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Constraints on chiral operators in $\mathcal{N} = 2$ SCFTs

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ABSTRACT: We study certain higher-spin chiral operators in $\mathcal{N} = 2$ superconformal field theories (SCFTs). In Lagrangian theories, or in theories related to Lagrangian theories by generalized Argyres-Seiberg-Gaiotto duality ("type A" theories in our classification), we give a simple superconformal representation theory proof that such operators do not exist. This argument is independent of the details of the superconformal index. We then use the index to show that if a theory is not of type A but has an $\mathcal{N} = 2$-preserving deformation by a relevant operator that takes it to a theory of this type in the infrared, the ultraviolet theory cannot have these higher-spin operators either. As an application of this discussion, we give a simple prescription to extract the $2a - c$ conformal anomaly directly from the superconformal index. We also comment on how this procedure works in the holographic limit.

Contents

1 Introduction

In the space of quantum field theories (QFTs), conformal field theories (CFTs) form a special subspace of enhanced symmetry. The resulting conformal symmetry gives rise to important simplifications. For instance, conformal invariance allows us to describe a CFT by a tightly constrained set of data: its spectrum and operator product expansion (OPE) coefficients.

While it is usually difficult to solve a given CFT (by which we mean to derive its spectrum and OPE coefficients), one can use general principles to restrict the space of allowed CFTs. For example, one can study the constraints imposed by associativity of the OPE and find bounds on operator dimensions and OPE coefficients; see, e.g., [1, 2]. The restrictions on the space of superconformal field theories (SCFTs) are potentially even more powerful; see, e.g., [3, 4] and references therein.

In this note, we describe new constraints (not derived from associativity of the OPE) on the operator spectra of three very broad (and overlapping) classes of four-dimensional $\mathcal{N} = 2$ SCFTs:

(A) Theories, $\mathcal{T}$, that have Lagrangian descriptions or theories related to Lagrangian SCFTs by generalized Argyres-Seiberg-Gaiotto duality [5, 6]: in the latter case, we gauge a global symmetry G of $\mathcal{T}$ in an exactly marginal fashion (adding additional matter or additional interacting SCFTs charged under the gauge group as necessary) and find a dual description in terms of a Lagrangian.

(B) $\mathcal{N} = 2$ SCFTs with a Coulomb branch.

(C) $\mathcal{N} = 2$ SCFTs admitting deformations by relevant couplings that trigger $\mathcal{N} = 2$-preserving renormalization group (RG) flows to Lagrangian theories (or, more generally, theories of type A) in the infrared (IR).

Although it is clear that many theories of type A are also theories of class $\mathcal{S}$ (the T_N theories are a prototypical set of examples), we will not assume that this is the case in general. Furthermore, as far as we are aware, all known $\mathcal{N} = 2$ SCFTs (whether they are in class $\mathcal{S}$ or not) are of type B, with the exception of theories consisting of free hypermultiplets. Note that one class of type B theories not of type A is the set of (G, G') (generalized) Argyres-Douglas (AD) SCFTs [7–10].[1] Finally, the class of theories of type C is also very broad. For example, it includes all the (G, G') AD SCFTs. In fact, it is not immediately clear to us if there are *any* theories that are of type B but not of type A or of type C.

The main claims of our paper are:

1. Theories of type A and C do not have a certain class of higher-spin chiral (and anti-chiral) primaries.[2]

2. If these operators are present in theories of type B, then they necessarily satisfy a non-trivial set of operator relations we will describe in detail.

3. The pole structure of a particular chiral limit of the superconformal index encodes the $2a - c$ conformal anomaly in a very broad class of $\mathcal{N} = 2$ theories and, possibly, all $\mathcal{N} = 2$ SCFTs.

Our operators of interest in items 1, 2 above are defined by the shortening conditions

$$\left[\widetilde{\mathcal{Q}}_{I\dot{\alpha}}, \mathcal{O}_{\alpha_1 \cdots \alpha_{2j_1}} \right\} = 0, \quad I = 1, 2, \tag{1.1}$$

where we have a commutator or anti-commutator depending on whether the $\mathrm{SU}(2)_1$ Lorentz spin, j_1, is even or odd, and I is an $\mathrm{SU}(2)_R$ index labeling the two sets of Poincaré supercharges. An interesting subset of the operators we study satisfy an additional constraint

$$\left[\mathcal{Q}^{I\alpha}, \mathcal{O}_{\alpha \alpha_1 \cdots \alpha_{2j_1-1}} \right\} = 0, \qquad j_1 > 0, \qquad I = 1, 2,$$
$$\left\{ \mathcal{Q}^{I\alpha}, \left[\mathcal{Q}^J_\alpha, \mathcal{O} \right] \right\} = 0, \qquad j_1 = 0, \qquad I, J = 1, 2. \tag{1.2}$$

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[1]This statement is rigorously true for theories with at least one chiral operator with non-integer scaling dimension.

[2]An argument against the existence of such operators in certain theories of class $\mathcal{S}$ was given in [11].

When $j_1 = 0$, the constraint in (1.2) implies that $\mathcal{O}$ is a free scalar satisfying the Klein-Gordon equation (it is the $\mathcal{N} = 2$ primary for a multiplet that contains a free U(1) field strength). In non-abelian Lagrangian theories, with vector multiplets involving scalars Φ in the adjoint of the gauge group, the $j_1 = 0$ primaries that satisfy (1.1) but not (1.2) have the form $\mathcal{O} = \text{Tr} \, \Phi^k$ (with $k \geq 2$). Hereafter we will refer to $\mathcal{N} = 2$ primaries with $j_1 > 0$ satisfying (1.1) as "exotic chiral primaries".[3]

In some sense it is surprising that exotic chiral primaries do not exist in such broad classes of theories.[4] For example, operators satisfying (1.1) but not (1.2) with $j_1 = 0$ are ubiquitous in all known $\mathcal{N} = 2$ SCFTs (with the exception of free hypermultiplet theories): their vevs parameterize the Coulomb branch, and so it might have been natural to imagine that exotic chiral operators describe some other geometrical aspects of theories of type A and C. In order to determine if such operators exist at all, it would be interesting to understand whether there are theories of type B that are not of type A or type C (or, if there are any theories outside of our classification), and, if so, whether one can use the operator relations we derive to prove that these operators do not exist in such theories. One might also attempt to study such operators using the conformal bootstrap.

Let us briefly describe the plan of the rest of the paper. In the next section, we discuss certain short multiplets of the four-dimensional $\mathcal{N} = 2$ superconformal algebra, highlighting our multiplets of interest. We then give a general argument that theories of type A do not have exotic chiral operators. Our proof relies only on superconformal representation theory (and the dynamical assumption of a duality with a Lagrangian theory). In order to set the stage for our discussion of theories of type B and C, we then introduce the superconformal index and a particularly useful limit of the index — the "Coulomb branch" or "chiral U(1)$_R$" index — that captures only contributions from operators of the general type defined in (1.1) and (1.2). We then study a special limit of the chiral U(1)$_R$ index and use the structure of the resulting divergences to argue that if theories of type B have exotic chiral operators, then they necessarily satisfy certain operator relations. In the next section we move on to theories of type C and show that exotic chiral operators do not exist in such theories. Finally, we conclude with an application of our results to an index-based computation of the $2a - c$ conformal anomaly.

2 $\mathcal{N} = 2$ SCFT generalities and the chiral U(1)$_R$ sector

Any four-dimensional $\mathcal{N} = 2$ SCFT, $\mathcal{T}$, has a conserved and traceless stress tensor along with two conserved supercurrents and conserved currents for the SU(2)$_R \times$ U(1)$_R$ superconformal R symmetry. Such theories typically admit a zoo of short representations (note that we will refer to "semi-short" representations as short representations) with $\mathcal{N} = 2$ primaries that are annihilated by certain combinations of the Poincaré supercharges [12, 14–16]. One universal short representation that is present in any $\mathcal{T}$ is the multiplet J that contains the aforementioned stress tensor, supercurrents, and U(1)$_R \times$ SU(2)$_R$ currents. In the notation

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[3]In the classification of [12], operators satisfying (1.1) and not (1.2) are denoted as $\overline{\mathcal{E}}_{r(j_1, 0)}$. Operators satisfying both (1.1) and (1.2) are denoted by $\overline{D}_{0(j_1, 0)}$.

[4]For example, in $\mathcal{N} = 1$ theories it is common to find chiral primaries that have non-zero spin [13].

of [3, 11, 12],[5] the J multiplet is of type $\widehat{C}_{0(0,0)}$ (it has zero $U(1)_R \times SU(2)_R$ charge, is a Lorentz scalar, and has scaling dimension two). Superconformal representation theory allows various short higher spin cousins of this multiplet, $\widehat{C}_{0(j_1,j_2)}$, that contain conserved higher spin currents and are therefore not present unless $\mathcal{T}$ has a sector of free fields.[6]

If $\mathcal{T}$ has a continuous flavor symmetry, G, (i.e., a continuous symmetry that commutes with the $\mathcal{N} = 2$ superconformal algebra), then the corresponding spin one symmetry current is a descendant in a short multiplet of type $\widehat{B}_1$.[7] The primary, L_G^{IJ}, is a dimension two scalar of $SU(2)_R$ spin one and $U(1)_R$ charge zero satisfying

$$L_G^{IJ} = L_G^{JI} , \quad \left(L_G^{IJ}\right)^\dagger = \epsilon_{IK}\epsilon_{JL}L_G^{KL} , \quad \left[\mathcal{Q}_\alpha^1, L_G^{11}\right] = \left[\widetilde{\mathcal{Q}}_{2\dot\alpha}, L_G^{11}\right] = 0 , \qquad (2.1)$$

where $I, J = 1, 2$ are $SU(2)_R$ indices. Defining $L_G^A := -\frac{i}{2}\sigma_I^{AJ}(L_G)_J^I$, where the σ^A are the $SU(2)_R$ Pauli matrices, we see from (2.1) that L_G^3 is the real moment map corresponding to G. This fact will be useful for us in our discussion below.

The main focus of our paper is on the set of short multiplets of the $\mathcal{N} = 2$ superconformal algebra whose primaries are annihilated by both sets of anti-chiral Poincaré supercharges. Indeed, such operators are necessarily of the type we discussed in (1.1)

$$\left[\widetilde{\mathcal{Q}}_{I\dot\alpha}, \mathcal{O}_{\alpha_1\cdots\alpha_{2j_1};\dot\alpha_1\cdots\dot\alpha_{2j_2}}^{I_1\cdots I_{2R}}\right\} = 0 \quad \Rightarrow \quad j_2 = R = 0 , \quad E = -r \geq 1 + j_1 , \qquad (2.2)$$

where R is the $SU(2)_R$ spin, E is the scaling dimension, r is the $U(1)_R$ charge, and the implication in (2.2) follows from unitarity. If the operator does not satisfy any additional shortening conditions, it is referred to as being of type $\overline{\mathcal{E}}_{r(j_1,0)}$. Note that the inequality in (2.2) is saturated if and only if the operator also satisfies (1.2) — such operators are referred to as being of type $\overline{D}_{0(j_1,0)}$. In a slight abuse of terminology, we will refer to the operators in (2.2) as constituting the "chiral $U(1)_R$" sector of the theory.[8] We will be particularly interested in understanding the "exotic" chiral $U(1)_R$ sector of the theory — namely, those operators satisfying (2.2) with $j_1 > 0$.

The $U(1)_R$ chiral sector operators have two special properties that will be important in what follows:

(i) The $\overline{\mathcal{E}}_{r(j_1,0)}$ and $\overline{D}_{0(j_1,0)}$ multiplets cannot combine with other short representations of the $\mathcal{N} = 2$ superconformal algebra to form long representations.

(ii) The $\overline{\mathcal{E}}_{r(j_1,0)}$ and $\overline{D}_{0(j_1,0)}$ operators are singlets under any flavor symmetries of the theory.

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[5]We provide a short summary of these conventions in appendix A.

[6]All Lagrangian $\mathcal{N} = 2$ SCFTs have a special limit of measure zero on their conformal manifolds where free fields emerge, but away from this limit the various $\widehat{C}_{0(j_1,j_2)}$ multiplets combine according to the rules given in [12] to become long multiplets with non-zero anomalous dimension.

[7]Note that while Lagrangian $\mathcal{N} = 2$ SCFTs necessarily have flavor symmetries, the same is not true in general for interacting theories. For example, the original AD theory in [8] does not have any flavor symmetry.

[8]One sometimes refers to primaries that have $j_2 = 0$ as chiral primaries. Here we only refer to operators annihilated by the full set of anti-chiral supercharges as being in the chiral $U(1)_R$ sector of the theory. Note that the superconformal algebra allows other short multiplets charged under $U(1)_R$ but not under $SU(2)_R$ that have $j_2 = 0$ but are not annihilated by the anti-chiral supercharges (e.g., the $\overline{\mathcal{C}}_{0,r(j_1,0)}$ multiplets).

The first property follows from the analysis in [12]. We will prove the second property by contradiction. To that end, let us assume that the $\mathrm{U}(1)_R$ chiral sector operators are charged under some flavor symmetry. Then, since the $\overline{\mathcal{E}}_{r(j_1,0)}$ and $\overline{D}_{0(j_1,0)}$ operators are chiral with respect to an $\mathcal{N} = 1 \subset \mathcal{N} = 2$ sub-algebra, it must be the case that the chiral-anti-chiral OPE includes terms of the type

$$\mathcal{O}_{(a)\alpha_1\cdots\alpha_{2j}}(x)\,\mathcal{O}^\dagger_{(\tilde{b})\dot\alpha_1\cdots\dot\alpha_{2j}}(0) \supset I_{\alpha_1\cdots\alpha_{2j},\dot\alpha_1\cdots\dot\alpha_{2j}}\left(\frac{1}{x^{2E}}\delta_{a\tilde{b}} + \frac{1}{x^{2E-2}}T^{\widehat{A}}_{a\tilde{b}}\cdot L^3_{\widehat{A}}(0)\right),$$

$$I_{\alpha_1\cdots\alpha_{2j},\dot\alpha_1\cdots\dot\alpha_{2j}} = \frac{1}{(2j)!x^{2j}}\left(\prod_i x_{\alpha_i\dot\alpha_i} + \mathrm{perm}\right), \tag{2.3}$$

where a runs over the full set of $\mathcal{O}$ with the same $\mathcal{N} = 2$ superconformal quantum numbers, $L^3_{\widehat{A}}$ is the real moment map corresponding to a flavor symmetry generator labeled by $\widehat{A}$ (see the discussion below (2.1)), and

$$T^{\widehat{A}}_{a\tilde{b}} = \frac{1}{4\pi^2}\gamma^{\widehat{A}\widehat{B}}(t_{\widehat{B}})_a{}^c\delta_{c\tilde{b}}, \quad \langle L^A_{\widehat{A}}(x)L^B_{\widehat{B}}(0)\rangle = \frac{1}{x^4}\delta^{AB}\gamma_{\widehat{A}\widehat{B}}\,. \tag{2.4}$$

