\mathrm{Frac}\,A)$. For $(u,v) = (m,p)^+$ the matrix is given by $\big(X_{ij}^{(m,p)^+}\big)_{ij} := \big(X_{ij}\big)_{ij}$. Then, assuming that the matrix $\big(X_{ij}^{(u,v)^+}\big)_{ij}$ is known, the matrix $\big(X_{ij}^{(u,v)}\big)_{ij}$ is obtained as follows:

$$X_{ij}^{(u,v)} := \begin{cases} X_{ij}^{(u,v)^+} & \text{if } i \geq u \text{ or } j \geq v, \\ X_{ij}^{(u,v)^+} - X_{iv}^{(u,v)^+} X_{uj}^{(u,v)^+} \big(X_{uv}^{(u,v)^+}\big)^{-1} & \text{if } i < u \text{ and } j < v. \end{cases}$$

Set $(u,v)^- := (u^-, v^-)$. The subalgebra of Frac $A$ generated by the $X_{ij}^{(u,v)}$ is denoted by $C_{(u,v)}$. This is an iterated Poisson-Ore extension of the form:

$$C_{(u,v)} = \mathbb{K}[X_{11}^{(u,v)}]\cdots[X_{u^-v^-}^{(u,v)};\alpha'_{u^-v^-},\delta'_{u^-v^-}]_P[X_{uv}^{(u,v)};\alpha'_{uv}]_P\cdots[X_{mp}^{(u,v)};\alpha'_{mp}]_P,$$

for suitable maps $\alpha'_{ij}$ and $\delta'_{ij}$. We have $C_{(m,p)^+} = A$ and we set $\overline{A} := C_{(1,2)}$, and $\big(T_{ij}\big)_{ij} := \big(X_{ij}^{(1,2)}\big)_{ij}$. In particular $\overline{A}$ is a Poisson affine space in $mp$ indeterminates. For all $w \in W$ we set:

$$J_w := \langle T_{ij} \mid (i,j) \in w\rangle \in \text{P.Spec}_w(\overline{A}).$$

### 7.3.2 Some reminders

In this section we recall some results from Section 5.4 adjusting notation to the context of matrix Poisson varieties. Let $(1,2) \le (u,v) \le (m,p)$. There is an injective map $\varphi_{(u,v)}$ from P.Spec $(C_{(u,v)^+})$ to P.Spec $(C_{(u,v)})$ defined by:

$$\varphi_{(u,v)}(P) = \begin{cases} PS_{(u,v)}^{-1} \cap C_{(u,v)} & \text{if } X_{uv}^{(u,v)^+} \notin P, \\ g_{(u,v)}^{-1}\big(P/\langle X_{uv}^{(u,v)^+}\rangle_P\big) & \text{if } X_{uv}^{(u,v)^+} \in P, \end{cases}$$

where $S_{(u,v)}$ is the multiplicative set of $C_{(u,v)^+}$ generated by $X_{uv}^{(u,v)^+} = X_{uv}^{(u,v)}$, and where:

$$\begin{array}{cccl} g_{(u,v)} & : & C_{(u,v)} & \longrightarrow \dfrac{C_{(u,v)^+}}{\langle X_{uv}^{(u,v)^+}\rangle_P} \\ & & X_{ij}^{(u,v)} & \longmapsto X_{ij}^{(u,v)^+} + \langle X_{uv}^{(u,v)^+}\rangle_P, \end{array}$$

is a surjective Poisson algebra homomorphism. By setting $\varphi := \varphi_{(1,2)} \circ \cdots \circ \varphi_{(m,p)}$, we obtain the canonical embedding $\varphi$ : P.Spec $(A) \to$ P.Spec $(\overline{A})$. We also define for all $(1,1) \le (u,v) \le (m,p)$ an injective map $f_{(u,v)}$ from P.Spec $(C_{(u,v)^+})$ to P.Spec $(\overline{A})$ by setting $f_{(1,1)} := \text{id}_{\text{P.Spec}(\overline{A})}$ and $f_{(u,v)} := f_{(u,v)^-} \circ \varphi_{(u,v)}$. For $Q \in \text{Im}(\varphi)$, we set $P_{(u,v)} := f_{(u,v)^-}^{-1}(Q) \in \text{P.Spec}(C_{(u,v)})$ for all $(1,2) \le (u,v) \le (m,p)^+$ and $P := P_{(m,p)^+} = \varphi^{-1}(Q)$. Note that we have $P_{(1,2)} = Q$. We now recall the following membership criterion for $\text{Im}(\varphi_{(u,v)})$, where $N_{(u,v)} := \ker(g_{(u,v)})$.

**Proposition 7.3.2.** [Lemma 5.4.7] *Let $Q \in \text{P.Spec}(C_{(u,v)})$. Then:*

$$Q \in \text{Im}(\varphi_{(u,v)}) \iff \big(\text{ either } X_{uv}^{(u,v)} = X_{uv}^{(u,v)^+} \notin P_{(u,v)}, \text{ or } N_{(u,v)} \subseteq Q\ \big).$$

**Corollary 7.3.3.** [Corollary 5.4.11] *Let* $(1,1) \leq (u,v) \leq (m,p)$*, and* $Q \in \mathrm{Im}(\varphi)$*. We have the following equivalence:*

$$T_{uv} = X_{uv}^{(1,2)} \in Q \iff X_{uv}^{(u,v)^+} \in P_{(u,v)^+}.$$

We will often use this corollary without reference in the following.

### 7.3.3 Cauchon diagrams for $\mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$

In this section we give a combinatorial description of the set $W'_P$ of Cauchon diagrams for the Poisson algebra $\mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$. More precisely we prove that $\xi(W'_P) = \mathcal{G}$. The following definition will make the next proofs more readable.

**Definition 7.3.4.** Let $R$ be a $\mathbb{K}$-algebra. The matrix $M = (m_{ij}) \in M_{m,p}(R)$ is a *Cauchon matrix* provided that for all $(i,j) \in [\![1,m]\!] \times [\![1,p]\!]$ we have:

$$\big(m_{ij} = 0\big) \iff \big(m_{kj} = 0 \text{ for all } k \leq i, \text{ or } m_{il} = 0 \text{ for all } l \leq j\big).$$

Note that to a Cauchon matrix $M = (m_{ij}) \in M_{m,p}(R)$ we can associate a unique element $\mathcal{C}_M$ of $\mathcal{G}$ (of size $m \times p$) by colouring in black the box in position $(i,j)$ if and only if $m_{ij} = 0$. We are just saying that the shape of the 0s in $M$ forms an element of $\mathcal{G}$.

**Proposition 7.3.5.** *We have* $\xi(W'_P) \subseteq \mathcal{G}$*.*

*Proof.* Let $w \in W'_P$. We have to show that $\mathcal{C}_w := \xi(w) \in \mathcal{G}$. Since $w \in W'_P$, there exists an ideal $P_w \in \mathrm{Im}(\varphi) \cap \mathrm{P.Spec}_{\,w}(\overline{A})$, and we set $P_{(u,v)} := f^{-1}_{(u,v)^-}(P_w)$ for all $(1,2) \leq (u,v) \leq (m,p)^+$, so that $P_{(1,2)} = P_w$. Note that $\mathcal{C}_w \in \mathcal{G}$ if and only if $\big(T_{ij} + P_w\big)_{ij} \in M_{m,p}\big(\overline{A}/P_w\big)$ is a Cauchon matrix. We prove, by a decreasing induction on $(u,v)$, that the matrix $\big(X_{ij}^{(u,v)} + P_{(u,v)}\big)_{ij} \in M_{m,p}\big(C_{(u,v)}/P_{(u,v)}\big)$ is a Cauchon matrix for all $(m,p)^+ \geq (u,v) \geq (1,2)$. The case $(u,v) = (1,2)$ will give the result.
<u>Initialisation:</u> Assume $(u,v) = (m,p)^+$. If $X_{ij}^{(m,p)^+} \in P_{(m,p)^+}$, then:

$$\{X_{ij}^{(m,p)^+}, X_{st}^{(m,p)^+}\} = -2X_{it}^{(m,p)^+} X_{sj}^{(m,p)^+} \in P_{(m,p)^+}$$

for all $s < i$ and all $t < j$. On the other hand, if $X_{sj}^{(m,p)^+} \notin P_{(m,p)^+}$ for some $s < i$, then we have $X_{it}^{(m,p)^+} \in P_{(m,p)^+}$ for all $t < j$ by primality of $P_{(m,p)^+}$. This concludes the initialisation.

