Wilson loops in 3-dimensional N=6 supersymmetric Chern-Simons Theory and their string theory duals

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Abstract

We study Wilson loops in the three-dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons theory recently constructed by Aharony, Bergman, Jafferis and Maldacena, that is conjectured to be dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. We construct loop operators in the Chern-Simons theory which preserve 1/6 of the supercharges and calculate their expectation value up to 2-loop order at weak coupling. The expectation value at strong coupling is found by constructing the string theory duals of these operators. For low dimensional representations these are fundamental strings, for high dimensional representations these are D2-branes and D6-branes. In support of this identification we demonstrate that these string theory solutions match the symmetries, charges and the preserved supersymmetries of their Chern-Simons theory counterparts.

1 Introduction

This work focuses on supersymmetric Wilson loop operators in the three-dimensional Chern-Simons (CS) theory of Aharony, Bergman, Jafferis and Maldacena [1]. This theory is conjectured to represent the low-energy dynamics of N coincident M2-branes at a $\mathbb{Z}_k$ orbifold of the transverse $\mathbb{R}^8$ space. This in turn has an alternative description as a weakly coupled type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ (or more generally M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$).

There are several reasons to focus on Wilson loop operators. They can be defined in any gauge theory and in the case of pure Chern-Simons theory, which is topological, they are the principal observables. While the theory of [1] includes additional matter fields, Wilson loops are still very natural observables. Furthermore, these operators play an important role in the AdS/CFT correspondence [2], since they are dual to semiclassical strings in the dual supergravity background [3,4]. Lastly, in the case of $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory in four dimensions, the expectation value of the 1/2 BPS circular Wilson loop is a non-trivial function of the 't Hooft coupling λ and the rank of the gauge group N, yet it can be calculated exactly and matched with string theory [5–7]. It is therefore interesting to see if an analog observable exists in the 3-dimensional theory.

The supersymmetric Chern-Simons theory has two gauge groups of equal rank N and opposite level k and $-k$. In addition to the gauge fields there are bosonic and fermionic fields C_I and ψ_I respectively in the bi-fundamental $(\mathbf{N}, \bar{\mathbf{N}})$ representation of the gauge groups and their complex conjugates.

With two gauge groups and this matter content there are quite a few possibilities to construct gauge-invariant Wilson loop operators. One choice would simply be the standard Wilson loop operator in one of the gauge groups (with gauge field A_μ or $\hat{A}_\mu$)

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp\left(i \int A_\mu dx^\mu \right). \tag{1.1}$$

Our experience from $\mathcal{N}=4$ SYM in 4-dimensions suggests that such a Wilson loop is not supersymmetric, which can be verified by a direct calculation.

In the four dimensional theory a supersymmetric Wilson loop couples also to an adjoint scalar field [4,8]. Here there are no adjoint fields, but we can use two bi-fundamental fields to construct a composite in the adjoint

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \int \left(i A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| M_J^I C_I \bar{C}^J \right) ds\,. \tag{1.2}$$

M_J^I is a matrix whose properties will be determined by supersymmetry. This is the Wilson loop operator we shall focus on.

With the appropriate choice of M_J^I, this Wilson loop will turn out to preserve 1/6 of the supercharges (4 out of 24) when the path of the loop is a straight line or a circle. In the first case it has a trivial expectation value, but not in the case of the circle, where we calculate it to 2-loop order in the gauge theory and to leading order at strong coupling. For arbitrary shape it will not preserve global supersymmetry, but we still expect it to be the natural observable with a simple description in the string theory dual.

The next section studies the Wilson loop in the gauge theory and the following section does the same from the string-theory side.

In the course of this work we have learnt that some of our results were independently obtained by several other groups [9–11].

2 Gauge theory construction

In this section we study the Wilson loop (1.2). We classify the conditions for it to be supersymmetric, derive the perturbative expression for this Wilson loop and calculate it at two loop order in an expansion in the 't Hooft coupling $\lambda = N/k$.

2.1 Supersymmetry

The $\mathcal{N}=6$ CS theory has 12 Poincaré supercharges $(Q_{IJ})_\alpha = -(Q_{JI})_\alpha$, where $I, J = 1, \dots, 4$, and the spinor index takes the values $\alpha = 1, 2$. Along with the 12 superconformal supercharges S_{IJ}, to be discussed below, these make up the 24 supersymmetries of the theory. From [13] we have the supersymmetry transformations of the bosonic fields of the theory

$$\begin{aligned}
\delta C_K &= (\theta^{IJ} Q_{IJ})\, C_K = \theta^{IJ}\, \varepsilon_{IJKL} \bar{\psi}^L ,\\
\delta \bar{C}^K &= (\theta^{IJ} Q_{IJ})\, \bar{C}^K = \theta^{IJ} \left(\delta_I^K \psi_J - \delta_J^K \psi_I\right),\\
\delta A_\mu &= (\theta^{IJ} Q_{IJ})\, A_\mu = \frac{2\pi i}{k}\, \theta^{IJ}\, \sigma_\mu (C_I \psi_J - C_J \psi_I + \varepsilon_{IJKL} \bar{\psi}^K \bar{C}^L),\\
\delta \hat{A}_\mu &= (\theta^{IJ} Q_{IJ})\, \hat{A}_\mu = \frac{2\pi i}{k}\, \theta^{IJ}\, \sigma_\mu (\psi_J C_I - \psi_I C_J + \varepsilon_{IJKL} \bar{C}^L \bar{\psi}^K).
\end{aligned} \tag{2.1}$$

with the Poincaré supersymmetry parameter $(\theta^{IJ})^\alpha$. We note the complex conjugation properties $\bar{C}^K = (C_K)^\dagger$, $\bar{\psi}^K = (\psi_K)^\dagger$ and $(\theta^{IJ})^\dagger = \frac{1}{2} \varepsilon_{IJKL}\, \theta^{KL}$.

Let us then consider the supersymmetry variation of the Wilson loop (1.2) and demand that it vanishes for a suitable choice of the θ^{IJ}. One then finds the following condition

$$\begin{aligned}
\delta W \sim\ & \theta_\alpha^{IJ} \left[-\dot{x}_\mu\, \sigma^\mu_{\alpha\beta}\, \delta_I^P + |\dot{x}|\, \delta_{\alpha\beta}\, M_I^P\, \right] C_p\, (\psi_J)_\beta \\
& + \epsilon_{IJKL}\, \theta_\alpha^{IJ} \left[\dot{x}_\mu\, \sigma^\mu_{\alpha\beta}\, \delta_P^K + |\dot{x}|\, \delta_{\alpha\beta}\, M_P^K\, \right] (\bar{\psi}^L)_\beta\, \bar{C}^P = 0
\end{aligned} \tag{2.2}$$

For a supersymmetric loop both terms in the above have to vanish seperately. Let us then consider a straight space-like Wilson line in the 1 direction, i.e. $x^\mu(s) = \delta^{1\mu}\, s$ and decompose the above equation with respect to the projectors $P_\pm = \frac{1}{2}(1 \pm \sigma^1)$. We then find from (2.2) the conditions

$$\theta_+^{IJ} \, (-\delta_I^P + M_I^P) + \theta_-^{IJ} \, (\delta_I^P + M_I^P) = 0\,, \tag{2.3}$$

$$\varepsilon_{IJKL}\, \theta_+^{IJ} \, (\delta_P^K + M_P^K) + \varepsilon_{IJKL}\, \theta_-^{IJ} \, (-\delta_P^K + M_P^K) = 0\,, \tag{2.4}$$

where $\theta^{IJ}_\pm = P_\pm \theta^{IJ}$. To analyze the possible solutions it is simplest to start with one specific supercharge, parameterized without loss of generality by a non-vanishing θ^{12}_+. This choice implies

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad M^I_J C_I \bar{C}^J = C_1 \bar{C}^1 + C_2 \bar{C}^2 - C_3 \bar{C}^3 - C_4 \bar{C}^4 \,. \tag{2.5}$$

It is simple to see that this choice of M^I_J then allows for one more independent non-vanishing supercharge, parameterized by θ^{34}_-.

This Wilson loop operator is therefore invariant under two out of the 12 Poincaré supersymmetries, i.e. 1/6 of the super-Poincaré generators are preserved[1].

Let us now turn to the 12 super-conformal symmetries $(S^{IJ})_\alpha = -(S^{JI})_\alpha$. The superconformal transformations of the $\mathcal{N}=6$ CS theory have been constructed recently in [14]. For the transformations of the bosonic fields the only change with respect to (2.1) is the replacement $\theta^{IJ} \to x \cdot \sigma\, \eta^{IJ}$, while the super-conformal transformations of the fermionic fields receive an additional contribution. This additional term, however, does not affect the variation of the Wilson loop operator (1.2) and the super-conformal analogue of the above Wilson line analysis then results in the simple replacement of $\bar{\theta}^{IJ} \to \bar{\eta}^{IJ} s\, \sigma^1$ in (2.3) and (2.4). Hence, also two of the 12 super-conformal symmetries are intact and we indeed find that the Wilson line operator (1.2) is 1/6 BPS.

This analysis is valid for an infinite straight line. Under a conformal transformation a line will be mapped to a circle, which will therefore posses the same number of supersymmetries. The conformal transformation mapping the line to the circle mixes the super-Poincaré and superconformal charges, hence the circular Wilson loop is invariant under a linear combination of $Q^{IJ} \pm S^{IJ}$.

These Wilson loops are invariant also under some bosonic symmetries, part of the $SO(4,1) \times SO(6)$ symmetry of the vacuum. There is an $SL(2,\mathbb{R}) \times U(1)$ subgroup of the conformal group comprised, in the case of the line, of translations along the line P_1, dilation D, a special conformal transformation K_1 and a rotation around the line, J. These generators combine with the supercharges to form the supergroup $OSp(2|2)$ (with a non-compact $Sp(2)$). In addition there is an extra $SU(2) \times SU(2)$ subgroup of the $SO(6)$ R-symmetry group rotating $C_1 \leftrightarrow C_2$ and $C_3 \leftrightarrow C_4$ that leaves M^I_J, and hence the Wilson loop, invariant. The supercharges, being in the antisymmetric representation of the R-symmetry group are neutral under this extra bosonic symmetry.

Thus far we have discussed space-like Wilson loops. For a straight time-like Wilson loop we find similar conditions, only that the matrix M will be imaginary. For a straight light-like line the scalar contribution to (1.2) vanishes, but the loop is still supersymmetric. In this case it is invariant under half of the super-Poincaré charges and all the super-conformal ones. The

[1] Note that if the sign of the δ^K_P terms in (2.4) was the opposite, the choice $M^I_J = \delta^I_J$ would preserve half the supercharges. Alas, this is not the case.

fact that the scalar coupling is real for a space-like curve, imaginary for a time-like one and vanishes for a light-like curve is familiar from $\mathcal{N}=4$ SYM in four dimensions [8].

Given a choice of supercharges it is an interesting question to ask what is the most general loop preserving it. We saw that the basic Wilson loop (1.2) with the geometry of a line or a circle preserves four real supercharges. Under this choice of supercharges the matrix M_J^I was fixed, as was the value of $\dot{x}_\mu$. So the loop is restricted to be a line in a fixed direction. Parallel lines will preserve the same super-Poincaré charges, but different superconformal ones.

