Wavy Wilson line and AdS / CFT

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Abstract

Wilson loops which are small deviations from straight, infinite lines, called wavy lines, are considered in the context of the AdS/CFT correspondence. A single wavy line and the connected correlation function of a straight and wavy line are considered. It is argued that, to leading order in "waviness", the functional form of the loop is universal and the coefficient, which is a function of the 't Hooft coupling, is found in weak coupling perturbation theory and the strong coupling limit using the AdS/CFT correspondence. Supersymmetric arguments are used to simplify the computation and to show that the straight line obeys the Migdal-Makeenko loop equation.

1 Introduction and Summary

The AdS/CFT correspondence has provided a fascinating array of relationships between gauge field theories, string theory and supergravity[1]-[9]. One of the natural objects of gauge theory which couples directly to strings is the Wilson loop. The study of Wilson loops has provided an interesting approach to extracting information from the AdS/CFT correspondence [10]-[41].

In $\mathcal{N} = 4$ supersymmetric Yang-Mills theory[1], the Wilson loop of most interest contains both the gauge field and a scalar field in the exponent and has the form in Euclidean space[10]

$$\mathrm{Tr}\left(\mathcal{P}\ e^{\oint ds(iA_\mu(x)\dot{x}^\mu(s)+\Phi(x)\cdot\theta|\dot{x}(s)|)}\right) \tag{1}$$

This loop measures the holonomy of the wave-function of a heavy W-boson which occurs when the gauge symmetry of super Yang-Mills theory is realized in a Higgs phase with the unit vector θ^I related to the condensate, $\left\langle \Phi^I(x)\right\rangle \sim \theta^I$.

Computations of the expectation value of this Wilson loop have proven to be tractable in certain special geometries. For example, the infinite straight line, or any array of infinite parallel straight lines form a BPS object and it is expected that all radiative corrections cancel, so that the Wilson loop corresponding to them has expectation value exactly equal to one.

The expectation value of the circular loop, which is also a BPS object closely related to the straight line, is conjectured to be known exactly[42]. In that case ladder diagrams can be summed explicitly. The sum can be extrapolated to strong coupling and compared with the predictions of the AdS/CFT correspondence where it agrees beautifully. It is conjectured that all corrections to ladder diagrams cancel. This has been demonstrated to leading and next-to-leading orders [42, 43, 44] and there are other arguments to support it[45]. Similar observations have been made for the correlators of chiral primary operators with the circular Wilson loop[46].

Polyakov and Rychkov [47, 48, 49] have discussed Wilson loops which were small deviations from straight lines, their so-called "wavy lines". There, they observed some interesting structures which gave some hope that the area Ansätz at strong coupling actually satisfied the loop equations of the gauge theory.

In this Letter, we shall present some preliminary results of our investigation of wavy lines. We begin by reviewing some preliminaries.

[1]The action and other conventions are summarized in the Appendix.

1.1 Preliminaries

We will be entirely concerned with Wilson loops in four-dimensional Euclidean space. There is some closely related and very interesting work on Minkowski space loops[34]. Apparent differences between those and the present work are attributable to the richer array of boundary conditions which can be imposed in Minkowski space.

A wavy line deviates by a small amount from an infinite straight line. We shall describe it using the Monge gauge parameterization

$$x^\mu(s) = (s, \xi(s)) \quad s \in (-\infty, +\infty) \tag{2}$$

The three-dimensional vector $\xi(s)$ is a smooth function of the curve parameter s with small magnitude.

The expectation value of the Wilson loop is a functional of the geometry of the loop. We will consider the leading order of this functional, which is quadratic in $\xi(s)$. This leading order is restricted by the spacetime symmetries of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Rotation and translation invariance dictate that it has the form

$$\int ds \int ds' \dot{\xi}^i(s) K(s - s') \dot{\xi}^i(s') \tag{3}$$

Scale invariance indicates that the kernel $K(s - s')$ has dimension 1/distance^2 and is therefore of the form

$$K(s - s') \sim \frac{1}{(s - s')^2}$$

However the integration then diverges linearly. We do not expect that such divergences appear in the supersymmetric Wilson loop that we are considering. Therefore, the kernel must be a distribution. There are two distributions with the correct dimensions,

$$\frac{d}{ds}\delta(s - s') \quad , \quad \frac{d}{ds}\frac{\mathbf{P}}{s - s'}$$

The second of these, the derivative of the principal value distribution, is an even function, so we must chose it. By adding terms which integrate to zero, we can then write the functional in a more manifestly finite form,

$$\int ds \int ds' \left\{ 2\dot{\xi}(s) \cdot \dot{\xi}(s') - 2\dot{\xi}(s')^2 \right\} \frac{d}{ds}\frac{\mathbf{P}}{(s - s')} \;=\; \int ds \int ds' \frac{\left(\dot{\xi}(s) - \dot{\xi}(s')\right)^2}{(s - s')^2} \tag{4}$$

We shall see that this is precisely the form that we obtain for the wavy line, both in perturbation theory and in the strong coupling limit using AdS/CFT. In the leading

order in perturbation theory, the part in (3) comes from the gauge interactions, whereas the extra terms needed to make (4) come from the scalar fields.

