Moduli Spaces of Lumps on Real Projective Space

Steffen Krusch[*] and *Abera A. Muhamed*[†]
School of Mathematics, Statistics and Actuarial Sciences
University of Kent, Canterbury, UK

December 9, 2014

Abstract

Harmonic maps that minimize the Dirichlet energy in their homotopy classes are known as lumps. Lump solutions on real projective space are explicitly given by rational maps subject to a certain symmetry requirement. This has consequences for the behaviour of lumps and their symmetries. An interesting feature is that the moduli space of charge three lumps is a D_2-symmetric 7-dimensional manifold of cohomogeneity one. In this paper, we discuss the charge three moduli spaces of lumps from two perspectives: discrete symmetries of lumps and the Riemann-Hurwitz formula. We then calculate the metric and find explicit formula for various geometric quantities. We also discuss the implications for lump decay.

1 Introduction

Rational maps of degree N are solutions of the Bogomolny equation of the $O(3)$ sigma model with topological charge N and energy $2\pi|N|$. Belavin and Polyakov [14] studied the Bogomolyni equations by change of variables and explored the Lagrangian density of the classically equivalent $\mathbb{C}P^1$ sigma model. The algebraic topology of rational maps and the construction of harmonics between surfaces have been studied by Segal [17] and by Eells and Lemaire [4], respectively. Speight and Sadun [18] showed the moduli space for a compact Riemann surface is geodesically incomplete. The metric on the space of holomorphic maps is given by restricting the kinetic energy term where the moduli space coordinates are allowed to depend

[*]S.Krusch@kent.ac.uk
[†]A.A.Muhamed@kent.ac.uk

on time. This metric is Kähler [15]. The low energy dynamics of a $\mathbb{C}P^1$ lump on the space-time $S^2 \times \mathbb{R}$ [20] and the geometry of a space of rational maps of degree N [13] have been studied. The Fubini-Study metric γ_{FS} of rational maps of degree one has been studied by Krusch and Speight [7]. Here a rational map was identified with the projective equivalence classes of its coefficient such that Rat_1 is an open subset of $\mathbb{C}P^3$ which is equipped with the Fubini-Study metric of constant holomorphic sectional curvature 4. Lumps can decay but it has been shown in Refs. [10, 23] that the scattering of lumps takes place before lump decay using the geodesic approximation. A head-on collisions between lumps in the $2+1$-dimensional $\mathbb{C}P^1$ model on a flat torus has been studied numerically by Cova and Zakrzewski [3] and analytically by Speight [22]. Rational maps also play an important role in related models like the Skyrme model [19]. For example, the rational map ansatz [6] gives a good approximations for the symmetries of Skyrme configurations, and the Finkelstein-Rubinstein constraints can be calculated directly from this ansatz using homotopy theory [8]. We study the symmetries of rational maps to understand the geometry and dynamics of lumps.

Harmonic maps are solutions of Laplace's equation on Riemannian manifolds and are usually known as lumps. Denote by M_N the moduli space of degree N harmonic maps. M_N is a $2N+1$-dimensional smooth complex Riemannian manifold. There is a natural Riemannian metric on M_N, which is called the L^2 metric. The L^2 metric is well defined on a compact Riemann surface as the non-normalizable zero modes are absent [13]. For the $\mathbb{C}P^1$ model, one can have an explicit expression of harmonic maps in terms of rational maps given by the ratio of two polynomials with no common roots. Lumps and their symmetries can be understood in terms of rational maps on the projective plane. Speight [21] studied the L^2 metric on the moduli spaces of degree 1 harmonic maps on both S^2 and $\mathbb{R}P^2$ and obtained an explicit formula. We focus mainly on charge three rational maps between $\mathbb{R}P^2$, acquiring a detailed and careful understanding of their L^2 geometry. The L^2 metric plays an important role in slow lump dynamics just as Samols [16] metric does for vortices.

In section 2, we discuss the $O(3)$ and $\mathbb{C}P^1$-sigma models on a Riemann surface. In section 3, we derive families of symmetric rational maps in real projective space. We then apply the Riemann-Hurwitz formula. We also study an $SO(3) \times SO(3)$ invariant angular integral of rational maps which plays an important role in our understanding of the moduli space of charge three lumps. In section 4, we discuss the moduli space metric of charge three lumps on projective space. We evaluate the metric coefficients explicitly and calculate various geometric quantities. We end with a conclusion.

2 The $O(3)$-sigma model on a Riemann surface

In $(d+1)$-dimensional space time, the non-linear sigma model on the space Σ with target space Y is defined by the Lagrangian

$$L = \frac{1}{2}\int_{\Sigma} d\mu_g \partial_\mu \phi_l \partial^\mu \phi_m H^{lm}, \tag{2.1}$$

where $d\mu_g$ is the volume element of Σ, g is the Riemannian metric on Σ, $\partial^\mu = \eta^{\mu\nu}\partial_\nu$ and $\eta^{\mu\nu}$ are the components of the inverse of the Lorentzian metric

$$\eta = dt^2 - g$$

on the space-time $\mathbb{R}\times\Sigma$, and H_{lm} is the metric on Y. The $O(3)$-sigma model is a famous example of a non-linear sigma model. In the $O(3)$-sigma model, the field can be parameterized as a three-component unit vector, $\phi = (\phi_1, \phi_2, \phi_3)$ with $\phi\cdot\phi = 1$, and the Lagrangian is given by

$$L = \frac{1}{4}\int_{\Sigma} d\mu_g \partial_\mu \phi \cdot \partial^\mu \phi. \tag{2.2}$$

Thus, the target space can be identified with a Riemann sphere S^2. The sigma model can be formulated in terms of fields (ϕ_1, ϕ_2, σ) such that $\sigma = \pm\sqrt{1-(\phi_1^2+\phi_2^2)}$ where ϕ_1 and ϕ_2 are locally unconstrained [11]. We consider $\Sigma = S^2$ and $\Sigma = \mathbb{R}P^2$, and target space $Y = \Sigma$, but other examples have been discussed in the literature [21]. Denote by M_N the moduli space of degree N static solutions of the $\mathbb{C}P^1$ model on Σ and let $\phi : \Sigma \longrightarrow \mathbb{C}P^1$. The kinetic energy functional induces a natural metric γ on M_N which is a finite dimensional, smooth Riemannian manifold. The moduli space M_N is geodesically incomplete with respect to the metric γ induced by the kinetic energy functional for any degree N static solutions of the $\mathbb{C}P^1$ model on Σ [18]. The homotopy classes of a continuous map ϕ, by the Hopf degree theorem [2], are labeled by the topological degree of ϕ.

In the following we consider $\Sigma = S^2$. Let R be the stereographic coordinate image of ϕ on the target space. The coordinate R is given by $R = \frac{\phi_1+i\phi_2}{1+\phi_3}$ and let the local complex coordinate $z = x^1 + ix^2$ and its conjugate $\bar{z} = x^1 - ix^2$. One can then explicitly express ϕ in terms of R as

$$\phi = \left(\frac{R+\bar{R}}{1+|R|^2}, \frac{-i\left(R-\bar{R}\right)}{1+|R|^2}, \frac{1-|R|^2}{1+|R|^2}\right).$$

Since $R = R(t, z, \bar{z})$ is the function of t, z and $\bar{z}$, the Lagrangian (2.2) becomes

$$L = \int_{S^2} dS \frac{\partial_\mu R \partial^\mu \bar{R}}{(1+|R|^2)^2}, \tag{2.3}$$

where $\mu = t, z, \bar{z}$. This Lagrangian is referred to as the $\mathbb{C}P^1$ sigma model. The $\mathbb{C}P^1$ sigma model in $d = 2+1$ dimensions is a non-linear field theory possessing topological solitons, called lumps.

