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Divergence Cancellation and Loop Corrections in String Field Theory on a Plane Wave Background

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Abstract: We investigate the one-loop energy shift δE to certain two-impurity string states in light-cone string field theory on a plane wave background. We find that there exist logarithmic divergences in the sums over intermediate mode numbers which cancel between the cubic Hamiltonian and quartic "contact term". Analyzing the impurity non-conserving channel we find that the non-perturbative $\mathcal{O}(g_2^2\sqrt{\lambda'})$ contribution to $\delta E/\mu$ predicted in [33] is in fact an artifact of these logarithmic divergences and vanishes with them, leaving an $\mathcal{O}(g_2^2\lambda')$ contribution. Exploiting the supersymmetry algebra, we present a form for the energy shift which appears to be manifestly convergent and free of non-perturbative terms. We use this form to argue that $\delta E/\mu$ receives $\mathcal{O}(g_2^2\lambda')$ contributions at every order in intermediate state impurities.

Keywords: String Field Theory, AdS/CFT Correspondence, pp-wave background.

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Contents

1. Preamble

The AdS/CFT correspondence asserts an exact duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and IIB superstring theory on $AdS_5 \times S^5$ background [1, 2, 3]. Attempts to test and exploit this strong coupling – weak coupling duality have led to a number of interesting insights about both string theory and gauge theory.

The Penrose limit of $AdS_5 \times S^5$ which produces a pp-wave geometry [4, 5] and the corresponding BMN limit of super-Yang-Mills theory [6] provide one of the most important tests of the AdS/CFT correspondence. Non-interacting type IIB string theory can be solved on the pp-wave background using the light-cone gauge [7]. Operators of Yang-Mills theory which correspond to the free string states can be identified. The light-cone momenta of string theory are

$$p^- = \mu\,(\Delta - J) \tag{1.1}$$

$$p^+ = \frac{\Delta + J}{2\mu\sqrt{g_{YM}^2 N}\alpha'} \tag{1.2}$$

where Δ is the dilatation operator, J is a U(1) R-charge and μ is a parameter of the pp-wave metric. The limits $N, \Delta, J \to \infty$, must be taken in such a way that (p^+, p^-) remain finite. Eq. (1.1) relates eigenvalues of the light-cone Hamiltonian to the eigenvalues of the dilatation operator of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. Eq. (1.2), together with the relation between coupling constants, $g_{YM}^2 = 4\pi g_s$, yield two effective couplings

$$\frac{1}{(\mu\alpha' p^+)^2} = \frac{g_{YM}^2 N}{J^2} \equiv \lambda' \quad , \quad 4\pi g_s \left(\mu\alpha' p^+\right)^2 = \frac{J^2}{N} \equiv g_2 \quad , \quad N, J \to \infty \tag{1.3}$$

Here, λ' is related to the string tension, and g_2 weights the genus of the string worldsheet. If g_2 is put to zero, strings still propagate on the pp-wave background, but are non-interacting. This free string limit coincides with the planar limit, or large N 't Hooft limit of Yang-Mills theory. This is consistent with the fact that the remaining parameter λ' depends on the g_{YM} and N only through the 't Hooft coupling $g_{YM}^2 N$.

There is a beautiful agreement between the spectra of free strings propagating on the plane-wave background and the eigenvalues of the dilatation operator in planar super-Yang-Mills theory in the BMN limit [6, 8, 9]. This agreement has been extended to scenarios beyond the BMN limit [10, 11, 12, 13, 14] and to the non-perturbative sector [15, 16]. It has thus led to many promising insights.

Non-planar corrections in Yang-Mills theory should correspond to string loop corrections in string theory. In Yang-Mills theory in the BMN limit, these were studied early on [17, 18, 19, 20, 21] and predictions of string loop corrections to the spectra of some string states were computed. For example, in a double expansion in λ' and g_2, the spectrum of a particular two impurity (or two oscillator) state of the string is [22, 23]

$$\Delta - J = 2 + n^2\lambda' - \frac{1}{4}n^4\lambda'^2 + \frac{1}{8}n^6\lambda'^3 ... + \frac{g_2^2}{4\pi^2}\left(\frac{1}{12} + \frac{35}{32n^2\pi^2}\right)\left(\lambda' - \frac{1}{2}\lambda'^2 n^2\right) + \dots \tag{1.4}$$

There have been many attempts to reproduce these corrections within string theory [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. This would constitute a highly nontrivial check of the AdS/CFT correspondence at the level of interacting strings. Despite much optimism, the present status of this work is that the order $g_2^2\lambda'$ and order $g_2^2\lambda'^2$ terms in the predicted spectrum (1.4) have not yet been computed using string theory techniques alone.

Computations of string theory interactions on pp-wave backgrounds necessarily involve light-cone string field theory. Due to complications with the Ramond-Ramond background field, a conformal field theory approach is not available. Light-cone string field theory begins with constructing the light-cone Hamiltonian and the supercharges of the residual supersymmetry of the light-cone frame. The quadratic, "free" part of the Hamiltonian and supercharges are straightforward to obtain. They are simply a summary of the known spectrum of free strings.

It is necessary to find the interaction parts of the Hamiltonian and similarly the non-quadratic parts of the supercharges. The guide to finding these is that they must respect the symmetries of the background at the quantum level. For the closed bosonic string, this is achieved by a local three-string vertex. In that case, there is a further check of the correctness of the ansatz in that it is known that the integration over string interactions maps onto conformal field theory integrals over the moduli of punctured Riemann surfaces with the appropriate vertex operators inserted at the punctures [40].

It has also been successful in superstring theory on the background of Minkowski space where the supersymmetry algebra, together with locality are sufficient to fix the interaction Hamiltonian. In that case, the three-string vertex has complicated pre-factors and four- and higher-point contact interactions appear in the Hamiltonian. One important role of the contact interactions is to cancel divergences which occur at the boundaries of the integration over the parameters of string diagrams which are constructed using the cubic vertex. Another intuitive reason for why they should be there is the expectation that the vacuum state of supersymmetric string field theory is stable.

The problem of the pp-wave background is that, though there is a beautiful and unambiguous free string theory available in the light-cone gauge, all of the details of the interaction Hamiltonian are not fixed by the supersymmetry algebra of the background. An unknown pre-factor of the three-string vertex and the contact interactions remains undetermined.

In the current state of the art, computations make use of an un-justified truncation of the string spectrum to the impurity preserving channel, which for a "two-impurity" external state amounts in keeping intermediate states with only two impurities. It has been observed that some multi-impurity states actually have contributions which are of lower than the leading order in the coupling constant, like $g_2^2\sqrt{\lambda'}$. When the coupling λ' is small, this contribution is larger than the expected perturbative shift, which should be of order $g_2^2\lambda'$. It is argued that, since it is not analytic in the coupling, it corresponds to a non-perturbative correction to Yang-Mills theory.

In this Paper, we make two bits of progress toward matching string field theory and Yang-Mills theory in the BMN limit. First of all, we observe that, in string field theory, individual perturbative contributions to some processes have logarithmic divergences when summed over intermediate states. These quantities appear to be finite on the Yang-Mills theory side. Then, we note that, when all contributions are assembled, these divergences cancel, leaving a finite result. We present an algebraic proof of cancellation of divergences for some two-impurity states. In fact for the spectrum of these states, the leading order contribution of order $g_2^2\sqrt{\lambda'}$ cancels along with the logarithmic divergences, leaving the natural leading order of λ'. Unfortunately, at this point we cannot go beyond this observation. The terms of order λ' seem to obtain contribution from intermediate states with any number of impurities, making their precise computation a formidable task.

It is worth noting that in the DLCQ of type-IIB superstring on the plane-wave background, for which there exists a dual gauge theory [41], and the mass shift corrections to two impurity operators have been computed [42], it was shown that energy shifts and corrections appear in the combination $(4\pi g_s\mu\alpha' p^+)^2 = g_2^2\lambda'$ [43].

We use the notation and conventions of ref. [38] and reproduce them in appendices A through D.

2. Invitation: impurity conserving channel

The shift in energy of a string state is computed using quantum mechanical perturbation theory. The Hamiltonian has a known quadratic part H_2 and interaction terms $g_2 H_3$, $g_2^2 H_4$, etc. $g_2^2 H_4$ and higher order are "contact terms". The cubic interaction H_3 has non-zero matrix elements only between states with n strings and $n \pm 1$ strings. For this reason, the linear order in perturbative correction to a single string state vanishes,

$$\delta E_n^{(1)} = \langle \phi_n^A | H_3 | \phi_n^B \rangle = 0$$

where $|\phi_n^A\rangle$ and $|\phi_n^B\rangle$ are one-string states and

$$H_2 |\phi_n^A\rangle = E_n^{(0)} |\phi_n^A\rangle$$

If the states are degenerate, an ortho-normal basis is labeled by A and B.

The leading non-vanishing correction is of second order,

$$\delta E_n^{(2)AB} = g_2^2 \left(\langle \phi_n^A | H_3 \frac{\mathcal{P}}{E_n^{(0)} - H_2} H_3 | \phi_n^B \rangle + \langle \phi_n^A | H_4 | \phi_n^B \rangle \right) \tag{2.1}$$

where $\mathcal{P}$ is a projection operator onto the orthogonal complement of the states spanned by $|\phi_n^A\rangle$. It will also be used to enforce level matching of intermediate states.

In previous literature, it is common to consider an approximation to (2.1) which restricts to two-oscillator intermediate states. This is done by replacing $\mathcal{P}$ in the first term with a projection onto level-matched two oscillator, two-string states. In the second term, supersymmetry is used to factor H_4 into a product of super-charges and a similar projector is inserted (see equation (B.21)). This is the so-called "two-impurity channel", or "impurity conserving channel". It is known that, although they are very similar in form, the correction obtained in this channel does not match the predictions of Yang-Mills theory [38].

One source of discrepancy is that the expression (2.1) could obtain contributions from other than just two-impurity intermediate states. In fact, it was noted in ref. [33] that the contribution of the four impurity channel to the mass shift of the string state[1]

$$|[\mathbf{9}, \mathbf{1}]\rangle^{(ij)} = \frac{1}{\sqrt{2}} \left(\alpha_n^{\dagger i} \alpha_{-n}^{\dagger j} + \alpha_n^{\dagger j} \alpha_{-n}^{\dagger i} - \frac{1}{2} \delta^{ij} \alpha_n^{\dagger k} \alpha_{-n}^{\dagger k} \right) |\alpha\rangle$$

appeared to diverge if the large μ limit was taken prior to summing over intermediate mode numbers. The authors noted that summing the expressions at finite μ regularized the divergence but then resulted in a $\sqrt{\lambda'}$ contribution to $\delta E^{(2)}/\mu$. For small λ', such a contribution is more important than the leading contributions which arose from the two-impurity-channel, for example, where the leading terms were of order λ', and it is hard to see how it could ever arise in Yang-Mills theory.

We will show that, in fact, this $\sqrt{\lambda'}$ contribution comes from mode-number sums which are logarithmically divergent. The existence of logarithmic divergences is counter to the philosophy that string field theory loop corrections should be finite. Upon further investigation, we shall see that, in this four-impurity channel case, the logarithmic divergences actually cancel. Along with the logarithmic divergences, the contributions of order $\sqrt{\lambda'}$ also cancel, leaving what one expects, a leading contribution of order λ'.

[1] The normalization of this state is $1 + \frac{1}{2}\delta^{ij}$

The cancellation of logarithmic divergences is between contributions from the H_3 vertex and the contact term. This is in line with the known fact that the role of the contact term is to cancel divergences of this kind, which also arise in the conformal field theory computation of superstring amplitudes on Minkowski space. There, the contact terms cancel divergent surface terms which appear upon integration by parts in the integrals of correlators of vertex operators over the moduli of Riemann surfaces [44].

Indeed logarithmic divergences of precisely the same nature, and a similar cancellation mechanism, can already be seen in a much simpler case: a careful calculation of the two-impurity channel contribution to the mass shift of the normalized bosonic trace state

$$|[\mathbf{1},\mathbf{1}]\rangle = \frac{1}{2}\alpha_n^{i\dagger}\alpha_{-n}^{i\dagger}|\alpha\rangle \tag{2.2}$$

In [34], this calculation was performed by taking the large μ limit first, then summing over mode numbers. That procedure found a finite result. However, it is not quite legitimate. If μ is kept finite, there are logarithmically divergent summations which must be dealt with before the large μ limit is taken. In the following we will re-examine this question and observe that the logarithmically divergent pieces would make the mass shift infinite even when μ is finite. Happily, we shall find that they cancel when all terms are taken into account.