Induction step: Assume now that $\left(X_{ij}^{(u,v)^+} + P_{(u,v)^+}\right)_{ij} \in M_{m,p}\big(C_{(u,v)^+}/P_{(u,v)^+}\big)$ is a Cauchon matrix for some $(m,p) \geq (u,v) \geq (1,2)$ and let us show that $\left(X_{ij}^{(u,v)} + P_{(u,v)}\right)_{ij} \in M_{m,p}\big(C_{(u,v)}/P_{(u,v)}\big)$ is also a Cauchon matrix. Assume that $X_{ij}^{(u,v)} \in P_{(u,v)}$.

Case 1: If $(i,j) < (u,v)$, then we conclude in the same manner as in the initialisation.

Case 2: If $(i,j) = (u,v)$, then we have $X_{ij}^{(i,j)} = X_{uv}^{(i,j)} \in P_{(i,j)}$ so $X_{ij}^{(i,j)^+} \in P_{(i,j)^+}$ and the induction hypothesis tells us that $X_{it}^{(i,j)^+} \in P_{(i,j)^+}$ for all $1 \leq t \leq j$, or $X_{sj}^{(i,j)^+} \in P_{(i,j)^+}$ for all $1 \leq s \leq i$.

Since $X_{ij}^{(i,j)^+} \in P_{(i,j)^+}$, Lemma 5.4.18 gives us a Poisson algebra isomorphism between $C_{(i,j)^+}/P_{(i,j)^+}$ and $C_{(i,j)}/P_{(i,j)}$ sending the coset of $X_{ab}^{(i,j)^+}$ to the coset of $X_{ab}^{(i,j)}$ for all $(1,1) \leq (a,b) \leq (m,p)$. Therefore we have $X_{it}^{(i,j)} \in P_{(i,j)}$ for all $1 \leq t \leq j$, or $X_{sj}^{(i,j)} \in P_{(i,j)}$ for all $1 \leq s \leq i$, and this concludes this case.

Case 3: If $(i,j) > (u,v)$, then we distinguish between 2 cases.

Case 3.1: If $X_{uv}^{(u,v)^+} \in P_{(u,v)^+}$, then the isomorphism $C_{(u,v)^+}/P_{(u,v)^+} \cong C_{(u,v)}/P_{(u,v)}$ allows us to conclude in the same manner as in Case 2 above.

Case 3.2: Assume now that $X_{uv}^{(u,v)^+} \notin P_{(u,v)^+}$. Since $(i,j) > (u,v)$ we have $i \geq u$, and:

$$X_{ij}^{(u,v)^+} = X_{ij}^{(u,v)} \in P_{(u,v)} \cap C_{(u,v)} \subseteq P_{(u,v)^+}.$$

The induction hypothesis tells us that $X_{it}^{(u,v)^+} \in P_{(u,v)^+}$ for all $1 \leq t \leq j$, or $X_{sj}^{(u,v)^+} \in P_{(u,v)^+}$ for all $1 \leq s \leq i$. Note that these two cases are not symmetric.

Case 3.2.1: If $X_{it}^{(u,v)^+} \in P_{(u,v)^+}$ for all $1 \leq t \leq j$, then, since $i \geq u$, we have:

$$X_{it}^{(u,v)} = X_{it}^{(u,v)^+} \in P_{(u,v)^+} \cap C_{(u,v)} \subseteq P_{(u,v)} \text{ for all } 1 \leq t \leq j.$$

This concludes this case.

Case 3.2.2: If $X_{sj}^{(u,v)^+} \in P_{(u,v)^+}$ for all $1 \leq s \leq i$, then for all $1 \leq s \leq i$ we have:

$$X_{sj}^{(u,v)} = \begin{cases} X_{sj}^{(u,v)^+} & \text{when } s \geq u \text{ or } j \geq v, \\ X_{sj}^{(u,v)^+} - X_{sv}^{(u,v)^+} X_{uj}^{(u,v)^+} \big(X_{uv}^{(u,v)^+}\big)^{-1} & \text{when } s < u \text{ and } j < v. \end{cases}$$

So when $s \geq u$ or $j \geq v$, we have $X_{sj}^{(u,v)} \in P_{(u,v)^+} \cap C_{(u,v)} \subseteq P_{(u,v)}$ for all $1 \leq s \leq i$. Otherwise, when $s < u$ and $j < v$, we have:

$$X_{sj}^{(u,v)} = X_{sj}^{(u,v)^+} - X_{sv}^{(u,v)^+} X_{uj}^{(u,v)^+} \big(X_{uv}^{(u,v)^+}\big)^{-1} \in P_{(u,v)^+} \big(S_{(u,v)^+}\big)^{-1} \cap C_{(u,v)} = P_{(u,v)}$$

since $X_{sj}^{(u,v)^+}$ and $X_{uj}^{(u,v)^+}$ belong to $P_{(u,v)^+}$ as $s, u \leq i$ (recall that $S_{(u,v)^+}$ is the multiplicative set of $C_{(u,v)^+}$ generated by $X_{uv}^{(u,v)^+}$). So $X_{uj}^{(u,v)} \in P_{(u,v)}$ for all $1 \leq s \leq i$ and this concludes the induction. □

Conversely, we will prove that $\xi^{-1}(\mathcal{G}) \subseteq W'_P$ in the next proposition. For this purpose, we use Proposition 7.3.2 which requires an explicit description of $N_{(u,v)}$. This is the goal of the next lemma. Recall that $\operatorname{char} \mathbb{K} \neq 2$.

**Lemma 7.3.6.** *We have*

$$N_{(u,v)} = \begin{cases} \langle X_{uv}^{(u,v)} \rangle & \textit{if } u = 1 \textit{ or } v = 1, \\ \langle X_{uv}^{(u,v)}, X_{u-l,v}^{(u,v)} X_{u,v-k}^{(u,v)} \mid (l,k) \in [\![1, u-1]\!] \times [\![1, v-1]\!] \rangle & \textit{otherwise.} \end{cases}$$

*Proof.* For convenience of notation we write $Y_{ij} = X_{ij}^{(u,v)^+}$ for all $(1,1) \leq (i,j) \leq (m,p)$. By Remark 5.4.9 if we show that the ideal:

$$I := < Y_{uv}, \delta_{uv}(Y_{kl}) \mid (k,l) \leq (u,v)^- >$$

is a Poisson ideal of $C_{(u,v)^+}$, then the lemma is proved since we have:

$$I = \begin{cases} < Y_{uv} > & \text{if } u = 1 \text{ or } v = 1, \\ < Y_{uv}, Y_{kv} Y_{ul} \mid (k,l) \in [\![1, u-1]\!] \times [\![1, v-1]\!] > & \text{otherwise.} \end{cases}$$

Firstly it is clear that $\{Y_{uv}, a\} \in I$ for all $a \in C_{(u,v)^+}$. This solves the case where $u = 1$ or $v = 1$. We now assume that $u$ and $v$ are both strictly greater than 1. It remains to prove that $\{Y_{kv} Y_{ul}, Y_{ij}\} \in I$ for all $(k,l) \in [\![1, u-1]\!] \times [\![1, v-1]\!]$ and all $(i,j) \in [\![1, m]\!] \times [\![1, p]\!]$, by considering all the possible cases. If $(i,j) > (u,v)$, then an easy computation prove that $\{Y_{kv}Y_{ul}, Y_{ij}\} \in I$ for all $(k,l) \in [\![1, u-1]\!] \times [\![1, v-1]\!]$. Now if $(i,j) \leq (u,v)$, then we set:

$$f := \{Y_{kv}Y_{ul}, Y_{ij}\} = Y_{kv}\{Y_{ul}, Y_{ij}\} + Y_{ul}\{Y_{kv}, Y_{ij}\}.$$

We will prove that $f \in I$ by examining all possible cases for $(1,1) \leq (i,j) \leq (u,v)$.

