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Wilson Loops in $\mathcal{N}=2$ Superconformal Yang-Mills Theory

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Abstract

We present a three-loop ($\mathcal{O}(g^6)$) calculation of the difference between the expectation values of Wilson loops evaluated in $\mathcal{N}=4$ and superconformal $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ using dimensional reduction. We find a massive reduction of required Feynman diagrams, leaving only certain two-matter-loop corrections to the gauge field and associated scalar propagator. This "diagrammatic difference" leaves a finite result proportional to the bare propagators and allows the recovery of the $\zeta(3)$ term coming from the matrix model for the 1/2 BPS circular Wilson loop in the $\mathcal{N}=2$ theory. The result is valid also for closed Wilson loops of general shape. Comments are made concerning light-like polygons and supersymmetric loops in the plane and on S^2.

Contents

1 Introduction and results

The study of supersymmetric Wilson loops has enjoyed exciting development since the very early days of AdS/CFT, when the basic object and string dual were identified [1, 2]. Standing at the forefront of these investigations has been the 1/2 BPS circle of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM). This object is intimately related to the trivial 1/2 BPS infinite line, through the singular conformal inversion $x^\mu \to x^\mu/x^2$, which leaves only the point at infinity for the site of non-trivial dynamics, which are therefore captured by a 0-dimensional quantum field theory - the celebrated matrix model of [3, 4]. Perhaps the most potent feature of this matrix model is that it captures three very different regimes in the dual string theory, corresponding to the scaling of the rank R of the representation which the trace is taken in, with respect to N, the rank of the gauge group. For $R \sim N^0$, one has a semi-classical fundamental string describing a minimal surface in $AdS_5 \times S^5$, for $R \sim N^1$ the string becomes a D-brane (or collection thereof) again in $AdS_5 \times S^5$, while for $R \sim N^2$ the back-reaction of the branes deform the background geometry and $AdS_5 \times S^5$ is replaced by a new space. These are a very rich set of phenomena, and the fact that they can be reduced to a relatively simple 0-dimensional theory is astounding. Perhaps more astounding is that the matrix model should also describe the full quantum, string-loop-corrected versions of these objects.

Recently the precise way in which the 1/2 BPS Wilson loop comes to be described by the matrix model has been understood through the techniques of localization [5]. Beyond providing a previously lacking proof of the equivalence between the matrix model and the Wilson loop, this work has opened up the study of Wilson loops into exciting new avenues [6–10]. One of the basic extensions provided by [5] is to the description of the 1/2 BPS circular loop in $\mathcal{N} = 2$ SYM. In the superconformal case, when $N_f = 2N$ fundamental hypermultiplets are coupled to the theory, the matrix model is modified with respect to the $\mathcal{N} = 4$ case by the insertion of a determinant

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factor[1]. This contribution was worked out in detail in [5], for the specific case of $SU(2)$, where it was shown that the effect of the determinant factor in a perturbative expansion was the addition of a term at $\mathcal{O}(g^6)$, proportional to $\zeta(3)$. Using the explicit expression for the determinant factor provided in [5], it is a trivial matter to generalize the calculation for $SU(N)$, as we do in section 2, and the additional $\zeta(3)$ term remains at $\mathcal{O}(g^6)$, albeit with a generalized coefficient.

The purpose of this paper is to recover this $\zeta(3)$ term from perturbation theory. The technique we use is dimensional reduction. Our strategy is to take the "diagrammatic difference" of the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ results. In so doing we can prove that the calculations cancel up to $\mathcal{O}(g^4)$, in agreement with the matrix model result. Further, a massively reduced set of Feynman diagrams remains at $\mathcal{O}(g^6)$, all of which are two-loop matter-corrections to the $\mathcal{N} = 2$ adjoint gauge and scalar field propagators. Of these, only two give $\zeta(3)$ contributions, and are responsible for the exact match with the matrix model. We find a complete cancellation of divergences, which are generically $\mathcal{O}(1/\epsilon^2)$ where the dimension is taken as $4 - 2\epsilon$. The result is proportional to the bare gauge field and real scalar propagator, and therefore is directly applicable to *any* closed[2] Wilson loop in the $\mathcal{N} = 2$ theory of the form

$$W = \frac{1}{N} \operatorname{Tr} P \exp \oint d\tau \big(i\dot{x}^\mu(\tau) A_\mu + |\dot{x}(\tau)| \Theta^I(\tau) \Phi_I \big), \qquad I = 1, 2, \qquad (1)$$

where the Φ_I are the two real adjoint scalars in the gauge multiplet. The result may be compactly expressed in the following way

$$\langle W \rangle_{\mathcal{N}=4} - \langle W \rangle_{\mathcal{N}=2} =$$
$$g^6 \left[\frac{12\zeta(3)}{(4\pi)^4} (N^2 + 1) \right] \frac{N^2 - 1}{2N} \frac{1}{2!} \oint d\tau_1 \oint d\tau_2 \frac{|\dot{x}_1||\dot{x}_2|\Theta_1 \cdot \Theta_2 - \dot{x}_1 \cdot \dot{x}_2}{4\pi^2 (x_1 - x_2)^2} + \mathcal{O}(g^8), \qquad (2)$$

where the bracketed expression is the dressing the propagators receive, while the remainder of the expression is the standard expansion of the Wilson loop to second order.