In addition, we shall consider the connected correlation function of a wavy line with a straight line. In that case, the correlator depends on the distance, L, between the two lines. We will compute the leading order, which varies as $\frac{1}{L^2}$. Then, the same reasoning and counting of dimensions tells us that the correlator must be of the form

$$\frac{1}{L^2}\int ds \int ds' \dot{\xi}(s)\cdot\dot{\xi}(s')\ln[\Lambda^2(s-s')^2] \tag{5}$$

Here, Λ is a constant with the dimension of inverse length.

1.2 Results

Our results can be summarized as follows:

- Unlike the circle and other loops that have been computed in the past, where ladder diagrams were the most important, the wavy line gets all of its corrections from internal loops. Any ladder diagrams either cancel or vanish identically (beyond the trivial leading order for the single wavy line).

- We shall find that the wavy line and the connected correlation function of a wavy line with a straight line indeed have the universal forms, (4) and (5), respectively. We show this to leading order in weak coupling perturbation theory and we confirm it at strong coupling using the AdS/CFT correspondence. For the single wavy line, we also confirm that it is so to next-to-leading order at weak coupling.

- In the universal form (4), to leading orders in perturbation theory, the role of the scalar field in the Wilson loop is minimal. It serves to regulate divergences and define the distribution in the kernel. This is consistent with the results of Polyakov and Rychkov[47, 49] who applied similar ideas to non-supersymmetric loops.

- The coefficients of the universal functionals are nontrivial functions of the coupling constant which we expect obtain contributions from all orders in perturbation theory.

- One way that a scale dependence could creep into the Wilson loop is if the power law in (4) is corrected by logarithms in higher orders of perturbation theory. In

the case of the wavy line, we shall confirm to the next to leading order that it is not corrected by logarithms. We also use AdS/CFT to compute the strong coupling limit and find the same functional form, suggesting that logarithms do not appear at any order.

- The above statement is even more interesting in the case of (5) where, in our explicit computations, a logarithm of the cutoff Λ in fact appears. The integral, however, is insensitive to the appearance of this logarithm. It can be removed by adding a term which is a total derivative. If higher orders in logarithms appeared there, it is hard to see how the cutoff dependence could be removed in this way. We confirm using AdS/CFT that, at strong coupling, there is indeed only this single logarithm. In that case, the ultraviolet cutoff, Λ, is replaced by the inverse of an infrared cutoff, a symptom of the interchange of ultraviolet and infrared behaviors which occurs in the AdS/CFT correspondence in general.

- We use supersymmetry to simplify the computation of correlation functions and put them in a form where further computations can be done more readily.

- We use supersymmetry to find that the infinite straight line obeys the Migdal-Makeenko[50, 51]loop equation of gauge theory. This is beautifully consistent with the results of and Drukker, Gross and Ooguri[45] and Polyakov and Rychkov[47, 49] who showed that the strong coupling Ansätz obeys the loop equation. For a wavy line, this can actually be deduced directly from the fact that it has the functional form in (4) where the kernel does not contain the delta function singularity which would be identified with the loop operator in the quadratic variation of the loop.

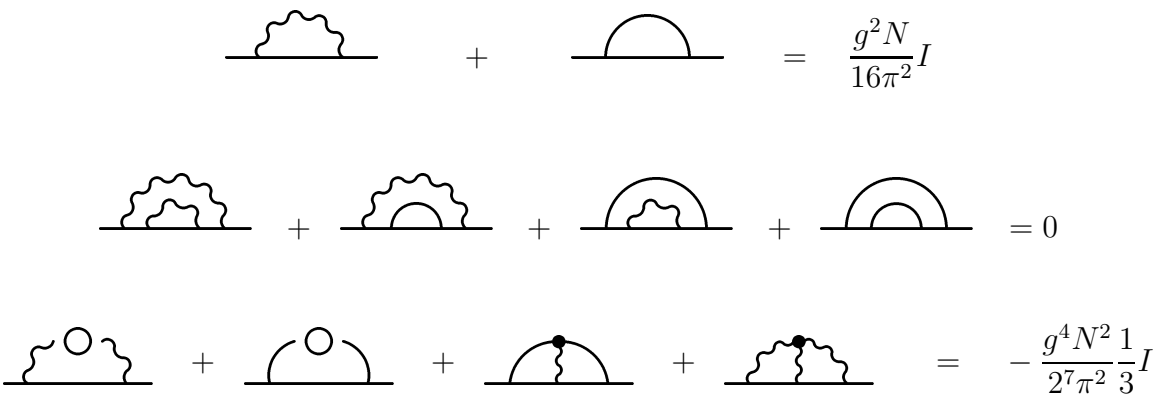
- A local limit of the waviness can be taken so that one could in principle use the operator product expansion to compute the correlation functions of gauge invariant operators with the straight line Wilson loop. We hope to report results in the near future.