- If $(i,j) = (u,v)$, then $f = 2Y_{kv}Y_{ul}Y_{uv} \in I$.
- If $i = u$ and $l < j < v$, then $f = Y_{kv}Y_{ul}Y_{uj} \in I$.
- If $(i,j) = (u,l)$, then $f = 0 \in I$.
- If $i = u$ and $1 \leq j < l$, then $f = -Y_{kv}Y_{ul}Y_{uj} \in I$.
- If $k < i < u$ and $v < j \leq p$, then $f = 2Y_{kj}Y_{ul}Y_{iv} \in I$, since $Y_{ul}Y_{iv} \in I$.
- If $k < i < u$ and $j = v$, then $f = Y_{kv}Y_{ul}Y_{iv} \in I$.
- If $k < i < u$ and $l < j < v$, then $f = 0 \in I$.
- If $k < i < u$ and $j = l$, then $f = -Y_{kv}Y_{ul}Y_{il} \in I$.
- If $k < i < u$ and $1 \leq j < l$, then $f = -2Y_{uj}Y_{kv}Y_{il} \in I$ since $Y_{uj}Y_{kv} \in I$.
- If $i = k$ and $v < j \leq p$, then $f = Y_{kv}Y_{ul}Y_{kj} \in I$.
- If $(i,j) = (k,v)$, then $f = 0$.
- If $i = k$ and $l < j < v$, then $f = -Y_{kv}Y_{ul}Y_{kj} \in I$.
- If $(i,j) = (k,l)$, then $f = -2Y_{kv}Y_{ul}Y_{kl} \in I$.
- If $i = k$ and $1 \leq j < l$, then $f = -2Y_{kv}Y_{uj}Y_{kl} - Y_{kv}Y_{ul}Y_{kj} \in I$ since $Y_{kv}Y_{uj} \in I$.
- If $1 \leq i < k$ and $v < j \leq p$, then $f = 0$.
- If $1 \leq i < k$ and $j = v$, then $f = -Y_{kv}Y_{ul}Y_{iv} \in I$.
- If $1 \leq i < k$ and $l < j < v$, then $f = -2Y_{ul}Y_{iv}Y_{kj} \in I$ since $Y_{ul}Y_{iv} \in I$.
- If $1 \leq i < k$ and $j = l$, then $f = -2Y_{ul}Y_{iv}Y_{kj} - Y_{kv}Y_{ul}Y_{il} \in I$ since $Y_{ul}Y_{iv} \in I$.

- If $1 \le i < k$ and $1 \le j < l$ then $f = -2Y_{kv}Y_{uj}Y_{il} - 2Y_{ul}Y_{iv}Y_{kj} \in I$ since $Y_{kv}Y_{uj} \in I$ and $Y_{ul}Y_{iv} \in I$.

□

We are now ready to prove the remaining inclusion.

**Proposition 7.3.7.** *We have $\xi^{-1}(\mathcal{G}) \subseteq W'_P$.*

*Proof.* Let $\mathcal{C} \in \mathcal{G}$. We want to show that $w_{\mathcal{C}} := \xi^{-1}(\mathcal{C}) \in W'_P$. Note that to show that $w_{\mathcal{C}} \in W'_P$ it is enough to show that $J_{w_{\mathcal{C}}} \in \mathrm{Im}(\varphi)$. We show by induction that $J_{w_{\mathcal{C}}} \in \mathrm{Im}(f_{(u,v)})$ for all $(1,1) \le (u,v) \le (m,p)$. The case $(u,v) = (m,p)$ will then give the result since $f_{(m,p)} = \varphi$.

If $(u,v) = (1,1)$ we have $f_{(u,v)} = \mathrm{id}_{\mathrm{P.Spec}\,(\overline{A})}$ and the result is trivial.

Assume that $J_{w_{\mathcal{C}}} \in \mathrm{Im}(f_{(u,v)^-})$ for some $(1,2) \le (u,v) \le (m,p)$, and set:

$$P_{(u,v)} := f^{-1}_{(u,v)^-}(J_{w_{\mathcal{C}}}) \in \mathrm{P.Spec}\,(C_{(u,v)}).$$

Since $f_{(u,v)} = f_{(u,v)^-} \circ \varphi_{(u,v)}$ it is enough to prove that $P_{(u,v)} \in \mathrm{Im}(\varphi_{(u,v)})$.

Case 1: If $T_{uv} \notin J_{w_{\mathcal{C}}}$, then $T_{uv} = X_{uv}^{(u,v)^+} = X_{uv}^{(u,v)} \notin P_{(u,v)}$ and $P_{(u,v)} \in \mathcal{P}^0_{(u,v)}(C_{(u,v)})$. So by Proposition 7.3.2 we have indeed $P_{(u,v)} \in \mathrm{Im}(\varphi_{(u,v)})$.

Case 2: If $T_{uv} \in J_{w_{\mathcal{C}}}$, then $T_{uv} = X_{uv}^{(u,v)^+} = X_{uv}^{(u,v)} \in P_{(u,v)}$ and $P_{(u,v)} \in \mathcal{P}^1_{(u,v)}(C_{(u,v)})$. By Proposition 7.3.2, we have $P_{(u,v)} \in \mathrm{Im}(\varphi_{(u,v)})$ if and only if $N_{(u,v)} \subseteq P_{(u,v)}$. Therefore, by Lemma 7.3.6, we will get the result if we show that $X_{ul}^{(u,v)} \in P_{(u,v)}$ for all $1 \le l \le v$, or that $X_{kv}^{(u,v)} \in P_{(u,v)}$ for all $1 \le k \le u$. Since $\mathcal{C} \in \mathcal{G}$, the matrix $\left(T_{ij} + J_{w_{\mathcal{C}}}\right)_{ij}$ is a Cauchon matrix. Therefore, $T_{uv} \in J_{w_{\mathcal{C}}}$ implies that $T_{ul} \in J_{w_{\mathcal{C}}}$ for all $1 \le l \le v$, or $T_{kv} \in J_{w_{\mathcal{C}}}$ for all $1 \le k \le u$.

Case 2.1: If $T_{ul} \in J_{w_{\mathcal{C}}}$ for all $1 \le l \le v$, then we prove by induction on $j$ that $X_{ul}^{(u,j)} \in P_{(u,j)}$ for all $l \le j \le v$. If $j = l$, the result comes from the equivalence:

$$T_{ul} \in J_{w_{\mathcal{C}}} \iff X_{ul}^{(u,l)^+} = X_{ul}^{(u,l)} \in P_{(u,l)} \quad \text{(Corollary 7.3.3)}.$$

Assume now that $X_{ul}^{(u,j)} \in P_{(u,j)}$ for some $j \in \{l, \ldots, v-1\}$. We show that $X_{ul}^{(u,j)^+} \in P_{(u,j)^+}$.

