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Chiral primary one-point functions in the D3-D7 defect conformal field theory

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Abstract

We compute the one-point functions of chiral primary operators in the non-supersymmetric defect conformal field theory that is dual to the IIB string theory on $AdS_5 \times S^5$ background with a probe D7 brane with internal gauge field flux, both in perturbative Yang-Mills theory and in the string theory dual. The former is expected to be accurate at weak coupling whereas the latter should be accurate in the planar strong coupling limit of the gauge theory. We consider the distinct cases where the D7 brane has geometry $AdS_4 \times S^4$ with an instanton bundle of the worldvolume gauge fields on S^4 and $AdS_4 \times S^2 \times S^2$ with Dirac monopole bundles on each S^2. The gauge theory computation and the string theory computation can be compared directly in the planar limit and then a subsequent limit where the worldvolume flux is large. We find that there is exact agreement between the two in the leading order.

1 Introduction

Probe branes and defect conformal field theories have been widely studied in the context of AdS/CFT holography [1]-[23]. The classic example is the D3-D5 intersection [3, 4] where the relative orientation of the D3 and D5 branes is shown in table 1. In the appropriate limit, this system becomes $AdS_5 \times S^5$ bisected by the probe D5 brane with world volume geometry $AdS_4 \times S^2$. The original idea [1, 2, 5] was that, in a sense, the AdS/CFT correspondence acts twice. The near horizon geometry of the D3-D5 system contains two sets of excitations, those of closed IIB superstrings occupying AdS_5, which are dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on the R^4 boundary of AdS_5, and open strings connecting the D3 and D5 branes, which are dual to field theory excitations on the R^3 boundary of AdS_4, the latter forming a

co-dimension one defect in R^4. The resulting field theory dual is then $\mathcal{N} = 4$ super-symmetric Yang-Mills theory occupying 3+1 dimensions and interacting with a field theory living on a 2+1-dimensional defect. The D3-D5 system preserves half of the supersymmetries of the bulk $\mathcal{N} = 4$ theory and the degrees of freedom which live on the defect are a bi-fundamental hypermultiplet which transforms in the fundamental representation of the bulk gauge group and the fundamental representation of the defect gauge group. The defect theory reduces the conformal symmetry of $\mathcal{N} = 4$ Yang-Mills theory to the $SO(2,3)$ conformal symmetry of the defect. It also reduces the $SO(6)$ R-symmetry to $SO(3) \times SO(3)$ and the superalgebra from $PSU(2,2|4)$ to $OSP(4|4)$.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$D3$	×	×	×	×						
$D5$	×	×	×		×	×	×			
$D7$	×	×	×		×	×	×	×	×	

Table 1 : **D3**, **D5 and D7 orientation**

An interesting variant of the D3-D5 system is where the worldvolume gauge fields of the probe D5 brane have a monopole bundle with a quantized U(1) magnetic flux. The geometry of the brane is still $AdS_4 \times S^2$ and the flux is on the S^2. As depicted in Figure 1, in the field theory dual, the defect separates 3+1-dimensional space-time into regions where the gauge group of $\mathcal{N} = 4$ Yang-Mills theory has different ranks, N on one side of the defect and $N - k$ on the other side of the defect, with k the number of units of Dirac monopole flux. In the string theory this corresponds to the situation where k of the N D3 branes end on the worldvolume of a D5 brane. The low energy action of the D5 brane contains a Wess-Zumino term proportional to $\int_{D5} C^{(4)} \wedge F$, with $C^{(4)}$ the Ramond-Ramond 4-form carrying the D3 brane charge, and F the field strength of the worldvolume gauge field. The D3 branes intersect the D5 brane on a subvolume which can be linked by a 2-sphere. The first Chern class of the gauge field, the integral of F over the 2-sphere must be equal to the number of $D3$ branes which terminate there. The holographic system with this flux has some interesting new features which have been exploited in holographic constructions of 2+1-dimensional physical systems [11, 20]. In the limits which are taken in holographic theories, the magnetic flux has to be large, $k \sim \sqrt{\lambda}$, where λ is the 't Hooft coupling of $\mathcal{N} = 4$ Yang-Mills theory, in order to have an effect on the geometry of the D5 brane and to be visible in the physics of the holographic dual.

In an interesting recent paper [25], Nagasaki and Yamaguchi showed that the one-point functions of chiral primary operators could be computed for the D3-D5 defect field theory in the limit of large monopole number k. The defect preserves an $SO(3) \times SO(3)$ subgroup of the $SO(6)$ R-symmetry and, in the defect conformal field theory, an $SO(3) \times SO(3)$ invariant primary operator $\mathcal{O}_\Delta(x)$ can have a one-point function

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{C_\Delta}{|z|^\Delta}, \tag{1}$$
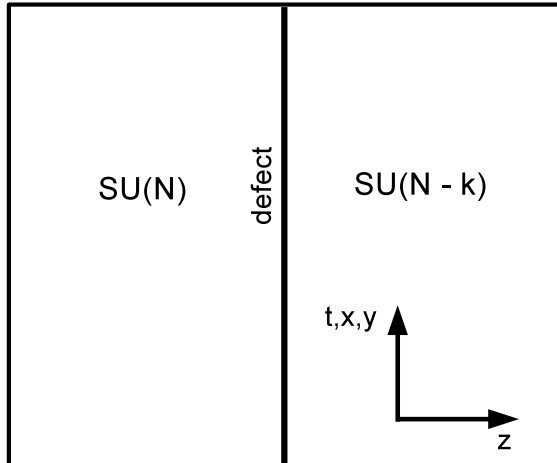
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Figure 1: **Defect conformal field theory**: The defect is depicted by the vertical line through the center of the diagram. The defect occupies 2+1-dimensions and divides the 3+1-dimensional space-time into two different regions which are occupied by $\mathcal{N} = 4$ supersymmetric Yang-Mills theories with gauge group $SU(N)$ on the left and $SU(N - k)$ on the right. The integer k is the number of D3 branes which end on the D5 or D7 brane. In the latter cases, k should be replaced by $k_1 k_2$ or d_G.

where Δ is the conformal dimension of the operator in the bulk and $|z|$ is the distance between x and the defect. The result for the one-point function of the chiral primary operator with conformal dimension Δ in the limit of large $\frac{k}{\sqrt{\lambda}}$ [25] is

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{k}{\sqrt{\Delta}} \left(\frac{2\pi^2 k^2}{\lambda} \right)^{\Delta/2} Y_\Delta(0) \frac{1}{|z|^\Delta}, \tag{2}$$

where Δ must be an even integer and $Y_\Delta(\psi)$ is the unique $SO(3) \times SO(3)$ symmetric spherical harmonic at level Δ. (See Section 2, equation (26)). In equation (2), it is evaluated at $\psi = 0$, where the S^2 has maximum volume.

What is remarkable is that this result is obtained on both the gauge theory and the string theory side. In both cases, it is of leading order in the parameter $\frac{\sqrt{\lambda}}{k}$ which is assumed to be small on both sides. It is one of the few instances where the gauge and string theory computations can be compared directly. Moreover, it is a test of the AdS/CFT correspondence in the presence of probe branes complementary to previous tests which have examined the open D3-D5 string sector in the plane wave BMN limit [26] [27]-[30]. The accuracy of the BMN limit in the string theory depends on a large angular momentum on S^5 whereas, in the present case, it is k, the number of D3 branes which end on the D5 brane that is large.