For $\Sigma = S^2$, the energy E and the topological charge N are given by

$$E = 2\int \frac{(|\partial_z R|^2 + |\partial_{\bar{z}} R|^2)(1+|z|^2)^2}{(1+|R|^2)^2} \frac{dzd\bar{z}}{(1+|z|^2)^2}, \tag{2.4}$$

$$N = \frac{1}{\pi}\int \frac{(|\partial_z R|^2 - |\partial_{\bar{z}} R|^2)(1+|z|^2)^2}{(1+|R|^2)^2} \frac{dzd\bar{z}}{(1+|z|^2)^2}, \tag{2.5}$$

where $\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$. Now one can show that the energy E and the topological degree N satisfy the the Bogomolny bound $E \geq 2\pi N$. Equality holds if and only if the Cauchy-Riemann equation is satisfied, namely $\partial_{\bar{z}} R = 0$, whose solutions are holomorphic functions $R(z)$. We can do a similar calculation to obtain $E \geq -2\pi N$ with equality when $\partial_z R = 0$, which is satisfied by antiholomorphic functions $R(\bar{z})$. In summary, the energy E is minimized to $2\pi|N|$ in each topological class by a solution of the Cauchy-Riemann equations

$$\begin{cases} \partial_{\bar{z}} R = 0 & \text{if } N \geq 0, \\ \partial_z R = 0 & \text{if } N \leq 0. \end{cases}$$

Without loss of generality we will focus on holomorphic maps. For the complex coordinates z and R on the domain and codomain, the general degree N rational map is

$$R(z) = \frac{p(z)}{q(z)} = \frac{a_1 + a_2 z + ... + a_{N+1} z^N}{a_{N+2} + a_{N+3} z + ... + a_{2N+2} z^N},$$

where $a_i \in \mathbb{C}$ are constants and a_{N+1} and a_{2N+2} do not both vanish simultaneously, and $p(z)$ and $q(z)$ have no common roots. Suppose $a_{2N+2} \neq 0$ and define a complex coordinate $b_\alpha = \frac{a_\alpha}{a_{2N+2}}, \alpha = 1, ..., 2N+1$. The inclusion property $M_N \subset \mathbb{C}P^{2N+1}$ implies that the metric γ is Kähler in this coordinate system. Thus, γ is given by

$$\gamma = \gamma_{\alpha\beta} db^\alpha \overline{db^\beta}, \tag{2.6}$$

where repeated indices are summed over. The metric $\gamma_{\alpha\beta}$ can be written as

$$\gamma_{\alpha\beta} = \int_{\mathbb{C}} \frac{dzd\bar{z}}{(1+|z^2|)^2(1+|R|^2)^2} \frac{\partial R}{\partial b^\alpha} \overline{\frac{\partial R}{\partial b^\beta}}, \tag{2.7}$$

and $R(z)$ is given by

$$R(z) = \frac{b_1 + b_2 z ... + b_{N+1} z^N}{b_{N+2} + b_{N+3} z + ... + z^N}.$$

3 Symmetries of rational maps on projective space

In the following, we derive families of symmetric rational maps in real projective space. We start by considering rational maps between Riemann spheres. A rational map $R(z)$ has a discrete symmetry if

$$M_1(R(M_2(z))) = R(z), \tag{3.1}$$

where M_1 and M_2 are Möbius transformations. M_1 is a rotation in target space, also known as an isorotation, whereas M_2 is a rotation in space. Note if $R(z)$ has symmetry (3.1) then the rational map $\tilde{R}(z) = \hat{M}_1(R(\hat{M}_2(z)))$ has the symmetry

$$\tilde{R}(z) = \tilde{M}_1(\tilde{R}(\tilde{M}_2(z))),$$

where $\tilde{M}_1(z) = \hat{M}_1(M_1(\hat{M}_1^{-1}(z)))$ and $\tilde{M}_2 = \hat{M}_2^{-1}(M_2(\hat{M}_2(z)))$. So, by change of coordinates in domain and target, we can choose our symmetry to be around convenient axes.

We define a C_n^k symmetry of a rational map as a rotation in space by $\alpha = 2\pi/n$ followed by a rotation in target space by $\beta = k\alpha$. The following lemma classifies which C_n^k symmetries are allowed for a rational map of degree N.

Lemma 1. *A rational map of degree N can have a C_n^k symmetry if and only if $N \equiv 0 \mod n$ or $N \equiv k \mod n$.*

For a proof see [9]. Note that S^2 is the universal covering space of $\mathbb{R}P^2$ and $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$ [12]. The map $\phi : S^2 \to S^2$ projects to a harmonic map $\tilde{\phi} : \mathbb{R}P^2 \to \mathbb{R}P^2$ if and only if $p \circ \phi = \phi \circ p$ [4, 20], where $p : S^2 \longrightarrow S^2$, $z \mapsto -\bar{z}^{-1}$ is the antipodal map in stereographic coordinates. Hence, in real projective space, rational maps must satisfy the additional constraint

$$R\left(-\frac{1}{\bar{z}}\right) = -\frac{1}{\bar{R}(z)}. \tag{3.2}$$

Then $R(z)$ can be written in the general form as

$$R(z) = \frac{\sum\limits_{k=0}^{N} a_k z^k}{\sum\limits_{k=0}^{N} (-1)^k \bar{a}_{N-k} z^k}, \tag{3.3}$$

where $a_k \in \mathbb{C}$. Alternatively, we can write the general form of this rational map as

$$R(z) = e^{i\phi} \frac{(z - z_1)(z - z_2)...(z - z_N)}{(1 + \bar{z}_1 z)(1 + \bar{z}_2 z)...(1 + \bar{z}_N z)}, \tag{3.4}$$

where $z_1, z_2, ..., z_n \in \mathbb{C}$, and $\phi \in [0, 2\pi)$. This severely restricts the number of allowed symmetries. Denote by $\widetilde{Rat}_N$ the degree N rational maps in real projective space which is a submanifold of Rat_N.

Lemma 2. *A rational map of degree N which satisfies (3.2) can have a C_n^k symmetry if and only if $N \equiv k \mod n$. If $n \geq N$ then the rational map has D_∞ symmetry.*

Proof:
Without loss of generality we choose coordinates such that one C_n^k rotation is around the x_3-axis in space and target space.

First consider $N \equiv 0 \mod n$. Then $N = nl$, and a C_n^k rational map can be written as

$$R(z) = \frac{r(z^n)}{z^{n-k} s(z^n)},$$

where $r(z)$ has degree l and $s(z)$ has degree less than l. On the other hand, given $r(z)$ the constraint (3.2) fixes the coefficients of the denominator. In particular, only coefficients of powers of z^n will be non-zero. Hence, the only compatible solution is $k = 0$.

Consider the case $N \equiv k \mod n$ which includes the $k = 0$ case for $N \equiv 0 \mod n$. Set $k = N \mod n$ and $s = (N-k)/n$, then the rational map is given by

$$R(z) = \frac{\sum_{j=0}^{s} a_j z^{jn+k}}{\sum_{j=0}^{s} b_j z^{jn}}. \tag{3.5}$$

The inversion symmetry (3.2) leads to the following two constraints on the coefficients

$$(-1)^{nj} \bar{b}_{s-j} = \lambda a_j \tag{3.6}$$

and

$$(-1)^{k+1} \bar{a}_{s-j} (-1)^{nj} = \lambda \bar{b}_j, \tag{3.7}$$

where λ takes account of the fact that numerator and denominator can be multiplied with a common factor. Taking the modulus, we obtain that $|\lambda| = 1$, so that $\bar{\lambda} = 1/\lambda$. By relabelling $j \mapsto s - j$, equation (3.7) becomes

$$a_j = \bar{\lambda} (-1)^{k+1+n(s-j)} \bar{b}_{s-j}. \tag{3.8}$$

This is compatible with equation (3.6) provided $ns + k = N$ is odd. For $N = n$, we obtain the map

$$R(z) = \lambda \frac{a_1 z^n + a_0}{-\bar{a}_0 z^n + \bar{a}_1}. \tag{3.9}$$

Performing a Möbius transformation in target space to remove the phase λ this is equivalent to a Möbius transformation of the axial map.

Similarly, for $n > N$, the rational map (3.5) reduces to

$$R(z) = \frac{\lambda a_0}{\bar{a}_0} z^N,$$

since in this case $N = k$ and $s = 0$. This is again the axial map. $\square$

For $N = 1$, the only rational maps compatible with the constraint (3.2) is

$$R(z) = \frac{az + b}{-\bar{b}z + \bar{a}}, \tag{3.10}$$

where $|a|^2 + |b|^2 \neq 1$. Hence, this is isomorphic to $PU(2) \cong SO(3)$, and the moduli space for charge one is $SO(3)$, as pointed out in Ref. [21].

