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Satisficing behavior with a secondary criterion

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Abstract Using the techniques of revealed preference analysis, we study a two-stage model of choice behavior. In the first stage, the decision maker maximizes a menu-dependent binary relation encoding preferences that are imperfectly perceived. In the second, a menu-independent binary relation is maximized over the subset of alternatives that survive the first stage. This structure can support various interpretations, including those of salience effects, positive action, and surface characteristics. We characterize the model behaviorally both in ordinal form and in terms of the corresponding numerical representations.

1 Introduction

1.1 The secondary criterion

This paper uses the techniques of revealed preference analysis to study a two-stage model of choice behavior. In the first stage, the decision maker (henceforth "DM") maximizes a menu-dependent binary relation encoding preferences that are imperfectly perceived due to cognitive or information-processing constraints. As detailed in [35], this mechanism leads to a form of satisficing in the sense of Simon [32].[1] In the second stage of the model, a menu-independent binary relation (termed the "secondary criterion") is maximized over the subset of alternatives that survive the first stage.

At the formal level, adding a second stage to the basic satisficing model changes the implied constraints on behavior, and our main result will identify

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[1] Satisficing has been defined by Reber et al. [21, p. 701] as "accept[ing] a choice or judgement as one that is good enough, one that satisfies."

these constraints. At the level of interpretation, the secondary criterion can be used to capture a variety of additional factors that may influence the DM's choices. For example:

- *Salience effects.* After the DM has used his deliberative resources to identify a subset of alternatives as good, if not necessarily the best, attention effects may influence his final selection from among these options that pass the satisficing threshold.[2] Here the secondary relation would be interpreted as a measure of salience—the property of standing out from the rest.[3]
- *Positive action.* Legal frameworks that prohibit most employment discrimination may nevertheless permit or even mandate "positive action" aimed at increasing participation by groups deemed to be disadvantaged for historical or other reasons.[4] Importantly, such positive discrimination is allowed only if the individual to be given favorable treatment is "as qualified as" other candidates. In a maximization setting, the latter proviso means that action can be taken only to break indifference between candidates who are exactly equally qualified. But if the employer engages in satisficing at the merit-evaluation (i.e., first) stage, there is more room left for group-identity (i.e., second-stage) criteria to influence employment outcomes.
- *Surface characteristics.* In many situations the DM cares about two distinct aspects of the choice problem, one of which is more important than the other but at the same time requires more effort to rank the alternatives. For instance, in the choice of a new car a frequent commuter may value reliability more than appearance. But appearance is a surface characteristic, while reliability is to some extent hidden. Hence the DM may seek to identify a subset of cars that are "reliable enough" (without ranking the options extremely precisely on this dimension), and may then simply choose the most visually attractive car from within this set.[5]

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[2] Attention is a core topic in cognitive psychology; see, e.g., Anderson [1, pp. 72–105]. In the context of salience effects it is understood that we refer to attention allocated involuntarily, rather than consciously.

[3] Combining satisficing and salience effects is natural in that both are responses to the same problem: Human cognitive capabilities are limited, while the environment in which choices are made can be highly complex. Satisficing deals with this problem by allowing a margin of error in the attempt to find an optimal alternative. Salience, meanwhile, focuses attention on aspects of the environment that we are predisposed to find interesting or enticing. The present framework provides a model of how these two coping mechanisms could interact.

[4] Positive (or "affirmative") action legislation specifies certain "protected characteristics," such as age, disability, marital status, pregnancy and maternity, race and national origin, sex and sexual orientation; which by law may not normally play a role in employment decisions. If a group is considered to suffer a disadvantage related to one of these characteristics, then an employer may (according to the U.K. Equality Act 2010, Part 11, Chapter 2) "[treat] a person (A) more favourably in connection with recruitment or promotion than another person (B) because A has the protected characteristic but B does not."

[5] Similarly, a reader selecting a novel to take on an extended beach vacation may value literary merit (a hidden characteristic) more than the number of pages (a surface characteristic), and a voter may care more about candidates' policy preferences (hidden) than about their party affiliations (surface).

Our goal is to develop an abstract model that encompasses these and similar phenomena within the framework of axiomatic choice theory.

The combination of hidden characteristics at the initial stage and surface characteristics at the second stage is at the core of our model, and applies also to the first two interpretations above. In the case of salience effects, the hidden criterion is the DM's utility (i.e., his maximand in the absence of cognitive constraints) while the surface criterion is the ability of the alternatives to attract attention. In the case of positive action, the hidden criterion is the employee's merit while the surface criterion is his degree of membership in the favored group. Note that for our model to be suitable these criteria must be applied lexicographically, since allowing for tradeoffs would lead to quite different forms of behavior.[6]

Since the interpretation of the model is not fixed, some important conceptual questions cannot be answered until we commit to a particular viewpoint. For example, the DM's welfare could be measured by the primary criterion alone (as in the case of salience effects when salience per se is not valued) or by the lexicographic composition of the primary and second criteria (as in the case of surface characteristics such as the visual attractiveness of cars). Other issues that hinge on interpretation include whether or not the second-stage maximization should be thought of as a deliberate action by the DM—arguably so under the positive action interpretation but not under the salience effects interpretation—and the conceptual relationship of our theory to other models of choice.[7]

1.2 Components of the model

To see how our model works, imagine a setting where alternatives are drawn from the set $X = xyz$.[8] (A more elaborate four-alternative example appears below in Figure 1.) The DM will have two complete and transitive relations over X, namely a primary relation R^1 and a secondary relation R^2; and the symmetric and asymmetric parts of each R^k will be denoted by I^k and P^k, respectively. For example, the primary and secondary rankings could be $x\mathrm{P}^1y\mathrm{P}^1z$ (implying $x\mathrm{P}^1z$) and $z\mathrm{I}^2y\mathrm{P}^2x$ (implying $z\mathrm{P}^2x$).

In the absence of cognitive constraints, the DM would optimize R^1 at the first stage and then R^2 at the second stage, amounting to a procedure of lexicographic preference maximization. Indeed, this is precisely how our DM will behave when faced with "binary" choice problems containing just two alternatives. For instance, when facing the menu xy, a DM with the two relations

[6] In connection with the lexicographic nature of the model, see Manzini and Mariotti's [14, p. 1825] discussion of "sequential noncompensatory heuristics" and the associated psychology literature.

[7] For further discussion of these points, see Section 3 regarding welfare analysis and Section 4 regarding the relationship to other models.

[8] Note the multiplicative notation for enumerated sets, which we shall use whenever convenient.

specified above will apply the strict primary preference $x\mathrm{P}^1y$ and eliminate alternative y from consideration, at which point the secondary relation R^2 is irrelevant. The rationale here is that binary menus are particularly simple in terms of cognition, and so it is in these contexts that the DM's true primary rankings are most likely to be reflected accurately.

In larger and thus more complex problems we shall allow cognitive constraints to bind at the first stage. This will mean that strict primary rankings perceived in the relevant binary problem need not be perceived when additional options are present. For instance, in problem xyz it could be that the strict preferences $x\mathrm{P}^1z$ and $y\mathrm{P}^1z$ are perceived but the strict preference $x\mathrm{P}^1y$ is *not* perceived, even though the latter is perceived in the simpler problem xy. In problem xyz the DM will then be left "pseudo-indifferent" between x and y after the first stage (alternative z having been eliminated), and so the secondary relation will become relevant. Indeed, since $y\mathrm{P}^2x$ we can conclude that x will be eliminated from consideration in the second stage and y will be chosen (despite being R^1-suboptimal).[9]

To keep a record of when different rankings are and are not perceived, we shall make explicit the menu dependence of the relations R^1, I^1, and P^1; writing $x\mathrm{P}^1_{xy}y$ to indicate perception of the strict preference $x\mathrm{P}^1y$ in problem xy, and writing $x\mathrm{I}^1_{xyz}y$ to indicate its non-perception (resulting in pseudo-indifference) in problem xyz.[10] Since by assumption cognitive constraints do not affect the second stage, the relations R^2, I^2, and P^2 will remain menu-independent. When facing an arbitrary problem A, our model thus imagines the DM maximizing a menu-dependent relation R^1_A and then maximizing a menu-independent relation R^2 over the alternatives that remain.

Observe that the relations R^1 and R^2 in our example can be represented numerically by any utility assignments with $f_1(x) > f_1(y) > f_1(z)$ and $f_2(z) = f_2(y) > f_2(x)$. Moreover, for problem xyz we can set the first-stage satificing threshold $\theta_1(xyz)$ equal to the *lowest* f_1-utility assigned to an alternative in the *highest* pseudo-indifference class according to R^1_{xyz}. Since the preferences perceived here are $xy\mathrm{P}^1_{xyz}z$, the highest pseudo-indifference class is the subset xy and the threshold is $\theta_1(xyz) = f_1(y) < f_1(x)$. Assigning thresholds in the same way for each problem A, the DM will then be willing to choose $w \in A$ if and only if it maximizes $f_2(w)$ subject to $f_1(w) \geq \theta_1(A)$.[11]

[9] For completeness, the full mapping from choice problems to surviving alternatives in this example is $xy \mapsto x$, $xz \mapsto x$, $yz \mapsto y$, and $xyz \mapsto y$.

[10] Permitting arbitrary menu dependence at the first stage would make our theory hard to falsify, but we shall impose additional restrictions in the form of the "nestedness" hypothesis on perceived preferences discussed in Section 2.2 below (and in [35, pp. 54–56]).

[11] Of course, the DM need not solve this constrained maximization problem consciously or intentionally. Indeed, if the satisficing constraint results from cognitive limitations then a literal understanding of the numerical formulation of our model is not entirely natural: In order to check the constraint the DM would need to know f_1, but then he would maximize this function perfectly and ignore the threshold. This is why the relation R^1 and the associated f_1 should be interpreted as objects that the DM would maximize *in the absence of cognitive constraints*, and why the numerical representation $\langle f_1, \theta_1, f_2\rangle$ is best viewed as a convenient analytical tool rather than as a procedural model.

In summary, our theory has three interlinked components. Its mathematical primitive is the DM's choice behavior, and our main result (Theorem 2.9) will concern when this behavior possesses a certain underlying structure. In contrast, our *conceptual* primitive is the ordinal model (comprising R^1, P^2, R^1_{xy}, I^1_{xyz}, etc.) formalizing our assumptions about how choices are generated, and it is to this model that we should turn when considering whether the theory is plausible introspectively.[12] Finally, the third component—which is not in any sense primitive—is the numerical representation of the DM's behavior (comprising f_1, θ_1, and f_2). This representation may be useful in applications of the theory, and a subsidiary result (Theorem 2.11) will translate our characterization of the model into numerical terms.

The remainder of the paper is organized as follows. In Section 2 we define our model formally (Sections 2.1–2.2), introduce our axioms (Section 2.3), and state our main and subsidiary characterization results (Sections 2.4–2.5). Section 3 discusses welfare analysis, while Section 4 reviews two strands of related work. All proofs not in the text appear in Appendix A.

