Cosmological constant from quantum spacetime

Majid, S; Tao, W-Q

PHYSICAL REVIEW D **91**, 124028 (2015)

Cosmological constant from quantum spacetime

Shahn Majid* and Wen-Qing Tao

Queen Mary University of London, School of Mathematical Sciences,
Mile End Road, London E1 4NS, United Kingdom

(Received 12 December 2014; published 9 June 2015)

We show that a hypothesis that spacetime is quantum with coordinate algebra $[x^i, t] = \lambda_P x^i$, and spherical symmetry under rotations of the x^i, essentially requires in the classical limit that the spacetime metric is the Bertotti-Robinson metric, i.e., a solution of Einstein's equations with a cosmological constant and a non-null electromagnetic field. Our arguments do not give the value of the cosmological constant or the Maxwell field strength, but they cannot both be zero. We also describe the quantum geometry and the full moduli space of metrics that can emerge as classical limits from this algebra.

DOI: 10.1103/PhysRevD.91.124028 PACS numbers: 04.60.-m, 02.20.Uw, 02.40.Gh

I. INTRODUCTION

Recently in [1], a new phenomenon was uncovered whereby the constraints of noncommutative algebra force the form of quantum metric and hence of its classical limit. Put another way, if a spacetime is quantized, as is by now widely accepted as a plausible model of quantum gravity effects, then this would be visible classically as quantizability conditions [2] on the classical spacetime metric so as to extend to the quantum algebra. Thus the quantum spacetime hypothesis potentially has strong and observable consequences for classical general relativity (GR).

Specifically, Ref. [1] looked at the most popular quantum spacetime algebra, the bicrossproduct or Majid-Ruegg model [3] with generators $x^i, t,\ i = 1, \ldots, n-1$, and relations

$$[x^i, x^j] = 0, \qquad [x^i, t] = \lambda x^i, \tag{1}$$

where $\lambda = \imath\lambda_P$ and λ_P is a real quantization parameter, usually assumed in this context to be the Planck time. Here $n = 4$ but we will consider other dimensions also. The paper [1] showed that in the 2D case the quantizability constraints force a strong gravitational source or an expanding universe depending on a sign degree of freedom. This provided a toy model, but in 4D the constraints were so strong that there was no fully invertible quantum metric at all. The analysis depended on the differential structure on the algebra, and we used the standard one as in Refs. [4,5].

In the present paper, we will now consider the same phenomenon for another natural choice of differential structure on (1), which we call the "α family" and which we show, in Sec. II, is the only good alternative that treats the x^i equally in the sense of rotationally invariant and works in all dimensions. The relations for this differential calculus first appeared in Ref. [6] as did those for another, which we call the "β family" and which generalizes the standard one. In our case, we come to these same differential calculi out of a systematic classification theory [7] based on pre-Lie algebras. Remarkably. we then find for the α family, in Sec. III, that this time there *is* a moduli of quantum metrics, and in Sec. IV we consider their classical limits and show that in the spherically symmetric case they are all locally of the form $S^{n-2} \times dS_2$ or $S^{n-2} \times \mathrm{AdS}_2$ depending on the sign of one of the two curvature-scale parameters $\delta, \bar{\delta}$. This means that they are the Levi-Bertotti-Robinson metric [8–11], which has been of interest in a number of contexts in GR and is known to solve Einstein's equation with a cosmological constant and Maxwell field. We can write the value of the cosmological constant here as

$$\Lambda = \frac{(n-2)(n-3)}{2}\delta - q^2 G_N,$$
$$q^2 G_N = \frac{1}{2}((n-3)\delta - \bar{\delta}),$$

where q is the Maxwell field coupling in suitable units. In our context $\delta > 0$, so that for small q we are forced to $\Lambda > 0$. Moreover, the arguments that force us to this form of metric depend on the structure of the differential algebra when spacetime is noncommutative, which is believed to be a quantum gravity effect. In 2D, there is no S^{n-2} factor and being the limit of a quantum metric in the α family in 2D forces the metric to be de Sitter or anti–de Sitter for some scale $\bar{\delta}$.

The further noncommutative Riemannian geometry for our quantum metrics in the α family is obtained by the same methods as in Ref. [1], and a brief outline of this is included in Sec. V. We work in this paper with one particular algebra (1) assumed to be some local model of quantum spacetime. The general analysis at lowest order in λ, i.e., at the level of a general Poisson structure on spacetime and the constraints on the classical metric arising from being quantizable along

*On leave at the Mathematical Institute, Oxford, United Kingdom.
s.majid@qmul.ac.uk

with it, can be found in Ref. [2]. Some more remarks about this are in the conclusions in Sec. VI.

An earlier model where vacuum energy was speculated to arise from noncommutative geometry of the quantum spacetime (1) was the nonrelativistic gravity model in Ref. [12]. A cosmological constant is also needed for quantum Born reciprocity in 3D quantum gravity [13], which also shows how the 3D version of (1) can arise there.

II. CHOICE OF DIFFERENTIAL STRUCTURE

Differential structure classically turns a topological space into something where we can define vector fields and differential forms. This is something that tends to be taken for granted in physics but is nevertheless an ingredient. Differential structure on an algebra means for us similarly a specification of the exterior algebra of "differential forms" or in practice the commutation relations between differentials $\mathrm{d}x^i, \mathrm{d}t$ and quantum spacetime coordinates. The exterior derivative d on arbitrary noncommutative functions in the coordinates is then defined by the Leibniz rule. We look for differential structures that are (i) connected, meaning only constant functions are killed by d and (ii) translation invariant with respect to the additive coproduct on (1). The latter says that as a differential space this is much like $\mathbb{R}^n$ in the same way that a classical manifold has local coordinates where the differentials $\mathrm{d}x^i, \mathrm{d}t$ are related to the standard translation-invariant Lebesgue measure.