We calculate the following matrix element for the state $|[\mathbf{1},\mathbf{1}]\rangle$

$$\begin{aligned}\langle\alpha_3|\frac{1}{2}\alpha_n^i\alpha_{-n}^i\,\langle\widetilde{\alpha}_2|\langle\widetilde{\alpha}_1|\alpha_p^K\alpha_{-p}^L|H_3\rangle = -g_2\,\frac{r\,(1-r)}{8}&\left[8\left(\frac{\omega_n^{(3)}}{\alpha_3}+\frac{\omega_p^{(1)}}{\alpha_1}\right)\widetilde{N}^{3\,1}_{-n\,p}\widetilde{N}^{3\,1}_{n\,p}\,\delta^{kl}\right.\\ &\left.+16\,\frac{\omega_n^{(3)}}{\alpha_3}\widetilde{N}^{3\,3}_{n\,n}\widetilde{N}^{1\,1}_{p\,-p}\,\delta^{KL}+16\,\frac{\omega_p^{(1)}}{\alpha_1}\widetilde{N}^{3\,3}_{n\,-n}\widetilde{N}^{1\,1}_{p\,p}\,\Pi^{KL}\right]\end{aligned} \tag{2.3}$$

where the index $i=1,\ldots,4$ is summed over. Note that $K,L=1,\ldots,8$, while δ^{kl} is non-zero only for $k=l=1,\ldots,4$. The matrix Π^{KL} is given by

$$\Pi^{KL} = \mathrm{diag}(1,1,1,1,-1,-1,-1,-1)$$

When calculating the H_3 contribution to the mass shift it is only the very last term in (2.3) which is divergent. Singling-out its contribution, one finds[2],

$$\delta E_{H_3}^{\mathrm{div}} = \int_0^1 dr\,\left(g_2\frac{r(1-r)}{8}\right)^2\,\frac{-\alpha_3}{2\,r\,(1-r)}\sum_{KL}\sum_{p=-\infty}^{\infty}\frac{\left[16\,\frac{\omega_p}{-r\,\alpha_3}\widetilde{N}^{3\,3}_{n\,-n}\,\widetilde{N}^{1\,1}_{p\,p}\,\Pi^{KL}\right]^2}{2\,\omega_n-2\,r^{-1}\omega_p} \tag{2.4}$$

A quick inspection of the forms of the Neumann matrices (see Appendix D) reveals that the numerator in (2.4) goes like a constant for large $|p|$, and thus the sum as a whole

[2] We use the following intermediate state projector:

$$\mathbf{1}_B = \int_0^1 \frac{dr}{2\,r(1-r)}\sum_p \alpha_p^{\dagger\,K}\,\alpha_{-p}^{\dagger\,L}\,|\widetilde{\alpha}_1\rangle\,|\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\,\langle\widetilde{\alpha}_1|\,\alpha_{-p}^L\,\alpha_p^K$$

where $\widetilde{\alpha}_1 \equiv -\alpha_3\,r$ and $\widetilde{\alpha}_2 \equiv -\alpha_3(1-r)$. Note that oscillators act only on the vacuum closest to them.

goes like $1/|p|$ for $|p| \gg |\mu\alpha_3|$. This is a logarithmically diverging sum. In [34] the strict large μ limit was taken for the energy denominator, leading to a convergent $1/p^2$ behavior instead. Here we will stick with the finite μ expressions and show that the divergence is removed by the contact term. Note that a double fermionic impurity intermediate state also contributes to the H_3 piece, however it does not display any divergent behavior. Further, the $\alpha_0^\dagger|\alpha_1\rangle\alpha_0^\dagger|\alpha_2\rangle$ intermediate state is unimportant to us for the same reason.

The contribution from the contact term stems from the following matrix element,

$$\left(g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^{-1}\langle\alpha_3|\frac{1}{2}\alpha_n^i\alpha_{-n}^i\,\langle\widetilde{\alpha}_2|\langle\widetilde{\alpha}_1|\alpha_p^K\beta_{-p}^{\Sigma_1\,\Sigma_2}|Q_{3\,\beta_1\dot{\beta}_2}\rangle$$
$$=\left(G_{|p|}^{(1)}\,K_n^{(3)}\widetilde{N}_{n\,p}^{3\,1}+G_{|p|}^{(1)}\,K_{-n}^{(3)}\widetilde{N}_{-n\,p}^{3\,1}\right)(\sigma^k)_{\beta_1}^{\sigma_1}\delta_{\dot{\beta}_2}^{\dot{\sigma}_2}+4\,G_{|p|}^{(1)}\,K_{-p}^{(1)}\widetilde{N}_{n\,-n}^{3\,3}(\sigma^K)_{\beta}^{\Sigma}\delta_{\beta}^{\Sigma} \quad (2.5)$$

along with a similar element with $|Q_{3\,\dot{\beta}_1\beta_2}\rangle$. Here $K = 1,\ldots,8$ while the Σ and β indices are either dotted or undotted as required by the particular SO(4) representation indicated by K. The last term in (2.5) gives rise to a log-divergent sum. For large positive p, $(K_{-p}^{(1)})^2$ goes as a constant, and so the sum is controlled by $(G_{|p|}^{(1)})^2$ which goes as $1/p$, and hence diverges logarithmically. For p negative, the sum converges. Thus, the divergent contribution to $\delta E^{(2)}$ is found to be[3],

$$\delta E_{H_4}^{\rm div} = 8\int_0^1 dr\,\left(g_2\frac{1}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^2\frac{1}{r(1-r)}\sum_{p=1}^{\infty}\left(4\,G_{|p|}^{(1)}\,K_{-p}^{(1)}\widetilde{N}_{n\,-n}^{3\,3}\right)^2 \quad (2.6)$$

The leading factor of 8 comes from the sum over K. Note that two factors of 2 from the delta function (in Pauli indices) and the (squared) Pauli matrix trace cancel the two factors of 1/8 coming from the two terms of the contact term, $Q_{3\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2}$ and $Q_{3\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2}$ (See equation (B.21)). Again the intermediate state $\alpha_0^\dagger|\alpha_1\rangle\beta_0^\dagger|\alpha_2\rangle$ is unimportant to convergence and is ignored here.

In taking the large p limits of the summands in (2.4) and (2.6), one finds,

$$\delta E_{H_3}^{\rm div} \sim -\frac{1}{2}\int_0^1 dr\,\frac{g_2^2\,r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}_{n\,-n}^{3\,3}\right)^2\,\frac{1}{|p|} \quad (2.7)$$

$$\delta E_{H_4}^{\rm div} \sim +\int_0^1 dr\,\frac{g_2^2\,r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}_{n\,-n}^{3\,3}\right)^2\,\frac{1}{p} \quad (2.8)$$

Noting that in the H_3 contribution the divergence is found for both positive and negative p, while in the H_4 contribution the divergence occurs only for positive p, and hence a relative factor of 2 is induced in the H_3 term, one sees that the logarithmically

[3]We use the following intermediate state projector:

$$\mathbf{1}_F = \int_0^1\frac{dr}{r(1-r)}\sum_p\alpha_p^{\dagger\,K}\,\beta_{-p}^{\dagger\,\Sigma_1\,\Sigma_2}\,|\widetilde{\alpha}_1\rangle\,|\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\,\langle\widetilde{\alpha}_1|\,\beta_{-p}^{\Sigma_1\,\Sigma_2}\,\alpha_p^K$$

divergent sums cancel identically between the H_3 and contact terms, leaving a convergent sum.

This cancellation fixes the relative weight of the H_3 and contact terms to that employed in [38]. It differs by a factor of 1/2 from the weight originally given in [33], where it was argued to be a reflection symmetry factor.

The singlet state $|[\mathbf{1},\mathbf{1}]\rangle$ is more generally constructed also in terms of fermionic oscillators, e.g. $(\beta_n^\dagger)_{\alpha_1\alpha_2}(\beta_n^\dagger)^{\alpha_1\alpha_2}$. These states start to mix with the bosonic trace state (2.2) at loops higher than one. Therefore the two impurity channel contribution to the mass shift of the state (2.2), can be computed, up to one loop, by assembling the finite parts of all the possible two impurity intermediate states, both in H_3 and in H_4, without bothering about state mixing. With the relative coefficient between H_3 and H_4 fixed by the requirement of the cancellation of divergences, the result reads

$$\frac{\delta E_n^{(2)}}{\mu} = \frac{g_2^2\lambda'}{4\pi^2}\left(\frac{1}{24}+\frac{65}{64n^2\pi^2}\right) \tag{2.9}$$

This is in agreement with the order λ' result of ref. [38] for the traceless symmetric state $\big|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ and in disagreement with what is expected from the gauge theory. In the next section we will discuss a possible explanation for this disagreement [4].

3. Four impurity channel

We now consider the mass shift of the $\big|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ string state due to intermediate states which contain four impurities.

In the explicit expression for the matrix element to be quoted below, we shall see that the parameter $\mu\alpha_3$ occurs only in combinations involving ω_p and there is a duality between the large p and the large $\mu\alpha_3$ limits. Therefore, since a logarithmic divergence in the sums indicates that the summands have as many (inverse) powers of the summation variables as there are summation variables, this translates into vanishing $\mu\alpha_3$ dependence for this contribution to $\delta E^{(2)}$, leaving $\delta E^{(2)}/\mu \sim \sqrt{\lambda'}$. It is thus seen that $\sqrt{\lambda'}$ behavior is simply the result of log divergences, which should, if pp-wave light-cone string field theory is to make any sense, cancel out entirely.

We begin with the H_3 contribution to the mass shift. We consider the following intermediate state,

$$\mathbf{1}_B = \int_0^1 \frac{dr}{4!\,r(1-r)} \sum_{p_1\,p_2\,p_3\,p_4} \alpha_{p_1}^{\dagger\,K}\,\alpha_{p_2}^{\dagger\,L}\,\alpha_{p_3}^{\dagger\,M}\,\alpha_{p_4}^{\dagger\,N}\,|\tilde\alpha_1\rangle\,|\tilde\alpha_2\rangle\langle\tilde\alpha_2|\,\langle\tilde\alpha_1|\,\alpha_{p_4}^{N}\,\alpha_{p_3}^{M}\,\alpha_{p_2}^{L}\,\alpha_{p_1}^{K} \tag{3.1}$$

where the sum over mode numbers is restricted by the level matching condition $\sum_i p_i = 0$ and $\tilde\alpha_1 \equiv -\alpha_3\, r$, $\tilde\alpha_2 \equiv -\alpha_3(1-r)$.

Although there are many possible contractions of this state with the oscillators in $|H_3\rangle$, we will only be concerned with those which lead to log divergent sums. These are the ones

[4] We note that in fact it is not hard to show that the trace (but not the $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$) state receives an order λ' contribution from the zero impurity channel.

where the $\alpha^\dagger$ in the prefactor of $|H_3\rangle$ contracts with one of the oscillators in $\mathbf{1}_B$. We find this contribution to $\delta E^{(2)}$ to be[5],

$$\delta E^{\rm div}_{H_3} = \int_0^1 \frac{dr}{4!\, r\,(1-r)} \left(g_2 \frac{r(1-r)}{4} \right)^2 \sum_{p_2\, p_3\, p_4} \frac{-\alpha_3\, r}{2\,\omega_n\, r - \sum_{i=1}^4 \omega_{p_i}} \times$$

$$\left(2\, \frac{\omega_{p_1} + \omega_{p_2}}{-r\,\alpha_3}\, \widetilde{N}^{1\,1}_{-p_1\, p_2} \right)^2 \left\{ 8 \cdot 12\, \left(\widetilde{N}^{3\,1}_{n\, p_3} \widetilde{N}^{3\,1}_{-n\, p_4} \right)^2 + 6\, \widetilde{N}^{3\,1}_{n\, p_3} \widetilde{N}^{3\,1}_{-n\, p_3} \widetilde{N}^{3\,1}_{n\, p_4} \widetilde{N}^{3\,1}_{-n\, p_4} \right\} \quad (3.2)$$

where $p_1 = -(p_2 + p_3 + p_4)$. The factors of 6 and 12 are combinatoric and count the number of ways equivalent contractions can be made. The factor of 8 comes from a sum over the spacetime indices of $\mathbf{1}_B$ and only affects squared terms. It is easy to see that in the above, the sum over p_2 is log divergent. In fact, it is the very same form as appears in (2.4). Using the techniques described in Appendix E, one sees that $\delta E^{\rm div}_{H_3} \sim$ constant, and therefore $\delta E^{(2)}/\mu \sim \sqrt{\lambda'}$. There are also contributions from intermediate states which contain two bosonic and two fermionic impurities, however these produce convergent sums and $\mathcal{O}(\lambda')$ contributions to $\delta E^{(2)}/\mu$.