2 Characterization results

2.1 Preliminaries

Fix a set X of *alternatives* and a *domain* $\mathfrak{D} \subseteq \{A \subseteq X : A \neq \emptyset\} =: \mathfrak{F}$. A map $C : \mathfrak{D} \to \mathfrak{F}$ is a *choice function* if $\forall A \in \mathfrak{D}$ we have $C(A) \subseteq A$. Here each $A \in \mathfrak{D}$ is a *menu* and $C(A)$ is the corresponding *choice set* containing the alternatives that might be chosen from A given some cognitive hypothesis. We shall assume that $\{xy : x, y \in X\} \subseteq \mathfrak{D}$, so that C associates a choice set with each binary (or unary) menu, and also for convenience that $X \in \mathfrak{D}$. But $\mathfrak{D}$ will be otherwise unrestricted and need not be the full domain $\mathfrak{F}$.[13]

A (*binary*) *relation* R on X is a subset of $X \times X$, with $\langle x, y\rangle \in \mathrm{R}$ commonly written as $x\mathrm{R}y$. Such a relation is a *complete preorder* if it is complete ($\neg[x\mathrm{R}y]$ only if $y\mathrm{R}x$) and transitive ($x\mathrm{R}y\mathrm{R}z$ only if $x\mathrm{R}z$); a *tournament* if it is complete and antisymmetric ($x\mathrm{R}y\mathrm{R}x$ only if $x = y$); and a *complete order* if it is complete, transitive, and antisymmetric.

[12] Indeed, the purpose of choice-theoretic analysis is to draw logical connections between observable behavior and hypotheses about cognition. The related point that theories of choice cannot be based only on "internal consistency" has been made by Sen [31] and Rubinstein and Salant [26, pp. 118–120]. Note also that it is not our goal to explain the origin of the DM's primary or secondary preferences, nor why these preferences are or are not perceived in different choice problems. In particular, we do not attempt to embed preference perception in an explicit optimization model with, for example, information-gathering or contemplation costs (see [35, pp. 64–65]). As is typical in axiomatic choice theory, we take cognitive structures as given and subject only to consistency or regularity assumptions.

[13] The inclusion of all binary menus in $\mathfrak{D}$ is essential for our analysis. However, this restriction is weak by the standards of axiomatic choice theory, where $\mathfrak{D} = \mathfrak{F}$ is commonly assumed (despite exceptions such as, e.g., Bossert et al. [8]). A penetrating analysis of domain specifications in choice-theoretic models of bounded rationality is provided by de Clippel and Rozen [10].

Notation 2.1. Write $G(A, \mathrm{R}) := \{x \in A : \forall y \in A,\ x\mathrm{R}y\}$.

Recall that classical choice theory (see, e.g., Samuelson [28] and Arrow [3]) imagines the DM's behavior to be determined entirely by his or her preferences among alternatives. Writing $x\mathrm{R}^1y$ when x is considered at least as good as y ("weak preference"), $x\mathrm{P}^1y$ when $x\mathrm{R}^1y$ and $\neg[y\mathrm{R}^1x]$ ("strict preference"), and $x\mathrm{I}^1y$ when $x\mathrm{R}^1y\mathrm{R}^1x$ ("indifference"), this hypothesis becomes the requirement that $\forall A \in \mathfrak{D}$ we have $C(A) = G(A, \mathrm{R}^1)$, expressed more compactly as $C = G(\cdot, \mathrm{R}^1)$. When R^1 is a complete preorder, C then simply selects from each menu the highest I^1-equivalence class of alternatives according to P^1.

2.2 Relation systems and nestedness

The model of decision making studied here differs from the classical model in two respects. Firstly, primary preference maximization is imperfect, and may become increasingly so as the menu becomes more complex. And secondly, the initial satisficing stage is followed by maximization of an independent secondary relation. We proceed now to develop the first-stage structure, before turning in Sections 2.3–2.4 to the characterization of the full composite model.

In our formulation, satisficing is represented cognitively as menu dependence of the "perceived preferences" guiding choice behavior. That is to say, for each $A \in \mathfrak{D}$ we have a separate relation R^1_A, with associated P^1_A and I^1_A, that incorporates both the DM's true primary preferences over A (encoded in R^1) and the resolution at which these preferences are perceived. Assembling the menu-specific relations into a vector then yields the DM's "primary preference system" $\mathcal{R}^1$, a type of construct that we now define more formally.

Definition 2.2. **A.** A *relation system* $\mathcal{R} = \langle \mathrm{R}_A \rangle_{A \in \mathfrak{D}}$ on $\mathfrak{D}$ is a vector of relations on the menus in $\mathfrak{D}$. **B.** A *system of complete preorders* is a relation system each component of which is a complete preorder.

To require that the primary preference system $\mathcal{R}^1$ be made up of complete preorders is to assume that while its components may be incomplete in the sense of reflecting the DM's true preferences only coarsely, each R^1_A must be both complete and transitive in the relation-theoretic sense. It follows that menu A is partitioned into well-defined "pseudo-indifference" (i.e., I^1_A-equivalence) classes, and maximization of perceived preferences then amounts to selecting the highest such class according to P^1_A.[14]

Notation 2.3. Write $G(A, \mathcal{R}) := G(A, \mathrm{R}_A) = \{x \in A : \forall y \in A,\ x\mathrm{R}_Ay\}$.

[14] As emphasized by the associate editor, the complete preorder assumption on perceived preferences is debatable for a DM with cognitive limitations. In particular, cognitive constraints could plausibly be manifested as incompleteness of R^1_A or intransitivity of I^1_A. For the first stage of our model in isolation, this issue has been considered in [35, pp. 60–61], where characterization results are given for perceived preferences with less stringent ordering properties. Determining which of these results can be extended to two stages is, however, beyond the scope of the present paper. Here our starting point is the (one-stage) satisficing model in [35], for which the full complement of standard ordering properties is needed.

In addition to the intramenu ordering requirement in Definition 2.2B, we shall impose the following pair of intermenu consistency properties on the primary preference system.

Definition 2.4. **A.** A relation system $\mathcal{R}$ is *nested* if $\forall x, y \in A, B \in \mathfrak{D}$ such that $A \subseteq B$ we have $x\mathrm{P}_B y$ only if $x\mathrm{P}_A y$. **B.** A relation system $\mathcal{R}$ is *binary transitive* if $\forall x, y, z \in X$ we have $x\mathrm{R}_{xy}y\mathrm{R}_{yz}z$ only if $x\mathrm{R}_{xz}z$.

Nestedness captures an assumption that the DM can discriminate among alternatives at least as precisely when the menu on which they appear is smaller. Since the default relationship between any two alternatives is pseudo-indifference, it is the agent's *strict* preferences that his cognitive faculties seek to uncover. Given $x, y \in A \subseteq B$, we posit that if the strict preference $x\mathrm{P}^1 y$ is perceived in the context of problem B, written $x\mathrm{P}^1_B y$, then this same strict preference should also be perceived in the context of the (weakly) simpler problem A, written $x\mathrm{P}^1_A y$.[15] This is equivalent (when $\mathcal{R}^1$ is a system of complete preorders) to $y\mathrm{R}^1_A x \Longrightarrow y\mathrm{R}^1_B x$, but does *not* guarantee the converse.[16]

The second intermenu consistency property concerns "binary" choice problems with either one or two alternatives. Intuitively, our assumption is that the DM fully perceives his primary preferences when facing these very simple menus. That is to say, we have $x\mathrm{P}^1_{xy} y \Longleftrightarrow x\mathrm{P}^1 y$, or equivalently $y\mathrm{R}^1_{xy} x \Longleftrightarrow y\mathrm{R}^1 x$. Of course, we also assume that the true primary relation R^1 is a complete preorder, and in particular that it is transitive. Since true and binary perceived preferences are identical, this amounts to the binary transitivity property $x\mathrm{R}^1_{xy}y\mathrm{R}^1_{yz}z \Longrightarrow x\mathrm{R}^1_{xz}z$ in Definition 2.4B.

For $x, y \in A$, note that $x\mathrm{P}^1_A y \Longrightarrow x\mathrm{P}^1_{xy} y \Longleftrightarrow x\mathrm{P}^1 y$, where the first implication follows from nestedness. This means that a strict preference perceived in any choice problem is always genuine, in the sense that it would be affirmed by the DM if he were cognitively unconstrained. In contrast $x\mathrm{R}^1_A y$ does *not* in general ensure that $x\mathrm{R}^1 y$, so weak perceived preferences need not be genuine. For this reason we refer to the assertion $x\mathrm{R}^1_A y$ as a "pseudo-preference," just as $x\mathrm{I}^1_A y$ (equivalent to $x\mathrm{R}^1_A y\mathrm{R}^1_A x$) is a statement of "pseudo-indifference."[17]

In summary, the first stage of our model describes a DM possessing true primary preferences of the classical sort; who perceives these preferences fully in binary problems but (perhaps) imperfectly in larger ones; whose perceived preferences partition each menu into well-defined pseudo-indifference classes; and who perceives a strict preference in a given problem only if he also perceives it in each smaller problem in which it is relevant.

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[15] The implicit assumption that the relative complexity of two menus is aligned with set inclusion is discussed at length in [35, pp. 54–56].

[16] The intuition for nestedness can be understood in terms of an analogy (suggested by Robert Wilson) to either mapmaking or telescopic vision. The larger the area one wishes to depict on one's map or view through one's telescope, the lower will be the resolution of the resulting image. Zooming in on a particular region—analogous to removing alternatives from a menu—will improve the level of detail but at the cost of narrower scope.

[17] Here the asymmetry between strict and weak perceived preference results from pseudo-indifference being the default relationship between options, which can be overturned by the DM's cognitive efforts.

2.3 Weak Congruence and Base Transitivity

When maximization of the secondary relation R^2 is appended to the first-stage model described above, the choice set for a given $A \in \mathfrak{D}$ is determined as

$$C(A) = G(G(A, \mathcal{R}^1), \mathrm{R}^2). \tag{1}$$

Our task is to characterize this model behaviorally for the case of R^2 a complete preorder and $\mathcal{R}^1$ a nested, binary transitive system of complete preorders.

To achieve the desired characterization, we shall need methods of deducing from raw choice data sufficient information about $\mathcal{R}^1$ and R^2, both assumed to be unobserved.[18] The first type of information is contained in the following pair of revealed relation systems, which for a given menu B conduct a "local" (relative to B) search of the domain for direct or indirect evidence of a primary pseudo-preference.

Definition 2.5. Define the relation systems $\hat{\mathcal{R}}^{1\mathrm{d}}$ and $\hat{\mathcal{R}}^{1\mathrm{i}}$ as follows. For $x, y \in B \in \mathfrak{D}$: **A.** Let $x\hat{\mathrm{R}}^{1\mathrm{d}}_B y$ if $\exists A \in \mathfrak{D}$ such that both $y \in A \subseteq B$ and $x \in C(A)$. **B.** Let $x\hat{\mathrm{R}}^{1\mathrm{i}}_B y$ if $\exists n \geq 2$ and $z_1, z_2, \ldots, z_n \in B$ such that $x = z_1\hat{\mathrm{R}}^{1\mathrm{d}}_B z_2\hat{\mathrm{R}}^{1\mathrm{d}}_B \cdots \hat{\mathrm{R}}^{1\mathrm{d}}_B z_n = y$.