The $t_{\widehat{B}}$ matrix in (2.4) is the representation matrix that $\mathcal{O}_{(a)\alpha_1\cdots\alpha_{2j}}$ transforms under, when acted on by the symmetry generator labeled by $\widehat{B}$. It appears in the OPE

$$j_{\mu,\widehat{B}}(x)\mathcal{O}_{(a)\alpha_1\cdots\alpha_{2j}}(0) \supset -i(t_{\widehat{B}})_a{}^b\frac{x_\mu}{2\pi^2 x^4}\mathcal{O}_{(b)\alpha_1\cdots\alpha_{2j}}(0), \tag{2.5}$$

where $j_{\mu,\widehat{B}}$ is the symmetry current corresponding to the symmetry generator $\widehat{B}$ (note that (2.5) is equivalent to the statement that the charge, $Q_{\widehat{B}} \equiv \int d^3x\, j_{0,\widehat{B}}$, acts on the exotic chiral operator in the way we claim).[9] Now, since $L^3_{\widehat{A}}$ is part of an $\mathrm{SU}(2)_R$ triplet

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[9]We can further justify this statement as follows. Our discussion can be rephrased in terms of the following superconformally covariant three-point function (we refer the reader to [17] for further details on notation and conventions)

$$\langle\mathcal{O}^i_{(a)}(z_a)\mathcal{O}^{\dagger j}_{(\tilde{b})}(z_{\tilde{b}})L^3_{\widehat{C}}(z_{\widehat{C}})\rangle = \frac{I^{i\bar{i}}_{(a)}(x_{a\overline{\widehat{C}}},x_{\overline{a}\widehat{C}})I^{j\bar{j}}_{(\tilde{b})}(x_{\tilde{b}\overline{\widehat{C}}},x_{\overline{\tilde{b}}\widehat{C}})}{x^{2\overline{q}_a}_{a\overline{\widehat{C}}}x^{2q_a}_{\overline{\widehat{C}}a}x^{2\overline{q}_{\tilde{b}}}_{\tilde{b}\overline{\widehat{C}}}x^{2q_{\tilde{b}}}_{\overline{\widehat{C}}\tilde{b}}}\cdot t_{\overline{i}\overline{j}}(X_{\widehat{C}},\Theta_{\widehat{C}},\overline{\Theta}_{\widehat{C}}),$$

$$= \frac{I^{i\bar{i}}_{(a)}(x_{a\overline{\widehat{C}}})I^{j\bar{j}}_{(\tilde{b})}(x_{\overline{\tilde{b}}\widehat{C}})}{x^{2q_a}_{\overline{\widehat{C}}a}x^{2\overline{q}_{\tilde{b}}}_{\tilde{b}\overline{\widehat{C}}}}\cdot t_{\overline{i}\overline{j}}(X_{\widehat{C}},\Theta_{\widehat{C}},\overline{\Theta}_{\widehat{C}}), \tag{2.6}$$

where i and j represent the spins (i.e., $i \sim \alpha_1\cdots\alpha_{2j}$). Note that the second equality follows from the fact that $\overline{q}_a = q_{\tilde{b}} = 0$ for our chiral operators ($\overline{q}_a = 0$ is equivalent to $E = -r$ in (2.2); the equation $q_{\tilde{b}} = 0$ follows from the analogous equation for anti-chiral operators) and the fact that the $I^{i\bar{i}}_{(a)}$ may be constructed from products of the $I_{\alpha\dot\alpha}$ in (3.24) of [17] which only depend on $x_{a\overline{\widehat{C}}}$ (and analogously for $I^{j\bar{j}}_{(\tilde{b})}$).

In this language, to show that the OPE coefficients of the moment maps, $L^3_{\widehat{C}}$, in (2.3) and the OPE coefficients in (2.5) are related as in (2.4), it is sufficient to show that $t_{\overline{i}\overline{j}}$ does not depend on the Grassmann parameters $\Theta_{\widehat{C}}$ and $\overline{\Theta}_{\widehat{C}}$ (since the current, $j_{\mu,\widehat{C}}$, is a superconformal descendant of the moment map, $L^3_{\widehat{C}}$; indeed, in $\mathcal{N} = 1$ language, we have that $L^3_{\widehat{C}}$ is the primary of a superfield that has the form $L^3_{\widehat{C}} + \cdots - \theta\sigma^\mu\overline{\theta}j_{\mu,\widehat{C}} + \cdots$). To prove this latter statement, we use the fact that $\overline{D}^{\dot\alpha}_{1a}$ annihilates the l.h.s. of (2.6) to conclude, via (6.1) and (6.2) of [17], that $t_{\overline{i}\overline{j}}$ is a function only of $\overline{X}_{\widehat{C}}$ and $\overline{\Theta}_{\widehat{C}}$. Finally, using the fact that $D^1_{\tilde{b}\alpha}$ annihilates the l.h.s. of (2.6), and again applying (6.1) and (6.2) of [17], we find that $t_{\overline{i}\overline{j}}$ is independent of $\overline{\Theta}_{\widehat{C}}$ as well.

and $\mathcal{O}_{(a)\alpha_1\cdots\alpha_{2j}}$ is an SU(2)$_R$ singlet, we see that $T^{\widehat{A}}_{a\bar{b}} = 0$. As a result, from (2.4) and unitarity, $(t_{\widehat{B}})_a{}^b = 0$, and so the chiral U(1)$_R$ sector operators cannot be charged under any flavor symmetries.

3 Constraints on exotic chiral operators

In this section we study constraints on the exotic chiral operators we introduced above. In the first subsection, we apply our previous results to conclude that theories of type A do not have such operators. In the next subsection we introduce the superconformal index and a particularly useful limit of it in order to set the stage for our discussion of theories of type B and C.

3.1 Theories of type A

For Lagrangian SCFTs it is straightforward to prove that there are no exotic chiral operators since such theories have exactly marginal gauge couplings that can be tuned to zero. By property (i) of the previous section, we see that superconformal representation theory prevents the $\overline{\mathcal{E}}_{r(j_1,0)}$ and $\overline{D}_{0(j_1,0)}$ operators from pairing up to form long representations, and so it suffices to study the free theory. At zero coupling, we immediately see that the only possible chiral operators have the form $\mathcal{O} = \text{Tr}\,\Phi^k$, where Φ is a chiral adjoint, and so $j_1 = 0$.

Let us now consider a non-Lagrangian theory, $\mathcal{T}$, that we can deform to a Lagrangian one. We assume that such a deformation does not induce an RG flow, and so it must proceed by an exactly marginal gauging of some global symmetry group of $\mathcal{T}$, $G_0 \subset G_{\mathcal{T}}$ (see, for example, the appendix of [18]). In order to ensure that the gauging is exactly marginal, we may have to add another SCFT, $\mathcal{T}'$, with $G_0 \simeq G_0' \subset G_{\mathcal{T}'}$ ($\mathcal{T}'$ may be a collection of decoupled SCFTs; note that $\mathcal{T}'$ may just be a set of free hypermultiplets). In the spirit of [5, 6] we may then be able to find a duality

$$\mathcal{L} \quad \leftrightarrow \quad \mathcal{T} - G - \mathcal{T}', \quad G = \text{diag}\left(G_0 \times G_0'\right), \tag{3.1}$$

where $\mathcal{L}$ is a Lagrangian theory, and we have identified the gauge group G with the diagonal subgroup of $G_0 \times G_0'$. In general, the remaining flavor symmetry, $G_{\mathcal{L}}$, is just the maximal commuting subgroup with G in $G_{\mathcal{T}} \times G_{\mathcal{T}'}$

$$G_{\mathcal{L}} = \left\{ T \in G_{\mathcal{T}} \times G_{\mathcal{T}'} \,\Big|\, \left[T, \widehat{T}\right] = 0 \ \forall \ \widehat{T} \in G \right\}. \tag{3.2}$$

In writing (3.2), we have used the fact that flavor symmetries are non-anomalous in $\mathcal{N} = 2$ theories since the moment maps satisfy (see, e.g. [18, 19])

$$L_{\widehat{A}}^A(x) L_{\widehat{B}}^B(0) \supset \frac{\kappa_{\widehat{A}\widehat{B}}{}^{\widehat{C}}}{x^2} \epsilon^{AB}{}_C L_{\widehat{C}}^C(0), \tag{3.3}$$

where κ is the (putative) flavor anomaly and ϵ the Levi-Civita tensor. In particular, we see that if $A = B = C = 3$, then the r.h.s. of (3.3) vanishes and the symmetries must be non-anomalous (by the $\mathcal{N} = 1$ arguments in [18, 19]).

Let us now study the chiral $U(1)_R$ sector of the theories appearing in (3.1). In particular, let us take the limit of zero gauge coupling for the G gauge group, $\tau_G \to i\infty$. In this limit, it is clear that the only effect of the gauging on $\mathcal{T}$ and $\mathcal{T}'$ is to eliminate non gauge-invariant operators. However, by property (ii) in the previous section, we see that the $\overline{\mathcal{E}}_{r(j_1,0)}$ and $\overline{D}_{0(j_1,0)}$ operators are not charged under G. Therefore, such operators persist on the right hand side of (3.1), and we must have

$$\left\{\overline{\mathcal{E}}^{(\mathcal{L})}_{r(0,0)}, \overline{D}^{(\mathcal{L})}_{0(0,0)}\right\} = \left\{\overline{\mathcal{E}}^{(G)}_{r(0,0)}\right\} \cup \left\{\overline{\mathcal{E}}^{(\mathcal{T})}_{r(j_1,0)}, \overline{D}^{(\mathcal{T})}_{0(j_1,0)}\right\} \cup \left\{\overline{\mathcal{E}}^{(\mathcal{T}')}_{r(j_1,0)}, \overline{D}^{(\mathcal{T}')}_{0(j_1,0)}\right\} \quad \Rightarrow \quad j_1 = j_1' = 0\,,$$
(3.4)

where the implication that there are no exotic chiral primaries in $\mathcal{T}$ or $\mathcal{T}'$ follows from the fact that there are no exotic chiral primaries in the Lagrangian sectors $\mathcal{L}$ and G.

As a simple example, we can consider the original duality of [5]. In that case, $\mathcal{L}$ is the $SU(3)$ theory with six flavors, $\mathcal{T}$ is the E_6 theory of [20], $G = SU(2)$, and $\mathcal{T}'$ is a collection of two hypermultiplets (so $G_{\mathcal{T}} = E_6$, $G_{\mathcal{T}'} = Sp(2)$, and $G_{\mathcal{L}} = U(1) \times SU(6)$). By our above reasoning, we see that the chiral $U(1)_R$ sector of the pure E_6 theory (without gauging an $SU(2) \subset E_6$) is fully described by a non-exotic dimension three operator, $\mathcal{O}$, dual to the $\operatorname{Tr} \Phi^3$ operator in $\mathcal{L}$.

Let us again emphasize that while many constructions of the type (3.1) are in class $\mathcal{S}$, we do not need to assume that this is the case more generally for our above logic to hold. Finally, let us mention that there are large classes of theories in which (3.1) does not hold because (among other possibilities) there are no appropriate symmetries to gauge (such theories are of type B or C but not A). One example is the original AD theory considered in [8] (and many of the generalizations in [10]). To constrain the existence of chiral exotics in such theories, we will need some more powerful tools which we introduce in the next subsection.

3.2 The $\mathcal{N} = 2$ superconformal index

The superconformal index [21, 22] counts short representations of the superconformal algebra modulo combinations of short representations that can pair up to form long representations (the index is therefore invariant under exactly marginal deformations). The counting is taken with respect to some supercharge, $\mathcal{Q}$, and is weighted by fugacities for the symmetries that commute with $\mathcal{Q}$. In what follows, we choose the supercharge $\widetilde{\mathcal{Q}}_{2\dot{-}}$ (other choices lead to equivalent constructions of the index) and define the index to be[10]

$$\mathcal{I}(p,q,t) = \operatorname{Tr}_{\mathcal{H}} (-1)^F e^{-\beta\widetilde{\delta}_{2\dot{-}}} p^{j_2+j_1} q^{j_2-j_1} t^R \left(\frac{pq}{t}\right)^{-r}\,,$$
$$\widetilde{\delta}_{2\dot{-}} := 2\left\{\widetilde{\mathcal{Q}}_{2\dot{-}}, \widetilde{\mathcal{S}}^{2\dot{-}}\right\} = E - 2j_2 - 2R + r\,,$$
(3.5)

where p, q, t are complex fugacities, $j_{1,2}$ are Cartans for the $SU(2)_{1,2}$ isometry, r is the superconformal $U(1)_R$ charge, and R is the Cartan of $SU(2)_R$. In (3.5), the trace is understood as being over the full Hilbert space of the theory. However, the only contributions come

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[10]We will use the same letter for both the Cartan generators and the associated charges, hoping that this will not cause confusion. We review our conventions for the superconformal algebra in appendix A.

from operators in $\mathbf{R}^4$ (or, equivalently, the corresponding states on S^3) that are annihilated by $\widetilde{\mathcal{Q}}_{2\dot{-}}$ and $\widetilde{\mathcal{S}}^{2\dot{-}}$; such operators (or states) necessarily have $E = 2j_2 + 2R - r$.[11]

In a Lagrangian theory, the gauge and matter sectors contribute to the index as follows

$$
\begin{aligned}
\mathcal{I}_V^{\text{s.l.}}(p,q,t) &= -\frac{p}{1-p} - \frac{q}{1-q} + \frac{\frac{pq}{t} - t}{(1-p)(1-q)}\,, \\
\mathcal{I}_{\frac{H}{2}}^{\text{s.l.}}(p,q,t) &= \frac{\sqrt{t} - \frac{pq}{\sqrt{t}}}{(1-p)(1-q)}\,,
\end{aligned}
\tag{3.6}
$$

where the subscripts "V" and "$\frac{H}{2}$" refer to contributions from an $\mathcal{N}=2$ vector multiplet and half hypermultiplet respectively; the superscript "s.l." stands for "single letter" and indicates that we should appropriately exponentiate the corresponding contribution to the index in order to compute the contributions from all short multiplets built out of products of V, H, and their derivatives. For simplicity, we will specialize our Lagrangian discussion to a U(1)^N gauge theory with free (uncharged) hypermultiplets (the non-abelian and charged-matter generalizations are straightforward) and we will drop all flavor fugacities. In this case, we find

$$
\begin{aligned}
\mathcal{I} &= \prod_{a=1}^{N} \text{P.E.}(I_{V_a}^{\text{s.l.}}) \prod_i \text{P.E.} \left(I_{\frac{H_i}{2}}^{\text{s.l.}} \right) \\
&:= \prod_{a=1}^{N} \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} I_{V_a}^{\text{s.l.}}(p^n, q^n, t^n) \right) \prod_i \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} I_{\frac{H_i}{2}}^{\text{s.l.}}(p^n, q^n, t^n) \right)\,,
\end{aligned}
\tag{3.7}
$$

where the product is over all the abelian U(1)'s and matter fields, and "P.E." stands for the "plethystic exponential."

Via the state-operator map, one can also understand the index as being equivalent — modulo regularization scheme-dependent pre-factors — to a partition function for the theory on $S^1 \times S^3$ with twisted boundary conditions for the various fields according to their charges [21–24].[12] By assigning a chemical potential to each fugacity, $x = e^{-\beta v_x}$, we can write (3.5) as

$$
\begin{aligned}
\mathcal{I} &= \text{Tr}_{\mathcal{H}}(-1)^F \exp\big[-\beta(E - 2j_2 - 2R + r \\
&\qquad\qquad\qquad + v_p(j_1 + j_2 - r) + v_q(j_2 - j_1 - r) + v_t(R + r)\big] \\
&=: \text{Tr}_{\mathcal{H}}(-1)^F \exp\left[-\beta(E + \alpha)\right]\,,
\end{aligned}
\tag{3.8}
$$

where note that $[E + \alpha, \widetilde{\mathcal{Q}}_{2\dot{-}}] = 0$. The twisted boundary conditions can be replaced with periodic ones as $x_4 \sim x_4 + \beta$, by additionally shifting the derivative along the circle [25]

$$
\partial_4 \to \partial_4 - \alpha =: D_4\,.
\tag{3.9}
$$

---

[11]Note that we can often add various flavor fugacities to (3.5), but we will not do so in what follows since our primary interest is in operators that we have seen are flavor neutral.