We now show that the contact term contribution stemming from the following intermediate state,

$$\mathbf{1}_F = \int_0^1 \frac{dr}{3!\, r(1-r)} \sum_{p_1\, p_2\, p_3\, p_4} \beta^{\dagger\, a}_{p_1}\, \alpha^{\dagger\, L}_{p_2}\, \alpha^{\dagger\, M}_{p_3}\, \alpha^{\dagger\, N}_{p_4}\, |\widetilde{\alpha}_1\rangle\, |\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\, \langle\widetilde{\alpha}_1|\, \alpha^N_{p_4}\, \alpha^M_{p_3}\, \alpha^L_{p_2}\, \beta^a_{p_1} \quad (3.3)$$

cancels the divergent piece coming from the H_3 contribution, leaving an $\mathcal{O}(\lambda')$ contribution to $\delta E^{(2)}/\mu$. In the above a is an SO(8) index and thus represents both dotted and undotted indices in the language of [38]. The log divergent piece comes from contractions where the $\alpha^\dagger$ in the prefactor of $|Q_3\rangle$ is joined with one of the bosonic oscillators in $\mathbf{1}_F$. One finds,

$$\delta E^{\rm div}_{H_4} = \int_0^1 \frac{dr}{3!\, r\,(1-r)} \left(g_2 \frac{1}{4} \sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}} \right)^2 \sum_{p_2\, p_3\, p_4} (2\, G_{p_1} K_{-p_2})^2$$

$$\times \left\{ 8 \cdot 6\, \left(\widetilde{N}^{3\,1}_{n\, p_3} \widetilde{N}^{3\,1}_{-n\, p_4} \right)^2 + 3\, \widetilde{N}^{3\,1}_{n\, p_3} \widetilde{N}^{3\,1}_{-n\, p_3} \widetilde{N}^{3\,1}_{n\, p_4} \widetilde{N}^{3\,1}_{-n\, p_4} \right\} \quad (3.4)$$

In the above one sees the very same pattern as was seen in section 2. The sum over p_2 is divergent on the positive side, and cancels the divergence in (3.2). The remaining (convergent) expression gives an $\mathcal{O}(\lambda')$ contribution to $\delta E^{(2)}/\mu$. Again, there is a non-divergent contribution from the intermediate state with three fermionic and one bosonic impurity which is not considered here.

The cancellation exposed here is also found for the following remaining pairs of intermediate states,

$$\mathbf{1}_B = \int_0^1 \frac{dr}{3!\, r(1-r)} \sum_{p_1\, p_2\, p_3} \alpha^{\dagger\, K}_{p_1}\, \alpha^{\dagger\, L}_{p_2}\, \alpha^{\dagger\, M}_{p_3}\, |\widetilde{\alpha}_1\rangle\, \alpha^{\dagger\, N}_0\, |\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\, \alpha^N_0\, \langle\widetilde{\alpha}_1|\, \alpha^M_{p_3}\, \alpha^L_{p_2}\, \alpha^K_{p_1}$$

$$\mathbf{1}_F = \int_0^1 \frac{dr}{2!\, r(1-r)} \sum_{p_1\, p_2\, p_3} \beta^{\dagger\, a}_{p_1}\, \alpha^{\dagger\, L}_{p_2}\, \alpha^{\dagger\, M}_{p_3}\, |\widetilde{\alpha}_1\rangle\, \alpha^{\dagger\, N}_0\, |\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\, \alpha^N_0\, \langle\widetilde{\alpha}_1|\, \alpha^M_{p_3}\, \alpha^L_{p_2}\, \beta^a_{p_1} \quad (3.5)$$

[5]The normalization $1 + \frac{1}{2}\delta^{ij}$ of the external state has been suppressed here.

where $\sum_{i=1}^{3} p_i = 0$ and,

$$\mathbf{1}_B = \int_0^1 \frac{dr}{2\cdot(2!)^2\, r(1-r)} \sum_{p_1\,p_2} \alpha^{\dagger\,K}_{p_1}\,\alpha^{\dagger\,L}_{-p_1}\,|\widetilde{\alpha}_1\rangle\,\alpha^{\dagger\,M}_{p_2}\,\alpha^{\dagger\,N}_{-p_2}\,|\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\,\alpha^{N}_{-p_2}\,\alpha^{M}_{p_2}\,\langle\widetilde{\alpha}_1|\,\alpha^{L}_{-p_1}\,\alpha^{K}_{p_1}$$
$$\mathbf{1}_F = \int_0^1 \frac{dr}{2!\, r(1-r)} \sum_{p_1\,p_2} \alpha^{\dagger\,K}_{p_1}\,\alpha^{\dagger\,L}_{-p_1}\,|\widetilde{\alpha}_1\rangle\,\alpha^{\dagger\,M}_{p_2}\,\beta^{\dagger\,a}_{-p_2}\,|\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\,\beta^{a}_{-p_2}\,\alpha^{M}_{p_2}\,\langle\widetilde{\alpha}_1|\,\alpha^{L}_{-p_1}\,\alpha^{K}_{p_1} \quad (3.6)$$

and so we find that the entire contribution to $\delta E^{(2)}/\mu$ from the four impurity channel is convergent / leads as λ'. It is not hard to generalize the above argument to $\mathbf{1}_B$'s containing an arbitrary number of bosonic impurities and no fermionic impurities. The divergent expressions cancel against contact interactions with $\mathbf{1}_F$'s containing one fermionic and the same number (less-one) of bosonic oscillators as $\mathbf{1}_B$. Adding fermionic impurities is far less trivial because the full forms [36] of $|H_3\rangle$ and $|Q_3\rangle$, given in Appendix B, must be used for the calculation[6]. In the next section, however, a more elegant argument is presented which claims the absence of log divergences for arbitrary impurity intermediate states.

4. Generalizing to arbitrary impurities

It is possible to formally manipulate the contact term in such a way that the H_3 portion of the energy shift is canceled entirely, leaving a convergent expression, which appears devoid of any $\sqrt{\lambda'}$ contributions to $\delta E^{(2)}/\mu$. H_3 arises from the anti-commutators derived from the dynamical constraints up to order g_2 (cfr. Appendix B)

$$\begin{aligned}\left\{Q_{2\alpha_1\dot{\alpha}_2}, Q_{3\beta_1\dot{\beta}_2}\right\} + \left\{Q_{3\alpha_1\dot{\alpha}_2}, Q_{2\beta_1\dot{\beta}_2}\right\} &= -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}H_3\ ,\\ \left\{Q_{2\dot{\alpha}_1\alpha_2}, Q_{3\dot{\beta}_1\beta_2}\right\} + \left\{Q_{3\dot{\alpha}_1\alpha_2}, Q_{2\dot{\beta}_1\beta_2}\right\} &= -2\epsilon_{\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\alpha_2\beta_2}H_3\end{aligned} \quad (4.1)$$

Analogously to order g_2^2 one has

$$\left\{Q_{3\alpha_1\dot{\alpha}_2}, Q_{3\beta_1\dot{\beta}_2}\right\} + \left\{Q_{2\alpha_1\dot{\alpha}_2}, Q_{4\beta_1\dot{\beta}_2}\right\} + \left\{Q_{4\alpha_1\dot{\alpha}_2}, Q_{2\beta_1\dot{\beta}_2}\right\} = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}H_4 \quad (4.2)$$
$$\left\{Q_{3\dot{\alpha}_1\alpha_2}, Q_{3\dot{\beta}_1\beta_2}\right\} + \left\{Q_{2\dot{\alpha}_1\alpha_2}, Q_{4\dot{\beta}_1\beta_2}\right\} + \left\{Q_{4\dot{\alpha}_1\alpha_2}, Q_{2\dot{\beta}_1\beta_2}\right\} = -2\epsilon_{\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\alpha_2\beta_2}H_4 \quad (4.3)$$

To get H_3 and H_4 the first of the equations in both (4.1) and (4.3) should be multiplied by $\epsilon^{\alpha_1\beta_1}\epsilon^{\dot{\beta}_2\dot{\alpha}_2}$ and the second by $\epsilon^{\dot{\alpha}_1\dot{\beta}_1}\epsilon^{\beta_2\alpha_2}$. On the left hand sides of the equations the epsilons just raise indices, on the right hand sides they give -4. The anti-commutators in eq.(4.1) thus give

$$\left\{Q_{2\beta_1\dot{\beta}_2}, Q_3^{\beta_1\dot{\beta}_2}\right\} = +4H_3 \quad , \quad \left\{Q_{2\dot{\beta}_1\beta_2}, Q_3^{\dot{\beta}_1\beta_2}\right\} = +4H_3 \quad (4.4)$$

Those in eq.(4.3) give the contact Hamiltonian

$$\begin{aligned}H_4 = \frac{1}{8}Q_{3\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2} + \frac{1}{8}Q_{3\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2} + \frac{1}{8}Q_{4\beta_1\dot{\beta}_2}Q_2^{\beta_1\dot{\beta}_2} + \frac{1}{8}Q_{4\dot{\beta}_1\beta_2}Q_2^{\dot{\beta}_1\beta_2}\\ +\frac{1}{8}Q_{2\beta_1\dot{\beta}_2}Q_4^{\beta_1\dot{\beta}_2} + \frac{1}{8}Q_{2\dot{\beta}_1\beta_2}Q_4^{\dot{\beta}_1\beta_2}\end{aligned} \quad (4.5)$$

[6] We remind the reader that the $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ state receives no contributions to its energy shift from the zero impurity channel.

Using these formulas, the contribution of H_4 to $\delta E^{(2)}$ can be rewritten as a sum of a term which cancels the H_3 contribution plus other pieces which all contain Q_2 acting on one of the external states. Taking the expectation value of part of (4.5), and introducing a representation of unity, we have,

$$\frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2}+Q_{3\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2}\right\rangle = \frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}P\frac{E_0-H_2}{E_0-H_2}Q_3^{\beta_1\dot{\beta}_2}\right\rangle \tag{4.6}$$

$$+\frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}P\frac{E_0-H_2}{E_0-H_2}Q_3^{\dot{\beta}_1\beta_2}\right\rangle \tag{4.7}$$

It could be that the energy denominator which we have introduced here will have a zero. In that case, the projector P is a reminder to define the singularity using a principle value prescription [7].

Equation (4.6) can be written as

$$= -\frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}\left[H_2,Q_3^{\beta_1\dot{\beta}_2}\right]\right\rangle - \frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}\left[H_2,Q_3^{\dot{\beta}_1\beta_2}\right]\right\rangle \tag{4.10}$$

Up to order g_2 the following equation holds

$$\left[H_2,Q_3^{\beta_1\dot{\beta}_2}\right] = \left[Q_2^{\beta_1\dot{\beta}_2},H_3\right] \tag{4.11}$$

so that (4.10) becomes

$$= \frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}\left[H_3,Q_2^{\beta_1\dot{\beta}_2}\right]\right\rangle + \frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}\left[H_3,Q_2^{\dot{\beta}_1\beta_2}\right]\right\rangle \tag{4.12}$$

Since Q_2 commutes with H_2 one has

$$\begin{aligned}= &+\frac{1}{8}\left\langle Q_{2\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}H_3\right\rangle + \frac{1}{8}\left\langle Q_{2\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}H_3\right\rangle \\ &+\frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}H_3Q_2^{\beta_1\dot{\beta}_2}\right\rangle + \frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}H_3Q_2^{\dot{\beta}_1\beta_2}\right\rangle \\ &-\left\langle H_3\frac{P}{E_0-H_2}H_3\right\rangle\end{aligned} \tag{4.13}$$

[7] There is one additional subtlety, the intermediate states must each obey the level-matching condition. This condition can be enforced by inserting a projection operator. For example, for two-string intermediate states, we can combine such a projector with the energy denominator as

$$\frac{P}{E_0-H_2} = \int_0^\infty d\tau\ e^{E_0\tau}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ e^{-H_2^{(1)}\tau+i\theta_1N^{(1)}}\ e^{-H_2^{(2)}\tau+i\theta_2N^{(2)}} \tag{4.8}$$

where

$$N^{(r)} = \sum_n n\left(a_n^{I(r)\dagger}a_n^{I(r)}+b_{an}^{(r)\dagger}b_{an}^{(r)}\right) \tag{4.9}$$

with $r=1,2$ are the level number operators for the two intermediate strings. The net effect of the operators in the above equation is to make the replacement $\left(a_n^{(r)\dagger},b_n^{(r)\dagger}\right)\to\left(e^{-\omega_n\tau+in\theta_{(r)}}a_n^{(r)\dagger},e^{-\omega_n\tau+in\theta_{(r)}}b_n^{(r)\dagger}\right)$ for all creation operators which lie to the right of the projector. Then, after the matrix element is computed, we multiply it by $e^{E_0\tau}$ and integrate over τ and θ_r. Any potential divergences come from the region near $\tau=0$.

and the last term cancels the H_3 contribution to the energy shift. The final expression for the energy shift is

$$\begin{aligned}\delta E^{(2)} = &+ \frac{1}{8}\left\langle Q_{2\beta_1\dot\beta_2} Q_3^{\beta_1\dot\beta_2} \frac{P}{E_0 - H_2} H_3 \right\rangle + \frac{1}{8}\left\langle Q_{2\dot\beta_1\beta_2} Q_3^{\dot\beta_1\beta_2} \frac{P}{E_0 - H_2} H_3 \right\rangle \\ &+ \frac{1}{8}\left\langle Q_{3\beta_1\dot\beta_2} \frac{P}{E_0 - H_2} H_3 Q_2^{\beta_1\dot\beta_2} \right\rangle + \frac{1}{8}\left\langle Q_{3\dot\beta_1\beta_2} \frac{P}{E_0 - H_2} H_3 Q_2^{\dot\beta_1\beta_2} \right\rangle \\ &+ \frac{1}{4}\left\langle Q_{2\beta_1\dot\beta_2} Q_4^{\beta_1\dot\beta_2} \right\rangle + \frac{1}{4}\left\langle Q_{2\dot\beta_1\beta_2} Q_4^{\dot\beta_1\beta_2} \right\rangle \\ &+ \frac{1}{4}\left\langle Q_{4\beta_1\dot\beta_2} Q_2^{\beta_1\dot\beta_2} \right\rangle + \frac{1}{4}\left\langle Q_{4\dot\beta_1\beta_2} Q_2^{\dot\beta_1\beta_2} \right\rangle \end{aligned} \tag{4.14}$$

It is amusing to note that the vanishing energy correction for a supersymmetric external state is manifest in (4.14), since if Q_2 annihilates the external state, all of the terms are identically zero. Please note the discussion above equation (B.21), where it is explained that for calculations Q_4 is set to zero.

