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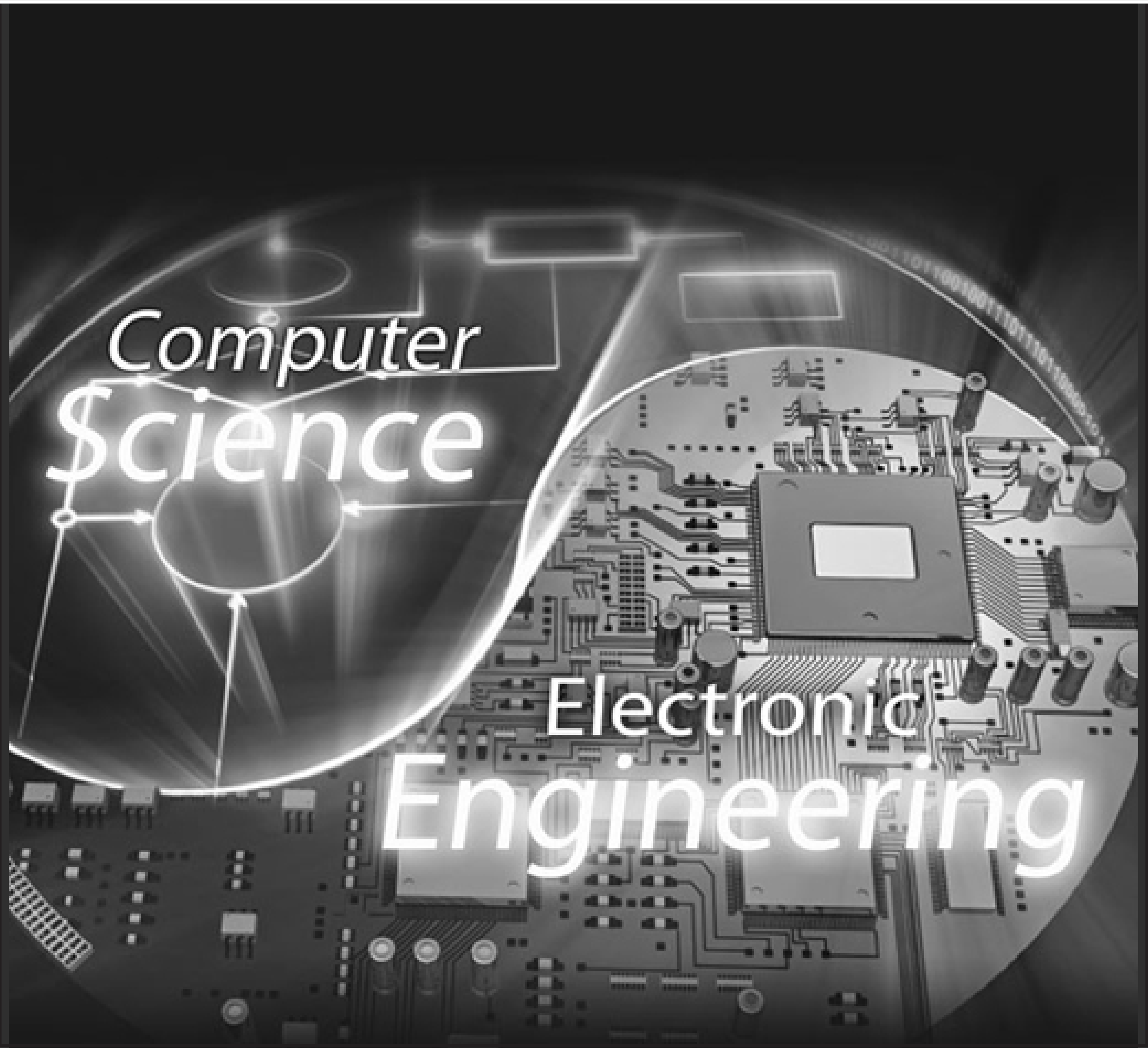

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Roland Badeau, Mark D. Plumbley

***Abstract*—Several probabilistic models involving latent components have been proposed for modelling time-frequency (TF) representations of audio signals such as spectrograms, notably in the nonnegative matrix factorization (NMF) literature. Among them, the recent high resolution NMF (HR-NMF) model is able to take both phases and local correlations in each frequency band into account, and its potential has been illustrated in applications such as source separation and audio inpainting. In this paper, HR-NMF is extended to multichannel signals and to convolutive mixtures. The new model can represent a variety of stationary and non-stationary signals, including autoregressive moving average (ARMA) processes and mixtures of damped sinusoids. A fast variational expectation-maximization (EM) algorithm is proposed to estimate the enhanced model. This algorithm is applied to a stereophonic piano signal, and proves capable of accurately modelling reverberation and restoring missing observations.**