Thus the choice of four supercharges completely fixes the geometry of the loop. However, this does not mean that there is only a unique Wilson loop preserving these supercharges, there are different ones with the same geometry but in different representations of the gauge groups.

In (1.2) we chose one of the gauge groups, but a similar operator exists also in the other group. In that case instead of $C_I\bar{C}^J$ the scalar bilinear will be of the opposite order $\bar{C}^J C_I$. More generally, we can take the Wilson loop to be in any representation of each of the gauge groups, so the most general Wilson loop will be characterized by a pair of Young tableau for the representations R and $\hat{R}$

$$W^{\pm}_{R\hat{R}} = \frac{1}{2}\left[\mathrm{Tr}_R\mathcal{P}e^{\int\left(iA_\mu\dot{x}^\mu+\frac{2\pi}{k}|\dot{x}|M_J^I C_I\bar{C}^J\right)ds} \pm \widehat{\mathrm{Tr}}_{\hat{R}}\mathcal{P}e^{\int\left(i\hat{A}_\mu\dot{x}^\mu+\frac{2\pi}{k}|\dot{x}|\hat{M}_J^I\bar{C}^J C_I\right)ds}\right]. \tag{2.6}$$

This in fact over-counts the number of Wilson loops. Recall that in Chern-Simons theory there are 't Hooft vertices which are in the k'th symmetric representation [15, 16]. These are important to create some of the local gauge invariant states in the theory [1], but they also affect the Wilson loops. Since they can be added freely, they essentially identify representations which are related to each-other by multiplication by the k'th symmetric representation. Thus they reduce the number of distinguished Wilson loop observables to be those given by Young tableau with fewer than k columns.

Furthermore, it could also be quite difficult to find all of those different Wilson loops in the supergravity limit. In similar cases (like in orbifolds of $\mathcal{N}=4$ of SYM) only the Wilson loops that are symmetric under interchange of the gauge groups have a known simple description. In this theory the most natural operator of the type (2.6) is the one that is symmetric under the exchange of the two gauge groups, while exchanging also the representation with its conjugate (since the matter is in the fundamental - anti-fundamental).

We expect therefore our string theory solutions presented in Section 3 to correspond to this linear combination of Wilson loops in the two gauge groups. The leading planar contribution will be a single string, dual to a single-trace Wilson loop (or a multiply wrapped Wilson loop). For very large representations the planar approximation breaks down and the fundamental string should be supplanted by D-branes.

2.2 Perturbative calculation

Let us now turn to the perturbative evaluation in $\lambda = N/k$ of the 1/6 BPS Wilson loop (1.2) for circular and straight line contours. We shall work in Euclidean space. At leading order in λ

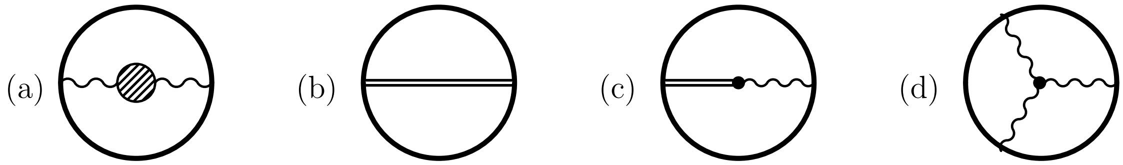

Figure 1: The two-loop Feynman diagrams contributing to a circular $\langle W\rangle$. The bold circular line represents the Wilson loop contour, whereas wiggly lines denote gluon and straight lines scalar propagators.

the only possible contribution is from a tree-level gluon exchange which is identical to that in pure CS theory. The result is rather subtle and depends on the "framing" of the Wilson loop, which is extra information needed to define it beyond the path of the loop (c.f. [17–19]). We will take a slightly naive approach; since the gluon propagator is proportional to the antisymmetric epsilon tensor, it vanishes for all loops lying in a plane. This corresponds to zero framing. A possible additional subtlety arises from the self contraction of the two scalars fields at leading order. We take them to be defined as normal ordered. Hence there is no contribution at leading order in λ.

Expanding W to second order we need the one-loop corrected Feynman gauge gluon propagator and the bare scalar propagator calculated in Appendix A (see also [20])

$$\begin{aligned}\langle A_\mu(x)_{ij}A_\nu(y)_{kl}\rangle &= \delta_{ik}\delta_{jl}\frac{1}{k}\left[-i\frac{\varepsilon_{\mu\nu\rho}(x-y)^\rho}{2|x-y|^3}+\frac{N}{k}\left(\frac{\delta_{\mu\nu}}{|x-y|^2}-\partial_\mu\partial_\nu\ln|x-y|\right)\right],\\ \langle (C_I)_{i\hat{i}}(x)\,(\bar{C}^J)_{\hat{j}j}(y)\rangle &= \delta_I^J\,\delta_{ij}\,\delta_{\hat{i}\hat{j}}\,\frac{1}{4\pi|x-y|}\,.\end{aligned} \tag{2.7}$$

At this two-loop order one finds that in the loop-to-loop propagator the propagator of the composite scalar $M_J^I\,\bar{C}^J C_I$, diagram (b), combines with the one-loop piece of the gauge field propagator, diagram (a), to give

$$\mathcal{D}[x_1(\tau_1),x_2(\tau_2)] \equiv -\frac{N^3}{k^2}\left[\frac{\dot{x}_1\cdot\dot{x}_2-|\dot{x}_1||\dot{x}_2|}{(x_1-x_2)^2}-\partial_{\tau_1}\partial_{\tau_2}\ln|x_1-x_2|\right]. \tag{2.8}$$

We would like to point out a subtlety in the last term, which being a total derivative can be removed by a gauge transformation – albeit a singular one. Depending on the regularization it may lead to divergences along the loop, as we do not expect divergencies for the supersymmetric Wilson loop we conclude that it should be dropped. Also note that the scalar contribution is insensitive to the choice of signs in the ± 1 entries of the diagonal M_J^I as these come in squares. One sees that for a straight line this yields a vanishing effective propagator, while for the circle it gives a constant propagator $\mathcal{D}=N^3/(2k^2)$ somewhat similar to the situation in four dimensional $\mathcal{N}=4$ super Yang-Mills. Thus this contribution gives at $\mathcal{O}(k^{-2})$

$$\frac{1}{N}\frac{1}{2!}\oint d\tau_1\oint d\tau_2\,\frac{N^3}{2k^2}=\frac{\pi^2N^2}{k^2}\,. \tag{2.9}$$

There are two other diagrams contributing at $\mathcal{O}(k^{-2})$. The diagram (c) is the interaction between a scalar bilinear $C_I\,\bar{C}^J$ and a gauge field

$$\begin{aligned}&\left\langle \frac{2}{N}\mathrm{Tr}\oint d\tau_1\,d\tau_2\,i\dot{x}_1^\mu|\dot{x}_2|\frac{M_J^I}{k}A_\mu(x_1)C_I\bar{C}^J(x_2)\int d^3w\,\mathrm{Tr}\,\left(i\partial_\rho C_K A_\rho \bar{C}^K - i\partial_\rho \bar{C}^K C_K A_\rho\right)\right\rangle \\ &\qquad\propto \oint d\tau_1\,d\tau_2\int d^3w\,\varepsilon_{\mu\nu\rho}\,\dot{x}_1^\mu\,|\dot{x}_2|\,(x_1-w)^\rho\frac{1}{|x_1-w|^3|x_2-w|}\frac{\partial}{\partial w^\nu}\frac{1}{|x_2-w|}=0\,.\end{aligned} \tag{2.10}$$

It is zero because the integrand is odd in the third component of w, i.e. the component orthogonal to the plane of the circular loop.

The remaining diagram (d) is an interaction of three gauge fields through the Chern-Simons interaction. This graph appears also in pure Chern-Simons theory and its value depends only on the topology of the loop. The circle is an "unknot", for which the result is $-N^2\pi^2/(6k^2)$ [18].

Putting together the $\mathcal{O}(k^{-2})$ contributions we find

$$\langle W\rangle = 1+\frac{\pi^2N^2}{k^2}-\frac{\pi^2N^2}{6k^2}+\mathcal{O}(k^{-3})\,, \tag{2.11}$$

where we have separated the $O(k^{-2})$ contribution into two terms, one from the combined gauge-field and scalar exchange and the second, the topological contribution identical to pure Chern-Simons.

So far we discussed the Wilson loop in one of the two $U(N)$ factors, but it makes sense to consider the linear combination of the operators in the two groups (2.6). In particular we expect the string theory duals to be symmetric under the exchange of the two groups. As mentioned before, one would be lead to take the Wilson loop in the conjugate representation, which can be simply expressed as the usual Wilson loop with an overall sign reversed.

The perturbative calculation for the second gauge group is identical to the first up to some sign changes. The sign of the level k is reversed, which will change the signs of the propagators and the interaction vertex. The total number of them in all of the graphs of order λ^2 is always even, so that will not create any change. But the overall sign in the Wilson loop is also reversed which will affect the signs of the graphs where the loop was expanded to odd-order. In our case there is only one such graph, Fig. 1d. This is the graph that gave the pure CS contribution.

Therefore at the 2-loop order if we consider the two possible linear combinations of the loops in the two gauge groups in the fundamental and anti-fundamental representations, the sum of the two will not include the CS term and the difference will include only the CS contribution[2].

For the 1/2 BPS circular Wilson loop in $\mathcal{N}=4$ in four-dimensions the gauge field and scalar propagators combined to a constant, similar to what we have found here at order λ^2. In four dimensions the interactions also cancel and the full answer is given by summing over the free constant propagators, i.e. a zero-dimensional Gaussian matrix model [5, 6]. In that case the result in the planar approximation can be expressed in terms of a Bessel function

$$\langle W_{\mathcal{N}=4}\rangle_{\mathrm{planar}}\sim\frac{2}{\sqrt{\lambda}}\,I_1(\sqrt{\lambda})=\begin{cases}1+\frac{1}{8}\,\lambda+\dots & \text{for}\quad \lambda\ll 1\\ e^{\sqrt{\lambda}} & \text{for}\quad \lambda\gg 1\end{cases} \tag{2.12}$$

[2] The possibility for such a cancelation was first observed in [11], though for a somewhat different construction. See also [12]

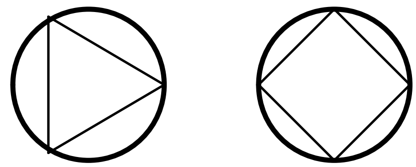

Figure 2: Some examples of higher level N-gon tree graphs.

In the case at hand it is clear that interactions do contribute. For one, the constant propagator $\mathcal{D} = N^3/(2k^2)$ emerged from the sum of the one-loop corrected gluon self-energy and the tree-level scalar exchange (2.8). Furthermore independent of that we also found the result of pure Chern-Simons. Another novel feature is that the tree-level graphs do not only have ladder structure. Rather, there will be in general tree level graphs of N-gon topology due to the biscalar coupling in the loop, see figure 2.

Let us also note that in the case of $\mathcal{N} = 4$ SYM in four dimensions there are other BPS Wilson loops preserving fewer supercharges whose perturbative expansions are rather complicated and do include interacting graphs. Still there is some evidence that they are given by the same answer as the circular Wilson loop (2.12), only with a rescaled coupling [23–26].