Using the $\big|[\mathbf{9},\mathbf{1}]\big\rangle^{(ij)}$ external state, we can check that what is left is manifestly convergent for the four impurity channel, and then show that the addition of impurities will not disturb this, leaving $\mathcal{O}(\lambda')$ contributions at every order in impurities. We have two sorts of terms in (4.14), which we can represent schematically as follows,

$$\delta E_1 = \sum_I \frac{\Big(\langle\Phi|\langle I|Q_3\rangle\Big)\Big(\langle\Psi|\langle I|H_3\rangle\Big)^*}{E_\Phi - E_I} \qquad \delta E_2 = \sum_I \frac{\Big(\langle\Phi|\langle I|H_3\rangle\Big)\Big(\langle\Psi|\langle I|Q_3\rangle\Big)^*}{E_\Phi - E_I} \tag{4.15}$$

where $|\Phi\rangle$ is the $\big|[\mathbf{9},\mathbf{1}]\big\rangle^{(ij)}$ external state, $|\Psi\rangle = Q_2|\Phi\rangle$, and $|I\rangle$ is a level-matched, two-string intermediate state. In order to evaluate the convergence and large μ behavior of these terms, we can be entirely schematic. We take,

$$|\Psi\rangle \sim \sqrt{-\mu\alpha_3}\,\beta_n^\dagger\,\alpha_{-n}^\dagger|\widetilde{\alpha}_3\rangle \qquad |\Phi\rangle \sim \alpha_n^\dagger\,\alpha_{-n}^\dagger|\widetilde{\alpha}_3\rangle \tag{4.16}$$

while for the purpose of evaluating convergence we can take

$$G_p^{(1)} \sim \frac{1}{\sqrt{p}} \qquad K_{-p}^{(1)} \sim \text{constant} \qquad \widetilde{N}^{3\,r}_{n\,p} \sim \frac{1}{p} \qquad \widetilde{N}^{r\,s}_{q\,p} \sim \frac{1}{p+q} \tag{4.17}$$

where we take all integers to be positive. Let us begin with the first type of term in (4.15), we have two choices for four impurity intermediate states,

$$\begin{aligned}|I\rangle &\sim \alpha_{p_1}^\dagger \beta_{p_2}^\dagger \alpha_{p_3}^\dagger \alpha_{p_4}^\dagger |\widetilde{\alpha}_1\rangle|\widetilde{\alpha}_2\rangle \\ |I\rangle &\sim \alpha_{p_1}^\dagger \beta_{p_2}^\dagger \beta_{p_3}^\dagger \beta_{p_4}^\dagger |\widetilde{\alpha}_1\rangle|\widetilde{\alpha}_2\rangle\end{aligned} \tag{4.18}$$

We can proceed with the first one, which will give,

$$\begin{aligned}&\delta E_1 \sim \sqrt{x} \sum_{p_1\,p_2\,p_3\,p_4} \frac{1}{2\,r\,\omega_n - \sum_{i=1}^4 \omega_{p_i}} \cdot \\ &\langle\widetilde{\alpha}_3|\alpha_n\,\alpha_{-n}\langle\widetilde{\alpha}_2|\langle\widetilde{\alpha}_1|\alpha_{p_1}\,\beta_{p_2}\,\alpha_{p_3}\,\alpha_{p_4}\,|Q_3\rangle\Big(\langle\widetilde{\alpha}_3|\beta_n\,\alpha_{-n}\langle\widetilde{\alpha}_2|\langle\widetilde{\alpha}_1|\alpha_{p_1}\,\beta_{p_2}\,\alpha_{p_3}\,\alpha_{p_4}\,|H_3\rangle\Big)^*\end{aligned} \tag{4.19}$$

where $x = -\mu\alpha_3$ and $\sum_i p_i = 0$. Before continuing with contractions we should note that because of the appearance of multiple fermionic oscillators we should be using the forms of $|H_3\rangle$ and $|Q_3\rangle$ given in Appendix B. We refer the reader there for these expressions. There are two general ways in which we can contract the $\beta^{(r)}$'s. They can connect to factors of $\sum_m G_m \beta^\dagger_m$ in the prefactors of $|H_3\rangle$ and $|Q_3\rangle$, or they can pair-up to bring down a factor of $\widetilde{Q}^{r\,s}_{m\,p} - \widetilde{Q}^{r\,s}_{p\,m}$ from $|E_\beta\rangle$ (see Appendix C). As far as convergence and large x power-counting is concerned however, $G^{(r)}_m G^{(s)}_p$ is equivalent to $\widetilde{Q}^{r\,s}_{m\,p} - \widetilde{Q}^{s\,r}_{p\,m}$, and so we will simply use the former. When contracting $\beta^{(3)}$'s there is a fundamental difference between $G^{(3)}_n G^{(r)}_p$ and $\widetilde{Q}^{3\,r}_{n\,p} - \widetilde{Q}^{r\,3}_{p\,n}$, as far as large x behavior is concerned, because of the pole in the latter. In fact $\widetilde{Q}^{3\,r}_{n\,p} - \widetilde{Q}^{r\,3}_{p\,n}$ is essentially equivalent to $\widetilde{N}^{3\,r}_{n\,p}$ and therefore the two can be interchanged in this analysis.

Because K_{-p} goes as a constant for large p, the worst convergence will always be realized by contracting the intermediate bosonic impurities with the prefactors of $|H_3\rangle$ and $|Q_3\rangle$. These contractions will yield[8],

$$\delta E_1 \sim \sqrt{x} \sum_{p_1\,p_2\,p_3\,p_4} \frac{G^{(1)}_{p_2} \widetilde{N}^{3\,1}_{-n\,p_1} K^{(1)}_{-p_3} \widetilde{N}^{3\,1}_{n\,p_4} \times K^{(1)}_{-p_3} K^{(1)}_{p_4} \widetilde{N}^{3\,1}_{-n\,p_1} \begin{cases} \widetilde{Q}^{3\,1}_{n\,p_2} - \widetilde{Q}^{1\,3}_{p_2\,n} \\ G^{(3)}_n G^{(1)}_{p_2} \end{cases}}{2\,r\,\omega_n - \sum_{i=1}^4 \omega_{p_i}} \tag{4.20}$$

Taking $p_4 = -(p_1 + p_2 + p_3)$, and using (4.17) we see that,

$$\delta E_1 \sim \sum_{p_1\,p_2\,p_3} \frac{1}{(p_1+p_2+p_3)^2} \frac{1}{p_1^2} \begin{cases} \frac{1}{p_2^{3/2}} \\ \frac{1}{p_2} \end{cases} \tag{4.21}$$

where all p_i are considered absolute valued, or equivalently the sum considered over positive integers. This is manifestly convergent. Continuing on to evaluate the leading x dependence, for the top choice in (4.20) we have poles for all three summation variables, while in the large x limit the K's go as constants, $G \sim 1/\sqrt{x}$ and the energy denominator is linear in x, thus giving $\delta E_1 \sim 1/x$. For the bottom choice in (4.20), p_1 and p_3 have poles, while the sum over p_2 must be executed using (E.2). The scaling turns out identical however. Thus $\delta E_1/\mu$ is convergent and $\mathcal{O}(\lambda')$. One can repeat this argumentation for the second intermediate state in (4.18) and find the same behavior. Also the entire exercise may be repeated for δE_2 using the following intermediate states,

$$\begin{aligned} |I\rangle &\sim \alpha^\dagger_{p_1} \alpha^\dagger_{p_2} \alpha^\dagger_{p_3} \alpha^\dagger_{p_4} |\widetilde{\alpha}_1\rangle |\widetilde{\alpha}_2\rangle \\ |I\rangle &\sim \alpha^\dagger_{p_1} \alpha^\dagger_{p_2} \beta^\dagger_{p_3} \beta^\dagger_{p_4} |\widetilde{\alpha}_1\rangle |\widetilde{\alpha}_2\rangle \end{aligned} \tag{4.22}$$

[8]Note that any contraction which would yield a delta function on the external state's spacetime indices is naturally zero here because we have chosen to analyze the traceless symmetric $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ state. It is a simple matter to analyze the trace state of section 2 here, and one finds convergence as well, however the number of (inverse) powers of summation variables will be 4 in the worst case, and thus the convergence is marginal (see discussion below (4.22)). In no case does $\sqrt{\lambda'}$ behavior occur here.

and one discovers the same behavior. The essential point is that we will always have at least 5 (inverse) powers of the summation variables, while the number of summation variables is 3. Alternate positionings of the oscillators in the intermediate states such as $|I\rangle \sim \alpha^\dagger_{p_1}\alpha^\dagger_{p_2}|\widetilde{\alpha}_1\rangle\, \alpha^\dagger_{p_3}\alpha^\dagger_{p_4}|\widetilde{\alpha}_2\rangle$ only improves the convergence, since level matching removes one more summation variable in these cases.

We can now consider adding additional pairs of fermionic and bosonic impurities to the intermediate state $|I\rangle$. This will add two factors of $\widetilde{N}^{1\,1}_{p_i\,p_j}$ or two factors of $G^{(1)}_{p_i}G^{(1)}_{p_j}$ (or equivalently two factors of $\widetilde{Q}^{1\,1}_{p_ip_j} - \widetilde{Q}^{1\,1}_{p_jp_i}$). Either way the number of powers of summation variables increases in concert with the number of summation variables, preserving the convergence. Similarly the leading behavior in λ' is unaffected. So it would seem that there are $\mathcal{O}(\lambda')$ contributions to $\delta E^{(2)}/\mu$ at every order in impurities, however any non-perturbative $\sqrt{\lambda'}$ behavior is absent.

5. Conclusions

In this paper we have discovered logarithmic divergences in the one-loop mass shift of two-impurity string states on the plane wave background. As superstring amplitudes should be finite, these divergences ought to cancel, and we find that they do, via a cancellation between the H_3 vertex and the contact term. This is reminiscent of similar results for string amplitudes in Minkowski space.

Further we have shown that the apparent non-perturbative $\sqrt{\lambda'}$ behavior of contributions to the mass shift from an impurity non-conserving channel (where the number of impurities is increased by two in the intermediate states) is in fact an artifact of these logarithmic divergences and vanishes with them, leaving an $\mathcal{O}(\lambda')$ contribution. We have also given arguments which generalize the above statements to intermediate states with an arbitrary number of impurities, and up to a possible role played by an as yet unknown Q_4, have derived a formula (equation (4.14)) for the mass shift which appears to be manifestly devoid of any $\sqrt{\lambda'}$ or non-convergent behavior.

We have also shown that generically, every order in impurities contributes an $\mathcal{O}(\lambda')$ piece to the mass shift, making the prospects for computing this quantity, so that it can be matched to Yang-Mills theory, rather disappointing. On the other hand it is heartening that the string amplitudes appear to match the Yang-Mills results in terms of the leading power of λ', for any intermediate state.

A formal evaluation of the expectation values in equation (4.14) may be possible, and we hope to publish further work in this direction soon.