The assertion $x\hat{\mathrm{R}}^{1\mathrm{d}}_B y$ means that x is choosable in the presence of y in at least one problem $A \subseteq B$. This implies that $x\mathrm{R}^1_A y$ (since otherwise x would have been eliminated in the first stage) and so $x\mathrm{R}^1_B y$ by nestedness. But then $x = z_1\hat{\mathrm{R}}^{1\mathrm{d}}_B z_2\hat{\mathrm{R}}^{1\mathrm{d}}_B \cdots \hat{\mathrm{R}}^{1\mathrm{d}}_B z_n = y$ implies $x = z_1\mathrm{R}^1_B z_2\mathrm{R}^1_B \cdots \mathrm{R}^1_B z_n = y$, whereupon $x\mathrm{R}^1_B y$ since R^1_B is transitive. We conclude that either $x\hat{\mathrm{R}}^{1\mathrm{d}}_B y$ or $x\hat{\mathrm{R}}^{1\mathrm{i}}_B y$ is valid evidence that $x\mathrm{R}^1_B y$, and it is worth noting that this remains true regardless of what occurs later in the decision-making process.

Next we deduce secondary preferences from the choice data, collecting this information in menu-independent binary relations and conducting a "global" search of the domain.

Definition 2.6. Define the relations $\hat{\mathrm{R}}^{2\mathrm{d}}$ and $\hat{\mathrm{R}}^{2\mathrm{i}}$ as follows. For $x, y \in X$: **A.** Let $x\hat{\mathrm{R}}^{2\mathrm{d}}y$ if $\exists A \in \mathfrak{D}$ such that $y\hat{\mathrm{R}}^{1\mathrm{i}}_A x$ and $x \in C(A)$. **B.** Let $x\hat{\mathrm{R}}^{2\mathrm{i}}y$ if $\exists n \geq 2$ and $z_1, z_2, \ldots, z_n \in X$ such that $x = z_1\hat{\mathrm{R}}^{2\mathrm{d}}z_2\hat{\mathrm{R}}^{2\mathrm{d}} \cdots \hat{\mathrm{R}}^{2\mathrm{d}}z_n = y$.

The assertion $x\hat{\mathrm{R}}^{2\mathrm{d}}y$ means that x is choosable in the presence of y in at least one problem A for which $y\hat{\mathrm{R}}^{1\mathrm{i}}_A x$. Since the latter implies that $y\mathrm{R}^1_A x$ and since x is not eliminated in the first stage, for each $z \in A$ we have $y\mathrm{R}^1_A x\mathrm{R}^1_A z$ and thus $y\mathrm{R}^1_A z$ since R^1_A is transitive. It follows that y too survives the first stage, and since x is choosable we can conclude that $x\mathrm{R}^2 y$. In short, we have that $x\hat{\mathrm{R}}^{2\mathrm{d}}y$ implies $x\mathrm{R}^2 y$. But then $x = z_1\hat{\mathrm{R}}^{2\mathrm{d}}z_2\hat{\mathrm{R}}^{2\mathrm{d}} \cdots \hat{\mathrm{R}}^{2\mathrm{d}}z_n = y$ implies $x = z_1\mathrm{R}^2 z_2\mathrm{R}^2 \cdots \mathrm{R}^2 z_n = y$, whereupon $x\mathrm{R}^2 y$ since R^2 is transitive. We conclude that either $x\hat{\mathrm{R}}^{2\mathrm{d}}y$ or $x\hat{\mathrm{R}}^{2\mathrm{i}}y$ is valid evidence that $x\mathrm{R}^2 y$.

[18] In some applications the secondary criterion may be observed or otherwise known to the modeler, in which case information about this relation no longer needs to be inferred from choice behavior. For example, R^2 may reflect the relative salience of the alternatives as determined by observable advertising. Such scenarios lead to a different revealed preference exercise, of the sort studied by Manzini et al. [16].

We shall use Definitions 2.5–2.6 to state the first of two axioms needed to characterize our model. Given $x, y \in A$, let $x \in C(A)$ and $y\hat{\mathrm{R}}^{1\mathrm{i}}_A x$, so that y survives the first stage (as shown above). If also $y\hat{\mathrm{R}}^{2\mathrm{i}}x$ then we know that $y\mathrm{R}^2 x$ (again as shown above), and since x survives the second stage it follows that $y \in G(G(A, \mathcal{R}^1), \mathrm{R}^2) = C(A)$. The following property of the choice function is therefore a necessary condition for the model.

Condition 2.7 (Weak Congruence)**.** Given $x, y \in A \in \mathfrak{D}$, if $x \in C(A)$, $y\hat{\mathrm{R}}^{1\mathrm{i}}_A x$, and $y\hat{\mathrm{R}}^{2\mathrm{i}}x$, then $y \in C(A)$.

In words, Weak Congruence states that if alternative x is choosable and alternative y is revealed to be (perceived to be) at least as good as x at both stages of the model, then y too must be choosable.[19]

The second axiom we shall employ imposes transitivity on the DM's binary choices.

Condition 2.8 (Base Transitivity)**.** Given $x, y, z \in X$, if $x \in C(xy)$ and $y \in C(yz)$ then $x \in C(xz)$.

In binary problems the DM applies the true first-stage relation R^1 (since true and binary perceived preferences are identical), followed by the secondary relation R^2. Furthermore, since both of these relations are complete preorders, their lexicographic composition (see Definition 3.1) will inherit this property, and thus choices must satisfy Base Transitivity.

2.4 Ordinal characterization

Our main characterization result states that Weak Congruence and Base Transitivity together are necessary and sufficient for the two-stage model under investigation.

Theorem 2.9. *There exist a nested, binary transitive system of complete preorders $\mathcal{R}^1$ and a complete preorder R^2 such that $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ if and only if Weak Congruence and Base Transitivity hold.*

As usual, moving from the cognitive model to the axioms is the more straightforward exercise. Indeed, we have already seen rough arguments for the necessity of Weak Congruence and Base Transitivity when choices are generated as in Equation 1. Conversely, the heart of the proof of Theorem 2.9 lies in showing that $C = G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \hat{\mathrm{R}}^{2\mathrm{i}})$; i.e., that the indirect revealed

[19] The name "Weak Congruence" refers to Richter's [23, p. 637] classical Congruence axiom, which can be stated as follows: Given $x, y \in A \in \mathfrak{D}$, if $x \in C(A)$ and $y\hat{\mathrm{R}}^{1\mathrm{i}}_X x$, then $y \in C(A)$. The conclusion of this implication is retained in Condition 2.7 while the hypotheses are strengthened—thereby weakening its logical force—in two distinct ways. First, the stage-one hypothesis $y\hat{\mathrm{R}}^{1\mathrm{i}}_A x$ replaces $y\hat{\mathrm{R}}^{1\mathrm{i}}_X x$, so that the revealed preference must be found within menu A. And second, the stage-two hypothesis $y\hat{\mathrm{R}}^{2\mathrm{i}}x$ is added.
Note also that, while weaker than Congruence, Weak Congruence continues to imply Sen's [29, p. 384] so-called "β" condition: Given $x, y \in A, B \in \mathfrak{D}$, if $A \subseteq B$, $x, y \in C(A)$, and $y \in C(B)$, then $x \in C(B)$.

<table>
<tr><td colspan="12">$C(wxyz) = x$
$z\mathrm{I}^1_{wxyz}y\mathrm{I}^1_{wxyz}x\mathrm{P}^1_{wxyz}w$
$x\mathrm{P}^2w\mathrm{P}^2y\mathrm{I}^2z$</td></tr>
<tr><td colspan="3">$C(wxy) = x$
$y\mathrm{I}^1_{wxy}x\mathrm{P}^1_{wxy}w$
$x\mathrm{P}^2w\mathrm{P}^2y$</td><td colspan="3">$C(wxz) = z$
$z\mathrm{P}^1_{wxz}x\mathrm{P}^1_{wxz}w$
$x\mathrm{P}^2w\mathrm{P}^2z$</td><td colspan="3">$C(wyz) = yz$
$z\mathrm{I}^1_{wyz}y\mathrm{P}^1_{wyz}w$
$w\mathrm{P}^2y\mathrm{I}^2z$</td><td colspan="3">$C(xyz) = z$
$z\mathrm{P}^1_{xyz}y\mathrm{P}^1_{xyz}x$
$x\mathrm{P}^2y\mathrm{I}^2z$</td></tr>
<tr><td colspan="2">$C(wx) = x$
$x\mathrm{P}^1_{wx}w$
$x\mathrm{P}^2w$</td><td colspan="2">$C(wy) = y$
$y\mathrm{P}^1_{wy}w$
$w\mathrm{P}^2y$</td><td colspan="2">$C(wz) = z$
$z\mathrm{P}^1_{wz}w$
$w\mathrm{P}^2z$</td><td colspan="2">$C(xy) = y$
$y\mathrm{P}^1_{xy}x$
$x\mathrm{P}^2y$</td><td colspan="2">$C(xz) = z$
$z\mathrm{P}^1_{xz}x$
$x\mathrm{P}^2z$</td><td colspan="2">$C(yz) = z$
$z\mathrm{P}^1_{yz}y$
$y\mathrm{I}^2z$</td></tr>
</table>

Fig. 1 A choice function satisfying the conditions in Theorem 2.9. The first entry in each cell shows the subset of choosable options, and the second and third entries show the relevant ranking information for the two stages of the model. Here the DM's primary and secondary rankings are $z\mathrm{P}^1y\mathrm{P}^1x\mathrm{P}^1w$ and $x\mathrm{P}^2w\mathrm{P}^2y\mathrm{I}^2z$. Moreover, the assertion $z\mathrm{P}^1_{xyz}y$ means that the strict primary ranking $z\mathrm{P}^1y$ is perceived in problem xyz, while $z\mathrm{I}^1_{wxyz}y$ means that the same strict ranking is not perceived in problem $wxyz$. In each problem A the choice set $C(A)$ is obtained by maximizing R^1_A followed lexicographically by R^2.

primary preference system $\hat{\mathcal{R}}^{1\mathrm{i}}$ and secondary relation $\hat{\mathrm{R}}^{2\mathrm{i}}$ can stand in for the unobserved structures $\mathcal{R}^1$ and R^2.[20]

An example of a choice function that satisfies the conditions in Theorem 2.9 appears in Figure 1, in which each cell corresponds to a choice problem drawn from $wxyz$. Here the first entry in a given cell shows which alternatives on the menu are deemed choosable (e.g., $C(wyz) = yz$), the second shows the perceived primary preference rankings (e.g., $z\mathrm{I}^1_{wyz}y\mathrm{P}^1_{wyz}w$), and the third shows the relevant secondary rankings (e.g., $w\mathrm{P}^2y\mathrm{I}^2z$). It is easily verified in this case that $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$, in accordance with our result.