While there is no strong evidence for a simple cancelation, we still find it conceivable that the 1/6 BPS circular Wilson loop will also have an exact perturbative result that can be resummed to all orders, like the supersymmetric Wilson loops in four-dimensions. If indeed so, then to get a match with the string theory result in the next section the coupling in the analogeous matrix model result (2.12) would certainly have to be renormalized in some way.

3 String theory description

The three-dimensional $\mathcal{N} = 6$ CS theory is conjectured to be dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. To understand the action of the $\mathbb{Z}_k$ orbifold, one should write S^7 as a circle fibration over complex projective space $\mathbb{CP}^3$, where the orbifold acts on the fiber (see (3.6) below). For large k the radius of this "M-theory circle" becomes small, so the theory can be described in terms of type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ with string-frame metric

$$ds^2 = \frac{R^3}{4k}\left(ds^2_{AdS_4} + 4ds^2_{\mathbb{CP}^3}\right) . \tag{3.1}$$

We choose in this paper to work in the string theory picture, but all the solutions we describe below should also have an uplift to the full M-theory.

For the AdS_4 part we may use the global Lorentzian metric

$$ds^2_{AdS_4} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\big(d\theta^2 + \sin^2\theta\, d\psi^2\big) . \tag{3.2}$$

or as foliated by AdS_2 slices

$$ds^2_{AdS_4} = du^2 + \cosh^2 u\, ds^2_{AdS_2} + \sinh^2 u\, d\phi^2\,.$$
$$ds^2_{AdS_2} = \begin{cases} d\rho^2 - \cosh^2\rho\, dt^2\,, & \text{appropriate for a time-like line,} \\ d\rho^2 + \sinh^2\rho\, d\psi^2\,, & \text{appropriate for a space-like circular loop.} \end{cases} \tag{3.3}$$

The metric on $\mathbb{CP}^3$ can be written in terms of four complex projective coordinates z_i as

$$ds^2_{\mathbb{CP}^3} = \frac{1}{\rho^2}\sum_{i=1}^{4} dz_i\, d\bar{z}_i - \frac{1}{\rho^4}\left|\sum_{i=1}^{4} z_i\, d\bar{z}_i\right|^2, \qquad \rho^2 = \sum_{i=1}^{4}|z_i|^2\,. \tag{3.4}$$

In the following we choose a specific representations in terms of angular coordinates (used also in [27, 21]). We start by parametrizing $S^7 \subset \mathbb{C}^4$ as

$$\begin{aligned} z_1 &= \cos\frac{\alpha}{2}\cos\frac{\vartheta_1}{2}\, e^{i(2\varphi_1+\chi+\zeta)/4}\,, \\ z_2 &= \cos\frac{\alpha}{2}\sin\frac{\vartheta_1}{2}\, e^{i(-2\varphi_1+\chi+\zeta)/4}\,, \\ z_3 &= \sin\frac{\alpha}{2}\cos\frac{\vartheta_2}{2}\, e^{i(2\varphi_2-\chi+\zeta)/4}\,, \\ z_4 &= \sin\frac{\alpha}{2}\sin\frac{\vartheta_2}{2}\, e^{i(-2\varphi_2-\chi+\zeta)/4}\,, \end{aligned} \tag{3.5}$$

The metric on S^7 is then given by

$$\begin{aligned} ds^2_{S^7} =& \frac{1}{4}\Bigg[d\alpha^2 + \cos^2\frac{\alpha}{2}(d\vartheta_1^2 + \sin^2\vartheta_1^2\, d\varphi_1^2) + \sin^2\frac{\alpha}{2}(d\vartheta_2^2 + \sin^2\vartheta_2^2\, d\varphi_2^2) \\ & + \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(d\chi + \cos\vartheta_1\, d\varphi_1 - \cos\vartheta_2\, d\varphi_2)^2 + \frac{1}{4}(d\zeta + A)^2\Bigg]\,, \end{aligned} \tag{3.6}$$
$$A = \cos\alpha\, d\chi + 2\cos^2\frac{\alpha}{2}\cos\vartheta_1\, d\varphi_1 + 2\sin^2\frac{\alpha}{2}\cos\vartheta_2\, d\varphi_2\,. \tag{3.7}$$

The angle ζ appears only in the last term and if we drop it we end up with the metric on $\mathbb{CP}^3$

$$\begin{aligned} ds^2_{\mathbb{CP}^3} = \frac{1}{4}\Bigg[& d\alpha^2 + \cos^2\frac{\alpha}{2}(d\vartheta_1^2 + \sin^2\vartheta_1^2\, d\varphi_1^2) + \sin^2\frac{\alpha}{2}(d\vartheta_2^2 + \sin^2\vartheta_2^2\, d\varphi_2^2) \\ & + \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(d\chi + \cos\vartheta_1\, d\varphi_1 - \cos\vartheta_2\, d\varphi_2)^2\Bigg]. \end{aligned} \tag{3.8}$$

The ranges of the angles are $0 \le \alpha, \vartheta_1, \vartheta_2 \le \pi$, $0 \le \varphi_1, \varphi_2 \le 2\pi$ and $0 \le \chi \le 4\pi$.

In addition to the metric, the supergravity background has the dilaton, and the 2-form and 4-form field strengths from the Ramond-Ramond (RR) sector

$$e^{2\Phi} = \frac{R^3}{k^3}\,, \qquad F_4 = \frac{3}{8}R^3\, d\Omega_{AdS_4}\,, \qquad F_2 = \frac{k}{4}\, dA\,. \tag{3.9}$$

Here $d\Omega_{AdS_4}$ is the volume form on AdS_4 and F_2 is proportional to the Kähler form on $\mathbb{CP}^3$.

To write down the general D-brane action in this background one also needs the potentials for these forms. The one-form potential is, up to gauge transformations

$$C_1 = \frac{k}{4}\,A\,, \tag{3.10}$$

With A defined in (3.7).

C_3, the three-form potential for F_4 will actually not play a role in our current calculations, but we write it down for completeness. The forms are defined in principle only up to a gauge choice, but since C_3 involves the non-compact directions and it may couple to branes that approach the boundary of space, one should impose a proper asymptotic behavior on it. It seems like the analog of choosing Fefferman-Graham coordinates [28] is to take the 3-form to not have any component in the du direction in the coordinate systems in (3.3). Such a prescription indeed gave the correct result in $\mathcal{N}=4$ SYM in four dimensions [29][3]. We therefore have for the three-form potential

$$C_3 = \frac{1}{8}R^3\cosh^3 u \times \begin{cases} \cosh\rho\, dt\wedge d\rho\wedge d\phi\,, & \text{appropriate for a time-like line} \\ \sinh\rho\, d\psi\wedge d\rho\wedge d\phi\,, & \text{appropriate for a space-like circular loop.} \end{cases} \tag{3.11}$$

The dual of F_4 is proportional to the volume form on $\mathbb{CP}^3$

$$F_6 = \star F_4 = \frac{3R^6}{2^8k}\sin^3\alpha\sin\vartheta_1\sin\vartheta_2\; d\alpha\wedge d\vartheta_1\wedge d\vartheta_2\wedge d\chi\wedge d\varphi_1\wedge d\varphi_2\,. \tag{3.12}$$

The five-form potential for F_6 can then be written as

$$C_5 = -\frac{R^6}{2^8k}\,(\sin^2\alpha\cos\alpha + 2\cos\alpha - 2)\sin\vartheta_1\sin\vartheta_2\; d\vartheta_1\wedge d\vartheta_2\wedge d\chi\wedge d\varphi_1\wedge d\varphi_2\,. \tag{3.13}$$

Here we chose a gauge that is regular at $\alpha=0$. Reversing the sign on the -2 term in the parentheses gives the gauge that is regular at $\alpha=\pi$.

The relation between the parameters of the string background and of the field theory are (for $\alpha'=1$)

$$\frac{R^3}{4k} = \pi\sqrt{\frac{2N}{k}} = \pi\sqrt{2\lambda}\,. \tag{3.14}$$

3.1 Fundamental string

In the strong coupling description of $\mathcal{N}=4$ SYM in terms of type IIB string theory on $AdS_5\times S^5$, a Wilson loop in the fundamental representation is given by a fundamental string ending along the path of the loop on the boundary of space. We expect this property to extend from $\mathcal{N}=4$ in four dimensions to our Wilson loops in the 3-dimensional CS theory.

In $\mathcal{N}=4$ SYM the natural Wilson loop carries an $SO(6)$ vector index, representing its position on S^5, the analog for $\mathbb{CP}^3$ would be the fundamental representation of $SU(4)$, though

[3] See a more detailed discussion in [30].

we saw that the 1/6 BPS Wilson loop couples to two scalars, one in the **4** representation and the other in the $\bar{\mathbf{4}}$ with a matrix M_J^I. This matrix breaks $SU(4) \to SU(2) \times SU(2)$, so the string theory dual should not be localized at a point on $\mathbb{CP}^3$ (which would break $SU(4) \to U(3)$) but rather smeared along a $\mathbb{CP}^1$.

Still, if the scalar couplings are constant along the loop, we can forget about the $\mathbb{CP}^3$ part of the σ-model and focus on AdS_4. Any known string solution found in AdS_5 which can be embedded within an AdS_4 subspace is immediately a solution for this theory. So, many results that were derived for Wilson loops in $\mathcal{N} = 4$ SYM are valid also for our Wilson loops in $\mathcal{N} = 6$ CS. For example the expressions for the anti-parallel lines ("quark - anti-quark potential" [3, 4]) and for the light-like cusp [31] are exactly the same in the planar limit up to the change $\lambda_{\mathcal{N}=4} \to 2\pi^2 \lambda_{CS}$. A similar result for the cusp anomalous dimension was obtained from rotating strings in [1].

In this paper we focus on supersymmetric configurations of straight lines or circles. The analog of the straight line on the $S^2 \times \mathbb{R}$ boundary of global AdS_4 (3.2) is a pair of anti-parallel lines at antipodal points on S^2 (or in the coordinate system (3.3) one sets $u = 0$). The string solution describing them is an AdS_2 subspace spanned by the coordinates ρ and t. After subtracting a divergence, the resulting action vanishes, meaning that the expectation value of the Wilson loop is unity.

To describe the circular Wilson loop one could use the Poincaré patch metric, as was done in [32, 8], or use global AdS_4 and for simplicity take the circle to wrap a big circle on S^2, i.e. $\theta = \pi/2$ at constant time t (or $u = 0$ in the metric (3.3)). The string solution will now be a Euclidean AdS_2 section spanned by ρ and ψ. The action is proportional to the area

$$\mathcal{S}_{\text{string, cl.}} = \frac{R^3}{8\pi k} \int_0^{2\pi} d\psi \int_0^{\rho_0} d\rho \, \sinh\rho = \pi\sqrt{2\lambda}\,(\cosh\rho_0 - 1)\ . \tag{3.15}$$

Here ρ_0 is a cutoff near the boundary of AdS_2 (which is also at the boundary of AdS_4) and we expect the divergent term to be removed by a boundary term as in [8]. Using the standard AdS/CFT dictionary we derive

$$\langle W \rangle_{\text{string}} \sim e^{\pi\sqrt{2\lambda}}\,. \tag{3.16}$$

As mentioned before, this string would not be localized on $\mathbb{CP}^3$, but has to be smeared on a $\mathbb{CP}^1$. This can be the sphere parameterized by ϑ_1 and φ_1 at $\alpha = 0$ in the coordinate system (3.8). As mentioned before, in the string theory picture there isn't a simple way of distinguishing between the two gauge groups. We expect this string (as well as the D-branes discussed below) to correspond to a linear combination of Wilson loops which is symmetric under the exchange of the two gauge groups. Note that this is also the combination in the gauge theory where the pure Chern-Simons term at order λ^2 dropped out.