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A. Free string on the pp-wave

The light-cone action in the pp-wave background is

$$S_b = \frac{e(\alpha)}{4\pi\alpha'}\int d\tau \int_0^{2\pi|\alpha|} d\sigma \left(\partial_\tau X^I \partial_\tau X^I - \partial_\sigma X^I \partial_\sigma X^I - \mu^2 X^I X^I\right) + \\ + \frac{1}{8\pi}\int d\tau \int_0^{2\pi|\alpha|} d\sigma \left(i\bar{\vartheta}\partial_\tau\theta + i\vartheta\partial_\tau\bar{\vartheta} - \vartheta\partial_\sigma\bar{\vartheta} + \bar{\vartheta}\partial_\sigma\vartheta - 2\mu\bar{\vartheta}\Pi\vartheta\right) \tag{A.1}$$

where $I = 1, ..., 8$, $e(\alpha) = \mathrm{sign}(\alpha)$, $\alpha = \alpha' p^+$, θ is an 8-component positive chirality spinor of SO(8) and $\Pi = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$ is a symmetric, traceless projection operator, $\Pi^2 = 1$. We use the convention that $p^+ < 0$ for incoming strings and $p^+ > 0$ for outgoing strings. The mode expansions for the bosonic and ferminonic coordinates and their conjugate momenta are

$$X^I(\sigma) = x_0^I + \sqrt{2}\sum_{n=1}^{\infty}\left(x_n^I \cos\frac{n\sigma}{|\alpha|} + x_{-n}^I \sin\frac{n\sigma}{|\alpha|}\right) \tag{A.2}$$

$$P^I(\sigma) = \frac{1}{2\pi|\alpha|}\left[p_0^I + \sqrt{2}\sum_{n=1}^{\infty}\left(p_n^I \cos\frac{n\sigma}{|\alpha|} + p_{-n}^I \sin\frac{n\sigma}{|\alpha|}\right)\right] \tag{A.3}$$

$$\vartheta^a(\sigma) = \vartheta_0^a + \sqrt{2}\sum_{n=1}^{\infty}\left(\vartheta_n^a \cos\frac{n\sigma}{|\alpha|} + \vartheta_{-n}^a \sin\frac{n\sigma}{|\alpha|}\right) \tag{A.4}$$

$$\lambda^a(\sigma) = \frac{1}{2\pi|\alpha|}\left[\lambda_0^a + \sqrt{2}\sum_{n=1}^{\infty}\left(\lambda_n^a \cos\frac{n\sigma}{|\alpha|} + \lambda_{-n}^a \sin\frac{n\sigma}{|\alpha|}\right)\right] \tag{A.5}$$

where $2\lambda_n^a = |\alpha|\bar{\vartheta}_n^a$ and a is an $SO(8)$ spinor index. The non-vanishing (anti-)commutators of the Fourier modes are

$$\left[x_m^I, p_n^J\right] = i\delta^{IJ}\delta_{mn} \quad , \quad \left\{\vartheta_m^a, \lambda_n^b\right\} = \delta^{ab}\delta_{mn} \tag{A.6}$$

and lead to

$$\left[x^I(\sigma), p^J(\sigma')\right] = i\delta^{IJ}\delta(\sigma-\sigma') \quad , \quad \left\{\vartheta^a(\sigma), \lambda^b(\sigma')\right\} = \delta^{ab}\delta(\sigma-\sigma') \tag{A.7}$$

The modes can also be written in terms of oscillators as

$$x_n^I = i\sqrt{\frac{\alpha}{2\omega_n}}\left(a_n^I - a_n^{I\dagger}\right) \quad , \quad p_n^I = i\sqrt{\frac{\alpha}{2\omega_n}}\left(a_n^I + a_n^{I\dagger}\right) \quad , \quad \left[a_m^I, a_n^{J\dagger}\right] = i\delta^{IJ}\delta_{mn} \tag{A.8}$$

$$\vartheta_n^a = \frac{c_n}{\sqrt{|\alpha|}}\left[\left(1+\rho_n\Pi\right) b_n^a + e(n\alpha)\left(1-\rho_n\Pi\right) b_{-n}^{a\dagger}\right] \tag{A.9}$$

$$\lambda_n^a = \frac{\sqrt{|\alpha|}c_n}{2}\left[\left(1+\rho_n\Pi\right) b_n^{a\dagger} + e(n\alpha)\left(1-\rho_n\Pi\right) b_{-n}^{a}\right] \tag{A.10}$$

$$\left\{b_m^a, b_n^{b\dagger}\right\} = \delta^{ab}\delta_{mn} \tag{A.11}$$

with $\omega_n = \sqrt{n^2 + (\mu\alpha)^2}$, $\rho_n = \frac{\omega_n - |n|}{\mu\alpha}$, $c_n = \frac{1}{\sqrt{1+\rho_n^2}}$.

The free string Hamiltonian for the r-th string

$$\begin{aligned} H_2^{(r)} &= \frac{1}{2}\int_0^{2\pi|\alpha_r|} d\sigma \left[2\pi\alpha' P^{(r)2} + \frac{1}{2\pi\alpha'}(\partial_\sigma X^{(r)})^2 + \frac{1}{2\pi\alpha'}\mu^2 X^{(r)2}) \right] \\ &+ \frac{1}{2}\int_0^{2\pi|\alpha_r|} d\sigma \left[-2\pi\alpha'\lambda^{(r)}\partial_\sigma\lambda^{(r)} + \frac{1}{2\pi\alpha'}\theta^{(r)}\partial_\sigma\theta^{(r)} + 2\mu\lambda^{(r)}\Pi\theta^{(r)} \right] \end{aligned} \tag{A.12}$$

in this Fock space basis reduces to

$$H_2^{(r)} = \sum_{n=-\infty}^{\infty} \frac{\omega_n^{(r)}}{|\alpha_r|}\left(a_n^{(r)\dagger} a_n^{(r)} + b_n^{(r)\dagger} b_n^{(r)} \right). \tag{A.13}$$

Isometries of the pp-wave background are generated by H, P^+, J^{+I}, J^{ij} and $J^{i'j'}$ where $i,j = 1,2,3,4$, $i'j' = 5,6,7,8$. The latter two are angular momentum generators of the transverse $SO(4)\times SO(4)$ symmetry. There are 32 conserved supercharges Q^+, $\bar{Q}^+$ and Q^-, $\bar{Q}^-$. These generators are divided into two groups, kinematical generators

$$P^I \ , \ P^+ \ , \ J^{+I} \ , \ J^{ij} \ , \ J^{i'j'} \ , \ Q^+ \ , \ \bar{Q}^+$$

which are not corrected when string interactions are introduced and the dynamical generators

$$H \ , \ Q^- \ , \ \bar{Q}^-$$

which get corrections from interactions. The quadratic parts of H is given in (A.13) above and the supercharges are given by

$$Q^+_{(r)} = \sqrt{\frac{2}{\alpha'}}\int_0^{2\pi|\alpha_r|} d\sigma_r \, \sqrt{2}\lambda_r \,, \tag{A.14}$$

$$Q^-_{(r)} = \sqrt{\frac{2}{\alpha'}}\int_0^{2\pi|\alpha_r|} d\sigma_r \, \left[2\pi\alpha' e(\alpha_r) p_r \gamma\lambda_r - i x'_r \gamma\bar{\lambda}_r - i\mu x_r \gamma\Pi\lambda_r \right] , \tag{A.15}$$

$\bar{Q}^\pm_{(r)} = e(\alpha_r)[Q^\pm_{(r)}]^\dagger$ and γ^I are the $SO(8)$ Weyl matrices.[9]

The mode expansion of Q^- is

$$\begin{aligned} Q^-_{(r)} &= \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\gamma\Big(\sqrt{\mu}\left[a_{0(r)}(1+e(\alpha_r)\Pi) + a^\dagger_{0(r)}(1-e(\alpha_r)\Pi) \right]\lambda_{0(r)} \\ &+ \sum_{n\neq 0}\sqrt{|n|}\left[a_{n(r)} P^{-1}_{n(r)} b^\dagger_{n(r)} + e(\alpha_r)e(n) a^\dagger_{n(r)} P_{n(r)} b_{-n(r)} \right]\Big) , \end{aligned} \tag{A.16}$$

where

$$P_{n(r)} \equiv \frac{1-\rho_{n(r)}\Pi}{\sqrt{1-\rho^2_{n(r)}}} = \frac{1+\Pi}{2}U^{1/2}_{|n|(r)} + \frac{1-\Pi}{2}U^{-1/2}_{|n|(r)} \,, \qquad U_{n(r)} \equiv \frac{\omega_{n(r)} - \mu\alpha_r}{n} . \tag{A.17}$$

[9]The $SO(8)$ gamma-matrices are $\Gamma^I = \begin{pmatrix} 0 & \gamma^I \\ \bar{\gamma}^I & 0 \end{pmatrix}$.

These operators generate the superalgebra

$$[H, P^I] = -i\mu^2 J^{+I}\,, \qquad [H, Q^+] = -\mu\Pi Q^+\,, \tag{A.18}$$

$$\{Q^-_{\dot{\alpha}}, \bar{Q}^-_{\dot{\beta}}\} = 2\delta_{\dot{a}\dot{b}} H - i\mu(\gamma_{ij}\Pi)_{\dot{a}\dot{b}} J^{ij} + i\mu(\gamma_{i'j'}\Pi)_{\dot{a}\dot{b}} J^{i'j'}\,. \tag{A.19}$$

The fermionic normal modes (A.9,A.10) break the $SO(8)$ symmetry to $SO(4)\times SO(4)$. To make this symmetry manifest it is convenient to label representations of $SO(4)_1\times SO(4)_2$ through $(SU(2)\times SU(2))_1 \times (SU(2)\times SU(2))_2$ spinor indices. With this decomposition of the R-charge index, the fermionic fields ϑ^a and λ^a, are expressed in terms of creation operators $b^\dagger_{\alpha_1\alpha_2}$ and $b^\dagger_{\dot{\alpha}_1\dot{\alpha}_2}$ which transform in the $(1/2, 0, 1/2, 0)$ and $(0, 1/2, 0, 1/2)$ representations of $(SU(2)\times SU(2))_1 \times (SU(2)\times SU(2))_2$, respectively; α_k,$\dot{\alpha}_k$ being two-component Weyl indices of $SO(4)_k$.

The $SO(8)$ vector index I splits into two $SO(4)\times SO(4)$ vector indices (i, i') so that we use vector index $i = 1, \ldots, 4$ and bi-spinor indices $\alpha_1, \dot{\alpha}_1 = 1, 2$ for the first $SO(4)$ and $(i', \alpha_2, \dot{\alpha}_2)$ for the second $SO(4)$. Vectors are constructed in terms of bi-spinor indices as $(a_n)_{\alpha_1\dot{\alpha}_1} = \sigma^i_{\alpha_1\dot{\alpha}_1} a^i_n/\sqrt{2}$, $(a_n)_{\alpha_2\dot{\alpha}_2} = \sigma^{i'}_{\alpha_2\dot{\alpha}_2} a^{i'}_n/\sqrt{2}$ and transform as $(1/2, 1/2, 0, 0)$ and $(0, 0, 1/2, 1/2)$, respectively. Here the σ-matrices consist of the usual Pauli-matrices together with the 2d unit matrix

$$\sigma^i_{\alpha\dot{\alpha}} = (i\tau^1, i\tau^2, i\tau^3, -1)_{\alpha\dot{\alpha}} \tag{A.20}$$

and satisfy the reality properties $[\sigma^i_{\alpha\dot{\alpha}}]^\dagger = \sigma^{i\,\dot{\alpha}\alpha}$, $[\sigma^{i\,\dot{\alpha}}{}_{\alpha}]^\dagger = -\sigma^{i\,\alpha}{}_{\dot{\alpha}}$.

Spinor indices are raised and lowered with the two-dimensional Levi-Civita symbols, $\varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for example

$$\sigma^i_{\alpha\dot{\alpha}} = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\sigma^{i\,\dot{\beta}\beta} \equiv \varepsilon_{\alpha\beta}\sigma^{i\,\beta}{}_{\dot{\alpha}} \equiv \varepsilon_{\dot{\alpha}\dot{\beta}}\sigma^{i\,\dot{\beta}}{}_{\alpha}\,. \tag{A.21}$$

The σ-matrices satisfy the relations

$$\sigma^i_{\alpha\dot{\alpha}}\sigma^{j\,\dot{\alpha}\beta} + \sigma^j_{\alpha\dot{\alpha}}\sigma^{i\,\dot{\alpha}\beta} = 2\delta^{ij}\delta^\beta_\alpha\,, \qquad \sigma^{i\,\dot{\alpha}\alpha}\sigma^j_{\alpha\dot{\beta}} + \sigma^{j\,\dot{\alpha}\alpha}\sigma^i_{\alpha\dot{\beta}} = 2\delta^{ij}\delta^{\dot{\alpha}}_{\dot{\beta}}\,. \tag{A.22}$$

Some other properties satisfied by these matrices are

$$\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} = \delta^\delta_\alpha\delta^\gamma_\beta - \delta^\gamma_\alpha\delta^\delta_\beta\,, \tag{A.23}$$

$$\sigma^i_{\alpha\dot{\beta}}\sigma^{j\,\dot{\beta}}{}_{\beta} = -\delta^{ij}\varepsilon_{\alpha\beta} + \sigma^{ij}_{\alpha\beta}\,, \qquad (\sigma^{ij}_{\alpha\beta} \equiv \sigma^{[i}_{\alpha\dot{\alpha}}\sigma^{j]\,\dot{\alpha}}{}_{\beta} = \sigma^{ij}_{\beta\alpha}) \tag{A.24}$$

$$\sigma^i_{\alpha\dot{\alpha}}\sigma^{j\,\alpha}{}_{\dot{\beta}} = -\delta^{ij}\varepsilon_{\dot{\alpha}\dot{\beta}} + \sigma^{ij}_{\dot{\alpha}\dot{\beta}}\,, \qquad (\sigma^{ij}_{\dot{\alpha}\dot{\beta}} \equiv \sigma^{[i}_{\alpha\dot{\alpha}}\sigma^{j]\,\alpha}{}_{\dot{\beta}} = \sigma^{ij}_{\dot{\beta}\dot{\alpha}}) \tag{A.25}$$

$$\sigma^k_{\alpha\dot{\alpha}}\sigma^k_{\beta\dot{\beta}} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\,, \tag{A.26}$$

$$\sigma^{kl}_{\alpha\beta}\sigma^{kl}_{\gamma\delta} = 4(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma})\,, \tag{A.27}$$

$$\sigma^{kl}_{\alpha\beta}\sigma^{kl}_{\dot{\gamma}\dot{\delta}} = 0\,, \tag{A.28}$$

$$2\sigma^i_{\alpha\dot{\alpha}}\sigma^j_{\beta\dot{\beta}} = \delta^{ij}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} + \sigma^{k(i}_{\alpha_1\beta_1}\sigma^{j)k}_{\dot{\alpha}_1\dot{\beta}_1} - \varepsilon_{\alpha\beta}\sigma^{ij}_{\dot{\alpha}\dot{\beta}} - \sigma^{ij}_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\,. \tag{A.29}$$