Case 2.1.1: If $X_{uj}^{(u,j)} \notin P_{(u,j)}$, then we have:

$$P_{(u,j)^+} = P_{(u,j)}\big(S_{(u,j)^+}\big)^{-1} \cap C_{(u,j)^+}.$$

Therefore:

$$X_{ul}^{(u,j)} = X_{ul}^{(u,j)^+} \in P_{(u,j)} \cap C_{(u,j)^+} \subset P_{(u,j)^+},$$

as required.

Case 2.1.2: If $X_{uj}^{(u,j)} \in P_{(u,j)}$, then we have:

$$P_{(u,j)} = g_{(u,j)}^{-1}\Big(\frac{P_{(u,j)^+}}{\langle X_{uj}^{(u,j)^+}\rangle_P}\Big).$$

In particular:

$$g_{(u,j)}(X_{ul}^{(u,j)}) \in \frac{P_{(u,j)^+}}{\langle X_{uj}^{(u,j)^+}\rangle_P},$$

i.e.

$$\Big(X_{ul}^{(u,j)^+} + \langle X_{uj}^{(u,j)^+}\rangle_P\Big) \in \frac{P_{(u,j)^+}}{\langle X_{uj}^{(u,j)^+}\rangle_P}.$$

Thus $X_{ul}^{(u,j)^+} \in P_{(u,j)^+}$ as desired.

So in all cases, by taking $j = v$, we get that $X_{ul}^{(u,v)} \in P_{(u,v)}$ for all $1 \leq l \leq v$, which proves the result in that case.

Case 2.2: If $T_{kv} \in J_{w_\mathcal{C}}$ for all $1 \leq k \leq u$, then we prove by induction on $(i,j)$ that $X_{kv}^{(i,j)} \in P_{(i,j)}$ for all $1 \leq k \leq u$ and all $(1,2) \leq (i,j) \leq (u,v)$. If $(i,j) = (1,2)$, we have $X_{kv}^{(1,2)} = T_{kv} \in J_{w_\mathcal{C}} = P_{(1,2)}$ for all $1 \leq k \leq u$. Assume now that for some $(1,2) \leq (i,j) < (u,v)$ we have $X_{kv}^{(i,j)} \in P_{(i,j)}$ for all $1 \leq k \leq u$, and let us show that $X_{kv}^{(i,j)^+} \in P_{(i,j)^+}$ for all $1 \leq k \leq u$. As usual we distinguish between 2 cases.

Case 2.2.1: If $X_{ij}^{(i,j)} \in P_{(i,j)}$, then by Lemma 5.4.18 we obtain a Poisson algebra isomorphism $C_{(i,j)}/P_{(i,j)} \cong C_{(i,j)^+}/P_{(i,j)^+}$ sending for all $(1,1) \leq (u,v) \leq (m,p)$ the coset of $X_{uv}^{(i,j)}$ to the coset of $X_{uv}^{(i,j)^+}$. Thus $X_{kv}^{(i,j)} \in P_{(i,j)}$ implies that $X_{kv}^{(i,j)^+} \in P_{(i,j)^+}$ for all $1 \leq k \leq u$, as desired.

Case 2.2.2: If $X_{ij}^{(i,j)} \notin P_{(i,j)}$, then $P_{(i,j)^+} = P_{(i,j)}\big(S_{(i,j)^+}\big)^{-1} \cap C_{(i,j)^+}$. Now for all $1 \leq k \leq u$ we have:

$$X_{kv}^{(i,j)} = \begin{cases} X_{kv}^{(i,j)^+} & \text{if } k \geq i \text{ or } v \geq j, \\ X_{kv}^{(i,j)^+} - X_{kj}^{(i,j)^+} X_{iv}^{(i,j)^+} \big(X_{ij}^{(i,j)^+}\big)^{-1} & \text{if } k < i \text{ and } v < j. \end{cases}$$

Since $X_{kj}^{(i,j)^+} = X_{kj}^{(i,j)}$, $X_{iv}^{(i,j)^+} = X_{iv}^{(i,j)}$ and $X_{ij}^{(i,j)^+} = X_{ij}^{(i,j)}$, we get:

$$X_{kv}^{(i,j)^+} = \begin{cases} X_{kv}^{(i,j)} & \text{if } k \geq i \text{ or } v \geq j, \\ X_{kv}^{(i,j)} + X_{kj}^{(i,j)} X_{iv}^{(i,j)} \big(X_{ij}^{(i,j)}\big)^{-1} & \text{if } k < i \text{ and } v < j. \end{cases}$$

Moreover $k$ and $i$ are smaller than $u$, so we have $X_{kv}^{(i,j)} \in P_{(i,j)}$ and $X_{iv}^{(i,j)} \in P_{(i,j)}$. Thus:

$$X_{kj}^{(i,j)} X_{iv}^{(i,j)} \big(X_{ij}^{(i,j)}\big)^{-1} \in P_{(i,j)}\big(S_{(i,j)^+}\big)^{-1}, \text{ and so:}$$

$$X_{kv}^{(i,j)^+} \in P_{(i,j)}\big(S_{(i,j)^+}\big)^{-1} \cap C_{(i,j)^+} = P_{(i,j)^+}.$$

This finishes the induction.

We conclude this case by taking $(i,j) = (u,v)$ to get $X_{kv}^{(u,v)} \in P_{(u,v)}$ for all $1 \leq k \leq u$, as desired. $\square$

By gathering Propositions 7.3.5 and 7.3.7 together, we get the following result:

**Theorem 7.3.8.** *Let $w \in W$. The following are equivalent:*

(1) $w \in W'_P$,

(2) $J_w \in \mathrm{Im}(\varphi)$,

(3) $C_w \in \mathcal{G}$,

(4) $\varphi^{-1}\big(\mathrm{P.Spec}_w(\overline{A})\big) \neq \emptyset$.

Since $J_w \subseteq Q$ for all $Q \in \mathrm{P.Spec}_w(A)$, Proposition 5.4.16 and the equivalence (1) $\iff$ (2) shows that for all $w \in W'_P$ we have $\mathrm{P.Spec}_w(\overline{A}) \subseteq \mathrm{Im}(\varphi)$. Therefore we have:

$$\varphi\big(\mathrm{P.Spec}_w(A)\big) = \mathrm{P.Spec}_w(\overline{A}) \quad \text{when} \quad w \in W'_P,$$

i.e. the inclusion of Theorem 5.4.14 is actually an equality in the case of matrix Poisson varieties. When $\mathbb{K}$ is infinite, this could also have been deduced from Theorem 5.5.6 since there exists a torus action on $A$ satisfying the appropriate assumptions.

Theorem 7.3.8 also shows in particular that there is a bijection between the sets $W'_P$ and $\mathcal{G}$. This answers positively Question 7.1.1 in this situation, since the set of Cauchon diagrams $W'$ for the algebra of $m \times p$ quantum matrices is also in bijection with $\mathcal{G}$ as shown in [9, Théorème 3.2.1].

It is conjectured in [16] that Spec $\big(\mathcal{O}_q(M_{m,p}(\mathbb{K})\big)$ and P.Spec $\big(\mathcal{O}(M_{m,p}(\mathbb{K})\big)$ should be homeomorphic. Our work on the combinatorial side goes in the same direction as this conjecture. In future work we would like to compare topologically these two spectra using both Cauchon's algorithm and our algorithm.