Our starting point is a recent theorem [7] that connected translation-invariant differential structures of the correct classical dimension on the enveloping algebra of a Lie algebra $\mathfrak{g}$ are in 1-1 correspondence with pre-Lie algebra structures on $\mathfrak{g}$. This means a map $\circ\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ such that $x\circ y - y\circ x$ recovers the given Lie algebra bracket and

$$(x\circ y)\circ z - (y\circ x)\circ z = x\circ(y\circ z) - y\circ(x\circ z).$$

Moreover, for a Lie algebra like (1) there is an algebraic method which provides all inequivalent pre-Lie algebra structures, which in the 2D case [14] over $\mathbb{C}$ gives two distinct families and three discrete choices at the algebraic level. The corresponding differential structures in the 2D case are computed in Ref. [7] and come out as

$$\text{(i)}\colon [t,\mathrm{d}x] = -\lambda\mathrm{d}x, \quad [t,\mathrm{d}t] = \lambda\alpha\mathrm{d}t,$$

$$\text{(ii)}\colon [x,\mathrm{d}t] = \lambda\beta\mathrm{d}x, \quad [t,\mathrm{d}x] = \lambda(\beta-1)\mathrm{d}x, \quad [t,\mathrm{d}t] = \lambda\beta\mathrm{d}t,$$

$$\text{(iii)}\colon [t,\mathrm{d}x] = -\lambda\mathrm{d}x, \quad [t,\mathrm{d}t] = \lambda(\mathrm{d}x-\mathrm{d}t),$$

$$\text{(iv)}\colon [x,\mathrm{d}x] = \lambda\mathrm{d}t, \quad [t,\mathrm{d}x] = -\lambda\mathrm{d}x, \quad [t,\mathrm{d}t] = -2\lambda\mathrm{d}t,$$

$$\text{(v)}\colon [x,\mathrm{d}t] = \lambda\mathrm{d}x, \quad [t,\mathrm{d}t] = \lambda(\mathrm{d}x+\mathrm{d}t).$$

In each case we have listed only the nonzero commutation relations. Of these, clearly, only (i) and (ii) immediately generalize to all dimensions, namely, as the α family

$$[t,\mathrm{d}x^i] = -\lambda\mathrm{d}x^i, \qquad [t,\mathrm{d}t] = \lambda\alpha\mathrm{d}t \tag{2}$$

and the β family

$$[x^i,\mathrm{d}t] = \lambda\beta\mathrm{d}x^i, \quad [t,\mathrm{d}x^i] = \lambda(\beta-1)\mathrm{d}x^i, \quad [t,\mathrm{d}t] = \lambda\beta\mathrm{d}t \tag{3}$$

for the nonzero relations; cf. Ref. [6], where there are some similar relations to these two families. The case $\beta = 1$ of the second family is the standard calculus used in Refs. [1,4,5]. It should also be noted that case (v) is equivalent to the standard calculus in 2D in case (ii) by a change of variables if we allow a sufficient class of functions. Likewise, case (iii) is equivalent to $\alpha = -1$ in case (i) if we allow a sufficient class of functions.

We consider only these two families (2), (3) in what follows: by our above results, they are the only connected translation-invariant differential structures in the quantum spacetime (1) that work in all dimensions including 2D. To fully classify all 4D calculi is also possible by using the algebraic method for pre-Lie algebras [15] and could include more exotic possibilities, but they are unlikely to treat the different x^i equally in the sense of spatial rotations as otherwise they would specialize to 2D.

The physical meaning of the real parameters α, β thrown up by our analysis is best seen from the formulas for the partial derivatives. For a normal ordered function $f(t,x)$ on the quantum spacetime where all the t's are to the left, say, one can deduce from the Leibniz rule and the relations (2) and (3), respectively, that

$$\mathrm{d}f(t,x) = \partial_0^\alpha f(t,x)\mathrm{d}t + \frac{\partial}{\partial x^i} f(t,x)\mathrm{d}x^i,$$

$$\mathrm{d}f(t,x) = \partial_0^\beta f(t,x)\mathrm{d}t + \frac{\partial}{\partial x^i} f(t-\lambda\beta,x)\mathrm{d}x^i$$

for the two cases, where

$$\partial_0^\alpha f(t) = \frac{f(t)-f(t-\lambda\alpha)}{\lambda\alpha}$$

and similarly for ∂_0^β. The spatial partial derivatives are unchanged, but the time one becomes a finite difference, with the parameters α, β giving the step size in units of λ. This is a typical feature of this class of models: spacetime is "fuzzy" due to finite λ in the commutation relations (1) but without having a lattice, while the differential calculus acquires finite differences. This phenomenon is well known for the standard $\beta = 1$ calculus and responsible for the variable speed of light prediction in Ref. [4]. In the $\beta = 1$ case it is slightly better to normal order with the t to the right, but for general β there is no advantage. The cases $\alpha = 0$ or $\beta = 0$ coincide, and in this case we have the classical time derivative.

III. QUANTUM METRICS FOR THE α AND β CALCULI

In both cases, we will tend to focus on the radial-time sector of the algebra. Here $r = \sqrt{\sum_i x^{i2}}$ and the inherited relations are $[r, t] = \lambda r$ as well as relations for $\mathrm{d}r$ of the same form as for $\mathrm{d}x^i$. In both cases, we let

$$\omega^i = \mathrm{d}x^i - \frac{x^i}{r}\mathrm{d}r,$$

which commutes with spatial variables. Here the sphere direction variables $z^i = \frac{x^i}{r}$ commute with x^i, t according to the relations (1) and obey $\sum_i z^{i2} = 1$, and $\mathrm{d}z^i = r^{-1}\omega^i$. The element $r^{-2}\sum_i \omega^i \otimes \omega^i$ makes sense generally and in our case, for the α and β calculi where spatial differentials and functions commute, behaves classically so long as $t, \mathrm{d}t$ are not involved. Within this sector (or, of course, in the classical limit $\lambda_P \to 0$), we can use standard polar coordinates, when it becomes

$$r^{-2}\sum_i \omega^i \otimes \omega^i = \mathrm{d}\Omega^2 = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi$$

in 4D. In n dimensions, $\mathrm{d}\Omega^2$ here is the metric on the unit sphere S^{n-2}. We use a more explicit notation for tensor products (over the coordinate algebra) than is usual in GR. Also, we will extend our noncommutative algebra to include, say, $r^{\pm\alpha}$, $r^{\pm\beta}$ and in the classical limit all smooth functions. We are working as in quantum mechanics with Hermitian $x^{i*} = x^i$ and $t^* = t$, and $r^* = r$. In the classical limit, the $*$ becomes a complex conjugation of functions, and these requirements become that our coordinates are real. The $*$ extends to the differentials with $\mathrm{d}* = *\mathrm{d}$.