3.1 $N = 3$ lumps

In the following, we will discuss the case $N = 3$ in more detail. According to Lemma 1 and 2, imposing C_n symmetry with $n \geq 3$, we obtain D_∞ symmetry, given by maps of the form

$$R(z) = \frac{az^3 + b}{-\bar{b}z^3 + \bar{a}}, \tag{3.11}$$

where $a, b \in \mathbb{C}$, and a and b are not both zero. Here the rotation axis in space has been chosen to be the x_3-axis. This choice corresponds to fixing two real parameters. Here, the symmetry is a C_3^0. Consider the case $a \neq 0$. Then the rational map (3.11) can be rewritten as

$$R(z) = \mathrm{e}^{i\psi} \frac{z^3 + c}{-\bar{c}z^3 + 1},$$

where $c = b/a$ and $e^{i\psi} = a/\bar{a}$. Hence, the moduli space of the symmetric lumps of (3.11) is parametrized by one complex number and a phase which together with the choice of axes gives real dimension 5. The moduli space can also be viewed as the orbit under rotations and isorotations of the map

$$R(z) = z^3.$$

Since, rotation and isorotations act independently apart from the axial symmetry around the third axis, the dimension of the moduli space is again 5.

The only rational maps that are compatible with a C_2 symmetry around the x_3-axis are given by

$$R(z) = \frac{az^3 + bz}{\bar{b}z^2 + \bar{a}}. \tag{3.12}$$

By Möbius transformations preserving this symmetry, namely rotations around the third axis in space and target space, the rational map can be brought into the form

$$R(z) = \frac{z^3 + cz}{cz^2 + 1}, \tag{3.13}$$

such that c is real and non-negative. The surprising fact is that this map has D_2 symmetry, since it is also symmetric under

$$R(z) = \frac{1}{R\left(\frac{1}{z}\right)}.$$

Hence, imposing a C_2 symmetry automatically gives a family of D_2 symmetric maps. Since rotations and isorotations act independently and cannot change the magnitude of the parameter c, the moduli space of the symmetric lumps of (3.13) has real dimension 7. Another way of calculating the dimension of the symmetry orbit of (3.13) is as follows. A general rational map can be written as equation (3.4) for $N = 3$:

$$R(z) = \mathrm{e}^{i\phi} \frac{(z - z_1)(z - z_2)(z - z_3)}{(1 + \bar{z}_1 z)(1 + \bar{z}_2 z)(1 + \bar{z}_3 z)}. \tag{3.14}$$

When we impose symmetry under a π rotation around the x_3-axes in space followed by an isorotation around the x_3-axis in target space, one zero has to be equal to zero and the other two map into each other under $z \mapsto -z$. The symmetric rational map is then given by

$$R(z) = \mathrm{e}^{i\phi} \frac{z(z - z_1)(z + z_1)}{(1 + \bar{z}_1 z)(1 - \bar{z}_1 z)}. \tag{3.15}$$

Hence, this rational map is parametrized by $\phi \in \mathbb{R}$ and $z_1 \in \mathbb{C}$, that is by 3 real parameters. A further 4 real parameters correspond to our choice of axes on $S^2 \times S^2$.

To list all families of degree 3 rational maps on the projective plane, we can also use the Riemann-Hurwitz formula [5]. For a degree N rational map $R(z)$ ramified at points p_i in S^2, the Riemann-Hurwitz formula is given by

$$\chi(S^2) = N\chi(S^2) - \sum_{p_i} (d_{p_i} - 1)\,, \tag{3.16}$$

where $\chi(S^2)$ is the Euler characteristic of S^2, and d_{p_i} is the ramification index of $R(z)$ at the point p_i. Thus, for $N = 3$, $\sum_{p_i} (d_{p_i} - 1) = 4$ implies that there are two possibilities. The first possibility is $d_{p_1} = 3$ which fixes $d_{p_2} = 3$. Hence, the first family is the symmetry orbit of rational maps z^3 which coincides with the family of maps (3.11). The second possibility is $d_{p_i} = 2$, for $i = 1, 2, 3, 4$. Choosing a critical point at $z = 0$ with $R(0) = 0$ and then using Möbius transformations, we can find the family of rational maps as

$$R(z) = \frac{z^2(z-a)}{1+az}, \quad a > 0. \tag{3.17}$$

Note that the sign of a can be changed by $R(z) \mapsto -R(-z)$. Here rotations and isorotations act independently and cannot change the magnitude of a, hence the space is 7-dimensional. This is compatible with writing the rational map as equation (3.14).

The angular integral of a degree N rational map R is given by

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{dz d\bar{z}}{(1+|z|^2)^2}. \tag{3.18}$$

It can be shown that $\mathcal{I} \geq N^2$ [11]. This quantity $\mathcal{I}$ is invariant under rotations in space and rotations in target space. Therefore, it distinguishes between maps not related by the symmetry group. In Fig.1, we display the angular integral $\mathcal{I}$ for the rational map R_c in (3.13) as a function of c. For $c = 0$, we obtain the axial map $R(z) = z^3$, for which the angular integral can be evaluated explicitly to $\mathcal{I}_0 = \frac{81+16\sqrt{3}\pi}{9}$. As $c \to 1$, the integral $\mathcal{I}$ diverges as two lumps become spiky, see Fig.2, and for $c = 1$ the rational map becomes $R(z) = z$. For $c = 3$, the angular integral $\mathcal{I}$ again takes the value $\mathcal{I}_0$. This can be understood as follows: Since the D_2 symmetry fixes the three axes x_1, x_2 and x_3, we can permute the (x_1, x_2, x_3) axes cyclically and obtain, after a suitable isorotation, another map of the form (3.13), but with a new value $\tilde{c} = \frac{3-c}{c+1}$. This relates $c = 0$ to $\tilde{c} = 3$, which is the axial map in disguise. Furthermore, the interval $0 \leq c < 1$ is mapped to $1 < \tilde{c} \leq 3$. Similarly, we can map the interval $0 \leq c < 1$ to $3 \leq \hat{c} < \infty$, where $\hat{c} = \frac{c+3}{1-c}$. Thus, the angular integral of (3.13) for $0 < c < 1$ and $1 < \tilde{c} < 3$ are identical, and similarly for $3 < \hat{c} < \infty$. Hence, it is sufficient to consider the interval $0 \leq c < 1$, and its symmetry orbit in order to parametrize $\widetilde{Rat}_3$.

The integral $\mathcal{I}$ as a function of a for R_a in (3.17) is also displayed in Fig.1. This indicates that the maps $R_a \in \widetilde{Rat}_3$ and $R_c \in \widetilde{Rat}_3$ with the same value of $\mathcal{I}$ are related to each other via Möbius transformations. In fact, this can be checked using Maple by explicitly computing the relevant Möbius transformations. Hence the moduli space of lumps of (3.13) and (3.17) parametrize the same space, and therefore (3.17) has a hidden D_2 symmetry.

The maps (3.17) and (3.13) can also be used to discuss interesting lump decay channels. First, consider maps $R_a \in \widetilde{Rat}_3$ of the form (3.17). Following the zeros and poles for $a \in [0, \infty]$, we start with the axial map with three zeros at the origin 0 and three poles at ∞. Then one zero moves from 0 to ∞ along the positive

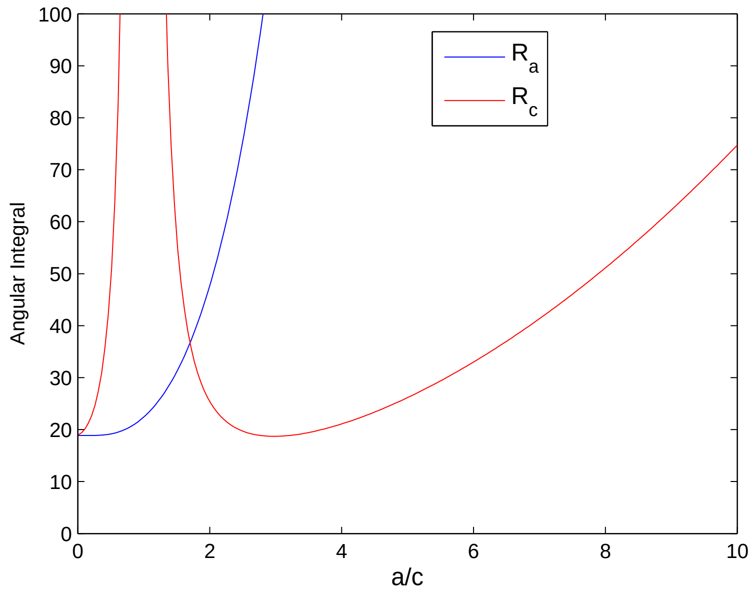


Figure 1: Plots of the angular integral $\mathcal{I}$ in (3.18) as a function of a and c for R_a and R_c, respectively.

real axis while one pole moves from ∞ to 0 along the negative real axis. The zero cancels with a pole at ∞ while the pole cancels with a zero at 0.