Starting with the choice function in Figure 1, a violation of Weak Congruence can be manufactured by changing the data point $C(wxyz) = x$ to $C(wxyz) = y$.[21] To confirm the violation, observe that since $y \in C(xy)$ we have $y\hat{\mathrm{R}}^{1\mathrm{d}}_{wxy}x$, and since $x \in C(wxy)$ we then have both $x\hat{\mathrm{R}}^{1\mathrm{d}}_{wxyz}y$ and $x\hat{\mathrm{R}}^{2\mathrm{d}}y$. But now, since $x\hat{\mathrm{R}}^{1\mathrm{i}}_{wxyz}y$, $x\hat{\mathrm{R}}^{2\mathrm{i}}y$, and $y \in C(wxyz)$, Weak Congruence demands that $x \in C(wxyz)$, when in fact $C(wxyz) = y$. By Theorem 2.9 the new (post-modification) choice function is thus inconsistent with our model.

2.5 Numerical representations

In this section we examine numerical representations of the cognitive model characterized in Theorem 2.9 under the simplifying assumption that X is fi-

[20] Furthermore, the revealed constructs are shown to have the required nestedness and ordering properties, except that $\hat{\mathrm{R}}^{2\mathrm{i}}$ need not be complete. This difficulty is overcome by using Szpilrajn's Theorem [34] to replace $\hat{\mathrm{R}}^{2\mathrm{i}}$ with a complete preorder that generates the same behavior when composed with $\hat{\mathcal{R}}^{1\mathrm{i}}$.

[21] Base Transitivity of course continues to hold after this modification.

nite.[22] Clearly the operation $G(\cdot, \mathrm{R}^2)$ can then be represented by maximization of a real-valued function, while $G(\cdot, \mathcal{R}^1)$ does not in general have this property. Instead, the latter operation admits the following more elaborate type of representation, which makes its satisficing interpretation apparent.

Definition 2.10. **A.** A *threshold structure* $\langle f_1, \theta_1 \rangle$ is a pair of functions $f_1 : X \to \mathbb{R}$ and $\theta_1 : \mathfrak{D} \to \mathbb{R}$ such that $\forall x, y \in X$ we have $\theta_1(xy) = \max f_1[xy]$. **B.** A threshold structure $\langle f_1, \theta_1 \rangle$ is said to be *expansive* if $\forall A, B \in \mathfrak{D}$ such that $A \subseteq B$ and $\max f_1[A] \geq \theta_1(B)$ we have $\theta_1(A) \geq \theta_1(B)$.

Here f_1 represents the true primary relation R^1, while θ_1 associates with each menu A an f_1-threshold $\theta_1(A)$ for viability as a potential choice. The alternatives that progress to the second stage are those whose primary utilities fall on the interval between $\theta_1(A)$ and $\max f_1[A]$, which is another way to describe the highest pseudo-indifference class of options according to P^1_A. The $\theta_1(xy) = \max f_1[xy]$ requirement then enforces our assumption that the agent perceives his true primary preferences in binary choice problems.[23]

Nestedness of the perceived preference system $\mathcal{R}^1$ translates into the requirement that the threshold representation be "expansive."[24] To understand this property, let $A \subseteq B$ and suppose (contrary to Definition 2.10B) that $\max f_1[A] \geq \theta_1(B) > \theta_1(A)$. Selecting $x \in A$ such that $f_1(x) = \max f_1[A]$ and assuming without loss of generality that $\exists y \in A$ with $f_1(y) = \theta_1(A)$, we then have $f_1(x) \geq \theta_1(B) > \theta_1(A) = f_1(y)$. In this case the strict preference $x\mathrm{P}^1 y$ is perceived in problem B but not in $A \subseteq B$, violating nestedness. The expansiveness property thus requires θ_1 to be in a sense conditionally decreasing: Larger menus must be assigned lower thresholds, but only if some option on the smaller menu achieves the threshold for the larger one.[25]

We can now state a version of the characterization in Theorem 2.9 in which the DM's satisficing with respect to the primary criterion and subsequent maximization of the secondary criterion appear explicitly.

Theorem 2.11. *Let X be finite. Then there exist an expansive threshold structure $\langle f_1, \theta_1 \rangle$ and an $f_2 : X \to \mathbb{R}$ such that $\forall A \in \mathfrak{D}$ we have*

$$C(A) = \operatorname*{arg\,max}_{x \in A \,\wedge\, f_1(x) \geq \theta_1(A)} f_2(x) \tag{2}$$

[22] Without the finiteness restriction, necessary and sufficient conditions for a binary relation to admit a utility representation are provided by Fishburn [12, p. 27]. The relationship between threshold structures (see Definition 2.10) and relation systems can also be investigated in the general case, but this essentially technical issue is not pursued here.

[23] For example, the first-stage constructs R^1 and $\mathcal{R}^1$ in Figure 1 admit representations $\langle f_1, \theta_1 \rangle$ such that $f_1(z) > f_1(y) > f_1(x) > f_1(w)$; $\theta_1(wx) = \theta_1(wxy) = \theta_1(wxyz) = f_1(x)$; $\theta_1(wy) = \theta_1(xy) = \theta_1(wyz) = f_1(y)$; and $\theta_1(wz) = \theta_1(xz) = \theta_1(yz) = \theta_1(wxz) = \theta_1(xyz) = f_1(z)$.

[24] This terminology originated in [35, p. 59], where the property was linked to the "Strong Expansion" axiom on the choice function (see Sen [30, p. 66]).

[25] Note that since $\theta_1(A)$ equals the lowest f_1-value in the highest pseudo-indifference class of alternatives in A, we always have $\max f_1[A] \geq \theta_1(A)$ and at least one option survives the first stage. Of course, there do exist threshold structures for which this inequality fails for one or more menus, but these structures do not yield well-defined choice functions. (Recall that we assume that C is nonempty-valued.)

if and only if Weak Congruence and Base Transitivity hold.

Since the conditions on the choice function remain the same as in the earlier result, this establishes (for the case of finite X) a three-way equivalence between the cognitive model, the behavioral restrictions, and the numerical representation in Equation 2.

3 Welfare analysis

3.1 Alternative conceptions of well-being

The problem of welfare analysis for boundedly rational and other nonstandard models of decision making has been discussed at length in the literature. In a detailed survey of the issue, Bernheim [5] contrasts three methodological approaches, which identify welfare with "revealed well-being," "measured well-being," or "choice." Our model will be suited to the first of these approaches once we have agreed which of its components is the indicator of well-being.[26] However, there are at least two good candidates for this role.

1. *The primary criterion.*
   Under some interpretations of the model, R^1 will be the appropriate welfare standard. In the case of positive action, for example, it is natural to suppose that the employer's well-being (e.g., profit) is determined entirely by the employee's merit, and that group identity is taken into account in the second stage only because of legal or social pressure. Similarly, in the case of salience effects we would normally assume that salience (the secondary criterion) neither benefits nor harms the DM directly.
2. *The lexicographic composition of the primary and secondary criteria.*
   In the case of surface characteristics, both R^1 and R^2 are relevant to well-being. Welfare is therefore indicated by the composition of these two criteria, which is the relation that the DM would maximize in the absence of first-stage satisficing. This welfare order would also apply if we were to posit that group identity directly benefits the employer (but cannot substitute for higher merit), or that higher salience directly benefits the DM.

Taking the second of these two candidates first, and recalling that R^1 is fully perceived in binary problems, we can define the lexicographic composition of the primary and secondary criteria more precisely as follows.

Definition 3.1. Given a relation system $\mathcal{R}^1$ and a relation R^2, let $x\mathrm{L}^{12}y$ if and only if either $x\mathrm{P}^1_{xy}y$ or $y\mathrm{I}^1_{xy}x\mathrm{R}^2y$.

[26] Specifically, our model is of the sort that Bernheim [5, p. 274] describes as "allowing for divergences between preferences and behavior," a strategy advocated by Rubinstein and Salant [26, pp. 120–122].

When our model holds we then have $\forall x, y \in X$ that

$$G(xy, \mathrm{L}^{12}) = G(G(xy, \mathrm{R}^1_{xy}), \mathrm{R}^2) = G(G(xy, \mathcal{R}^1), \mathrm{R}^2) = C(xy), \tag{3}$$

and thus $x\mathrm{L}^{12}y \iff x \in C(xy)$.[27] It follows that the DM's well-being is revealed precisely by his behavior in binary problems.

Proposition 3.2. *Given a nested, binary transitive system of complete preorders $\mathcal{R}^1$ and a complete preorder R^2, let $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$. Then $\forall x, y \in X$ we have $x\mathrm{L}^{12}y$ if and only if $x \in C(xy)$.*

We can therefore conclude that if L^{12} is the appropriate welfare order then welfare analysis is straightforward under our model—at least provided binary choice data are available.

Now suppose instead that the primary criterion R^1 is the welfare standard. In this case by Equation 3 we have $G(xy, \mathrm{R}^1_{xy}) \supseteq C(xy)$ and thus $x \in C(xy) \Longrightarrow x\mathrm{R}^1_{xy}y$, but we lack the converse implication $x\mathrm{R}^1_{xy}y \Longrightarrow x \in C(xy)$. Of course, the latter assertion is equivalent to $x \notin C(xy) \Longrightarrow y\mathrm{P}^1_{xy}x$. Hence we know that revealed weak base preferences of the form $x \in C(xy)$ are always genuine weak primary preferences, but we do not yet know when revealed *strict* base preferences of the form $C(xy) = x \neq y$ are genuine *strict* primary preferences.

To address this question, observe first that $C(xy) = x$ implies $x\mathrm{R}^1_{xy}y$. If in addition $y\hat{\mathrm{R}}^{1\mathrm{d}}_X x$, then $\exists A \in \mathfrak{D}$ such that $x \in A$ and $y \in C(A) = G(G(A, \mathcal{R}^1), \mathrm{R}^2)$. From $x\mathrm{R}^1_{xy}y$ and the nestedness of $\mathcal{R}^1$ we have also $x\mathrm{R}^1_A y$, so that $x \in G(A, \mathcal{R}^1)$ and $y\mathrm{R}^2 x$. And to ensure that $y \notin C(xy)$ we must then have $x\mathrm{P}^1_{xy}y$. This argument shows that any revealed strict base preference standing *in opposition to* a $\hat{\mathrm{R}}^{1\mathrm{d}}_X$-ranking is certain to be genuine.