For a quantum metric, we take something of the form $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu$, where $x^0 = t$ and x^i are the spatial coordinates, the coefficients $g_{\mu\nu}$ are elements of the quantum coordinate algebra, and the subscript 1 reminds us that this is the quantum tensor product, i.e., over the quantum coordinate algebra. We require the quantum metric to be invertible in the sense of an inner product (,) on the space of 1-forms, which behaves well (is "strongly tensorial") with respect to multiplication by coordinates in the sense [1]

$$f(\omega,\eta) = (f\omega,\eta), \quad (\omega f,\eta) = (\omega, f\eta), \quad (\omega,\eta f) = (\omega,\eta)f$$

for all elements f of the quantum coordinate algebra and all 1-forms ω, η. It is shown in Ref. [1] that this requires g to commute with elements of the quantum coordinate algebra. We also require that g is "quantum symmetric" in the sense

$$\wedge(g) = 0. \tag{4}$$

The quantum wedge product here is an extension of the 1-forms to an associative algebra of forms of all degree and to which d extends. In our case, the basic 1-forms $\mathrm{d}x^i$, $\mathrm{d}t$ obey the usual exterior or Grassmann algebra (they anticommute), and hence (4) says that the matrix of g in this basis is symmetric. In more complicated models, the wedge product need not take such a standard form, but (4) still makes sense as the appropriate basis-independent concept.

Finally, we need a condition that expresses the reality of the metric coefficients, which we express as [1,16]

$$(* \otimes_1 *)\mathrm{flip}(g) = g, \tag{5}$$

where "flip" swaps the factors of $\otimes_1$. We will in practice omit the subscript on the tensor product as this should be clear from context.

A. Quantum metrics for the α calculus

For the α family (2), we consider a quantum metric of the arbitrary form

$$g = \sum_{i,j}^{n-1} a_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + \sum_i^{n-1} b_i(\mathrm{d}x^i \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^i) + c\mathrm{d}t \otimes \mathrm{d}t,$$

where the coefficients a_{ij}, b_i, c are all elements in the quantum spacetime algebra and obey $a_{ij} = a_{ji}$. This form is dictated by "quantum symmetry" in the form $\wedge(g) = 0$. Using the Leibniz rule and the relation (2), we have

$$\begin{aligned}[g,t] &= \sum_{i,j}^{n-1}([a_{ij},t] + 2\lambda a_{ij})\mathrm{d}x^i \otimes \mathrm{d}x^j \\ &\quad + \sum_i^{n-1}([b_i,t] - \lambda(\alpha-1)b_i)(\mathrm{d}x^i \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^i) \\ &\quad + ([c,t] - 2\lambda\alpha c)\mathrm{d}t \otimes \mathrm{d}t, \\ [g,x^k] &= \sum_{i,j}^{n-1}[a_{ij},x^k]\mathrm{d}x^i \otimes \mathrm{d}x^j \\ &\quad + \sum_i^{n-1}[b_i,x^k](\mathrm{d}x^i \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^i) \\ &\quad + [c,x^k]\mathrm{d}t \otimes \mathrm{d}t.\end{aligned}$$

This means that g central amounts to

$$[a_{ij},t] = -2\lambda a_{ij} \quad \forall\ i,j, \qquad [b_i,t] = \lambda(\alpha-1)b_i \quad \forall\ i,$$
$$[c,t] = 2\lambda\alpha c, \qquad [a_{ij},x^k] = 0 \quad \forall\ i,j,k,$$
$$[b_i,x^k] = 0 \quad \forall\ i,k, \qquad [c,x^k] = 0 \quad \forall\ k.$$

By solving this, we see that our requirements are that a_{ij}, b_i, c are all functions only of x and have scaling degree $-2, \alpha - 1, 2\alpha$, respectively. Hence, there is a larger moduli of metrics for this differential calculus; we just have to make sure that the coefficients are homogeneous of the appropriate degree.

If we look among spherically symmetric quantum metrics, which seems natural from the form of the algebra (1), then we have

$$g = \delta^{-1} r^{-2} \sum_i \omega^i \otimes \omega^i + a r^{-2} \mathrm{d}r \otimes \mathrm{d}r + b r^{\alpha-1}(\mathrm{d}r \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}r) + c r^{2\alpha} \mathrm{d}t \otimes \mathrm{d}t \quad (6)$$

for $\delta, a, b, c \in \mathbb{R}$, which by the above is central. Here $\delta > 0$ could be normalized to $\delta = 1$, but we have refrained from this as it is dimensionful with dimensions of inverse square length. The quantum metric is quantum symmetric in the sense (4) and obeys the "reality" condition (5) given that r commutes with $\mathrm{d}x^i, \mathrm{d}t$ in this calculus.

B. Quantum metrics for the β calculus

The β family (3) contains the standard calculus at $\beta = 1$, and we find basically the same result as for that in Ref. [1]. We will omit the details and the proof as the methods are the same, but the result is *for the β family calculi in dimension $n > 2$, there are no central quantum metrics among a reasonable class of coefficient functions*.

One can, however, consider metrics that are spherically symmetric and commute with functions of r, t. To do this, let us first note that the elements

$$u = r^{\beta-1}\mathrm{d}r, \qquad v = r^{\beta-1}(r\mathrm{d}t - \beta t \mathrm{d}r), \qquad r^{\beta-1}\omega_i$$

commute with r, t. Also,

$$u^* = u, \qquad v^* = \lambda\beta(\beta-2)u + v, \qquad \omega_i^* = \omega_i$$

by using the commutation relations. Looking in the 2D $r - t$ sector, the element

$$g_{2D} = v^* \otimes v + \beta\lambda(u \otimes v - v^* \otimes u) - \gamma_1(u \otimes v + v^* \otimes u) + \gamma_2 u \otimes u$$

then manifestly commutes with t, r and is "real" in the Hermitian sense provided γ_1, γ_2 are real, and also manifestly obeys $\wedge(g) = 0$. Now let $t' = t + \frac{\gamma_1}{\beta}$, so $\mathrm{d}t' = \mathrm{d}t$, $v' = r^{\beta-1}(r\mathrm{d}t' - \beta t' \mathrm{d}r) = v - \gamma_1 u$, and thus

$$g_{2D} = v'^* \otimes v' + \beta\lambda(u \otimes v' - v'^* \otimes u) + \gamma u \otimes u,$$

where $\gamma = \gamma_2 - {\gamma_1}^2$ is a real parameter. Therefore, we can assume that the time variable has been shifted to eliminate the γ_1 term as the expense of the γ_2 term. We now combine this information with the angular part of the metric, so

$$g = r^{2\beta-2} \sum_i \omega^i \otimes \omega^i + a u \otimes u + b v^* \otimes v + b\lambda\beta(u \otimes v - v^* \otimes u), \quad (7)$$

for $a, b \in \mathbb{R}$, $a, b \neq 0$, commutes with r, t. One could insert an overall normalization to fix the dimensions of g. The additional angular term commutes, has zero wedge product, and obeys the reality condition, so these features all still hold for g. This metric generalizes the one in Ref. [1] from the case $\beta = 1$. Using the same methods as in Ref. [1], we can show that, up to a shift in the t variable, this is the most general form of spherically symmetric metric that commutes with r, t and involves a reasonable class of functions.