2 Weak Coupling

2.1 Single Line

We have calculated the expectation value of a single wavy line to second order in the 't Hooft coupling. The calculation involves evaluating various Feynman diagrams. In

the following the horizontal line denotes the Wilson line, the wiggly line denotes the gauge field A_μ, while the solid line denotes the scalar field Φ_J.



where,

$$I = \oint ds_1\, ds_2 \frac{\left[\dot{\xi}(s_1) - \dot{\xi}(s_2)\right]^2}{2\left(s_1 - s_2\right)^2} \tag{6}$$

The first two diagrams in the last line represent the one-loop corrected exchange of a single particle. These diagrams are divergent, but a divergent piece from the following diagrams (those with an internal vertex) cancel these divergences exactly, leaving a finite result. The diagrams in the second line are zero at second order in waviness individually. Summarizing, we find

$$\langle W(C)\rangle = 1 + \left[\frac{g^2N}{2^4\pi^2} - \frac{g^4N^2}{3\cdot 2^7\pi^2} + \ldots\right] \oint ds_1\, ds_2 \frac{\left[\dot{\xi}(s_1) - \dot{\xi}(s_2)\right]^2}{2\left(s_1 - s_2\right)^2} + \ldots . \tag{7}$$

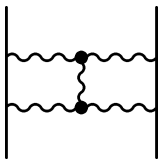
2.2 Line-Line Correlator

A Wilson loop which consists of an array of infinite parallel straight lines is a BPS object. As an operator it commutes with half of the supercharges and one might expect that, like the single straight line, it is protected from quantum corrections. Explicit computations to a few orders in perturbation theory indeed show that lower order corrections cancel and one might conjecture that they do to all orders. This is consistent with what is found in the strong coupling limit using the AdS/CFT

correspondence. It has the physical interpretation that an array of static heavy quarks do not interact.

An interesting variant of this configuration is a combination of an infinite straight line and a wavy deformation of another parallel straight line. In this case, the connected correlation function measures the interaction energy of two quarks which is induced by the slight motion of one of them.

We can compute the correlation function of an infinite straight line and a wavy line in perturbation theory. The leading contribution turns out to be at cubic order in the 't Hooft coupling and is given by the following diagram:



where the two Wilson loops are represented as vertical lines. The other "H-diagrams" obtained by all possible substitutions of scalar for gauge fields also contribute.

In the limit of large separation L, we find:

$$\langle W_1 W_2 \rangle_{\text{connected}} = -\frac{1}{N^2}\,\frac{1}{L^2}\,\left[\frac{g^6 N^3}{2^{11}\pi^4} + \ldots\right] \oint ds\, ds'\, \dot{\xi}(s)\cdot\dot{\xi}(s')\, \ln(\Lambda^2 (s-s')^2) + \ldots \quad (8)$$

The parameter Λ in the logarithm is an ultraviolet cutoff. Note that the result is finite. The cutoff disappears when we integrate by parts in s or s'.

3 Strong Coupling

The AdS/CFT correspondence hypothesizes that the strong coupling limit of the Wilson loop is found by computing the regularized minimal area of a surface in $AdS_5 \times S^5$ whose boundary is the curve of the Wilson loop, embedded in the boundary of the space[10].

We use the metric of $AdS_5 \times S^5$,

$$ds^2 = R^2 \frac{dx^\mu dx^\mu + dy^I dy^I}{y^I y^I}$$

where, according to the AdS/CFT correspondence, the radius of curvature is $R = (g^2 N)^{1/4}\sqrt{\alpha'}$. We shall consider surfaces which are located on a single point in S^5, given by $y^I = y\theta^I$. The boundary of the space is located at $y \to 0$.

3.1 Single Line

The straight line is the boundary of a surface which is orthogonal to the boundary of AdS, which we parameterize using the coordinates (s, t). The parametric embedding of the surface is

$$(x^\mu, y^I) = (s, 0, 0, 0, t\theta^I) \tag{9}$$

This surface is itself AdS_2, with metric

$$ds^2 = R^2 \frac{ds^2 + dt^2}{t^2}$$

It has infinite area,

$$\mathcal{A}_0 = \int_{-\infty}^{\infty} ds \int_0^{\infty} dt \frac{R^2}{t^2}$$

which must be defined by regularization and the infinity must be subtraced to obtain the final result. The regularization is normally carried out by cutting off the integrals

$$\mathcal{A}_0 = \int_{-L/2}^{L/2} ds \int_{\epsilon}^{\infty} dt \frac{R^2}{t^2} = R^2 L/\epsilon$$

The subtraction of the infinite part is implemented by operating $\left(1 + \epsilon \frac{d}{d\epsilon}\right)$ which has an interesting interpretation as a Legendre transform[15, 26]. In the case of an infinite straight line this subtracts everything

$$\mathcal{A}_{0\text{Reg}} = \left(1 + \epsilon \frac{d}{d\epsilon}\right) R^2 \frac{L}{\epsilon} = 0$$

so the result is

$$\langle W(\text{straight line}) \rangle = e^{-\frac{R^2}{2\pi\alpha'} \mathcal{A}_{0\text{Reg}}} = 1$$

To describe a wavy line, we must find a minimal surface whose boundary is the wavy line. We describe the surface using the embedding coordinates

$$(x^\mu, y^I) = (s,\, \Delta_j(t, s),\, t\theta^I) \quad \text{where} \quad \Delta_j(0, s) = \xi_j(s) \tag{10}$$

The three components Δ_j are the small deviation from AdS_2 induced by the waviness of the line.

