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ISSN 1470-5559

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RR-06-08 November 2006

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November 3, 2006

1 Introduction

1.1 Excuses

Davidson (following Austin (Austin, 1956–7)) considers the following pattern of reasoning.

> 'I didn't know that it was loaded' belongs to one standard pattern of excuse. I do not deny that I pointed the gun and pulled the trigger, nor that I shot the victim. My ignorance explains how it happens that I pointed the gun and pulled the trigger intentionally, but did not shoot the victim intentionally. ... The logic of this sort of excuse includes, it seems, at least this much structure: I am accused of doing b, which is deplorable. I admit I did a, which is excusable. My excuse for doing b rests upon my claim that I did not know that $a = b$. (Davidson, 1980d, p. 109)

Davidson, then, is arguing for two things:

1. that equalities between actions are meaningful, and
2. that they are used in common-sense reasoning about action.

Our goal in this article will be to investigate these claims by presenting a logic which will allow us to formalise this sort of equational reasoning, which will turn out to be powerful and interesting. In order to formalise the equational reasoning we need a background logic for reasoning about actions: we will use ideas derived from Reiter's work on actions (Reiter, 2001).

1.2 States and Possible Worlds

Reasoning about action has two sides, which we will, following philosophical terminology, call the intensional and the extensional. The intensional side is the agent's view

*Submitted to *Commonsense'07*

of actions: what actions are performed, in what sequence, and so on. It is this view of actions which is sometimes referred to as the "knowledge level" (Newell, 1982). We can think of this view as giving us a labelled transition system: the nodes of the system will be called *states* (in AI terminology, *situations*), the arrows will be, in philosophical terminology, action *tokens*, and the arrows will be labelled with action *types* (for the type/token distinction, see Davidson (Davidson, 1980c; Davidson, 1980b), Hornsby (Hornsby, 1999; Hornsby, 1998), and Wetzel (Wetzel, 1998)). Our actions will be deterministic – that is, there will be at most one action token of a given action type starting from a given state.

However, as well as their intensional aspect, actions also have an effect on the world. This is the *extensional* side of action and it will be important also to talk about it: for example, the misfortune in the above scenario stems directly from the gap between the intensional and the extensional. We will represent the extensional side of actions by propositional assertions about states. Our underlying logic will, following Reiter, be classical, so "the way the world is" can be described by assigning truth values to propositions: that is, by what is called, in logical jargon, a *possible world*, and we can, therefore, think of the effect of an action as a function from possible worlds to possible worlds.

Now actions, as Reiter emphasises (Reiter, 2001), are not usually performable in all circumstances: furthermore, whether an action is performable or not will, in general, depend on circumstances unknown to the agent (for example, I may try to open a door, not knowing whether it is open or closed). So whether an action is performable or not is a matter of the extensional side of things, in which we are representing actions as functions from possible worlds to possible worlds: and we can conveniently represent this by having these functions be partially defined. An action will be performable in precisely those worlds in which the corresponding function is defined.

Extensions and intensions will be related as follows. States encode intensional information, and such information will, in general, only yield partial knowledge of the world: thus, each (intensional) state will, in general, correspond to several different possible worlds. However, the information that we encode in states will be part of the world, so that each possible world will correspond to a unique such state. So, each state will have, associated to it, a set of possible worlds, and these sets of possible worlds will be disjoint. We will talk of a possible world x *at* a state s, and we will write this $x : s$.

1.3 Formalism

1.3.1 Davidson

Let us suppose, then, that, for each state s, we have variables available, which we will write $x : s$, and which will range over partial worlds at s. Suppose we have an action α between states s and t (we write this $s \xrightarrow{\alpha} t$): as we have argued above, we can regard this as a partial function from possible worlds at s to possible worlds at t. We can, then, write α in the form $\alpha(x : s)$, and view this function as delivering a result of type t.

1.3.2 Reiter

As we have said, the background treatment for action will be Reiter's. The fundamental problem for Reiter is what is called *regression*: that is, given a transition $s \overset{\alpha}{\frown} t$ between situations s and t, and given a proposition P at t – what Reiter would describe as a *fluent* – the regression problem is to find a proposition P' at s which which will be true iff P is true at t. In our terms, this regression operator will be given by substitution: P can be regarded as a function $P(x : t)$ from possible worlds at t to truth values, the action as a partially defined function $\alpha(x : s)$ from possible worlds at s to possible worlds at t. The regression operator will be given by substitution: the proposition P' will be the function $P(\alpha(s))$, defined on the possible worlds at s where α can be executed.

There is another important concept, known as *progression*: that is, given an action $s \overset{\alpha}{\frown} t$, and given a theory P describing the state s, find the theory Q describing t. Reiter does, in fact, use this notion: however, his insight was that regression is technically simpler to work with, and that progression can be characterised in terms of it.