The outline of this paper is as follows. We present the result stemming from the matrix model for general $SU(N)$ in section 2. In section 3 we describe the structure of the perturbation theory calculation, giving details in appendix A. Finally in section 4 we discuss the implications of our result for other well-known Wilson loops, including the Zarembo loops [11], the longitudes of [12], and the light-like polygonal Wilson loop.

2   Results from localization

In this section we derive the result for the circular Wilson loop expectation value in superconformal $SU(N)$ $\mathcal{N} = 2$ SYM, coming from the matrix model of Pestun

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[1]And also instanton contributions, which will not concern us here.
[2]That the Wilson loop be closed is important for the results of section 3.1.

[5]. In particular we are interested in the $\zeta(3)$ term occurring at $\mathcal{O}(g^6)$. We take coordinates on the Cartan sub-algebra of $SU(N)$, $\vec{a}$, which is an $(N-1)$-component vector, and the weights of the fundamental representation $\vec{w}_i$, $i = 1, \ldots, N$. The roots are given by $\vec{w}_{ij} \equiv \vec{w}_i - \vec{w}_j$. The Wilson loop expectation value, excluding instanton contributions, is then given by

$$\langle W \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} d^{N-1}\vec{a} \prod_{i \neq j} (\vec{w}_{ij} \cdot \vec{a}) \, \mathcal{Z} \, e^{-4\pi^2 \vec{a}^2/g^2} \frac{1}{N} \sum_i e^{2\pi \vec{w}_i \cdot \vec{a}}, \tag{3}$$

where Z is the integral without the Wilson loop insertion $\frac{1}{N} \sum_i e^{2\pi \vec{w}_i \cdot \vec{a}}$ included. The determinant factor $\mathcal{Z}$ is absent in the $\mathcal{N} = 4$ case, and is given by

$$\mathcal{Z} = \prod_{i \neq j} H(i\vec{w}_{ij} \cdot \vec{a}) \left(\prod_i H(i\vec{w}_i \cdot \vec{a}) \right)^{-2N}, \tag{4}$$

where $H(x) \equiv G(1 + x)G(1 - x)$, where $G(x)$ is the Barnes G-function. We will require the perturbative expansion of $\mathcal{Z}$, and it is simplest to expand its logarithm using

$$\log H(x) = -(1 + \gamma)x^2 - \sum_{n=2}^{\infty} \zeta(2n - 1) \frac{x^{2n}}{n}. \tag{5}$$

Using the property $\sum_i \vec{w}_i = 0$, stemming from the tracelessness of the group generators, one finds that the first correction is quartic in $\vec{a}$

$$\log \mathcal{Z} = \zeta(3) \left[N \sum_i (\vec{w}_i \cdot \vec{a})^4 - \sum_{i<j} (\vec{w}_{ij} \cdot \vec{a})^4 \right] + \mathcal{O}(\vec{a}^6). \tag{6}$$

Using the explicit construction of $SU(N)$ weights

$$\vec{w}_1 = (\frac{1}{2}, \frac{1}{\sqrt{12}}, \ldots, \frac{1}{\sqrt{2N(N-1)}}),$$
$$\vec{w}_k = (0, \ldots, 0, \frac{-(k-1)}{\sqrt{2k(k-1)}}, \frac{1}{\sqrt{2k(k+1)}}, \ldots, \frac{1}{\sqrt{2N(N-1)}}), \tag{7}$$

(6) may be further simplified to

$$\log \mathcal{Z} = -\frac{3}{4}\zeta(3) \left(\vec{a}^2 \right)^2 + \mathcal{O}(\vec{a}^6). \tag{8}$$

We can then express this factor as a derivative by the coupling acting on the quadratic action in the matrix model

$$\left(\vec{a}^2 \right)^2 = \left(\frac{g^2}{4\pi^2} \right)^2 \left[\frac{d^2}{dq^2} e^{4\pi^2 q \vec{a}^2/g^2} \right]_{q=1}. \tag{9}$$

The first contribution of $\mathcal{Z}$ to the Wilson loop's expectation value may then be expressed as

$$\langle W \rangle = \frac{\left(1 + \alpha \, \partial_q^2\right) (g^2/q)^{\frac{N^2-1}{2}} \left[1 + \frac{N^2-1}{8N} g^2/q + \ldots\right]}{\left(1 + \alpha \, \partial_q^2\right) (g^2/q)^{\frac{N^2-1}{2}}} \Bigg|_{q=1} \tag{10}$$
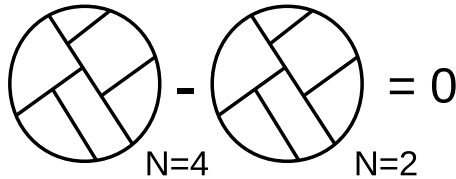
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Figure 1: "Tree" type diagrams are identical in the two theories, and so their difference vanishes.