# Appendix A

# The algorithm in an example

In this appendix, we apply the Poisson deleting derivations algorithm to a polynomial Poisson algebra of dimension 5. In this example there are 3 steps in the algorithm. We express the generators of the Poisson algebras obtained at each step in terms of the generators of the Poisson algebra we started with. Then, using the properties of the canonical embedding derived in Chapter 5, we compute the set of Cauchon diagrams for this Poisson algebra. Let $\mathbb{K}$ be an arbitrary field.

We endow the polynomial algebra $A = \mathbb{K}[X_1, \ldots, X_5]$ with the Poisson structure given by:

$$\begin{aligned} \{X_1, X_2\} &= X_1X_2, & \{X_2, X_4\} &= X_3, \\ \{X_1, X_3\} &= X_2, & \{X_2, X_5\} &= 1 - X_2X_5, \\ \{X_1, X_4\} &= 1 - X_1X_4, & \{X_3, X_4\} &= X_3X_4, \\ \{X_1, X_5\} &= -X_1X_5, & \{X_3, X_5\} &= X_4, \\ \{X_2, X_3\} &= X_2X_3, & \{X_4, X_5\} &= X_4X_5. \end{aligned}$$

The algebra $A$ is an iterated Poisson-Ore extension:

$$A = \mathbb{K}[X_1][X_2; \alpha_2]_P[X_3; \alpha_3, \delta_3]_P[X_4; \alpha_4, \delta_4]_P[X_5; \alpha_5, \delta_5]_P,$$

where:

$$\alpha_i(X_{i-1}) = -X_{i-1}, \qquad \delta_i(X_{i-1}) = 0,$$

$$\alpha_i(X_{i-2}) = 0, \qquad \delta_i(X_{i-2}) = -X_{i-1},$$

$$\alpha_i(X_{i-3}) = X_{i-3}, \qquad \delta_i(X_{i-3}) = -1,$$

$$\alpha_i(X_{i-4}) = X_{i-4}, \qquad \delta_i(X_{i-4}) = 0,$$

with the convention that $\alpha_i$ and $\delta_i$ are defined on $X_{i-j}$ only when $1 \leq j < i \leq 5$. It is shown in Section 4.2.6 that the Poisson algebra $A$ is the semiclassical limit of a suitable iterated Ore extension. It is then easy to show that $A \in \mathcal{P}$. In particular, for $3 \leq i \leq 5$, the derivation $\delta_i$ extends to an iterative, locally nilpotent higher $(1, \alpha_i)$-skew Poisson derivation $(D_{i,k})_{k=0}^{\infty}$ on $\mathbb{K}[X_1, \dots, X_{i-1}]$. Thanks to Proposition 4.1.1 we can be more explicit:

$$D_{i,k}(X_j) = \frac{\Delta_i^k(x_j)}{(t-1)^k (k)!_t}\Big|_{t=1}$$

for all $1 \leq j < i$, and all $k \geq 0$, where the map $\Delta_i$ is defined in Section 4.2.6. Thus, we can give the values of theses higher derivations on the generators of $A$. We obtain:

$$D_{3,k}(X_1) = \begin{cases} X_1 & k = 0 \\ -X_2 & k = 1 \\ 0 & k > 1, \end{cases}$$

and $D_{3,k}(X_2) = 0$ for all $k > 0$. Also:

$$D_{4,k}(X_1) = \begin{cases} X_1 & k = 0 \\ -1 & k = 1 \\ 0 & k > 1, \end{cases}$$

$$D_{4,k}(X_2) = \begin{cases} X_2 & k = 0 \\ -X_3 & k = 1 \\ 0 & k > 1, \end{cases}$$

and $D_{4,k}(X_3) = 0$ for all $k > 0$. Finally $D_{5,k}(X_1) = 0$ for all $k > 0$,

$$D_{5,k}(X_2) = \begin{cases} X_2 & k = 0 \\ -1 & k = 1 \\ 0 & k > 1, \end{cases}$$

$$D_{5,k}(X_3) = \begin{cases} X_3 & k = 0 \\ -X_4 & k = 1 \\ 0 & k > 1, \end{cases}$$

and $D_{5,k}(X_4) = 0$ for all $k > 0$.

## A.1 Poisson deleting derivations algorithm

Since $A \in \mathcal{P}$ we can apply the Poisson deleting derivations algorithm. First, for all $1 \leq i \leq 5$ we relabel the generators of $A$ by setting $X_{i,6} := X_i$, and we set $C_6 := A$. For each $2 \leq j \leq 5$ we will compute the generators of the Poisson algebra $C_j$. The first step of the algorithm corresponds to deleting the derivation $\delta_5$. We obtain the Poisson algebra $C_5$:

$$C_5 = \mathbb{K}[X_{1,5}][X_{2,5}; \alpha_2]_P[X_{3,5}; \alpha_3, \delta_3]_P[X_{4,5}; \alpha_4, \delta_4]_P[X_{5,5}; \alpha_5]_P,$$

where:

$$\begin{aligned} X_{5,5} &= X_{5,6} = X_5 \\ X_{4,5} &= X_{4,6} = X_4 \\ X_{3,5} &= X_{3,6} - X_{4,6}X_{5,6}^{-1} = X_3 - X_4X_5^{-1} \\ X_{2,5} &= X_{2,6} - X_{5,6}^{-1} = X_2 - X_5^{-1} \\ X_{1,5} &= X_{1,6} = X_1. \end{aligned}$$

These generators are obtained from the deleting derivation homomorphism formula (see Theorem 2.2.2) and the higher derivation $(D_{5,k})_k$ computed above. For instance, since

$\eta_5 = 1$, we have:

$$X_{3,5} = \sum_{k\geq 0} D_{5,k}(X_{3,6})X_{5,6}^{-k} = X_{3,6} - X_{4,6}X_{5,6}^{-1}.$$

Similarly we obtain the Poisson algebra:

$$C_4 = \mathbb{K}[X_{1,4}][X_{2,4};\alpha_2]_P[X_{3,4};\alpha_3,\delta_3]_P[X_{4,4};\alpha_4]_P[X_{5,4};\alpha_5]_P,$$

where:

$$\begin{aligned}
X_{5,4} &= X_{5,5} = X_5,\\
X_{4,4} &= X_{4,5} = X_4,\\
X_{3,4} &= X_{3,5} = X_3 - X_4X_5^{-1},\\
X_{2,4} &= X_{2,5} - X_{3,5}X_{4,5}^{-1} = X_2 - X_3X_4^{-1},\\
X_{1,4} &= X_{1,5} - X_{4,5}^{-1} = X_1 - X_4^{-1}.
\end{aligned}$$

The Poisson algebra $C_3$ is given by:

$$C_3 = \mathbb{K}[X_{1,3}][X_{2,3};\alpha_2]_P[X_{3,3};\alpha_3]_P[X_{4,3};\alpha_4]_P[X_{5,3};\alpha_5]_P,$$

where:

$$\begin{aligned}
X_{5,3} &= X_{5,4} = X_{5,5} = X_5,\\
X_{4,3} &= X_{4,4} = X_{4,5} = X_4,\\
X_{3,3} &= X_{3,4} = X_{3,5} = X_3 - X_4X_5^{-1},\\
X_{2,3} &= X_{2,4} = X_{2,5} - X_{3,5}X_{4,5}^{-1} = X_2 - X_3X_4^{-1},\\
X_{1,3} &= X_{1,4} - X_{2,4}X_{3,4}^{-1} = X_1 - \frac{X_2 - X_5^{-1}}{X_3 - X_4X_5^{-1}} = X_1 - \frac{X_2X_5 - 1}{X_3X_5 - X_4}.
\end{aligned}$$