Proposition 3.3. *Given a nested, binary transitive system of complete preorders $\mathcal{R}^1$ and a complete preorder R^2, let $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$. Then $\forall x, y \in X$ we have:* **A.** *$x \in C(xy)$ only if $x\mathrm{R}^1_{xy}y$.* **B.** *$C(xy) = x \neq y$ and $y\hat{\mathrm{R}}^{1\mathrm{d}}_X x$ only if $x\mathrm{P}^1_{xy}y$.*

If $y\hat{\mathrm{R}}^{1\mathrm{d}}_X x$ does not hold then $\forall A \in \mathfrak{D}$ with $x \in A$ we have $y \notin C(A)$, and in particular $y \notin C(xy)$. This is the case in which the revealed strict base preference $C(xy) = x \neq y$ does *not* stand in opposition to a $\hat{\mathrm{R}}^{1\mathrm{d}}_X$-ranking, and so Proposition 3.3B does not apply. Indeed, the conclusion $x\mathrm{P}^1_{xy}y$ need not hold in such situations.[28] The issue here is that our two-stage model has extra degrees of freedom relative to standard preference maximization: When y is never chosen in the presence of x, we lack direct evidence to determine whether this fact should be attributed to primary or secondary preference. As with most generalizations of the standard model, we require violations of classical rationality axioms (see, e.g., Footnote 19) to fully identify the components of our theory.[29]

[27] In Equation 3, the first equality comes from Definition 3.1, the second from Notation 2.3, and the third from Theorem 2.9.

[28] For example, if $\mathfrak{D} = \{x, y, xy\}$ then $C(xy) = x \neq y$ is consistent with $x\mathrm{I}^1_{xy}y$ and $x\mathrm{P}^2y$.

[29] For example, if $\mathfrak{D} = \{x, y, z, xy, xz, yz, xyz\}$ then $\mathcal{R}^1$ and R^2 are both fully identified by the choice data $C(xy) = xy$, $C(xz) = x$, $C(yz) = y$, and $C(xyz) = z$.

3.2 Single-valued choice sets

Proposition 3.2 shows that binary choice data provide reliable welfare comparisons when L^{12} is the appropriate ranking, while Proposition 3.3 shows that the same is true to a partial extent when R^1 is the appropriate ranking. To obtain a model in which L^{12} and R^1 are identical, and hence binary welfare comparisons are reliable in either case, it suffices to require that the primary criterion be a complete order. This can be implemented by imposing the following restriction on the preference system $\mathcal{R}^1$.

Definition 3.4. A relation system $\mathcal{R}$ is *binary antisymmetric* if $\forall x, y \in X$ we have $x\mathrm{R}_{xy}y\mathrm{R}_{xy}x$ only if $x = y$.

The observable consequence of this restriction is that menus with two alternatives will have choice sets with only one.

Condition 3.5 (Base Univalence)**.** For each $x, y \in X$ we have $|C(xy)| = 1$.

Proposition 3.6. *There exist a nested, binary transitive, binary antisymmetric system of complete preorders* $\mathcal{R}^1$ *and a complete preorder* R^2 *such that* $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ *if and only if Weak Congruence, Base Transitivity, and Base Univalence hold.*

Propositions 3.2–3.3 can then be merged and strengthened into the following result.

Proposition 3.7. *Given a nested, binary transitive, binary antisymmetric system of complete preorders* $\mathcal{R}^1$ *and a complete preorder* R^2*, suppose that* $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$*. Then* $\forall x, y \in X$ *we have* $x \in C(xy) \iff x\mathrm{L}^{12}y \iff x\mathrm{R}^1_{xy}y$*.*

Incidentally, for some purposes we may wish to require that C is globally single-valued; for example, to relate the theory to other models that generate unique choices.

Condition 3.8 (Univalence)**.** For each $A \in \mathfrak{D}$ we have $|C(A)| = 1$.

This has the incremental effect of imposing the complete order requirement also on the secondary relation.[30]

Theorem 3.9. *There exist a nested, binary transitive, binary antisymmetric system of complete preorders* $\mathcal{R}^1$ *and a complete order* R^2 *such that* $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ *if and only if Weak Congruence, Base Transitivity, and Univalence hold.*

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[30] Hence adding Univalence to the axioms in Theorem 2.9 does not reduce our model to standard preference maximization, as occurs under satisficing without a secondary criterion (see [35, pp. 61–62]).

3.3 The primacy of binary problems

To summarize, the message of Section 3 is that welfare analysis is quite tractable under our model if either R^1 or L^{12} is the appropriate welfare order. Conflicts between classical revealed preferences are resolved by favoring data from smaller menus in which—according to the model—the true, welfare-significant preferences are perceived with higher resolution. This leads us to take binary choices as the most reliable arbiter of well-being.

Our approach to "behavioral welfare economics" is in fact anticipated in Bernheim's survey [5, p. 299], which offers the following illustration of the methodology proposed by Bernheim and Rangel [6].

> For example, if it is known that an individual with well-behaved preferences sometimes "satisfices" when confronted with more than two alternatives, we would restrict [the welfare-relevant domain] to binary choice sets [i.e., menus], and thereby generate a welfare criterion that accurately reflects his well-being.

The same approach is valid in the context of Manzini and Mariotti's [15, p. 1160] model of choice with categorization, in which "[w]henever there is conflict between choice from a binary set and choice from a larger set, the choice from the binary set is driven by preference[.]" Indeed, binary choices can be used to infer welfare rankings in any setting where removing other options strips away complicating factors to expose the DM's underlying evaluation of two alternatives.[31]

4 Related work

4.1 Consideration-set models

This paper belongs to a substantial literature that studies the behavioral implications of nonstandard models of decision making. Without attempting a comprehensive survey, we now highlight some areas of related work.

Several recent papers introduce into choice theory versions of the "consideration set." Recall that this is the subset of available alternatives that is actively investigated by the DM, a concept discussed extensively in the marketing literature.[32] Writing $\sigma(A)$ for the consideration set associated with menu A and R^2 for the preference relation, such models assume that

$$C(A) = G(\sigma(A), \mathrm{R}^2). \tag{4}$$

[31] Note that this commonality of welfare rankings under different models allows a partial escape from the problem of "observational equivalence" emphasized by Bernheim [5, p. 279].

[32] See, e.g., Roberts and Lattin [24]. Versions of this idea have been incorporated into economic models by, among others, Eliaz and Spiegler [11], who examine the strategic interaction among firms that try to manipulate the consideration sets of their customers with "door opener" products and other costly marketing schemes; and by Armstrong et al. [2], who analyze a search model in which one seller is more "prominent" than its rivals and is therefore sampled first by potential buyers.

The empirical content of the theory is determined by the restrictions imposed on the map $\sigma : \mathfrak{D} \to \mathfrak{F}$ and the relation R^2, as in the following special cases.

1. Masatlioglu et al. [18] require that $\forall A, B \in \mathfrak{D}$ such that $\sigma(B) \subseteq A \subseteq B$ we have $\sigma(A) = \sigma(B)$, in which case σ is designated an "attention filter." In addition, they require that R^2 is a complete order.
2. Lleras et al. [13] impose the "consideration filter" assumption that $\forall A, B \in \mathfrak{D}$ with $A \subseteq B$ we have $\sigma(B) \cap A \subseteq \sigma(A)$, and assume also that R^2 is a complete order. The same structure describes Cherepanov et al.'s [9] "order rationalization theory."
3. In Manzini and Mariotti's [14] model of "rational shortlist methods," R^2 is a tournament and there exists a complete relation R^1 such that $\sigma = G(\cdot, \mathrm{R}^1)$.
4. Spears [33] requires that σ be a consideration filter and R^2 a tournament. This also describes Cherepanov et al.'s [9] notion of "basic rationalization" and Manzini and Mariotti's [15, p. 1160] "categorize-then-choose" theory.

It is apparent from the descriptions of these models that Cases 2 and 3 each separately imply Case 4, since any complete order is a tournament and for any R^1 the map $G(\cdot, \mathrm{R}^1)$ is a consideration filter. Cases 2 and 3 are independent of each other, however, and Case 1 is independent of all the others.[33]

Taking $\sigma = G(\cdot, \mathcal{R}^1)$, our model is also an instance of the consideration-set framework in Equation 4. Our hypothesis that R^2 is a complete preorder is weaker than the complete-order assumption and independent of the tournament assumption (since transitivity and antisymmetry are themselves independent). Moreover, for a relation system $\mathcal{R}^1$ possessing the properties specified in our model, the hypothesis that $\sigma = G(\cdot, \mathcal{R}^1)$ is independent of both the attention filter and consideration filter assumptions, as well as independent of the structure imposed on the map σ in Case 3 (where R^1 need not be transitive). In short, our theory of satisficing behavior with a secondary criterion is structurally unrelated to all of the above consideration-set models.[34]

Depending on the interpretation of our framework, it may or may not bear conceptual similarities to the theories cited above. On the one hand, the idea of surface characteristics resembles Manzini and Mariotti's [14] motivation for "sequential rationalizability" (see Footnote 6). The substantive difference between these two models lies in how the DM's bounded rationality is manifested in the lexicographic setting—whether by relaxing transitivity of R^1 in the case of rational shortlist methods, or by moving to a nested preference system $\mathcal{R}^1$ in the case of our theory.

[33] Note that all of the contributions cited in Cases 1–4 impose Univalence (Condition 3.8) as a background assumption, and all except for [15] assume that $\mathfrak{D} = \mathfrak{F}$.

[34] Indeed, it is shown in [35] that the consideration-filter property is precisely the feature of classical choice behavior that must be abandoned to allow for satisficing (as modeled in Section 2.2). Observe also that the independence of our theory from those in Cases 1–4 at the level of the cognitive model has its counterpart at the axiomatic level. For example, Weak Congruence and Base Transitivity together do not imply either of the "Weak WARP" and "Expansion" conditions used in Cases 3–4 (see [14, p. 1828]). And conversely, Weak Congruence is itself not implied by any of the other consideration-set structures.

On the other hand, if we interpret our model as one of salience effects, then conceptually there are clear distinctions from the above consideration-set theories. In Masatlioglu et al. [18] and Lleras et al. [13], for example, the options in $\sigma(A)$ are preselected from menu A by some cognitive mechanism (respectively, awareness and consideration), after which the DM applies a standard preference relation R^2. Contrastingly, in our model R^2 encodes comparative salience, while the preference relation R^1 used in the first stage is coarsened by satisficing into the nested relation system $\mathcal{R}^1$. Furthermore, these differences will persist when the theories are adapted for applied work: Whereas the structures in [18] and [13] are well-designed for modeling *informative* advertising, which (among other goals) seeks to bring alternatives into the consideration set, our model could be used to create a role for *non-informative* advertising that seeks to affect the relative salience of options between which a satisficing agent is pseudo-indifferent.[35]

In [36], the author provides an abstract theory of two-stage choice procedures that is general enough to encompass versions of the Lleras et al. [13] consideration-filter model and the Manzini and Mariotti [14] rational-shortlist model, as well as a weaker version of our model. This theory proceeds by formulating a generalization of the Weak Congruence condition that can characterize any procedure with a certain lattice structure. The abstract analysis adds little of value to the present paper, telling us only that removing Base Transitivity from the axioms in Theorem 2.9 is equivalent to relaxing binary transitivity of $\mathcal{R}^1$. In particular, it yields nothing resembling the numerical representation in Equation 2, for which Base Transitivity is essential. However, [36] does illustrate that multi-stage models of choice can share mathematical structure that allows us to transfer techniques between seemingly distinct frameworks.