where $\alpha \equiv -(3/4)\zeta(3)(g^2/(4\pi^2))^2$, and the series in square brackets is the expectation value of the circular Wilson loop in $\mathcal{N} = 4$ SYM, with coupling g^2/q. The result is

$$\langle W \rangle_{\mathcal{N}=4} - \langle W \rangle_{\mathcal{N}=2} = \frac{3\,\zeta(3)}{512\pi^4}\frac{(N^2-1)(N^2+1)}{N}\,g^6 + \mathcal{O}(g^8). \tag{11}$$

In the next section we will recover this result from perturbation theory.

3   Perturbation theory

We write the action of Euclidean $\mathcal{N} = 2$ superconformal Yang-Mills theory following [13], as the sum of $\mathcal{N} = 1$ SYM in 6-d dimensionally reduced to $4-2\epsilon$ dimensions, and $2N$ hypermultiplets in the fundamental. In this way, one obtains the action of $\mathcal{N} = 4$ SYM by restricting to one adjoint hypermultiplet as opposed to $2N$ fundamental ones. Let us write the actions for these two theories schematically as follows (see appendix A for details)

$$\begin{aligned} S_{\mathcal{N}=4} &= S_{\mathcal{N}=1}^{6\to4-2\epsilon} + S_{HM}^{1,\text{adj.}}, \\ S_{\mathcal{N}=2} &= S_{\mathcal{N}=1}^{6\to4-2\epsilon} + S_{HM}^{2N,\text{fund.}}. \end{aligned} \tag{12}$$

The Wilson loop under consideration does not contain couplings to the hypermultiplet fields, it is given by (1) where A_μ is the gauge field, and Φ_I are the $2+2\epsilon$ real scalar fields sitting in $S_{\mathcal{N}=1}^{6\to4-2\epsilon}$. We now consider the difference

$$\langle W \rangle_{\mathcal{N}=4} - \langle W \rangle_{\mathcal{N}=2}. \tag{13}$$

Let us begin at $\mathcal{O}(g^2)$. The only diagram is a single gauge-field or scalar exchange. It is clear that the hypermultiplets play no rôle. Therefore the difference at this order in perturbation theory is identically zero. We can generalize this logic in the following way. Since the "source" fields, i.e. those coupled in the Wilson loop are common between the two theories, all diagrams which do not contain loops vanish identically in the difference[3], see figure 1. Now let us consider the diagrams at $\mathcal{O}(g^4)$. There are three: the two-rung diagram, the trivalent graph consisting of a single cubic vertex with all three fields attached to the Wilson loop, and the one-loop-corrected one-rung

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[3]This is because the couplings are at least quadratic in the hypermultiplet fields, see (26), (27).
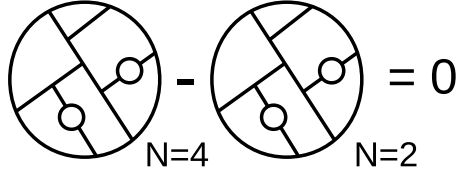
Figure 2: One-loop corrected tree-type diagrams are also identical in the two theories, and so their difference also vanishes.
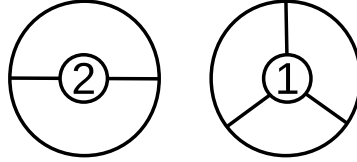

Figure 3: After application of rules depicted in figures 1 and 2, only the two-loop propagator, and 1-loop triple-vertex corrections remain at $\mathcal{O}(g^6)$.

diagram. By the logic just expounded upon, only the last diagram has a chance of surviving the difference. As we will now show, it too cancels-out. The colour factor in the one-loop correction to the gauge field A_μ (or real scalar Φ_I) propagator stemming from a loop of one adjoint field, or $2N$ fundamental fields is the same

$$
\begin{aligned}
&1 \text{ adjoint field} \rightarrow i^2 f^{qik} f^{kjq} = N\delta^{ij} \\
&2N \text{ fundamental fields} \rightarrow 2N \operatorname{Tr}(T^i T^j) = N\delta^{ij}.
\end{aligned}
\tag{14}
$$

Thus we are also free to decorate the diagrams of figure 1 with one-loop-corrected propagators, see figure 2.