The uplift of this string solution to M-theory is straight forward.

3.2 D2-brane

In $\mathcal{N} = 4$ SYM in four dimensions a Wilson loop in a low dimensional representation is well represented at strong coupling by a free string in $AdS_5 \times S^5$. For representations of dimension of order N a better description is in terms of D3-branes (the symmetric representation) or D5-branes (antisymmetric) [3,29,33–37]. This is the Wilson loop version of a giant-graviton [38–40], sometimes also referred to as "giant Wilson loop." For even higher dimensional representations the branes back-react on the geometry and one instead finds "bubbling geometries" [41–46].

In this subsection we present a D2-brane solution that is a possible candidate for a dual of Wilson loops. In the next subsection we present a D6-brane solution. In support of the identification with Wilson loop operators are their symmetries, their charges, classical action, and the supercharges they preserve.

Since the Wilson loop has an $SL(2,\mathbb{R})$ symmetry we expect the D2-brane to have an AdS_2 factor, which will be inside AdS_4. The third world-volume direction will be compact — a circle. We therefore take as world-volume coordinates ρ, t from (3.2) (or alternatively ρ and t, or ρ and ψ from (3.3) with $u = 0$) and a third world-volume coordinate τ of period 2π.

We have found a few different solutions to the equations of motion of the D2-brane with this circle made of the ϕ circle at non-zero u in AdS_4 (3.3) and/or a circle inside $\mathbb{CP}^3$ similar to those of [47]. While the experience from $AdS_5 \times S^5$ might lead one to suspect that the dual of the Wilson loop in the symmetric representation should have the circle inside AdS_4, these solutions have a different gauge-theory interpretation [48]. The most likely candidate for a dual of the Wilson loop has the circle inside $\mathbb{CP}^3$.

Since our Wilson loops have an $SU(2) \times SU(2)$ symmetry which acts by rotating z_1 into z_2 and z_3 into z_4, it is natural to take the circle to be in the χ direction, i.e. $\chi = -2\tau$ (recall that χ has period 4π and the choice of sign seems to be dictated by suspersymmetry). We would still need to set its location in terms of the other angles ϑ_1, φ_1, ϑ_2, φ_2 and α. For now we take all of them to be constants, which seems to be a consistent ansatz. At the end, in order to restore the $SU(2) \times SU(2)$ symmetry (and the correct supersymmetry) we will smear the brane over the ϑ_1, ϑ_2, φ_1 and φ_2 directions.

The action includes the Dirac-Born-Infeld (DBI) piece and the Wess-Zumino (WZ) coupling

$$\mathcal{S}_{\rm D2} = T_{\rm D2} \int e^{-\Phi} \sqrt{\det(g + 2\pi\alpha' F)} + T_{\rm D2} \int \Big[P[C_3] + 2\pi i \alpha' P[C_1] \wedge F \Big]. \tag{3.17}$$

Here g is the induced metric and F is the intrinsic field strength on the world-volume. To describe a Wilson loop, which carries electric charge the component $F_{t\rho} = E \cosh\rho$ will be non-zero, in the Lorentzian case. For the dual of the space-like circular loop, which is the case we work out in detail, it will instead be $F_{\psi\rho} = E \sinh\rho$. Being that it represents an electric field and that the signature is Euclidean, it is imaginary. $P[C_3]$ is the pullback of the RR three-form potential, which vanishes on our configuration and $P[C_1]$ is the pullback of the one-form. The last term comes with an i again due to the fact that we are in Euclidean signature.

After fixing all the other angles, the angles α and $\chi/2$ parameterize an S^2 of radius $1/2$. The field-strength F_2 in (3.9) is that of $k/2$ Dirac monopoles, but the one-form (3.10) with A

as in (3.7) is singular at both $\alpha = 0$ and $\alpha = \pi$. Instead we take

$$C_1 = \frac{k}{4}(\cos\alpha - 1)\, d\chi\,, \tag{3.18}$$

which is regular at $\alpha = 0$. The same expression with $(\cos\alpha + 1)$ will be regular at $\alpha = \pi$.

Plugging our ansatz in we find

$$\mathcal{S}_{\rm D2} = \frac{T_{\rm D2} R^3}{8} \int d\rho\, d\psi\, d\tau\, \sinh\rho \left[\sin\alpha\sqrt{1 + \beta^2 E^2} - i\beta E(\cos\alpha - 1)\right]\,, \tag{3.19}$$

with $\beta = 8\pi k/R^3 = \sqrt{2/\lambda}$ (setting $\alpha' = 1$). and note that we are using conventions where the D2-brane tension is $T_{\rm D2} = 1/4\pi^2$.

The equation of motion for α allows it to be an arbitrary constant but gives the relation

$$i\beta E = -\cos\alpha\,. \tag{3.20}$$

The gauge field is a cyclic variable and the flux through the brane is proportional to the conjugate momentum

$$p = -4\pi i\,\frac{\delta\mathcal{L}}{\delta F} = \frac{k}{2}\,. \tag{3.21}$$

Now we wish to evaluate the action on this classical solution. As is explained in [29], the action as it stands does not give the correct classical value, since it is a functional of the electric field and one should take a Legendre transform to replace E by p. The result is

$$\mathcal{S}_{\text{L.T, classical}} = \mathcal{S}_{\text{clasical}} - pE = \frac{R^3}{8}\int d\rho\, \sinh\rho = \frac{k}{2}\,\pi\sqrt{2\lambda}(\cosh\rho_0 - 1) \tag{3.22}$$

Once we remove the divergence from large ρ, we see that this solution agrees with that of $k/2$ fundamental strings.

The charge and action agree exactly with that of $k/2$ fundamental strings, while the angle α is completely arbitrary. To see if there are solutions with $|p| < k/2$ it is useful to consider the Legendre transform before solving the equations of motion. The action in terms of p is

$$\mathcal{S}_{\text{L.T.}} = \mathcal{S}_{\rm D2} - pE = \frac{T_{\rm D2}R^3}{8}\int d\rho\, d\psi\, d\tau\, \sinh\rho\sqrt{p^2 + \frac{k}{2p^2}(k - 2p)(1 - \cos\alpha)}\,. \tag{3.23}$$

The equation of motion for α gives

$$(k - 2p)\sin\alpha = 0\,, \qquad \mathcal{S}_{\text{L.T, classical}} = -p\,. \tag{3.24}$$

So either the solution has $p = k/2$ and arbitrary α or $\sin\alpha = 0$ and p is arbitrary. The first case is the solution presented before, while in the second it is not justified to use the D2-brane description, since it is singular, and a better description is in terms of p fundamental strings.

Note that the two gauge choices for C_1 change the string charge by k, meaning that the charge is defined only modulo k. This is in agreement with the expectation from the gauge theory, where the k-th symmetric representation is analogous to the trivial one by the inclusion of an 't Hooft vertex.

It seems like the only regular configuration describes $k/2$ coincident Wilson loops (or a Wilson loop in the $k/2$ symmetric representation). We found singular solutions for other charges, but it is possible that our ansatz was too restrictive and that there are other regular solutions for arbitrary charges. We note here that also in $AdS_5 \times S^5$, while there are many explicit solutions for giant gravitons with fewer than 16 supercharges (see e.g. [49]), only one class is known for 1/4 BPS Wilson loops [50], so it is not too surprising if we cannot classify all possible D-branes dual to the 1/6 BPS Wilson loops in the three-dimensional theory.

Furthermore note that usually the D-brane description of gauge theory operators is valid for representations of order N. The type IIA description is valid though for large $\lambda = N/k$, so a symmetric representation, whose dimension is capped by k, cannot approach N. This may explain why we find a regular solution only at the maximal value of p.

3.3 D6-brane

The D2-brane solution seems to correspond to a Wilson loop in the symmetric representation, similar to the D3-brane in $AdS_5 \times S^5$. There a Wilson loop in the anti-symmetric representation was described by a D5-brane, and the analog in our case is a D6-brane. We present the solution here.

This D6-brane will wrap a 5-dimensional submanifold of $\mathbb{CP}^3$, which we choose to have explicit $SU(2) \times SU(2)$ symmetry, as does the gauge theory operator.

Like the string and the D2-brane, the D6-brane will span an $AdS_2 \subset AdS_4$. As usual, for the time-like Wilson line on antipodal points on S^2 it is parameterized by ρ and t, while for the circular loop it is parameterized by ρ and ψ. Inside $\mathbb{CP}^3$ it will extend in the χ, ϑ_1, φ_1, ϑ_2 and φ_2 directions at constant α. We also turn on an electric flux proportional to the volume form on AdS_2, so either $F = E\cosh\rho\, dt \wedge d\rho$, or $F = E\sinh\rho\, d\psi \wedge d\rho$.

The straight-line case will give a zero answer while the circle should give a non-trivial result. Due to that and the fact that the calculations are essentially identical, we write here the details for the case of the circle.

The action for this brane will include the DBI piece, as usual, and the Wess-Zumino term coupling the pullback of C_5 (3.13) to the world-volume field strength $F_{\psi\rho} = E\sinh\rho$

$$\mathcal{S}_{\rm D6} = T_{\rm D6} \int \Big[e^{-\Phi} \sqrt{\det(g + 2\pi\alpha' F)} + 2\pi i P[C_5] \wedge F \Big] . \qquad (3.25)$$

Plugging in our ansatz we find

$$\mathcal{S}_{\rm D6} = \frac{R^9 T_{\rm D6}}{2^{10} k^2} \int \sinh\rho \sin\vartheta_1 \sin\vartheta_2 \Big[\sin^3\alpha \sqrt{1 + \beta^2 E^2} - i\beta E\Big((\sin^2\alpha + 2)\cos\alpha - 2 \Big) \Big] . \qquad (3.26)$$

Here $\beta = 8\pi k/R^3 = \sqrt{2/\lambda}$ and $T_{\rm D6} = 1/(2\pi)^6$.

Integrating over the five remaining coordinates on $\mathbb{CP}^3$ gives a factor of $2^6\pi^3$ and we are left with an effective theory on AdS_2. Now the equation of motion for α fixes the value of E

$$i\beta E = -\cos\alpha\,. \tag{3.27}$$

The string charge carried by the D6-brane is the conjugate to the gauge field

$$p = -i\frac{\delta S}{\delta E} = \frac{\pi^3 R^9 T_{\rm D6}\beta}{8k^2}(1-\cos\alpha) = \frac{N}{2}(1-\cos\alpha)\,, \tag{3.28}$$

where we used that $N = R^6/(32\pi^2 k)$. The value of p ranges between 0 and N, where the appearance of N is a manifestation of the "stringy exclusion principle," and is an indication that this D-brane represents Wilson loops in anti-symmetric representations.