[12]In line with [23], the version of the theory on $S^1 \times S^3$ has additional mass terms, coupling the scalars to the background curvature.

The corresponding Euclidean path integral, schematically $\int [\mathcal{D}\Phi] e^{-S_E[\Phi, D_4\Phi]}$, then counts the same $\frac{1}{8}$-BPS states as the index. This will provide us with an interesting alternative perspective below. Note that the index is by construction independent of continuous couplings. As a result, the evaluation of the twisted partition function for the $\mathcal{N} = 1$ chiral multiplets inside the $\mathcal{N} = 2$ vector or hypermultiplets can be carried out in the free limit, where it reduces to one-loop determinant factors for the scalars and fermions [22, 24, 26].[13]

It is interesting to point out that there is yet another, more geometric, interpretation of the index as corresponding (modulo pre-factors) to the partition function of the theory placed on a Hopf surface, M_4. The complex structure and holomorphic bundle moduli of M_4 are directly related to the fugacities of the index [27] (see also [28]).

3.3 The chiral $\widehat{U}(1)_R$ limit

Since we will be interested in analyzing theories of type B — i.e., theories with a Coulomb branch, $\mathcal{M}_{\mathcal{C}}$, in $\mathbf{R}^4$ —it is useful for us to ask how we can see such a moduli space from the index (3.7) in the simple case above (see also the general discussion of moduli spaces and the index in [29]). In this example, $\mathcal{M}_{\mathcal{C}}$ is parameterized by the $U(1)_R$-charged vector multiplet scalars, Φ_a. Therefore, away from the origin of $\mathcal{M}_{\mathcal{C}}$, the $U(1)_R$ symmetry is broken. As a result, if we wish to study the effects of such vacua, we should take a limit of fugacities $pq = t$ in (3.5) so that the $U(1)_R$ charge does not enter the index. We define

$$\mathcal{I}_{\widehat{U}(1)_R} := \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta \widetilde{\delta}_{2\dot{-}}} p^{j_2 + j_1 + R} q^{j_2 - j_1 + R}, \quad \frac{pq}{t} = 1 . \tag{3.10}$$

Using this index to evaluate the single letter contributions for vector multiplets and half-hypermultiplets, we find that (3.6) becomes

$$\mathcal{I}^{\mathrm{s.l.}}_{\widehat{U}(1)_R \; V}(p,q,t) = 1 ,$$
$$\mathcal{I}^{\mathrm{s.l.}}_{\widehat{U}(1)_R \; \frac{H}{2}}(p,q,t) = 0 . \tag{3.11}$$

The first contribution can be understood as coming from a Φ_a scalar with no derivatives acting on it. Note that the hypermultiplet contributions vanish, and our result depends only on operators in the chiral $U(1)_R$ sector; hence the subscript "$\widehat{U}(1)_R$" in (3.11). Plugging these results into (3.7), we find unsupressed contributions from all powers of Φ_a

$$\mathcal{I}_{\widehat{U}(1)_R} = \prod_a \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\right) = \infty . \tag{3.12}$$

This divergence describes the opening up of a Coulomb branch.

In fact, this discussion is more general. Indeed, we find the following single letter contributions to the index for any $\mathcal{N} = 2$ SCFT in the above limit:

$$\mathcal{I}^{\mathrm{s.l.}}_{\widehat{U}(1)_R \; \mathcal{C}_{R,r(j_1,j_2)}} = 0 ,$$
$$\mathcal{I}^{\mathrm{s.l.}}_{\widehat{U}(1)_R \; \widehat{\mathcal{C}}_{R(j_1,j_2)}} = 0 ,$$

---

[13]For the $\mathcal{N} = 1$ vector multiplet contributions one needs to take into account gauge field zero-modes that have to be dealt with exactly.

$$\mathcal{I}^{\text{s.l.}}_{\widehat{U}(1)_R \ \overline{\mathcal{E}}_{r(j_1,0)}} = (-1)^{2j_1} \chi_{j_1}\left(\sqrt{\frac{p}{q}}\right),$$

$$\mathcal{I}^{\text{s.l.}}_{\widehat{U}(1)_R \ \overline{\mathcal{D}}_{0(j_1,0)}} = (-1)^{2j_1} \frac{(1+pq)\chi_{j_1}\left(\sqrt{\frac{p}{q}}\right) - \sqrt{pq}\left(\chi_{j_1+\frac{1}{2}}\left(\sqrt{\frac{p}{q}}\right) + \chi_{j_1-\frac{1}{2}}\left(\sqrt{\frac{p}{q}}\right)\right)}{(1-p)(1-q)},$$

$$\mathcal{I}^{\text{s.l.}}_{\widehat{U}(1)_R \ \mathcal{D}_{0(0,j_2)}} = 0, \tag{3.13}$$

where $\chi_{j_1}(x) = \frac{x^{2j_1+1} - x^{-(2j_1+1)}}{x - x^{-1}}$ is a character for the spin j_1 representation of $SU(2)_1$. The subscripts denote contributions from the various short representations of the $\mathcal{N} = 2$ superconformal algebra. We review the various properties of these different multiplets in appendix A. However, suffice it to say that the only contributions to the index in (3.13) come from the chiral $U(1)_R$-sector operators $\overline{\mathcal{E}}_{r(j_1,0)}$ and $\overline{D}_{0(j_1,0)}$ we introduced above.

The divergence of the index in the above limit, as well as the absence of hypermultiplet contributions, have simple physical interpretations in terms of the path integral description. Note that this construction is useful for theories with Lagrangian limits. In order to proceed, let us first remind the reader that in the $M_4 \simeq S^1 \times S^3$ path integral language, for generic values of chemical potentials, the scalars Φ_a typically have a mass (since they couple to curvature). Due to this mass term, the theory on curved space has a unique vacuum, as opposed to the moduli space of vacua enjoyed by the theory in flat space. However, for the limit described in (3.10), we will find bosonic zero modes in the Φ_a.

The Lagrangian at quadratic order in the fields includes the following terms for the vector multiplet scalars

$$\mathcal{L}_{S^1 \times S^3} \supset \text{Tr}\,\overline{\Phi}\Big(-\Delta_{S^3} - D_4^2 + 1\Big)\Phi, \tag{3.14}$$

where Δ_{S^3} is the Laplacian on the three-sphere and the factor of 1 comes from the conformal coupling to the curvature of the S^3, in units of its radius [23]. One can read off the scalar mass from (3.14):

$$M_\phi^2 = (1+\alpha)(1-\alpha). \tag{3.15}$$

Further focusing on the constant mode (s-wave) for the vector multiplet scalars on $S^1 \times S^3$, such that $j_1 = j_2 = 0$, $r = -1$ and $R = 0$, produces

$$M_{\phi,s}^2 = (2 - v_p - v_q + v_t)(v_p + v_q - v_t). \tag{3.16}$$

Clearly, in the limit where $pq = t$, or equivalently $v_p + v_q - v_t = 0$ with β fixed, this mass goes to zero and the Coulomb branch opens up. Hence, the divergence in the $\mathcal{I}_{\widehat{U}(1)_R}$ index for $pq = t$, (3.12), can be understood as the result of integrating over this zero mode in field space. This insight can be applied more generally. Appropriately tuning the various fugacities can lead to opening up different flat directions; see [29] for a baryonic Higgs-branch example.

We can similarly interpret the absence of hypermultiplet contributions. In the $pq = t$ limit, the bosonic and fermionic contributions to the single-letter hypermultiplet index (3.11) canceled out. In the path integral description, this is suggestive of an additional

supersymmetric cancelation between chiral scalars and fermions. It can indeed be seen that

$$\lim_{pq \to t} [E + \alpha, \widetilde{\mathcal{Q}}_{1\dot{+}}] = 0 \,, \tag{3.17}$$

that is, in the limit of interest the partition function counts $\frac{1}{4}$-BPS states annihilated by both $\widetilde{\mathcal{Q}}_{2\dot{-}}$ and $\widetilde{\mathcal{Q}}_{1\dot{+}}$, thus excluding the hypermultiplets.

3.4   The chiral $U(1)_R$ limit

We have seen that the limit of the index (3.10) captures contributions only from Coulomb-branch operators. However, before proceeding to analyze the chiral $U(1)_R$ sector operators of interest, we would like to find a better-behaved (finite) limit of the index where most of the theory decouples and only chiral $U(1)_R$ sector operators contribute. It is crucial that the exotic chiral operators contribute as well, since we will use their putative contributions to argue that they must be absent in theories of type C and constrained in theories of type B. This limit will also immediately shed light on the order of the divergence in (3.7) and the nature of the divergent contribution in (3.11).

To that end, we consider the limit described in [11][14]

$$|p| \,, |q| \,, |t| \ll |z| \,, |w| \,, \quad |w| = 1, |z| \leq 1 \,, \quad w = \sqrt{\frac{p}{q}} \,, \quad z = \frac{pq}{t} \,, \tag{3.19}$$

where we neglect corrections of order $\mathcal{O}(|p|, |q|, |t|)$ to the various quantities we study. Here, z and w correspond to $U(1)_R$ and j_1 fugacities respectively. In this limit, we have

$$\mathcal{I}_{U(1)_R} := \operatorname{Tr}_{\mathcal{H}} (-1)^F e^{-\beta \widetilde{\delta}_{2\dot{-}}} z^{-(R+r)} w^{2j_1} \,, \tag{3.20}$$

with contributions only from states satisfying $E = -r$ and $j_2 = -R$. As we will see below, there are no $SU(2)_R$-charged states contributing and so we can take z to be a $U(1)_R$ fugacity.

We will refer to this limit as the "chiral $U(1)_R$" limit of the index. For a general SCFT we have

$$\begin{aligned}
\mathcal{I}_{U(1)_R}^{\text{s.l.}} \, \mathcal{C}_{R,r(j_1,j_2)} &= 0 \,, \\
\mathcal{I}_{U(1)_R}^{\text{s.l.}} \, \widehat{\mathcal{C}}_{R(j_1,j_2)} &= 0 \,, \\
\mathcal{I}_{U(1)_R}^{\text{s.l.}} \, \overline{\mathcal{E}}_{r(j_1,0)} &= (-1)^{2j_1} z^{-r} \chi_{j_1}(w) \,, \\
\mathcal{I}_{U(1)_R}^{\text{s.l.}} \overline{\mathcal{D}}_{0(j_1,0)} &= (-1)^{2j_1} z^{j_1+1} \chi_{j_1}(w) \,, \\
\mathcal{I}_{U(1)_R}^{\text{s.l.}} \, \mathcal{D}_{0(0,j_2)} &= 0 \,.
\end{aligned} \tag{3.21}$$

In this case, (3.11) becomes

$$\mathcal{I}_{U(1)_R \, V}^{\text{s.l.}}(p,q,t) = z \,, \quad \mathcal{I}_{U(1)_R \, \frac{H}{2}}^{\text{s.l.}}(p,q,t) = 0 \,. \tag{3.22}$$

---

[14]There this limit was written as

$$p, q, t \to 0 \,, \quad \text{with} \, \left\{ z = \frac{pq}{t}, w = \sqrt{\frac{p}{q}} \right\} \to \text{ fixed} \,. \tag{3.18}$$

As a result, the divergence in (3.7) is regulated and

$$\mathcal{I}_{\mathrm{U}(1)_R} = \prod_a \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} z^n\right) = \frac{1}{(1-z)^N} \ . \tag{3.23}$$

In particular, taking the additional limit $z \to 1$, once again opens up the Coulomb branch. We see that the divergence in this limit corresponds to a pole of order N (the complex dimension of the Coulomb branch, $\dim \mathcal{M}_{\mathcal{C}}$) due to contributions from the N different Φ_a scalars.

For a general Lagrangian SCFT, we have

$$\mathcal{I}_{\mathrm{U}(1)_R} = \prod_a \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} z^{E_a n}\right) = \prod_a \frac{1}{1 - z^{E_a}} \ , \tag{3.24}$$

where the E_a are the scaling dimensions of the Casimirs, $\mathcal{O}_a = \mathrm{Tr}\, \Phi^k$, and $a = 1, \ldots, N$ parameterizes the directions along the Coulomb branch. In deriving (3.24), we have used the relation $E_a = -r_a$ (see (2.2)), the fact that the Casimirs are good coordinates on $\mathcal{M}_{\mathcal{C}}$, and, more generally, the fact that there are no operator relations among the Casimirs. We see again that the order of the pole at $z = 1$ in (3.24) corresponds to the dimension of the Coulomb branch.

In a more general $\mathcal{N} = 2$ theory, the direct analogs of the $\mathcal{O}_a$ operators (i.e., the non-nilpotent (scalar) generators of the $\mathrm{U}(1)_R$ chiral ring) might obey various constraints (although we are not aware of any examples in the currently published literature; note that we are also not aware of any examples of nilpotent $\mathrm{U}(1)_R$ chiral ring generators). Such constraints are implemented by setting independent holomorphic functions, $f_i(\mathcal{O}_a)$, with a well-defined $\mathrm{U}(1)_R$ charge to zero

$$f_i(\mathcal{O}_a) = 0 \,, \quad r(f_i) = -E_i < 0 \ . \tag{3.25}$$

When we write the right hand side of the first equation in (3.25), we work modulo descendants. We can include such constraints via a meromorphic factor, $\widetilde{I}(z)$

$$\mathcal{I}_{\mathrm{U}(1)_R} = \prod_a \frac{1}{1 - z^{E_a}} \cdot \widetilde{\mathcal{I}}(z) \ . \tag{3.26}$$

If there are no additional relations among the $f_i(\mathcal{O}_a)$, then one can capture the full effect of the constraints via a "wrong-sign" contribution of the form

$$\widetilde{\mathcal{I}}(z) = \prod_i \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} z^{E_i}\right) = \prod_i (1 - z^{E_i}) \ . \tag{3.27}$$

In this case, each constraint contributes a zero to the index in the limit $z \to 1$, and we interpret it as "lifting" a direction of the moduli space. If the number of $\mathcal{O}_a$, N_a, is larger

than the number of f_i, N_i, then we find a pole of order $N_a - N_i$ as $z \to 1$, and we interpret the order of this pole as the complex dimension of the U(1)$_R$ moduli space.[15]

If our theory has exotic chiral operators, we get contributions to the index of the form

$$\mathcal{I}_{\mathrm{U(1)}_R, \, j_1} = \prod_{a=1}^{N_{j_1}} \prod_{j=-j_1}^{j_1} \frac{1}{1 - z^{E_a} w^{2j}}, \quad j_1 \in \mathbf{Z},$$

$$\mathcal{I}_{\mathrm{U(1)}_R, \, j_1'} = \prod_{a=1}^{N_{j_1'}} \prod_{j=-j_1'}^{j_1'} (1 - z^{E_a} w^{2j}), \quad j_1' \in \mathbf{Z} + \frac{1}{2}, \tag{3.28}$$

where the first line in (3.28) comes from N_{j_1} spin j_1 chiral bosons, and the second line in (3.28) comes from $N_{j_1'}$ spin j_1' exotic chiral fermions. We can implement higher-spin constraints via a meromorphic function, $\widetilde{\mathcal{I}}(z, w)$, and find the general result

$$\mathcal{I}_{\mathrm{U(1)}_R} = \prod_I \mathcal{I}_{\mathrm{U(1)}_R, j_I} \cdot \widetilde{\mathcal{I}}(z, w) \,. \tag{3.29}$$

Before proceeding, let us note that $\widetilde{\mathcal{I}}(z, w)$ must be such that we have

$$\mathcal{I}_{\mathrm{U(1)}_R} = \sum_B z^{E_B} \chi_{j_B}(w) - \sum_F z^{E_F} \chi_{j_F}(w), \tag{3.30}$$

where B and F run over the full set of bosonic and fermionic chiral operators that contribute to the index taking into account all chiral operator relations. Crucially for us below, it follows that in any $\mathcal{N} = 2$ SCFT

$$\partial_w \left(\lim_{z \to 1} \mathcal{I}_{\mathrm{U(1)}_R} \right) \neq 0 \quad \Leftrightarrow \quad \exists \, \mathcal{O}_{\alpha_1 \cdots \alpha_{2j_1}} \text{ s.t. } j_1 > 0 \,. \tag{3.31}$$

In other words, the $z \to 1$ limit of the chiral U(1)$_R$ index is a non-trivial function of w if and only if we have exotic chiral operators.