It is worth underscoring at this point that we have now found agreement with the matrix model results presented in section 2 at the first two consecutive orders of perturbation theory, without evaluating a single Feynman diagram. At the next order, $\mathcal{O}(g^6)$, we will have to do more work. Applying the rules depicted in figures 1 and 2, the only diagrams remaining are bona fide two-loop matter[4] corrections to the gauge/scalar propagator and bona fide one-loop matter corrections to the triple vertex, see figure 3. Let us concentrate on the former. We can reduce this class of diagram even further. Introducing a fat graph notation, where fundamental fields are represented by single lines, and adjoint ones by double lines, we find that the following topology of diagram cancels between the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ theories

 $\quad \sim 2N \, i f^{qik} \operatorname{Tr}(T^k T^j T^q) = \frac{N^2}{2}\delta^{ij}$

---

[4]By "matter" we mean the fields in the hypermultiplet, whether they are in the adjoint or fundamental representation.

whilst the adjoint counter-part has the same colour factor

$$i^4 f^{qik} f^{klr} f^{rjm} f^{mlq} = \frac{N^2}{2} \delta^{ij}. \tag{15}$$

For the two-loop matter corrections to the propagator, we find no further cancellations. We are left with eight diagrams which are collected and evaluated in appendix A.

It turns out that $\zeta(3)$ is very hard to come by in these Feynman diagrams. In fact, the only time it appears is from the well-known integral

$$\mathcal{I} = \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{k^2 q^2 (k-p)^2 (q-p)^2 (k-q)^2} = \frac{1}{(4\pi)^4 p^2} 6\,\zeta(3), \tag{16}$$

arising solely from the topology:
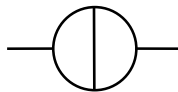
which typically also contains other terms (owing to numerators), however these other terms do not contain $\zeta(3)$. The general structure of these diagrams is as follows

$$\frac{(p^2)^{1-2\epsilon}}{(4\pi)^{4-2\epsilon}} \left[A_1 \left(\frac{1}{\epsilon^2} - \zeta(2) \right) + \frac{A_2}{\epsilon} + A_3 + A_4 \zeta(3) + \mathcal{O}(\epsilon) \right], \tag{17}$$

where p is the external momentum, and where the A_i are rational numbers[5]. This structure is also found for all the other two-loop matter correction diagrams, albeit with $A_4 = 0$. In summing the contributions from all diagrams we find that the coefficients A_1, A_2, and A_3 sum to zero, and so all divergences and non-$\zeta(3)$ terms cancel entirely. The details of the calculation are collected in appendix A.

The diagrams responsible for the $\zeta(3)$ terms are shown below, where the solid (dashed) lines in the loop indicate the scalar (fermion) fields of the hypermultiplet. The external lines represent the adjoint gauge field (wiggly) or real scalar field (straight)[6]. Let us begin by calculating the colour factor associated with these diagrams. We are interested in the difference between taking the matter in the adjoint and $2N$ times in the fundamental, the result being

$$i^4 f^{qik} f^{klr} f^{rjm} f^{mlq} - 2N \operatorname{Tr}[T^i T^k T^j T^k] = \frac{N^2}{2} \delta^{ij} - 2N \frac{-\delta^{ij}}{4N} = \frac{N^2 + 1}{2} \delta^{ij}. \tag{18}$$

Since in the perturbative expansion of the Wilson loop to second order, i.e. two fields emanating from the loop, one gains a factor of $\operatorname{Tr}(T^i T^i)/N \sim (N^2 - 1)/N$, one can already verify that the correct colour factor has emerged for a match to (11).

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[5]The Euler-gamma terms have been removed through the usual $e^{2\epsilon\gamma}$ factor.

[6]Note that for fermion loops the real scalar field is also exchanged in the loop. For convenience we have let the vertical wiggly line represent both gauge and scalar exchange in this instance.

Explicitly we find that the colour-stripped diagram-differences yield the following results

$$\cdots\bigcirc\cdots - \cdots\bigcirc\cdots = \frac{4}{3}p^4\left(\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\mathcal{I} + \text{non-}\zeta(3)$$

$$\cdots\bigcirc\cdots - \cdots\bigcirc\cdots = \frac{8}{3}p^4\left(\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\mathcal{I} + \text{non-}\zeta(3)$$

$$=\bigcirc = - =\bigcirc = = 4\,p^4\,\delta^{IJ}\,\mathcal{I} + \text{non-}\zeta(3)$$

where $\mathcal{I}$ is the expression given in (16), and the "non-$\zeta(3)$" terms are the divergences and constant terms shown schematically in (17), and which cancel identically against the rest of the diagrams, which have no $\zeta(3)$ contribution. It is clear that the gauge field and real scalar field propagator contributions are equal, as they are in the celebrated one-loop calculation for the $\mathcal{N} = 4$ theory presented in [3]. Adding the external propagators to these amputated diagrams also reveals that the result is proportional to the bare propagators[7] $\delta^{\mu\nu}/p^2$ and δ^{IJ}/p^2, for the gauge, and real scalar fields respectively. Fourier transforming back to position space and evaluating the expectation value of the Wilson loop, one obtains (2). Plugging in the circular contour $x^\mu = (\cos\tau, \sin\tau, 0, 0)$, and $\Theta^I(\tau) = \delta^{I1}$, one obtains

$$\frac{|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2}{4\pi^2(x_1 - x_2)^2} = \frac{1}{4\pi^2}\frac{1}{2}. \tag{19}$$

The result is that (2) is exactly the expression given in (11), namely

$$\frac{3\,\zeta(3)}{512\pi^4}\frac{(N^2 - 1)(N^2 + 1)}{N}, \tag{20}$$

and so we have recovered the matrix model result from perturbation theory.