The regularized area to second order in Δ is given by

$$\mathcal{A}_{\text{Reg}} = -\int \frac{dt\, ds}{t^2} \left\{ \frac{1}{2} (\partial_t \Delta)^2 + \frac{1}{2} (\partial_s \Delta)^2 \right\} \tag{11}$$

The variation of this functional gives an equation of motion for Δ, the solution of which, with the boundary condition in (10) is

$$\Delta(t,s) = \int \frac{ds'\xi(s')}{\pi} \frac{2t^3}{((s-s')^2+t^2)^2} \tag{12}$$

When plugged back into (11) we find the area

$$\mathcal{A}_{\rm Reg}[\xi] = -\frac{1}{2\pi}\int ds'ds \frac{\left[\dot{\xi}(s)-\dot{\xi}(s')\right]^2}{2(s-s')^2} \tag{13}$$

and the strong coupling limit of the wavy line Wilson loop is

$$\langle W(-)\rangle = \exp\left(\frac{\sqrt{g^2N}}{4\pi^2}\int ds'ds \frac{\left[\dot{\xi}(s)-\dot{\xi}(s')\right]^2}{2(s-s')^2} + \dots\right)$$

Note that, as was expected, this procedure gives the same functional of the deviation from the straight line as we found at weak coupling. This supports the idea that the power law behavior is not corrected by logarithms at intermediate orders in perturbation theory. The coefficient seems to be a non-trivial function of the coupling, interpolating between a linear function at weak coupling and the square root at strong coupling.

3.2 Line-Line Correlator

If we consider an array of parallel straight lines, they are the boundaries of a set of sheets in $AdS_5 \times S^5$ similar to (9). If two lines are anti-parallel, they can be joined by a single sheet which dominates their connected correlation function. Since, in the case of interest to us, the lines are parallel, rather than anti-parallel, there is no single sheet whose boundary is more than one of the lines. This means that in the leading order, the lines do not interact, i.e. their connected correlation function vanishes. This is consistent with the weak coupling expansion where we saw that the connected correlation function indeed has a coefficient proportional to $1/N^2$ (times a function of the 't Hooft coupling g^2N) which indicates that it arises from higher genus Feynman diagrams. In the strong coupling limit, we expect this to translate to higher genus surfaces.

Still, we expect that, for exactly parallel lines, because of their BPS nature, higher genus contributions cancel exactly. We emphasize that we do not know an explicit proof of this statement. It can be checked to leading orders in weak coupling and it

appears to occur there. However, at strong coupling, we are not even able to check it to the leading order, but we shall assume that it is the case.

It is possible to take into account higher genus surfaces to the straight line-wavy line correlation function if we consider the limit where the lines are far apart compared to the distance scale of the waviness. In this case, at higher genus, the lines are boundaries of two infinite sheets which are connected by thin tubes, which are formed by the exchange of the light particles in the spectrum of supergravity linearized about the $AdS_5 \times S^5$ background.

This idea was used in [52] to compute the correlator of widely separated circular Wilson loops. The Euler character $\chi = -2$ worldsheet then has the area

$$\mathcal{A}_{\chi=-2} = \frac{g^2 N}{4\pi^2} \int_{\Omega_1} \int_{\Omega_2} \mathcal{V}_1 \, P \, \mathcal{V}_2 \tag{14}$$

where Ω_i refers to the worldsheet domains, $\mathcal{V}_i$ refers to the vertex operator on that worldsheet, and P denotes the propagator for the supergravity field which travels between the worldsheets. The factor in front comes from two powers of the coefficient of the worldsheet action, $(R^2/2\pi\sqrt{\alpha'})^2 = g^2 N/4\pi^2$.

The lightest mode comes from the perturbation of the AdS_5 metric, which is expressed in terms of a Kaluza-Klein scalar (in the following equation greek indices refer to the five coordinates on AdS_5)

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{10k}{3} g_{\mu\nu} \, s^k \, Y^k + \frac{4}{k+1} D_{(\mu} D_{\nu)} \, s^k \, Y^k + \frac{32k}{15} g_{\mu\nu} \, s^k \, Y^k, \tag{15}$$

where Y^k is the spherical harmonic on S^5, and s^k is the Kaluza-Klein scalar field described in section 4.2 of [52]. The lightest mode of s^k is for $k = 2$. The resulting perturbation in the Nambu action on AdS_5 yields the vertex operator. At zeroth order in waviness the vertex operator is

$$\mathcal{V}_{\Delta=0} = -\frac{12}{5}\frac{1}{t^2} - \frac{4}{15}\vec{\nabla}^2 + \frac{4}{5}\frac{1}{t}\partial_t + \frac{2}{5}(\partial_t^2 + \partial_s^2) \tag{16}$$

where s and t are the embedding coordinates as in (10), i.e. the worldsheet coordinates. The derivatives will act on the propagator. In the limit of large separation the propagator is

$$P = \frac{9}{8N^2} \frac{t^2 t'^2}{\left[(s-s')^2 + (t-t')^2 + (\vec{x}-\vec{x}')^2\right]^2} \Bigg|_{\vec{x}=\vec{x}'+\vec{L}} \tag{17}$$

where the separation is given by $\vec{L}$. As for the vertex operator of the wavy line, i.e. the vertex operator terms which are second order in waviness, we keep only those

terms which contain derivatives in t', other terms producing results subleading in the separation. We find

$$\begin{aligned}\mathcal{V}(s',t') = &\left(\frac{1}{2}\dot{\Delta}^2+\frac{1}{2}\Delta'^2\right)\frac{1}{t'^2}\left\{-\frac{12}{5}-\frac{4}{15}\left[t'^2\partial_{t'}^2-3t'\partial_{t'}\right]\right\}\\ &+\frac{1}{3}\left(-\dot{\Delta}^2\partial_{t'}^2+\Delta'^2\partial_{t'}^2\right)-\frac{4}{3}\frac{1}{t'}\dot{\Delta}^2\partial_{t'}\end{aligned} \tag{18}$$