Since there are no more derivations to delete, the algorithm terminates. So we obtain the Poisson affine space $\overline{A} = C_2 = C_3$ generated by:

$$\begin{aligned}
T_5 &= X_5,\\
T_4 &= X_4,\\
T_3 &= X_3 - X_4X_5^{-1},
\end{aligned}$$

$$T_2 = X_2 - X_3X_4^{-1},$$
$$T_1 = X_1 - \frac{X_2X_5 - 1}{X_3X_5 - X_4}.$$

The Poisson brackets between these generators are given by:

$$\begin{aligned} \{T_5, T_4\} &= -T_4T_5, & \{T_4, T_2\} &= 0, \\ \{T_5, T_3\} &= 0, & \{T_4, T_1\} &= T_1T_4, \\ \{T_5, T_2\} &= T_2T_5, & \{T_3, T_2\} &= -T_2T_3, \\ \{T_5, T_1\} &= T_1T_5, & \{T_3, T_1\} &= 0, \\ \{T_4, T_3\} &= -T_3T_4, & \{T_2, T_1\} &= -T_1T_2. \end{aligned}$$

## A.2 Cauchon diagrams

We compute the set of Cauchon diagrams $W'_P \subseteq W = \mathscr{P}([\![1,5]\!])$ for the Poisson algebra $A$. Recall that we have the following injective maps:

$$\varphi_5 : \text{P.Spec}\,(C_6) \longrightarrow \text{P.Spec}\,(C_5),$$
$$\varphi_4 : \text{P.Spec}\,(C_5) \longrightarrow \text{P.Spec}\,(C_4),$$
$$\varphi_3 : \text{P.Spec}\,(C_4) \longrightarrow \text{P.Spec}\,(C_3).$$

It is easy to check that $\langle X_{5,6}\rangle_P = C_6$, $\langle X_{4,5}\rangle_P = C_5$ and $\langle X_{3,4}\rangle_P = \langle X_{2,4}, X_{3,4}\rangle$. Therefore by Lemma 5.4.7 we have:

$$\begin{aligned} P \in \text{P.Spec}\,(C_5) \cap \text{Im}(\varphi_5) &\iff X_{5,6} = X_{5,5} \notin P, \\ Q \in \text{P.Spec}\,(C_4) \cap \text{Im}(\varphi_4) &\iff X_{4,5} = X_{4,4} \notin Q, \\ I \in \text{P.Spec}\,(C_3) \cap \text{Im}(\varphi_3) &\iff \big(X_{3,4} = T_3 \notin I \quad \text{or} \quad \langle T_2, T_3\rangle \subseteq I\big). \end{aligned} \tag{A.1}$$

We have $\text{P.Spec}\,(A) = \text{P.Spec}\,(C_6)$ and $\text{P.Spec}\,(\overline{A}) = \text{P.Spec}\,(C_3)$, and the canonical embedding is the injective map $\varphi$ from $\text{P.Spec}\,(A)$ to $\text{P.Spec}\,(\overline{A})$ defined by $\varphi := \varphi_3 \circ \varphi_4 \circ \varphi_5$.

- Firstly it is easy to show that:

$$\big\{\emptyset, \{1\}, \{2\}, \{1,2\}\big\} \subseteq W'_P \subseteq \mathscr{P}([\![1,3]\!]).$$

The first inclusion comes from Proposition 5.4.17 since $\delta_1 = \delta_2 = 0$. Moreover suppose that $5 \in w \in W'_P$ and let $P \in \text{P.Spec}_w(A)$. Then $T_5 \in \varphi(P)$, and by Corollary 5.4.11 we have $X_{5,6} \in \varphi_5(P) \in \text{Im}(\varphi_5)$, a contradiction to (A.1). We show similarly that $4 \notin w$ for all $w \in W'_P$.

- Moreover one can show that if $3 \in w \in W'_P$, then 2 must belong to $w$, i.e. that we have $\{3\} \notin W'_P$, and $\{1,3\} \notin W'_P$. Suppose that $3 \in w \in W'_P$ and let $P \in \text{P.Spec}_w(A)$. We have $T_3 = X_{3,4} \in \varphi(P)$. Since $\varphi(P) \in \text{Im}(\varphi_3)$, we must have $\langle T_2, T_3 \rangle \subseteq \varphi(P)$. This implies that $T_2 \in \varphi(P)$ and that $2 \in w$.

- We reduced our investigation to the following situation:

$$\{\emptyset, \{1\}, \{2\}, \{1,2\}\} \subseteq W'_P \subseteq \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}.$$

We will show that the undecided sets $\{2,3\}$ and $\{1,2,3\}$ also belong to $W'_P$. For, we exhibit ideals belonging to the appropriate strata. The details are left to the reader; methods from Section 7.2 can be used for instance. We have:

$$\begin{aligned}\varphi(\langle X_2X_5 - 1, X_3X_5 - X_4 \rangle) &= \varphi_3 \circ \varphi_4(\langle X_{2,5}, X_{3,5} \rangle) \\ &= \varphi_3(\langle X_{2,4}, X_{3,4} \rangle) \\ &= \langle T_2, T_3 \rangle,\end{aligned}$$

so that $\langle X_2X_5 - 1, X_3X_5 - X_4 \rangle \in \text{P.Spec}_{\{2,3\}}(A)$. Also we have:

$$\begin{aligned}\varphi(\langle X_1X_4 - 1, X_2X_5 - 1, X_3X_5 - X_4 \rangle) &= \varphi_3 \circ \varphi_4(\langle X_{1,5}X_{4,5} - 1, X_{2,5}, X_{3,5} \rangle) \\ &= \varphi_3(\langle X_{1,4}, X_{2,4}, X_{3,4} \rangle) \\ &= \langle T_1, T_2, T_3 \rangle,\end{aligned}$$

so that $\langle X_1X_4 - 1, X_2X_5 - 1, X_3X_5 - X_4 \rangle \in \text{P.Spec}_{\{1,2,3\}}(A)$. One can also find elements of the strata associated to $\{1\}, \{2\}$ and $\{1,2\}$:

$$\begin{aligned}\varphi(\langle X_1X_3X_5 - X_2X_5 - X_1X_4 + 1 \rangle) &= \varphi_3 \circ \varphi_4(\langle X_{1,5}X_{3,5} - X_{2,5} \rangle) \\ &= \varphi_3(\langle X_{1,4}X_{3,4} - X_{2,4} \rangle) \\ &= \langle T_1 \rangle,\end{aligned}$$

$$\begin{aligned}\varphi(\langle X_2X_4 - X_3\rangle) &= \varphi_3 \circ \varphi_4(\langle X_{2,5}X_{4,5} - X_{3,5}\rangle) \\ &= \varphi_3(\langle X_{2,4}\rangle) \\ &= \langle T_2\rangle,\end{aligned}$$

$$\begin{aligned}\varphi(\langle X_1X_4 - 1, X_2X_4 - X_3\rangle) &= \varphi_3 \circ \varphi_4(\langle X_{1,5}X_{4,5} - 1, X_{2,5}X_{4,5} - X_{3,5}\rangle) \\ &= \varphi_3(\langle X_{1,4}, X_{2,4}\rangle) \\ &= \langle T_1, T_2\rangle.\end{aligned}$$

We conclude that $W'_P = \big\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\big\}$. Note that in this example we have $\overline{W} \subsetneq W'_P$.