3.1 One-loop corrected trivalent graph

We now turn our attention to the one-matter-loop corrected trivalent graph, contributions to which are shown in figure 4. This diagram presents an interesting manifestation of the difference between $SU(2)$ and $SU(N)$ for $N > 2$. Let us look at the

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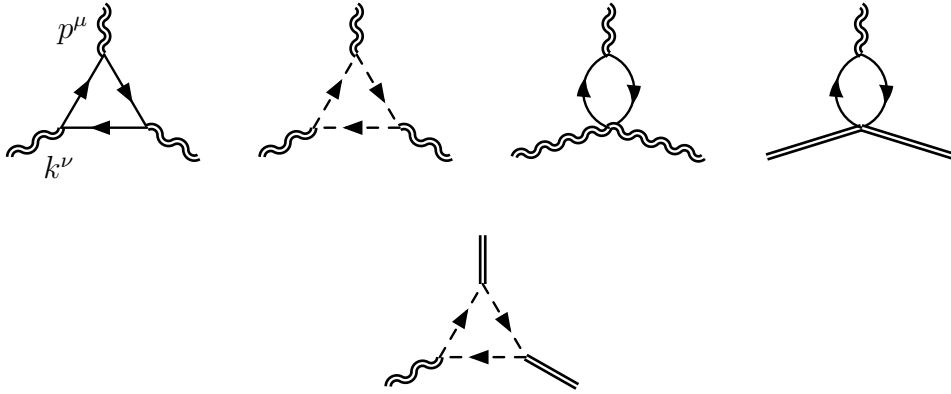[7]Up to the extra $p^\mu p^\nu/p^2$ factors, which may be removed by a gauge transformation.

Figure 4: One-matter-loop corrections to the triple vertex, shown (to reduce clutter) for the $\mathcal{N} = 2$ theory only. In the $\mathcal{N} = 4$ case internal lines are doubled.

colour factors arising from the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ versions of this graph. We have

$$
\begin{aligned}
\mathcal{N} = 4 \;\; &\rightarrow \;\; i^3 f^{qim} f^{mjn} f^{nkq} = \frac{N}{2} i f^{ijk}, \\
\mathcal{N} = 2 \;\; &\rightarrow \;\; 2N \operatorname{Tr}(T^i T^j T^k) = \frac{N}{2} i f^{ijk} + \frac{N}{2} d^{ijk}.
\end{aligned}
\tag{21}
$$

We see immediately that for $SU(2)$, where $d^{ijk} = 0$, these diagrams cancel identically. But for $SU(N)$ with $N > 2$, we are left with a term proportional to the totally symmetric structure constant d^{ijk}. This immediately implies that the result must also be symmetric in interchange between the two momenta p^μ and k^ν, see figure 4. In the Wilson loop expanded to 3$^{\rm rd}$ order, we will encounter another $\operatorname{Tr}(T^m T^n T^q) \sim i f^{mnq} + d^{mnq}$. Clearly only the d^{mnq} can survive, as the trace will be contracted with the d^{ijk} coming from the loop-corrected vertex. This leaves us with a completely symmetrized sum of path orderings in the Wilson loop expansion, which means that the path-ordering is removed, and we have complete integrals over each of the insertion points. The situation is most easily seen for the triple vertex with two real scalar fields and one gauge field[8]. It is clear that the corrected vertex must be of the form

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[8]Generalizing to the case of three external gauge fields does not alter the result.

where $F(p, k) = F(k, p)$. Decorating with propagators and Fourier transforming to position space, we will have, schematically

$$\partial_{x_3^\mu} \int d^4p \, d^4k \, e^{ip\cdot(x_1-x_3)+ik\cdot(x_2-x_3)} \frac{F(p, k)}{p^2 k^2 (p+k)^2}. \tag{22}$$

But then this will be integrated over in the Wilson loop

$$\oint d\tau_3 \, \dot{x}_3 \cdot \partial_{x_3} \int d^4p \, d^4k \, e^{ip\cdot(x_1-x_3)+ik\cdot(x_2-x_3)} \frac{F(p, k)}{p^2 k^2 (p+k)^2} = 0 \tag{23}$$

which is the integral of a total-derivative and therefore vanishes. Note that this cancellation is strictly only true for a closed loop, an open loop could give boundary contributions.