We shall, however, depart from Reiter in not identifying, as he does, states with sequences of actions; the intensional side of our system can be *any* labelled transition system, rather than the tree freely generated by action types which Reiter uses. We will, rather deliberately, use the word 'state' rather than 'situation', in order to avoid confusion.

2 The System

2.1 Primitives

As we describe above (p. 2), we describe actions by means of partially defined functions which take possible worlds, at the state before the action is performed, to possible worlds at the state after the action is performed.

We will, loosely following Scott (Scott, 1979), use a partially defined equality relation to reason about terms which may be undefined. Our intended notion of partial equality will be as follows:

Definition 1. Let $\boldsymbol{\alpha}(\boldsymbol{x})$ and $\boldsymbol{\beta}(\boldsymbol{y})$ be two partial functions whose values are possible worlds at the same tuple of states: then

$$\boldsymbol{\alpha}(\boldsymbol{x}) \simeq \boldsymbol{\beta}(\boldsymbol{y})$$

is true at worlds $\boldsymbol{x}$ and $\boldsymbol{y}$ iff

1. $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both defined at those worlds, and
2. the values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are equal.

Note that we do not give truth values to such equalities when either $\boldsymbol{\alpha}(\boldsymbol{x})$ or $\boldsymbol{\beta}(\boldsymbol{y})$ are undefined: we are not doing many valued logic here.

So, if we have two actions, $\alpha(x : s)$ and $\beta(y : t)$, whose values are possible worlds at the same state; then $\alpha(x) \simeq \beta(y)$ is a partially defined binary relation between possible worlds at different states, s and t. Clearly, if we go on like this, we will need

n-ary relations between possible worlds: it is easiest to work with n-tuples of worlds, of the form

$$\langle x_1 : s_1, \ldots, x_n : s_n \rangle,$$

and we shall write such a tuple, in boldface, as $\boldsymbol{x} : \boldsymbol{s}$, or, where the states are clear from the context, as $\boldsymbol{x}$. We will also be interested in k-tuples of actions

$$\langle \alpha_1(x_{i_1}), \ldots, \alpha_k(x_{i_k}) \rangle,$$

or $\boldsymbol{\alpha}(\boldsymbol{x})$, where each of the α_i will have an argument which is one of the x_i.

Notice that the equality $\boldsymbol{f}(\boldsymbol{x}) \simeq \boldsymbol{f}(\boldsymbol{x})$, which we shall abbreviate to $\boldsymbol{f}\downarrow$, is true iff the functions of $\boldsymbol{f}$ are all defined at $\boldsymbol{x}$.

2.2 Formulae, Contexts, and Sequents

In the previous section, we used sets of variables together with constraints given by partially defined equalities. We call such a collection of data a context:

Definition 2. A *context*, $\vartheta(\boldsymbol{x})$, will consist of the following data:

1. a tuple of variables, $\boldsymbol{x} : \boldsymbol{s}$, and
2. a set of *partial equations*, $\boldsymbol{\alpha}(\boldsymbol{x}) \simeq \boldsymbol{\beta}(\boldsymbol{x})$.

We will write contexts in the form $\vartheta(\boldsymbol{x})$, where $\boldsymbol{x}$ is the tuple of variables involved. From the semantics for partial equations given in Definition 1, we get a notion of semantic validity:

Definition 3. We say that an entailment $\vartheta(\boldsymbol{x}) \models_{\simeq} \vartheta(\boldsymbol{x})$ is *semantically valid* iff, for each interpretation of the functions occurring in ϑ and ϑ' as partial functions, whenever all of the members of ϑ are true then so are all of the members of ϑ'.

An axiom system for this notion of entailment can be found in (Palmgren & Vickers, 2005, pp. 4ff.): we will assume that this, or some other equivalent axiomatisation, is used, but the details will not concern us.

Contexts will represent the domains over which the free variables of formulae are supposed to range; we will write them as $\vartheta(\boldsymbol{x})$ or, occasionally, as $\{\boldsymbol{x}|\vartheta(\boldsymbol{x})\}$ for clarity. The only propositions that we have so far defined are those constructed out of our partial equalities. But it makes sense also to introduce relations $P(\boldsymbol{x}|\boldsymbol{\vartheta}(\boldsymbol{x}))$, where $\boldsymbol{x}$ is, as before, a k-tuple of variables, and where $\vartheta(\boldsymbol{x})$ is a set of equations of the form $\boldsymbol{\alpha}(\boldsymbol{x}) \simeq \boldsymbol{\beta}(\boldsymbol{x})$: such a relation will be defined on the subset of values for which the members of $\boldsymbol{\vartheta}(\boldsymbol{x})$ are all true.