IV. CLASSICAL LIMITS

We now look at the classical limits of the spherically symmetric quantum metrics allowed in Sec. III. This means we set $\lambda = 0$ so that our spacetime coordinates x^i, t are the usual commutative ones with their usual differentials and the metric is now understood as a classical one.

A. Emergence of the Bertotti-Robinson metric from the α family

Here we look at the classical limit of the metric in Sec. III A, namely,

$$g = \delta^{-1}\mathrm{d}\Omega^2 + a r^{-2}\mathrm{d}r \otimes \mathrm{d}r + b r^{\alpha-1}(\mathrm{d}r \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}r) + c r^{2\alpha}\mathrm{d}t \otimes \mathrm{d}t, \quad (8)$$

where $a, b, c \in \mathbb{R}$, $\delta > 0$ and we need $b^2 - ac > 0$ for a Minkowski signature. The first thing we do is define the combination

$$\bar{\delta} = \frac{c\alpha^2}{b^2 - ac}$$

and compute for $n \geq 3$ that the Einstein tensor is

$$G = -\frac{(n-2)(n-3)}{2}\delta g + ((n-3)\delta - \bar{\delta})\delta^{-1}\mathrm{d}\Omega^2. \quad (9)$$

We also mention the scalar curvature

$$S = (n-2)(n-3)\delta + 2\bar{\delta} \quad (10)$$

and that the metric is conformally flat for $n < 4$, while for $n = 4$ it is conformally flat when $\delta + \bar{\delta} = 0$.

Our first observation is that this G can never match a perfect fluid other than the vacuum energy case given by $(n-3)\delta = \bar{\delta}$. This is because the one-upper index Einstein tensor $\underline{G}$ is diagonal in our coordinate basis with eigenvalues

$$-\frac{(n-2)(n-3)}{2}\delta, \qquad -\bar{\delta}-\frac{(n-3)(n-4)}{2}\delta, \tag{11}$$

where the first eigenspace is spanned by the t, r directions and the other eigenspace is spanned by the angular directions. Now if the two eigenvalues of $\underline{G}$ are distinct, then we cannot have $\underline{G} = 8\pi G_N(p\mathrm{id} + (p+\rho)U \otimes u)$ for a timelike 1-form u and associated vector field U, because this would require u to have only one nonzero entry (since otherwise $U \otimes u$ would have off diagonals), and in that case adding $U \otimes u$ can only change the eigenvalue in a one-dimensional subspace, contradicting the equality of the eigenvalues in the r, t subspace.

Next, we define a Maxwell field strength

$$F = q\sqrt{b^2 - acr^{\alpha-1}}(\mathrm{d}t \otimes \mathrm{d}r - \mathrm{d}r \otimes \mathrm{d}t) \tag{12}$$

when viewed as a tensor product of 1-forms. Its stress-energy tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F^{\alpha}{}_{\nu} - \frac{F^2}{4}g_{\mu\nu}\right)$$

works out as

$$T = -\frac{q^2}{4\pi}\left(\frac{g}{2} - \delta^{-1}\mathrm{d}\Omega^2\right)$$

after a short computation. Here $F^2 = -2q^2$ so that when present (i.e., when $q \neq 0$) the electromagnetic type is non-null. Comparing with (9), we obey Einstein's equation with cosmological constant Λ if we set

$$\Lambda = \frac{(n-2)(n-3)}{2}\delta - q^2 G_N,$$
$$q^2 G_N = \frac{1}{2}((n-3)\delta - \bar{\delta}),$$

which entails $\bar{\delta} \leq (n-3)\delta$, with the case of equality being the vacuum energy solution already noted. This also implies that

$$\Lambda = \frac{1}{2}((n-3)^2\delta + \bar{\delta}), \tag{13}$$

$$\left(\frac{n-2}{2}\right)\bar{\delta} \leq \Lambda \leq \frac{(n-2)(n-3)}{2}\delta. \tag{14}$$

In the important case of $n = 4$, we see that the cosmological constant vanishes exactly in the conformally flat case $\bar{\delta} = -\delta$, while the Maxwell field strength vanishes exactly in the case $\bar{\delta} = \delta$.

These computations are a generalization of Ref. [11] for the standard form of Bertotti-Robinson metric if we take $\alpha = -1,\ \delta = q = q'^{-2},\ a = c_0^2 q'^2,\ b = 0,\ c = -q'^2$, and $\bar{\delta} = -c_0^{-2}q'^{-2}$ in terms of the notation there, denoting q in Ref. [11] as q' to avoid confusion with our q. On the other hand, we now show that all cases of (8), even as we vary α, are locally equivalent to the Bertotti-Robinson metric up to a change of variables; i.e., the moduli space is in fact only the two real parameters $\delta, \bar{\delta}$. We treat the different signs of $\bar{\delta}$ separately.

(i) If $\bar{\delta} > 0$, then this implies $c, a + \frac{\alpha^2}{\bar{\delta}}, \alpha^2 > 0$. We define a change of variables

$$t' = \frac{\alpha}{\sqrt{\bar{\delta}}}\ln r, \qquad r' = \sqrt{c}t - \frac{\sqrt{a + \frac{\alpha^2}{\bar{\delta}}}}{\alpha r^{\alpha}}$$

when $b > 0$ and the opposite sign in the second term of r' when $b < 0$. Then our metric becomes

$$g = \delta^{-1}\mathrm{d}\Omega^2 + e^{2t'\sqrt{\bar{\delta}}}\mathrm{d}r'^2 - \mathrm{d}t'^2, \tag{15}$$

which is a known form of the Bertotti-Robinson metric. Indeed, comparing to 2D de Sitter in the flat slicing

$$g_{dS} = e^{2t\sqrt{\bar{\delta}}}\mathrm{d}x - \mathrm{d}t^2,$$

we see that the metric is that of a part of $S^{n-2} \times dS_2$ with respective curvature scales $\delta, \bar{\delta}$.

(ii) If $\bar{\delta} = 0$ and $\alpha^2 > 0$, we have $c = 0$ and we use a different change of variables: if $a > 0$, say,

$$r' = \alpha r^{\alpha} - \frac{1}{\alpha r^{\alpha}} + \frac{2b}{a}t, \qquad t' = \alpha r^{\alpha} + \frac{1}{\alpha r^{\alpha}} - \frac{2b}{a}t,$$
$$g = \delta^{-1}\mathrm{d}\Omega^2 + \frac{a}{4\alpha^2}(\mathrm{d}r'^2 - \mathrm{d}t'^2).$$

If $a < 0$, we use the same but swap the roles of t', r'. If $\alpha = 0$, we have a different change of variables given by linear combinations of $\ln r, t$, with a similar conclusion. In all cases, we have the metric of a part of $S^{n-2} \times \mathbb{R}^2$ with sphere curvature scale δ.