Let us next switch to the path integral interpretation. The limit (3.19) involved taking:

$$|p|, |q|, |t| \ll |z|, |w|, \quad |w| = 1, |z| \leq 1, \quad w = \sqrt{\frac{p}{q}}, \quad z = \frac{pq}{t} \,. \tag{3.32}$$

We found that only vector multiplets contribute (finitely), while the hypermultiplets gave zero for all fields without the need for mutual cancelations. In terms of chemical potentials, and specializing to real and positive p, q, t, (3.32) can be recast into

$$v_z = v_p + v_q - v_t \ll 1, \quad v_w = \frac{1}{2} v_p - v_q \ll 1,$$

$$\text{with} \quad v_p \sim \mathcal{O}(1) \quad \text{and} \quad \beta \gg 1 \,. \tag{3.33}$$

---

[15]As an aside, we observe that for type A theories, the limit of the index we are studying is equivalent to the Hilbert series of the Coulomb branch. Various examples of Coulomb branch Hilbert series have been studied in [30]. Note that this relation between the index and the Hilbert series seems to be more robust than the relation between the Hall-Littlewood index and the Hilbert series of the Higgs branch (see e.g., [31]), which only holds for special genus zero classes of type A theories [11].

In order to proceed, consider

$$
\begin{aligned}
\alpha &= \Big(-2j_2 - 2R + r + v_p(j_1 + j_2 - r) + v_q(j_2 - j_1 - r) + v_t(R + r) \Big) \\
&= \Big(-2j_2 - 2R + r + v_z(j_2 - r) + 2v_w j_1 + v_t(R + j_2) \Big)
\end{aligned}
\tag{3.34}
$$

and focus once again on the masses for the constant modes. Using (3.34) one has

$$
\begin{aligned}
M_s^2 &= (1 - \alpha_s^2) \\
&= (1 - 2R + r - v_z r + v_t R)(1 + 2R - r + v_z r - v_t R) \,.
\end{aligned}
\tag{3.35}
$$

Then for the vector multiplet and hypermultiplet scalars, (Φ, Q), for which $r = -1$, $R = 0$ and $r = 0$, $R = \frac{1}{2}$ respectively, we obtain

$$
\begin{aligned}
M_{\Phi,s}^2 &= v_z(2 - v_z) \sim 2v_z \ll 1 \\
M_{Q,s}^2 &= v_t \left(1 - \frac{v_t}{4} \right) \sim \mathcal{O}(1) \,.
\end{aligned}
\tag{3.36}
$$

Note that the vector multiplet scalars are parametrically lighter than the hypermultiplets. When we set $v_z = 0$ to get $z = 1$, we see that $M_{\Phi,s}^2 = 0$, and the Coulomb branch opens up.

For a theory in flat space, one can integrate down to an arbitrary momentum shell and obtain a two-derivative effective Lagrangian. However, for a theory on curved space one can never integrate out states at scales lower than the inverse radius of curvature and remain with an effective two-derivative theory. The reason is that there is no way to separate higher derivative terms in the effective Lagrangian from terms suppressed by the inverse radius of curvature [23]. This picture self-consistently reverts to the flat space intuition when the volume — and hence the radius of curvature — goes to infinity.

We will hence compare the masses (3.36) against the curvature, which for $S^1 \times S^3$ — and in units of the S^3 radius — simply reads $\mathcal{R} \propto 1$. For some fixed $0 < v_t < 4$ this leads to a hierarchy of scales:[16]

$$
M_{\Phi,s}^2 \ll \mathcal{R} \sim M_{Q,s}^2 \,.
\tag{3.37}
$$

As a result, in this second limit of chemical potentials, the hypermultiplets can be integrated out and the index will once again only receive contributions from the vector multiplets. Since the index only depends on the parameter β (in units of the S^3 radius) and chemical potentials, and in particular is independent of all other continuous parameters including the RG scale [23], we can conclude that there will be no contributions from the hypermultiplets at any energy.

It may be useful to point out that a rescaling of both the S^1 and S^3 radii by κ, which does not affect the index since their ratio is kept fixed, will not disrupt the scale hierarchy. Although the curvature changes as $\mathcal{R} \to \kappa^{-2}\mathcal{R}$, the masses (3.36) are defined in units of the S^3 radius and will also scale by $M^2 \to \kappa^{-2}M^2$ leaving (3.37) unchanged.

---

[16]We have chosen the value of v_t so that it leads to a sensible field theory with real hypermultiplet masses. Having said that, the index is well defined for all $v_t > 0$.

3.5 Theories of type B

We now have the necessary tools to analyze theories of type B. Let us begin by discussing $SU(N)$ $\mathcal{N} = 2$ superconformal quantum chromodynamics (SCQCD) in more detail. Although we can immediately conclude from the preceding discussion that this theory does not have exotic chiral operators, a careful analysis of $SU(N)$ SCQCD and its Coulomb branch furnishes some important intuition about the behavior of the index in more general theories. From (3.24) we have

$$\mathcal{I}_{\mathrm{U}(1)_R} = \prod_{a=2}^{N} \frac{1}{1 - z^a} \ . \tag{3.38}$$

In the limit $z \to 1$, we find bosonic zero modes corresponding to the opening up of a Coulomb branch, and

$$\lim_{z \to 1} \mathcal{I}_{\mathrm{U}(1)_R} = |W_{\mathrm{SU}(N)}|^{-1} (1 - z)^{-\mathrm{rank}(\mathrm{SU}(N))} + \mathcal{O}\left((1 - z)^{-\mathrm{rank}(\mathrm{SU}(N))+1}\right) , \tag{3.39}$$

where $|W_{\mathrm{SU}(N)}| = N!$ is the order of the Weyl group of $SU(N)$, and $\mathrm{rank}(SU(N)) = N - 1$ is the rank of the gauge group.

To understand this result, recall that the Coulomb branch of the $SU(N)$ theory is $\mathcal{M}_{\mathcal{C}} = \mathbf{C}^{N-1}/W_{\mathrm{SU}(N)}$. Furthermore, the $z \to 1$ limit of the $SU(N)$ partition function on $M_4 \simeq S^3 \times S^1$ is dominated by field configurations with large vevs for the bosonic zero modes[17] — precisely the regime where the theory should behave like a $\mathrm{U}(1)^{N-1}$ theory on $\mathcal{M}_C$. We divide by the order of the Weyl group since we should only integrate over the vector multiplet scalars modulo the Weyl identification. Therefore, (3.39) conforms to our expectations: the $z \to 1$ limits of the $SU(N)$ partition function and index are $1/N!$ times the $z \to 1$ limits of the corresponding quantities for a $\mathrm{U}(1)^{N-1}$ theory.

It is also interesting to consider the generalization of the $|W_{\mathrm{SU}(N)}|^{-1}$ factor to non-Lagrangian theories (where an interpretation in terms of the order of the Weyl group is no longer possible): it is precisely the geometrical factor we get from restricting the partition function to field configurations that parameterize $\mathcal{M}_{\mathcal{C}}$. For example, consider the following rank 1 non-Lagrangian theories: the AD theories discussed in [8] derived from the $N_f = 1, 2, 3$ $SU(2)$ gauge theory — let us label these theories $\mathrm{AD}_{1,2,3}$ — and the $E_{6,7,8}$ theories of Minahan and Nemeschansky [20, 32] — let us label these theories as $\mathrm{MN}_{6,7,8}$. The corresponding complex one-dimensional Coulomb branches, $\mathcal{M}_{\mathcal{C}}^{\mathrm{AD}i}$ and $\mathcal{M}_{\mathcal{C}}^{\mathrm{MN}i}$, are just complex cones with opening angles (in units of 2π) of $5/6, 3/4, 2/3$ and $1/3, 1/4, 1/6$ respectively (they are part of a classification due to Kodaira [33, 34]). The values of $|W|^{-1}$ required to appropriately restrict the partition function are given in table 1. These are precisely the inverse dimensions of the Coulomb branch operators in these theories, E^{-1}, and so they can also be understood as arising from the $z \to 1$ limit of $\mathcal{I}_{\mathrm{U}(1)_R} \supset \frac{1}{1-z^E} \sim E^{-1} \frac{1}{1-z}$.[18] One interesting check of this story is to note that the Coulomb branch of

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[17]Similar reasoning was used in [29] to study the effect of hypermultiplet zero modes on the superconformal index.

[18]This discussion also applies to the new rank one theories in [35], since these latter theories have the same Coulomb branch spectrum as the corresponding MN theories.

$\mathcal{N} = 2$ Theory	$|W|^{-1}$
SU(N)	$1/N!$
AD$_1$	$5/6$
AD$_2$	$3/4$
AD$_3$	$2/3$
MN$_6$	$1/3$
MN$_7$	$1/4$
MN$_8$	$1/6$

Table 1. Values for the leading-divergence index coefficient.

SU(2) SCQCD is given by the cone $\mathbf{C}/W_{\mathrm{SU}(2)} \simeq \mathbf{C}/\mathbf{Z}_2$ and that therefore we should have $|W_{\mathrm{SU}(2)}|^{-1} = 1/2$ by our reasoning. Indeed, this result agrees with (3.38) and (3.39).

We see that this argument is quite general and so we expect that for any theory, $\mathcal{T}$, of type B there is a pole as $z \to 1$ of order the complex dimension of the Coulomb branch multiplying a term describing the geometry on $\mathcal{M}_{\mathcal{C}}$.[19] Since the theories away from the origin of the Coulomb branch are Lagrangian (with the possible exception of theories living on certain complex co-dimension one or higher sub-manifolds of $\mathcal{M}_{\mathcal{C}}$), this term should be independent of w. Furthermore, we do not expect divergent contributions to the index coming purely from exotic operators since there is no supersymmetric moduli space associated with them.[20]

In other words, we should have that

$$\lim_{z \to 1} \mathcal{I}_{\mathrm{U}(1)_R}(\mathcal{T})(z, w) = \lim_{z \to 1} \left(\frac{\kappa}{(1-z)^{d_{\mathcal{C}}}} + \cdots \right), \tag{3.40}$$

where $d_{\mathcal{C}} = \dim\mathcal{M}_{\mathcal{C}}(\mathcal{T})$, κ is a constant, and the leading term on the right hand side describes the Lagrangian bulk of the Coulomb branch (if the theory has more than one Coulomb branch, then we have a similar term for each branch). The ellipsis in (3.40) contains various lower-order terms (including any contributions in w).

One immediate consequence of (3.40) is that if a theory of type B has exotic chiral operators, then for each exotic chiral operator, $\mathcal{O}_{A\alpha_1 \cdots \alpha_{2j_A}}$, there exist non-negative integers, $N_{A,a,b}$, and dimensionless constants c_b such that

$$\mathcal{O}_{A\alpha_1 \cdots \alpha_{2j_A}} \cdot \sum_b \left(c_b \prod_a \mathcal{O}_a^{N_{A,a,b}} \right) = 0, \quad \sum_b \left(c_b \prod_a \mathcal{O}_a^{N_{A,a,b}} \right) \neq 0, \tag{3.41}$$

where the $\mathcal{O}_a$ are Coulomb branch operators (more precisely, we mean the non-nilpotent scalar generators of the U(1)$_R$ chiral ring). As in (3.25), we work modulo descendants (if

---

[19]If the Coulomb branch consists of multiple sub-branches — potentially of different dimensions — then we expect one such term for each sub-branch.

[20]As we will see below, it is possible to relax this assumption at the cost of making some of our arguments more complicated.

there are multiple Coulomb branches, then we have an equation of the form (3.41) for each corresponding set of Coulomb branch operators).

Let us now briefly discuss the lower-order divergences in the index as $z \to 1$. A priori, it is possible that sub-leading divergences in $1 - z$ are w-dependent since there can be co-dimension one or higher sub-manifolds of $\mathcal{M}_{\mathcal{C}}$ that correspond to non-Lagrangian (or, more generally, to non-type A) theories. In the next section, however, we will rule out any w-dependence in the superconformal indices of theories that can flow to type A theories by turning on a relevant $\mathcal{N} = 2$-preserving coupling.

From the above discussion, it is not too difficult to see that the superconformal index of a type B theory can be written as follows

$$
\begin{aligned}
\mathcal{I}_{\mathrm{U}(1)_R} = &\sum_{a_1, \cdots, a_{d_{\mathcal{C}}}} \frac{1}{1 - z^{E_{a_1}}} \cdots \frac{1}{1 - z^{E_{a_{d_{\mathcal{C}}}}}} \cdot \widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z) \\
&+ \sum_{a_1, \cdots, a_{d_{\mathcal{C}}-1}} \frac{1}{1 - z^{E_{a_1}}} \cdots \frac{1}{1 - z^{E_{a_{d_{\mathcal{C}}-1}}}} \cdot \widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}-1}}(z, w) \\
&+ \cdots + \sum_{a_1} \frac{1}{1 - z^{E_{a_1}}} \widetilde{\mathcal{I}}_{a_1}(z, w) + \widetilde{\mathcal{I}}(z, w) \,,
\end{aligned}
\tag{3.42}
$$

where $a_i \in \{1, \cdots, d_{\mathcal{C}}, d_{\mathcal{C}} + 1, \cdots, d_{\mathcal{C}} + n\}$ runs over the Coulomb branch generators, $\mathcal{O}_{a_i}$, of dimension E_{a_i} (let us note again that we mean the non-nilpotent scalar generators of the $\mathrm{U}(1)_R$ chiral ring — more precisely, a minimal subset of these generators that give all the infinite-order contributions to the $\mathrm{U}(1)_R$ chiral ring). If there are Coulomb branches of various dimensions, then $d_{\mathcal{C}}$ refers to the maximal dimensional Coulomb branch(es). Note that in type A theories, $n = 0$ and $a_i \in \{1, \cdots, d_{\mathcal{C}}\}$.