where $\Delta'(t',s') = \partial_{s'}\Delta(t',s')$ and $\dot{\Delta}(t',s') = \partial_{t'}\Delta(t',s')$, and where derivatives act on the numerator of the propagator only, else leading to subleading terms. Plugging everything into (14), we find to leading order in the separation

$$\mathcal{A}_{\chi=-2} = \frac{1}{L^2}\frac{1}{N^2}\frac{g^2N}{8\pi^2}\frac{3}{20}\int ds\, ds'\, \dot{\xi}(s)\cdot\dot{\xi}(s')\, \ln(\mu^2\,(s-s')^2) \tag{19}$$

This has the same dependence on ξ as the weak coupling limit, supporting our expectation that the logarithm is not modified by loop corrections and it remains the same universal form at all orders. The coefficient is indeed of order $1/N^2$ and the coupling constant appears to be nontrivial.

4 Supersymmetry, Simplifications, and the Loop Equation

Consider the expansion of the single wavy line to two orders in the function ξ,

$$\begin{aligned}\delta^2 W(C) =&\frac{1}{N}\,\mathrm{Tr}\int ds\int dt\;\mathcal{P}e^{\int_s^\infty ds_1 E(s_1)}\mathcal{O}_1(s)\mathcal{P}e^{\int_t^s ds_2 E(s_2)}\mathcal{O}_1(t)\mathcal{P}e^{\int_{-\infty}^t ds_3 E(s_3)}\\ &+\frac{1}{N}\,\mathrm{Tr}\int ds\mathcal{P}e^{\int_s^\infty ds_1 E(s_1)}\mathcal{O}_2(s)\mathcal{P}e^{\int_{-\infty}^s ds_2 E(s_2)},\end{aligned} \tag{20}$$

where,

$$E(s) = iA_0(x(s)) + \Phi(x(s))\cdot\theta \tag{21}$$

$$\mathcal{O}_1(s) = \xi_j(s)\left[\, iF_{j0}(x(s)) + D_j\Phi(x(s))\cdot\theta\,\right] \tag{22}$$

$$\mathcal{O}_2(s) = \xi_k(s)\xi_j(s)\left[\, iD_jF_{k0}(x(s)) + D_jD_k\Phi(x(s))\cdot\theta\,\right]. \tag{23}$$

In the following we will show that the second term in (20) is the result of acting the loop space Laplace operator on the Wilson loop and vanishes identically.

We shall also find that the wavy line operators (22, 23) can be described, via supersymmetry, in terms of fermionic operators. This provides a simplification of the Feynman diagrams involved in the various computations, as well as a proof that (23) vanishes to all orders in perturbation theory, which means that the straight line Wilson loop obeys the loop equation.

We use the ten dimensional supersymmetry transformations, as $D = 4, \mathcal{N} = 4$ super-Yang-Mills theory is just a dimensional reduction of $D = 10, \mathcal{N} = 1$ super-Yang-Mills theory. The supersymmetry transformations are

$$\delta A^\mu = \frac{i}{2}\bar{\epsilon}\gamma^\mu\psi \quad \delta\Phi^m = \frac{i}{2}\bar{\epsilon}\Gamma^m\psi \quad \delta\psi = -\frac{1}{4}\Gamma^{MN}F_{MN}\epsilon \quad \Gamma^{MN} = \frac{1}{2}[\Gamma^M, \Gamma^N] \tag{24}$$

where $M = (0, i, m)$ so that $i = 1, 2, 3$, $m = 4, ..., 9$, and $\mu = 0, ..., 3$. The 10-D gamma matrices are $\Gamma^M = (\gamma^\mu, \Gamma^m)$ and ψ is a 10-D Majorana-Weyl fermion. The generalized field strength F^{MN} is understood as being built from the 10-D gauge field $A^M = (A^\mu, \Phi^m)$.

We identify a projected supersymmetry transformation which commutes with the straight line Wilson loop, that is with the exponent (21)

$$\bar{\epsilon} = \bar{\eta}\,(i\gamma^0 + \Gamma\cdot\theta). \tag{25}$$

Let the supercharge responsible for this subset of transformations be called Q_p, while the full supercharge we will call Q. We find that

$$\mathcal{O}_1 = \frac{i}{4}\,\mathrm{tr}\left(\left\{Q_p, \left[\bar{Q}, \xi_i A_i\right]\right\}\right) \qquad \mathcal{O}_2 = \frac{i}{4}\,\mathrm{tr}\left(\left\{Q_p, \left[\bar{Q}, \xi_i\xi_j D_{(i}A_{j)}\right]\right\}\right) \tag{26}$$

where the trace is over Dirac indices. This allows us to write the following identity