We define formulae with specified contexts according to the rules in Table 1, and we also define sequents. Here $\Gamma(\boldsymbol{x}|\vartheta(\boldsymbol{x}))$ and $\Delta(\boldsymbol{x}|\vartheta(\boldsymbol{x}))$ stand for sets of formulae, all of which have domain $\boldsymbol{x}|\vartheta(\boldsymbol{x})$. Note that a sequent always has a specified context, and that the formation rules for sequents ensure that the relations in a sequent are defined over all of its context.

2.2.1 Substitution

We also have a notion of substitution, defined likewise in Table 1: if we have a predicate P, defined on the context $\vartheta'(\boldsymbol{y})$, and if we have an appropriately typed tuple of functions $\boldsymbol{\alpha}$ – that is, the target of $\boldsymbol{\alpha}$ should have the same type as $\boldsymbol{y}$ – then we can

formulae

$$\frac{}{\top(\boldsymbol{x}|\vartheta(\boldsymbol{x}))} \qquad \frac{}{\bot(\boldsymbol{x}|\vartheta(\boldsymbol{x}))} \qquad \frac{P(\boldsymbol{x}|\vartheta x)}{\neg P(\boldsymbol{x}|\vartheta x)}$$

$$\frac{P(\boldsymbol{x}|\vartheta(\boldsymbol{x})) \quad Q(\boldsymbol{x}|\vartheta(\boldsymbol{x}))}{(P \wedge Q)(\boldsymbol{x}|\vartheta(\boldsymbol{x}))} \qquad \frac{P(\boldsymbol{x}|\vartheta(\boldsymbol{x})) \quad Q(\boldsymbol{x}|\vartheta(\boldsymbol{x}))}{(P \vee Q)(\boldsymbol{x}|\vartheta(\boldsymbol{x}))}$$

$$\frac{P(\boldsymbol{x},\boldsymbol{y}|\vartheta(\boldsymbol{x}))}{\exists \boldsymbol{y}.P(\boldsymbol{x}|\vartheta(\boldsymbol{x}))} \qquad \frac{P(\boldsymbol{x},\boldsymbol{y}|\vartheta(\boldsymbol{x}))}{\forall \boldsymbol{y}.P(\boldsymbol{x}|\vartheta(\boldsymbol{x}))}$$

substitution

$$\frac{\vartheta(\boldsymbol{x}:\boldsymbol{s}) \models_{\simeq} \vartheta'(\boldsymbol{\alpha}(\boldsymbol{x}:\boldsymbol{s}):\boldsymbol{t}) \quad P(\boldsymbol{y}:\boldsymbol{t}|\vartheta'(\boldsymbol{y}))}{P(\boldsymbol{\alpha}(\boldsymbol{x}:\boldsymbol{s}))|\vartheta(\boldsymbol{x}))}$$

sequents

$$\frac{\Gamma(\boldsymbol{x}|\vartheta(\boldsymbol{x})) \quad \Delta(\boldsymbol{x}|\vartheta(\boldsymbol{x}))}{\vartheta|\Gamma \vdash \Delta}$$

Table 1: Formation Rules for Formulae and Sequents

substitute $\boldsymbol{\alpha}$ to obtain a predicate on the domain $\vartheta(\boldsymbol{x})$; we can write this, in the usual way, $P[\boldsymbol{\alpha}(\boldsymbol{x})/\boldsymbol{y}]$, but usually the more informal notion $P(\boldsymbol{\alpha}(\boldsymbol{x}))$ will suffice.

Substitutions, however, have to be well defined: in the case we are considering, this will be so provided that $\boldsymbol{\alpha}$ is well defined on $\vartheta(\boldsymbol{x})$ and that $\boldsymbol{\alpha}(\boldsymbol{x})$ is in the domain of definition of P (namely, $\{\boldsymbol{y}|\vartheta'(\boldsymbol{y})\}$) for each $\boldsymbol{x}$ with $\vartheta(\boldsymbol{x})$. The following lemma, whose proof follows easily from the definitions, shows us how to represent these conditions as an entailment:

Lemma 1. *Given a predicate P defined in context $\vartheta'(\boldsymbol{y})$, and an entailment of contexts*

$$\vartheta(\boldsymbol{x}:\boldsymbol{s}) \models_{\simeq} \vartheta'(\boldsymbol{\alpha}(\boldsymbol{x}:\boldsymbol{s}):\boldsymbol{t}).$$

Then $P[\boldsymbol{\alpha}(\boldsymbol{x})/\boldsymbol{y}]$ is well defined on $\{\boldsymbol{x}|\vartheta(\boldsymbol{x})\}$.