In writing (3.42), we have factored out the infinite-order contributions from the Coulomb branch operators, and so the $\widetilde{\mathcal{I}}$ are non-singular. Furthermore, we take the $\widetilde{\mathcal{I}}$ to be symmetric in their indices, and if two indices take on the same value, then the corresponding $\widetilde{\mathcal{I}}$ vanishes (so as not to over-count the contributions from Coulomb branch operators).

For completeness, let us examine the $\widetilde{\mathcal{I}}$ in more detail. We define the $\widetilde{\mathcal{I}}$ so that the chiral operators contributing to $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z)$ satisfy $\widetilde{\mathcal{O}} \cdot \mathcal{O} \neq 0$ for any $\mathcal{O} \neq 0$ in the ring generated by $\{\mathcal{O}_{a_1}, \cdots, \mathcal{O}_{a_{d_{\mathcal{C}}}}\}$. It therefore follows from (3.41) that $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z)$ is independent of w. There is, however, some ambiguity in the z-dependent contributions to $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z)$, since we may, for example, decide to include the contribution from $\widetilde{\mathcal{O}}$ itself in some lower-order $\widetilde{\mathcal{I}}$ function, or — if $\widetilde{\mathcal{O}} \cdot \mathcal{O}' \neq 0$ for any $\mathcal{O}' \neq 0$ in the ring generated by some other generators $\{\mathcal{O}_{a'_1}, \cdots \mathcal{O}_{a'_{d_{\mathcal{C}}}}\}$ — in $\widetilde{\mathcal{I}}_{a'_1 \cdots a'_{d_{\mathcal{C}}}}(z)$.

If we include the contribution of $\widetilde{\mathcal{O}}$ in $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z)$, then we find a corresponding term of the form

$$
\frac{1}{d_{\mathcal{C}}!} z^{\widetilde{E}} \subset \widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z) \,,
\tag{3.43}
$$

where $\widetilde{E}$ is the scaling dimension of $\widetilde{\mathcal{O}}$ (and the factor $(d_{\mathcal{C}}!)^{-1}$ comes from the symmetrization in the a_i indices). On the other hand, if we choose to include the contribution from

$\widetilde{\mathcal{O}}$ in some other $\widetilde{\mathcal{I}}$ function, then we should replace (3.43) with

$$\frac{1}{d_{\mathcal{C}}!} z^{\widetilde{E}} \cdot \left(\sum_{a_i} z^{E_{a_i}} - \sum_{(a_{i_1}, a_{i_2})} z^{E_{a_{i_1}} + E_{a_{i_2}}} + \sum_{(a_{i_1}, a_{i_2}, a_{i_3})} z^{E_{a_{i_1}} + E_{a_{i_2}} + E_{a_{i_3}}} - \cdots \right. \tag{3.44}$$
$$\left. + (-1)^{d_{\mathcal{C}}+1} z^{\sum_{a_i} E_{a_i}} \right) \subset \widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z) \,,$$

where $(a_{i_1}, \cdots, a_{i_k})$ takes values in the $\frac{d_{\mathcal{C}}!}{(d_{\mathcal{C}}-k)! k!}$ unordered k-tuples built out of $\{a_1, \cdots, a_{d_{\mathcal{C}}}\}$.

It follows from our discussion that $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z)$ is independent of the superconformal $U(1)_R$ charge when we set $z = 1$ (the only dependence of (3.43) and (3.45) on the superconformal $U(1)_R$ is through the scaling dimension of the chiral operators). Furthermore, $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(1)$ is independent of whether we choose to include the contribution of $\widetilde{\mathcal{O}}$ in $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(1)$ or not — i.e., the left hand sides of (3.43) and (3.45) are both equal to $+1$ when evaluated at $z = 1$ (this matching is guaranteed by the fact that removing a finite number of operator contributions cannot change the leading divergence of the index).[21] In other words, we see that the various $\widetilde{\mathcal{I}}$ functions that multiply the leading divergence in (3.42) are well-defined (finite and independent of the ambiguities discussed above)[22] and universal (independent of the superconformal $U(1)_R$ at $z = 1$, with

$$d_{\mathcal{C}}! \cdot \widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(1) = \sum_B 1 = n_B > 0 \,. \tag{3.45}$$

Note that the general form of $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(1)$, as a sum over positive contributions from contributing operators, is independent of which minimal subset of generators we choose to use in writing the infinite order contributions in (3.42).

Next, let us proceed to the order $d_{\mathcal{C}} - 1$ terms (i.e., the $\widetilde{\mathcal{I}}$ in the second sum in (3.42)). These terms may a priori receive contributions from exotic chiral operators. Since we have factored out the infinite-order contributions from the Coulomb branch operators, the $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}-1}}(z, w)$ are also finite. Furthermore, these functions are independent of the superconformal $U(1)_R$ charge when we set $z = 1$. Finally, in analogy to the discussion above, $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}-1}}(1, w)$ is independent of whether we choose to include the contribution of a particular exotic operator, $\widetilde{\mathcal{O}}_{\alpha_1 \cdots \alpha_{2j}}$, (or even an infinite but subleading number of similar contributions from other exotic operators) in it or not since we either have

$$\frac{(-1)^{2j}}{(d_{\mathcal{C}} - 1)!} z^{\widetilde{E}} \chi_j(w) \subset \widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}-1}}(z, w) \,, \tag{3.46}$$

---

[21]More generally, we may choose to remove an infinite (but sub-leading) number of contributions in $\widetilde{\mathcal{O}}$ from $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z)$. For example, we can remove all contributions that are not of the form $\widetilde{\mathcal{O}}\mathcal{O}_{a_1}\mathcal{O}$ by replacing (3.43) with $z^{\widetilde{E} + E_{a_1}} \subset \widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}$. Note that again $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}(z)$ is independent of this choice.

[22]We have made the physical assumption that the divergences of the index/partition function arise from the existence of moduli spaces of vacua. Therefore, we have assumed that the theory does not contain an infinite number of chiral operators that do not form a ring, as there is no moduli space associated with them. If the theory has an infinite number of Coulomb branches, the index will of course not be finite, although the origin of the divergence is different from the one we are considering here.

or

$$\frac{(-1)^{2j}}{(d_{\mathcal{C}}-1)!} z^{\widetilde{E}} \chi_j(w) \cdot \left(\sum_{a_i} z^{E_{a_i}} - \sum_{(a_{i_1}, a_{i_2})} z^{E_{a_{i_1}} + E_{a_{i_2}}} + \sum_{(a_{i_1}, a_{i_2}, a_{i_3})} z^{E_{a_{i_1}} + E_{a_{i_2}} + E_{a_{i_3}}} - \cdots \right.$$
$$\left. + (-1)^{d_{\mathcal{C}}+1} z^{\sum_{a_i} E_{a_i}} \right) \subset \widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}-1}}(z, w) . \tag{3.47}$$

Clearly, (3.46) and (3.47) are both equal to $\chi_j(w)$ at $z = 1$. Therefore, we find that the exotic operator contributions to $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}-1}}(1, w)$ are unambiguously given by

$$(d_{\mathcal{C}}-1)! \cdot \widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}-1}}(1, w)|_{j \neq 0} = \sum_{j \in \mathbf{Z} > 0} n_{(j)a_1 \cdots a_{d_{\mathcal{C}}-1}} \cdot \chi_j(w) - \sum_{j \in \mathbf{Z} + \frac{1}{2} > 0} n_{(j)a_1 \cdots a_{d_{\mathcal{C}}-1}} \cdot \chi_j(w) ,$$
$$\tag{3.48}$$

where the $n_{(j)}$ are non-negative integers.

In particular, it is clear that the order $d_{\mathcal{C}} - 1$ contribution to the index depends on w if and only if this is true in the limit $z \to 1$. Furthermore, if the order $d_{\mathcal{C}} - 1$ contribution is independent of w, then the order $d_{\mathcal{C}} - 2$ contribution is independent of w if and only if this is true in the limit $z \to 1$ (and so on sequentially down to the finite contributions). Finally, let us note that the divergences multiplying the various $\widetilde{\mathcal{I}}$ depend on the superconformal $\mathrm{U}(1)_R$ only through a positive overall factor (times $(1-z)^{-n}\widetilde{\mathcal{I}}$) given by the inverse product of (minus) the $\mathrm{U}(1)_R$ charges of the corresponding Coulomb branch operators.

3.6 Theories of type C

In this subsection, we wish to discuss theories that flow to type A theories when we turn on a deformation by a relevant coupling. Since we want to preserve $\mathcal{N} = 2$ invariance, we can turn on relevant deformations of the prepotential

$$\delta\mathcal{L} = \int d^4\theta \mathcal{O} + \text{h.c.}, \tag{3.49}$$

where $d^4\theta$ is the $\mathcal{N} = 2$ chiral integration measure, and $\mathcal{O}$ is a generator of the chiral ring with dimension $E(\mathcal{O}) < 2$. In addition, we can consider deformations of the superpotential by a moment map, μ, corresponding to some symmetry, G, of the UV SCFT

$$\delta\mathcal{L} = \int d^2\theta \mu + \text{h.c.} . \tag{3.50}$$

As far as we are aware, all known $\mathcal{N} = 2$ SCFTs admit deformations by relevant operators that preserve $\mathcal{N} = 2$ SUSY (for example, while the T_N theories do not have relevant deformations of the type described in (3.49), we can deform their superpotentials as in (3.50)). Note that we will not consider deformations by FI terms (in any case, the corresponding $\mathrm{U}(1)$ gauge factors are not UV-complete).

Both types of relevant deformations in (3.49) and (3.50) break the $\mathrm{U}(1)_R$ symmetry of the UV SCFT (but preserve the $\mathrm{SU}(2)_R$ symmetry). Therefore, if we wish to compute the supersymmetric index of the interpolating theory, we should turn off the fugacity

corresponding to $U(1)_R$, i.e., we should take $z \to 1$. It is again useful to study the $U(1)_R$ limit of the index. In the UV we find that the index is

$$\mathcal{I}_{U(1)_R} = \sum_{a_1,\cdots,a_{d_{\mathcal{C}}}} \frac{1}{1-z^{E_{a_1}}} \cdots \frac{1}{1-z^{E_{a_{d_{\mathcal{C}}}}}} \cdot \widetilde{\mathcal{I}}_{a_1\cdots a_{d_{\mathcal{C}}}}(z,w)$$

$$+ \sum_{a_1,\cdots,a_{d_{\mathcal{C}}-1}} \frac{1}{1-z^{E_{a_1}}} \cdots \frac{1}{1-z^{E_{a_{d_{\mathcal{C}}-1}}}} \cdot \widetilde{\mathcal{I}}_{a_1\cdots a_{d_{\mathcal{C}}-1}}(z,w)$$

$$+ \cdots + \sum_{a_1} \frac{1}{1-z^{E_{a_1}}} \widetilde{\mathcal{I}}_{a_1}(z,w) + \widetilde{\mathcal{I}}(z,w) . \tag{3.51}$$

This formula for the index is very similar to the one in (3.42), but we allow for the leading term to be w-dependent since we do not assume that the $U(1)_R$ moduli space, $\mathcal{M}_{U(1)_R}$, parameterized by the vevs of the (non-nilpotent) chiral generators, $\mathcal{O}_{a_i}$, is a Coulomb branch (i.e., we allow for a fully interacting theory along the whole moduli space, although we do not know of any such examples). In this case, by reasoning similar to that used in the previous subsection, the leading $\widetilde{\mathcal{I}}$ functions have the following general form at $z = 1$

$$d_{\mathcal{C}}! \cdot \widetilde{\mathcal{I}}_{a_1\cdots a_{d_{\mathcal{C}}}}(1,w) = \sum_{j\in\mathbf{Z}\geq 0} n_{(j)a_1\cdots a_{d_{\mathcal{C}}}} \cdot \chi_j(w) - \sum_{j\in\mathbf{Z}+\frac{1}{2}>0} n_{(j)a_1\cdots a_{d_{\mathcal{C}}}} \cdot \chi_j(w), \tag{3.52}$$

where the $n_{(j)}$ are non-negative integers.

As we discussed in the previous subsection, the index in (3.51) will generally diverge as we take $z \to 1$. However, as we mentioned above, the $\widetilde{\mathcal{I}}$ functions are finite and do not depend on the (broken) superconformal $U(1)_R$ symmetry when we set $z = 1$. We can therefore associate a universal meaning to (3.52) at all length scales, including in the deep IR. A useful way to extract the $\widetilde{\mathcal{I}}$ is to consider the following contour integrals around $z = 1$:

$$\mathcal{A}_{n+1}(w) = \oint \frac{dz}{2\pi i} (1-z)^n \cdot \mathcal{I}_{U(1)_R}(z,w) . \tag{3.53}$$

We can then pick out w-dependent quantities in the $\mathcal{A}_{n+1}$ by defining

$$\mathcal{B}_{j_i,n+1} = \oint \frac{dw}{2\pi i w} \Delta(w)\chi_{j_i}(w)\mathcal{A}_{n+1}(w), \quad \Delta(w) = \frac{1}{2}(1-w^2)(1-w^{-2}) . \tag{3.54}$$

Let us now compute the various $\mathcal{A}_{n+1}$ in the UV and compare them to the corresponding quantities in the IR. We begin with $\mathcal{A}_{d_{\mathcal{C}}}$ (for $n + 1 > d_{\mathcal{C}}$, the $\mathcal{A}_{n+1}$ vanish)

$$\mathcal{A}_{d_{\mathcal{C}}}^{UV}(w) = - \sum_{a_1,\cdots,a_{d_{\mathcal{C}}}} \left(\prod_i E_{a_i}^{-1} \cdot \widetilde{\mathcal{I}}_{a_1\cdots a_{d_{\mathcal{C}}}}(1,w) \right) . \tag{3.55}$$

The superconformal $U(1)_R$ dependence of the above expression is contained in the inverse UV scaling dimensions of the $\mathcal{O}_{a_i}$. In the IR we can then compute the same quantity[23] and find

$$\mathcal{A}_{d_{\mathcal{C}}}^{IR}(w) = - \sum_{\widehat{a}_1,\cdots,\widehat{a}_{d_{\mathcal{C}}}} \left(\prod_i \widehat{E}_{\widehat{a}_i}^{-1} \cdot \widetilde{\mathcal{I}}_{\widehat{a}_1\cdots \widehat{a}_{d_{\mathcal{C}}}}(1,w) \right) , \tag{3.56}$$

---

[23]Note that the number of non-nilpotent $\mathcal{O}_{a_i}$ generators do not change along our class of RG flows (although their dimensions will generally change, and operators parameterizing the same branch will generally mix). The total $n_{(j)}$ parameterizing a given branch are also invariant.

where $\widehat{E}_{\widehat{a}_i}$ are the (generally different) scaling dimensions of the IR $\widehat{\mathcal{O}}_{\widehat{a}_i}$ operators. While it is usually true that $\mathcal{A}_{d_{\mathcal{C}}}^{UV}(w) \neq \mathcal{A}_{d_{\mathcal{C}}}^{IR}(w)$, it is also clear that since the $E_{a_i}, \widehat{E}_{\widehat{a}_i} > 0$ and the $\widetilde{\mathcal{I}}_{a_1 \cdots a_{d_{\mathcal{C}}}}$ have the form given in (3.52), then $\mathcal{A}_{d_{\mathcal{C}}}^{UV}$ is w-dependent if and only if $\mathcal{A}_{d_{\mathcal{C}}}^{IR}$ is w-dependent. Said differently, $\mathcal{B}_{j_i,d_{\mathcal{C}}}^{UV} \neq 0$ if and only if $\mathcal{B}_{j_i,d_{\mathcal{C}}}^{IR} \neq 0$.