$$\begin{aligned}
0 &= \frac{i}{4}\,\mathrm{tr}\left\langle\left\{Q_p, \frac{1}{N}\,\mathrm{Tr}\int ds\,dt\;\mathcal{P}e^{\int E}\,\mathcal{O}_1(s)\,\mathcal{P}e^{\int E}\,[\bar{Q}, A_i](t)\,\mathcal{P}e^{\int E}\right\}\right\rangle \\
&= \left\langle\frac{1}{N}\,\mathrm{Tr}\int ds\,dt\;\mathcal{P}e^{\int E}\mathcal{O}_1(s)\mathcal{P}e^{\int E}\mathcal{O}_1(t)\mathcal{P}e^{\int E}\right\rangle \\
&- \left\langle\frac{1}{16}\,\mathrm{tr}\,\frac{1}{N}\,\mathrm{Tr}\int ds\,dt\;\mathcal{P}e^{\int E}\,\xi_i(s)\bar{\psi}(x(s))\gamma^i\,\mathcal{P}e^{\int E}\,\dot{\xi}_j(t)(i\gamma^0 + \Gamma\cdot\theta)\gamma^j\psi(x(t))\,\mathcal{P}e^{\int E}\right\rangle
\end{aligned} \tag{27}$$

and so we have found a fermionic representation of the first term in (20). This affords a considerable simplification of the Feynman diagrams involved in calculating the expectation values because the fermions have less couplings than the gauge fields and

scalars. For example, in the second appendix, we give the computation of the leading term in the wavy line, which is now just the free field limit of (27).

Applying the same argument to the operator $\mathcal{O}_2$

$$\begin{aligned} 0 &= \frac{i}{4}\,\mathrm{tr}\left\langle\left\{Q_p, \frac{1}{N}\,\mathrm{Tr}\int ds\,\mathcal{P}e^{\int_s^\infty ds_1 E(s_1)}\,\xi_i(s)\,\xi_j(s)[\bar{Q}, D_{(i}A_{j)}](s)\,\mathcal{P}e^{\int_{-\infty}^s ds_2 E(s_2)}\right\}\right\rangle \\ &= \left\langle\frac{1}{N}\,\mathrm{Tr}\int ds\,\mathcal{P}e^{\int_s^\infty ds_1 E(s_1)}\,\mathcal{O}_2(s)\,\mathcal{P}e^{\int_{-\infty}^s ds_2 E(s_2)}\right\rangle \end{aligned} \tag{28}$$

we find that $\mathcal{O}_2$ does not contribute at any order in perturbation theory. As is explained in [47], the loop operator $\hat{L}$ acting on the Wilson loop W is defined as the coefficient of $\delta(s-s')$ in the expression for

$$\frac{\delta^2 W}{\delta x_\mu(s)\delta x_\mu(s')} \tag{29}$$

According to (20), we have

$$\hat{L}\,W = \left\langle\frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}e^{\int E}\,[\,iD_jF_{j0}(x(s)) + D_jD_j\Phi(x(s))\cdot\theta\,]\,\mathcal{P}e^{\int E}\right\rangle \;=\; 0 \tag{30}$$

The infinite straight line is a solution of the loop equation. Note that in this case it is not a simple consequence of the equations of motion, as the potential terms for scalars and the fermionic currents are absent. It is, on the other hand, a result of the 1/2 BPS nature of the Wilson loop. It would be interesting to see whether other partially supersymmetric loops [27] also obey the loop equation.

Acknowledgments

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A Conventions

The action of $\mathcal{N}=4$ supersymmetric Yang-Mills theory that we use for our perturbative computations is

$$S = \int d^{2\omega}x\frac{1}{2g^2}\left\{\frac{1}{2}(F^a_{\mu\nu})^2 + (\partial_\mu\Phi^a + f^{abc}A^b_\mu\Phi^c)^2 + \bar{\psi}^a i\gamma^\mu\partial_\mu\psi^a +\right.$$

$$+\sum_{I<J=1}^{10-2\omega}\left(f^{abc}\Phi^{bI}\Phi^{Jc}\right)^2+f^{abc}\bar{\psi}^a i\left(\gamma^\mu A^b_\mu+\Gamma^I\Phi^{bI}\right)\psi^c\Bigg\}$$

where

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+f^{abc}A^b_\mu A^c_\nu$$

All variables are $N\times N$ Hermitian matrices which can be expanded in SU(N) Lie algebra generators,

$$A_\mu(x)=\sum_{a=1}^{N^2-1}A^a_\mu T^a$$

The generators are normalized so that

$$\mathrm{Tr}\left(T^aT^b\right)=\frac{1}{2}\delta^{ab}$$

We use regularization by dimensional reduction where use 10-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory dimensionally reduced to 2ω spacetime dimensions. There are 2ω components of the vector field A_μ and $10-2\omega$ scalar fields Φ^I. The fermions always have 16 real components and the Dirac matrices are appropriate to a Majorana-Weyl spinor in 10-dimensions.

All of our computations are done in the Feynman gauge where the free field correlation functions are

$$\left\langle A^a_\mu(x)A^b_\nu(y)\right\rangle_0=g^2\Delta(x-y)\delta_{\mu\nu}\delta^{ab}\quad,\quad\left\langle\Phi^{Ia}(x)\Phi^{Jb}(y)\right\rangle_0=g^2\Delta(x-y)\delta^{IJ}\delta^{ab}\quad,$$

$$\left\langle\psi^a(x)\bar{\psi}^b(y)\right\rangle_0=g^2 i\gamma^\mu\partial_\mu\Delta(x-y)\delta^{ab}$$

where

$$\Delta(x)=\frac{\Gamma(\omega-1)}{4\pi^\omega}\frac{1}{[(x-y)^2]^{\omega-1}}$$

We use dimensional reduction of $\mathcal{N}=1$ supersymmetric Yang-Mills theory in 10-dimensions to 2ω-dimensions. The physical dimension is $2\omega=4$. Note the factors of the coupling constant, which come from our normalization of the action.