In this Paper, we will extend and elaborate on this interesting result. In particular, we will extend it to the non-supersymmetric D3-D7 system. We will consider two different cases, the first with $SO(3) \times SO(3)$ symmetry where the probe brane geometry is $AdS_4 \times S^2 \times S^2$ and where the worldvolume U(1) gauge field of the D7

brane has k_1 and k_2 units of magnetic flux on the two 2-spheres, and the second with $SO(5)$ symmetry, where the probe brane geometry has is $AdS_4 \times S^4$ and the worldvolume gauge fields form a certain instanton bundle on the S^4 with instanton number d_G. The situation on the gauge theory side is again as illustrated in Figure 1, only in the first case the parameter k gets replaced by $k_1 k_2$ and in the second case it gets replaced by d_G. Remarkably, even in these non-supersymmetric systems, the one-point functions of chiral primary operators on the string and gauge theory sides agree in the leading order of a large $\frac{k_1}{\sqrt{\lambda}}, \frac{k_2}{\sqrt{\lambda}}$ limit of the $SO(3) \times SO(3)$ symmetric system,

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{k_1 k_2}{\sqrt{\Delta}} \left(\frac{2\pi^2(k_1^2 + k_2^2)}{\lambda} \right)^{\Delta/2} Y_\Delta(\arctan(k_2/k_1)) \frac{1}{|z|^\Delta}, \quad (z < 0), \quad (3)$$

or the limit of large $\frac{n}{\sqrt{\lambda}}$, with instanton number $d_G = \frac{1}{6}(n+1)(n+2)(n+3)$, in the $SO(5)$ symmetric system,

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{n^3}{6\sqrt{\Delta}} \left(\frac{\pi^2 n^2}{\lambda} \right)^{\frac{\Delta}{2}} \mathcal{Y}_\Delta(0) \frac{1}{|z|^\Delta}, \quad (z < 0), \quad (4)$$

where $\mathcal{Y}_\Delta(\theta)$ is the unique $O(5)$ symmetric spherical harmonic at level Δ. (See Section 4.1.1). Here it is evaluated at $\theta = 0$ where the size of the S^4 is maximal. The second remarkable fact is that a discontinuity in what is meant by chiral primary operator that is expected to occur as one crosses the defect will also turn out to be visible in the supergravity computation. On the gauge theory side, it is easily seen. The Yang-Mills theory computations which result in (3) and (4) are semi-classical, obtained by simply substituting a classical configuration of the scalar fields of the $\mathcal{N} = 4$ Yang-Mills theory. These fields are non-zero only on the side of the defect with $SU(N)$ gauge group, they vanish on the side with $SU(N - k_1 k_2)$ or $SU(N - d_G)$ gauge group. The Yang-Mills theory therefore tells us that the leading contribution to the one-point function (3) and (4) is nonzero only on the side of the defect where the gauge group is larger (thus the qualification $z < 0$ in that formula). It is interesting to ask whether this is reflected in the strong coupling, string theory computation. We shall find that indeed it is. There is a discontinuity in the leading order of the one-point function as the point crosses the D7 brane, with the value being much smaller on the side of the brane with smaller D3 brane flux.

The D3-D7 system is of interest as a holographic dual of a defect field theory where the degrees of freedom living on the 2+1-dimensional defect are fermions [9]-[23]. The flat space orientation of the D3 and D7 branes is displayed in Table 1. This is an $\#_{ND} = 6$ configuration and it is not supersymmetric [24]. Nevertheless, in flat space, there is no tachyon in the spectrum of the D3-D7 open strings and the only zero modes are in the Ramond sector. The light degrees of freedom are thus chiral fermions inhabiting the 2+1-dimensional overlap of the D3 and D7 worldvolumes.

However, the most straightforward constructions of the D3-D7 system, where the D7 brane worldvolume is either $AdS_4 \times S^4$ or $AdS_5 \times S^2 \times S^2$ without fluxes are unstable [9, 10, 11, 12]. The systems are non-supersymmetric and the D3 branes and

D7 brane repel each other. The symptoms of the resulting instability are tachyonic modes which violate the Breitenholder-Freedman bound for fluctuations of the embeddings. They are stabilized by adding the topological fluxes, the instanton bundle on the S^4 when the instanton number is large enough [11] or Dirac monopole bundles on either one or both of the S^2's in $S^2 \times S^2$, when the flux is large enough [12]. This lower bound on the flux is, in the first case, where the instanton number (which we shall call d_G) exceeds a number of the order of $\lambda^{3/2}$ and, in the second case, where the monopole numbers k_1 and/or k_2 are of order $\sqrt{\lambda}$. In this Paper, we shall be interested in the limits where $d_G \gg \lambda^{3/2}$ or k_1 and/or $k_2 \gg \sqrt{\lambda}$ as it is in this region where the gauge theory and string theory computations can be compared.

In Section 2, we present the Yang-Mills theory computation of the one-point function of chiral primary operators in the case where $SO(3) \times SO(3)$ R-symmetry is preserved. In Section 3 we present the same computation in supergravity. In Section 4 we examine the case where the $SO(5)$ subgroup of R-symmetry is preserved. Section 5 contains a discussion and conclusions.

2   Defect $\mathcal{N} = 4$ Yang-Mills Theory

2.1   Classical solution

We begin with a classical solution of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, where we set the vector and spinor fields to zero, $A_\mu = 0$, $\psi = 0$ and look for a solution of the remaining equations for the scalar fields,

$$\nabla^2 \phi_i - \sum_{j=1}^{6} [\phi_j, [\phi_j, \phi_i]] = 0, \tag{5}$$

$$\sum_{i=1}^{6} [\phi_i, \nabla \phi_i] = 0. \tag{6}$$

Equation (6) is the condition that the $SU(N)$ color current is zero. A solution of these equations with the appropriate symmetries is

$$\phi_i(z) = -\frac{1}{z}(t_i^{k_1} \otimes 1_{k_2 \times k_2}) \oplus 0_{(N-k_1 k_2) \times (N-k_1 k_2)}, \qquad \text{for} \quad i = 1, 2, 3, \tag{7}$$

$$\phi_i(z) = -\frac{1}{z}(1_{k_1 \times k_1} \otimes t_i^{k_2}) \oplus 0_{(N-k_1 k_2) \times (N-k_1 k_2)}, \qquad \text{for} \quad i = 4, 5, 6. \tag{8}$$

Here $t_i^{k_1}$ for $i = 1, 2, 3$ are generators of the k_1-dimensional irreducible representation of $SU(2)$ and $t_i^{k_2}$ for $i = 4, 5, 6$ are generators of the k_2-dimensional irreducible representation of $SU(2)$. Furthermore, $1_{k_1 \times k_1}$ and $1_{k_2 \times k_2}$ are unit matrices of dimension $k_1 \times k_1$ and $k_2 \times k_2$ respectively. This generalizes the solution that was used in reference [25] that was suitable for a single S^2 to a solution suitable for the product $S^2 \times S^2$.

The matrices ϕ_i have the following property

$$\phi_1^2 + \phi_2^2 + \phi_3^2 \;=\; \frac{1}{4z^2}(k_1^2 - 1)\mathbb{1}_{k_1 k_2 \times k_1 k_2} \oplus 0_{(N-k_1 k_2)\times(N-k_1 k_2)}, \tag{9}$$

$$\phi_4^2 + \phi_5^2 + \phi_6^2 \;=\; \frac{1}{4z^2}(k_2^2 - 1)\mathbb{1}_{k_1 k_2 \times k_1 k_2} \oplus 0_{(N-k_1 k_2)\times(N-k_1 k_2)}, \tag{10}$$

and hence

$$\mathrm{Tr}(\phi_1^2 + \phi_2^2 + \phi_3^2) \;=\; \frac{1}{4z^2}(k_1^2 - 1)\,k_1 k_2, \tag{11}$$

$$\mathrm{Tr}(\phi_4^2 + \phi_5^2 + \phi_6^2) \;=\; \frac{1}{4z^2}(k_2^2 - 1)\,k_1 k_2. \tag{12}$$

We will use these equations for computing the classical limit of the one-point function of a chiral primary operator, which we shall define in the next subsection.

2.2 The Chiral Primary Operators with $SO(3) \times SO(3)$ symmetry

The chiral primary operators of $\mathcal{N} = 4$ Yang-Mills theory are

$$\mathcal{O}_{\Delta I}(x) \equiv \frac{(8\pi^2)^{\frac{\Delta}{2}}}{\lambda^{\frac{\Delta}{2}}\sqrt{\Delta}} C_I^{i_1 i_2 \dots i_\Delta}\,\mathrm{Tr}\left(\phi_{i_1}(x)\phi_{i_2}(x)\dots\phi_{i_\Delta}(x)\right), \tag{13}$$

where the ϕ_i's can be any of the six real scalar fields. Here Δ is an integer which counts the number of fields and it is also equal to the bulk scaling dimension of the operator which does not depend on the coupling constant. The tensors $C_I^{i_1 i_2 \dots i_\Delta}$ are symmetric and traceless in the indices $i_1 \dots i_\Delta$ and they can be chosen to satisfy an orthogonality relation

$$\sum_{i_1 \dots i_\Delta = 1}^{6} C_{I_1}^{i_1 i_2 \dots i_\Delta} C_{I_2}^{i_1 i_2 \dots i_\Delta} = \delta_{I_1 I_2}\quad. \tag{14}$$

The indices I_a label different such tensors which are equal in number to the dimension of the totally symmetric and traceless irreducible representation of $SO(6)$ with Δ indices, $I_a = 1, \dots, (3+\Delta)(2+\Delta)^2(1+\Delta)/12$. With the normalization in (13) and (14), the planar limit of two point functions of the theory without the defect are unit normalized as

$$\langle \mathcal{O}_{\Delta_1 I_1}(x)\mathcal{O}_{\Delta_2 I_2}(y)\rangle = \frac{\delta_{I_1 I_2}\delta_{\Delta_1 \Delta_2}}{|x-y|^{2\Delta_1}}. \tag{15}$$