Suppose now that the IR theory is type A. In this case, we know that $\mathcal{B}_{j_i,d_{\mathcal{C}}}^{IR} = 0$ for all $j_i \neq 0$. Therefore, the same must hold in the UV. In other words, (3.52) is independent of w. Proceeding sequentially down the list of $\mathcal{B}_{j_i,n+1}^{UV,IR}$, we find that if the IR theory is of type A, then the UV theory has $\mathcal{B}_{j_i,n+1}^{UV} = 0$ for all n. In other words, the UV theory cannot have exotic operators.[24]

3.7 Examples of type C theories

Let us now consider some particular cases of non-Lagrangian type C theories. For example, we can consider the $N_f = 2$ AD theory discussed in [8]. This theory has a single relevant prepotential coupling. It corresponds to an operator, $\mathcal{O}$, of dimension $4/3$. At the superconformal point, the Seiberg-Witten curve is

$$x^2 = z^4 \ . \tag{3.57}$$

Deforming the theory by $\delta\mathcal{L} \sim \int d^4\theta \, u_2 \mathcal{O} + \text{h.c.}$ is equivalent to deforming (3.57) as follows

$$x^2 = z^4 + u_2 z^2 \ . \tag{3.58}$$

In the deep IR we recover (in addition to a decoupled U(1))

$$x^2 = z^2 \ , \tag{3.59}$$

which is the curve of a free Lagrangian theory and is therefore of type A.

As a result, we conclude that the $N_f = 2$ AD theory does not contain exotic chiral operators. More generally, we can take any of the infinitely many (G, G') AD SCFTs for $G, G' = A_n, D_n, E_n$ [9, 36] and sequentially deform them to Lagrangian theories via relevant deformations (the theory in (3.57) is the (A_1, A_3) theory). We explain this in some detail in appendix B. From our above reasoning, we conclude that these theories cannot have exotic chiral operators. It would be interesting to understand the full set of theories that can be deformed to type A ones — we suspect that this is a very large class.

4 Conformal anomalies and the index

As we discussed above, the superconformal index is a robust observable in any SCFT, $\mathcal{T}$, since it is invariant under exactly marginal deformations. Moreover, the superconformal index captures global information about the protected sector of the theory. Similarly, the a and c conformal anomalies are invariant under exactly marginal deformations and describe global information about the theory (since they measure $\mathcal{T}$'s response to a background

---

[24]In the discussion so far, we have assumed that the exotic chiral operators give finite contributions to the superconformal index (since they are not associated with a moduli space). If we relax this assumption, we can repeat the discussion around (3.42) and (3.51) with the corresponding additional poles factored out.

metric). Therefore, it is natural to imagine that there is some relation between the index and the conformal anomalies (see also the recent holographic discussion in [37, 38] and the more general analysis in [39]).[25]

This idea can be made more concrete in an $\mathcal{N} = 2$ SCFT, $\mathcal{T}$, with a freely generated Coulomb branch (i.e., a Coulomb branch with no relations between the generators; as far as we are aware, all known $\mathcal{N} = 2$ SCFTs are of this type) and a Seiberg-Witten description by recalling that [35, 41]

$$2a - c = \frac{1}{2} \sum_a \left(E_a - \frac{1}{2} \right) , \tag{4.1}$$

where a and c are the anomaly coefficients of $\mathcal{T}$, and the E_a are the scaling dimensions of the $\overline{\mathcal{E}}_{r(0,0)}$ operators. We have seen that the $\overline{\mathcal{E}}_{r(0,0)}$ operators contribute to the chiral $U(1)_R$ index with a weight determined by their conformal dimension. Thus, it is clear that we should be able to extract the right hand side of (4.1), and therefore $2a - c$, from the index.

In order to understand how to make contact with (4.1) using the index, it turns out to be useful to recall how the authors of [41] derived (4.1) by building on the work of [42, 43]. The main strategy of [41] was to topologically twist $\mathcal{T}$ and couple it to a background metric. Under these conditions, the anomalous shift in the $U(1)_R$-charge of the vacuum becomes[26]

$$\Delta r = -(2a - c)\chi - \frac{3}{2}c\sigma + \frac{1}{2}k_G n , \tag{4.2}$$

where χ, σ, n, and k_G correspond to the Euler character, the signature, the instanton number of the background, and the flavor anomaly respectively. Since we are interested in $2a - c$, we will focus on the term in (4.2) proportional to χ.

In an $\mathcal{N} = 2$ SCFT with a Coulomb branch, $\mathcal{M}_{\mathcal{C}}(\mathcal{T})$, we can easily compute (4.2) at a generic point on $\mathcal{M}_{\mathcal{C}}(\mathcal{T})$. Indeed, since the theory at such a point consists of $d_{\mathcal{C}} = \dim \mathcal{M}_{\mathcal{C}}(\mathcal{T})$ weakly coupled abelian vector multiplets, we simply have that

$$2a - c = \frac{d_{\mathcal{C}}}{4} . \tag{4.3}$$

On the other hand, when we approach the (strongly coupled) $\mathcal{N} = 2$ SCFT point it is more difficult to compute (4.2). However, it turns out that the correction to (4.3) from the additional massless matter of the SCFT is encoded in the topologically twisted partition function, Z.

To see this, note that Z reduces to an integral over the (generically abelian) low energy effective theory

$$Z = \int [du][dq] A^\chi B^\sigma C^n e^{-S_{\text{low energy}}} , \tag{4.4}$$

---

[25]Note that unlike in the interesting recent work [28], here we have in mind a direct relation between the index — not the $M_4 \simeq S^3 \times S^1$ partition function — and the conformal anomalies. Note also that we are extracting information from the full index, as opposed to the gauge integrand considered e.g. in [40], as this approach is useful even in the context of non-Lagrangian theories.

[26]Our conventions for the $\mathcal{N} = 2$ $U(1)_R$ charge differ from those in [41]. In particular, we have $r_{\text{here}} = -\frac{1}{2} r_{\text{there}}$ while keeping our conventions for a and c as well as the $SU(2)_R$ and $\mathcal{N} = 1$ $U(1)_R$ symmetries the same as in [41].

where $[du]$ and $[dq]$ correspond to the vector multiplets and hypermultiplets that parameterize the abelian moduli space (i.e., these fields are massless everywhere), and the holomorphic measure factors A, B, and C give corrections to the low energy theory arising from matter that becomes massless only at special points on the moduli space. In particular, we have that $2a - c$ at the superconformal point is just

$$2a - c = -r(A) + \frac{d_{\mathcal{C}}}{4} = E(A) + \frac{d_{\mathcal{C}}}{4} \, , \tag{4.5}$$

where r is the $\mathrm{U}(1)_R$ charge of A and E is its scaling dimension. The beautiful observation of the authors in [41, 42] is that A can be determined in terms of the data of the low energy theory. In particular, A is just the Jacobian

$$A \sim \left(\det \frac{\partial u_a}{\partial a^I} \right)^{\frac{1}{2}} \, , \tag{4.6}$$

where the u_a are the gauge and monodromy-invariant coordinates and the a^I are the special electric coordinates. The correction to $2a - c$ can then be computed since the dimensions of the u_a are just the dimensions, E_a, of the $\overline{\mathcal{E}}_{r(0,0)}$ multiplets of the SCFT (here we assume that the Coulomb branch is freely generated), while the a^I have dimension one. In particular, we have

$$E(A) = \frac{1}{2} \sum_a (E_a - 1) \, . \tag{4.7}$$

We then see that (4.7) together with (4.5) imply (4.1).

Let us now see how to make contact with (4.1) using the index. Surprisingly, even though our theory is not topologically twisted, there is, as in (4.5), a natural index decomposition of the contributions to $2a - c$ from weakly coupled vector multiplets that are massless in the bulk of $\mathcal{M}_{\mathcal{C}}$ and corrections from the SCFT.[27] Just as in [41], we will suppose that the Coulomb branch is freely generated. For simplicity, let us also suppose that the theory is of type A or type C so that we can drop any potential exotic contributions to the index (this result then applies to a broad class of theories including the T_N theories and the AD theories with a Seiberg-Witten description; however, the expressions we will derive make sense for any type A or C $\mathcal{N} = 2$ theory). Under these assumptions, the general $\mathrm{U}(1)_R$ index in (3.42) becomes

$$\mathcal{I}_{\mathrm{U}(1)_R} = \prod_a \frac{1}{1 - z^{E_a}} \, . \tag{4.8}$$

Since the derivation of [41] used properties of the theory on the Coulomb branch, it is natural to take the limit $z \to 1$ in (4.8). In this limit, we find that

$$\lim_{z \to 1} \mathcal{I}_{\mathrm{U}(1)_R} = \prod_a E_a^{-1} \cdot (1 - z)^{-d_{\mathcal{C}}} + \mathcal{O}\left((1 - z)^{-d_{\mathcal{C}} + 1} \right) \, , \tag{4.9}$$

---

[27]Perhaps this is due to some relation between topological twisting and the holomorphic twisting we employ in constructing the index.

where the number of degrees of freedom in the bulk of the Coulomb branch, $d_{\mathcal{C}}$, can be read off from the leading divergence of the index — in particular, we see that we should include $d_{\mathcal{C}}$ vector multiplets in computing the weakly coupled contribution to $2a - c$. This result is as we expect since the theory should behave to first approximation as the abelian theory on $\mathcal{M}_{\mathcal{C}}$.

Just as in [41], we would like to see if it is possible to extract the SCFT corrections to $2a - c$. Therefore, it is natural to consider the leading corrections to (4.9)

$$\lim_{z \to 1} \mathcal{I}_{U(1)_R} = \prod_a E_a^{-1} \left((1-z)^{-d_{\mathcal{C}}} - E(A) \cdot (1-z)^{-d_{\mathcal{C}}+1} \right) + \mathcal{O}\left((1-z)^{-d_{\mathcal{C}}+2} \right) \ . \quad (4.10)$$

In particular, we see that the leading correction precisely captures the scaling dimension of the Jacobian (4.6) that describes the change in coordinates between the free $U(1)$ variables and vevs of SCFT operators.

We can then give a compact formula for $2a - c$ in terms of contour integrals of the index around the point $z = 1$. Recalling the definitions in (3.53)

$$\mathcal{A}_{n+1} = \oint \frac{dz}{2\pi i} (1-z)^n \cdot \mathcal{I}_{U(1)_R}(z) \,, \quad (4.11)$$

it follows that

$$d_{\mathcal{C}} = n_{\max} = \max\{n | \mathcal{A}_n \neq 0\} \,, \ E(A) = \frac{\mathcal{A}_{n_{\max}-1}}{\mathcal{A}_{n_{\max}}} \ . \quad (4.12)$$

Therefore, we arrive at the result

$$2a - c = \frac{\mathcal{A}_{n_{\max}-1}}{\mathcal{A}_{n_{\max}}} + \frac{n_{\max}}{4} \ . \quad (4.13)$$

As a final aside, let us note that our results also apply to the subclass of the above theories with good holographic duals, where $2a - c \sim \mathcal{O}(N^2) \gg 1$. Although the index itself scales as $\mathcal{O}(N^0)$, and the residues we compute in (4.11) are heavily suppressed in the $1/N$ expansion (note, however, that the ratio of residues in (4.13) is not suppressed), we can read off the conformal anomaly by varying the fugacities so that the index approaches a pole in the z plane. The precise way in which the index behaves in the vicinity of the pole encodes the conformal anomalies (at least one linear combination of them).

5 Summary and conclusions

In this paper we have derived constraints on higher-spin chiral operators in large classes of $\mathcal{N} = 2$ SCFTs. The first family, which we denoted "type A", consisted of theories related to Lagrangian SCFTs by generalized Argyres-Gaiotto-Seiberg duality. We used this duality in combination with general symmetry arguments to argue that type A theories do not contain the above "exotic" chiral operators in their spectrum.

We then considered a "chiral $U(1)_R$" limit of the $\mathcal{N} = 2$ superconformal index, which only receives contributions from Coulomb branch operators and their cousins, as a tool towards investigating theories of "type B" — generic theories with a Coulomb branch —

and theories of "type C", which flow to type A via a deformation by a relevant coupling. In doing so, we elaborated on properties of the chiral $U(1)_R$ index from the perspective of its $S^1 \times S^3$ path-integral interpretation. We subsequently used this index to argue that theories of type C do not possess exotic operators.

In the final section, we used the formula $2a - c = \frac{1}{2}\sum_a (E_a - \frac{1}{2})$ of [41], for theories of type A or type C with a freely generated Coulomb branch and a Seiberg-Witten description,[28] to give a prescription for reading off $2a - c$ directly from the chiral $U(1)_R$ index. Our expressions were also valid for examples of such theories with good holographic duals.

Note that our formula in (4.13) is actually well-defined for any type A or C $\mathcal{N} = 2$ SCFT (more generally, we are not aware of any concrete theory for which it is ill-defined). It would therefore be interesting to study whether our prescription for extracting $2a - c$ applies directly to theories with a non-freely generated Coulomb branch (assuming such theories exist). If our prescription applies to this class of theories, then we can immediately constrain the possible set of Coulomb branch operator relations. For example, let us consider an abstract SCFT with Coulomb branch operators $\mathcal{O}_{1,2}$ satisfying the constraint $\mathcal{O}_1^M = \mathcal{O}_2^N$. A direct application of our formula leads to a violation of the Hofman-Maldacena bound [44] unless $M = 1$ or $N = 1$ — this reasoning would suggest that such constraints with $M, N \neq 1$ should not be allowed in a unitary $\mathcal{N} = 2$ SCFT.

Finally, it is natural to ask whether, in direct analogy to the above case of $2a - c$, one could use the index to also determine the $a - c$ linear combination of conformal anomaly coefficients (see also the discussion in [39]). The latter can often be related to the number of effective vector and hypermultiplets in the theory, which in turn is proportional to the dimension of the Higgs branch (when it exists), $a - c = \frac{1}{24}(n_V - n_H) = -\frac{1}{24}\dim\mathcal{H}$. It would then be reasonable to look for a limit of the superconformal index that also counts Higgs-branch operators and attempt to extract $a - c$ along the lines that we have discussed (although we should be careful to note that one must deal with theories that don't have genuine Higgs branches). This is another question that we will leave open as an attractive direction for future research.

Acknowledgments

A $\quad \mathcal{N} = 2$ superconformal multiplet conventions

In this appendix we provide a brief summary of our conventions for quick reference. We follow Dolan and Osborn [12]. A review of their results[29] can be found in the appendices of [3], to which we point the interested reader for a concise account.

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[28]This class of theories is very large and includes the T_N theories as well as the infinite set of (A_N, A_M) AD theories.

[29]These are based on earlier work of [14–16], which established the classification of unitary representations of $\mathfrak{su}(2, 2|N)$ for all N, and studied their unitarity bounds and shortening conditions.