B Example

Consider the wavy line written in the form

$$\begin{aligned}&\langle W(-)\rangle=1+\\+\;&\frac{1}{32N}\int dsdt\left\langle\mathrm{Tr}\,\mathcal{P}e^{\int E}\bar{\psi}(x(s))\gamma\cdot\xi(s)e^{\int E}(i\gamma^0+\Gamma\cdot\theta)\gamma\cdot\dot{\xi}(t)\psi(x(t))e^{\int E}\right\rangle\\+\;&\ldots\end{aligned}\tag{31}$$

where $x(s) = (s, 0, 0, 0)$. In the leading order, we insert the free field fermion propagator:

$$\langle W(-)\rangle = 1 + \frac{g^2 N}{256\pi^2}\int dsdt\xi_i(s)\dot{\xi}_j(t)\,\mathrm{Tr}\left[\gamma^i\gamma^j(i\gamma^0 + \Gamma\cdot\theta)(-i\gamma^0)\right]\frac{d}{ds}\frac{1}{(s-t)^2} + \dots \tag{32}$$

which can be written as

$$\begin{aligned}\langle W(-)\rangle &= 1 + \frac{g^2 N}{16\pi^2}\int dsdt\xi(s)\cdot\dot{\xi}(t)\frac{d}{ds}\frac{1}{(t-s)^2} + \dots \\ &= 1 + \frac{g^2 N}{16\pi^2}\int dsdt\frac{[\dot{\xi}(s) - \dot{\xi}(t)]^2}{2(s-t)^2} + \dots\end{aligned} \tag{33}$$

which is identical to our previous result.

References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[4] A. M. Polyakov, "The wall of the cave," Int. J. Mod. Phys. A **14**, 645 (1999) [arXiv:hep-th/9809057].

[5] J. L. Petersen, "Introduction to the Maldacena conjecture on AdS/CFT," Int. J. Mod. Phys. A **14**, 3597 (1999) [arXiv:hep-th/9902131].

[6] P. Di Vecchia, "Large N gauge theories and ADS/CFT correspondence," arXiv:hep-th/9908148.

[7] E. T. Akhmedov, "Introduction to the AdS/CFT correspondence," arXiv:hep-th/9911095.

[8] I. R. Klebanov, "TASI lectures: Introduction to the AdS/CFT correspondence," arXiv:hep-th/0009139.

[9] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[10] J. M. Maldacena, "Wilson loops in large N field theories," Phys. Rev. Lett. **80**, 4859 (1998) [arXiv:hep-th/9803002].

[11] S. J. Rey and J. T. Yee, "Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity," Eur. Phys. J. C **22**, 379 (2001) [arXiv:hep-th/9803001].

[12] S. J. Rey, S. Theisen and J. T. Yee, "Wilson-Polyakov loop at finite temperature in large N gauge theory and anti-de Sitter supergravity," Nucl. Phys. B **527**, 171 (1998) [arXiv:hep-th/9803135].

[13] U. H. Danielsson and A. P. Polychronakos, "Quarks, monopoles and dyons at large N," Phys. Lett. B **434**, 294 (1998) [arXiv:hep-th/9804141].

[14] D. J. Gross and H. Ooguri, "Aspects of large N gauge theory dynamics as seen by string theory," Phys. Rev. D **58**, 106002 (1998) [arXiv:hep-th/9805129].

[15] N. Drukker, D. J. Gross and H. Ooguri, "Wilson loops and minimal surfaces," Phys. Rev. D **60**, 125006 (1999) [arXiv:hep-th/9904191].

[16] J. Greensite and P. Olesen, "Remarks on the heavy quark potential in the supergravity approach," JHEP **9808**, 009 (1998) [arXiv:hep-th/9806235].

[17] E. Witten, "Branes And The Dynamics Of QCD," Nucl. Phys. Proc. Suppl. **68**, 216 (1998).

[18] O. Aharony and E. Witten, "Anti-de Sitter space and the center of the gauge group," JHEP **9811**, 018 (1998) [arXiv:hep-th/9807205].

[19] Y. Kinar, E. Schreiber and J. Sonnenschein, "Q anti-Q potential from strings in curved spacetime: Classical results," Nucl. Phys. B **566**, 103 (2000) [arXiv:hep-th/9811192].

[20] U. H. Danielsson, E. Keski-Vakkuri and M. Kruczenski, "Vacua, propagators, and holographic probes in AdS/CFT," JHEP **9901** (1999) 002 [arXiv:hep-th/9812007].

[21] K. Zarembo, "String breaking from ladder diagrams in super-Yang-Mills theory theory," JHEP **0103**, 042 (2001) [arXiv:hep-th/0103058].

[22] J. Polchinski and M. J. Strassler, "The string dual of a confining four-dimensional gauge theory," arXiv:hep-th/0003136.

[23] R. A. Janik, "String fluctuations, AdS/CFT and the soft pomeron intercept," Phys. Lett. B **500**, 118 (2001) [arXiv:hep-th/0010069].

[24] J. Polchinski and M. J. Strassler, "Hard scattering and gauge/string duality," Phys. Rev. Lett. **88**, 031601 (2002) [arXiv:hep-th/0109174].