In contrast, Manzini et al. [17] focus on the numerical representation, axiomatizing the existence of functions $f_1, f_2 : X \to \mathbb{R}$ and $\theta_1 : \mathfrak{D} \to \mathbb{R}$ such that Equation 2 holds. Since they impose neither expansiveness of the threshold structure $\langle f_1, \theta_1 \rangle$ nor the requirement (part of Definition 2.10A) that each $\theta_1(xy) = \max f_1[xy]$, this is weaker than the model characterized in Theorem 2.9. In fact, Manzini et al. show that it is *much* weaker, to the extent that Equation 2 is completely vacuous for choice functions satisfying Univalence. Since our theory significantly constrains choice behavior with or without Univalence, this implies that most of the logical strength of our axioms is captured not by Equation 2 itself, but rather by the additional restrictions we impose.[36]

[35] For instance, a "*Coca-Cola!*" billboard is unlikely to bring the ubiquitous carbonated drink into the consideration sets of many consumers previously unaware of its existence. The billboard might, however, make Coke an especially salient product for those consumers who have no perceivable preference between various brands of cola which both taste and cost very much the same. Non-informative advertising is by no means rare: Resnik and Stern [22] review 378 commercials broadcast on American network television in the year 1975, and conclude that "less than half of the sample's advertisements met the liberal criteri[on] of possessing [*any*] useful informational cues." (For a survey of economic justifications for non-informative, or "persuasive" advertising, see Bagwell [4].)

[36] It is straightforward to show that the axiom used by Manzini et al. [17] is implied by the conjunction of Weak Congruence and Base Transitivity.

4.2 Models with framing effects

A second area of related research studies the impact of "frames" on decision making. In the broadest sense, a frame is any aspect of a choice problem other than the available alternatives and their payoff-relevant characteristics that may affect the DM's behavior. Examples include the order in which the options are presented and the moment in time when the choice is made.[37]

Salant and Rubinstein [27] allow for framing effects by conditioning the choice function on the new, payoff-irrelevant information. Denoting the set of frames by F, the choice set associated with menu A in $f \in F$ is written $c(A, f)$, and $C(A) = \bigcup_{f \in F} c(A, f)$ then contains the alternatives that are choosable from A in at least one frame. Assuming for simplicity that each $|c(A, f)| = 1$, the authors investigate the relationship between the "extended choice function" $c : \mathfrak{D} \times F \to \mathfrak{F}$ and the induced $C : \mathfrak{D} \to \mathfrak{F}$. In particular, they show when C will satisfy classical rationality axioms.

If we interpret our model in terms of salience effects or in terms of positive action, then the payoff-irrelevant information affecting behavior is contained in the secondary relation R^2. Writing F for the set of all complete orders on X and taking some $\mathrm{R}^2 \in F$, Equation 1 can be rephrased as $c(A, \mathrm{R}^2) = G(G(A, \mathcal{R}^1), \mathrm{R}^2)$. Moreover, since here $C(A) = \bigcup_{\mathrm{R}^2 \in F} G(G(A, \mathcal{R}^1), \mathrm{R}^2) = G(A, \mathcal{R}^1)$, the models of satisficing behavior with and without a secondary criterion are linked in Salant and Rubinstein's formulation.

Extended choice functions could also be used to give our theory additional structure. Suppose, for example, that $\rho \in F \subseteq \mathbb{R}$ measures the cognitive resources available to the DM. This endowment will affect his perceived primary preference system, now written $\mathcal{R}^1(\rho)$, yielding conditional choice sets $c(A, \rho) = G(G(A, \mathcal{R}^1(\rho)), \mathrm{R}^2)$. The rationale for nestedness (see Section 2.2) will then apply not only to changes in the menu but also to changes in the resource allocation: Given $x, y \in A \in \mathfrak{D}$ and $\rho_1, \rho_2 \in F$ such that $\rho_1 \leq \rho_2$, we should have $x\mathrm{P}^1_A(\rho_1)y$ only if $x\mathrm{P}^1_A(\rho_2)y$. This extended nestedness assumption opens the door to revealed-preference deductions across different values of ρ, and to comparative statics with respect to the resource endowment.

One example of framing offered by Salant and Rubinstein [27, p. 1289] is a satisficing procedure in the context of choice from lists.[38] Here once again F is the set of all complete orders on X, and $\mathrm{R}^2 \in F$ has the interpretation that $x\mathrm{R}^2y$ whenever x is no later than y in the list order. The DM has utility function $f_1 : X \to \mathbb{R}$ and "aspiration threshold" $\bar{\theta} \in \mathbb{R}$, which we can think of as a constant map $\theta_1 : \mathfrak{D} \to \mathbb{R}$. Given a menu A, if any available alternative achieves the utility threshold then the DM chooses the first such option (according to the list order), and otherwise he chooses the last available alter-

[37] An important branch of the literature on framing seeks to model status-quo or other reference-point effects. Among numerous other contributions in this area, Bossert and Sprumont [7] propose a theory in which the status-quo alternative is "exogenous" (i.e., objective), while Ok et al. [19] consider the "endogenous" (i.e., subjective) case.

[38] See also Rubinstein and Salant [25, p. 5]. Papi [20] studies a related but considerably more flexible model of satisficing in which alternatives need not be examined one at a time.

native regardless of its utility. Taking any representation $f_2 : X \to \mathbb{R}$ of the list order R^2, this means that the DM will solve the constrained optimization problem in Equation 2 when facing any menu A such that $\max f_1[A] \geq \bar{\theta}$. At least over the latter subdomain of choice problems, the list order here plays the same role as salience or group identity in our theory. But of course in our model the map θ_1 need not be constant, and the thresholds assigned to menus are by construction always achievable.

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A Appendix

Given a binary relation R on X, its *transitive closure* R^* is defined by $x\mathrm{R}^*y$ if and only if $\exists n \geq 2$ and $z_1, z_2, \ldots, z_n \in X$ such that $x = z_1 \mathrm{R} z_2 \mathrm{R} \cdots \mathrm{R} z_n = y$. Furthermore, R has *symmetric part* $\mathrm{R}^\bullet$ defined by $x\mathrm{R}^\bullet y$ if and only if $x\mathrm{R}y\mathrm{R}x$, and *asymmetric part* R° defined by $x\mathrm{R}^\circ y$ if and only if both $x\mathrm{R}y$ and $\neg[y\mathrm{R}x]$.

A relation is a *strict partial order* if it is irreflexive ($x\mathrm{R}y$ only if $x \neq y$) and transitive; a *linear order* if it is irreflexive, transitive, and weakly complete ($x \neq y$ only if $x\mathrm{R}y$ or $y\mathrm{R}x$); and an *equivalence* if it is reflexive ($x = y$ only if $x\mathrm{R}y$), symmetric ($x\mathrm{R}y$ only if $y\mathrm{R}x$), and transitive. An equivalence Q is a *congruence with respect to* R whenever we have $w\mathrm{Q}x\mathrm{R}y\mathrm{Q}z$ only if $w\mathrm{R}z$.

The following fact about binary relations is adapted from Richter [23, pp. 639–640].

Lemma A.1. *Any reflexive relation* R *admits a complete preorder* $\mathrm{Q} \supseteq \mathrm{R}^*$ *such that* $x\mathrm{Q}y$ *only if* $x\mathrm{R}^*y$ *or* $\neg[y\mathrm{R}^*x]$.

Proof. Since R is reflexive, $[\mathrm{R}^*]^\bullet$ is a congruence with respect to the strict partial order $[\mathrm{R}^*]^\circ$. Write $\phi(x)$ for the $[\mathrm{R}^*]^\bullet$-equivalence class containing $x \in X$, and define a strict partial order $\gg$ on $\Phi = \{\phi(x) : x \in X\}$ by $\phi(x) \gg \phi(y)$ if and only if $x[\mathrm{R}^*]^\circ y$. Szpilrajn's Theorem [34] then allows us to embed $\gg$ in a linear order $\ggg$ on Φ, and we can proceed to define a complete preorder Q by $x\mathrm{Q}y$ if and only if $\neg[\phi(y) \ggg \phi(x)]$. It follows that $x\mathrm{R}^*y$ only if either $\phi(x) \gg \phi(y)$ or $\phi(x) = \phi(y)$. But then $\phi(x) \ggg \phi(y)$ or $\phi(x) = \phi(y)$, and in either case we have $\neg[\phi(y) \ggg \phi(x)]$ and $x\mathrm{Q}y$. Hence $\mathrm{R}^* \subseteq \mathrm{Q}$. Moreover, if $x\mathrm{Q}y$ then $\neg[\phi(y) \gg \phi(x)]$ and hence $\neg[y[\mathrm{R}^*]^\circ x]$, which means that $x\mathrm{R}^*y$ or $\neg[y\mathrm{R}^*x]$. □

Lemma A.2. **A.** $\hat{\mathcal{R}}^{1\mathrm{d}}$ *is a nested system of complete relations, and* $\hat{\mathcal{R}}^{1\mathrm{i}}$ *is a nested system of complete preorders.* **B.** $C \subseteq G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}})$. **C.** *If* $C \subseteq G(\cdot, \mathcal{R}^1)$ *for some nested system of complete preorders* $\mathcal{R}^1$, *then* $\hat{\mathcal{R}}^{1\mathrm{i}} \subseteq \mathcal{R}^1$.

Proof. **A.** The nestedness of $\hat{\mathcal{R}}^{1\mathrm{d}}$ is immediate. Also, given $x, y \in A \in \mathfrak{D}$ we have both $A \supseteq xy \in \mathfrak{D}$ and $C(xy) \neq \emptyset$, and therefore $\hat{\mathrm{R}}^{1\mathrm{d}}_A$ is complete. The nestedness of $\hat{\mathcal{R}}^{1\mathrm{i}}$ and completeness of $\hat{\mathrm{R}}^{1\mathrm{i}}_A$ follow, respectively, from the nestedness of $\hat{\mathcal{R}}^{1\mathrm{d}}$ and completeness of $\hat{\mathrm{R}}^{1\mathrm{d}}_A$. Moreover, $\hat{\mathrm{R}}^{1\mathrm{i}}_A$ is transitive by construction and is thus a complete preorder.

B. Given $x \in A \in \mathfrak{D}$, let $x \notin G(A, \hat{\mathcal{R}}^{1\mathrm{i}})$. Then $\exists y \in A$ such that $\neg[x\hat{\mathrm{R}}^{1\mathrm{i}}_A y]$, so $\neg[x\hat{\mathrm{R}}^{1\mathrm{d}}_A y]$ and $x \notin C(A)$.