Now we evaluate the classical action by performing a Legendre transform, replacing the electric field with its conjugate p. We also integrate over AdS_2 which gives a divergent answer, but whose regularized area is -2π

$$\mathcal{S}_{\rm L.T.} = \mathcal{S} - ipE = -\frac{\pi^4 R^9 T_{\rm D6}}{8k^2}\,\sin^2\alpha = -\pi\sqrt{2\lambda}\,\frac{p\,(N-p)}{N}\,. \tag{3.29}$$

This is indeed symmetric under $p \leftrightarrow N-p$, as should be the case of the antisymmetric representation. Also for small p it agrees with the result of p fundamental strings.

This construction is very similar to the D5-brane in $AdS_5\times S^5$ but some of the details are different. Here the relation between the charge p and the angle α is trigonometric, while in the other case it is transcendental. Also the final answer (3.29) is much simpler in this case. Note that the Gaussian matrix model reproduced the D5-brane result, so any modification of it to match the Wilson loop in $\mathcal{N}=6$ CS should reproduce (3.29), once the relevant limit is taken (including non-planar corrections).

3.4 Supersymmetry

We turn now to checking the number of supersymmetries preserved by our string and D-brane solutions. We work in this section in Lorentzian signature and take the Wilson loop (and resulting AdS_2 surfaces) to be timelike.

As a first step one needs to choose a set of elfbeine and find the Killing spinors. This is done in Appendix B, where the Killing spinors of M-theory on $AdS_4\times S^7$ in our coordinate system are found to be

$$e^{\frac{\alpha}{4}(\hat\gamma\gamma_4-\gamma_{7\natural})}e^{\frac{\vartheta_1}{4}(\hat\gamma\gamma_5-\gamma_{8\natural})}e^{\frac{\vartheta_2}{4}(\gamma_{79}+\gamma_{46})}e^{-\frac{\xi_1}{2}\hat\gamma\gamma_\natural}e^{-\frac{\xi_2}{2}\gamma_{58}}e^{-\frac{\xi_3}{2}\gamma_{47}}e^{-\frac{\xi_4}{2}\gamma_{69}}e^{\frac{\rho}{2}\hat\gamma\gamma_1}e^{\frac{t}{2}\hat\gamma\gamma_0}e^{\frac{\theta}{2}\gamma_{12}}e^{\frac{\phi}{2}\gamma_{23}}\epsilon_0 = \mathcal{M}\epsilon_0\,, \tag{3.30}$$

ϵ_0 is a constant 32-component spinor and the Dirac matrices satisfy $\gamma_{0123456789\natural}=1$.

The angles ξ_i are the phases of z_1, z_2, z_3, z_4 from (3.5)

$$\xi_1 = \frac{2\varphi_1+\chi+\zeta}{4}\,,\qquad \xi_2 = \frac{-2\varphi_1+\chi+\zeta}{4}\,,\qquad \xi_3 = \frac{2\varphi_2-\chi+\zeta}{4}\,,\qquad \xi_4 = \frac{-2\varphi_2-\chi+\zeta}{4}\,. \tag{3.31}$$

Here ζ is the fiber direction on which the $\mathbb{Z}_k$ orbifold acts.

To see which Killing spinors survive the orbifolding, we write the spinor ϵ_0 in a basis which diagonalizes

$$i\hat{\gamma}\gamma_\natural\epsilon_0 = s_1\epsilon_0\,, \qquad i\gamma_{58}\epsilon_0 = s_2\epsilon_0\,, \qquad i\gamma_{47}\epsilon_0 = s_3\epsilon_0\,, \qquad i\gamma_{69}\epsilon_0 = s_4\epsilon_0\,. \tag{3.32}$$

All the s_i take values ± 1 and by our conventions on the product of all the Dirac matrices, the number of negative eigenvalues is even. Now consider a shift along the ζ circle, which changes all the angles by $\xi_i \to \xi_i + \delta/4$, the Killing spinors transform as

$$\mathcal{M}\epsilon_0 \to \mathcal{M}e^{i\frac{\delta}{8}(s_1+s_2+s_3+s_4)}\epsilon_0\,. \tag{3.33}$$

This transformation is a symmetry of the Killing spinor when two of the s_i eigenvalues are positive and two negative and not when they all have the same sign (unless δ is an integer multiple of 4π). Note that on S^7 the radius of the ζ circle is 8π, so the $\mathbb{Z}_k$ orbifold of S^7 is given by taking $\delta = 8\pi/k$. The allowed values of the s_i are therefore

$$(s_1, s_2, s_3, s_4) \in \begin{Bmatrix} (+,+,-,-),\ (+,-,+,-),\ (+,-,-,+), \\ (-,+,+,-),\ (-,+,-,+),\ (-,-,+,+) \end{Bmatrix} \tag{3.34}$$

Each configuration represents four supercharges, so the orbifolding breaks 1/4 of the supercharges (except for $k = 1, 2$) and leaves 24 unbroken supersymmetries.

3.4.1 Fundamental string

Let us look now at the supersymmetries preserved by the string solution in Section 3.1. The string solution spans the ρ and t coordinates at some fixed values of the angles θ and ϕ (which we set to zero for simplicity).

The supersymmetry condition for the fundamental string is[4]

$$(1-\Gamma)\,\mathcal{M}\,\epsilon_0 = 0 \qquad \text{with} \quad \Gamma = \frac{1}{\mathcal{L}}\Gamma_{t\rho}\gamma_\natural = \gamma_{01\natural}\,. \tag{3.35}$$

It is simple to rewrite the equation after multiplying from the left by $\mathcal{M}^{-1}$. Setting $\alpha = 0$ we have

$$\mathcal{M}^{-1}\,\Gamma\,\mathcal{M} = \Gamma\left(\cos^2\frac{\vartheta_1}{2} + \cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_1}{2}\,e^{\xi_1\hat{\gamma}\gamma_\natural+\xi_2\gamma_{58}}(\hat{\gamma}\gamma_5 - \gamma_{8\natural}) - \sin^2\frac{\vartheta_1}{2}\hat{\gamma}\gamma_{58\natural}\right) \tag{3.36}$$

At the point $\vartheta_1 = 0$ we get the projector equation

$$(1-\Gamma)\epsilon_0 = 0 \tag{3.37}$$

Note that Γ is an independent operator which commutes with γ_{47}, γ_{58}, γ_{69} and $\hat{\gamma}\gamma_\natural$, so it will break half the supersymmetries. A localized string solution will therefore preserve 12 out of the 24 supercharges.

[4]We denote $\Gamma_\mu := e_\mu^a\,\gamma_a$ where e_μ^a is the elfbein with μ a curved and a a tangent space index.

As mentioned above, we expect the dual of the 1/6 BPS Wilson loop to be a string smeared along a $\mathbb{CP}^1$ subspace of $\mathbb{CP}^3$. This is given by taking arbitrary ϑ_1, ξ_1 and ξ_2 at $\alpha = 0$. We can see that if we impose the condition

$$(\hat{\gamma}\gamma_5 - \gamma_{8\natural})\epsilon_0 = 0\,, \tag{3.38}$$

then together with (3.37) this will guarantee that the equation $(1-\Gamma)\epsilon = 0$ is satisfied at any point with $\alpha = 0$.

The equation (3.38) is identical to the requirement $s_1 = s_2$ (3.32). Therefore out of the six allowed eigenvalue combinations of the s_i in (3.34) only two survive: $(+,+,-,-)$ and $(-,-,+,+)$. Together with (3.37) this means that four supercharges are preserved, as was found also in the gauge theory calculation.

3.4.2 D2-brane

The supersymmetries preserved by a D2-brane are determined by solving the following equation on the D2-brane solution

$$\Gamma\,\epsilon = \epsilon\,, \tag{3.39}$$

where $\epsilon = \mathcal{M}\,\epsilon_0$ is the Killing spinor of the background, and where Γ for our D2-brane solution is given by (see e.g. [51])

$$\Gamma = \frac{1}{\mathcal{L}_{DBI}}\left(\Gamma^{(3)} + 2\pi\alpha' F_{t\rho}\,\Gamma^{(1)}\gamma_\natural\right)\,. \tag{3.40}$$

Here

$$\Gamma^{(3)} = \Gamma_{\mu_1\mu_2\mu_3}\frac{\partial x^{\mu_1}}{\partial\sigma^1}\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3}\,, \tag{3.41}$$

is the pullback of the curved space-time Dirac matrices in all world-volume directions and $\Gamma^{(1)}$ is the same, excluding the directions of the field strength $F_{t\rho}$. Plugging in our choice of coordinates and the details of the solution discussed in Section 3.2 we find

$$\begin{aligned}
\Gamma^{(3)} &= -\frac{R^3}{16}\cosh\rho\,\sin\alpha\,\gamma_{017}\,,\\
2\pi\alpha' F_{t\rho}\Gamma^{(1)} &= \frac{R^3}{16}\cosh\rho\,\sin\alpha\,\cos\alpha\,\gamma_7\,,\\
\mathcal{L}_{DBI} &= \frac{R^3}{16}\cosh\rho\,\sin^2\alpha\,.
\end{aligned} \tag{3.42}$$

And we therefore find that (3.39) reads

$$\Big(\gamma_{01} + \cos\alpha\,\gamma_\natural\Big)\gamma_7\,\epsilon = -\sin\alpha\,\epsilon. \tag{3.43}$$

While we expect the D2-brane dual to the Wilson loop to be smeared over the directions parameterized by ϑ_1, φ_1, ϑ_2 and vp_2, we start by considering a brane localized at the point where all these angles vanish. With $\vartheta_1 = \vartheta_2 = \varphi_1 = \varphi_2 = 0$ the Killing spinor is greatly simplified

$$\epsilon|_{\vartheta_1=\vartheta_2=\varphi_1=\varphi_2=0} = e^{\frac{\alpha}{4}(\hat{\gamma}\gamma_4-\gamma_{7\natural})}e^{-\frac{\chi}{4}(s_1+s_2)}e^{\frac{\rho}{2}\hat{\gamma}\gamma_1}e^{\frac{t}{2}\hat{\gamma}\gamma_0}\epsilon_0\,, \tag{3.44}$$

where we remind the reader that the s_i are c-numbers obeying $s_1+s_2+s_3+s_4=0$. We then rewrite (3.43) in a suggestive manner

$$e^{\alpha\,\gamma_{7\natural}}\,\epsilon=\gamma_{01\natural}\,\epsilon\,. \tag{3.45}$$

Next we note that $\hat{\gamma}\gamma_0$ and $\hat{\gamma}\gamma_1$ commute with both $\gamma_{7\natural}$ and $\gamma_{01\natural}$, and therefore the ρ and t terms from the Killing spinor trivially cancel.