The irreducible representations of $SO(6)$ and the components of states in each representation are in one-to-one correspondence with spherical harmonics on S^5. Here, we shall follow the notation of reference [31]. Using the same tensor as in (13), a spherical harmonic on S^5 can be written as

$$Y_{\Delta I} = C_I^{i_1 i_2 \dots i_\Delta}\hat{x}_{i_1}\hat{x}_{i_2}\dots\hat{x}_{i_\Delta}, \tag{16}$$

where the $\hat{x}_i$'s are the components of a unit vector coordinate for the embedding of S^5 in R^6,

$$\hat{x}_1^2 + \hat{x}_2^2 + \ldots + \hat{x}_6^2 = 1. \tag{17}$$

Equation (16) plus the fact that the tensors $C^{i_1 i_2 \ldots i_\Delta}$ are traceless and symmetric and the identity $\nabla_j \hat{x}_i = \frac{1}{|x|}(\delta_{ij} - \hat{x}_i \hat{x}_j)$ where $\hat{x}_i = x_i/|x|$, can easily be used to show that

$$-\vec{\nabla}^2 Y_{\Delta I} = \frac{\Delta(\Delta + 4)}{|x|^2} Y_{\Delta I}, \tag{18}$$

and, remembering that $-\vec{\nabla}^2 = -\frac{1}{x^5}\frac{d}{dx}x^5\frac{d}{dx} + \frac{L_{ij}^2}{x^2}$, where $L_{ij} = -i(x_i \nabla_j - x_j \nabla_i)$, we see that $Y_{\Delta I}$ are eigenfunctions of the Laplacian, L_{ij}^2, on S^5 with eigenvalues $\Delta(\Delta + 4)$. The index I runs over the $(3 + \Delta)(2 + \Delta)^2(1 + \Delta)/12$ linearly independent spherical harmonics at level Δ. From these degenerate states, we must identify those which are symmetric under an $SO(3) \times SO(3)$ subgroup of $SO(6)$. Following reference [25], it is convenient to use coordinates on S^5 where

$$ds^2 = d\psi^2 + \cos^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2\right) + \sin^2\psi \left(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2\right). \tag{19}$$

Here, S^5 is constructed as two S^2's fibered over an interval $\psi \in [0, \frac{\pi}{2}]$. Then, knowing that the $SO(3) \times SO(3)$ invariant spherical harmonic will not depend on any of the coordinates of the S^2's, that is, it will depend only on ψ, it is clear that $SO(3) \times SO(3)$ invariant spherical harmonic, which we shall denote by $Y_\Delta(\psi)$ must obey the equation

$$\frac{1}{\cos^2\psi \sin^2\psi}\frac{d}{d\psi}\cos^2\psi \sin^2\psi \frac{d}{d\psi}Y_\Delta(\psi) = -\Delta(\Delta + 4)Y_\Delta(\psi). \tag{20}$$

We shall see shortly that, with regularity conditions at $\psi = 0$ and $\psi = \pi/2$, there is a non-singular solution of this equation when Δ is an even integer. This implies that, when Δ is even, there is only one $SO(3) \times SO(3)$ invariant state and when Δ is odd, there are no such states.

With the change of variables

$$z = e^{2i\psi}, \tag{21}$$

the differential equation (20) becomes

$$\left[-\left(z\frac{d}{dz}\right)^2 + \left(1 + \frac{\Delta}{2}\right)^2\right][(z - 1/z)Y_\Delta(z)] = 0, \tag{22}$$

which is solved by

$$Y_\Delta(z) = \frac{(-1)^{\Delta/2}}{2^{(\Delta-1)/2}\sqrt{(\Delta + 1)(\Delta + 2)}}\frac{z^{1+\Delta/2} - z^{-1-\Delta/2}}{z - 1/z}, \quad \Delta \text{ even} \tag{23}$$

$$= \frac{1}{2^{(\Delta-1)/2}\sqrt{(\Delta + 1)(\Delta + 2)}}\left(z^{\Delta/2} + z^{\Delta/2-1} + \ldots + z^{-\Delta/2}\right), \tag{24}$$

where we have chosen the solution which is regular at $z = 1$. It is also regular at $z = -1$ only when Δ is an even integer, in which case the solution is a polynomial, and we have normalized so that the spherical harmonic has its usual normalization

$$\int_{S^5} |Y_\Delta(\psi)|^2 = \frac{1}{2^{\Delta-1}(\Delta+1)(\Delta+2)} \int_{S^5} 1. \tag{25}$$

Using a binomial expansion of $z^{1+\Delta/2} = (\cos\psi + i\sin\psi)^{2+\Delta}$, the spherical harmonic can be presented as an even homogeneous polynomial of order Δ in $\cos\psi$ and $\sin\psi$,[1]

$$Y_\Delta(\psi) = \frac{(2+\Delta)!}{2^{(\Delta+1)/2}\sqrt{(\Delta+1)(\Delta+2)}} \sum_{p=0}^{\Delta/2} \frac{(-1)^p \sin^{\Delta-2p}\psi \cos^{2p}\psi}{(2p+1)!(1+\Delta-2p)!}. \tag{26}$$

When the classical solution (7) and (8) of the classical field equations are substituted into a trace over gauge group indices such as that in equation (13), the only components which survive are those which are invariant under the $SO(3) \times SO(3)$ subgroup of $SO(6)$. Therefore, only the components of the chiral primary operator having this symmetry will have non-vanishing one-point functions. Since the chiral primary operators are in one-to-one correspondence with the S^5 spherical harmonics, and, in the above, we have shown that there is a unique $SO(3) \times SO(3)$ invariant spherical harmonic for each even integer value of Δ, we can conclude that there is one unique $SO(3) \times SO(3)$ symmetric chiral primary operator for each even Δ. We shall denote this operator by $\mathcal{O}_\Delta(x)$. It is a specific linear combination of the operators $\mathcal{O}_{\Delta I}(x)$ in a general basis which we defined in (13). We assume that $\mathcal{O}_\Delta(x)$ is normalized as in equation (15).

To relate the trace in (13) to the spherical harmonic Y_Δ, we simply factor out the normalization of the six-dimensional vector $(\phi_1, \ldots, \phi_6)$, so that it behaves like a unit vector. From equations (11) and (12) we see that this results in an overall factor of the normalization to the power of the number of fields $\frac{(k_1^2+k_2^2-2)^{\Delta/2}}{(2|z|)^\Delta} \approx \frac{(k_1^2+k_2^2)^{\Delta/2}}{(2|z|)^\Delta}$ where we shall use the limit where k_1 and k_2 are large. Then, the trace produces the spherical harmonic $Y_\Delta(\psi)$ where, from (11) and (12), we can identify $\cos\psi = \frac{k_1}{\sqrt{k_1^2+k_2^2}}$ and $\sin\psi = \frac{k_2}{\sqrt{k_1^2+k_2^2}}$ as the relative radii of the two spheres. Then, the final trace will

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[1]The first few of the $SO(3) \times SO(3)$ invariant spherical harmonics are

$$Y_0 = 1,$$

$$Y_2 = \sqrt{\frac{1}{6}} \left[\sin^2\psi - \cos^2\psi \right],$$

$$Y_4 = \sqrt{\frac{3}{80}} \left[\sin^4\psi - \frac{10}{3}\sin^2\psi\cos^4\psi + \cos^4\psi \right],$$

$$Y_6 = \sqrt{\frac{1}{112}} \left[\sin^6\psi - 7\sin^4\psi\cos^2\psi + 7\sin^2\psi\cos^4\psi - \cos^6\psi \right],$$

$$Y_8 = \sqrt{\frac{5}{2304}} \left[\sin^8\psi - 12\sin^6\psi\cos^2\psi + \frac{126}{5}\sin^4\psi\cos^4\psi - 12\sin^2\psi\cos^6\psi + \cos^8\psi \right].$$

produce a factor of $k_1 k_2$. Putting these factors together results in the expression (3) which we recopy here for the reader's convenience[2]

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{k_1 k_2}{\sqrt{\Delta}} \left(\frac{2\pi^2(k_1^2 + k_2^2)}{\lambda} \right)^{\Delta/2} Y_\Delta(\arctan(k_2/k_1)) \frac{1}{|z|^\Delta}, \quad (z < 0).$$

In the next Section, we shall show how the same formula for the one-point function can be obtained from the IIB string theory on the $AdS_5 \times S^5$ background with a probe D7 brane. Normally, one would expect that the classical limit of string theory computes the strong coupling planar limit of the gauge theory, and the computation which we have performed in the gauge theory is perturbative, valid only at weak coupling. In this case, however, there are additional parameters k_1 and k_2 and in the limit where these parameters are large, there is an effective coupling $\sqrt{\frac{\lambda}{k_1^2 + k_2^2}}$ which can be small in both the gauge theory and the classical limit of the string theory. It is the leading order of the chiral primary one-point functions in an asymptotic expansion in this parameter that we are comparing.