4 Comments on light-like loops and loops on $\mathbb{R}^2$ and S^2

As stressed in the introduction, the difference between the Wilson loop in $\mathcal{N} = 4$ and superconformal $\mathcal{N} = 2$ SYM appears as a term proportional to the bare gauge field (and associated real scalar) propagator. This fact is independent of the shape of the Wilson loop, although we were primarily interested in the circle for obvious reasons. In this section we would like to point-out a couple of implications of this result.

4.1 Zarembo loops in the plane

The first is that we could consider Wilson loops of various shape, as long as the scalar coupling $\Theta^I(\tau)$ remains on an S^1. This is because in the $\mathcal{N} = 2$ theory, we only have two real scalar fields Φ_I. This class includes the planar Zarembo loops [11, 14] of arbitrary shape, and the longitudes Wilson loop of [12]. The Zarembo loops are defined by

$$|\dot{x}| \, \Theta^I(\tau) = M_\mu^I \, \dot{x}^\mu, \quad M_\mu^I M_\nu^I = \delta_{\mu\nu}, \tag{24}$$

and so have the property that the combined gauge field and scalar exchange, i.e. the LHS of (19), is identically zero. This leads to the result $\langle W_{\text{Zarembo}} \rangle = 1$, which is true to all orders in perturbation theory [15, 16]. It is clear that the triviality of the expectation value will not be disturbed in the superconformal $\mathcal{N} = 2$ case at $\mathcal{O}(g^6)$, since the corrected scalar and gauge field propagators remain equal. This makes sense, as the supersymmetry respected by this Wilson loop in the $\mathcal{N} = 2$ theory is the same as that in the $\mathcal{N} = 4$ theory.

4.2 Longitudes Wilson loop

The longitudes Wilson loop is given by an "orange wedge", descending from the north pole of an S^2 along a great circle to the south pole, and then returning along a second

longitude shifted by an azimuthal angle α

$$
\begin{aligned}
x^{\mu} &= \begin{cases} (\sin\tau, 0, \cos\tau, 0), & 0 \leq \tau \leq \pi, \\ (-\cos\alpha\sin\tau, -\sin\alpha\sin\tau, \cos\tau, 0), & \pi \leq \tau \leq 2\pi. \end{cases} \\
|\dot{x}|\Theta^{I}\Phi_{I} &= \begin{cases} \Phi_{2}, & 0 \leq \tau \leq \pi, \\ -\Phi_{2}\cos\alpha + \Phi_{1}\sin\alpha, & \pi \leq \tau \leq 2\pi. \end{cases}
\end{aligned}
\tag{25}
$$

As conjectured in [12, 17, 18], and backed-up in [19–24], it seems very certain that these Wilson loops in $\mathcal{N} = 4$ SYM are captured completely by pure two-dimensional Yang-Mills theory, and therefore enjoy invariance under area-preserving diffeomorphisms. This results in an expectation value which depends only on the area enclosed by the longitudes. Therefore the $\mathcal{O}(g^2)$ term is proportional to this area, and so then is the correction introduced by the $\mathcal{N} = 2$ superconformal theory at $\mathcal{O}(g^6)$. This observation leads one to the possibility that there may exist some deformation of pure 2-d Yang-Mills or of the correspondence between it and the Wilson loops of [12] which would accommodate the $\mathcal{N} = 2$ superconformal analogue, perhaps as a 1-loop determinant factor to be introduced into the localized path integral of [24].

4.3 Light-like Wilson loops and scattering amplitudes

As a final remark it is interesting to consider light-like polygonal Wilson loops in the superconformal $\mathcal{N} = 2$ theory, and any possible connection they may have to scattering amplitudes. We have proven that at $\mathcal{O}(g^4)$ there is no difference between the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ results for Wilson loops. However, in the $\mathcal{N} = 4$ theory, and in the planar limit, we know that a light-like Wilson loop at $\mathcal{O}(g^4)$ is equivalent to a gluon scattering amplitude at two-loops [25]. It is interesting to ask whether gluon scattering in the $\mathcal{N} = 2$ superconformal theory at large-N, i.e. in the Veneziano limit, is modified with respect to $\mathcal{N} = 4$ SYM, and more importantly at which order in perturbation theory. In the work [26], it was shown that there is no modification at one-loop, see [27] for related work. Based on our considerations of section 3, at one-loop, we showed that propagator loop corrections cancel, but the corrected triple vertex does not necessarily cancel for $N > 2$. It would be interesting to understand how this becomes consistent with [26].

In order for there to be an analogous scattering amplitude/Wilson loop duality for the $\mathcal{N} = 2$ theory at large-N, the gluon scattering amplitudes would have to be the same in the two theories at two-loops, and different (by exactly the term given in (2), evaluated for a light-like polygonal contour) at three-loops. It would be necessary to have higher loop versions of the results in [26] in order to check this. In any case, it is interesting to continue to investigate gluon scattering amplitudes in the $\mathcal{N} = 2$ theory. There are indications that the theory may be integrable in the planar limit [28], and there is also work to identify a string dual [29].