It is worth pointing out that there are degenerate, but significant, cases of the substitution rule. The first is where $\boldsymbol{\alpha}$ is trivial: if α is the identity, then we start with a predicate $P(\boldsymbol{x}|\vartheta(\boldsymbol{x}))$, and, given the side condition $\vartheta'(\boldsymbol{x}) \models_{\simeq} \vartheta(\boldsymbol{x})$, substitution yields a predicate on the context $\boldsymbol{x}|\vartheta(\boldsymbol{x})$. We write the result of this sort of substitution in the form

$$P(\boldsymbol{x}|\vartheta(\boldsymbol{x}))_{|\vartheta'(\boldsymbol{x})}$$

The second case is where α is simply the projection onto a sub-tuple of variables: in this case, we write

$$P(\boldsymbol{x}|\vartheta(\boldsymbol{x}))_{|\vartheta(\boldsymbol{x}),\vartheta'(\boldsymbol{x},\boldsymbol{y})}$$

2.3 The Sequent Calculus

The sequent calculus is given in Table 2.

$$\frac{\vartheta'(\boldsymbol{x}) \models_{\bumpeq} \vartheta(\boldsymbol{\alpha}(\boldsymbol{x})) \quad \vartheta(\boldsymbol{y}) \mid \Gamma(\boldsymbol{y}) \vdash \Delta(\boldsymbol{y})}{\vartheta'(\boldsymbol{\alpha}(\boldsymbol{x})) \mid \Gamma(\boldsymbol{\alpha}(\boldsymbol{x})) \vdash \Delta(\boldsymbol{\alpha}(\boldsymbol{x}))}\,\text{subs} \qquad \frac{\vartheta \models_{\bumpeq} \boldsymbol{\alpha}(\boldsymbol{x}) \bumpeq \boldsymbol{\beta}(\boldsymbol{y}) \quad \vartheta|\Gamma[\boldsymbol{\alpha}(\boldsymbol{x})/\boldsymbol{z}] \vdash \Delta[\boldsymbol{\alpha}(\boldsymbol{x})/\boldsymbol{z}]}{\vartheta|\Gamma[\boldsymbol{\beta}(\boldsymbol{y})/\boldsymbol{z}] \vdash \Delta[\boldsymbol{\beta}(\boldsymbol{y})/\boldsymbol{z}]}\,\text{rep}$$

$$\begin{array}{c} \vartheta, \boldsymbol{\alpha} \bumpeq \boldsymbol{\beta} \mid \Gamma_{|\vartheta,\boldsymbol{\alpha}\bumpeq\boldsymbol{\beta}} \vdash \Delta_{|\vartheta,\boldsymbol{\alpha}\bumpeq\boldsymbol{\beta}} \\ \hline\hline \vartheta, \boldsymbol{\alpha}{\downarrow}, \boldsymbol{\beta}{\downarrow} \mid \boldsymbol{\alpha} \bumpeq \boldsymbol{\beta}, \Gamma \vdash \Delta \end{array}\,| \qquad \frac{}{\vartheta(\boldsymbol{x})|P(\boldsymbol{x}|\vartheta(\boldsymbol{x})) \vdash P(\boldsymbol{x}|\vartheta(\boldsymbol{x}))}\,\text{Ax}$$

$$\frac{\vartheta, \boldsymbol{\alpha}{\downarrow}|\Gamma \vdash \Delta}{\vartheta, \boldsymbol{\alpha}{\downarrow}|\Gamma, P(\boldsymbol{\alpha}(\boldsymbol{x})) \vdash \Delta}\,\text{LW} \qquad \frac{\vartheta, \boldsymbol{\alpha}{\downarrow}|\Gamma \vdash \Delta}{\vartheta, \boldsymbol{\alpha}{\downarrow}|\Gamma \vdash Q(\boldsymbol{\alpha}(\boldsymbol{x})), \Delta}\,\text{RW}$$

$$\frac{\vartheta|\Gamma, P, P \vdash \Delta}{\vartheta|\Gamma, P \vdash \Delta}\,\text{LC} \qquad \frac{\vartheta|\Gamma \vdash Q, Q, \Delta}{\vartheta|\Gamma \vdash Q, \Delta}\,\text{RC}$$

$$\frac{}{\vartheta|\Gamma, \perp(\boldsymbol{\alpha}(\boldsymbol{x})) \vdash \Delta}\,\perp\text{L} \qquad \frac{}{\vartheta|\Gamma \vdash \top(\boldsymbol{\alpha}(\boldsymbol{x})), \Delta}\,\top\text{R}$$

$$\frac{\vartheta|\Gamma, P_1 \vdash \Delta \quad \vartheta|\Gamma, P_2 \vdash \Delta}{\vartheta|\Gamma, P_1 \vee P_2 \vdash \Delta}\,\vee\text{L} \qquad \frac{\vartheta|\Gamma \vdash \Delta, Q1, Q2}{\vartheta|\Gamma \vdash \Delta, Q_1 \vee Q_2}\,\vee\text{R}$$