In this basis the gamma matrices have the following representation

$$\gamma^i_{a\dot a} = \begin{pmatrix} 0 & \sigma^i{}_{\alpha_1\dot\beta_1}\delta^{\beta_2}_{\alpha_2} \\ \sigma^{i\,\dot\alpha_1\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} & 0 \end{pmatrix} , \qquad \gamma^i_{\dot a a} = \begin{pmatrix} 0 & \sigma^i{}_{\alpha_1\dot\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} \\ \sigma^{i\,\dot\alpha_1\beta_1}\delta^{\beta_2}_{\alpha_2} & 0 \end{pmatrix} , \tag{A.30}$$

$$\gamma^{i'}_{a\dot a} = \begin{pmatrix} -\delta^{\beta_1}_{\alpha_1}\sigma^{i'}{}_{\alpha_2\dot\beta_2} & 0 \\ 0 & \delta^{\dot\alpha_1}_{\dot\beta_1}\sigma^{i'\,\dot\alpha_2\beta_2} \end{pmatrix} , \qquad \gamma^{i'}_{\dot a a} = \begin{pmatrix} -\delta^{\beta_1}_{\alpha_1}\sigma^{i'\,\dot\alpha_2\beta_2} & 0 \\ 0 & \delta^{\dot\alpha_1}_{\dot\beta_1}\sigma^{i'}{}_{\alpha_2\dot\beta_2} \end{pmatrix} . \tag{A.31}$$

and the projector reads

$$\Pi_{ab} = \begin{pmatrix} (\sigma^1\sigma^2\sigma^3\sigma^4)_{\alpha_1}^{\beta_1}\delta^{\beta_2}_{\alpha_2} & 0 \\ 0 & (\sigma^1\sigma^2\sigma^3\sigma^4)^{\dot\alpha_1}_{\dot\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} \end{pmatrix} = \begin{pmatrix} \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} & 0 \\ 0 & -\delta^{\dot\alpha_1}_{\dot\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} \end{pmatrix} , \tag{A.32}$$

so that $(1 \pm \Pi)/2$ projects onto $(1/2, 0, 1/2, 0)$ and $(0, 1/2, 0, 1/2)$, respectively.

The supercharge $Q^-_{\alpha_1\dot\beta_2}$ is a $(1/2, 0, 0, 1/2)$ and $Q^-_{\dot\alpha_1\beta_2}$ is a $(0, 1/2, 1/2, 0)$ representation. In this notation it is convenient to define the linear combinations of the free supercharges

$$\sqrt{2}\eta Q \equiv Q^- + i\bar{Q}^- \quad , \quad \sqrt{2}\bar\eta\widetilde{Q} \equiv Q^- - i\bar{Q}^- \tag{A.33}$$

where $\eta = e^{i\pi/4}$. On the space of physical state they satisfy the dynamical constraints

$$\begin{aligned} \left\{Q_{\alpha_1\dot\alpha_2}, Q_{\beta_1\dot\beta_2}\right\} &= \left\{\widetilde{Q}_{\alpha_1\dot\alpha_2}, \widetilde{Q}_{\beta_1\dot\beta_2}\right\} = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}H \\ \left\{Q_{\alpha_1\dot\alpha_2}, \widetilde{Q}_{\beta_1\dot\beta_2}\right\} &= -\mu\epsilon_{\dot\alpha_2\dot\beta_2}\left(\sigma^{ij}\right)_{\alpha_1\beta_1}J^{ij} + \mu\epsilon_{\alpha_1\beta_1}\left(\sigma^{i'j'}\right)_{\dot\alpha_2\dot\beta_2}J^{i'j'} \end{aligned} \tag{A.34}$$

and similarly for $Q_{\dot\alpha_1\alpha_2}$ and $\widetilde{Q}_{\dot\beta_1\beta_2}$.

Among states that are created by two oscillators, the state with quantum numbers $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$ which are created by two bosons have no analogs amongst the two oscillator states containing either one or two fermions. Thus, they are not mixed with other members of the supermultiplet. These states in the main text are denoted $\big|[\mathbf{9}, \mathbf{1}]\rangle^{(ij)}$ and $|[\mathbf{1}, \mathbf{9}]\rangle^{(i'j')}$ in SO(8) notation.

B. Solving the dynamical constraints

When string interactions are considered, the dynamical generators of the superalgebra receive g_2 corrections so that they can be generally written in a perturbative g_2 expansion

$$\begin{aligned} &H = H_2 + g_2 H_3 + g_2^2 H_4 + \dots \ , \\ &Q_{\alpha_1\dot\alpha_2} = Q_{2\alpha_1\dot\alpha_2} + g_2 Q_{3\alpha_1\dot\alpha_2} + g_2^2 Q_{4\alpha_1\dot\alpha_2} + \dots \end{aligned} \tag{B.1}$$

H_3, Q_3 are the operators representing a three string interaction and H_4, Q_4 are contact term interactions. As we shall see H_4 is induced by cubic supercharges. Such an expansion can be used to solve perturbatively the dynamical constraints (A.34) At order $\mathcal{O}(g_2)$ the dynamical constraints (A.34) become

$$\{Q_{2\,\alpha_1\dot\alpha_2}, Q_{3\,\beta_1\dot\beta_2}\} + \{Q_{3\,\alpha_1\dot\alpha_2}, Q_{2\,\beta_1\dot\beta_2}\} = -2\varepsilon_{\alpha_1\beta_1}\varepsilon_{\dot\alpha_2\dot\beta_2}H_3 \,, \tag{B.2}$$

$$\{\widetilde{Q}_{2\,\alpha_1\dot\alpha_2}, \widetilde{Q}_{3\,\beta_1\dot\beta_2}\} + \{\widetilde{Q}_{3\,\alpha_1\dot\alpha_2}, \widetilde{Q}_{2\,\beta_1\dot\beta_2}\} = -2\varepsilon_{\alpha_1\beta_1}\varepsilon_{\dot\alpha_2\dot\beta_2}H_3 \,, \tag{B.3}$$

$$\{Q_{2\,\alpha_1\dot\alpha_2}, \widetilde{Q}_{3\,\beta_1\dot\beta_2}\} + \{Q_{3\,\alpha_1\dot\alpha_2}, \widetilde{Q}_{2\,\beta_1\dot\beta_2}\} = 0 \,. \tag{B.4}$$

It is convenient [45] to express H_3 and Q_3 as states $|H_3\rangle$ and $|Q_3\rangle$ in the multi-string Hilbert space and work in the number basis where the dynamical generators can be written as $\mathcal{P}|V\rangle$, where $\mathcal{P}$ are prefactors determined by imposing the dynamical constraints and $|V\rangle$ is the kinematical part of the vertex and implements the continuity conditions. Equations (B.2-B.4) become

$$\sum_{r=1}^{3} Q_{(r)\,\alpha_1\dot\alpha_2}|Q_{3\,\beta_1\dot\beta_2}\rangle + \sum_{r=1}^{3} Q_{(r)\,\beta_1\dot\beta_2}|Q_{3\,\alpha_1\dot\alpha_2}\rangle = -2\varepsilon_{\alpha_1\beta_1}\varepsilon_{\dot\alpha_2\dot\beta_2}|H_3\rangle\,, \tag{B.5}$$

$$\sum_{r=1}^{3} Q_{(r)\,\dot\alpha_1\alpha_2}|Q_{3\,\dot\beta_1\beta_2}\rangle + \sum_{r=1}^{3} Q_{(r)\,\dot\beta_1\beta_2}|Q_{3\,\dot\alpha_1\alpha_2}\rangle = -2\varepsilon_{\dot\alpha_1\dot\beta_1}\varepsilon_{\alpha_2\beta_2}|H_3\rangle\,, \tag{B.6}$$

$$\sum_{r=1}^{3} Q_{(r)\,\alpha_1\dot\alpha_2}|Q_{3\,\dot\beta_1\beta_2}\rangle + \sum_{r=1}^{3} Q_{(r)\,\dot\beta_1\beta_2}|Q_{3\,\alpha_1\dot\alpha_2}\rangle = 0\,, \tag{B.7}$$

and analogously for $Q \to \widetilde{Q}$.

Making an ansatz for, say $Q_{3\,\alpha_1\dot\alpha_2}$, compatible with the requirement that the Hamiltonian prefactor in its functional form is quadratic in derivatives [36], into (B.5) and demanding that the result only involves the tensor $\varepsilon_{\alpha_1\beta_1}\varepsilon_{\dot\alpha_2\dot\beta_2}$ fixes $Q_{3\,\alpha_1\dot\alpha_2}$ and consequently also H_3 up to their normalization. The same procedure applies to $\widetilde{Q}_{3\,\alpha_1\dot\alpha_2}$ and requires that its normalization is the same as of $Q_{3\,\alpha_1\dot\alpha_2}$.
A possible choice for the three-string vertex and dynamical supercharges [10] that solves the dynamical constraints up to order g_2 is [36]

$$\begin{aligned}
|H_3\rangle =\ & -g_2\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{\alpha'}{8\,\alpha_3^3}\Big[(K_i\widetilde{K}_j - \frac{\mu\kappa}{\alpha'}\delta_{ij})v^{ij} - (K_{i'}\widetilde{K}_{j'} - \frac{\mu\kappa}{\alpha'}\delta_{i'j'})v^{i'j'}\\
&-K^{\dot\alpha_1\alpha_1}\widetilde{K}^{\dot\alpha_2\alpha_2}s_{\alpha_1\alpha_2}(Y)s^*_{\dot\alpha_1\dot\alpha_2}(Z) - \widetilde{K}^{\dot\alpha_1\alpha_1}K^{\dot\alpha_2\alpha_2}s^*_{\alpha_1\alpha_2}(Y)s_{\dot\alpha_1\dot\alpha_2}(Z)\Big]|V\rangle\,,\\
|Q_{3\,\beta_1\dot\beta_2}\rangle =\ & g_2\,\eta\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s_{\dot\gamma_1\dot\beta_2}(Z)t_{\beta_1\gamma_1}(Y)\widetilde{K}^{\dot\gamma_1\gamma_1}\\
&+is_{\beta_1\gamma_2}(Y)t^*_{\dot\beta_2\dot\gamma_2}(Z)\widetilde{K}^{\dot\gamma_2\gamma_2}\Big)|V\rangle\,,\\
|Q_{3\,\dot\beta_1\beta_2}\rangle =\ & g_2\,\eta\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s^*_{\gamma_1\beta_2}(Y)t^*_{\dot\beta_1\dot\gamma_1}(Z)\widetilde{K}^{\dot\gamma_1\gamma_1}\\
&+is^*_{\dot\beta_1\dot\gamma_2}(Z)t_{\beta_2\gamma_2}(Y)\widetilde{K}^{\dot\gamma_2\gamma_2}\Big)|V\rangle\,.
\end{aligned} \tag{B.8}$$

where $\kappa \equiv \alpha_1\alpha_2\alpha_3$, $\alpha_3 < 0$ for the incoming and $\alpha_{1,2} > 0$ for the outgoing strings, K^I, $\widetilde{K}^I$ are defined in (D.11), Y and Z in (D.14) and

$$\widetilde{K}^{\dot\gamma_1\gamma_1} \equiv \widetilde{K}^i\sigma^{i\,\dot\gamma_1\gamma_1}\,, \qquad \widetilde{K}^{\dot\gamma_2\gamma_2} \equiv \widetilde{K}^{i'}\sigma^{i'\,\dot\gamma_2\gamma_2}\,, \tag{B.9}$$

Moreover

$$v^{ij} = \delta^{ij}\Big[1 + \frac{1}{12}(Y^4 + Z^4) + \frac{1}{144}Y^4Z^4\Big]$$

[10] The three-string vertex and dynamical supercharges in the open string case have been constructed in [46].

$$-\frac{i}{2}\Big[Y^{2ij}(1+\frac{1}{12}Z^4)-Z^{2ij}(1+\frac{1}{12}Y^4)\Big]+\frac{1}{4}[Y^2Z^2]^{ij}\,, \tag{B.10}$$

$$v^{i'j'}=\delta^{i'j'}\Big[1-\frac{1}{12}(Y^4+Z^4)+\frac{1}{144}Y^4Z^4\Big]$$

$$-\frac{i}{2}\Big[Y^{2i'j'}(1-\frac{1}{12}Z^4)-Z^{2i'j'}(1-\frac{1}{12}Y^4)\Big]+\frac{1}{4}[Y^2Z^2]^{i'j'}\,. \tag{B.11}$$