3 One-point function from supergravity

In this section we will obtain the one-point function (3) from a dual supergravity calculation. We begin by reviewing the problem of embedding a probe D7 brane with internal flux into $AdS_5 \times S^5$ and studying the limit where the flux is large.

3.1 Probe D7 brane

We shall study the D3-D7 system in the probe limit where the number of D7 branes N_7 is much smaller than N, the number of D3 branes. In principle, with multiple D7 branes, the coordinates are matrices and the worldvolume gauge fields have $U(N_7)$ gauge group. Since the non-abelian structure plays no role in this Section, we shall set $N_7 = 1$ (but of course will need to restore it in the following sections where we discuss non-Abelian worldvolume gauge fields). We shall be interested in the limit of the string theory which coincides with the planar limit of the gauge theory and, after the planar limit is taken, the large λ strong coupling limit. In this limit, the string

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[2] For the first few values of Δ, this expression is

$$\langle \mathcal{O}_2(x) \rangle = \sqrt{\frac{1}{12}} k_1 k_2 \left(\frac{2\pi^2}{\lambda} \right) \left[k_2^2 - k_1^2 \right] \frac{1}{|z|^2},$$

$$\langle \mathcal{O}_4(x) \rangle = \sqrt{\frac{3}{320}} k_1 k_2 \left(\frac{2\pi^2}{\lambda} \right)^2 \left[k_2^4 - \frac{10}{3} k_2^2 k_1^2 + k_1^4 \right] \frac{1}{|z|^4},$$

$$\langle \mathcal{O}_6(x) \rangle = \sqrt{\frac{1}{672}} k_1 k_2 \left(\frac{2\pi^2}{\lambda} \right)^3 \left[k_2^6 - 7k_2^4 k_1^2 + 7k_2^2 k_1^4 - k_1^6 \right] \frac{1}{|z|^6},$$

$$\langle \mathcal{O}_8(x) \rangle = \sqrt{\frac{5}{18432}} k_1 k_2 \left(\frac{2\pi^2}{\lambda} \right)^4 \left[k_2^8 - 12k_2^6 k_1^2 + \frac{126}{5} k_2^4 k_1^4 - 12k_2^2 k_1^6 + k_1^8 \right] \frac{1}{|z|^8}.$$

theory is classical, and the problem of including a D7 brane in the $AdS_5 \times S^5$ geometry reduces to that of finding an extremum of the Dirac-Born-Infeld and Wess-Zumino actions,

$$S = \frac{T_7}{g_s} \int d^8\sigma \left[-\sqrt{-\det(g + 2\pi\alpha'\mathcal{F})} + \frac{(2\pi\alpha')^2}{2} C^{(4)} \wedge \mathcal{F} \wedge \mathcal{F} \right], \qquad (27)$$

where g_s is the closed string coupling constant, which is related to the $\mathcal{N} = 4$ Yang-Mills coupling by $4\pi g_s = g_{YM}^2$, σ^a are the coordinates of the D7 brane worldvolume, $g_{ab}(\sigma)$ is the induced metric, $C^{(4)}$ is the 4-form of the $AdS_5 \times S^5$ background, $\mathcal{F}$ is the worldvolume gauge field and $T_7 = \frac{1}{(2\pi)^7\alpha'^4}$ is the D7 brane tension. We shall work with coordinates where the metric of $AdS_5 \times S^5$ is

$$ds^2 = \sqrt{\lambda}\alpha' \left[r^2(-dt^2 + dx^2 + dy^2 + dz^2) + \frac{dr^2}{r^2} + \right.$$
$$\left. + d\psi^2 + \cos^2\psi(d\theta^2 + \sin^2\theta d\phi^2) + \sin^2\psi(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2) \right]. \qquad (28)$$

Here, (t, x, y, z, r) are coordinates of the Poincare patch of AdS_5. The boundary of AdS_5 is located at $r \to \infty$ and the Poincare horizon at $r \to 0$. The coordinates of S^5 are identical to those which we used in the previous section. The Ramond-Ramond 4-form is

$$C^{(4)} = \lambda\alpha'^2 \left[r^4 dt \wedge dx \wedge dy \wedge dz + \frac{c(\psi)}{2} d\cos\theta \wedge d\phi \wedge d\cos\tilde{\theta} \wedge d\tilde{\phi} \right], \qquad (29)$$

with $\partial_\psi c(\psi) = 8\sin^2\psi\cos^2\psi$. The dynamical variables are the ten functions of eight worldvolume coordinates which embed the D7 brane in $AdS_5 \times S^5$, as well as the eight worldvolume gauge fields. The equations of motion for the embedding can be solved by a worldvolume geometry which is $AdS_4 \times S^2 \times S^2$ and which covers the whole range of the coordinates $(t, x, y, r, \theta, \phi, \tilde{\theta}, \tilde{\phi})$. It sits at a particular value of ψ, which must satisfy equation (32) below, and lies on the curve

$$z = -\frac{\Lambda}{r}, \qquad \Lambda \equiv \frac{f_1 f_2}{\sqrt{(f_1^2 + 4\cos^4\psi)(f_2^2 + 4\sin^4\psi) - f_1^2 f_2^2}}. \qquad (30)$$

The parameters f_1 and f_2 are fluxes of the worldvolume gauge fields corresponding to Dirac monopole on the S^2's, related to the monopole numbers k_1 and k_2 by $(k_1, k_2) = \frac{\sqrt{\lambda}}{2\pi}(f_1, f_2)$, so that

$$\mathcal{F} = \frac{1}{2} \left(k_1 d\cos\theta \wedge d\phi + k_2 d\cos\tilde{\theta} \wedge d\tilde{\phi} \right)$$
$$= \frac{\sqrt{\lambda}}{4\pi} \left(f_1 d\cos\theta \wedge d\phi + f_2 d\cos\tilde{\theta} \wedge d\tilde{\phi} \right). \qquad (31)$$

The equations of motion require that the angle ψ solves the equation

$$(f_1^2 + 4\cos^4\psi)\sin^2\psi = (f_2^2 + 4\sin^4\psi)\cos^2\psi, \qquad (32)$$

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The D7 brane worldvolume metric is

$$ds^2 = \sqrt{\lambda}\alpha' \left[r^2(-dt^2 + dx^2 + dy^2) + \frac{dr^2}{r^2}(1 + \Lambda^2) + \right.$$

$$\left. + \cos^2\psi(d\theta^2 + \sin^2\theta d\phi^2) + \sin^2\psi(d\tilde\theta^2 + \sin^2\tilde\theta d\tilde\phi^2) \right], \tag{33}$$

which is a product of AdS_4 with radius of curvature squared $\sqrt{\lambda}\alpha'(1 + \Lambda^2)$ and two S^2's with radii squared $\sqrt{\lambda}\alpha'$. In the limit where $f_1 \gg 1$ and $f_2 \gg 1$,

$$\Lambda \approx \sqrt{f_1^2 + f_2^2}/2 = \frac{\pi}{\sqrt{\lambda}}\sqrt{k_1^2 + k_2^2},$$

and, upon rescaling r, the worldvolume metric is

$$ds^2 = \frac{\pi^2}{\sqrt{\lambda}}\left(k_1^2 + k_2^2\right)\alpha'\left[r^2(-dt^2 + dx^2 + dy^2) + \frac{dr^2}{r^2} \right] +$$

$$+ \sqrt{\lambda}\alpha'\left[\frac{k_1^2}{k_1^2 + k_2^2}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{k_2^2}{k_1^2 + k_2^2}(d\tilde\theta^2 + \sin^2\tilde\theta d\tilde\phi^2) \right]. \tag{34}$$

where we have used the fact that, in this limit, the angle is given by

$$\tan\psi = \frac{k_2}{k_1}, \tag{35}$$

which is the same angle as we fixed in the previous section if we identify the number of units of flux with the dimension of the representation of the SO(3) representations. We see that, in the limit that we are considering, the S^2's have radius of curvature squared of order the $\sqrt{\lambda}\alpha'$ of the $AdS_5 \times S^5$ background whereas the AdS_4 is much flatter, with radius of curvature squared $\frac{\pi^2}{\sqrt{\lambda}}\left(k_1^2 + k_2^2\right)\alpha'$. Both of these must be large in order for the string sigma model to be semi-classical on this background.