C. Let $C \subseteq G(\cdot, \mathcal{R}^1)$ for some nested system of complete preorders $\mathcal{R}^1$. Given $x, y \in B \in \mathfrak{D}$, the assertion $y\hat{\mathrm{R}}^{1\mathrm{d}}_B x$ means that $\exists A \in \mathfrak{D}$ such that $x \in A \subseteq B$ and $y \in C(A) \subseteq G(A, \mathcal{R}^1)$. We then have $y\mathrm{R}^1_A x$ and hence $y\mathrm{R}^1_B x$ since $\mathcal{R}^1$ is nested. Thus $\hat{\mathcal{R}}^{1\mathrm{d}} \subseteq \mathcal{R}^1$, and it follows that $\hat{\mathcal{R}}^{1\mathrm{i}} = [\hat{\mathcal{R}}^{1\mathrm{d}}]^* \subseteq [\mathcal{R}^1]^* \subseteq \mathcal{R}^1$ since $\mathcal{R}^1$ is a system of transitive relations. □

Definition A.3. For $x, y \in X$: **A.** Let $x\hat{\mathrm{R}}^{\mathrm{b}}y$ if $x \in C(xy)$. **B.** Let $x\hat{\mathrm{P}}^{\mathrm{b}}y$ if $y \notin C(xy)$.

Lemma A.4. **A.** *Base Transitivity implies that $\hat{\mathcal{R}}^{1\mathrm{i}}$ is binary transitive.* **B.** *Base Univalence implies that $\hat{\mathcal{R}}^{1\mathrm{i}}$ is binary antisymmetric.*

Proof. **A.** Given $x, y, z \in X$, if $x\hat{\mathrm{R}}^{1\mathrm{i}}_{xy}y\hat{\mathrm{R}}^{1\mathrm{i}}_{yz}z$ then $x\hat{\mathrm{R}}^{1\mathrm{d}}_{xy}y\hat{\mathrm{R}}^{1\mathrm{d}}_{yz}z$ and therefore $x\hat{\mathrm{R}}^{\mathrm{b}}y\hat{\mathrm{R}}^{\mathrm{b}}z$. But then $x\hat{\mathrm{R}}^{\mathrm{b}}z$ by Base Transitivity, in which case $x\hat{\mathrm{R}}^{1\mathrm{d}}_{xz}z$ and $x\hat{\mathrm{R}}^{1\mathrm{i}}_{xz}z$.

B. Given $x, y \in X$, if $x\hat{\mathrm{R}}^{1\mathrm{i}}_{xy}y\hat{\mathrm{R}}^{1\mathrm{i}}_{xy}x$ then $x\hat{\mathrm{R}}^{1\mathrm{d}}_{xy}y\hat{\mathrm{R}}^{1\mathrm{d}}_{xy}x$ and therefore $x\hat{\mathrm{R}}^{\mathrm{b}}y\hat{\mathrm{R}}^{\mathrm{b}}x$. But then $x, y \in C(xy)$, and so $x = y$ by Base Univalence. □

Definition A.5. Define the binary relation $\hat{\mathrm{Q}}^{2\mathrm{d}}$ as follows. For $x, y \in X$, let $x\hat{\mathrm{Q}}^{2\mathrm{d}}y$ if $\forall A \in \mathfrak{D}$ such that $x\hat{\mathrm{R}}^{1\mathrm{i}}_{A}y$ and $y \in C(A)$ we have $x \in C(A)$.

Lemma A.6. **A.** $\hat{\mathrm{R}}^{2\mathrm{d}}$ *is reflexive.* **B.** $C \subseteq G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \hat{\mathrm{R}}^{2\mathrm{i}})$. **C.** *If* $C \subseteq G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ *for some nested system of complete preorders* $\mathcal{R}^1$ *and complete preorder* R^2*, then* $\hat{\mathrm{R}}^{2\mathrm{i}} \subseteq \mathrm{R}^2$. **D.** *If* $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ *for some nested system of complete preorders* $\mathcal{R}^1$ *and complete preorder* R^2*, then* $\mathrm{R}^2 \subseteq \hat{\mathrm{Q}}^{2\mathrm{d}}$.

Proof. **A.** Given $x \in X$, we have $\{x\} \in \mathfrak{D}$ and $C(x) = x$, so $x\hat{\mathrm{R}}^{2\mathrm{d}}x$ and $\hat{\mathrm{R}}^{2\mathrm{d}}$ is reflexive.

B. Given $x \in A \in \mathfrak{D}$, let $x \notin G(G(A, \hat{\mathcal{R}}^{1\mathrm{i}}), \hat{\mathrm{R}}^{2\mathrm{i}})$. If $x \notin G(A, \hat{\mathcal{R}}^{1\mathrm{i}})$, then $x \notin C(A)$ by Lemma A.2B. If $x \in G(A, \hat{\mathcal{R}}^{1\mathrm{i}})$ then $\exists y \in G(A, \hat{\mathcal{R}}^{1\mathrm{i}})$ such that $\neg[x\hat{\mathrm{R}}^{2\mathrm{i}}y]$, and so $\neg[x\hat{\mathrm{R}}^{2\mathrm{d}}y]$. Since $y\hat{\mathrm{R}}^{1\mathrm{i}}_{A}x$, this implies once again that $x \notin C(A)$.

C. Let $C \subseteq G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ for some nested system of complete preorders $\mathcal{R}^1$ and complete preorder R^2. Given $x, y \in X$, the assertion $y\hat{\mathrm{R}}^{2\mathrm{d}}x$ means that $\exists A \in \mathfrak{D}$ such that $x\hat{\mathrm{R}}^{1\mathrm{i}}_{A}y$ and $y \in C(A) \subseteq G(G(A, \mathcal{R}^1), \mathrm{R}^2)$. We then have $x\mathrm{R}^1_A y$ by Lemma A.2C. Moreover, since $y \in G(A, \mathcal{R}^1)$ and $\mathcal{R}^1$ is a system of complete preorders, we have $x \in G(A, \mathcal{R}^1)$ and it follows that $y\mathrm{R}^2x$. Thus $\hat{\mathrm{R}}^{2\mathrm{d}} \subseteq \mathrm{R}^2$, and so we can conclude that $\hat{\mathrm{R}}^{2\mathrm{i}} = [\hat{\mathrm{R}}^{2\mathrm{d}}]^* \subseteq [\mathrm{R}^2]^* \subseteq \mathrm{R}^2$ since R^2 is transitive.

D. Let $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ for some nested system of complete preorders $\mathcal{R}^1$ and complete preorder R^2. Given $x, y \in X$, the assertion $\neg[y\hat{\mathrm{Q}}^{2\mathrm{d}}x]$ means that $\exists A \in \mathfrak{D}$ such that $y\hat{\mathrm{R}}^{1\mathrm{i}}_{A}x$ and $x \in C(A) = G(G(A, \mathcal{R}^1), \mathrm{R}^2) \not\ni y$. We then have $y\mathrm{R}^1_A x$ by Lemma A.2C. Moreover, since $x \in G(A, \mathcal{R}^1)$ and $\mathcal{R}^1$ is a system of complete preorders, it follows that $y \in G(A, \mathcal{R}^1)$ and hence $\neg[y\mathrm{R}^2x]$ since R^2 is a complete preorder. But then $\mathrm{R}^2 \subseteq \hat{\mathrm{Q}}^{2\mathrm{d}}$ by contraposition. □

Proof of Theorem 2.9. Let Weak Congruence and Base Transitivity hold and suppose for some $x \in A \in \mathfrak{D}$ that $x \notin C(A)$. Then $\exists y \in C(A)$, and we have $y \in G(G(A, \hat{\mathcal{R}}^{1\mathrm{i}}), \hat{\mathrm{R}}^{2\mathrm{i}})$ by Lemma A.6B. If $x \in G(A, \hat{\mathcal{R}}^{1\mathrm{i}})$ then $y\hat{\mathrm{R}}^{2\mathrm{i}}x$, and since $x\hat{\mathrm{R}}^{1\mathrm{i}}_{A}y$ we have also $\neg[x\hat{\mathrm{Q}}^{2\mathrm{d}}y]$. In this case $\neg[x\hat{\mathrm{R}}^{2\mathrm{i}}y]$ by Weak Congruence (which is equivalent to $\hat{\mathrm{R}}^{2\mathrm{i}} \subseteq \hat{\mathrm{Q}}^{2\mathrm{d}}$). Defining S^2 by $w\mathrm{S}^2z \iff [w\hat{\mathrm{R}}^{2\mathrm{i}}z \vee \neg[z\hat{\mathrm{R}}^{2\mathrm{i}}w]]$, it follows that $\neg[x\mathrm{S}^2y]$ and hence $x \notin G(G(A, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{S}^2)$. But then $G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{S}^2) \subseteq C$ by contraposition. Moreover, $\hat{\mathrm{R}}^{2\mathrm{d}}$ is reflexive by Lemma A.6A, and so by Lemma A.1 there exists a complete preorder $\mathrm{T}^2 \supseteq \hat{\mathrm{R}}^{2\mathrm{i}}$ with $\mathrm{T}^2 \subseteq \mathrm{S}^2$. We then have that $C \subseteq G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \hat{\mathrm{R}}^{2\mathrm{i}}) \subseteq G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{T}^2) \subseteq G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{S}^2) \subseteq C$, using Lemma A.6B, and hence $C = G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{T}^2)$. Finally, by Lemmas A.2A and A.4A we have that $\hat{\mathcal{R}}^{1\mathrm{i}}$ is a nested, binary transitive system of complete preorders.

Conversely, suppose that there exist a nested, binary transitive system of complete preorders $\mathcal{R}^1$ and a complete preorder R^2 such that $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$. We then have $\hat{\mathrm{R}}^{2\mathrm{i}} \subseteq \mathrm{R}^2 \subseteq \hat{\mathrm{Q}}^{2\mathrm{d}}$ by Lemma A.6C–D, which implies Weak Congruence (i.e., $\hat{\mathrm{R}}^{2\mathrm{i}} \subseteq \hat{\mathrm{Q}}^{2\mathrm{d}}$). Moreover, given $x, y, z \in X$, if $x\hat{\mathrm{R}}^{\mathrm{b}}y\hat{\mathrm{R}}^{\mathrm{b}}z$ then $x\mathrm{R}^1_{xy}y\mathrm{R}^1_{yz}z$ since $C \subseteq G(\cdot, \mathcal{R}^1)$, and hence $x\mathrm{R}^1_{xz}z$ since $\mathcal{R}^1$ is binary transitive. Since $C \subseteq G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ we have $x\mathrm{I}^1_{xy}y \implies x\mathrm{R}^2y$, and similarly $y\mathrm{I}^1_{yz}z \implies y\mathrm{R}^2z$. It follows that $x\mathrm{I}^1_{xz}z \implies x\mathrm{R}^2y\mathrm{R}^2z \implies x\mathrm{R}^2z$ since R^2 is transitive. But then $x\hat{\mathrm{R}}^{\mathrm{b}}z$ since $G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2) \subseteq C$, and so Base Transitivity holds. □

Proof of Theorem 2.11. Let Weak Congruence and Base Transitivity hold, so that by Theorem 2.9 there exist a nested, binary transitive system of complete preorders $\mathcal{R}^1$ and a complete preorder R^2 such that $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$. Since R^2 is a complete preorder (and X is finite) it admits a representation $f_2 : X \to \mathbb{R}$. Moreover, by Base Transitivity we have that $\hat{\mathrm{R}}^{\mathrm{b}}$ is a complete preorder and so it too admits a representation $f_1 : X \to \mathbb{R}$.