Here we defined

$$Y^{2ij}\equiv\sigma^{ij}_{\alpha_1\beta_1}Y^{2\alpha_1\beta_1}\,,\quad Z^{2ij}\equiv\sigma^{ij}_{\dot\alpha_1\dot\beta_1}Z^{2\dot\alpha_1\dot\beta_1}\,,\quad (Y^2Z^2)^{ij}\equiv Y^{2k(i}Z^{2j)k} \tag{B.12}$$

and analogously for the primed indices. We have also introduced the following quantities quadratic and cubic in Y and symmetric in spinor indices

$$Y^2_{\alpha_1\beta_1}\equiv Y_{\alpha_1\alpha_2}Y_{\beta_1}^{\alpha_2}\,,\qquad Y^2_{\alpha_2\beta_2}\equiv Y_{\alpha_1\alpha_2}Y_{\beta_2}^{\alpha_1}\,, \tag{B.13}$$

$$Y^3_{\alpha_1\beta_2}\equiv Y^2_{\alpha_1\beta_1}Y^{\beta_1}_{\beta_2}=-Y^2_{\beta_2\alpha_2}Y^{\alpha_2}_{\alpha_1}\,, \tag{B.14}$$

and quartic in Y and antisymmetric in spinor indices

$$Y^4_{\alpha_1\beta_1}\equiv Y^2_{\alpha_1\gamma_1}Y^{2\gamma_1}{}_{\beta_1}=-\frac{1}{2}\varepsilon_{\alpha_1\beta_1}Y^4\,,\qquad Y^4_{\alpha_2\beta_2}\equiv Y^2_{\alpha_2\gamma_2}Y^{2\gamma_2}{}_{\beta_2}=\frac{1}{2}\varepsilon_{\alpha_2\beta_2}Y^4\,, \tag{B.15}$$

where

$$Y^4\equiv Y^2_{\alpha_1\beta_1}Y^{2\alpha_1\beta_1}=-Y^2_{\alpha_2\beta_2}Y^{2\alpha_2\beta_2}\,. \tag{B.16}$$

The spinorial quantities s and t are defined as

$$s(Y)\equiv Y+\frac{i}{3}Y^3\ ,\qquad t(Y)\equiv\varepsilon+iY^2-\frac{1}{6}Y^4\,. \tag{B.17}$$

Analogous definitions can be given for Z. The other dynamical supercharges can be obtained from $|\widetilde{Q}\rangle=|Q^*\rangle$.

The normalization of the dynamical generators is not fixed by the superalgebra at order $\mathcal{O}(g_2)$ and can be an arbitrary (dimensionless) function $f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})$ of the light-cone momenta and μ due to the fact that P^+ is a central element of the algebra. Indeed, it does not seem that further consistency conditions at higher orders in g_2 would allow to fix f.

Other solutions of the dynamical constraints to this order in g_2 can be found and have been provided in [35, 47].

Consider now the constraints at order $\mathcal{O}(g_2^2)$. These are,

$$\{Q_{2\,\alpha_1\dot\alpha_2},Q_{4\,\beta_1\dot\beta_2}\}+\{Q_{4\,\alpha_1\dot\alpha_2},Q_{2\,\beta_1\dot\beta_2}\}+\{Q_{3\,\alpha_1\dot\alpha_2},Q_{3\,\beta_1\dot\beta_2}\}=-2\varepsilon_{\alpha_1\beta_1}\varepsilon_{\dot\alpha_2\dot\beta_2}H_4 \tag{B.18}$$

$$\{\widetilde{Q}_{2\,\alpha_1\dot\alpha_2},\widetilde{Q}_{4\,\beta_1\dot\beta_2}\}+\{\widetilde{Q}_{4\,\alpha_1\dot\alpha_2},\widetilde{Q}_{2\,\beta_1\dot\beta_2}\}+\{\widetilde{Q}_{3\,\alpha_1\dot\alpha_2},\widetilde{Q}_{3\,\beta_1\dot\beta_2}\}=-2\varepsilon_{\alpha_1\beta_1}\varepsilon_{\dot\alpha_2\dot\beta_2}H_4 \tag{B.19}$$

$$\{Q_{2\,\alpha_1\dot\alpha_2},\widetilde{Q}_{4\,\beta_1\dot\beta_2}\}+\{Q_{4\,\alpha_1\dot\alpha_2},\widetilde{Q}_{2\,\beta_1\dot\beta_2}\}+\{Q_{3\,\alpha_1\dot\alpha_2},\widetilde{Q}_{3\,\beta_1\dot\beta_2}\}=0, \tag{B.20}$$

from these we derive the contact term used in the calculations. As it stands Q_4 is unknown and the custom in the literature is to set it to zero. Note that this choice is not inconsistent with the constraints listed above. Setting Q_4 to zero, we arrive at the following form of H_4 which is used in the contact term calculations of sections 2 and 3,

$$H_4=\frac{1}{8}Q_{3\beta_1\dot\beta_2}Q_3^{\beta_1\dot\beta_2}+\frac{1}{8}Q_{3\dot\beta_1\beta_2}Q_3^{\dot\beta_1\beta_2}. \tag{B.21}$$

C. BMN basis

In the calculations we use for the oscillators the BMN basis which is related to the exponential basis by

$$
\begin{aligned}
&\sqrt{2}a_n^i \equiv \alpha_n^i + \alpha_{-n}^i\,, \quad i\sqrt{2}a_{-n}^i \equiv \alpha_n^i - \alpha_{-n}^i\\
&\sqrt{2}a_n^{i'} \equiv \alpha_n^{i'} + \alpha_{-n}^{i'}\,, \quad i\sqrt{2}a_{-n}^{i'} \equiv \alpha_n^{i'} - \alpha_{-n}^{i'}\,,\\
&\sqrt{2}b_n^{\alpha_1\alpha_2} \equiv \beta_n^{\alpha_1\alpha_2} + \beta_{-n}^{\alpha_1\alpha_2}\,, \quad i\sqrt{2}b_{-n}^{\alpha_1\alpha_2} \equiv \beta_n^{\alpha_1\alpha_2} - \beta_{-n}^{\alpha_1\alpha_2}\,,\\
&i\sqrt{2}b_n^{\dot\alpha_1\dot\alpha_2} \equiv -\beta_n^{\dot\alpha_1\dot\alpha_2} + \beta_{-n}^{\dot\alpha_1\dot\alpha_2}\,, \quad \sqrt{2}b_{-n}^{\dot\alpha_1\dot\alpha_2} \equiv \beta_n^{\dot\alpha_1\dot\alpha_2} + \beta_{-n}^{\dot\alpha_1\dot\alpha_2}
\end{aligned}
\tag{C.1}
$$

for $n > 0$, and

$$
a_0^i \equiv \alpha_0^i \quad a_0^{i'} \equiv \alpha_0^{i'} \quad b_0^{\alpha_1\alpha_2} \equiv \beta_0^{\alpha_1\alpha_2} \quad \sqrt{2}b_{-n}^{\dot\alpha_1\dot\alpha_2} \equiv \beta_n^{\dot\alpha_1\dot\alpha_2} + \beta_{-n}^{\dot\alpha_1\dot\alpha_2}
\tag{C.2}
$$

for $n = 0$.

The commutation relation for the oscillators are

$$
[\alpha_m^i, \alpha_n^{\dagger\, j}] = \delta_{mn}\delta^{ij}\,, \quad \{(\beta_m)_{\alpha_1\alpha_2}, (\beta_n^\dagger)^{\beta_1\beta_2}\} = \delta_{mn}\delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2}\,.
\tag{C.3}
$$

The following relations are useful

$$
[(\alpha_m)_{\alpha_1\dot\alpha_1}, (\alpha_n^\dagger)^{\dot\beta_1\beta_1}] = \delta_{mn}\delta_{\alpha_1}^{\beta_1}\delta_{\dot\alpha_1}^{\dot\beta_1}\,, \qquad [(\alpha_m)_{\alpha_2\dot\alpha_2}, (\alpha_n^\dagger)^{\dot\beta_2\beta_2}] = \delta_{mn}\delta_{\alpha_2}^{\beta_2}\delta_{\dot\alpha_2}^{\dot\beta_2}\,.
\tag{C.4}
$$

where

$$
(\alpha_n^\dagger)_{\alpha_1\dot\alpha_1} \equiv \frac{1}{\sqrt{2}}(\sigma^i)_{\alpha_1\dot\alpha_1}\alpha_n^{\dagger\, i}\,, \quad (\alpha_n^\dagger)_{\alpha_2\dot\alpha_2} \equiv \frac{1}{\sqrt{2}}(\sigma^{i'})_{\alpha_2\dot\alpha_2}\alpha_n^{\dagger\, i'}
\tag{C.5}
$$

The free light-cone Hamiltonian (A.13) becomes

$$
H_{2(r)} = \frac{1}{\alpha_r}\sum_{n\in\mathcal{Z}}\omega_{n(r)}N_{n(r)}
\tag{C.6}
$$

where $N_{n(r)}$ is the number operator

$$
N_{n(r)} = \alpha_{n(r)}^{\dagger\, i}\alpha_{n(r)}^i + \alpha_{n(r)}^{\dagger\, i'}\alpha_{n(r)}^{i'} + (\beta_{n(r)}^\dagger)^{\alpha_1\alpha_2}(\beta_{n(r)})_{\alpha_1\alpha_2} + (\beta_{n(r)}^\dagger)^{\dot\alpha_1\dot\alpha_2}(\beta_{n(r)})_{\dot\alpha_1\dot\alpha_2}\,.
\tag{C.7}
$$

The ground state is defined as

$$
\alpha_{n(r)}|\alpha_r\rangle = 0\,, \quad \beta_{n(r)}|\alpha_r\rangle = 0\,, \quad n \in \mathcal{Z}\,.
\tag{C.8}
$$

The free dynamical supercharges (A.16) are given by

$$
\begin{aligned}
\sqrt{\frac{|\alpha|}{2}}Q^-_{\alpha_1\dot\alpha_2} =& -\frac{\sqrt{\mu|\alpha|}}{2\sqrt{2}}(1-e(\alpha))\left[\alpha_{0\,\alpha_1}^{\;\dot\beta_1}\beta_{0\,\dot\beta_1\dot\alpha_2}^\dagger + \alpha_{0\,\dot\alpha_2}^{\dagger\,\beta_2}\beta_{0\,\alpha_1\beta_2}\right]\\
&+\sum_{k\neq 0}\Big[\sqrt{\omega_k+\mu\alpha}\,\alpha_{k\,\alpha_1}^{\dagger\,\dot\beta_1}\beta_{k\,\dot\beta_1\dot\alpha_2} - ie(\alpha k)\sqrt{\omega_k-\mu\alpha}\,\alpha_{k\,\alpha_1}^{\;\dot\beta_1}\beta_{k\,\dot\beta_1\dot\alpha_2}^\dagger\\
&\quad -e(\alpha)\left(\sqrt{\omega_k+\mu\alpha}\,\alpha_{k\,\dot\alpha_2}^{\;\beta_2}\beta_{k\,\alpha_1\beta_2}^\dagger - ie(\alpha k)\sqrt{\omega_k-\mu\alpha}\,\alpha_{k\,\dot\alpha_2}^{\dagger\,\beta_2}\beta_{k\,\alpha_1\beta_2}\right)\Big]\,,\\
\sqrt{\frac{|\alpha|}{2}}Q^-_{\dot\alpha_1\alpha_2} =& \frac{\sqrt{\mu|\alpha|}}{2\sqrt{2}}(1+e(\alpha))\left[\alpha_{0\,\dot\alpha_1}^{\;\beta_1}\beta_{0\,\beta_1\alpha_2}^\dagger + \alpha_{0\,\alpha_2}^{\dagger\,\dot\beta_2}\beta_{0\,\dot\alpha_1\dot\beta_2}\right]
\end{aligned}
$$

$$+\sum_{k\neq 0}\Big[\sqrt{\omega_k+\mu\alpha}\,\alpha^{\dagger\,\dot\beta_2}_{k\,\alpha_2}\beta_{k\,\dot\alpha_1\dot\beta_2}-ie(\alpha k)\sqrt{\omega_k-\mu\alpha}\,\alpha_{k\,\alpha_2}^{\;\dot\beta_2}\beta^{\dagger}_{k\,\dot\alpha_1\dot\beta_2}$$
$$+e(\alpha)\left(\sqrt{\omega_k+\mu\alpha}\,\alpha_{k\,\dot\alpha_1}^{\;\beta_1}\beta^{\dagger}_{k\,\beta_1\alpha_2}-ie(\alpha k)\sqrt{\omega_k-\mu\alpha}\,\alpha^{\dagger\,\beta_1}_{k\,\dot\alpha_1}\beta_{k\,\beta_1\alpha_2}\right)\Big] \tag{C.9}$$

and $\bar{Q}^- = e(\alpha)[Q^-]^\dagger$.