Acknowledgements

A   Two-loop matter corrections

We write the action of Euclidean $\mathcal{N} = 4$ SYM following [13], as the sum of an $\mathcal{N} = 1$ SYM in six dimensions (consisting of a 6-d gauge field A_M and an 8-component Majorana-Weyl spinor λ) dimensionally reduced to $d = 4 - 2\epsilon$, and a single adjoint hypermultiplet consisting of 2 complex scalar fields q^α, $\alpha = 1, 2$, $(q^\alpha)^\dagger = q_\alpha$, and a complex four-dimensional (four-component) spinor Ψ

$$
\begin{aligned}
S_{\mathcal{N}=4} = \frac{2}{g^2} \operatorname{Tr} \int d^d x \Bigg[&\frac{1}{4} F_{MN}^2 + \frac{i}{2} \bar{\lambda} \Gamma^M D_M \lambda + D_M q^\alpha D_M q_\alpha + i \bar{\Psi} \Gamma^M D_M \Psi \\
&+ \bar{\lambda} \gamma^\alpha [q_\alpha, \Psi] + \bar{\Psi} \gamma_\alpha [q^\alpha, \lambda] + [q_\alpha, q^\beta][q_\beta, q^\alpha] - \frac{1}{2} [q_\alpha, q^\alpha][q_\beta, q^\beta] \Bigg],
\end{aligned}
\tag{26}
$$

where the γ^α are a set of gamma matrices obeying $\{\gamma^\alpha, \gamma_\beta\} = 2\delta_\beta^\alpha$ and anti-commuting with all Γ^M, see [13] for details. To deform this theory to the $\mathcal{N} = 2$ superconformal SYM we take $2N$ hypermultiplets instead of one, so that the q and Ψ fields earn a flavour index and become vectors, and we take them in the fundamental, instead of the adjoint representation

$$
\begin{aligned}
S_{\mathcal{N}=2} = \frac{1}{g^2} \int d^d x \Bigg[&2 \operatorname{Tr} \left(\frac{1}{4} F_{MN}^2 + \frac{i}{2} \bar{\lambda} \Gamma^M D_M \lambda \right) + D_M \vec{q}^\alpha \cdot D_M \vec{q}_\alpha + i \bar{\vec{\Psi}} \cdot \Gamma^M D_M \vec{\Psi} \\
&- \bar{\lambda}^i \gamma^\alpha \vec{q}_\alpha \cdot T^i \vec{\Psi} - \bar{\vec{\Psi}} \cdot \gamma_\alpha T^i \vec{q}^\alpha \lambda^i \\
&+ \left(\vec{q}_\alpha \cdot T^i \vec{q}^\beta \right) \left(\vec{q}_\beta \cdot T^i \vec{q}^\alpha \right) - \frac{1}{2} \left(\vec{q}_\alpha \cdot T^i \vec{q}^\alpha \right) \left(\vec{q}_\beta \cdot T^i \vec{q}^\beta \right) \Bigg],
\end{aligned}
\tag{27}
$$

where we have introduced the standard $SU(N)$ generators T^i, $i = 1, \ldots, N^2 - 1$, obeying

$$
T^i T^i = \frac{N^2 - 1}{2N} \mathbf{1}, \quad \operatorname{Tr}(T^i T^j) = \frac{1}{2} \delta^{ij}, \quad [T^i, T^j] = i f^{ijk} T^k, \quad f^{ijk} f^{ijl} = N \delta^{kl},
$$
$$
\{T^i, T^j\} = \frac{1}{N} \delta^{ij} \mathbf{1} + d^{ijk} T^k,
\tag{28}
$$

and the covariant derivatives act as follows

$$
\begin{aligned}
D_M \vec{q}^\alpha &= \partial_M \vec{q}^\alpha - i A_M^i T^i \vec{q}^\alpha, \quad D_M \vec{q}_\alpha = \partial_M \vec{q}_\alpha + i \vec{q}_\alpha T^i A_M^i, \\
D_M \vec{\Psi} &= \partial_M \vec{\Psi} - i A_M^i T^i \vec{\Psi}.
\end{aligned}
\tag{29}
$$

The pure gauge portion of the two actions (26) and (27) is exactly the same. This means that in the diagrammatic difference, ghosts cancel trivially. We are free to work in Feynman gauge where the propagators are

$$\langle q^\alpha q_\beta \rangle = g^2 \frac{\delta_\beta^\alpha}{p^2}, \quad \langle \Psi \bar{\Psi} \rangle = -g^2 \frac{\Gamma^\mu p_\mu}{p^2},$$

$$\langle A_M^i A_N^j \rangle = g^2 \frac{\delta^{ij} \delta_{MN}}{p^2}, \quad \langle \lambda^i \bar{\lambda}^j \rangle = -g^2 \delta^{ij} \frac{\Gamma^\mu p_\mu}{p^2}, \tag{30}$$

and where there is an implied delta function on the hypermultiplet propagators for either adjoint, or fundamental and flavour indices.