Second, consider maps $R_c \in \widetilde{Rat}_3$ of the form (3.13), as displayed in Fig.2: For $c = 0$ we obtain the axial maps with three zeros at the origin and three poles at ∞. For $0 < c < 1$ one zero remains fixed while one zero moves up and one moves down along the imaginary axis. Also, one pole remains fixed at ∞ while two pole travel towards 0 along the positive and negative imaginary axis, respectively. For $c = 1$ two poles and two zeros cancel. For $1 < c < \infty$ the poles move towards 0 while the poles move towards ∞ where they cancel.

Note that since a zero and a corresponding pole are opposite each other, a single lump cannot decay, and the module space is $\widetilde{Rat}_1 \cong SO(3)$. Similarly, the axial symmetry D_∞ fixes all N zeros and N poles so that axial symmetry prevents lump decay, and the moduli space of axially symmetric lumps is the 5-dimensional symmetry orbit of z^N.

3.2 $N = 5$ lumps

In the following we discuss the symmetries of $\widetilde{Rat}_5$. According to Lemma 2, when a C_n symmetry is imposed with $n \geq 5$ then the rational map has a D_∞ symmetry and is of the form

$$R(z) = \frac{az^5 + b}{-\bar{b}z^5 + \bar{a}}, \tag{3.19}$$

where a and b are complex and do not both vanish simultaneously. This moduli space of axially symmetric rational maps has dimension 5 and can be viewed as the symmetry orbit of z^5.

The rational map which is grouped to the C_2^1 symmetry family is given by

$$R(z) = \frac{z(a_0 + a_1 z^2 + a_2 z^4)}{\bar{a}_2 - \bar{a}_1 z^2 + \bar{a}_0 z^4}. \tag{3.20}$$

An interesting example of this kind of rational map which satisfies the D_2 symmetry is given by

$$R(z) = \frac{z(1 + iaz^2 + bz^4)}{b + iaz^2 + z^4}, \; a, b \in \mathbb{R}. \tag{3.21}$$

When $b = 1$, lump decay can be observed as a function of a since four zeros cancel with four poles at 0 and four poles cancel with four zeros at ∞. Furthermore, when $a = 0$, $R(z)$ has the symmetry of a square which give rise to C_4^1 symmetry.

Another family of maps has C_3^2 symmetry, and the corresponding rational map is given by

$$R(z) = \frac{z^2(a_0 + a_1 z^3)}{\bar{a}_1 - \bar{a}_0 z^3}. \tag{3.22}$$

By Möbius transformations preserving this symmetry, namely rotations around the third axis in space and target space, the rational map can be brought into the form

$$R(z) = \frac{z^2(z^3 + a)}{1 - az^3}, \; a \in [0, \infty), \tag{3.23}$$

assuming $a_1 \neq 0$. This rational map is also symmetric under C_2^1 rotations via

$$R\left(-\frac{1}{z}\right) = -\frac{1}{R(z)}, \tag{3.24}$$

hence (3.23) has D_3 symmetry.

The remaining family of maps with a rotational C_n^k symmetry is given by

$$R(z) = \frac{z(z^4 + a)}{1 + az^4}, \; a \in (0, 1) \cup (1, \infty), \tag{3.25}$$

which is C_4^1 symmetric. This map also has the C_2^1 symmetry (3.24), hence (3.25) is a D_4-symmetric map. If $a = -5$, then the map has an additional C_3 symmetry, the symmetry group is enhanced to octahedral symmetry O [11].

In the moduli space identified by the rational map (3.25), lump decay can be observed. For $a = 0$ we have the axial maps with 5 zeros at the origin and 5 poles at ∞. For $0 < a < 1$ one zero remains fixed while one zero moves up, one down along the imaginary axis, one zero moves right along the positive real axis and one zero moves to the left along the negative real axis. Also, one pole remains fixed at ∞ while two poles travel towards 0 along the positive and negative imaginary axis and two poles travel to 0 along the positive and negative real axis, respectively. For $a = 1$ four poles and four zeros cancel. For $1 < a < \infty$ the poles move towards 0 while the zeros move towards ∞ where they cancel.

Proposition 1. *If $N = n + k$ with $k < n$, then the moduli space of rational maps with C_n^k symmetry is 7-dimensional and has D_n symmetry.*

Proof. A rational map of degree $N = n + k$ with C_n^k symmetry is given by

$$R(z) = \frac{z^k(a_1 z^n + a_0)}{(-1)^n \bar{a}_0 z^n + \bar{a}_1}. \tag{3.26}$$

By Möbius transformation, the rational map can be brought into the form

$$R(z) = \frac{z^k(z^n + a)}{1 + az^n},\ a \in [0, \infty). \tag{3.27}$$

The map (3.27) satisfies the C_2^1 symmetry given by (3.24). Hence, the moduli space of the rational map has D_n symmetry. ☐

In the following, we explore the Riemann-Hurwitz formula [5] for charge five lumps. Thus, for $N = 5$, $\sum_{p_i}(d_{p_i} - 1) = 8$ implies that there are five possibilities. The first possibility is $d_{p_1} = 5$ which fixes $d_{p_2} = 5$. Then we obtain the rational map z^5 which coincides with (3.19). The second possibility is $d_{p_1} = d_{p_2} = 4$ which fix $d_{p_3} = d_{p_4} = 2$. In this case, the family of rational map is given by

$$R(z) = \frac{z^4(z + a)}{1 - az}. \tag{3.28}$$

The third possibility is $d_{p_1} = d_{p_2} = 3$ and $d_{p_i} = 2$ for $i = 3, 4, 5, 6$ which gives rational maps of the form

$$R(z) = \frac{z^3(z^2 + az + b)}{(1 - \bar{a}z + bz^2)}, \tag{3.29}$$

where $b \neq 0$, $|a|^2(b^2 + 1) + (b^2 - 1)^2 + b(a^2 + \bar{a}^2) \neq 0$ and the polynomial

$$P(z) = 3bz^4 + (2ab - 4\bar{a})z^3 + (5 + b^2 - 3|a|^2)z^2 - (2\bar{a}b - 4a)z + 3b$$

has four simple zeros. The fourth possibility is $d_{p_i} = 3$ for $i = 1, 2, 3, 4$. Then, the rational map is given by

$$R(z) = \frac{z^3(z^2 + az + b)}{(1 - az + bz^2)}, \tag{3.30}$$

where $b \neq 0$, $a^2(b+1)^2 + (b^2 - 1)^2 \neq 0$ and the polynomial

$$P(z) = 3bz^4 + (2ab - 4a)z^3 + (5 + b^2 - 3|a|^2)z^2 - (2ab - 4a)z + 3b$$

has two double zeros.

Finally, the remaining possibility is $d_{p_i} = 2$, $i = 1, ..., 8$, with rational map

$$R(z) = \frac{z^2(z^3 + az^2 + bz + c)}{1 - \bar{a}z + \bar{b}z^2 - cz^3}, \tag{3.31}$$

where the parameter a, b and c are coefficients such that the determinant of the Sylvester matrix

$$\begin{bmatrix} -1 & -a & -b & -c & 0 & 0 \\ 0 & -1 & -a & -b & -c & 0 \\ 0 & 0 & -1 & -a & -b & -c \\ c & -\overline{b} & \overline{a} & -1 & 0 & 0 \\ 0 & c & -\overline{b} & \overline{a} & -1 & 0 \\ 0 & 0 & c & -\overline{b} & \overline{a} & -1 \end{bmatrix}$$

is never zero, and furthermore the Wronskian of (3.31) has to have 7 simple roots.