(iii) If $\bar{\delta} < 0$, then $c, a + \frac{\alpha^2}{\bar{\delta}} < 0 < \alpha^2$, and we define

$$r' = \frac{\alpha}{\sqrt{-\bar{\delta}}}\ln r, \qquad t' = \sqrt{-c}t + \frac{\sqrt{-a - \frac{\alpha^2}{\bar{\delta}}}}{\alpha r^{\alpha}}$$

when $b > 0$ and the opposite sign in the second term of t' when $b < 0$. Then our metric becomes

$$g = \delta^{-1}\mathrm{d}\Omega^2 - e^{2r'\sqrt{-\bar{\delta}}}\mathrm{d}t'^2 + \mathrm{d}r'^2. \tag{16}$$

This should be compared with 2D anti–de Sitter space metric, a part of which in certain coordinates can be written as

$$g_{AdS} = -e^{2v\sqrt{-\bar{\delta}}}\mathrm{d}t^2 + \mathrm{d}v^2.$$

We see that the metric is that of a part of $S^{n-2} \times AdS_2$ with respective curvature scales $\delta, \bar{\delta}$.

In summary, up to local changes of coordinates, classical metrics which are classical limits of quantum metrics for the α calculus and which are spherically symmetric are given by two parameters $\delta, \bar{\delta}$ and equivalent to the Bertotti-Robinson metric.

If we drop the spherical symmetry assumption, i.e., we just ask for classical metrics that are limits of quantum ones, then we have the allowed form

$$\begin{aligned} g = h &+ r^{-1}(\eta \otimes \mathrm{d}r + \mathrm{d}r \otimes \eta) + r^{\alpha}(\zeta \otimes \mathrm{d}t + \mathrm{d}t \otimes \zeta) \\ &+ ar^{-2}\mathrm{d}r \otimes \mathrm{d}r + br^{\alpha-1}(\mathrm{d}r \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}r) \\ &+ cr^{2\alpha}\mathrm{d}t \otimes \mathrm{d}t \end{aligned}$$

due to the degree requirements in Sec. III A and our polar decomposition, where $h = h_{ij}(z)\mathrm{d}z^i \otimes \mathrm{d}z^j$ is now a general metric on S^{n-2}, a, b, c are now functions in S^{n-2}, and η, ζ are further possible 1-forms on S^{n-2}. The Einstein tensor is now typically much more complicated. This polar decomposition also applies in the quantum case of Sec. III A.

We note in passing that in $n = 4$ we can consider classical metrics like (8) but replace S^2 with $\delta^{-1}\mathrm{d}\Omega$ by a general surface Σ with metric h_Σ and set $\delta = S_\Sigma/2$ according to its Ricci scalar curvature. We keep a, b, c constant. Then the calculations above go through in the same way, and the Einstein tensor suggestively matches the stress energy of a Maxwell field (12) and Λ, $q^2 G_N$ given by the same formulas as before but typically now varying on Σ on account of δ varying. The constant case $H^2 \times dS_2$ or $H^2 \times AdS_2$ where we use the hyperboloid H^2 with curvature scale $\delta < 0$ gives constants and completes the Bertotti-Robinson family.

B. Emergence of flat metric from the β family

The results are again much the same as in Ref. [1] for $\beta = 1$. The conceptually new result is that for a different choice of β we can, however, see flat spacetime as emerging from our algebraic considerations.

We will again be interested in matching to a perfect fluid. If so, then this implies

$$\underline{G}.U = -8\pi G_N \rho U,$$

so that U is a timelike eigenvector with eigenvalue $-8\pi G_N \rho$. We look for this first as a necessary but not sufficient condition for matching to a perfect fluid. Having identified the possible values of $-8\pi G_N \rho$, we look in its eigenspace for a timelike vector U such that the original Einstein equation holds. In this case,

$$p = \frac{1}{3}\left(\rho + \frac{\mathrm{Trace}\underline{G}}{8\pi G_N}\right)$$

is also necessary, and we see for what parameter values G now obeys Einstein's equation. We focus on the 4D case. The 2D case, of course, automatically has $G = 0$.

The classical limit of the metric in Sec. III B is

$$g = r^{2\beta-2}(r^2\mathrm{d}\Omega^2 + (a + \beta^2 bt^2)\mathrm{d}r^2 - 2brt\mathrm{d}t\mathrm{d}r + br^2\mathrm{d}t^2).$$

From the determinant of g, we need $a + (\beta^2 - 1)bt^2$ and b to have opposite sign to have the possibility of Minkowski signature, which looking at small t means a, b have opposite sign and looking for large t means $\beta^2 \le 1$. We find Ricci scalar

$$S = 2\frac{a(a - \beta(4\beta + 3)) + (\beta^2 - 1)bt^2(2a - 3\beta^2 + (\beta^2 - 1)bt^2)}{r^{2\beta}(a + (\beta^2 - 1)bt^2)^2}.$$

As far as matching a perfect fluid is concerned, there are three distinct eigenvalues of the Einstein tensor. One of these gives us

$$8\pi G_N \rho = -\frac{\beta(\beta + 1)(2a + \beta(\beta - 1)bt^2)}{r^{2\beta}(a + (\beta^2 - 1)bt^2)^2}$$

with the null space of $\underline{G} + 8\pi G_N \rho$ being the angular directions. These are not timelike. The other choices of eigenvalue have

$$\begin{aligned} 8\pi G_N \rho = &\frac{a^2 - \beta(2\beta + 1)a + (\beta^2 - 1)bt^2((\beta^2 - 1)bt^2 - 2\beta^2 + 2a)}{r^{2\beta}(a + (\beta^2 - 1)bt^2)^2} \\ &\pm \frac{\beta(\beta + 1)\sqrt{a^2 + 6\beta(\beta - 1)abt^2 + (\beta - 1)^2\beta(4 + 5\beta)b^2t^4}}{r^{2\beta}(a + (\beta^2 - 1)bt^2)^2}, \end{aligned}$$

and of these only the $+$ sign has a timelike vector in the null space of $\underline{G} + 8\pi G_N \rho$. Taking this, we then require Einstein's equation to hold, and this fixes $a = \beta^2, -\beta(\beta + 2)$ (looking at the $t = 0$ term in an expansion of Einstein's equation); this then fixes $\beta = \pm 1$ (looking at higher powers of t).