A.1   The diagrams

There are 8 diagrams which do not cancel trivially between the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ theories. In this section we list the results for each of them. Every diagram gives a common factor of

$$\frac{(p^2)^{1-2\epsilon}}{(4\pi)^d}(N^2 + 1)\delta^{ij}, \tag{31}$$

where δ^{ij} is the delta function on the colour indices. We suppress this factor below. The results are given in terms of the scale-free integrals defined by [30]

$$G(n_1, n_2, n_3, n_4, n_5) \equiv \frac{1}{\pi^d} \int \frac{d^d k \, d^d q}{(k^2)^{n_1}(q^2)^{n_2}((k-p)^2)^{n_3}((q-p)^2)^{n_4}((q-k)^2)^{n_5}} \tag{32}$$

where in the above expression we replace $p^2 \to 1$. These integrals may be reduced to products of one-loop integrals using well-known techniques [30][9]. In the case of corrections to the gauge field propagator, which we present first, we take the trace over the external space-time indices μ, and ν. We have verified that the projection onto p_μ, p_ν yields the same cancellation of non-$\zeta(3)$ terms, and yields no $\zeta(3)$ contribution. Note that the comments of footnote 6 apply equally to the diagrams below.

The external lines are either wiggly (gauge field A_μ) or straight (scalar field Φ_I), while the internal lines carry arrows and are straight for the hypermultiplet scalar q^α and dashed for the hypermultiplet fermion Ψ. The dotted line denotes the gaugino λ. We take the "diagrammatic difference" between the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ theories, and thus report the differences between the diagrams. The fundamental representation is indicated by single, as opposed to double, lines.

---

[9] The integral from which the $\zeta(3)$ comes, i.e. (16), is $G(1, 1, 1, 1, 1)$.

$$= 4G(-1,1,1,1,1) + 4G(0,1,1,0,1)$$

$$- 12G(0,1,1,1,1) + 2G(1,1,1,1,-1) + 5G(1,1,1,1,0) + 2G(1,1,1,1,1)$$

$$= 4[-2(\epsilon-1)G(0,1,1,0,1) + 4(\epsilon-1)G(0,1,1,1,1)$$

$$- (\epsilon-1)G(1,1,1,1,1) + 2[G(1,1,1,1,-1) + G(1,1,1,1,0)]]$$

$$= 16(G(0,1,1,0,1) - G(1,1,1,1,-1) - G(1,1,1,1,0))$$

$$= -8(\epsilon-2)G(0,1,1,0,1)$$

$$= -4(G(-1,1,1,1,1) + 5G(0,1,1,0,1) - 2G(0,1,1,1,1))$$

$$= 3(2G(1,1,1,1,-1) + G(1,1,1,1,0))$$

$$= 4G(0,1,1,1,1) - 8G(0,1,1,0,1)$$

$$= 32(\epsilon-1)[G(-1,1,1,1,1) + G(0,1,1,0,1)$$

$$- G(0,1,1,1,1)]$$

$$= 2\delta^{IJ}(2G(0,1,1,0,1) - 4G(0,1,1,1,1) + G(1,1,1,1,1))$$

$$= 4\delta^{IJ}G(0,1,1,0,1)$$

$$= -16\delta^{IJ}(G(-1,1,1,1,1) + G(0,1,1,0,1)$$

$$- G(0,1,1,1,1))$$

In summing these contributions we find that all non-$\zeta(3)$ contributions, including divergences, canel identically. The 1-loop corrections to the hypermultiplet propagators indicated above, are given explicitly by the following diagrams (we show only the $\mathcal{N}=4$ case)

The various G functions may be reduced to one-loop forms as follows [30],

$$G(1,1,1,1,-1) = -\frac{1}{2}G(1,1)^2,$$

$$G(0,1,1,1,1) = G(1,1)G(1,1+\epsilon),$$

$$G(1,1,1,1,0) = G(1,1)^2,$$

$$G(1,1,1,1,1) = -\frac{1}{\epsilon}(G(1,1)G(2,1) - G(1,1)G(2,1+\epsilon)),$$

$$G(0,1,1,0,1) = G(1,1)G(1,\epsilon), \tag{33}$$

$$G(-1,1,1,1,1) = \frac{1}{2}G(1,1)(G(1,1+\epsilon) - G(1,\epsilon)),$$

$$G(-1,1,1,0,1) = \frac{1}{2}G(1,1)(G(1,\epsilon) - G(1,\epsilon-1)),$$

$$G(0,1,1,-1,1) = \frac{1}{2}G(1,1)(G(1,\epsilon) - G(1,\epsilon-1)),$$

where

$$G(n_1, n_2) = \frac{\Gamma(n_1 + n_2 - d/2)\Gamma(d/2 - n_1)\Gamma(d/2 - n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d - n_1 - n_2)}, \qquad d = 4 - 2\epsilon. \tag{34}$$

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