3.2 The one-point function

The calculation involves computing the fluctuation δS of the action of a $D7$ brane due to fluctuations in the background supergravity fields due to the insertion of a source on the boundary corresponding to the operator $\mathcal{O}_\Delta$ defined in (13), see Figure 2. This calculation has been performed for the case of a $D5$ brane in [25]; our calculation detailed below proceeds in a directly analogous manner.

We will be interested in the large f_1 and large f_2 limit, with

$$\xi \equiv \frac{f_2}{f_1}, \tag{36}$$

held fixed. In this limit $\psi = \arctan\xi$. We begin by writing down the induced metric-plus-worldvolume-field-strength

$$H = 2\pi\alpha'\mathcal{F} + \partial_a X^M \partial_b X_M = \sqrt{\lambda}\alpha'\text{diag}\left(r^2, r^2, r^2, \frac{1 + \Lambda^2}{r^2}, B_1, B_2 \right), \tag{37}$$
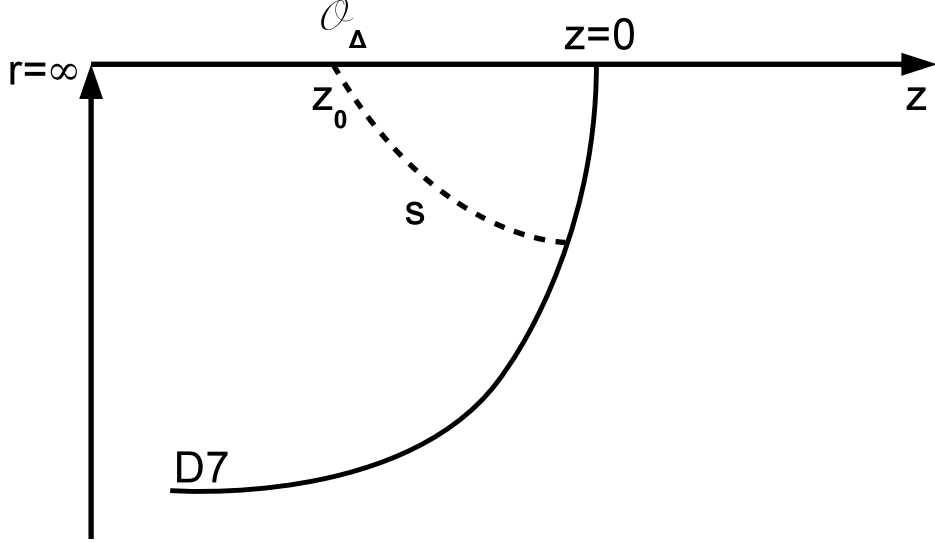
Figure 2: **Supergravity computation**: The insertion of the operator $\mathcal{O}_\Delta$ on the boundary ($r = \infty$) at $z = z_0$ is dual to a supergraviton s which propagates from the boundary insertion point into the bulk, where it is integrated over the $D7$ brane world volume.

where

$$B_1 = \begin{pmatrix} \cos^2\psi & -\frac{f_1}{2}\sin\theta \\ \frac{f_1}{2}\sin\theta & \cos^2\psi\sin^2\theta \end{pmatrix}, \quad B_2 = \begin{pmatrix} \sin^2\psi & -\frac{f_2}{2}\sin\tilde\theta \\ \frac{f_2}{2}\sin\tilde\theta & \sin^2\psi\sin^2\tilde\theta \end{pmatrix}. \tag{38}$$

For simplicity, we will set the factor of $\sqrt{\lambda}\alpha'$ in equation (37) to one and take it into account by multiplying the D7 brane action by $\lambda^2\alpha'^4$. The inverse of H is given by

$$H^{-1} = \mathrm{diag}\left(r^{-2}, r^{-2}, r^{-2}, \frac{r^2}{1+\Lambda^2}, B_1^{-1}, B_2^{-1}\right), \tag{39}$$

where

$$\begin{aligned}
B_1^{-1} &= \frac{1}{\sin^2\theta(\cos^4\psi + f_1^2/4)}\begin{pmatrix} \cos^2\psi\sin^2\theta & \frac{f_1}{2}\sin\theta \\ -\frac{f_1}{2}\sin\theta & \cos^2\psi \end{pmatrix}, \\
B_2^{-1} &= \frac{1}{\sin^2\tilde\theta(\sin^4\psi + f_2^2/4)}\begin{pmatrix} \sin^2\psi\sin^2\tilde\theta & \frac{f_2}{2}\sin\tilde\theta \\ -\frac{f_2}{2}\sin\tilde\theta & \sin^2\psi \end{pmatrix}.
\end{aligned} \tag{40}$$

The background metric fluctuations h associated to the insertion of the operator at the boundary give rise to fluctuations of the induced metric

$$\begin{aligned}
\mathcal{H}_{\alpha\beta} = \partial_a X^M \partial_b X^N h_{MN} = \mathrm{diag}\Big(h_{tt}, h_{xx}, h_{yy}, h_{rr}+2\frac{\Lambda}{r^2}h_{zr} + \frac{\Lambda^2}{r^4}h_{zz}, \\
h_{\theta\theta}, h_{\phi\phi}, h_{\tilde\theta\tilde\theta}, h_{\tilde\phi\tilde\phi}\Big),
\end{aligned} \tag{41}$$

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where h and the fluctuation of the background 4-form potential a are given by [31]

$$
\begin{aligned}
h_{\mu\nu}^{AdS_5} &= -\frac{2\Delta(\Delta-1)}{\Delta+1} g_{\mu\nu}^{AdS_5} s + \frac{4}{\Delta+1}\nabla_\mu\nabla_\nu s, \\
h_{\alpha\beta}^{S^5} &= 2\Delta g_{\alpha\beta}^{S^5} s, \\
a_{\mu\nu\rho\sigma}^{AdS_5} &= 4i\sqrt{g^{AdS_5}}\epsilon_{\mu\nu\rho\sigma\omega}\nabla^\omega s,
\end{aligned}
\tag{42}
$$

where s is a field on $AdS_5 \times S^5$, which is decomposed as a sum of Kaluza-Klein modes on the five-sphere (in the following Ω denotes coordinates on S^5 while X denotes coordinates on AdS_5)

$$
s(X,\Omega) = \sum_\Delta \sum_I s_{\Delta I}(X) Y_{\Delta I}(\Omega),
\tag{43}
$$

where the total spin (i.e. SO(6) Casimir) is represented by Δ, while the other quantum numbers defining the state are collected in the index I. Each mode $s_{\Delta I}(X)$ has mass-squared $= \Delta(\Delta-4)$ and corresponds, via the AdS/CFT dictionary, to a specific chiral primary operator of dimension Δ in the gauge theory, namely to that operator defined by the spherical harmonic $Y_{\Delta I}$, where, as in section 2, the embedding coordinates of the five-sphere are replaced by the six scalar fields of the gauge theory. Thus, for our purposes, we will be interested in the specific modes (i.e. single terms in the sum (43)) corresponding to the operators defined beneath (26), i.e. those with spherical harmonic (26).

We find the following relevant Christoffel symbols (i denotes coordinates t, x, y, z)

$$
\Gamma_{ii}^r = -r^3, \qquad \Gamma_{rr}^r = -\frac{1}{r}, \qquad \Gamma_{zr}^z = \frac{1}{r}.
\tag{44}
$$

The fluctuation of the Euclidean action is then given by

$$
\begin{aligned}
\delta S &= \delta S_{DBI} + \delta S_{WZ} \\
&= \frac{T_7}{g_s}\lambda^2\alpha'^4 \int d^8\sigma \left[\frac{\sqrt{H}}{2}\mathrm{Tr}\left(H^{-1}\mathcal{H}\right) + i\frac{(2\pi\alpha')^2}{2}a \wedge \mathcal{F} \wedge \mathcal{F}\right].
\end{aligned}
\tag{45}
$$

The pull-back of a is

$$
\begin{aligned}
&4i\sqrt{g^{AdS_5}}\left(\epsilon_{txyzr}z'r^2\partial_r + \epsilon_{txyrz}\frac{1}{r^2}\partial_z\right)s \\
&= 4ir^3\left(\Lambda\partial_r - \frac{1}{r^2}\partial_z\right)s,
\end{aligned}
\tag{46}
$$

giving

$$
\delta S_{WZ} = -\frac{T_7}{g_s}\lambda^2\alpha'^4 f_1 f_2 \int d^8\sigma\, r^3 \sin\theta \sin\vartheta \left(\Lambda\partial_r - \frac{1}{r^2}\partial_z\right)s.
\tag{47}
$$