$$\frac{\vartheta|\Gamma, P_1, P_2 \vdash \Delta}{\vartheta|\Gamma, P_1 \wedge P_2 \vdash \Delta}\,\wedge\text{L} \qquad \frac{\vartheta|\Gamma \vdash Q_1, \Delta \quad \vartheta|\Gamma \vdash Q_2, \Delta}{\vartheta|\Gamma \vdash Q_1 \wedge Q_2, \Delta}\,\wedge\text{R}$$

$$\frac{\vartheta, \Gamma \vdash Q, \Delta}{\vartheta|\Gamma, \neg Q \vdash \Delta}\,\neg\text{L} \qquad \frac{\vartheta|\Gamma, P \vdash \Delta}{\vartheta|\Gamma \vdash \neg Q, \Delta}\,\neg\text{R}$$

$$\frac{\vartheta(\boldsymbol{x}), \vartheta'(\boldsymbol{x}, \boldsymbol{y}) \mid \Gamma(\boldsymbol{x})_{|\vartheta(\boldsymbol{x}),\vartheta'(\boldsymbol{x},\boldsymbol{y})}, P(\boldsymbol{x}, \boldsymbol{y}) \vdash \Delta(\boldsymbol{x})_{|\vartheta(\boldsymbol{x}),\vartheta'(\boldsymbol{x},\boldsymbol{y})}}{\vartheta(\boldsymbol{x}) \mid \Gamma(\boldsymbol{x}),\ \exists \boldsymbol{y}.P(\boldsymbol{x}, \boldsymbol{y}|\vartheta(\boldsymbol{x}), \vartheta'(\boldsymbol{x}, \boldsymbol{y}))\ \vdash \Delta(\boldsymbol{x})}\,\exists\text{L}$$

$$\frac{\vartheta(\boldsymbol{x}) \models_{\bumpeq} \vartheta'(\boldsymbol{x}, \boldsymbol{\gamma}(\boldsymbol{x})) \quad \vartheta(\boldsymbol{x}) \mid \Gamma(\boldsymbol{x}) \vdash Q(\boldsymbol{x}, \boldsymbol{\gamma}(\boldsymbol{x})), \Delta(\boldsymbol{x})}{\vartheta(\boldsymbol{x}) \mid \Gamma(\boldsymbol{x}) \vdash \exists \boldsymbol{y}.Q(\boldsymbol{x}, \boldsymbol{y}|\vartheta(\boldsymbol{x}), \vartheta'(\boldsymbol{x}, \boldsymbol{y})),\ \Delta(\boldsymbol{x})}\,\exists\text{R}$$

$$\frac{\vartheta(\boldsymbol{x}) \models_{\bumpeq} \vartheta'(\boldsymbol{x}, \boldsymbol{\gamma}(\boldsymbol{x})) \quad \vartheta(\boldsymbol{x}) \mid \Gamma(\boldsymbol{x}),\ P(\boldsymbol{x}, \boldsymbol{\gamma}(\boldsymbol{x})) \vdash \Delta(\boldsymbol{x})}{\vartheta(\boldsymbol{x}) \mid \Gamma(\boldsymbol{x}),\ \forall \boldsymbol{y}.P(\boldsymbol{x}, \boldsymbol{y}) \vdash \Delta(\boldsymbol{x})}\,\forall\text{L}$$

$$\frac{\vartheta(\boldsymbol{x}), \vartheta'(\boldsymbol{x}, \boldsymbol{y}) \mid \Gamma(\boldsymbol{x})_{|\vartheta(\boldsymbol{x}),\vartheta'(\boldsymbol{x},\boldsymbol{y})} \vdash P(\boldsymbol{x}, \boldsymbol{y}), \Delta(\boldsymbol{x})_{|\vartheta(\boldsymbol{x}),\vartheta'(\boldsymbol{x},\boldsymbol{y})}}{\vartheta(\boldsymbol{x}) \mid \Gamma(\boldsymbol{x}) \vdash \forall \boldsymbol{y}.P(\boldsymbol{x}, \boldsymbol{y}), \Delta(\boldsymbol{x})}\,\forall\text{R}$$

$$\frac{\vartheta(\boldsymbol{x})|\Gamma(\boldsymbol{x}) \vdash P(\boldsymbol{x}), \Delta(\boldsymbol{x}) \quad \vartheta(\boldsymbol{x})|\Gamma'(\boldsymbol{x}), P(\boldsymbol{x}) \vdash \Delta'(\boldsymbol{y})}{\vartheta(\boldsymbol{x})|\Gamma(\boldsymbol{x}), \Gamma'(\boldsymbol{x}) \vdash \Delta(\boldsymbol{x}), \Delta'(\boldsymbol{x})}\,\text{cut}$$

Table 2: The Davidsonian Sequent Calculus

3 Semantics

3.1 Semantic Values

Semantics for our calculus will be as follows.