[25] R. A. Janik and R. Peschanski, "Reggeon exchange from AdS/CFT," Nucl. Phys. B **625**, 279 (2002) [arXiv:hep-th/0110024].

[26] G. W. Semenoff and K. Zarembo, "Wilson loops in super-Yang-Mills theory theory: From weak to strong coupling," Nucl. Phys. Proc. Suppl. **108**, 106 (2002) [arXiv:hep-th/0202156].

[27] K. Zarembo, "Supersymmetric Wilson loops," Nucl. Phys. B **643**, 157 (2002) [arXiv:hep-th/0205160].

[28] J. Polchinski and M. J. Strassler, "Deep inelastic scattering and gauge/string duality," JHEP **0305**, 012 (2003) [arXiv:hep-th/0209211].

[29] M. Kruczenski, "A note on twist two operators in N = 4 SYM and Wilson loops in Minkowski signature," JHEP **0212**, 024 (2002) [arXiv:hep-th/0210115].

[30] Y. Makeenko, "Light-cone Wilson loops and the string / gauge correspondence," JHEP **0301**, 007 (2003) [arXiv:hep-th/0210256].

[31] A. Mikhailov, "Special contact Wilson loops," arXiv:hep-th/0211229.

[32] A. V. Kotikov, L. N. Lipatov and V. N. Velizhanin, "Anomalous dimensions of Wilson operators in N = 4 super-Yang-Mills theory theory," Phys. Lett. B **557**, 114 (2003) [arXiv:hep-ph/0301021].

[33] A. V. Belitsky, A. S. Gorsky and G. P. Korchemsky, "Gauge / string duality for QCD conformal operators," Nucl. Phys. B **667**, 3 (2003) [arXiv:hep-th/0304028].

[34] A. Mikhailov, "Nonlinear waves in AdS/CFT correspondence," arXiv:hep-th/0305196.

[35] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, "Spinning strings in AdS(5) x S**5 and integrable systems," Nucl. Phys. B **671**, 3 (2003) [arXiv:hep-th/0307191].

[36] A. Gorsky, "Spin chains and gauge / string duality," arXiv:hep-th/0308182.

[37] A. V. Belitsky, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, "Superconformal operators in N = 4 super-Yang-Mills theory," arXiv:hep-th/0311104.

[38] M. Kruczenski, "Wilson loops and anomalous dimensions in cascading theories," Phys. Rev. D **69**, 106002 (2004) [arXiv:hep-th/0310030].

[39] V. Pestun and K. Zarembo, "Comparing strings in AdS(5) x S**5 to planar diagrams: An example," Phys. Rev. D **67**, 086007 (2003) [arXiv:hep-th/0212296].

[40] M. Bianchi, M. B. Green and S. Kovacs, "Instanton corrections to circular Wilson loops in N = 4 supersymmetric Yang-Mills," JHEP **0204**, 040 (2002) [arXiv:hep-th/0202003].

[41] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, "Three-loop universal anomalous dimension of the Wilson operators in N = 4 arXiv:hep-th/0404092.

[42] J. K. Erickson, G. W. Semenoff and K. Zarembo, "Wilson loops in N = 4 supersymmetric Yang-Mills theory," Nucl. Phys. B **582**, 155 (2000) [arXiv:hep-th/0003055].

[43] J. Plefka and M. Staudacher, "Two loops to two loops in N = 4 supersymmetric Yang-Mills theory," JHEP **0109**, 031 (2001) [arXiv:hep-th/0108182].

[44] G. Arutyunov, J. Plefka and M. Staudacher, "Limiting geometries of two circular Maldacena-Wilson loop operators," JHEP **0112**, 014 (2001) [arXiv:hep-th/0111290].

[45] N. Drukker and D. J. Gross, "An exact prediction of N = 4 supersymmetryM theory for string theory," J. Math. Phys. **42**, 2896 (2001) [arXiv:hep-th/0010274].

[46] G. W. Semenoff and K. Zarembo, "More exact predictions of SUSYM for string theory," Nucl. Phys. B **616**, 34 (2001) [arXiv:hep-th/0106015].

[47] A. M. Polyakov and V. S. Rychkov, "Gauge fields - strings duality and the loop equation," Nucl. Phys. B **581**, 116 (2000) [arXiv:hep-th/0002106].

[48] V. Kazakov, private communication.

[49] A. M. Polyakov and V. S. Rychkov, "Loop dynamics and AdS/CFT correspondence," Nucl. Phys. B **594**, 272 (2001) [arXiv:hep-th/0005173].

[50] Y. M. Makeenko and A. A. Migdal, "Exact Equation For The Loop Average In Multicolor QCD," Phys. Lett. B **88**, 135 (1979) [Erratum-ibid. B **89**, 437 (1980)].

[51] Y. Makeenko and A. A. Migdal, "Quantum Chromodynamics As Dynamics Of Loops," Nucl. Phys. B **188**, 269 (1981) [Sov. J. Nucl. Phys. **32**, 431.1980 YAFIA,32,838 (1980 YAFIA,32,838-854.1980)].

[52] D. Berenstein, R. Corrado, W. Fischler and J. M. Maldacena, "The operator product expansion for Wilson loops and surfaces in the large N Phys. Rev. D **59**, 105023 (1999) [arXiv:hep-th/9809188].