We find the following expression for the fluctuation of the Dirac-Born-Infeld action

$$\delta S_{DBI} = \frac{T_7}{2g_s}\lambda^2\alpha'^4 \int d^8\sigma\, r^2\sqrt{1+\Lambda^2}\sin\theta\sin\vartheta\sqrt{(\cos^4\psi + f_1^2/4)(\sin^4\psi + f_2^2/4)}\Bigg\{$$

$$\frac{4}{\Delta+1}\left[\frac{1}{r^2}\left(\partial_t^2 + \partial_x^2 + \partial_y^2\right) + 4r\partial_r + \frac{r^2}{1+\Lambda^2}\left(\partial_r^2 + 2\frac{\Lambda}{r^2}\left(\partial_z\partial_r - \frac{1}{r}\partial_z\right) + \frac{\Lambda^2}{r^4}\partial_z^2\right)\right]$$

$$+ 4\Delta\left(\frac{\cos^4\psi}{\cos^4\psi + f_1^2/4} + \frac{\sin^4\psi}{\sin^4\psi + f_2^2/4}\right) - 8\frac{\Delta(\Delta-1)}{\Delta+1}\Bigg\}\, s.$$

$$(48)$$

The field s is replaced by its bulk-to-boundary propagator, corresponding to a delta-function source s_0 on the boundary at $((0,0,0,z_0)$

$$s \to \frac{Y_\Delta(\psi)\, c_\Delta}{r^\Delta\left(\rho^2 + (z-z_0)^2 + 1/r^2\right)^\Delta}, \qquad (49)$$

where $Y_\Delta(\psi)$ is the spherical harmonic (26) (which, we remind the reader, is not dependent on $\{\theta, \tilde{\theta}, \phi, \tilde{\phi}\}$), while $\rho^2 = t^2 + x^2 + y^2$, and

$$c_\Delta \equiv \frac{\Delta+1}{2^{2-\Delta/2}N\sqrt{\Delta}}. \qquad (50)$$

This normalization is needed to give the bulk two-point function of the chiral primary unit normalization in the usual supergravity computation [31]. It is also important that we take the operator insertion point at $z_0 < 0$. We will also consider the other sign later, and we will find a different answer.

The worldvolume integrations over $\theta, \phi, \tilde{\theta}, \tilde{\phi}$ are trivial, as only the volume measure $\sin\theta\sin\tilde{\theta}$ depends upon them, leading to a factor of $(4\pi)^2$. The integration over t, x, y are simple radial integrals in the ρ variable. We replace $dt\, dx\, dy \to 4\pi\rho^2 d\rho$ and use

$$\int_0^\infty d\rho\, \frac{\rho^a}{(\rho^2 + r^{-2}(1 + (\Lambda - rz_0)^2))^b} = \left(r^{-2}(1 + (\Lambda - rz_0)^2)\right)^{(1+a-2b)/2}$$

$$\times \frac{\Gamma\big(b - (1+a)/2\big)\Gamma\big((1+a)/2\big)}{2\Gamma(b)}. \qquad (51)$$

The remaining integral over r is non-trivial, but may be done in closed form. We then take the $f_1 \to \infty$ limit, keeping f_2/f_1 fixed. It seems that the $\Delta = 2$ case is pathological in that the worldvolume integration over r does not converge. It would be interesting to understand the interpretation of this pathology further. On the gauge theory side, there is nothing special about the $\Delta = 2$ case.

3.3   Results for $z_0 < 0$

We first note the following identification of parameters between gauge and string theory

$$k_i = \frac{\sqrt{\lambda}}{2\pi}f_i, \qquad (52)$$

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and then, recalling that $\psi = \arctan(k_2/k_1)$, find the following result

$$\langle \mathcal{O}_\Delta(x) \rangle = -\left.\frac{\delta S}{\delta s_0}\right|_{s_0=0} = \frac{k_1 k_2}{\sqrt{\Delta}} \left(\frac{2\pi^2(k_1^2 + k_2^2)}{\lambda}\right)^{\Delta/2} Y_\Delta\big(\arctan(k_2/k_1)\big) \frac{1}{z_0^\Delta}$$
$$+ \text{ sub-leading in } \frac{k_i}{\sqrt{\lambda}}, \tag{53}$$

where the spherical harmonic Y_Δ corresponding to the operator $\mathcal{O}_\Delta$ is given by (26) and where z_0 should be equated to z. We therefore see that we have obtained a match with the gauge theory result (3).

3.4 Results for $z_0 > 0$

Here we find that the leading behaviour of both δS_{DBI} and δS_{WZ} goes as

$$\frac{\delta S}{\delta s_0} \sim \lambda^{\Delta/2-2} \frac{k_1 k_2}{(k_1^2 + k_2^2)^{\Delta/2-2}} \frac{Y_\Delta}{z_0^\Delta} + \text{ sub-leading in } \frac{k_i}{\sqrt{\lambda}}. \tag{54}$$

However these leading terms cancel identically between the Dirac-Born-Infeld and Wess-Zumino parts of the action. Even if they didn't, they are already suppressed with respect to the $z_0 < 0$ case for all $\Delta \geq 4$. The cancellation persists to an additional order in k_i^2/λ, bringing the result down by λ^2/k_i^4, so that for $\Delta = 4$, for example, it goes as λ/k_i^2. This further cancellation ensures that $\delta S/\delta s_0$ is suppressed with respect to the $z_0 < 0$ case for all $\Delta \geq 2$. This suppression is consistent with the result in the gauge theory, where we find zero.

4 $SO(5)$ symmetric solution

As well as the $SO(3) \times SO(3)$ symmetric configuration of the D7 brane that we have studied in the previous Sections, there is also a solution of the D3-D7 brane intersection which has $SO(5)$ symmetry.

4.1 $\mathcal{N} = 4$ Yang-Mills theory defect with $SO(5)$ symmetry

On the gauge theory side, there is an $SO(5)$ symmetric solution of the Yang-Mills theory field equations (6) which represents D3 branes ending on D7 branes in such a way that an $SO(5)$ subgroup of the $SO(6)$ R-symmetry is preserved. It has the form of a fuzzy funnel where, as $|z|$ decreases, a certain number of D3 branes blow up into a fuzzy 4-sphere,

$$\phi_i(x) = \frac{G_i}{\sqrt{8}\, z} \quad , \quad i = 1, \ldots, 5 \; , \quad \phi_6 = 0. \tag{55}$$

The fuzzy 4-sphere uses five matrices G_i with $i = 1, \ldots, 5$ which are the sum of totally symmetric direct products[3]

$$G_i = [\gamma_i \otimes \mathcal{I}_{4\times 4} \ldots \otimes \mathcal{I}_{4\times 4} + \mathcal{I}_{4\times 4} \otimes \gamma_i \otimes \ldots \otimes \mathcal{I}_{4\times 4} + \ldots + \mathcal{I}_{4\times 4} \otimes \ldots \otimes \gamma_i]_{\text{sym}}$$
$$\oplus \, 0_{(N-d_G)\times(N-d_G)}, \tag{56}$$

where γ_i are the 4×4 (Hermitian) Euclidean Dirac matrices and each direct product has n factors, the direct sums has n terms and where sym means restriction to the completely symmetrized tensor product space. These matrices are described in detail in references [32] and [33]. They have dimension

$$d_G = \frac{1}{6}(n+1)(n+2)(n+3), \tag{57}$$

and

$$G_i G_i = c_G \, \mathcal{I}_{d_G \times d_G} \oplus 0_{(N-d_G)\times(N-d_G)} \quad , \quad c_G(n+4). \tag{58}$$

Also, the spin matrix

$$G_{ij} = \frac{1}{4}[G_i, G_j], \tag{59}$$

generates $SO(5)$ rotations,

$$[G_i, G_{jk}] = (\delta_{jk} G_i - \delta_{ik} G_j), \tag{60}$$
$$[G_{ij}, G_{kl}] = (\delta_{jk} G_{il} + \delta_{il} G_{jk} - \delta_{ik} G_{jl} - \delta_{jl} G_{ik}). \tag{61}$$

Now, we can form the classical contribution to the one-point function of the chiral primary operator by simply plugging the classical solution (55) into equation (13) to obtain,

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{d_G}{\sqrt{\Delta}} \left(\frac{\pi^2 c_G}{\lambda}\right)^{\frac{\Delta}{2}} \mathcal{Y}_\Delta(0) \frac{1}{|z|^\Delta}, \tag{62}$$

where $\mathcal{Y}_\Delta(\theta)$ is the $O(5)$ symmetric spherical harmonic (which we shall derive below), evaluated at $\theta = 0$, the latitude where the S^4 is maximal (since $\phi_6 = 0$). In the limit