The case $\beta = 1$ is the case covered in Ref. [1], and these are the two choices $a = 1 (b < 0)$ and $a = -3 (b > 0)$ in this case that were found there, respectively, a strong gravitational source with positive pressure but zero density and a cosmological solution with negative pressure and positive density (quintessence ratio $-\frac{1}{2}$).

The case $\beta = -1$ has allowed values $a = 1$, and this turns out to correspond precisely to flat space. This is realized by the change of variables

$$r' = r^{-1}, \qquad x'^i = r^{-2} x^i, \qquad t' = r^{-1} t$$

given in Ref. [1] which rendered g manifestly flat up to a factor $r'^{-4} = r^4$. The above metric is r^{-4} times the metric in Ref. [1] so now becomes the flat metric $g = \mathrm{d}x'^{i2} + b\mathrm{d}t'^2$, where $b < 0$.

We conclude that in the 3-parameter space of a, b, β there are precisely three cases where the Einstein tensor matches a perfect fluid, namely, the two cases already in Ref. [1] where $\beta = 1$ and the new case $\beta = -1$ which has the flat metric when $a = 1$. Thus, while the flat metric does not extend to a full quantum metric, it does extend to the class that partially commutes, namely, with r, t.

V. QUANTUM GEOMETRY

Finally, we show that the quantum metric found in Sec. III A indeed leads to quantum Riemannian geometry in the formalism of Refs. [1,16], and we find, remarkably, that the change of variables that diagonalized our classical metric in Sec. IV A also provides canonically conjugate variables for the quantum algebra; i.e., the quantum spacetime in the radial-time sector is a Heisenberg algebra as in ordinary quantum mechanics.

We start with the $n = 2$ case so we are doing "quantum de Sitter space," leaving out the $\mathrm{d}\Omega^2 = r^{-2} \sum \omega^i \otimes \omega^i$ term from the quantum metric (6). In the classical limit in Sec. IV A, we used a change of variables (15) to convert this to de Sitter spacetime for some scale $\bar{\delta} = c\alpha^2/(b^2 - ac)$. We focus on $\bar{\delta} > 0$, but the anti–de Sitter case can be handled similarly.

Since r commutes with both $\mathrm{d}t$ and $\mathrm{d}r$, the change of variable we used classically works just as well in the quantum case. So, working in the quantum algebra as in Sec. III A, we set

$$T = \frac{\alpha}{\sqrt{\bar{\delta}}} \ln r, \qquad R = \sqrt{c}t - \frac{\sqrt{a + \frac{\alpha^2}{\bar{\delta}}}}{\alpha r^\alpha},$$
$$g = e^{2T\sqrt{\bar{\delta}}} \mathrm{d}R \otimes \mathrm{d}R - \mathrm{d}T \otimes \mathrm{d}T,$$

where

$$\mathrm{d}T = \frac{\alpha}{r\sqrt{\bar{\delta}}} \mathrm{d}r, \qquad \mathrm{d}R = \sqrt{c}\mathrm{d}t + \frac{\sqrt{a + \frac{\alpha^2}{\bar{\delta}}}}{r^{\alpha+1}} \mathrm{d}r$$

works out in just the same way. But note that $[f(r), t] = \lambda r f'(r)$ in view of the commutation relations $[r, t] = \lambda r$. Hence, in terms of the new variables we have

$$[T, R] = \left[\frac{\alpha}{\sqrt{\bar{\delta}}} \ln r, \sqrt{c}t\right] = \lambda', \qquad \lambda' = \lambda\sqrt{b^2 - ac}. \tag{17}$$

In other words, T, R are a canonical conjugate pair with Heisenberg relations between them, for a modified parameter λ'. Similarly, using the relations of the α family calculus we find

$$[R, \mathrm{d}T] = \left[\sqrt{c}t, \frac{\alpha}{r\sqrt{\bar{\delta}}} \mathrm{d}r\right] = \frac{\alpha\sqrt{c}}{\sqrt{\bar{\delta}}} \left[t, \frac{\mathrm{d}r}{r}\right] = 0,$$
$$[R, \mathrm{d}R] = \left[\sqrt{c}t, \sqrt{c}\mathrm{d}t + \frac{\sqrt{a + \frac{\alpha^2}{\bar{\delta}}}}{r^{\alpha+1}} \mathrm{d}r\right]$$
$$= \lambda c\alpha \mathrm{d}t + \lambda\sqrt{a + \frac{\alpha^2}{\bar{\delta}}} \frac{\alpha}{r^{\alpha+1}} \mathrm{d}r = \lambda'\sqrt{\bar{\delta}}\mathrm{d}R$$

and more obviously $[T, \mathrm{d}T] = 0$, $[T, \mathrm{d}R] = 0$. So we have a closed algebra of the R, T and their differentials which we now adopt (we can regard the passage between the two sets of variables as formal). This is a nonstandard differential calculus on the familiar Heisenberg algebra. We also have $R^* = R$ and $T^* = T$, as our change of variables involved only real coefficients, and we suppose that we can extend our Heisenberg algebra to include exponentials of T, for example, in some operator realization. We can check our calculations by seeing that g is indeed central:

$$[R, g] = [R, e^{2T\sqrt{\bar{\delta}}}]\mathrm{d}R \otimes \mathrm{d}R$$
$$+ e^{2T\sqrt{\bar{\delta}}}([R, \mathrm{d}R] \otimes \mathrm{d}R + \mathrm{d}R \otimes [R, \mathrm{d}R]) = 0$$

using the Heisenberg relations for the first term and the $[R, \mathrm{d}R]$ relations for the second term.

Next, we write down the quantum Levi-Civita connection

$$\nabla \mathrm{d}R = -\sqrt{\bar{\delta}}(\mathrm{d}R \otimes \mathrm{d}T + \mathrm{d}T \otimes \mathrm{d}R),$$

$$\nabla \mathrm{d}T = -\sqrt{\bar{\delta}} e^{2T\sqrt{\bar{\delta}}} \mathrm{d}R \otimes \mathrm{d}R$$

modeled on the classical one. We have taken the same Christoffel symbols, just with the quantum tensor product. We then check that this extends as a left quantum connection in the sense $\nabla(f\omega) = \mathrm{d}f \otimes \omega + f\nabla\omega$ for all 1-forms ω and all elements f of our quantum algebra. Note that in physics we usually think of a connection as a covariant derivative (say, on 1-forms) along vector fields, but one can think of it equally as a map from 1-forms to a tensor product of 1-forms where the first tensor factor is waiting to evaluate against any vector field. It is the latter that we take as a definition of ∇ in the quantum case. Torsion-freeness holds in the sense $\wedge\nabla \mathrm{d}R = \wedge\nabla \mathrm{d}T = 0$ under the wedge product (here $\mathrm{d}T, \mathrm{d}R$ anticommute as usual from the fact that the $\mathrm{d}t, \mathrm{d}r$ do).