We work with the four-dimensional $\mathcal{N} = 2$ superconformal algebra $\mathfrak{su}(2,2|2)$ in terms of a spinorial notation with $\alpha = +,-$ and $\dot{\alpha} = \dot{+}, \dot{-}$. The purely bosonic subalgebra can be captured by the following expressions:

$$[\mathcal{M}_\alpha{}^\beta, \mathcal{M}_\gamma{}^\delta] = \delta_\gamma^\beta \mathcal{M}_\alpha{}^\delta - \delta_\alpha^\delta \mathcal{M}_\gamma{}^\beta \,, \qquad\qquad [\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \mathcal{M}^{\dot\gamma}{}_{\dot\delta}] = \delta_{\dot\delta}^{\dot\alpha} \mathcal{M}^{\dot\gamma}{}_{\dot\beta} - \delta_{\dot\beta}^{\dot\gamma} \mathcal{M}^{\dot\alpha}{}_{\dot\delta}$$

$$[\mathcal{M}_\alpha{}^\beta, \mathcal{P}_{\gamma\dot\gamma}] = \delta_\gamma^\beta \mathcal{P}_{\alpha\dot\gamma} - \frac{1}{2}\delta_\alpha^\beta \mathcal{P}_{\gamma\dot\gamma} \,, \qquad\qquad [\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \mathcal{P}_{\gamma\dot\gamma}] = \delta_{\dot\gamma}^{\dot\alpha} \mathcal{P}_{\gamma\dot\beta} - \frac{1}{2}\delta_{\dot\beta}^{\dot\alpha} \mathcal{P}_{\gamma\dot\gamma}$$

$$[\mathcal{M}_\alpha{}^\beta, \mathcal{K}^{\dot\gamma\gamma}] = -\delta_\alpha^\gamma \mathcal{K}_{\dot\gamma\beta} + \frac{1}{2}\delta_\alpha^\beta \mathcal{K}_{\dot\gamma\gamma} \,, \qquad\qquad [\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \mathcal{K}^{\dot\gamma\gamma}] = -\delta_{\dot\beta}^{\dot\gamma} \mathcal{K}^{\dot\alpha\gamma} + \frac{1}{2}\delta_{\dot\beta}^{\dot\alpha} \mathcal{K}^{\dot\gamma\gamma}$$

$$[\mathcal{K}^{\dot\alpha\alpha}, \mathcal{P}_{\beta\dot\beta}] = \delta_\beta^\alpha \delta_{\dot\beta}^{\dot\alpha} E + \delta_\beta^\alpha \mathcal{M}^{\dot\alpha}{}_{\dot\beta} + \delta_{\dot\beta}^{\dot\alpha} \mathcal{M}_\beta{}^\alpha \,, \qquad\qquad [\mathcal{R}^I{}_J, \mathcal{R}^K{}_L] = \delta_J^K \mathcal{R}^I{}_L - \delta_L^I \mathcal{R}^K{}_J$$

$$[E, \mathcal{P}_{\alpha\dot\alpha}] = \mathcal{P}_{\alpha\dot\alpha} \,, \qquad\qquad [E, \mathcal{K}^{\dot\alpha\alpha}] = -\mathcal{K}^{\dot\alpha\alpha} \,, \tag{A.1}$$

where

$$[\mathcal{M}_\alpha{}^\beta] = \begin{pmatrix} j_1 & j_1^+ \\ j_1^- & -j_1 \end{pmatrix} \,, \qquad [\mathcal{M}^{\dot\beta}{}_{\dot\alpha}] = \begin{pmatrix} j_2 & j_2^+ \\ j_2^- & -j_2 \end{pmatrix} \tag{A.2}$$

and

$$[\mathcal{R}^I{}_J] = \begin{pmatrix} R & R^+ \\ R^- & -R \end{pmatrix} + \frac{1}{2}\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \,, \tag{A.3}$$

such that

$$[R^+, R^-] = 2R \,, \qquad [R, R^\pm] = \pm R^\pm \,. \tag{A.4}$$

The Poincaré and superconformal supercharges obey the relations

$$\{\mathcal{Q}_\alpha^I, \widetilde{\mathcal{Q}}_{J\dot\alpha}\} = \delta_J^I \mathcal{P}_{\alpha\dot\alpha}$$

$$\{\widetilde{\mathcal{S}}^{I\dot\alpha}, \mathcal{S}_J^\alpha\} = \delta_J^I \mathcal{K}^{\dot\alpha\alpha}$$

$$\{\mathcal{Q}_\alpha^I, \mathcal{S}_J^\beta\} = \frac{1}{2}\delta_J^I \delta_\alpha^\beta E + \delta_J^I \mathcal{M}_\alpha{}^\beta - \delta_\alpha^\beta \mathcal{R}^I{}_J$$

$$\{\widetilde{\mathcal{S}}^{I\dot\alpha}, \widetilde{\mathcal{Q}}_{J\dot\beta}\} = \frac{1}{2}\delta_J^I \delta_{\dot\beta}^{\dot\alpha} E + \delta_J^I \mathcal{M}^{\dot\alpha}{}_{\dot\beta} + \delta_{\dot\beta}^{\dot\alpha} \mathcal{R}^I{}_J \,. \tag{A.5}$$

These in turn satisfy the hermiticity conditions

$$(\mathcal{Q}_\alpha^I)^\dagger = \mathcal{S}_I^\alpha \,, \quad (\widetilde{\mathcal{Q}}_{I\dot\alpha})^\dagger = \widetilde{\mathcal{S}}^{I\dot\alpha} \,, \quad (R^I{}_J)^\dagger = R^J{}_I \,, \quad (\mathcal{M}_\alpha{}^\beta)^\dagger = \delta^{\dot\beta\gamma} \mathcal{M}_\gamma{}^\delta \delta_{\delta\dot\alpha} \,. \tag{A.6}$$

For the bosonic and fermionic generators we also have

$$[\mathcal{M}_\alpha{}^\beta, \mathcal{Q}_\gamma^I] = \delta_\gamma^\beta \mathcal{Q}_\alpha^I - \frac{1}{2}\delta_\alpha^\beta \mathcal{Q}_\gamma^I \,, \qquad\qquad [\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \widetilde{\mathcal{Q}}_{I\dot\delta}] = \delta_{\dot\delta}^{\dot\alpha} \widetilde{\mathcal{Q}}_{I\dot\beta} - \frac{1}{2}\delta_{\dot\beta}^{\dot\alpha} \widetilde{\mathcal{Q}}_{I\dot\delta}$$

$$[\mathcal{M}_\alpha{}^\beta, \mathcal{S}_I^\gamma] = -\delta_\alpha^\gamma \mathcal{S}_I^\beta + \frac{1}{2}\delta_\alpha^\beta \mathcal{S}_I^\gamma \,, \qquad\qquad [\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \widetilde{\mathcal{S}}^{I\dot\gamma}] = -\delta_{\dot\beta}^{\dot\gamma} \widetilde{\mathcal{S}}^{I\dot\alpha} + \frac{1}{2}\delta_{\dot\beta}^{\dot\alpha} \widetilde{\mathcal{S}}^{I\dot\gamma}$$

$$[E, \mathcal{Q}_\alpha^I] = \frac{1}{2}\mathcal{Q}_\alpha^I \,, \qquad\qquad [E, \widetilde{\mathcal{Q}}_{I\dot\alpha}] = \frac{1}{2}\widetilde{\mathcal{Q}}_{I\dot\alpha}$$

$$[E, \mathcal{S}_I^\alpha] = -\frac{1}{2}\mathcal{S}_I^\alpha \,, \qquad\qquad [E, \widetilde{\mathcal{S}}^{I\dot\alpha}] = -\frac{1}{2}\widetilde{\mathcal{S}}^{I\dot\alpha}$$

$$[\mathcal{R}^I{}_J, \mathcal{Q}_\alpha^K] = \delta_J^K \mathcal{Q}_\alpha^I - \frac{1}{4}\delta_J^I \mathcal{Q}_\alpha^K \,, \qquad\qquad [\mathcal{R}^I{}_J, \widetilde{\mathcal{Q}}_{K\dot\alpha}] = -\delta_K^I \widetilde{\mathcal{Q}}_{J\dot\alpha} + \frac{1}{4}\delta_J^I \widetilde{\mathcal{Q}}_{K\dot\alpha}$$

$$[\mathcal{K}^{\dot\alpha\alpha}, \mathcal{Q}_\beta^I] = \delta_\beta^\alpha \widetilde{\mathcal{S}}^{I\dot\alpha} \,, \qquad\qquad [\mathcal{K}^{\dot\alpha\alpha}, \widetilde{\mathcal{Q}}_{I\dot\beta}] = \delta_{\dot\beta}^{\dot\alpha} \mathcal{S}_I^\alpha$$

$$[\mathcal{P}_{\alpha\dot\alpha}, \mathcal{S}_I^\beta] = -\delta_\alpha^\beta \widetilde{\mathcal{Q}}_{I\dot\alpha} \,, \qquad\qquad [\mathcal{P}_{\alpha\dot\alpha}, \widetilde{\mathcal{S}}^{I\dot\beta}] = -\delta_{\dot\alpha}^{\dot\beta} \mathcal{Q}_\alpha^I \,. \tag{A.7}$$

$\mathcal{Q}$	$R, r(j_1, j_2)$	$\delta := 2\{\mathcal{Q}, \mathcal{S}\}$	Commuting δs		
$\mathcal{Q}^1_+$	$\frac{1}{2}, \frac{1}{2}(\frac{1}{2}, 0)$	$E + 2j_1 - 2R - r$	$\delta^2_-,$	$\widetilde{\delta}_{2\dot+},$	$\widetilde{\delta}_{2\dot-}$
$\mathcal{Q}^1_-$	$\frac{1}{2}, \frac{1}{2}(-\frac{1}{2}, 0)$	$E - 2j_1 - 2R - r$	$\delta^2_+,$	$\widetilde{\delta}_{2\dot+},$	$\widetilde{\delta}_{2\dot-}$
$\mathcal{Q}^2_+$	$-\frac{1}{2}, \frac{1}{2}(\frac{1}{2}, 0)$	$E + 2j_1 + 2R - r$	$\delta^1_-,$	$\widetilde{\delta}_{1\dot+},$	$\widetilde{\delta}_{1\dot-}$
$\mathcal{Q}^2_-$	$-\frac{1}{2}, \frac{1}{2}(-\frac{1}{2}, 0)$	$E - 2j_1 + 2R - r$	$\delta^1_+,$	$\widetilde{\delta}_{1\dot+},$	$\widetilde{\delta}_{1\dot-}$
$\widetilde{\mathcal{Q}}_{1\dot+}$	$-\frac{1}{2}, -\frac{1}{2}(0, \frac{1}{2})$	$E + 2j_2 + 2R + r$	$\widetilde{\delta}_{2\dot-},$	$\delta^2_+,$	δ^2_-
$\widetilde{\mathcal{Q}}_{1\dot-}$	$-\frac{1}{2}, -\frac{1}{2}(0, -\frac{1}{2})$	$E - 2j_2 + 2R + r$	$\widetilde{\delta}_{2\dot+},$	$\delta^2_+,$	δ^2_-
$\widetilde{\mathcal{Q}}_{2\dot+}$	$\frac{1}{2}, -\frac{1}{2}(0, \frac{1}{2})$	$E + 2j_2 - 2R + r$	$\widetilde{\delta}_{1\dot-},$	$\delta^1_+,$	δ^1_-
$\widetilde{\mathcal{Q}}_{2\dot-}$	$\frac{1}{2}, -\frac{1}{2}(0, -\frac{1}{2})$	$E - 2j_2 - 2R + r$	$\widetilde{\delta}_{1\dot+},$	$\delta^1_+,$	δ^1_-

Table 2. List of the various quantum numbers for each supercharge $\mathcal{Q}$.

Using the above commutation relations, one can easily determine the quantum numbers associated with the various supercharges. These are given in table 2.

The full spectrum of operators can be built by identifying a "highest-weight state", which is a superconformal primary of the $\mathcal{N} = 2$ SCFT, and acting with the 8 Poincaré supercharges to create superconformal descendants. The most generic such (long) multiplet is denoted by $\mathcal{A}^E_{R,r(j_1,j_2)}$ and obeys a unitarity bound. Multiplets that are additionally annihilated by Poincaré supercharges are denoted as short and saturate the bound. The various shortening conditions can be summarized as

$$\mathcal{B}^I: \qquad [\mathcal{Q}^I_\alpha, \mathcal{O}_{\dot\alpha_1 \cdots \dot\alpha_{2j_2}}\} = 0, \qquad \text{for } \alpha = +, -$$

$$\overline{\mathcal{B}}_I: \qquad [\widetilde{\mathcal{Q}}_{I\dot\alpha}, \mathcal{O}_{\alpha_1 \cdots \alpha_{2j_1}}\} = 0, \qquad \text{for } \dot\alpha = \dot+, \dot-$$

$$\mathcal{C}^I: \qquad \begin{cases} \epsilon^{\alpha\beta}[\mathcal{Q}^I_\alpha, \mathcal{O}_{\beta\alpha_1 \cdots \alpha_{2j_1-1},\dot\alpha_1 \cdots \dot\alpha_{2j_2}}\} = 0, & \text{for } j_1 > 0 \\ \epsilon^{\alpha\beta}\{\mathcal{Q}^I_\alpha, [\mathcal{Q}^I_\beta, \mathcal{O}_{\dot\alpha_1 \cdots \dot\alpha_{2j_2}}]\} = 0, & \text{for } j_1 = 0 \end{cases}$$

$$\overline{\mathcal{C}}_I: \qquad \begin{cases} \epsilon^{\dot\alpha\dot\beta}[\widetilde{\mathcal{Q}}_{I\dot\alpha}, \mathcal{O}_{\alpha_1 \cdots \alpha_{2j_1},\dot\beta\dot\alpha_1 \cdots \dot\alpha_{2j_2-1}}\} = 0, & \text{for } j_2 > 0 \\ \epsilon^{\dot\alpha\dot\beta}\{\widetilde{\mathcal{Q}}_{I\dot\alpha}, [\widetilde{\mathcal{Q}}_{I\dot\beta}, \mathcal{O}_{\alpha_1 \cdots \alpha_{2j_1}}]\} = 0, & \text{for } j_2 = 0 \end{cases}, \qquad (A.8)$$

where the operators $\mathcal{O}$ above are understood to be in the highest weight representation of $SU(2)_R$.

A full classification of all such possibilities and the associated short multiplet structure is listed in table 3.

One conventionally incorporates most of the above into multiplets of type $\mathcal{C}, \overline{\mathcal{C}}, \widehat{\mathcal{C}}$ by allowing for "spin $-\frac{1}{2}$" representations as follows [3, 12]:

$$\mathcal{C}_{R,r(-\frac{1}{2},j_2)} := \mathcal{B}_{R+\frac{1}{2},r+\frac{1}{2}(0,j_2)}, \qquad\qquad \overline{\mathcal{C}}_{R,r(j_1,-\frac{1}{2})} := \overline{\mathcal{B}}_{R+\frac{1}{2},r-\frac{1}{2}(j_1,0)}$$

$$\widehat{\mathcal{C}}_{R(-\frac{1}{2},j_2)} := \mathcal{D}_{R+\frac{1}{2}(0,j_2)}, \qquad\qquad \widehat{\mathcal{C}}_{R(j_1,-\frac{1}{2})} := \overline{\mathcal{D}}_{R+\frac{1}{2}(j_1,0)}$$

$$\widehat{\mathcal{C}}_{R(-\frac{1}{2},-\frac{1}{2})} := \mathcal{D}_{R+\frac{1}{2}(0,-\frac{1}{2})} = \overline{\mathcal{D}}_{R+\frac{1}{2}(-\frac{1}{2},0)} = \widehat{\mathcal{B}}_{R+1} . \qquad (A.9)$$

We make use of this convention in the main part of this paper.