---

[3]More precisely, operating on a completely symmetric tensor $v^{\alpha_1 \alpha_2 \ldots \alpha_n}$,

$$G_i v^{\alpha_1 \alpha_2 \ldots \alpha_n} \equiv \gamma^{\alpha_1}_{\beta_1} v^{\beta_1 \alpha_2 \ldots \alpha_n} + \gamma^{\alpha_2}_{\beta_2} v^{\alpha_1 \beta_2 \ldots \alpha_n} + \ldots + \gamma^{\alpha_n}_{\beta_n} v^{\alpha_1 \alpha_2 \ldots \beta_n}.$$

The dimension of the space of completely symmetric tensors with n indices, with each index running from one to four is

$$d_G = \frac{(n+3)!}{3!n!} = \frac{1}{6}(n+1)(n+2)(n+3).$$

where the integer n is large, this reduces to[4]

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{n^3}{6\sqrt{\Delta}} \left(\frac{\pi^2 n^2}{\lambda} \right)^{\frac{\Delta}{2}} \mathcal{Y}_\Delta(0) \frac{1}{|z|^\Delta}. \tag{63}$$

4.1.1 $O(5)$ symmetric spherical harmonic at level Δ

Consider the coordinate system where the metric of the 5-sphere is

$$ds^2 = d\theta^2 + \cos^2 \theta d\Omega_4^2.$$

The spherical harmonic that we are interested in does not depend on any of the coordinates of S^4, and therefore satisfies the Laplace equation

$$\left[\frac{1}{\cos^4 \theta} \frac{d}{d\theta} \cos^4 \theta \frac{d}{d\theta} + \Delta(\Delta + 4) \right] \mathcal{Y}_\Delta(\theta) = 0,$$

which, using the variable

$$z = \frac{1 - \sin \theta}{2},$$

is the hypergeometric differential equation,

$$\left[z(1-z) \frac{d^2}{dz^2} + \left(\frac{5}{2} - 5z \right) \frac{d}{dx} + \Delta(\Delta + 4) \right] \mathcal{Y}_\Delta(z) = 0. \tag{64}$$

This equation is solved by the hypergeometric function

$$\mathcal{Y}_\Delta(z) = C_\Delta \, _2F_1(-\Delta, \Delta + 4; \frac{5}{2}; z), \tag{65}$$

which is a polynomial,

$$_2F_1(-\Delta, \Delta + 4; \frac{5}{2}; z) = \sum_{n=0}^{\Delta} \frac{\Gamma(5/2)}{\Gamma(5/2 + n)} \frac{(\Delta + 3 + n)! \Delta!}{(\Delta - n)!(\Delta + 3)! n!} \left(\frac{\sin \theta - 1}{2} \right)^n. \tag{66}$$

---

[4]Using the spherical harmonics which will be derived in the next subsecitonb, the explicit expressions for the first few values of Δ are

$$\langle \mathcal{O}_2(x) \rangle = -\sqrt{\frac{1}{60}} \frac{\pi^2 n^5}{\lambda}$$

$$\langle \mathcal{O}_4(x) \rangle = \sqrt{\frac{9}{11200}} \frac{\pi^2 n^7}{\lambda^2}$$

$$\langle \mathcal{O}_6(x) \rangle = -\sqrt{\frac{1}{14112}} \frac{\pi^2 n^9}{\lambda^3}$$

$$\langle \mathcal{O}_8(x) \rangle = \sqrt{\frac{5}{608256}} \frac{\pi^2 n^{11}}{\lambda^4}$$

Here C_Δ is a constant which must be fixed so that the spherical harmonic has its canonical normalization (25),

$$C_\Delta = \sqrt{\frac{(\Delta + 2)(\Delta + 3)}{3 \cdot 2^{\Delta+1}}}. \tag{67}$$

We are interested in these polynomials at $\theta = 0$ which can be found using Gauss' second summation theorem

$$_2F_1(a, b; \tfrac{1}{2}(a + b + 1); \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a + b + 1))}{\Gamma(\frac{1}{2}(1 + a))\Gamma(\frac{1}{2}(1 + b))}, \tag{68}$$

to find

$$\mathcal{Y}_\Delta(\theta = 0) = C_\Delta \, _2F_1(-\Delta, \Delta + 4; \tfrac{5}{2}; \tfrac{1}{2}) = C_\Delta \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2}(1 - \Delta))\Gamma(\frac{1}{2}(5 + \Delta))}. \tag{69}$$

This vanishes when $\Delta = 1, 3, 5, \dots$ and, when $\Delta = 2\ell$,

$$\mathcal{Y}_{2\ell}(\theta = 0) = C_{2\ell} \frac{(\frac{1}{2} - 1)(\frac{1}{2} - 2)\dots(\frac{1}{2} - \ell)}{(\frac{3}{2} + \ell)(\frac{1}{2} + \ell)\dots(\frac{3}{2} + 1)}. \tag{70}$$

The first few spherical harmonics are

$$\begin{aligned}
\mathcal{Y}_0 &= 1, \\
\mathcal{Y}_1 &= \sin\theta, \\
\mathcal{Y}_2 &= \sqrt{\frac{1}{30}}\left[5 - 6\cos^2\theta\right], \\
\mathcal{Y}_3 &= \sqrt{\frac{9}{40}}\left[-\sin\theta + \frac{8}{3}\sin^3\theta\right], \\
\mathcal{Y}_4 &= \sqrt{\frac{1}{2800}}\left[35 - 112\cos^2\theta + 80\cos^4\theta\right].
\end{aligned} \tag{71}$$

4.2 Supergravity computation

Now we shall consider the computation of the one-point function using supergravity. In the scenario where d_G D3 branes end on N_7 D7 branes, they act as a source of topological charge on the world volume of the D7 branes which is reflected in the second Chern class of the gauge fields on the worldvolume of the D7 branes,

$$d_G = \frac{1}{8\pi^2}\int_{S^4} \text{Tr}F \wedge F, \tag{72}$$

where the integral is over a 4-cycle in the D7 brane worldvolume which links the D3 brane endpoints.

In the large N planar limit and the subsequent large λ limit, the problem of embedding the D7 brane in $AdS_5 \times S^5$ reduces to finding an extremum of the Dirac-Born-Infeld action plus the Wess-Zumino term,

$$S = \frac{T_7}{g_s}\text{STr} \int d^8\sigma \left[\sqrt{\det(g + 2\pi\alpha'F)} + i\frac{(2\pi\alpha')^2}{2}C^{(4)} \wedge F \wedge F \right], \qquad (73)$$

where the trace STr is computed using the symmetrized trace prescription and we are assuming Euclidean signature of the metric. Moreover, the worldvolume gauge fields must take up a configuration such that their second Chern class is given by equation (72). Their appearance in the Wess-Zumino term then influences the geometry of the D7 brane. There is a solution of the equations of motion where the D7 brane is $AdS_4 \times S^4$ embedded in $AdS_5 \times S^5$ and the gauge fields are a "homogeneous instanton", which was constructed in reference [33] and which has $SO(5)$ symmetry. This instanton is obtained by simply replacing the Pauli matrix generators of $SU(2)$ in the usual Belavin-Polyakov-Schwarz-Tyupkin instanton (stereographically projected to S^4) by the $(n+2)$-dimensional irreducible representation of $SU(2)$, where n is the same integer which appears in equation (57) and then d_G in (72) is given by (57). This means that the number of D7 branes is $N_7 + 2$.

We shall use a coordinate system where the metric of $AdS_5 \times S^5$ is

$$ds^2 = r^2(-dt^2 + dx^2 + dy^2 + dz^2) + \frac{dr^2}{r^2} + d\theta^2 + \cos^2\theta d\Omega_4^2, \qquad (74)$$

where $d\Omega_4^2$ is the metric of the unit 4-sphere. The equations for the D7 brane embedding are solved by the space $AdS_4 \times S^4$ which covers the space spanned by the coordinates (t, x, y, r) and wraps the $S^4 \subset S^5$ located at angle $\theta = 0$ while the transverse coordinate is r-dependent,

$$z = -\frac{\Lambda}{r}, \qquad \Lambda \equiv \frac{Q}{\sqrt{1 + 2Q}}, \qquad Q = \frac{6\pi^2}{\lambda}\frac{d_G}{n+2} \ . \qquad (75)$$

The worldvolume metric is

$$ds^2 = \sqrt{\lambda}\alpha' \left[r^2(-t^2 + x^2 + y^2) + (1 + \Lambda^2)\frac{dr^2}{r^2} + d\Omega_4^2 \right]. \qquad (76)$$

We see that, as in the $SO(3) \times SO(3)$ symmetric case, the radius of curvature increases as a result of the back-reaction of the geometry of the D7 brane in the presence of the instanton flux. In the limit of large Q, it becomes much larger than the radius of curvature of the S^4. Even though we will shortly consider the case where Q and n are large, we note that what we have done so far is valid for any value of Q with one restriction. When Q is small, the D7 brane embedding is unstable to fluctuations of the latitude at which the S^4 is located on S^5 [10]. It is stable when $Q > \frac{7}{2}$ [11]. We caution the reader that this is a condition for stability to small fluctuations and it does not rule out the possibility of nonperturbative instabilities.