# Appendix B

# Question 7.1.1 for an algebra without torus action

In this appendix we answer positively Question 7.1.1 for the following algebra. Let $R_t$ be the iterated Ore extension:

$$R_t = \mathbb{K}[t^{\pm 1}][x_1, x_2][x_3, \sigma_3][x_4; \sigma_4, \Delta_4],$$

where:

- $\sigma_3$ is the automorphism of $\mathbb{K}[t^{\pm 1}][x_1, x_2]$ such that $\sigma_3(t) = t$, $\sigma_3(x_1) = tx_1$ and $\sigma_3(x_2) = tx_2$,
- $\sigma_4$ is the automorphism of $\mathbb{K}[t^{\pm 1}][x_1, x_2][x_3, \sigma_3]$ such that $\sigma_4(t) = t$, $\sigma_4(x_1) = t^{-1}x_1$, $\sigma_4(x_2) = t^{-1}x_2$ and $\sigma_4(x_3) = x_3$,
- $\Delta_4$ is the $\sigma_4$-derivation of $\mathbb{K}[t^{\pm 1}][x_1, x_2][x_3, \sigma_3]$ such that $\Delta_4(t) = \Delta_4(x_1) = \Delta_4(x_2) = 0$ and $\Delta_4(x_3) = (t-1)(x_1 + x_2)$.

It is easy to check that $R_t \in \mathcal{R}$. Suppose $\operatorname{char} \mathbb{K} = 0$. We now compute the sets $W'_P$ and $W'$ of Cauchon diagrams for the algebras $A := R_t/(t-1)R_t$ and $R_q := R_t/(t-q)R_t$, for a nonzero non root of unity $q \in \mathbb{K}$.

## B.1 Cauchon diagrams for $A := R_t/(t-1)R_t$

The Poisson algebra $A$ is the Poisson-Ore extension $A = \mathbb{K}_{\boldsymbol{\lambda}}[X_1, X_2, X_3][X_4; \alpha, \delta]_P$, where $\lambda_{12} = 0$, $\lambda_{13} = \lambda_{23} = -1$, $\alpha := -X_1 \frac{\partial}{\partial X_1} - X_2 \frac{\partial}{\partial X_2}$ and $\delta := (X_1 + X_2)\frac{\partial}{\partial X_3}$. We have:

$$\begin{aligned} \{X_4, X_1\} &= -X_1X_4, & \{X_3, X_1\} &= X_1X_3, \\ \{X_4, X_2\} &= -X_2X_4, & \{X_3, X_2\} &= X_2X_3, \\ \{X_4, X_3\} &= X_1 + X_2, & \{X_2, X_1\} &= 0. \end{aligned}$$

Note that this Poisson algebra appears in Example 5.4.15. Since the derivation $\delta$ is locally nilpotent and that we have $[\delta, \alpha] = \delta$, the derivation $\delta$ uniquely extends to an iterative, locally nilpotent higher $(1, \alpha)$-skew Poisson derivation:

$$(D_i)_i = \left(\frac{\delta^i}{i!}\right)_i$$

on $\mathbb{K}_{\boldsymbol{\lambda}}[X_1, X_2, X_3]$. The deleting derivations algorithm (there is only one step) leads to the Poisson affine space $\overline{A} = \mathbb{K}_{\boldsymbol{\lambda}'}[T_1, T_2, T_3, T_4]$ where:

$$\boldsymbol{\lambda}' = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix},$$

and where $T_1 := X_1$, $T_2 := X_2$, $T_3 := X_3 + (X_1 + X_2)X_4^{-1}$ and $T_4 := X_4$. The canonical embedding is the map $\varphi$ from P.Spec$(A)$ to P.Spec$(\overline{A})$ defined by:

$$P \longmapsto \begin{cases} PS^{-1} \cap \overline{A} & X_4 \notin P, \\ g^{-1}(P/\langle X_4 \rangle_P) & X_4 \in P, \end{cases}$$

where $S$ is the multiplicative set of $A$ generated by $X_4$, and where:

$$\begin{aligned} g \; : \quad \overline{A} &\longrightarrow \frac{A}{\langle X_4 \rangle_P}, \\ T_i &\longmapsto X_i + \langle X_4 \rangle_P. \end{aligned}$$

Let $w \in W$ and assume that $4 \notin w$. Then $\langle T_i \mid i \in w \rangle \in \mathrm{Im}(\varphi)$ by Lemma 5.4.7 and:

$$\varphi^{-1}(\langle T_i \mid i \in w \rangle) \in \mathrm{P.Spec}_w(A).$$

In particular this shows that $\mathscr{P}([\![1,3]\!]) \subseteq W'_P$ (alternatively, we obtain the same result from Proposition 5.4.17). We can actually be more precise and give the explicit preimages of the ideals $\langle T_i \mid i \in w \rangle$ for $w \in \mathscr{P}([\![1,3]\!])$:

$$\begin{aligned}
&\varphi^{-1}(\emptyset) = \emptyset, \\
&\varphi^{-1}(\langle T_1 \rangle) = \langle X_1 \rangle, \\
&\varphi^{-1}(\langle T_2 \rangle) = \langle X_2 \rangle, \\
&\varphi^{-1}(\langle T_3 \rangle) = \langle X_1 + X_2 + X_3X_4 \rangle, \\
&\varphi^{-1}(\langle T_1, T_2 \rangle) = \langle X_1, X_2 \rangle, \\
&\varphi^{-1}(\langle T_1, T_3 \rangle) = \langle X_1, X_2 + X_3X_4 \rangle, \\
&\varphi^{-1}(\langle T_2, T_3 \rangle) = \langle X_2, X_1 + X_3X_4 \rangle, \\
&\varphi^{-1}(\langle T_1, T_2, T_3 \rangle) = \langle X_1, X_2, X_3 \rangle.
\end{aligned}$$

The methods we use to prove these equalities are the same as the one used in Section 7.2 and in Appendix A.2. We illustrate the case $w = \{3\}$. Let $Q := \langle T_3 \rangle$. Since $T_4 \notin Q$ we have $Q \in \mathrm{Im}(\varphi)$ and:

$$\varphi^{-1}(Q) = QS^{-1} \cap A = \{fT_3T_4^{-i} \mid f \in \overline{A},\ i \geq 0\} \cap A.$$

Set $P := \langle X_1 + X_2 + X_3X_4 \rangle$. Since:

$$X_1 + X_2 + X_3X_4 = X_4\big(X_3 + (X_1 + X_2)X_4^{-1}\big) = T_4T_3 \in \varphi^{-1}(Q),$$

we have $P \subseteq \varphi^{-1}(Q)$. It is easy to see that $X_1 + X_2 + X_3X_4 \in Z_P(A)$, so the ideal $P$ is a Poisson ideal. Moreover one can check that $X_1 + X_2 + X_3X_4$ is irreducible in $A$, so $P$ is also a prime ideal. Therefore $P \in \mathrm{P.Spec}\,(A)$. It is clear that $X_4 \notin P$ so we have:

$$\varphi(P) = PS^{-1} \cap \overline{A} = \{f\big(X_1 + X_2 + X_3X_4\big)X_4^{-i} \mid f \in A,\ i \geq 0\} \cap \overline{A}.$$

Since:

$$T_3 = X_3 + (X_1 + X_2)X_4^{-1} = \big(X_3X_4 + X_1 + X_2\big)X_4^{-1} \in \varphi(P),$$

we have $Q \subseteq \varphi(P)$.