Then, multiplying from the left by $e^{-\frac{\alpha}{4}(\hat{\gamma}\gamma_4-\gamma_{7\natural})}$ and commuting it though $\gamma_{01\natural}$ we find the following dependence on the angle α

$$e^{\alpha\,\gamma_{7\natural}}\,\epsilon_0=e^{-\frac{\alpha}{2}(\hat{\gamma}\gamma_4-\gamma_{7\natural})}\,\gamma_{01\natural}\,\epsilon_0 \tag{3.46}$$

It is now clear that in order to solve (3.43) the following two conditions must be imposed upon ϵ_0,

$$\hat{\gamma}\gamma_4\,\epsilon_0=-\gamma_{7\natural}\,\epsilon_0,\qquad \gamma_{01\natural}\,\epsilon_0=\epsilon_0. \tag{3.47}$$

Since $i\hat{\gamma}\gamma_\natural\,\epsilon_0=s_1\,\epsilon_0$ and $i\gamma_{47}\,\epsilon_0=s_3\,\epsilon_0$, we see that the first of these two conditions is that $s_1=-s_3$, while the second condition, as we saw previously for the fundamental string, acts independently to halve the supersymmetries. Out of the six possible signs of the s_i in (3.34), the condition $s_1=-s_3$ chooses four: $(+,+,-,-)$, $(+,-,-,+)$, $(-,+,+,-)$, and $(-,-,+,+)$. Recall that each choice corresponds to 4 supersymmetries, all of which are halved by $\gamma_{01\natural}\,\epsilon_0=\epsilon_0$. We have therefore a total of 8 out of 24 supersymmetries preserved, i.e. the D2-brane at fixed ϑ_1, ϑ_2, φ_1 and ϑ_2 is 1/3 BPS.

A D2-brane localized at any other point will also preserve eight supercharges, we want to check which ones are shared by all of them. Consider then a D2-brane at the point $\vartheta_1=\pi$ and $\vartheta_2=\varphi_1=\varphi_2=0$. In this case the Killing spinor is

$$\epsilon\big|_{\vartheta_1=\pi,\ \vartheta_2=\varphi_1=\varphi_2=0}=e^{\frac{\alpha}{4}(\hat{\gamma}\gamma_4-\gamma_{7\natural})}e^{\frac{\pi}{4}(\hat{\gamma}\gamma_5-\gamma_{8\natural})}e^{-\frac{\chi}{4}(s_1+s_2)}e^{\frac{\rho}{2}\hat{\gamma}\gamma_1}e^{\frac{t}{2}\hat{\gamma}\gamma_0}\epsilon_0\,. \tag{3.48}$$

Using relations like

$$e^{-\frac{\pi}{4}\hat{\gamma}\gamma_5}e^{\frac{\alpha}{4}\hat{\gamma}\gamma_4}e^{\frac{\pi}{4}\hat{\gamma}\gamma_5}=e^{-\frac{\alpha}{4}\gamma_{45}}\,,\qquad\text{and}\qquad e^{\frac{\pi}{4}\gamma_{8\natural}}e^{-\frac{\alpha}{4}\gamma_{7\natural}}e^{-\frac{\pi}{4}\gamma_{8\natural}}=e^{-\frac{\alpha}{4}\gamma_{78}}\,, \tag{3.49}$$

transforms the projector equation to the form of (3.46) with the replacements $\hat{\gamma}\gamma_4\to-\gamma_{45}$, $\gamma_{7\natural}\to\gamma_{78}$, and $\gamma_{01\natural}\to s_1s_2\gamma_{01\natural}$ so the equation is solved for ϵ_0 satisfying

$$\gamma_{45}\,\epsilon_0=\gamma_{78}\,\epsilon_0,\qquad s_1s_2\gamma_{01\natural}\,\epsilon_0=\epsilon_0. \tag{3.50}$$

The first condition is analogous to imposing $s_2=-s_3$ and leaves the sign choices $(+,+,-,-)$, $(-,+,-,+)$, $(+,-,+,-)$, and $(-,-,+,+)$. The second condition is a modification of the usual one $(\gamma_{01\natural}-1)\epsilon_0=0$ for states with $s_1\neq s_2$.

Together with the previous condition, $s_1=-s_3$, for the D2-brane at $\vartheta_1=0$, this leaves only the two configurations $(+,+,-,-)$ and $(-,-,+,+)$. Now also $s_1=s_2$, so the second condition in (3.50) agrees with that in (3.46) giving a total of four real supercharges. These are the same supercharges preserved by the fundamental string after it was smeared on $\mathbb{CP}^1$.

A similar analysis can be done at any other value of the angles ϑ_1, ϑ_2, φ_1 and ϑ_2, but it is rather involved. A simpler route to the proof is to impose on the Killing spinor the conditions $s_1 = s_2 = -s_3 = -s_4$ which eliminates from the Killing spinor all dependence on these angles

$$\epsilon = e^{-\frac{\alpha}{2}\gamma_{7\natural}} e^{-\frac{\chi}{2}s_1} e^{\frac{\rho}{2}\hat{\gamma}\gamma_1} e^{\frac{t}{2}\hat{\gamma}\gamma_0}\epsilon_0 \,. \tag{3.51}$$

Commuting the Dirac matrices in the projector equation (3.45) through we find that after imposing $(\gamma_{01\natural} - 1)\epsilon_0 = 0$, the projector equation is satisfied.

We conclude that after smearing the D2-brane, we end up with a configuration which is 1/6-BPS, like the Wilson loop operators in the gauge theory.

3.4.3 D6-brane

The supersymmetry projector associated to the D6-brane is $\Gamma\,\epsilon = \epsilon$, where now (see e.g. [51])

$$\Gamma = \frac{1}{\mathcal{L}_{DBI}}\left(\Gamma^{(7)} + 2\pi\alpha'\, F_{t\rho}\,\Gamma^{(5)}\gamma_\natural\right)\,, \qquad \Gamma^{(7)} = \Gamma_{\mu_1\ldots\mu_7}\,\frac{\partial x^{\mu_1}}{\partial\sigma^1}\cdots\frac{\partial x^{\mu_7}}{\partial\sigma^7}\,. \tag{3.52}$$

$\Gamma^{(5)}$ again is the same as $\Gamma^{(7)}$, excluding the directions of the field strength $F_{t\rho}$. Plugging in our choice of coordinates and the details of the solution presented in Section 3.3, we find

$$\Big(\gamma_{01} + \cos\alpha\,\gamma_\natural\Big)\gamma_{56789}\,\epsilon = \sin\alpha\,\epsilon. \tag{3.53}$$

The form of this projector is quite easy to understand. $\Gamma^{(7)}$, $\Gamma^{(5)}$ and the Lagrangian share the same volume element on $\mathbb{CP}^3$ and with the field-strength also that of AdS_2. Then the remaining factors come from $\beta E = -\cos\alpha$ and a factor of $\sqrt{1-\beta^2E^2} = \sin\alpha$ in the DBI Lagrangian.

The equation is very similar to that in the D2-brane case. It needs to be checked for all values of ϑ_1, φ_1, ϑ_2, φ_2 and χ. One first chooses a pair of points and verifies that the same conditions as for the fundamental string and the D2-brane are necessary at those two points. Then we can use these conditions, in particular $s_1 = s_2 = -s_3 = -s_4 = \pm 1$ to express the Killing spinor as (3.51)

$$\epsilon = e^{-\frac{\alpha}{2}\gamma_{7\natural}} e^{-\frac{\chi}{2}s_1} e^{\frac{\rho}{2}\hat{\gamma}\gamma_1} e^{\frac{t}{2}\hat{\gamma}\gamma_0}\epsilon_0 \,. \tag{3.54}$$

Now we rewrite (3.53) as

$$e^{-\alpha\,\gamma_{56789\natural}}\,\epsilon = e^{\alpha\,\gamma_{7\natural}}\,\epsilon = \gamma_{01\natural}\,\epsilon\,. \tag{3.55}$$

Since $\hat{\gamma}\gamma_0$ and $\hat{\gamma}\gamma_1$ commute with $\gamma_{7\natural}$, γ_{69}, γ_{58}, and $\gamma_{01\natural}$, the ρ and t terms from the Killing spinor trivially cancel. We are left with

$$e^{\frac{\alpha}{2}\gamma_{7\natural}}\,\epsilon_0 = \gamma_{01\natural}\,e^{-\frac{\alpha}{2}\gamma_{7\natural}}\,\epsilon_0 \tag{3.56}$$

The alpha dependence drops since $\gamma_{01\natural}$ anti-commutes with $\gamma_{7\natural}$, so finally we are left with the condtion $\gamma_{01\natural}\,\epsilon_0 = \epsilon_0$. We have therefore found that the D6-brane preserves the same supersymmetries of the smeared fundamental string the D2-branes and of the Wilson loop operator.

4 Discussion

In this paper we studied supersymmetric Wilson loops in the $\mathcal{N}=6$ Chern-Simons theory constructed by Aharony et al. [1]. The natural Wilson loop observable couples to a bi-linear of the scalar fields and we studied the simplest such loops, with the geometry of a line or a circle both in the gauge theory (at order λ^2) and at strong coupling using fundamental strings, D2-branes and D6-branes in $AdS_4 \times \mathbb{CP}^3$.

In the maximally supersymmetric theory in four dimensions the circular Wilson loop has a non-trivial expression which can be matched between the gauge theory and string theory by an exact interpolating function. It would be interesting to see if such results apply also here, though the basic Wilson loop observable preserves 1/6 of the supercharges, not 1/2.

It is a rather puzzling fact that the natural Wilson loops preserve only four supercharges. A fundamental string ending along a straight line on the boundary of AdS_4 and localized on $\mathbb{CP}^3$ preserves 12 supercharges. In order to match with the gauge theory observable and its $SU(2) \times SU(2)$ symmetry we smeared the string over a $\mathbb{CP}^1$, and it indeed broke the supersymmetry down to 1/6. But the question remains what is the gauge theory dual to a *localized* fundamental string.

We comment below on some possibilities to construct such operators, but will not pursue them here further.

The Wilson loop (1.2) preserves 4 supercharges which match with 4 out of the 12 supercharges preserved by the localized fundamental string, but it breaks the other 8. Those other eight supercharges will not annihilate this loop, but transform it into a different loop and by repeated action one can generate a full multiplet of Wilson loops. This multiplet is closed under the action of all the required 12 supercharges, so the state created by integrating over all those Wilson loops with flat measure will necessarily preserve these 12 supercharges.

This is a standard way of enhancing symmetry, by integrating over the zero modes of the broken symmetry. It is guaranteed to give an object with at least the desired symmetry, but it might also lead to a trivial operator, the identity or 0. It would be interesting to construct this operator explicitly and study its properties.

Let us point out here another possible construction of a supersymmetric Wilson loop. Consider the purely bosonic operator (1.1) with the holonomy in *both* of the gauge groups and in opposite representations. Such a Wilson loop may be writen schematically as

$$W = \operatorname{Tr} \mathcal{P} \exp \left(i \int (A_\mu - \hat{A}_\mu) dx^\mu \right) . \tag{4.1}$$

The relative sign was put in by hand to represent the fact that if the first group is in the fundamental representation the second one is in the anti-fundamental, and hence the gauge fields act on the fields from the right, rather than the left.

A naive tree-level calculation of the supersymmetry variation of this loop will show that it is invariant under *all* the supersymmetries, the variation of $\hat{A}$ canceling that of A after taking the trace. We do not expect this to extend beyond the tree level, and indeed the expectation value of this loop will suffer from divergences at order λ^2. But it is possible that this loop will become supersymmetric once augmented with the correct scalar insertions.