To facilitate our computation, we rewrite the Dirac-Born-Infeld part of the action in the following way[5],

$$S_{\text{DBI}} = \frac{T_7}{g_s} \text{STr} \int d^8\sigma \ \sqrt{\det(g + 2\pi\alpha' F)}$$

$$= \frac{T_7}{g_s} \int d^8\sigma \sqrt{g_{\text{AdS}}}\sqrt{g_S} \left((n+2) + \frac{(2\pi\alpha')^2}{4} \text{Tr} \ F_{ab}F^{ab} \right). \tag{77}$$

This identity is valid when the D7 brane metric decomposes into two nonzero 4×4 blocks which, on the classical solution, become AdS_4 and S^4 (as in equation (76)). The metric blocks are therefore denoted g_{AdS} and g_S, respectively. Moreover, the gauge field has nonzero components only in the g_S block and it is self-dual with respect to the g_S geometry,

$$F_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}.$$

We note that, when the metrics of the blocks are set equal to the AdS_4 and S^4 metrics plus a perturbation by the supergraviton field $s(\sigma)$ which corresponds to the chiral primary operator (the explicit perturbations are $(h^{\text{AdS}}, h^{S^4})$ given in equation (41)), they retain their block diagonal form and

$$\delta\sqrt{g_{\text{AdS}}} = \lambda\alpha'^2\sqrt{1+\Lambda^2}r^2 \left(1 + \frac{1}{2}\text{tr} \ g_{\text{AdS}}^{-1}h^{\text{AdS}} \right), \tag{78}$$

$$\delta\sqrt{g_S} = \lambda\alpha'^2\sqrt{g_{S^4}} \left(1 + \frac{1}{2}\text{tr} \ g_{S^4}^{-1}h^{S^4} \right), \tag{79}$$

where g_{S^4} is the metric of the unit S^4. We also note that F_{ab} (i.e. with down-indices) is independent of the metric and therefore is independent of the supergraviton perturbation $s(\sigma)$. The integrand in equation (77) becomes

$$\delta\mathcal{L}_{\text{DBI}} = \lambda\alpha'^2\sqrt{1+\Lambda^2}r^2\sqrt{g_{S^4}} \left((n+2) + \frac{(2\pi\alpha')^2}{4}\text{Tr} \ F_{ab}F^{ab} \right) \frac{\text{tr}}{2}g_{\text{AdS}}^{-1}h^{\text{AdS}}$$

$$+ \lambda\alpha'^2\sqrt{1+\Lambda^2}r^2\sqrt{g_{S^4}} \left((n+2) + \frac{(2\pi\alpha')^2}{4}\text{Tr} \ F_{ab}F^{ab} \right) \frac{\text{tr}}{2}g_{S^4}^{-1}h^{S^4} \tag{80}$$

$$+ \lambda\alpha'^2\sqrt{1+\Lambda^2}r^2\sqrt{g_{S^4}}\frac{(2\pi\alpha')^2}{4}\text{Tr} \ F_{ab}F^{ab} \left(-\frac{\text{tr}}{2}g_{S^4}^{-1}h^{S^4} \right),$$

so that the last term cancels part of the second term.

Now, plugging in the expressions for $(h^{\text{AdS}}, h^{S^4})$ given in equation (41) and noting that $s(\sigma)$ will not depend on the coordinates on S^4, integration over the S^4 yields

$$\delta S_{DBI} = \frac{T_7}{g_s}(n+2) \int d^4\sigma \ r^2\sqrt{1+\Lambda^2}\frac{8\pi^2}{3}\frac{1+Q}{2} \left\{ \frac{4}{\Delta+1} \left[\frac{1}{r^2}\left(\partial_t^2 + \partial_x^2 + \partial_y^2\right) + \right. \right.$$

$$+ 4r\partial_r + \frac{r^2}{1+\Lambda^2}\left(\partial_r^2 + 2\frac{\Lambda}{r^2}\left(\partial_z\partial_r - \frac{1}{r}\partial_z\right) + \frac{\Lambda^2}{r^4}\partial_z^2\right) \bigg] - 8\frac{\Delta(\Delta-1)}{\Delta+1} \right\} s(\sigma)$$

$$+ \frac{T_7}{g_s}(n+2) \int d^4\sigma \ r^2\sqrt{1+\Lambda^2}\frac{8\pi^2}{3} 4\Delta s(\sigma).$$

---

[5]See equation (46) of reference [33].

where the unit 4-sphere volume is $\frac{8\pi^2}{3}$ and Q is defined in equation (75). The last term stemming from the remaining variation in g_S is sub-leading, since it does not have a factor of Q associated to it.

The Wess-Zumino term in the action has a structure similar to that in the two-flux case that we studied in the previous section, as $s(\sigma)$ appears only in the Ramond-Ramond potential, so that

$$\delta S_{WZ} = -\frac{T_7}{g_s}\frac{32\pi^2}{3}(n+2)Q \int d^4\sigma \, r^3 \left(\Lambda\partial_r - \frac{1}{r^2}\partial_z\right) s(\sigma), \tag{81}$$

where we have used (72). The remainder of the problem simply follows the procedure of the two-flux case that we used in the previous section. We find that, when $z_0 < 0$, and in the limit where n and Q are large,

$$\langle\mathcal{O}_\Delta(x)\rangle = \frac{1}{\sqrt{\Delta}}\frac{n^3}{6z_0^\Delta}\left(\frac{\pi^2 n^2}{\lambda}\right)^{\Delta/2} \mathcal{Y}_\Delta(0), \tag{82}$$

which is identical to the Yang-Mills theory result (63).

4.2.1 $z_0 > 0$

It is also easy to show that, on the other side of the D7 brane, where $z_0 > 0$,

$$\langle\mathcal{O}_\Delta(x)\rangle \sim Q^{1-\Delta/2}, \tag{83}$$

is much smaller than that on the $z_0 < 0$ side. This is consistent with the fact that, in the gauge theory, the leading order result vanishes.

5 Discussion

We have computed the one-point functions of chiral primary operators in two separate cases of the D3-D7 brane system. Both computations can be done in the gauge theory and in the string theory dual and, in a particular large flux limit, the computations can be compared and they indeed do match. It would be very interesting to check this agreement beyond the leading order that we have computed. In principle, the string theory computation can be done for any value of the monopole or instanton numbers. Accuracy of the computation depends only on $\sqrt{\lambda}$ being large. It would thus be easy to find the sub-leading orders of the computation. On the gauge theory side, on the other hand, the sub-leading orders would require computing perturbation theory in small λ. In both cases, however, one could generate an asymptotic series in $\sqrt{\lambda}/k$. It could happen that, like the BMN limit [26] of the string spectrum, such a comparison could fail at a higher order simply due to a non-commutativity of limits problem [34]. It would be very interesting to investigate this.

A computation of this kind could be helpful in constructing the quantum field theory on the defect. The bi-fundamental fields living on the defect should be chiral

fermions. Local self-interactions of fermions are irrelevant dimension four and greater operators. A candidate Lagrangian for the fermions is

$$S_{\text{defect}} = \int d^3x \left\{ \bar{\psi}(x) i \gamma^\mu (\partial_\mu - i A_\mu) \psi(x) + \ldots \right\},$$

where A is the bulk gauge field and its coupling to the defect fermions is fixed by gauge invariance. With a single D3 brane, the fermions are doublets of the SO(2,1) Lorentz symmetry and singlets of $SO(3) \times SO(3)$, so the above are apparently the only marginal operators which contain the defect fermions. We are also allowed to add operators made only from the bulk fields and comparison with the string theory could be a be a useful guide as to what these operators are. There is a possibility that they could also be checked by requiring that the resulting theory has conformal symmetry.

Finally, it would be interesting to examine similar issues for other defect theories, a good example being the theory based on the Janus solution [35, 36, 37], for which fully back-reacted backgrounds are also known [38, 39].

Acknowledgments

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