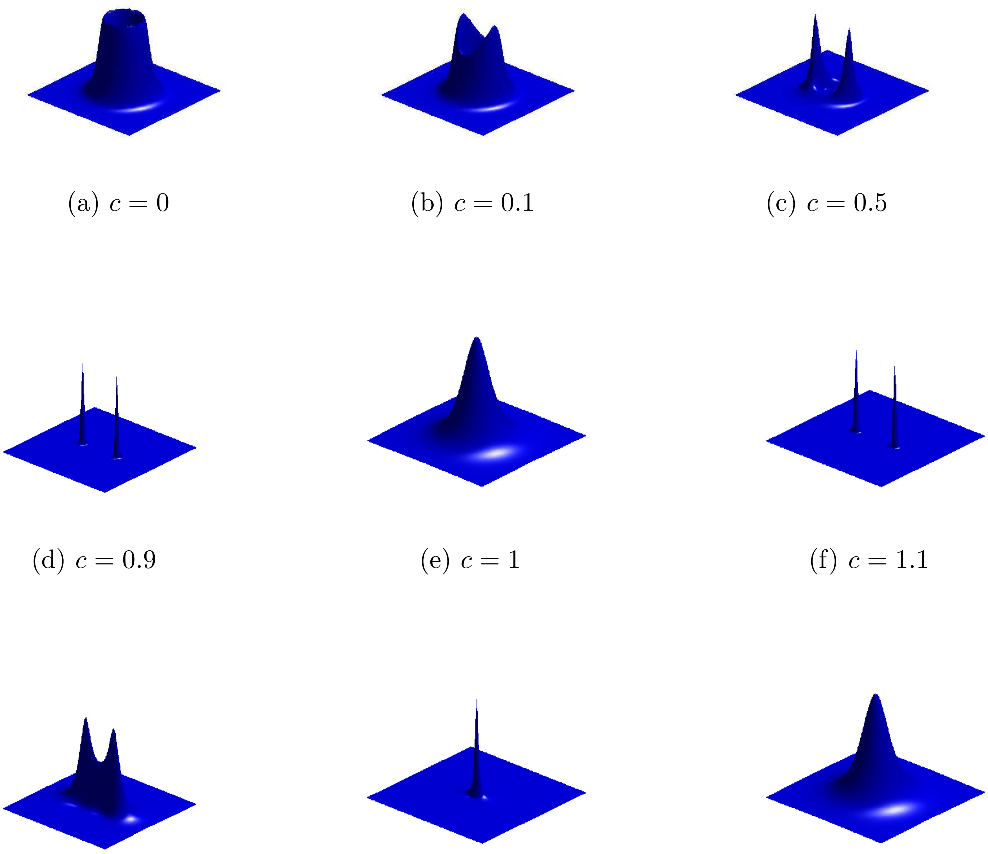
4 The moduli space $\widetilde{Rat}_3$ on $\mathbb{R}P^2$

In this section we discuss the moduli space of charge three lumps on real projective space. We calculate the metric and various geometric quantities. We first discuss maps of the form (3.13) which possess dihedral symmetry D_2. We describe the 7-dimensional symmetry orbit of $R_c \in \widetilde{Rat}_3$ and maps of the form (3.11) which possess axial symmetry D_∞.

Consider first the space of functions R_c given by (3.13). The Wronskian of R_c is a polynomial of degree 4 given by

$$w(z) = cz^4 + (3 - c^2)z^2 + c.$$

For $c \approx 1$, the poles and zeros of R_c come together and cancel each other, then R_c becomes a rational map of degree one which is z. For $c \approx \infty$, the poles and zeros come together and cancel each other and then R_c becomes the rational map $\frac{1}{z}$. One can see the energy density of this space in Fig.2 that shows the energy density is symmetric at $c = 0$, $c = 1$ and $c = \infty$. The energy densities dissociate and form spikes as c approaches 1 and ∞.



(a) $c = 0$ (b) $c = 0.1$ (c) $c = 0.5$

(d) $c = 0.9$ (e) $c = 1$ (f) $c = 1.1$

(g) $c = 2$ (h) $c = 10$ (i) $c = \infty$

Figure 2: These figures display the energy densities of charge three lumps given by R_c in (3.13) for different values of c.

Proposition 2. *$\widetilde{Rat_3}$ is a non-compact totally geodesic Lagrangian submanifold of the space of degree* 3 *lumps* Rat_3.

Proof. One can find a similar proof in Ref. [20]. $\square$

Our next step is to compute the metric on the moduli space of Rat_3. Consider the rotation group $SO(3)$ action on the coordinate systems z and R, $R \in Rat_3$. Consider first the action $z \mapsto Uz, U \in SO(3) \cong PU(2) \cong SU(2)/\mathbb{Z}_2$. We can expand the left invariant $1-$form $U^{-1}dU$ in terms of a convenient basis of the Lie algebra $\frac{i}{2}\tau_a$, $a = 1, 2, 3$, where τ_a are Pauli matrices:

$$U^{-1}dU = \sigma \cdot \frac{i}{2}\tau_a = \sigma_1 t_1 + \sigma_2 t_2 + \sigma_3 t_3, \tag{4.1}$$

where $d\sigma_i = \frac{1}{2}\varepsilon_{ijk}\sigma_j \wedge \sigma_k$. Similarly, for the action $R \mapsto MR, M \in SO(3)$, we have an expression in the Lie algebra $\frac{i}{2}\tau_a, a = 1, 2, 3$:

$$M^{-1}dM = \eta \cdot \frac{i}{2}\tau_a = \eta_1 t_1 + \eta_2 t_2 + \eta_3 t_3, \tag{4.2}$$

where $d\eta_i = \frac{1}{2}\varepsilon_{ijk}\eta_j \wedge \eta_k$. For example, consider $\tilde{M} \in SU(2)$ defined by

$$\tilde{M} = \begin{pmatrix} e^{\frac{i}{2}(\psi+\phi)}\cos(\frac{\theta}{2}) & e^{\frac{i}{2}(\psi-\phi)}\sin(\frac{\theta}{2}) \\ -e^{\frac{i}{2}(\phi-\psi)}\sin(\frac{\theta}{2}) & e^{-\frac{i}{2}(\psi+\phi)}\cos(\frac{\theta}{2}) \end{pmatrix}.$$

We can then see that $\tilde{M}^{-1}d\tilde{M} = \eta_1 t_1 + \eta_2 t_2 + \eta_3 t_3$ where the η_i's are computed as

$$\begin{aligned} \eta_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi \\ \eta_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi \\ \eta_3 &= d\psi + \cos\theta d\phi. \end{aligned}$$

Furthermore, let $R \in SU(2)$ and $M \mapsto MR, M \in SO(3)$. Then we can find that $\sigma \mapsto \mathcal{R}\sigma$ and $\eta \mapsto \mathcal{R}\eta$, where $\mathcal{R} \in SO(3)$ with matrix component $R_{ab} = \frac{1}{2}tr(\tau_a R^\dagger \tau_b R)$. Hence both σ and η transform as $3-$vectors under rotations. One can change from the coordinate basis on $SO(3)$, $\{d\alpha, d\beta, d\gamma\}$, to the left invariant 1-forms on $SO(3)$ which are given by (4.1) and (4.2) as before, but with the range of angles appropriate to $SO(3)$, $\alpha \in [0, \pi), \beta \in [0, 2\pi), \gamma \in [0, 2\pi)$.

The metric is invariant under spatial rotations. Then by considering dc, σ and η, we can construct the most general possible metric as

$$g = A_{ij}(c)\sigma_i\sigma_j + B_i(c)\sigma_i dc + C(c)d^2c + D_{ij}(c)\eta_i\eta_j + E_i(c)\eta_i dc + F_{ij}(c)\sigma_i\eta_j, \tag{4.3}$$

where $i, j = 1, 2, 3$ and each component function depends only on c and is independent of the Euler angles.