The calculations in sections 2 and 3 use the following expressions for the cubic vertex and supercharges, which can be deduced from their forms given in the previous Appendix and are valid for bosonic external and intermediate states for H_3 and intermediate states involving a single fermionic (and an arbitrary number of bosonic) impurities for H_4

$$|H_3\rangle = -g_2\frac{r(1-r)}{4}\sum_{s=1}^{3}\sum_{p=-\infty}^{\infty}\sum_{K,L=1}^{8}\frac{\omega_{p(s)}}{\alpha_s}\alpha^{\dagger K}_{p\,(s)}\alpha^{L}_{-p\,(s)}\,\Pi^{KL}|V\rangle \tag{C.10}$$

$$|Q_{3\,\beta_1\dot\beta_2}\rangle = g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\sum_{r,s=1}^{3}\sum_{p,q=-\infty}^{\infty}(\sigma^k)_{\beta_1}^{\;\dot\gamma_1}K_{p\,(s)}G_{|q|(r)}\,\alpha^{\dagger k}_{-p\,(s)}\beta^{\dagger}_{q\,(r)\,\dot\gamma_1\dot\beta_2}|V\rangle \tag{C.11}$$

$$|Q_{3\,\dot\beta_1\beta_2}\rangle = g_2\frac{\bar\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\sum_{r,s=1}^{3}\sum_{p,q=-\infty}^{\infty}(\sigma^k)_{\dot\beta_1}^{\;\gamma_1}K_{p\,(s)}G_{|q|(r)}\,\alpha^{\dagger k}_{-p\,(s)}\beta^{\dagger}_{q\,(r)\,\gamma_1\beta_2}|V\rangle \tag{C.12}$$

and similarly for $|\widetilde{Q}_3\rangle$. The kinematical part of the vertex $|V\rangle$ in the number basis is defined as follows,

$$|V\rangle = |E_\alpha\rangle|E_\beta\rangle\delta(\sum_{r=1}^{3}\alpha_r) \tag{C.13}$$

where $|E_\alpha\rangle$ and $|E_\beta\rangle$ are exponentials of bosonic and fermionic oscillators respectively

$$|E_\alpha\rangle = \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\alpha^{\dagger K}_{m\,(s)}\widetilde{N}^{st}_{mn}\alpha^{\dagger K}_{n\,(t)}\right)|\alpha\rangle_{123} \tag{C.14}$$

and

$$|E_\beta\rangle = \exp\left(\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}(\beta^{\alpha_1\alpha_2\,\dagger}_{m(r)}\beta^{\dagger}_{n(s)\,\alpha_1\alpha_2}-\beta^{\dot\alpha_1\dot\alpha_2\,\dagger}_{m(r)}\beta^{\dagger}_{n(s)\,\dot\alpha_1\dot\alpha_2})\widetilde{Q}^{rs}_{mn}\right)|\alpha\rangle_{123} \tag{C.15}$$

$|\alpha\rangle_{123} = |\alpha_1\rangle\otimes|\alpha_2\rangle\otimes|\alpha_3\rangle$ is the tensor product of three vacuum states. All the quantities appearing above are defined in Appendix D.

D. Neumann matrices and associated quantities

In this section we present the explicit expressions for the quantities appearing in the prefactors and exponentials part of $|H_3\rangle$ and $|Q_3\rangle$. Following the notation of [38], the Neumann matrices can be written as

$$\widetilde{N}^{st}_{mn} = \begin{cases} \frac{1}{2}\bar{N}^{st}_{|m||n|}\left(1 + U_{m(s)}U_{n(t)}\right) & , m, n \neq 0 \\ \frac{1}{\sqrt{2}}\bar{N}^{st}_{|m|0} \quad m \neq 0 \\ \bar{N}^{st}_{00} \end{cases} \tag{D.1}$$

with[11]

$$\bar{N}^{st}_{mn} = -(1 - 4\mu\kappa K)^{-1} \frac{\kappa}{\alpha_s \omega_{n(t)} + \alpha_t \omega_{m(s)}} \left[C U^{-1}_{(s)} C^{1/2}_{(s)} \bar{N}^s \right]_m \left[C U^{-1}_{(t)} C^{1/2}_{(t)} \bar{N}^t \right]_n \tag{D.2}$$

$$\bar{N}^{st}_{m0} = \sqrt{-2\mu\kappa(1 - \beta_t)}\sqrt{\omega_{m(s)}}\bar{N}^s_m\,, \qquad t \in \{1, 2\} \tag{D.3}$$

$$\bar{N}^{st}_{00} = (1 - 4\mu\kappa K)\left(\delta^{st} - \sqrt{\beta_s \beta_t}\right)\,, \qquad s, t \in \{1, 2\} \tag{D.4}$$

$$\bar{N}^{s3}_{00} = -\sqrt{\beta_s}\,, \qquad s \in \{1, 2\} \tag{D.5}$$

where

$$C_n = n\,, \qquad C_{n(s)} = \omega_{n(s)} \equiv \sqrt{n^2 + (\mu\alpha_s)^2}\,, \qquad \kappa \equiv \alpha_1\alpha_2\alpha_3 \tag{D.6}$$

$$U_{n(s)} = \frac{1}{n}(\omega_{n(s)} - \mu\alpha_s)\,, \qquad U^{-1}_{n(s)} = \frac{1}{n}(\omega_{n(s)} + \mu\alpha_s) \tag{D.7}$$

and [32]

$$1 - 4\mu\kappa K \approx -\frac{1}{4\pi r(1 - r)\mu\alpha_3} \tag{D.8}$$

$$\alpha_3 \bar{N}^3_n \approx -\frac{\sin(n\pi r)}{\pi r(1 - r)} \frac{1}{\omega_{n(3)}\sqrt{-2\mu\alpha_3(\omega_{n(3)} + \mu\alpha_3)}} \tag{D.9}$$

$$\alpha_3 \bar{N}^s_n \equiv \alpha_3 \bar{N}_n(\beta_s) \approx -\frac{\sqrt{\beta_s}}{2\pi r(1 - r)} \frac{1}{\omega_{n(s)}\sqrt{-2\mu\alpha_3(\omega_{n(s)} - \mu\alpha_3\beta_s)}} \tag{D.10}$$

up to exponential corrections $\sim \mathcal{O}(e^{-\mu\alpha_3})$ [12]. For the bosonic constituents of the prefactor one has

$$K^I = \sum_{s=1}^{3}\sum_{n\in\mathcal{Z}} K_{n(s)}\alpha^{I\,\dagger}_{n(s)}\,, \quad \widetilde{K}^I = \sum_{s=1}^{3}\sum_{n\in\mathcal{Z}} K_{n(s)}\alpha^{I\,\dagger}_{-n(s)} \tag{D.11}$$

where

$$K_{0(s)} = (1 - 4\mu\kappa K)^{1/2}\sqrt{-\frac{2\mu\kappa}{\alpha'}(1 - \beta_s)}\,, \qquad K_{0(3)} = 0 \tag{D.12}$$

and

$$K_{n(s)} = -\frac{\kappa}{\sqrt{2\alpha'}\alpha_s}(1 - 4\mu\kappa K)^{-1/2}(\omega_{n(s)} + \mu\alpha_s)\sqrt{\omega_{n(s)}}\bar{N}^s_{|n|}(1 - U_{n(s)}) \tag{D.13}$$

[11] To have a manifest symmetry in $1 \leftrightarrow 2$ we additionally redefined the oscillators as $(-1)^{s(n+1)}\alpha_{n(s)} \rightarrow \alpha_{n(s)}$ for $n \in \mathcal{Z}$, $s = 1, 2, 3$ and analogously for the fermionic oscillators.

[12] To compare with the definition used in [32] note that $\bar{N}^s_{n\,\text{here}} = (-1)^{s(n+1)} U_{n(s)} C^{-1/2}_{n(s)} \bar{N}^s_{n\,\text{there}}$.

For the fermionic constituents of the prefactor one has

$$Y^{\alpha_1\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathcal{Z}} G_{|n|(s)}\beta^{\dagger\,\alpha_1\alpha_2}_{n(s)}\,, \qquad Z^{\dot\alpha_1\dot\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathcal{Z}} G_{|n|(s)}\beta^{\dagger\,\dot\alpha_1\dot\alpha_2}_{n(s)}\,, \tag{D.14}$$

where

$$G_{0(s)} = (1-4\mu\kappa K)^{1/2}\sqrt{1-\beta_s}\,, \qquad G_{0(3)} = 0 \tag{D.15}$$

and

$$G_{n(s)} = \frac{e(\alpha_s)}{\sqrt{2|\alpha_s|}}\frac{\sqrt{-\kappa}}{(1-4\mu\kappa K)^{1/2}}\sqrt{(\omega_{n(s)}+\mu\alpha_s)\omega_{n(s)}}\bar{N}^s_{|n|} \tag{D.16}$$

where in the above expressions we have used $\beta_1 \equiv r$ and $\beta_2 \equiv 1-r$ (with $\beta_t \equiv -\alpha_t/\alpha_3$ and $\alpha_3 < 0$).

Let us now collect some expressions needed for the computations presented in the Paper. The Neumann matrices which couple the external string (labeled by 3) to the internal strings which are labeled by $r,s = 1,2$ are special in that they contain a pole proportional to the external state mode number. Taking a large μ limit of the expression given above we get,

$$\widetilde{N}^{3\,r}_{n\,p} \simeq e(n)\,\frac{\sin(|n|\pi r)}{2\pi\sqrt{\omega^{(3)}_n\,\omega^{(r)}_p}}\,\frac{\left(\omega^{(r)}_p+\beta_r\,\omega^{(3)}_n\right)}{p-\beta_r\,n} \tag{D.17}$$

where $\beta_1 = r$, $\beta_2 = (1-r)$ $(\beta_r = -\alpha_r/\alpha_3)$.

The internal-internal Neumann matrix is,

$$\widetilde{N}^{r\,s}_{n\,p} = \frac{\sqrt{\beta_r\,\beta_s}\left(\sqrt{\omega^{(r)}_n-\beta_r\mu\alpha_3}\sqrt{\omega^{(s)}_p-\beta_s\mu\alpha_3}+e(n\,p)\,\sqrt{\omega^{(r)}_n+\beta_r\mu\alpha_3}\sqrt{\omega^{(s)}_p+\beta_s\mu\alpha_3}\right)}{4\pi\sqrt{\omega^{(r)}_n\omega^{(s)}_p}\left(\beta_s\omega^{(r)}_n+\beta_r\omega^{(s)}_p\right)} \tag{D.18}$$

The K and G vectors from the prefactor are as follows,

$$K^{(r)}_{-p} = -\alpha_3\sqrt{\frac{r\,(1-r)}{4\pi\,\beta_r\,\alpha'}}\,\frac{\sqrt{\omega^{(r)}_p-\beta_r\mu\alpha_3}+e(p)\sqrt{\omega^{(r)}_p+\beta_r\mu\alpha_3}}{\sqrt{\omega^{(r)}_p}} \tag{D.19}$$

$$K^{(3)}_{-n} = -\alpha_3\,\sin(|n|\pi r)\sqrt{\frac{r\,(1-r)}{\pi\,\alpha'}}\,\frac{\sqrt{\omega^{(3)}_n+\mu\alpha_3}+e(n)\sqrt{\omega^{(3)}_n-\mu\alpha_3}}{\sqrt{\omega^{(3)}_n}} \tag{D.20}$$

$$G^{(r)}_p = \frac{1}{\sqrt{4\pi\,\omega^{(r)}_p}} \qquad G^{(3)}_n = -\frac{\sin(|n|\pi r)}{\sqrt{\pi\,\omega^{(3)}_n}} \tag{D.21}$$

Finally, the fermionic Neumann matrices are given by,

$$\widetilde{Q}^{r\,s}_{n\,p} = i\,\frac{\beta_s}{4\pi}\,\frac{n}{\sqrt{\omega^{(r)}_n\omega^{(s)}_p}\,\left(\beta_s\omega^{(r)}_n + \beta_r\omega^{(s)}_p\right)} \tag{D.22}$$

$$\widetilde{Q}^{3\,r}_{n\,p} - \widetilde{Q}^{r\,3}_{p\,n} = -i\frac{\sin(|n|\pi r)}{2\pi\sqrt{\omega^{(3)}_n\,\omega^{(r)}_p}}\,\frac{\left(\omega^{(r)}_p + \beta_r\,\omega^{(3)}_n\right)}{p-\beta_r\,n} \tag{D.23}$$

E. Leading μ dependence of sums

Sums are evaluated using the contour integral method,

$$\sum_{p=-\infty}^{\infty} f(p) = -\frac{i}{2}\oint dz\,f(z)\,\cot(\pi z) \tag{E.1}$$

rotating and scaling the integration variable through the substitution $z \to ixz$, where $x = -\mu\alpha_3$, turns the cotangent into $\coth(\pi xz)$ which can be set to one in the large x limit. If the summand $f(z)$ has no poles on the real axis, the procedure simply replaces p by $p' = x\,p$ and integrates,

$$\sum_{p=-\infty}^{\infty} f(p) = \int_{-\infty}^{\infty} dp' f(p') \tag{E.2}$$

yielding the large x behavior. If there are poles on the real axis, one must evaluate their residue using the integrand in (E.1) and then integrate along any cut which $f(z)$ may possess along the imaginary axis. The essential point here is that the cut integrals are always sub-leading compared to the residues, so that the sum is dominated by the summand's behavior at the poles. For the sums which concern us in this paper, poles come from factors of $\widetilde{N}^{3\,r}_{n\,p}$ which as far as power counting in x is concerned should simply be ignored, replacing the summation variable p with it's value at the pole everywhere in the summand and adding some factors of π and $\cot(\pi n)$. By contrast sums not involving $\widetilde{N}^{3\,r}_{n\,p}$ can be evaluated by straightforward application of (E.2).

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