Multiplet	Unitarity Bounds		Shortening
$\mathcal{A}^E_{R,r(j_1,j_2)}$	$E \geq 2 + 2j_1 + 2R + r$	$E \geq 2 + 2j_2 + 2R - r$	—
$\mathcal{B}_{R,r(0,j_2)}$	$E = 2R + r$	$j_1 = 0$	$\mathcal{B}^1$
$\overline{\mathcal{B}}_{R,r(j_1,0)}$	$E = 2R - r$	$j_2 = 0$	$\overline{\mathcal{B}}_2$
$\widehat{\mathcal{B}}_R$	$E = 2R$	$j_1 = j_2 = r = 0$	$\mathcal{B}^1 \cap \overline{\mathcal{B}}_2$
$\mathcal{C}_{R,r(j_1,j_2)}$	$E = 2 + 2j_1 + 2R + r$		$\mathcal{C}^1$
$\overline{\mathcal{C}}_{R,r(j_1,j_2)}$	$E = 2 + 2j_2 + 2R - r$		$\overline{\mathcal{C}}_2$
$\mathcal{C}_{0,r(j_1,j_2)}$	$E = 2 + 2j_1 + r$	$R = 0$	$\mathcal{C}^1 \cap \mathcal{C}^2$
$\overline{\mathcal{C}}_{0,r(j_1,j_2)}$	$E = 2 + 2j_2 - r$	$R = 0$	$\overline{\mathcal{C}}_1 \cap \overline{\mathcal{C}}_2$
$\widehat{\mathcal{C}}_{R(j_1,j_2)}$	$E = 2 + 2R + j_1 + j_2$	$r = j_2 - j_1$	$\mathcal{C}^1 \cap \overline{\mathcal{C}}_2$
$\mathcal{D}_{R(0,j_2)}$	$E = 1 + 2R + j_2$	$r = j_2 + 1$	$\mathcal{B}^1 \cap \overline{\mathcal{C}}_2$
$\overline{\mathcal{D}}_{R(j_1,0)}$	$E = 1 + 2R + j_1$	$-r = j_1 + 1$	$\overline{\mathcal{B}}_2 \cap \mathcal{C}^1$
$\mathcal{D}_{0(0,j_2)}$	$E = r = 1 + j_2$	$R = 0$	$\mathcal{B}^1 \cap \mathcal{B}^2 \cap \overline{\mathcal{C}}_I$
$\overline{\mathcal{D}}_{0(j_1,0)}$	$E = -r = 1 + j_1$	$R = 0$	$\overline{\mathcal{B}}_1 \cap \overline{\mathcal{B}}_2 \cap \mathcal{C}^I$
$\mathcal{E}_{r(0,j_2)}$	$E = r$	$R = 0$	$\mathcal{B}^1 \cap \mathcal{B}^2$
$\overline{\mathcal{E}}_{r(j_1,0)}$	$E = -r$	$R = 0$	$\overline{\mathcal{B}}_1 \cap \overline{\mathcal{B}}_2$

Table 3. The $\mathcal{N} = 2$ superconformal multiplet structure.

B   RG flow from (G, G') theories

Here we will explain that all the (G, G') theories defined in [9] are of type C in the sense of subsection 3.6. Let us first review the definition of (G, G') theories. We consider type IIB string theory on a local Calabi-Yau threefold given by the hypersurface singularity

$$W(x_1, \cdots, x_4) = 0 \,, \tag{B.1}$$

in $\mathbf{C}^4$. Here $W(x_1, \cdots, x_4)$ is a quasi-homogeneous polynomial such that

$$W(\zeta^{q_i} x_i) = \zeta W(x_i) \tag{B.2}$$

for all $\zeta \in \mathbf{C}^*$. We also assume that $dW = 0$ if and only if $x_i = 0$. The holomorphic 3-form on the threefold is given by $\Omega = \frac{dx_1 dx_2 dx_3 dx_4}{dW}$ up to rescaling. Since the Calabi-Yau threefold is non-compact, it engineers an $\mathcal{N} = 2$ gauge theory in the transverse four dimensions. If the singularity (B.1) arises at finite distance in the moduli space of some compact Calabi-Yau threefold, we can think of the 4d gauge theory as a rigid supersymmetry limit of some 4d supergravity. In terms of $\widehat{c} \equiv \sum_{i=1}^4 (1 - 2q_i)$, the necessary and sufficient condition for this is written as $2 - \widehat{c} > 0$ [45]. A class of $W(x_i)$ satisfying this condition is given by [9]

$$W(x_1, \cdots, x_4) = W_G(x_1, x_2) + W_{G'}(x_3, x_4) \,, \tag{B.3}$$

where $G, G' = A_n, D_n, E_n$ and

$$
\begin{aligned}
W_{A_n}(x,y) &= x^{n+1} + y^2 \,, \\
W_{D_n}(x,y) &= x^{n-1} + xy^2 \,, \\
W_{E_6}(x,y) &= x^3 + y^4 \,, \\
W_{E_7}(x,y) &= x^3 + xy^3 \,, \\
W_{E_8}(x,y) &= x^3 + y^5 \,.
\end{aligned}
\tag{B.4}
$$

The 4d theory engineered by the singularity (B.3) is called the (G, G') theory [9].

The BPS states of the 4d theory correspond to D3-branes wrapping on supersymmetric compact 3-cycles of the Calabi-Yau threefold. Since all the compact 3-cycles are now shrinking at $x_i = 0$, the 4d BPS states are all massless. This suggests that the (G, G') theory is a 4d superconformal filed theory of Argyres-Douglas type [7, 8]. The transformation $x_i \to \zeta^{q_i} x_i$, which keeps the equation $W(x_i) = 0$ invariant, corresponds to a scale transformation in four dimensions. To identify the scaling dimension of the variable x_i, let us consider a 4d particle corresponding to a D3-brane wrapping on a 3-cycle of the Calabi-Yau threefold. The central charge of the 4d particle is given by $\oint \Omega$, where the integration is taken over the 3-cycle. Now $x_i \to \zeta^{q_i} x_i$ induces

$$
\oint \Omega \ \to \ \zeta^{\frac{2-\widehat{c}}{2}} \oint \Omega \,.
\tag{B.5}
$$

Since the central charge has scaling dimension one, we interpret $\xi := \zeta^{\frac{2-\widehat{c}}{2}}$ as the dilatation factor of a 4d scale transformation. Then $\zeta^{q_i} = \xi^{2q_i/(2-\widehat{c})}$ implies that the scaling dimension of x_i is written as [36]

$$
[x_i] = \frac{2q_i}{2 - \widehat{c}} \,.
\tag{B.6}
$$

B.1 Deformation of the singularity

The deformations of the singularity (B.1) correspond to the complex structure moduli of the Calabi-Yau threefold. In particular, complex structure moduli associated with the vanishing 3-cycles govern the Coulomb branch of the 4d gauge theory. Varying such moduli replaces $W(x_i) = 0$ with

$$
W(x_i) + \delta W(x_i) = 0 \,,
\tag{B.7}
$$

for some lower order polynomial $\delta W(x_i)$. To find a basis of such deformations, let us consider the ring

$$
\mathcal{R} := \mathbf{C}[x_1, \cdots, x_4]/I_W \,,
\tag{B.8}
$$

where I_W is an ideal of $\mathbf{C}[x_1, \cdots, x_4]$ generated by all $\partial W/\partial x_i$. It is known [36, 46] that the number of independent vanishing 3-cycles is equal to $\dim_{\mathbf{C}} \mathcal{R}$ (as a vector space over $\mathbf{C}$). Let us take a set of polynomials $\{W_k\}$ which represent a basis of $\mathcal{R}$. Then the most general deformation is written as

$$
W(x_i) + \sum_{k=1}^{\dim_{\mathbf{C}} \mathcal{R}} c_k W_k(x_i) = 0
\tag{B.9}
$$

for $c_k \in \mathbf{C}$. We can think of $\{c_k\}$ as a local coordinate of the complex structure moduli near the singular point. In terms of 4d physics, they are interpreted as masses and Coulomb branch parameters of the engineered gauge theory. In particular, $c_k = 0$ corresponds to the superconformal point.

Let us take $W_k(x_i)$ to be monomials in x_i. There exists $Q_k \in \mathbf{Q}^+$ such that

$$W_k(\zeta^{q_i} x_i) = \zeta^{Q_k} W_k(x_i) \tag{B.10}$$

for all $\zeta \in \mathbf{C}^*$. The scaling dimension of c_k in 4d physics is then evaluated to be [36]

$$[c_k] = \frac{2(1 - Q_k)}{2 - \widehat{c}}. \tag{B.11}$$

For any c_k there exists c_ℓ such that $[c_k] + [c_\ell] = 2$, which is consistent with the arguments of [7]. If $[c_k] < 1$, the parameter c_k is interpreted as a coupling constant of the theory. The corresponding deformation of the theory is given by

$$\int d^4\theta \, c_k \mathcal{O}_k \,, \tag{B.12}$$

with some Coulomb branch operator $\mathcal{O}_k$ of dimension $2 - [c_k] = [c_\ell]$. Now c_ℓ is interpreted as the vev of the operator $\mathcal{O}_k$. On the other hand, if $[c_k] > 1$, the roles of c_k and c_ℓ are exchanged. If $[c_k] = 1$, the parameter c_k is interpreted as a mass deformation parameter.

B.2   Reduction to Seiberg-Witten description

If the polynomial $W(x_i)$ is of the form

$$W(x_i) = x_1^2 + x_2^2 + \widetilde{W}(x_3, x_4) \,, \tag{B.13}$$

the Coulomb branch of the 4d theory has a Seiberg-Witten description with the curve $\widetilde{W}(s, t) = 0$ and the 1-form $\lambda = sdt$ on it [9]. The reason for this is that any monomial containing x_1 or x_2 becomes trivial in $\mathcal{R}$; there is no non-trivial complex structure deformation in the $x_{1,2}$ directions. For example, any (A, A) theory has a Seiberg-Witten description. These types of theories were studied in [10] from the viewpoint of M5-branes on a Riemann surface, while their RG-flows investigated in [47]. On the other hand, if $W(x_i)$ cannot be expressed in the form (B.13) (even after changing variables), the corresponding 4d theory does not seem to have a Seiberg-Witten description [9]. For instance, the (D_n, D_m) theories for $n, m \geq 4$ and (E, E) theories provide such examples.

B.3   RG flow by relevant deformations

We now study RG flows from (G, G') theories to show that they are of type C in the sense of subsection 3.6. Let us start with the (A_n, A_m) theory. The corresponding Calabi-Yau singularity is described by

$$W(x, y, s, t) = x^{n+1} + y^2 + s^{m+1} + t^2 = 0 \,. \tag{B.14}$$

We have $q_x = \frac{1}{n+1}, q_s = \frac{1}{m+1}, q_y = q_t = \frac{1}{2}$, and therefore $2 - \hat{c} = 2(\frac{1}{n+1} + \frac{1}{m+1})$. The dimensions of the variables are

$$[x] = \frac{1}{1 + \frac{n+1}{m+1}}, \qquad [s] = \frac{1}{1 + \frac{m+1}{n+1}}, \qquad [y] = [t] = \frac{1}{2(\frac{1}{n+1} + \frac{1}{m+1})}. \tag{B.15}$$

The ring $\mathcal{R} = \mathbf{C}[x, y, s, t]/I_W$ is now $\mathbf{C}[x, y, s, t]/\langle x^n, y, s^m, t \rangle$. Since there is no complex structure deformation involving y or t, we have a Seiberg-Witten description with the curve

$$x^{n+1} + s^{m+1} = 0. \tag{B.16}$$

and the 1-form $\lambda = s dx$. The most general deformed curve in the Coulomb branch is written as

$$x^{n+1} + s^{m+1} + \sum_{k=0}^{n-1}\sum_{\ell=0}^{m-1} c_{k,\ell} x^k s^\ell = 0. \tag{B.17}$$

The dimensions of the parameters are

$$[c_{k,\ell}] = \frac{(m+1)(n+1) - k(m+1) - \ell(n+1)}{m+n+2}, \tag{B.18}$$

which satisfy $[c_{k,\ell}] + [c_{n-k-1,m-\ell-1}] = 2$.

In particular,

$$[c_{n-1,0}] = \frac{2(m+1)}{m+n+2}, \qquad [c_{0,m-1}] = \frac{2(n+1)}{m+n+2}. \tag{B.19}$$

If $n \geq m$, $0 < [c_{n-1,0}] \leq 1$ implies that $c_{n-1,0}$ is a relevant coupling or mass parameter. The corresponding deformation does not turn on any vev of the Coulomb branch operators, and leads to the deformed curve

$$x^{n+1} + s^{m+1} + c_{n-1,0}\, x^{n-1} = 0. \tag{B.20}$$

We now have several massive BPS states whose masses are proportional to $(c_{n-1,0})^{1/[c_{n-1,0}]}$. In the deep IR, all the massive degrees of freedom are integrated out. This corresponds to focusing on the region $|x| \ll \sqrt{|c_{n-1,0}|}$. Then the IR curve is written as

$$x^{n-1} + s^{m+1} = 0, \tag{B.21}$$

after rescaling x and s.[30] Since this is the curve of the (A_{n-2}, A_m) theory, we identify that the IR CFT is the (A_{n-2}, A_m) theory (there is also a decoupled U(1)). Thus, in the case $n \geq m$, the (A_n, A_m) theory can flow to the (A_{n-2}, A_m) theory via a relevant perturbation. On the other hand, if $n \leq m$, $c_{0,m-1}$ is a relevant coupling or mass parameter. The corresponding relevant deformation leads to the (A_n, A_{m-2}) theory (and a decoupled U(1)) in the infrared.

By repeating this procedure, the (A_n, A_m) theory eventually flows (up to several decoupled U(1) factors) to the (A_1, A_1), (A_1, A_0) or (A_2, A_0) theory via relevant deformations.

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[30]To be more precise, we rescale x and s so that $c_{n-1,0}x^{n-1} + s^{m+1} = 0$ becomes $x^{n-1} + s^{m+1} = 0$ while keeping $\lambda = s dx$ fixed. This changes the scaling dimensions of x and s.

Here the (A_1, A_0) and (A_2, A_0) theories are trivial theories while the (A_1, A_1) theory is a theory of a single free hypermultiplet. Therefore any (A_n, A_m) theory can flow to a Lagrangian theory via relevant deformations. This means that all the (A_n, A_m) theories are of type C in the sense of subsection 3.6.

It is straightforward to generalize this argument to the other (G, G') theories. For example, the (A_n, E_6) theory can flow to some (A_m, D_4) theory without turing on vev's of the Coulomb branch operators. Then the latter can further flow to some (A, A) theory via relevant deformations. Since all the (A, A) theories are of type C, the original (A_n, E_6) theory is also of type C. Exactly the same argument shows that all the (G, G') theories are of type C. Our discussion in subsection 3.6 then implies that (G, G') theories have no exotic chiral operators.

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