We know that $X_4 = T_4 \notin \varphi(P)$, and that $T_3 \in \varphi(P)$. One can check that $T_1$ and $T_2$ do not belong to $\varphi(P)$ as follows. Suppose that $T_1 = X_1 \in \varphi(P)$ and write:

$$T_1 = X_1 = f(X_1 + X_2 + X_3X_4)X_4^{-i},$$

for some $f \in A$ and some $i \geq 0$. We have:

$$X_1X_4^i = f(X_1 + X_2 + X_3X_4), \tag{B.1}$$

and by a degree argument we deduce that $f \in \mathbb{K}[X_2, X_3, X_4]$. By seeing equation (B.1) as an equality of two polynomials in the variable $X_1$ and with coefficients in $\mathbb{K}[X_2, X_3, X_4]$ we obtain:

$$f = X_4^i, \quad \text{and} \quad 0 = (X_2 + X_3X_4)f,$$

which is impossible. Thus $T_1 = X_1 \notin \varphi(P)$. We conclude similarly that we have $T_2 = X_2 \notin \varphi(P)$. Therefore we have $P \in \mathrm{P.Spec}_{\{3\}}(A)$. Note that by definition we have $\varphi^{-1}(Q) \in \mathrm{P.Spec}_{\{3\}}(A)$. Since $\varphi$ induces an homeomorphism from $\mathrm{P.Spec}_{\{3\}}(A)$ to $\varphi\big(\mathrm{P.Spec}_{\{3\}}(A)\big)$, the inclusions $P \subseteq \varphi^{-1}(Q)$ and $Q \subseteq \varphi(P)$ give us $\varphi(P) = Q$, i.e.:

$$\varphi^{-1}(\langle T_3 \rangle) = \langle X_1 + X_2 + X_3X_4 \rangle.$$

We now deal with the remaining elements of $W$. Assume that $4 \in w \in W'_P$ and let $Q \in \mathrm{P.Spec}_w(\overline{A}) \cap \mathrm{Im}(\varphi)$. By Lemma 5.4.7 we must have:

$$\langle T_1 + T_2, T_4 \rangle \subseteq Q,$$

or equivalently:

$$\langle X_1 + X_2, X_4 \rangle \subseteq \varphi^{-1}(Q).$$

Thus for $4 \in w \in W'_P$ and $Q \in \text{P.Spec}_w(\overline{A}) \cap \text{Im}(\varphi)$ we must have the following equivalence:

$$\big(1 \in w\big) \Leftrightarrow \big(T_1 \in Q\big) \Leftrightarrow \big(X_1 \in \varphi^{-1}(Q)\big) \Leftrightarrow \big(X_2 \in \varphi^{-1}(Q)\big) \Leftrightarrow \big(T_2 \in Q\big) \Leftrightarrow \big(2 \in w\big).$$

This means that the sets $\{1,4\}, \{2,4\}, \{1,3,4\}$ and $\{2,3,4\}$ do not belong to $W'_P$. On the other hand it is easy to see that:

$$\begin{aligned}
&\langle X_1 + X_2, X_4 \rangle \in \text{P.Spec}_{\{4\}}(A), \\
&\langle X_1 + X_2, X_3, X_4 \rangle \in \text{P.Spec}_{\{3,4\}}(A), \\
&\langle X_1, X_2, X_4 \rangle \in \text{P.Spec}_{\{1,2,4\}}(A), \\
&\langle X_1, X_2, X_3, X_4 \rangle \in \text{P.Spec}_{\{1,2,3,4\}}(A),
\end{aligned}$$

using the fact that when $X_4 \in P \in \text{P.Spec}(A)$, there is a Poisson algebra isomorphism from $A/P$ to $\overline{A}/\varphi(P)$ sending $X_i + P$ to $T_i + \varphi(P)$ for $1 \leq i \leq 4$ (Lemma 5.4.18). Therefore we obtain:

$$W'_P = \mathscr{P}([\![1,3]\!]) \cup \big\{\{4\}, \{3,4\}, \{1,2,4\}, \{1,2,3,4\}\big\}.$$

## B.2 Cauchon diagrams for $R_q = R_t/(t-q)R_t$

The algebra $R_q$ is given by generators $x_1, x_2, x_3, x_4$ and relations:

$$\begin{aligned}
x_2 x_1 &= x_1 x_2, & x_4 x_1 &= q^{-1} x_1 x_4, \\
x_3 x_1 &= q x_1 x_3, & x_4 x_2 &= q^{-1} x_2 x_4, \\
x_3 x_2 &= q x_2 x_3, & x_4 x_3 &= x_3 x_4 + (q-1)(x_1 + x_2).
\end{aligned}$$

$R_q$ can be expressed an in iterated Ore extension, and Cauchon's deleting derivations algorithm can be applied. The algebra $\overline{R_q}$ is the quantum affine space in the generators:

$$t_1 := x_1, \quad t_2 := x_2, \quad t_3 := x_3 + q(x_1 + x_2)x_4^{-1}, \quad \text{and} \quad t_4 := x_4,$$

associated to the matrix $\boldsymbol{\lambda}'$ defined in the previous section. We denote again by $\varphi$ the canonical embedding from $\text{Spec}(R_q)$ to $\text{Spec}(\overline{R_q})$ ([8, Definition 4.4.1]). By [8, Proposition 4.3.1] it is clear that all ideals in $\text{Spec}(\overline{R_q})$ which do not contain $t_4$ belong to $\text{Im}(\varphi)$. Thus if $4 \notin w \in W$ then $w \in W'$ since:

$$\langle t_i \mid i \in w \rangle \in \mathrm{Im}(\varphi).$$

Now suppose that $x_4 \in P \in \mathrm{Spec}\,(R_q)$. Then the image of $P$ by $\varphi$ is given by:

$$g^{-1}(P/\langle x_4 \rangle),$$

where $g$ is the surjective algebra homomorphism from $\overline{R_q}$ to $R_q/\langle x_4 \rangle$ sending $t_i$ to $x_i + \langle x_4 \rangle$ for all $i$ ([8, Notation 4.3.1]). Thus a prime ideal $\overline{R_q}$ containing $t_4$ will belong to $\mathrm{Im}(\varphi)$ if and only if it contains $\ker g$. There is a vector space isomorphism $\Psi$ from $\overline{R_q}$ to $R_q$ sending $t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4}$ to $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ for all $i_1, i_2, i_3, i_4 \geq 0$. So we can write $g = \pi \circ \Psi$ where $\pi$ is the canonical projection, and thus it is enough to understand $\langle x_4 \rangle$ since $\ker g = \Psi^{-1}(\langle x_4 \rangle)$. First note that we have:

$$x_1 + x_2 = \frac{[x_4, x_3]}{q-1} \in \langle x_4 \rangle.$$

One can check that $\langle x_4 \rangle = R_q x_4 + R_q(x_1 + x_2)$, where by $R_q x_4$ we denote the right ideal generated by $x_4$. It is now easy to see that:

$$\ker g = \overline{R_q} t_4 + \overline{R_q}(t_1 + t_2).$$

Thus a prime ideal in $\overline{R_q}$ containing $t_4$ belongs to $\mathrm{Im}(\varphi)$ if and only if it contains $t_1 + t_2$. This shows that if $4 \in w$, then $w \in W'$ if and only if $\big((1 \in w) \iff (2 \in w)\big)$. In particular we have:

$$\begin{aligned}
&\varphi^{-1}(\ker g) \in \mathrm{Spec}_{\{4\}}(R_q),\\
&\varphi^{-1}(\langle \ker g, t_3 \rangle) \in \mathrm{Spec}_{\{3,4\}}(R_q),\\
&\varphi^{-1}(\langle t_1, t_2, t_4 \rangle) \in \mathrm{Spec}_{\{1,2,4\}}(R_q),\\
&\varphi^{-1}(\langle t_1, t_2, t_3, t_4 \rangle) \in \mathrm{Spec}_{\{1,2,3,4\}}(R_q).
\end{aligned}$$

We conclude that:

$$W' = \mathscr{P}([\![1,3]\!]) \cup \big\{\{4\}, \{3,4\}, \{1,2,4\}, \{1,2,3,4\}\big\},$$

so that $W' = W'_P$, and Question 7.1.1 is verified for the algebra $R_t$.

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