We also have a right-hand connection rule $\nabla(\omega f) = \sigma(\omega\otimes \mathrm{d}f) + (\nabla\omega).f$ as in Refs. [1,16–18], where, in our case, σ is the flip map on $\mathrm{d}R, \mathrm{d}T$. These values of σ are determined from ∇ by the formula stated; we just have to check that it is well defined when extended "strongly tensorially" as a bimodule map, i.e., commuting with multiplication by elements of the quantum coordinate algebra from either side. On general 1-forms, it will not simply be a flip. For example, using the commutation relations for the calculus,

$$\begin{aligned}\sigma(\mathrm{d}RR\otimes \mathrm{d}R) &= \sigma(\mathrm{d}R\otimes R\mathrm{d}R)\\ &= \sigma(\mathrm{d}R\otimes \mathrm{d}R.R) + \lambda'\sqrt{\bar\delta}\sigma(\mathrm{d}R\otimes \mathrm{d}R)\\ &= \sigma(\mathrm{d}R\otimes \mathrm{d}R)R + \lambda'\sqrt{\bar\delta}\sigma(\mathrm{d}R\otimes \mathrm{d}R)\\ &= \mathrm{d}R\otimes \mathrm{d}R(R+\lambda'\sqrt{\bar\delta}).\end{aligned}$$

We also have ∇ real in the sense [1,16]

$$\nabla(\omega^*) = \sigma((*\otimes *)\mathrm{flip}\nabla\omega)$$

for all 1-forms ω. Finally, metric compatibility now makes sense in the form $\nabla g = 0$, where ∇ is computed by the rules above and extended to the two tensor factors in g by acting on each factor and using σ to flip the left output of ∇ to the far left. We compute

$$\begin{aligned}\nabla g = {}& \nabla(e^{2T\sqrt{\bar\delta}}\mathrm{d}R)\otimes \mathrm{d}R\\ & - \sqrt{\bar\delta}\sigma(e^{2T\sqrt{\bar\delta}}\mathrm{d}R\otimes \mathrm{d}R)\otimes \mathrm{d}T\\ & - \sqrt{\bar\delta}\sigma(e^{2T\sqrt{\bar\delta}}\mathrm{d}R\otimes \mathrm{d}T)\otimes \mathrm{d}R\\ & - \nabla \mathrm{d}T\otimes \mathrm{d}T + \sqrt{\bar\delta}\sigma(\mathrm{d}T\otimes e^{2T\sqrt{\bar\delta}}\mathrm{d}R)\otimes \mathrm{d}R\\ = {}& 0.\end{aligned}$$

Here the value of ∇ on the second tensor factor has been inserted, and σ is used to bring its left output to the far left. When the value of ∇ on the first tensor factor is also inserted and the rules for σ are used, we find all the terms cancel and we get zero. The curvature R_∇ as a 2-form valued operator on 1-forms can also be computed by using the definitions in Refs. [1,16], and one finds

$$\begin{aligned}R_\nabla \mathrm{d}R &= \bar\delta \mathrm{d}R\wedge \mathrm{d}T\otimes \mathrm{d}T,\\ R_\nabla \mathrm{d}T &= \bar\delta e^{2T\sqrt{\bar\delta}}\mathrm{d}R\wedge \mathrm{d}T\otimes \mathrm{d}R.\end{aligned}$$

Lifting the 2-forms to antisymmetric tensors and tracing, one then gets $\mathrm{Ricci} = \bar\delta g$ when normalized in a way that matches the classical conventions. For this, the inverse metric is $(\mathrm{d}R, \mathrm{d}R) = e^{-2T\sqrt{\bar\delta}}$, $(\mathrm{d}T, \mathrm{d}T) = -1$ extended as a bimodule map in the manner explained in Sec. II. These calculations for quantum de Sitter geometry would be much harder in the r, t algebra variables, but in the R, T ones, which are very close to classical, we see that they follow the classical form provided we are careful about some of the orderings.

The general case in $n\geq 4$ or quantum Bertotti-Robinson space is not really any different. In the quantum case, we do not want to work with angles but work with $z^i = \frac{x^i}{r}$. These commute with r, t and, in the α calculus, so do their differentials $\mathrm{d}z^i = \omega^i/r$ as we saw in Sec. III A. It is also true, again for the α calculus, that z^i commute with $\mathrm{d}r, \mathrm{d}t$. Hence, they describe an entirely classical S^{n-2} which commutes with R, T and their differentials as well. After our change of variables, the quantum metric in Sec. III A now becomes

$$g = \delta^{-1}\mathrm{d}\Omega^2 + e^{2T\sqrt{\bar\delta}}\mathrm{d}R\otimes \mathrm{d}R - \mathrm{d}T\otimes \mathrm{d}T$$

much as before. Now, because the z^i and their differentials decouple from the R, T sector as explained, one can show that the quantum Levi-Civita connection is given by that on the S^{n-2}, which is the same as classically, namely,

$$\nabla \mathrm{d}z^i = -z^i\delta^{-1}\mathrm{d}\Omega^2,$$

and the connection in the R, T sector already obtained above. In principle, there could also be other exotic quantum Levi-Civita connections with no classical limit, a phenomenon seen in the model in Ref. [1]. Also, we have only here done the algebraic level, and there may be issues at the operator algebras level. At that point, the model may usefully tie up with a different approach to noncommutative geometry in Ref. [19].

We have seen that the formulas for the above quantum Bertotti-Robinson space look deceptively like their classical counterparts. This is because the angular sector remains classical and decouples while in the radial-time sector the coefficients of the quantum metric in our basis involve only r (or T in the new variables) which, in the α calculus, commutes with itself and both differentials. As long as functions of t (or R) do not enter, we do not see the noncommutation relations nor do we see the finite-difference partial derivatives that were explained at the end of Sec. II. The same applies to the particular Maxwell field (12) in Sec. IV, so Einstein's equation holds at the quantum level if we take the same formula as before for the stress-energy tensor. This does not mean that the noncommutation relations do not show up when doing other computations, such as solving the wave equation, looking at other Maxwell configurations, or looking for a quantum isometry group of some kind. In that regard, we do have a classical $SO(n-2)$ acting on the sphere variables, and we have preserved this in our constructions, but the classical $SO(1,2)$ or $SL_2(\mathbb{R})$ on the de Sitter or anti–de Sitter part does not appear to quantize in any obvious way so as to respect the nontrivial commutation relations in the radial-time sector.