Definition 4. A *Davidsonian pre-model* for a transition system Σ will consist of the following:

1. for each state s, a set $[\![s]\!]$ of possible worlds at s,

2. for each action $s \xrightarrow{\alpha} t$, a partial function $[\![\alpha]\!]_s : [\![s]\!] \to [\![t]\!]$.

Definition 5. Given a Davidson pre-model, we can define semantic values for individual partial equalities:

$$[\![\alpha(x : s) \simeq \beta(y : t)]\!] = \Big\{ \langle x \in [\![s]\!], y \in [\![t]\!] \rangle \,\Big|\, \begin{array}{l} [\![\alpha]\!] \text{ is defined at } x, \\ [\![\beta]\!] \text{ is defined at } y, \\ [\![\alpha]\!](x) = [\![\beta]\!](y) \end{array} \Big\}$$

and also for contexts:

$$[\![\boldsymbol{\alpha}(\boldsymbol{x}) \simeq \boldsymbol{\beta}(\boldsymbol{x})]\!] = \bigcap [\![\alpha_i(x_i) \simeq \beta_i(x'_i)]\!]$$

where the intersection is taken over all the equations in the context

Having defined semantic values for contexts, we can now define models:

Definition 6. A *Davidsonian model* is a Davidsonian pre-model together with, for each primitive relation $P(\boldsymbol{x}|\vartheta(\boldsymbol{x}))$, a set of possible worlds

$$[\![P(\boldsymbol{x}|\vartheta(\boldsymbol{x}))]\!]_{\vartheta(\boldsymbol{x})} \subseteq [\![\vartheta(\boldsymbol{x})]\!]$$

We define semantic values for composite formulae in Table 3; again, we omit definitions which can be recovered from the duality of the system. The definition of semantic validity is now routine.

Definition 7. Let $\vartheta(\boldsymbol{x})|\Gamma(\boldsymbol{x}) \vdash \Delta(\boldsymbol{x})$ be a sequent. Then, given a model M, define the semantic value of Γ to be

$$[\![\Gamma(\boldsymbol{x})]\!]_{\vartheta(\boldsymbol{x})} = \bigcap_{P \in \Gamma} [\![P(\boldsymbol{x})]\!]_{\vartheta(\boldsymbol{x})}$$

and that of Δ to be

$$[\![\Delta(\boldsymbol{x})]\!]_{\vartheta(\boldsymbol{x})} = \bigcup_{Q \in \Delta} [\![P(\boldsymbol{x})]\!]_{\vartheta(\boldsymbol{x})}$$

$$
\begin{aligned}
[\![\top]\!]_{\vartheta(\boldsymbol{x})} &= [\![\vartheta(\boldsymbol{x})]\!] \\
[\![P(\boldsymbol{\alpha}(\boldsymbol{x})|\vartheta'(\boldsymbol{x}))]\!]_{\vartheta'(\boldsymbol{x})} &= \left\{\boldsymbol{x} \in [\![\vartheta'(\boldsymbol{x})]\!] \,\middle|\, \boldsymbol{\alpha}(\boldsymbol{x}) \in [\![P(\boldsymbol{y}|\vartheta(\boldsymbol{y}))]\!]_{\vartheta(\boldsymbol{y})}\right\} \\
[\![P \wedge Q]\!]_{\vartheta(\boldsymbol{x})} &= [\![P]\!]_{\vartheta(\boldsymbol{x})} \cap [\![Q]\!]_{\vartheta(\boldsymbol{x})} \\
[\![\neg P]\!]_{\vartheta(\boldsymbol{x})} &= [\![\vartheta(\boldsymbol{x})]\!] - [\![P]\!]_{\vartheta(\boldsymbol{x})} \\
[\![\exists \boldsymbol{y}.P(\boldsymbol{x},\boldsymbol{y}|\vartheta(\boldsymbol{x}),\vartheta'(\boldsymbol{x},\boldsymbol{y}))]\!]_{\vartheta(\boldsymbol{x})} &= \left\{\boldsymbol{x} \,\middle|\, \exists \boldsymbol{y} \in [\![\boldsymbol{t}]\!] \,.\, \langle \boldsymbol{x},\boldsymbol{y}\rangle \in [\![P(\boldsymbol{x},\boldsymbol{y}|\vartheta(\boldsymbol{x},\vartheta'(\boldsymbol{x},\boldsymbol{y})))]\!]_{\vartheta(\boldsymbol{x}),\boldsymbol{y},\vartheta'(\boldsymbol{x},\boldsymbol{y})}\right\}
\end{aligned}
$$