One can also use this operator to construct open Wilson loops, by putting a bi-fundamental field, say C_I at one end and an adjoint field $\bar{C}^J$ at the other. Furthermore, one can start the open Wilson-loop at one C_I and then continue to insert more C_I fields along its path. After each scalar field the representation of the Wilson loop will change (to a product representation of the fundamental-antifundamental). Because the operator $(C_I)^k$ is gauge invariant (with the inclusion of an 't Hooft vertex), after k insertions, the Wilson loop can end.

These are two ways of constructing open Wilson loops. While we don't expect a loop of finite extent to be supersymmetric, one can consider the infinite line with a distribution of bi-fundamental scalar field insertions. With an appropriate choice of scalars (the simplest being all identical), the same naive argument would lead one to conclude that this Wilson loop preserves some supersymmetries. We leave a closer examination of those Wilson loops to the future.

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A $\mathcal{N}=6$, $d=3$ super Cherns-Simons-matter action and Feynman rules

Here we summarize the action and conventions for the perturbative computations. The field content consists of two $U(N)$ gauge fields $(A_\mu)_{ij}$ and $(\hat{A}_\mu)_{\hat{i}\hat{j}}$, the complex fields $(C_I)_{i\hat{i}}$ and $(\bar{C}^I)_{\hat{i}i}$ as well as the fermions $(\psi_I)_{\hat{i}i}$ and $(\bar{\psi}^I)_{i\hat{i}}$ in the $(\mathbf{N},\bar{\mathbf{N}})$ and $(\bar{\mathbf{N}},\mathbf{N})$ of $U(N)$ respectively, $I=1,2,3,4$ is the $SU(4)_R$ index. We employ the covariant gauge fixing function $\partial_\mu A^\mu$ for both gauge fields and have two sets of ghosts $(\bar{c},c)$ and $(\bar{\hat{c}},\hat{c})$. We work with the Euclidian space action (see [52, 1, 53])

$$\begin{aligned} S_{\rm CS} &= -i\frac{k}{4\pi}\int d^3x\,\varepsilon^{\mu\nu\rho}\Big[\,{\rm Tr}\,(A_\mu\partial_\nu A_\rho+\frac{2}{3}\,A_\mu A_\nu A_\rho)-{\rm Tr}\,(\hat{A}_\mu\partial_\nu\hat{A}_\rho+\frac{2}{3}\,\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho)\Big] \\ S_{\rm gf} &= \frac{k}{4\pi}\int d^3x\,\Big[\,\frac{1}{\xi}\,{\rm Tr}\,(\partial_\mu A^\mu)^2+{\rm Tr}\,(\partial_\mu\bar{c}\,D_\mu c)-\frac{1}{\xi}\,{\rm Tr}\,(\partial_\mu\hat{A}^\mu)^2+{\rm Tr}\,(\partial_\mu\bar{\hat{c}}\,D_\mu\hat{c})\,\Big] \\ S_{\rm Matter} &= \int d^3x\,\Big[\,{\rm Tr}\,(D_\mu\,C_I\,D^\mu\bar{C}^I)+i\,{\rm Tr}\,(\bar{\psi}^I\,\not{D}\,\psi_I)\Big]+S_{\rm int} \end{aligned} \tag{A.1}$$

Here $S_{\rm int}$ are the sextic scalar potential and $\psi^2 C^2$ Yukawa type potentials spelled out in [1]. The matter covariant derivatives are defined as

$$\begin{aligned}
D_\mu C_I &= \partial_\mu C_I + i(A_\mu\, C_I - C_I\, \hat{A}_\mu) \\
D_\mu \bar{C}^I &= \partial_\mu \bar{C}^I - i(\bar{C}^I\, A_\mu - \hat{A}_\mu\, \bar{C}_I) \\
D_\mu \psi_I &= \partial_\mu \psi_I + i(\hat{A}_\mu\, \psi_I - \psi_I\, A_\mu) \\
D_\mu \bar{\psi}^I &= \partial_\mu \bar{\psi}^I - i(\bar{\psi}^I\, \hat{A}_\mu - A_\mu\, \bar{\psi}^I)\,.
\end{aligned} \tag{A.2}$$

From this we read off the momentum space propagators

$$\begin{aligned}
\langle (A_\mu)_{ij}(p)\, (A_\nu)_{kl}(-p)\rangle_0 &= \frac{2\pi}{k}\, \delta_{il}\, \delta_{jk} \left[\varepsilon_{\mu\nu\rho}\, p^\rho + \xi\, \frac{p_\mu p_\nu}{p^2}\right] \frac{1}{p^2} \\
\langle (\hat{A}_\mu)_{ij}(p)\, (\hat{A}_\mu)_{kl}(-p)\rangle_0 &= -\frac{2\pi}{k}\, \delta_{il}\, \delta_{jk} \left[\varepsilon_{\mu\nu\rho}\, p^\rho + \xi\, \frac{p_\mu p_\nu}{p^2}\right] \frac{1}{p^2}
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\langle (c)_{ij}(p)\, (\bar{c})_{kl}(-p)\rangle_0 &= \frac{2\pi}{k}\, \delta_{il}\, \delta_{jk}\, \frac{1}{p^2} \\
\langle (\hat{c})_{ij}(p)\, (\bar{\hat{c}})_{kl}(-p)\rangle_0 &= -\frac{2\pi}{k}\, \delta_{il}\, \delta_{jk}\, \frac{1}{p^2}
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\langle (C_I)_{i\hat{i}}(p)\, (\bar{C}^J)_{\hat{j}j}(-p)\rangle_0 &= \delta_I^J\, \delta_{ij}\, \delta_{\hat{i}\hat{j}}\, \frac{1}{p^2} \\
\langle (\psi_I)_{\hat{i}i}(p)\, (\bar{\psi}^J)_{j\hat{j}}(-p)\rangle_0 &= -\delta_I^J\, \delta_{ij}\, \delta_{\hat{i}\hat{j}}\, \frac{1}{\not{p}}
\end{aligned} \tag{A.5}$$

We also note the relevant Fourier transformations to configuration space:

$$\left[\frac{\delta_{\mu\nu}}{p} - \frac{p_\mu p_\nu}{p^3}\right]_{d=3} \to \frac{\delta_{\mu\nu}}{2\pi^2\, x^2} - \frac{1}{4\pi^2}\partial_\mu\partial_\nu \log x^2\,, \qquad \left[\frac{1}{p^2}\right]_{d=3} \to \frac{1}{4\pi}\,\frac{1}{x} \tag{A.6}$$

A.1 The gluon self-energy

The one-loop correction to the gluon self energy from gluon and ghost contributions is known to vanish (see e.g. [52]). We here evaluate the matter contribution.

For bosons in the loop there are two graphs to consider, the four-valent bubble vanishes in dimensional regularization. The other graph comes from expanding the cubic vertex to second order from $e^{-S_{\rm Matter}}$:

$$\Big\langle (\, i\, {\rm Tr}\, (A_\mu\, C_I\, \partial_\mu \bar{C}^I\; - \partial_\mu C_I \bar{C}^I A_\mu))\, (\, i\, {\rm Tr}\, (A_\mu\, C_I\, \partial_\mu \bar{C}^I\; - \partial_\mu C_I \bar{C}^I A_\mu))\Big\rangle \tag{A.7}$$

Contracting, Fourier transforming and amputating the gluon legs yields the self-energy contribution

$$\Pi^{(B)}_{\mu\nu}(p) = N\, \delta^I_I \int \frac{d^3k}{(2\pi)^3}\, \frac{(2k+p)_\mu (2k+p)_\nu}{k^2\, (p+k)^2}. \tag{A.8}$$

This is to be contracted with two gluon propagators (we use Landau gauge $\xi = 0$ from now on) to get the one-loop corrected gluon propagator

$$G^{(B,1)}_{\mu\nu}(p) = \left(\frac{2\pi}{k}\right)^2 \frac{\varepsilon_{\mu\rho\kappa} p^\kappa}{p^2}\, \Pi^{(B)}_{\rho\lambda}(p)\, \frac{\varepsilon_{\lambda\nu\delta} p^\delta}{p^2}\,, \tag{A.9}$$

In this expression we see that the term in the integral proportional to p_ν in (A.8) drops out. Performing the integral in (A.8) in dimensional regularization yields a finite result. Contracting with the two ε-tensors we find

$$G^{(B,1)}_{\mu\nu}(p) = \left(\frac{2\pi}{k}\right)^2 \frac{N\,\delta^I_I}{16}\,\frac{1}{p}\,\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) . \tag{A.10}$$

Turning to the fermionic contributions to the loop we need to contract

$$\Big\langle (\, i\,\mathrm{Tr}\,(\bar\psi^I\, i\not{A}\,\psi_I\,)\,(\, i\,\mathrm{Tr}\,(\bar\psi^I\, i\not{A}\,\psi_I\,)\Big\rangle \tag{A.11}$$

yielding

$$\Pi^{(F)}_{\mu\nu}(p) = -N\,\delta^I_I \int \frac{d^3k}{(2\pi^3)}\,\frac{\mathrm{tr}(\gamma_\mu\,(\not{p}+\not{k})\,\gamma_\nu\,\not{k})}{k^2\,(p+k)^2}. \tag{A.12}$$

We note

$$\mathrm{tr}(\gamma_\mu\not{p}\gamma_\nu\not{k}) = 2\,(-\delta_{\mu\nu}\,p\cdot k + k_\mu p_\nu + p_\mu k_\nu)\,, \tag{A.13}$$

where the last two terms vanish upon contraction with the epsilon tensors of the attached gluon propagators. This leaves the $p\cdot k$ term. Upon using a Feynman parameter α, only the momentum shift of $k \to k-(1-\alpha)p$ will survive integration. This gives

$$2\,N\delta^I_I\delta_{\mu\nu}\int\frac{d^3k}{(2\pi)^3}\int_0^1 d\alpha\frac{-p^2(1-\alpha)}{[k^2+\alpha(1-\alpha)p^2]^2} = -2\,N\delta^I_I\delta_{\mu\nu}\frac{p\pi^{3/2}\Gamma(1/2)}{(2\pi)^3}\int_0^1 d\alpha\sqrt{\frac{1-\alpha}{\alpha}} = \frac{-N\delta^I_I\delta_{\mu\nu}p}{8} \tag{A.14}$$

We must also consider the term:

$$\mathrm{tr}(\gamma_\mu\not{k}\gamma_\nu\not{k}) = 2\,(-\delta_{\mu\nu}\,k\cdot k + 2k_\mu k_\nu)\,. \tag{A.15}$$