VI. CONCLUSIONS

A secondary conclusion of the paper is that we have successfully quantized the Bertotti-Robinson spacetime, with quantum coordinate algebra, differentials, metric, and Maxwell field. This model has a modest amount of noncommutativity taking the familiar form of the canonical Heisenberg commutation relations (17) in the right coordinates. We did not find an obvious quantization of the classical isometry group, however, as we have come to the model from the quantum differential geometry rather than from its symmetries. Note that there is a bicrossproduct Poincaré quantum group acting on the algebra (1) as in Ref. [3], but this does not appear to be directly relevant, as it does not respect the differentials and is not the appropriate one for the metric either.

Our main conclusion, however, is in the other direction concerning a mechanism for how classical geometry can emerge out of quantum gravity. This is a central problem for most approaches to quantum gravity, and our answer is that if spacetime, as an intermediate stage, gets noncommutative coordinates due to quantum-gravity effects, then the metric can emerge from the noncommutative algebra. Specifically, starting with nothing but this quantum spacetime hypothesis in the form of a well-known quantum coordinate algebra (1) in the literature, translation symmetry in the differentials, and spherical symmetry in the metric, we were forced to a particular form of metric, namely, one obeying the Einstein equations with a cosmological constant and Maxwell field. The classical limit gives a new characterization of the Bertotti-Robinson metric as emerging from the quantum algebra (1) or, conversely, as quantizable to this algebra. This demonstrates that the quantum spacetime hypothesis can have not only Planck scale but *classical* consequences.

It is not the case that the emerging metric or constrained class of metrics always consists of solutions of Einstein's equation for some known form of stress tensor, but it seems that this is so in some contexts, as here. The Einstein equations themselves classically have their origin in diffeomorphism invariance, which is expressed in the quantum case as our differential algebra constructions after Sec. II being required to be well defined, independently of the choice of algebra generators as coordinates (actually writing down a quantum diffeomorphism group is rather harder and not understood in any generality). In Sec. II, we also required differentials to be similarly well defined even though we wrote them with particular generators. At semiclassical order, these requirements appear in the general analysis of Ref. [2] as classical geometric data subject to quantizability equations. The data for a quantum coordinate algebra is a Poisson tensor π, say, as usual. The data for a differential calculus in Sec. II at semiclassical order are a Poisson-compatible covariant derivative $\hat{\nabla}$ partially defined along Hamiltonian vector fields. Centrality of the quantum metric in Sec. III comes down to metric compatibility $\hat{\nabla}g = 0$. If we suppose this first, then the compatibility of $\hat{\nabla}$ with π is equivalent to [2]

$$\nabla_\rho \pi^{\mu\nu} + \pi^{\mu\alpha} S^\nu{}_{\alpha\rho} + \pi^{\alpha\nu} S^\mu{}_{\alpha\rho} = 0,$$

where $S = \nabla - \hat{\nabla}$ is the contorsion tensor as determined by the torsion of $\hat{\nabla}$, and ∇ is the Levi-Civita connection. This implies an integrability condition on the Riemann curvature of the Levi-Civita connection. Associativity of the quantum differential calculus further requires $\hat{\nabla}$ to be flat and also implies a condition [2] on the Riemann curvature in terms of S and its covariant derivatives. We speculate then that if the torsion of $\hat{\nabla}$ can be related in some context to stress energy, then these ideas may ultimately lead to a fuller understanding of when solutions of Einstein's equations are emergent from noncommutative geometry.

ACKNOWLEDGMENTS

W.-Q. T. was supported by the China Scholarship Council.

[1] E. J. Beggs and S. Majid, Gravity induced by quantum spacetime, Classical Quantum Gravity **31**, 035020 (2014).

[2] E. J. Beggs and S. Majid, Semiquantisation functor and Poisson Riemannian geometry, arXiv:1403.4231.

[3] S. Majid and H. Ruegg, Bicrossproduct structure of the κ-Poincaré group and noncommutative geometry, Phys. Lett. B **334**, 348 (1994).

[4] G. Amelino-Camelia and S. Majid, Waves on noncommutative spacetime and gamma-ray bursts, Int. J. Mod. Phys. A **15**, 4301 (2000).

[5] R. Oeckl, Classification of differential calculi on $U_q(b_+)$, classical limits and duality, J. Math. Phys. (N.Y.) **40**, 3588 (1999).

[6] S. Meljanac, S. Kresic-Juric, and R. Strajn, Differential algebras on κ-Minkowski space and action of the Lorentz algebra, Int. J. Mod. Phys. A **27**, 1250057 (2012).

[7] S. Majid and W. Tao, Noncommutative differentials on Poisson-Lie groups and pre-Lie algebras, arXiv:1412.2284.

[8] T. Levi-Civita, Realta fisica di alcuni spazi normali del Bianchi, Rend. Reale Accad. Lincei **26**, 519 (1917).

[9] B. Bertotti, Uniform electromagnetic field in the theory of general relativity, Phys. Rev. **116**, 1331 (1959).

[10] R. Robinson, A solution of the Maxwell-Einstein equations, Bull. Acad. Pol. Sci. **7**, 351 (1959).

[11] M. Gürses and Ö. Sarioglu, Acelerated Levi-Civita-Bertotti-Robinson metric in D-dimensions, Gen. Relativ. Gravit. **37**, 2015 (2005).

[12] S. Majid, Newtonian gravity on quantum spacetime, Eur. Phys. J. Web Conf. **70**, 00082 (2014).

[13] S. Majid and B. Schroers, q-Deformation and semidualisation in 3D quantum gravity, J. Phys. A **42**, 425402 (2009).

[14] D. Burde, Simple left-symmetric algebras with solvable Lie algebra, Manuscr. Math. **95**, 397 (1998).

[15] C. Bai, Left-symmetric algebras from linear functions, Journal of algebra **281**, 651 (2004).

[16] E. J. Beggs and S. Majid, Compatible connections in noncommutative Riemannian geometry, J. Geom. Phys. **61**, 95 (2011).

[17] M. Dubois-Violette and P. W. Michor, Connections on central bimodules in noncommutative differential geometry, J. Geom. Phys. **20**, 218 (1996).

[18] J. Mourad, Linear connections in noncommutative geometry, Classical Quantum Gravity **12**, 965 (1995).

[19] A. Connes, *Noncommutative Geometry* (Academic, New York, 1994).