The transformations $\rho : z \mapsto \bar{z}$ and $w : R \mapsto \bar{R}$ map lumps to anti-lumps because each reverses the sign of the topological degree, and so both are not an isometry of the moduli space. In fact, the composite transformation $w \circ \rho$ is an isometry. Consider the isometry transformation $U \mapsto \bar{U}$, where $U \in SO(3)$ as a $SU(2)$ Möbius transformation and suppose again $c \mapsto c$. Then

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) \mapsto (-\sigma_1, \sigma_2, -\sigma_3) \quad \text{and} \quad \eta = (\eta_1, \eta_2, \eta_3) \mapsto (-\eta_1, \eta_2, -\eta_3). \quad (4.4)$$

This isometry removes $B_i(c)$ and $E_i(c)$ for $i = 1, 3$ from the general possible metric equation (4.3) because for $c \mapsto c$, we have that

$$\begin{aligned} \sigma \cdot dc &\mapsto (-\sigma_1 dc, \sigma_2 dc, -\sigma_3 dc), \\ \eta \cdot dc &\mapsto (-\eta_1 dc, \eta_2 dc, -\eta_3 dc). \end{aligned}$$

The isometry (4.4) also results in $A_{12}(c) \equiv A_{21}(c) \equiv A_{23}(c) \equiv A_{32}(c) \equiv 0$ and $D_{12}(c) \equiv D_{21}(c) \equiv D_{23}(c) \equiv D_{32}(c) \equiv F_{12}(c) \equiv F_{21}(c) \equiv F_{13}(c) \equiv F_{31}(c) \equiv F_{23}(c) \equiv F_{32}(c) \equiv 0$. Furthermore, we can use the fact that $R \in \widetilde{Rat}_3$ has D_2 symmetry. Take a π rotation around the third axis

$$(\sigma_1, \sigma_2, \sigma_3) \mapsto (-\sigma_1, -\sigma_2, \sigma_3) \quad \text{and} \quad (\eta_1, \eta_2, \eta_3) \mapsto (-\eta_1, -\eta_2, \eta_3). \quad (4.5)$$

This isometry (4.5) gives $A_{13}(c) \equiv A_{31}(c) \equiv D_{13}(c) \equiv D_{31}(c) \equiv F_{13}(c) \equiv F_{31}(c) \equiv B_2(c) \equiv E_2(c) \equiv 0$ because we have that

$$\sigma_1\sigma_3 \mapsto -\sigma_1\sigma_3, \quad \eta_1\eta_3 \mapsto -\eta_1\eta_3, \quad \sigma_2 dc \mapsto -\sigma_2 dc \quad \text{and} \quad \eta_2 dc \mapsto -\eta_2 dc.$$

Hence, they can be removed from the general possible metric equation (4.3).

Our next task is finding the remaining metric functions of c by taking the appropriate Euler angles. Firstly, consider the parametrization of $SO(3)$ by

$$M(\alpha, \theta, \varphi) = \begin{pmatrix} \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\cos\theta & i\sin\frac{\alpha}{2}(\cos\varphi + i\sin\varphi)\sin\theta \\ i\sin\frac{\alpha}{2}(\cos\varphi - i\sin\varphi)\sin\theta & \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\cos\theta \end{pmatrix}. \quad (4.6)$$

Take first the action $R \mapsto R_\star = e^{i\alpha}R$. That is, we consider $\theta = 0$ in $M(\alpha, \theta, \varphi)$. Then we have a metric of the form $\gamma = \gamma_{\alpha\alpha}(c)d^2\alpha$, where

$$D_{33}(c) = \gamma_{\alpha\alpha}(c) = \int_D \frac{|\partial_\alpha R_\star|^2}{(1+|R_\star|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} = \int_D \frac{|R|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Now we can also evaluate $D_{11}(c)$ and $D_{22}(c)$. Suppose we are taking ($\theta = \frac{\pi}{2}$ and $\varphi = 0$) the action $R \mapsto R_\star = MR$, where M is given by the matrix (4.6). Then, we have a metric of the form $\gamma = \gamma_{\alpha\alpha}(c)d^2\alpha$ where

$$D_{11}(c) = \gamma_{\alpha\alpha}(a) = \int_D \frac{|\partial_\alpha R_\star|^2}{(1+|R_\star|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} = \frac{1}{4}\int_D \frac{|1-R^2|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Similarly, to find the expression for $D_{22}(c)$, take $\alpha = \frac{\pi}{2}$ and $\varphi = 0$ in our parametrization of $SO(3)$ in (4.6). Then the metric is of the form $\gamma = \gamma_{\theta\theta}(c)d^2\theta$ where

$$D_{22}(c) = \gamma_{\theta\theta}(c) = \int_D \frac{|\partial_\theta R_\star|^2}{(1+|R_\star|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} = \frac{1}{4}\int_D \frac{|1+R^2|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

To find the functions $A_{ii}(c), i = 1,2,3$, we can follow the same argument in evaluating the $D_{ii}(c), i = 1,2,3$. Suppose we are considering the same parametrization of $SO(3)$ as (4.6) and the $SO(3)$ action on z. For instance, let $z \mapsto e^{i\alpha}z$. Then $R \mapsto R_\star = R(e^{i\alpha}z)$. Therefore, we are now able to find $A_{33}(c)$ from the metric of the form $\gamma = \gamma_{\alpha\alpha}(c)d^2\alpha$ where

$$A_{33}(c) = \gamma_{\alpha\alpha}(c) = \int_D \frac{|\partial_\alpha R_\star|^2}{(1+|R_\star|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2} = \int_D \frac{|z|^2|\frac{dR}{dz}|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

We can also find the other two functions $A_{11}(c)$ and $A_{22}(c)$ by taking the Euler angles ($\theta = \frac{\pi}{2}$ and $\varphi = 0$) and ($\alpha = \frac{\pi}{2}$ and $\varphi = 0$), respectively. Considering the above Euler angles and from the metrics of the form $\gamma = \gamma_{\alpha\alpha}(c)d^2\alpha$ and $\gamma = \gamma_{\theta\theta}(c)d^2\theta$, we can find that

$$A_{11}(c) = \gamma_{\alpha\alpha}(c) = \frac{1}{4}\int_D \frac{|1-z^2|^2|\frac{dR}{dz}|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}$$

and

$$A_{22}(c) = \gamma_{\theta\theta}(c) = \frac{1}{4}\int_D \frac{|1+z^2|^2|\frac{dR}{dz}|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Similarly, we can find the following metric functions $F_{ii}(c), i = 1,2,3$ as

$$F_{11}(c) = \frac{1}{4}\int_D \frac{\Re\big((1-z^2)(1-\bar{R}^2)\frac{dR}{dz}\big)}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2},$$
$$F_{22}(c) = \frac{1}{4}\int_D \frac{\Re\big((1+z^2)(1+\bar{R}^2)\frac{dR}{dz}\big)}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2},$$
$$F_{33}(c) = \int_D \frac{\Re\big(z\bar{R}\frac{dR}{dz}\big)}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Finally, the function $C(c)$ is finite since it is bounded from above by 2π and given by

$$C(c) = \int_D \frac{|\partial_c R|^2}{(1+|R|^2)^2}\frac{dzd\bar{z}}{(1+|z|^2)^2}.$$

Hence, the metric on the 7-dimensional space of charge three lumps is given by

$$g = A_{ii}(c)\sigma_i^2 + C(c)d^2c + D_{ii}(c)\eta_i^2 + F_{ii}(c)\sigma_i\eta_i, \tag{4.7}$$

where $i = 1, 2, 3$. The coefficient functions of the metric (4.7) are displayed in Fig.3. As $c = 0$ is the axial map, we find that $A_{11}(0) = A_{22}(0)$, $D_{11}(0) = D_{22}(0)$ and $F_{11}(0) = F_{22}(0) = 0$.