Table 3: Semantic Values

and we say that $\vartheta(\boldsymbol{x})|\Gamma(\boldsymbol{x}) \vdash \Delta(\boldsymbol{x})$ is *valid in M* iff

$$[\![\Gamma(\boldsymbol{x})]\!]_{\vartheta(\boldsymbol{x})} \subseteq [\![\Delta(\boldsymbol{x})]\!]_{\vartheta(\boldsymbol{x})}$$

Finally, we say that our sequent is *semantically valid* iff it is valid in all models: we write

$$\vartheta(\boldsymbol{x})|\Gamma(\boldsymbol{x}) \vDash \Delta(\boldsymbol{x})$$

3.2 Soundness and Completeness

We can prove the following:

Theorem 1 (Soundness for the Davidsonian system)**.** *The calculus of Table 2 is sound for the semantics of this section.*

Theorem 2. *The semantics of this section is complete: that is, if a sequent is semantically valid, there is a proof according to the system of Table 2.*

Proofs are routine, but too large for this paper.

4 Applications

4.1 Davidson's Example

Example 1 (Davidson)**.** *We first deal with Davidson's example: we write his propositions in our logic, using partial equality and being careful to stipulate that actions are performable. Let* sh *stand for 'shoot' and* pt *stand for 'pull trigger'.*

$$\mathsf{sh}(x) \simeq \mathsf{sh}(x), \mathsf{pt}(x) \simeq \mathsf{pt}(x) \mid \mathsf{loaded}(x) \;\dashv\vdash\; \mathsf{pt}(x) \simeq \mathsf{sh}(x) \tag{1}$$

We also describe the effects of shooting as follows:

$$\mathsf{sh}(x) \simeq \mathsf{sh}(x) \mid \mathsf{alive}(x) \vdash \mathsf{dead}(\mathsf{sh}(x)) \tag{2}$$

Here we assume that loaded, alive *and* dead *are predicates which are defined for all values of x.* sh *and* pt, *on the other hand, are actions which may not be performable in all circumstances (there may not be a gun to hand, for example), and thus these entailment have non-trivial contexts.*

We prove the unfortunate consequence in Table 4.

$$
\cfrac{\cfrac{}{\vartheta_2 \mid \mathsf{loaded}(x) \vdash \mathsf{pt}(x) \simeq \mathsf{sh}(x)}\,(1) \qquad \cfrac{\cfrac{\cfrac{\cfrac{}{\vartheta_2 \mid \mathsf{alive}(x) \vdash \mathsf{dead}(\mathsf{sh}(x))}\,(2)}{\vartheta_1 \mid \mathsf{alive}(x)_{|\vartheta_1} \vdash \mathsf{dead}(\mathsf{sh}(x))_{|\vartheta_1}}\,\text{subs}}{\vartheta_1 \mid \mathsf{alive}(x)_{|\vartheta_1} \vdash \mathsf{dead}(\mathsf{pt}(x))_{|\vartheta_1}}\,\text{repl}}{\vartheta_2 \mid \mathsf{sh}(x) \simeq \mathsf{pt}(x), \mathsf{alive}(x) \vdash \mathsf{dead}(\mathsf{pt}(x))}\,|}{\vartheta_2 \mid \mathsf{loaded}(x), \mathsf{alive}(x) \vdash \mathsf{dead}(\mathsf{pt}(x))}\,\text{cut}
$$

We abbreviate the context $\mathsf{sh}(x)\downarrow, \mathsf{pt}(x)\downarrow$ to ϑ_2, and $\mathsf{pt}(x) \simeq \mathsf{sh}(x)$ to ϑ_1. Note that, by symmetry and transitivity, we have $\vartheta_1 \vDash_{\simeq} \vartheta_2$.

Table 4: The Davidsonian Scenario

4.2 Action Progression

We consider now the problem of what Reiter calls *action progression*: that is, given an axiomatisation of a state of affairs at a situation s, and an action $s \xrightarrow{\alpha} t$, to find an axiomatisation of the corresponding state of affairs at t. We will suppose that the axiomatisations in question can be represented by single propositions (the treatment could be extended to sets of propositions, but at a considerable cost in bureaucracy and little gain in insight).

First a definition, following (Lawvere, 1969).

Definition 8. Suppose that we have tuple of actions $\boldsymbol{s} \xrightarrow{\boldsymbol{\alpha}} \boldsymbol{t}$, a context $\vartheta(\boldsymbol{y})$ at $\boldsymbol{t}$, and a proposition $P(\boldsymbol{x})$ in the context $\vartheta(\boldsymbol{\alpha}(\boldsymbol{x})), \boldsymbol{\alpha}\downarrow$ Then define

$$
(\exists_{\boldsymbol{\alpha}} P)(\boldsymbol{y}|\vartheta(\boldsymbol{y})) \;=\; \exists \boldsymbol{x}.P(\boldsymbol{\alpha}(\boldsymbol{x})|\vartheta(\boldsymbol{y}), \boldsymbol{y} = \boldsymbol{\alpha}(\boldsymbol{x}))
$$

The proof of the following is routine.