Define $\theta_1 : \mathfrak{D} \to \mathbb{R}$ by setting each $\theta_1(A) = \min f_1[G(A, \mathcal{R}^1)]$, with the implication that $G(A, \mathcal{R}^1) \subseteq \{z \in A : f_1(z) \geq \theta_1(A)\}$. Now, given any $z \in A \in \mathfrak{D}$ for which $z \notin G(A, \mathcal{R}^1)$, select any $y \in G(A, \mathcal{R}^1)$ such that $f_1(y) = \min f_1[G(A, \mathcal{R}^1)]$. We then have $y\mathrm{P}^1_A z$ since R^1_A is a complete preorder, $y\mathrm{P}^1_{yz} z$ since $\mathcal{R}^1$ is nested, and $z \notin G(G(yz, \mathcal{R}^1), \mathrm{R}^2) = C(yz)$. It follows that $y\hat{\mathrm{P}}^{\mathrm{b}} z$, and therefore $f_1(z) < f_1(y) = \theta_1(A)$ since f_1 represents $\hat{\mathrm{R}}^{\mathrm{b}}$. Hence, by contraposition, we have $\{z \in A : f_1(z) \geq \theta_1(A)\} \subseteq G(A, \mathcal{R}^1)$, and so we can conclude that $G(A, \mathcal{R}^1) = \{z \in A : f_1(z) \geq \theta_1(A)\}$. But then Equation 2 holds for each $A \in \mathfrak{D}$, as desired. To confirm that $\langle f_1, \theta_1 \rangle$ is a valid threshold structure, take any $x, y \in X$ such that $f_1(x) \geq f_1(y)$, in which case $f_1(x) \geq \theta_1(xy)$. If $f_1(x) > \theta_1(xy) = f_1(y)$, then $x\hat{\mathrm{P}}^{\mathrm{b}} y$ since f_1 represents $\hat{\mathrm{R}}^{\mathrm{b}}$, $y \notin C(xy)$, and thus $f_1(y) < \theta_1(xy)$, a contradiction. Hence $\theta_1(xy) = f_1(x) = \max f_1[xy]$, as desired. Finally, to confirm that $\langle f_1, \theta_1 \rangle$ is expansive, let $A, B \in \mathfrak{D}$ be such that $A \subseteq B$ and $\max f_1[A] \geq \theta_1(B)$. Then $\exists y \in A \subseteq B$ such that $f_1(y) \geq \theta_1(B)$, and so $y \in \{z \in B : f_1(z) \geq \theta_1(B)\} = G(B, \mathcal{R}^1)$. For any $x \in G(A, \mathcal{R}^1)$ we have $x\mathrm{R}^1_A y$ and so $x\mathrm{R}^1_B y$ since $\mathcal{R}^1$ is nested. Since $y \in G(B, \mathcal{R}^1)$ and R^1_B is a complete preorder, this implies that $x \in G(B, \mathcal{R}^1)$. But then $G(A, \mathcal{R}^1) \subseteq G(B, \mathcal{R}^1)$ and therefore $\theta_1(A) = \min f_1[G(A, \mathcal{R}^1)] \geq \min f_1[G(B, \mathcal{R}^1)] = \theta_1(B)$, as desired.

Conversely, suppose that there exist an expansive threshold structure $\langle f_1, \theta_1 \rangle$ and an $f_2 : X \to \mathbb{R}$ such that Equation 2 holds for each $A \in \mathfrak{D}$. Define a relation system $\mathcal{R}^1$ as follows: For each $A \in \mathfrak{D}$, let $x\mathrm{R}^1_A y$ if and only if $\forall B \in \mathfrak{D}$ with $B \supseteq A$ and $f_1(y) \geq \theta_1(B)$ we have $f_1(x) \geq \theta_1(B)$. By construction $\mathcal{R}^1$ is then nested and each R^1_A is complete, and clearly $\forall A \in \mathfrak{D}$ we have $G(A, \mathcal{R}^1) \subseteq \{z \in A : f_1(z) \geq \theta_1(A)\}$. Now, given $x, y, z \in A \in \mathfrak{D}$ such that $x\mathrm{R}^1_A y\mathrm{R}^1_A z$, for any $B \in \mathfrak{D}$ with $B \supseteq A$ and $f_1(z) \geq \theta_1(B)$ we have $f_1(y) \geq \theta_1(B)$ since $y\mathrm{R}^1_A z$, and in turn $f_1(x) \geq \theta_1(B)$ since $x\mathrm{R}^1_A y$. It follows that $x\mathrm{R}^1_A z$, so we have that R^1_A is transitive and $\mathcal{R}^1$ is a system of complete preorders. Furthermore, given $x \in A \in \mathfrak{D}$ such that $x \notin G(A, \mathcal{R}^1)$, there exists a $y \in A$ such that $y\mathrm{P}^1_A x$. It follows that $\exists B \in \mathfrak{D}$ such that $B \supseteq A$ and $f_1(y) \geq \theta_1(B) > f_1(x)$, and since $\langle f_1, \theta_1 \rangle$ is expansive we have that $\theta_1(A) \geq \theta_1(B) > f_1(x)$ and $x \notin \{z \in A : f_1(z) \geq \theta_1(A)\}$. But then by contraposition we have $\{z \in A : f_1(z) \geq \theta_1(A)\} \subseteq G(A, \mathcal{R}^1)$ and thus $G(A, \mathcal{R}^1) = \{z \in A : f_1(z) \geq \theta_1(A)\}$. To confirm that $\mathcal{R}^1$ is binary transitive, let $x, y, z \in X$ be such that $x\mathrm{R}^1_{xy} y\mathrm{R}^1_{yz} z$. We then have $f_1(y) \geq \theta_1(xy) \Longrightarrow f_1(x) \geq \theta_1(xy)$ and similarly $f_1(z) \geq \theta_1(yz) \Longrightarrow f_1(y) \geq \theta_1(yz)$. Since also $\theta_1(xy) = \max f_1[xy]$ and similarly $\theta_1(yz) = \max f_1[yz]$, we can conclude that $f_1(x) \geq f_1(y) \geq f_1(z)$ and hence $x\mathrm{R}^1_{xz} z$, as desired. Finally, denoting by R^2 the complete preorder represented by f_2, we have that $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ and so Weak Congruence and Base Transitivity hold by Theorem 2.9. □

Proof of Proposition 3.6. If Weak Congruence, Base Transitivity, and Base Univalence hold, then $\hat{\mathcal{R}}^{1\mathrm{i}}$ is a nested, binary transitive, binary antisymmetric system of complete preorders by Lemmas A.2A and A.4. Moreover, it can be shown (see the above proof of Theorem 2.9) that there exists a complete preorder T^2 such that $C = G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{T}^2)$.

Conversely, if there exist a nested, binary transitive, binary antisymmetric system of complete preorders $\mathcal{R}^1$ and a complete preorder R^2 such that $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$, then Weak Congruence and Base Transitivity follow from Theorem 2.9 and Base Univalence follows from $C \subseteq G(\cdot, \mathcal{R}^1)$ and the binary antisymmetry of $\mathcal{R}^1$. □

Proof of Proposition 3.7. Suppose that $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ for some nested, binary transitive, binary antisymmetric system of complete preorders $\mathcal{R}^1$ and complete preorder R^2. For $x, y \in X$ we then have $x\mathrm{L}^{12} y \Longleftrightarrow x \in C(xy) \Longrightarrow x\mathrm{R}^1_{xy} y$ by Propositions 3.2 and 3.3A. Moreover, if $x\mathrm{R}^1_{xy} y$ then since $\mathcal{R}^1$ is binary antisymmetric we have $x\mathrm{P}^1_{xy} y$ or $x = y$, and in either case it follows that $x \in C(xy)$. □

Proof of Theorem 3.9. If Weak Congruence, Base Transitivity, and Univalence hold, then $\hat{\mathcal{R}}^{1\mathrm{i}}$ is a nested, binary transitive, binary antisymmetric system of complete preorders by Lemmas A.2A and A.4. Define a relation S^2 by $w\mathrm{S}^2 z \Longleftrightarrow w\hat{\mathrm{R}}^{2\mathrm{d}} z \neq w$. If $\exists x \in X$ with $x[\mathrm{S}^2]^* x$, then clearly $\exists y \in X$ such that $x\mathrm{S}^2 y[\mathrm{S}^2]^* x \neq y$. Hence $x\hat{\mathrm{R}}^{2\mathrm{d}} y\hat{\mathrm{R}}^{2\mathrm{i}} x$, and it follows that $\exists A \in \mathfrak{D}$ with $y\hat{\mathrm{R}}^{1\mathrm{i}}_A x$ and $x \in C(A)$. But then $y \in C(A)$ by Weak Congruence and so $y = x$ by Univalence, a contradiction. This establishes that $[\mathrm{S}^2]^*$ is irreflexive and hence a strict partial order. Szpilrajn's Theorem [34] allows us to embed $[\mathrm{S}^2]^*$ in a linear order, which can be reflexivized to yield a complete order $\mathrm{T}^2 \supseteq [\mathrm{S}^2]^* \supseteq \hat{\mathrm{R}}^{2\mathrm{i}}$. Using Lemma A.6B, it follows that $C \subseteq G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \hat{\mathrm{R}}^{2\mathrm{i}}) \subseteq G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{T}^2)$. Now, take any $x \in A \in \mathfrak{D}$ with $x \notin C(A)$. Using Lemma A.2B, we know that $\exists y \in C(A) \subseteq G(A, \hat{\mathcal{R}}^{1\mathrm{i}})$ such that $y \neq x$. If $x \in G(A, \hat{\mathcal{R}}^{1\mathrm{i}})$ then $x\hat{\mathrm{R}}^{1\mathrm{i}}_A y$ and hence $y\hat{\mathrm{R}}^{2\mathrm{d}}x$. But then $y\mathrm{T}^2x$, whereupon $\neg[x\mathrm{T}^2y]$ since $x \neq y$ and T^2 is a complete order. It follows that $x \notin G(G(A, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{T}^2)$, and thus we can conclude that $G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{T}^2) \subseteq C$ by contraposition. Hence $C = G(G(\cdot, \hat{\mathcal{R}}^{1\mathrm{i}}), \mathrm{T}^2)$.

Conversely, if there exist a nested, binary transitive, binary antisymmetric system of complete preorders $\mathcal{R}^1$ and a complete order R^2 such that $C = G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$, then Weak Congruence and Base Transitivity follow from Theorem 2.9 and Univalence follows from $C \subseteq G(G(\cdot, \mathcal{R}^1), \mathrm{R}^2)$ and the antisymmetry of R^2. □

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