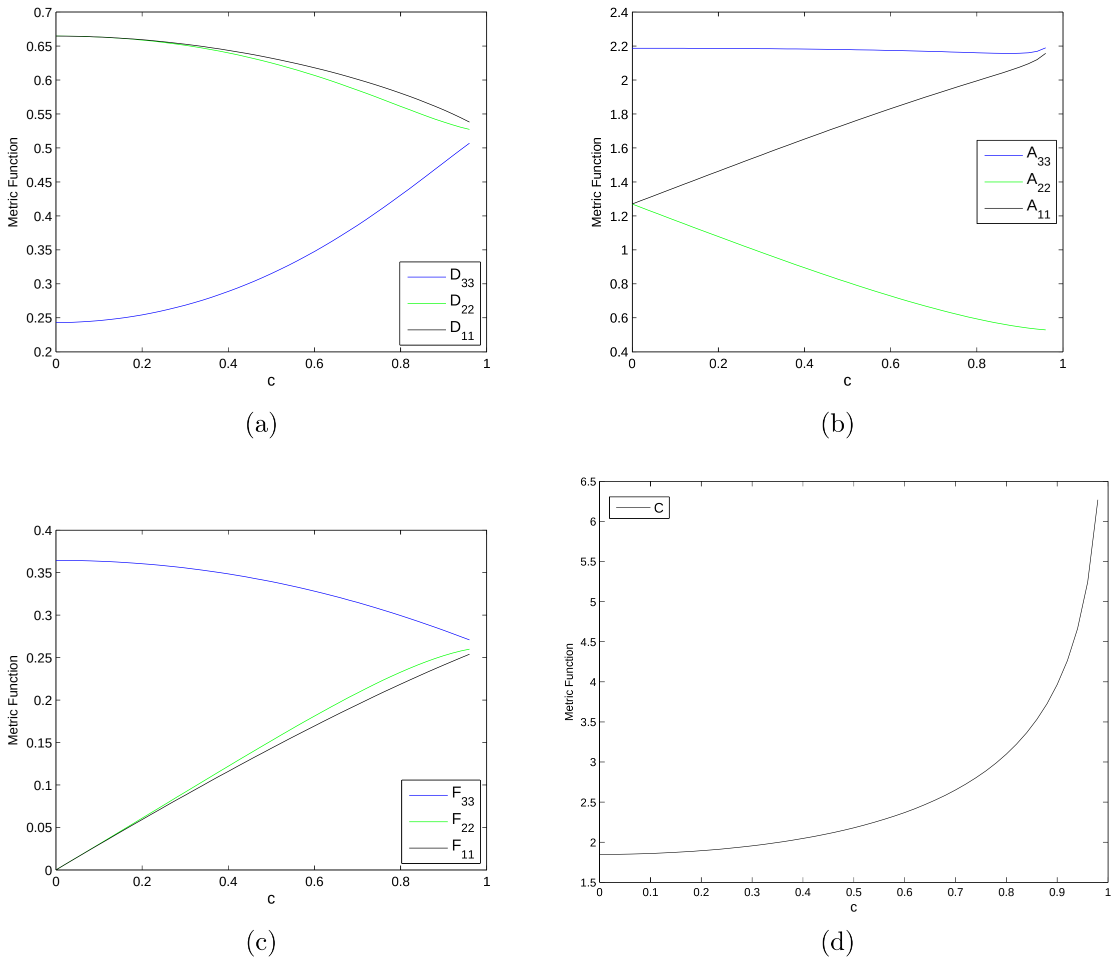


Figure 3: (a) The metric functions $D_{11}(c)$, $D_{22}(c)$, $D_{33}(c)$. (b) The metric functions $A_{11}(c)$, $A_{22}(c)$, $A_{33}(c)$. (c) The metric functions $F_{11}(c)$, $F_{22}(c)$, $F_{33}(c)$. (d) The metric function $C(c)$.

Proposition 3. *The moduli space $\widetilde{Rat_3}$ with respect to the general metric g has finite volume, that is, the volume of $(\widetilde{Rat_3}, g)$ is finite.*

Proof. The volume on the moduli space $\widetilde{Rat_3}$ is given by

$$Vol(\widetilde{Rat_3}) = \int_{SO(3)\times SO(3)\times\mathbb{R}} \sqrt{(|\det(g_{ij})|)} Vol_g(\widetilde{Rat_3}), \tag{4.8}$$

where $Vol_g(\widetilde{Rat_3})$ is the volume element on g is given by

$$Vol_g(\widetilde{Rat_3}) = \sqrt{|\det(g_{ij})|}\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge dc. \tag{4.9}$$

Note that $Vol(SO(3)) = \int_{SO(3)} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = 8\pi^2$ and similarly $\int_{SO(3)} \eta_1 \wedge \eta_2 \wedge \eta_3 = 8\pi^2$. The determinat of g_{ij} is calculated as

$$\det(g_{ij}) \leq (A_{11}A_{22}A_{33}D_{11}D_{22}D_{33}C)(c). \tag{4.10}$$

We can also find the following inequalities in the metric functions which are $A_{22} \leq \frac{4}{(1+c^2)^2}$ and both A_{11} and A_2 are bounded above by $\frac{2\pi}{3} + \frac{2\pi^2\sqrt{3}}{243}$. Similarly, we can see that $D_{33} \leq \frac{\pi}{3}$, $D_{22} \leq \frac{\pi}{3} + \frac{4\pi^2\sqrt{3}}{243}$ and $D_{11} \leq \frac{\pi}{3} + \frac{4\pi^2\sqrt{3}}{243}$. Hence, $\det(g_{ij}) \leq \frac{4}{(1+c^2)^2}$. Note that $\int_0^\infty \frac{2}{1+c^2}dc = \pi$ which implies the following integral

$$\begin{aligned} Vol(\widetilde{Rat_3}) &= 64\pi^4 \int_0^1 \sqrt{|\det(g_{ij})|}\, dc \\ &\leq 64\pi^4 \int_0^\infty \sqrt{|\det(g_{ij})|}\, dc \leq 64\pi^4 \int_0^\infty \frac{2}{1+c^2}\, dc = 64\pi^5. \end{aligned}$$

This proves the volume of the moduli space of charge three lumps is finite. □

Proposition 4. *The moduli space $(\widetilde{Rat_3}, \gamma_{cc})$ has a submanifold of finite length, where $\gamma_{cc}(c) = C(c)$.*

Proof. Since the integral $\int_0^1 \sqrt{\gamma_{cc}}dc < \infty$, it shows automaticaly the length is finite. Therefore, the boundary of $(\widetilde{Rat_3}, \gamma_{cc})$ at infinity lies at finite distance. Then the space $\widetilde{Rat_3}$ is geodesically incomplete. □

Finally, we consider rational functions of the form $R(z) = z^3$ and its symmetry orbit denoted by $\widetilde{Rat_3^0}$. The energy density is symmetric and its metric is equivalent to the metric on the moduli space $\{\xi z^n : \xi \in \mathbb{C}^\times\}$ with $\xi = 1$ and $n = 3$ which was shown in Ref. [13]. Note that $\widetilde{Rat_3^0}$ is a totally geodesic submanifold of $\widetilde{Rat_3}$. The general metric g_0 on $\widetilde{Rat_3^0}$ is given by

$$\begin{aligned} g_0 &= f_i\sigma_i^2 + h_i\eta_i^2 + F_{33}\sigma_3\eta_3, && (4.11) \\ &= f_1\sigma_1^2 + f_2\sigma_2^2 + f_3(\sigma_3 + 3\eta_3)^2 + h_1\eta_1^2 + h_2\eta_2^2, && (4.12) \end{aligned}$$

where

$$\begin{aligned} f_1 &= f_2 = \frac{2\pi^2\sqrt{3}}{27}, \quad h_1 = h_2 = \frac{\pi}{6} + \frac{2\pi^2\sqrt{3}}{243}, \\ f_3 &= \frac{3\pi}{2} - \frac{4\pi^2\sqrt{3}}{27}, \quad h_3 = \frac{\pi}{6} - \frac{4\pi^2\sqrt{3}}{243}, \quad F_{33} = \frac{\pi}{2} - \frac{4\sqrt{3}\pi^2}{81}. \end{aligned}$$

Proposition 5. *$\widetilde{Rat_3^0}$ has a finite volume with* $Vol(\widetilde{Rat_3^0}) = \frac{\pi^{13/2}}{19683}\left(4\,\pi\,\sqrt{3}+81\right)\sqrt{324-32\,\sqrt{3}}$.