Lemma 2. *Given, besides the above, a proposition $Q(y|\vartheta(\boldsymbol{y}))$ in the context $\vartheta(\boldsymbol{y} : \boldsymbol{t})$, the following rules are admissible:*

$$
\frac{\vartheta(\boldsymbol{\alpha}(\boldsymbol{x})), \boldsymbol{\alpha}\downarrow \mid P(\boldsymbol{x} : \boldsymbol{s}) \vdash Q(\boldsymbol{\alpha}(\boldsymbol{x} : \boldsymbol{s}) : \boldsymbol{t})}{\vartheta(\boldsymbol{y} : \boldsymbol{t}) \mid (\exists_{\boldsymbol{\alpha}} P)(\boldsymbol{y} : \boldsymbol{t}) \vdash Q(\boldsymbol{y} : \boldsymbol{t})}
$$

Now we argue as follows. Consider a Davidsonian model of our theory. Suppose that we have a proposition P at s which axiomatises some theory of how things are at s, and suppose that we have an action $s \xrightarrow{\alpha} t$ between s and t. We are interested in how things are at t after α, given that P is true at s: we may assume (restricting P if necessary) that α is performable in all of the worlds in the domain of definition of P. Then, given a possible world x at s at which α is performable, a proposition Q at t (defined on all of the worlds which result from executing α) is true in the possible world resulting from the execution of α iff $Q(\alpha x)$ is true at x: and this will be the case in all of the possible worlds for which P is true iff, in our model,

$$
[\![P(x)]\!]_{\vartheta(x)} \subseteq [\![Q(\alpha(x))]\!]_{\vartheta(x)\alpha\simeq\alpha}
$$

Now this will be the case in all models iff

$$\vartheta(x) \mid P(x) \vdash Q(\alpha(x))$$

However, by Lemma 2, this is equivalent to

$$\vartheta(x) \mid \exists_\alpha.P(x) \vdash Q(y);$$

so, $\exists_\alpha.P(x)$ is a proposition at t which describes the set of possible worlds resulting from the execution of α in all of the worlds for which P is true at s. That is, $\exists_\alpha.P(x)$ is, in Reiter's terms, the progression of P by the action α. There is a more formal treatment of this argument in (White, n.d., §4.1).

5 Conclusions

5.1 Comparison with Reiter's System

It may help to compare our system with Reiter's. His system has very similar notation: propositions about states have an argument place for holding state-related information (situations for him, possible worlds for us), and actions are represented as functions. However, there are differences: as well as the difference between situation arguments and our possible world arguments, his primitive relations have a single situation argument, whereas we allow genuine relations between different possible worlds at each state. He does use equalities between actions (see, for example, the formulae on (Reiter, 2001, pp. 28f)), but these are equalities between action tokens rather than between the values of the functions representing actions, as ours are. He represents regression more or less explicitly, using a defined operator on a fragment of his system, and this is more or less the same treatment as ours. However, he has no explicit progression operator: he has a criterion for when a particular formula is a progression of another formula, but he does not have possible world variables, and so cannot, as we do, define progression in terms of quantification.

5.2 Davidson

Davidson uses the admissibility of equations between actions to argue for some ambitious theses to do with the metaphysics of action: briefly, he wants action tokens to be first-class individuals because, according to Quine (Quine, 1960; Quine, 1969c; Quine, 1969a), the admissibility of equational reasoning about entities of a certain sort is the chief criterion for those entities to be first-class individuals. It is not clear that our work supports the conclusions that Davidson wants to draw: our system handles the sort of reasoning that Davidson uses as evidence, but it uses equations, not between action tokens, but between possible worlds.

5.3 The Theoretical Background of this Work

It might not be immediately clear how the sequent calculus was arrived at: it is a fairly complex thing, and writing down the rules is trickier than one might expect. However, the paper (White, n.d.) was actually written first: it was an attempt to understand Reiter's work, using the tools of fibred category theory, and Lemma 2 was one of the key ingredients. It turned out that, in order to have a proof theory the actions needed more structure than Reiter was using: this structure, when made suitably concrete, was precisely the equational reasoning that Davidson uses.

There is a moral here. The phenomenology of action is surprisingly difficult: we have few reliable intuitions about the structure of our actions. A good way of proceeding is to start with what we understand better, namely logic, and see if it is possible to construct a logic with good theoretical properties using some treatment of action as a basis. The mathematical structures that one has to postulate in order to set up such a logic may well have something significant to say about action: they should not, of course, be postulated on mathematical grounds alone, but the structures that mathematics suggests to us are certainly worth investigating to see if they might conceivably correspond to anything in reality.

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