Upon shifting the momentum k as above we obtain

$$-2\delta_{\mu\nu}\left[k^2+(1-\alpha)^2p^2\right] + 4\left[k_\mu k_\nu + (1-\alpha)^2 p_\mu p_\nu\right] \to -2\delta_{\mu\nu}\left[\frac{k^2}{3}+(1-\alpha)^2p^2\right] \tag{A.16}$$

where we have symmetrized the $k_\mu k_\nu$ integral, and removed the $p_\mu p_\nu$ term as it is killed by epsilon contractions. Integrating over k we find

$$-2N\delta^I_I\frac{\delta_{\mu\nu}}{(2\pi)^3}\int_0^1 d\alpha\Big[-\frac{1}{3}\delta_{\mu\nu}\frac{3}{2}\pi^{3/2}\Gamma(-1/2)\sqrt{\alpha(1-\alpha)}p - \delta_{\mu\nu}\frac{(1-\alpha)^{3/2}}{\sqrt{\alpha}}\pi^{3/2}\Gamma(1/2)p\Big] = \frac{N\delta^I_I\delta_{\mu\nu}p}{16} \tag{A.17}$$

We therefore have

$$G^{(F,1)}_{\mu\nu}(p) = \left(\frac{2\pi}{k}\right)^2 \frac{N\,\delta^a_a}{16}\,\frac{1}{p}\,\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) , \tag{A.18}$$

and so the combined bosonic and fermionic matter contributions yield

$$G^{(1)}_{\mu\nu}(p) = G^{(B,1)}_{\mu\nu}(p) + G^{(F,1)}_{\mu\nu}(p) = \left(\frac{2\pi}{k}\right)^2 \frac{1}{8}\,\frac{N\,\delta^a_a}{p}\,\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) . \tag{A.19}$$

B Killing spinors

In this appendix we derive an explicit form for the Killing spinors in the coordinate system where the metric on AdS_4 is (3.2)

$$ds^2_{AdS_4} = R^2\left[d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\big(d\theta^2 + \sin^2\theta\, d\phi^2\big)\right] . \tag{B.1}$$

and the metric on S^7 is given by (3.6)

$$\begin{aligned} ds^2_{S^7} = \frac{R^2}{4}\Bigg[& d\alpha^2 + \cos^2\frac{\alpha}{2}(d\vartheta_1^2 + \sin^2\vartheta_1^2\, d\varphi_1^2) + \sin^2\frac{\alpha}{2}(d\vartheta_2^2 + \sin^2\vartheta_2^2\, d\varphi_2^2) \\ &+ \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(d\chi + \cos\vartheta_1\, d\varphi_1 - \cos\vartheta_2\, d\varphi_2)^2 \\ &+ \left(\frac{d\zeta}{2} + \cos^2\frac{\alpha}{2}\cos\vartheta_1\, d\varphi_1 + \sin^2\frac{\alpha}{2}\cos\vartheta_2\, d\varphi_2 + \frac{\cos\alpha}{2}\, d\chi\right)^2\Bigg]. \end{aligned} \tag{B.2}$$

We take the elfbeine to be

$$\begin{aligned} &e^0 = \frac{R}{2}\cosh\rho\, dt\,, \quad e^1 = \frac{R}{2}\, d\rho\,, \quad e^2 = \frac{R}{2}\sinh\rho\, d\theta\,, \quad e^3 = \frac{R}{2}\sinh\rho\sin\theta\, d\phi\,, \\ &e^4 = \frac{R}{2}d\alpha, \qquad e^5 = \frac{R}{2}\cos\frac{\alpha}{2}\, d\vartheta_1, \qquad e^6 = \frac{R}{2}\sin\frac{\alpha}{2}\, d\vartheta_2, \\ &e^7 = \frac{R}{2}\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\Big(\cos\vartheta_1\, d\varphi_1 - \cos\vartheta_2\, d\varphi_2 + d\chi\Big), \\ &e^8 = \frac{R}{2}\cos\frac{\alpha}{2}\sin\vartheta_1\, d\varphi_1, \quad e^9 = \frac{R}{2}\sin\frac{\alpha}{2}\sin\vartheta_2\, d\varphi_2\,, \\ &e^\natural = -\frac{R}{4}\left(d\zeta + 2\cos^2\frac{\alpha}{2}\cos\vartheta_1\, d\varphi_1 + 2\sin^2\frac{\alpha}{2}\cos\vartheta_2\, d\varphi_2 + \cos\alpha\, d\chi\right). \end{aligned} \tag{B.3}$$

To find the relevant Killing spinor equation for this background we look at the supersymmetry transformation of the gravitino

$$\delta\Psi_\mu = D_\mu\epsilon - \frac{1}{288}\Big(\Gamma_\mu^{\ \nu\lambda\rho\sigma} - 8\delta_\mu^\nu\Gamma^{\lambda\rho\sigma}\Big)F_{\nu\lambda\rho\sigma}\epsilon\,, \qquad D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon\,. \tag{B.4}$$

The 4-form corresponding to the $AdS_4\times S^7$ solution is $F_{\nu\lambda\rho\sigma} = 6\,\varepsilon_{\nu\lambda\rho\sigma}$, where the epsilon symbol is the volume form on AdS_4 (so the indices take the values $0,1,2,3$). Plugging this into the variation above one finds the Killing spinor equation

$$D_\mu\epsilon = \frac{1}{2}\hat\gamma\gamma_\mu\epsilon \tag{B.5}$$

where μ runs over all 11 coordinates, and $\hat\gamma = \gamma^{0123}$. Note that small γ have tangent-space indices while capital Γ carry curved-space indices. Calculating the spin-connection for our chosen elfbeine, we find the following explicit Killing spinor equations

$$\begin{aligned} \partial_t\epsilon &= \frac{1}{2}\hat\gamma\gamma_1\, e^{\rho\,\hat\gamma\gamma_0}\epsilon\,, \\ \partial_\rho\epsilon &= \frac{1}{2}\hat\gamma\gamma_1\epsilon\,, \\ \partial_\theta\epsilon &= \frac{1}{2}\gamma_{12}\, e^{-\rho\,\hat\gamma\gamma_1}\epsilon\,. \\ \partial_\phi\epsilon &= \frac{1}{2}\left(\coth\rho\,\gamma_{13} + \cos\theta\,\gamma_{23} + \sinh\rho\sin\theta\,\hat\gamma\gamma_3\right)\epsilon = 0\,, \end{aligned} \tag{B.6}$$

and

$$\begin{aligned}
\partial_\alpha\epsilon =&\frac{1}{4}\,(\hat\gamma\gamma_4-\gamma_{7\natural})\epsilon\,,\\
\partial_{\vartheta_1}\epsilon =&\frac{1}{4}\,\left(\hat\gamma\gamma_5\,e^{-\frac{\alpha}{2}\hat\gamma\gamma_4}-\gamma_{8\natural}\,e^{\frac{\alpha}{2}\gamma_{7\natural}}\right)\epsilon\,,\\
\partial_{\vartheta_2}\epsilon =&\frac{1}{4}\,\left(\gamma_{46}\,e^{-\frac{\alpha}{2}\hat\gamma\gamma_4}+\gamma_{79}\,e^{\frac{\alpha}{2}\gamma_{7\natural}}\right)\epsilon\,,\\
\partial_{\varphi_1}\epsilon =&\frac{1}{4}\,\left(\cos\vartheta_1\,\gamma_{58}-\cos\vartheta_1\,\hat\gamma\gamma_\natural\,e^{\frac{\alpha}{2}(\gamma_{7\natural}-\hat\gamma\gamma_4)}+\sin\vartheta_1\left(\hat\gamma\gamma_8\,e^{-\frac{\alpha}{2}\hat\gamma\gamma_4}+\gamma_{5\natural}\,e^{\frac{\alpha}{2}\gamma_{7\natural}}\right)\right)\epsilon\,,\\
\partial_{\varphi_2}\epsilon =&\frac{1}{4}\,\left(\cos\vartheta_2\,\gamma_{69}-\cos\vartheta_2\,\gamma_{47}\,e^{\frac{\alpha}{2}(\gamma_{7\natural}-\hat\gamma\gamma_4)}+\sin\vartheta_2\left(\gamma_{49}\,e^{-\frac{\alpha}{2}\hat\gamma\gamma_4}+\gamma_{67}\,e^{\frac{\alpha}{2}\gamma_{7\natural}}\right)\right)\epsilon\,,\\
\partial_\chi\epsilon =&\frac{1}{8}\,\left((\gamma_{47}-\hat\gamma\gamma_\natural)e^{-\alpha\hat\gamma\gamma_4}+\gamma_{69}-\gamma_{58}\right)\epsilon\,,\\
\partial_\zeta\epsilon =&-\frac{1}{8}(\gamma_{58}+\gamma_{69}+\gamma_{47}+\hat\gamma\gamma_\natural)\epsilon\,.
\end{aligned} \tag{B.7}$$

These equations are solved by the following Killing spinor

$$e^{\frac{\alpha}{4}(\hat\gamma\gamma_4-\gamma_{7\natural})}e^{\frac{\vartheta_1}{4}(\hat\gamma\gamma_5-\gamma_{8\natural})}e^{\frac{\vartheta_2}{4}(\gamma_{79}+\gamma_{46})}e^{-\frac{\xi_1}{2}\hat\gamma\gamma_\natural}e^{-\frac{\xi_2}{2}\gamma_{58}}e^{-\frac{\xi_3}{2}\gamma_{47}}e^{-\frac{\xi_4}{2}\gamma_{69}}e^{\frac{\rho}{2}\hat\gamma\gamma_1}e^{\frac{t}{2}\hat\gamma\gamma_0}e^{\frac{\theta}{2}\gamma_{12}}e^{\frac{\phi}{2}\gamma_{23}}\epsilon_0=\mathcal{M}\epsilon_0\,, \tag{B.8}$$

where the ξ_i are given by

$$\xi_1=\frac{2\varphi_1+\chi+\zeta}{4}\,,\qquad \xi_2=\frac{-2\varphi_1+\chi+\zeta}{4}\,,\qquad \xi_3=\frac{2\varphi_2-\chi+\zeta}{4}\,,\qquad \xi_4=\frac{-2\varphi_2-\chi+\zeta}{4}\,. \tag{B.9}$$

In (B.8) ϵ_0 is a constant 32-component spinor and the Dirac matrices were chosen such that $\gamma_{0123456789\natural}=1$. A similar calculation in a different coordinate system was done in [54].

One may also consider the AdS_4 in terms of AdS_2 slices (3.3)

$$ds^2=du^2+\cosh^2u\Big(-\cosh^2\rho\,dt^2+d\rho^2\Big)+\sinh^2u\,d\phi^2, \tag{B.10}$$

a vierbein basis being given by

$$e^0=\frac{R}{2}\cosh u\cosh\rho\,dt,\quad e^1=\frac{R}{2}\cosh u\,d\rho,\quad e^2=\frac{R}{2}du,\quad e^3=\frac{R}{2}\sinh u\,d\phi, \tag{B.11}$$

leading to the following spin connection

$$\omega^{01}=\sinh\rho\,dt,\quad \omega^{02}=\sinh u\cosh\rho\,dt,\quad \omega^{12}=\sinh u\,d\rho,\quad \omega^{23}=-\cosh u\,d\phi. \tag{B.12}$$

In these coordinates the final four factors in the Killing spinor in (B.8) are replaced by

$$e^{\frac{u}{2}\hat\gamma\gamma_2}e^{\frac{\phi}{2}\gamma_{23}}e^{\frac{\rho}{2}\hat\gamma\gamma_1}e^{\frac{t}{2}\hat\gamma\gamma_0}. \tag{B.13}$$

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