Proof. Here to avoid over-counting, we divide the volume element by two since the space has an additional C_2 symmetry. As earlier in Proposition 3, we have that $Vol(SO(3)) = \int_{SO(3)} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = 8\pi^2$. From matrix (4.6), $\eta_1 \wedge \eta_2 = \sin(\theta) d\phi d\theta$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, the integral is $\int_{SO(3)/SO(2)} \eta_1 \wedge \eta_2 = 4\pi$. The volume is given by

$$Vol(\widetilde{Rat_3^0}) = \int \sqrt{|f_1 f_2 f_3 h_1 h_2|}\, \eta_1 \wedge \eta_2 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3,$$

which can be evaluated as

$$Vol(\widetilde{Rat_3^0}) = 32\pi^3 \sqrt{|f_1 f_2 f_3 h_1 h_2|} = \frac{4\pi^{13/2}}{19683}\left(4\,\pi\,\sqrt{3}+81\right)\sqrt{324-32\,\sqrt{3}}.$$

□

5 Conclusion

In this paper, we studied the moduli space of lumps on real projective space $\widetilde{Rat_N}$ which is given by the space of rational maps of degree N subject to symmetry requirements [21]. We examined this moduli space using two different approaches. First we classified all possible cyclic symmetries of $\widetilde{Rat_N}$. Then we analysed $\widetilde{Rat_N}$ using the Riemann-Hurwitz formula.

The symmetry requirements for $\widetilde{Rat_N}$, which from a geometric point of view mean that zeros and poles have to be opposite, greatly reduce the number of allowed cyclic symmetries C_n^k compared to general rational maps. The allowed symmetries are given in Lemma 2, and can be compared to Lemma 1, which states a similar results for general rational maps [8].

We then focused on the case $N = 3$. Imposing a C_2 symmetry automatically leads to a D_2 symmetric map, and a cyclic symmetry C_n for $n > 2$ results in an axially symmetric rational map with D_∞ symmetry. On $\widetilde{Rat_N}$, the symmetry group $SO(3) \times SO(3)$ of rotations and isorotations acts isometrically. Hence, the moduli space $\widetilde{Rat_3}$ consists of two orbits of the symmetry group, namely a 5-dimensional orbit of the axial map and a 7-dimensional orbit of dihedral symmetry. The Riemann-Hurwitz formula also decomposes $\widetilde{Rat_3}$ into a 5-dimensional space with two ramification points of index 3 and a 7-dimensional space with 4 ramification points of index 2, and we showed that these two points of view produce the same spaces. In summary, the moduli space of charge three lumps is a 7-dimensional manifold which can be described as the symmetry orbit of D_2 symmetric maps R_c in (3.13) where $c = 0$ denotes the axially symmetric map. Furthermore, as $c \to \infty$ two lumps become increasingly spiky and collapse, as two poles of R_c cancel with two zeros. The symmetry requirement that poles and zeros have to be opposite results in a more complicated lump decay, and in particular, D_∞ symmetry prevents lump decay.

The dihedral symmetry of the symmetry of $\widetilde{Rat_3}$ allowed us to find explicit expressions of the L^2 metric [21] that is induced on the moduli space by the kinetic energy. It is rare that the moduli space metric can be evaluated for topological charge greater than one. Recently, the metric of hyperbolic vortices of charge two has been calculated in Ref. [1]. We showed that the volume of $\widetilde{Rat_3}$ is finite. Furthermore, we constructed a geodesic that connects the axial map to the boundary of $\widetilde{Rat_3}$ which has finite length. This shows that $\widetilde{Rat_3}$ is geodesically incomplete, as shown in Ref. [21]. We also evaluated the volume of the space of axially symmetric maps.

When $N = 5$ the symmetry approach gives the following possible symmetries: D_∞, O, D_4, D_3, D_2 and C_2 and no symmetry. The metrics of submanifolds with D_n symmetry could be evaluated in a similar way to the metric of $\widetilde{Rat_3}$. The Riemann-Hurwitz formula provides an alternative decomposition into 5 different spaces, but now there is no obvious correspondence between the two approaches apart from the axially symmetric case. How these spaces are related is an interesting problem for further study.

Acknowledgment

SK would like to thank Martin Speight for suggesting to look into this problem and further useful discussions. SK is grateful to Jim Shank for stimulating discussions with regards to the geometry of rational maps. SK acknowledges the EPSRC for the grant EP/I034491/1. AM would like to thank Mareike Haberichter for assisting with Matlab plots and useful discussions. AM acknowledges SMSAS for a PhD studentship.

References

[1] L. S. Alqahtani and J. M. Speight, "Ricci magnetic geodesic motion of vortices and lumps," arXiv:1410.4698 [math-ph].

[2] R. Bott and L. W. Tu, "Differential forms in algebraic topology," Springer-Verlag, 1982.

[3] R. J. Cova and W. J. Zakrzewski, "Scattering of periodic solitons," Rev. Mex. Fis. **50** (2005) 527 [hep-th/0308161].

[4] J. Eells and L. Lemaire, "On the Construction of Harmonic and Holomorphic Maps between Surfaces," Math. Ann. **252** (1980) 27.

[5] W. Fulton and R. Weiss, "Algebraic Curves: an introduction to algebraic geometry," volume 30. Benjamin New York, 1969.

[6] C. J. Houghton, N. S. Manton and P. M. Sutcliffe, "Rational maps, monopoles and Skyrmions," Nucl. Phys. B **510** (1998) 507 [hep-th/9705151].

[7] S. Krusch and J. M. Speight, "Quantum lump dynamics on the two-sphere," Commun. Math. Phys. **322** (2013) 95.

[8] S. Krusch, "Homotopy of rational maps and the quantization of skyrmions," Annals Phys. **304** (2003) 103 [hep-th/0210310].

[9] S. Krusch, "Finkelstein-Rubinstein constraints for the Skyrme model with pion masses," Proc. Roy. Soc. Lond. A **462** (2006) 2001 [hep-th/0509094].

[10] R. A. Leese, "Low-energy Scattering of Solitons in the CP**1 Model," Nucl. Phys. B **344** (1990) 33.

[11] N. S. Manton and P. Sutcliffe, Topological solitons, Cambridge University Press, 2004.

[12] W. S. Massey, A Basic Course in Algebraic Toplogy, Springer-Verlag, New York, 1991.

[13] J. A. McGlade and J. M. Speight, "Slow equivariant lump dynamics on the two sphere," Nonlinearity **19** (2006) 441.

[14] A. M. Polyakov and A. A. Belavin, "Metastable States of Two-Dimensional Isotropic Ferromagnets," JETP Lett. **22** (1975) 245 [Pisma Zh. Eksp. Teor. Fiz. **22** (1975) 503].

[15] P. J. Ruback, "σ Model Solitons and Their Moduli Space Metrics," Commun. Math. Phys. **116** (1988) 645.

[16] T. M. Samols, "Vortex scattering," Commun. Math. Phys. **145** (1992) 149.

[17] G. Segal, "The Topology of Spaces of Rational Functions," Acta. Math. **143** (1979) 39.

[18] L. A. Sadun and J. M. Speight, "Geodesic incompleteness in the $\mathbb{C}P^1$ model on a compact Riemann surface," Lett. Math. Phys. **43** (1998) 329 [hep-th/9707169].

[19] T. H. R. Skyrme, "A nonlinear field theory, " Proc.Roy. Soc. A260 127.

[20] J. M. Speight, "Low-energy dynamics of a $\mathbb{C}P^1$ lump on the sphere," J. Math. Phys. **36** (1995) 796 [hep-th/9712089].

[21] J. M. Speight, "The L^2 geometry of spaces of harmonic maps $S^2 \to S^2$ and $\mathbb{R}P^2 \to \mathbb{R}P^2$," J. Geom. Phys. **47** (2003) 343.

[22] J. M. Speight, "Lump dynamics in the $\mathbb{C}P^1$ model on the torus," Commun. Math. Phys. **194** (1998) 513 [hep-th/9707101].

[23] R. S. Ward, "Slowly Moving Lumps in the $\mathbb{C}P^1$ Model in (2+1)-dimensions," Phys. Lett. B **158** (1985) 424.