***Index Terms*—Non-stationary signal modelling, Time-frequency analysis, Nonnegative matrix factorisation, Multichannel signal analysis, Variational EM algorithm.**

I. Introduction

NONNEGATIVE matrix factorisation was originally introduced as a rank-reduction technique, which approximates a non-negative matrix $\boldsymbol{V} \in \mathbb{R}^{F\times T}$ as a product $\boldsymbol{V} \approx \boldsymbol{W}\boldsymbol{H}$ of two non-negative matrices $\boldsymbol{W} \in \mathbb{R}^{F\times S}$ and $\boldsymbol{H} \in \mathbb{R}^{S\times T}$ with $S < \min(F,T)$ [1]. In audio signal processing, it is often used for decomposing a magnitude or power TF representation, such as a Fourier or a constant-Q transform (CQT) spectrogram. The columns of $\boldsymbol{W}$ are then interpreted as a dictionary of spectral templates, whose temporal activations are represented in the rows of $\boldsymbol{H}$. Several applications to audio have been addressed, such as multi-pitch estimation [2]–[4], automatic music transcription [5], [6], musical instrument recognition [7], and source separation [8]–[10].

In the literature, several probabilistic models involving latent components have been proposed to provide a probabilistic framework to NMF. Such models include NMF with additive Gaussian noise [11], probabilistic latent component analysis (PLCA) [12], NMF as a sum of Poisson components [13], and NMF as a sum of Gaussian components [14]. Although they have already proved successful in a number of audio applications such as source separation [11]–[13] and multipitch estimation [14], most of these models still lack of consistency in some respects.

Firstly, they focus on modelling a magnitude or power TF representation, and simply ignore the phase information. In an application of source separation, the source estimates are then obtained by means of Wiener-like filtering [8]–[10], which consists in applying a mask to the magnitude TF representation of the mixture, while keeping the phase field unchanged. It can be easily shown that this approach cannot properly separate sinusoidal signals lying in the same frequency band, which means that the frequency resolution is limited by that of the TF transform. In other respects, the separated TF representation is generally not consistent, which means that it does not correspond to the TF transform of a temporal signal, resulting in artefacts such as musical noise. Therefore enhanced algorithms are needed to reconstruct a consistent TF representation [15]. In the same way, in an application of model-based audio synthesis, where there is no available phase field to assign to the sources, reconstructing consistent phases requires employing ad-hoc methods [16], [17].

Secondly, these models generally focus on the spectral and temporal dynamics, and assume that all time-frequency bins are independent. This assumption is clearly not relevant in the case of sinusoidal or impulse signals for instance, and it is not consistent with the existence of spectral or temporal dynamics. Indeed, in the case of wide sense stationary (WSS) processes, spectral dynamics (described by the power spectral density) is closely related to temporal correlation (described by the autocovariance function). Reciprocally, in the case of uncorrelated processes (all samples are uncorrelated with different variances), temporal dynamics induces spectral correlation. In other respects, further dependencies in the TF domain may be induced by the TF transform, due to spectral and temporal overlap between TF bins.

In order to overcome the assumption of independent TF bins, Markov models have been introduced for taking the local dependencies between contiguous TF bins of a magnitude or power TF representation into account [18]–[20]. However, these models still ignore the phase information. Conversely, the complex NMF model [21], [22], which was explicitly designed to represent phases alongside magnitudes in a TF representation, is based on a deterministic framework that does not represent statistical correlations.

In order to model both phases and correlations within frequency bands in a principled way, we introduced in [23], [24] a new model called high resolution (HR) NMF. We showed

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that this model offers an improved frequency resolution, able to separate sinusoids within the same frequency band, and an improved synthesis capability, able to restore missing TF observations. It can be used with both complex-valued and real-valued TF representations, such as the short-time Fourier transform (STFT) and the modified discrete cosine transform (MDCT). It also generalizes some popular models, such as the Itakura-Saito NMF model (IS-NMF) [14], autoregressive (AR) processes [25], and the exponential sinusoidal model (ESM), commonly used in HR spectral analysis of time series [25].

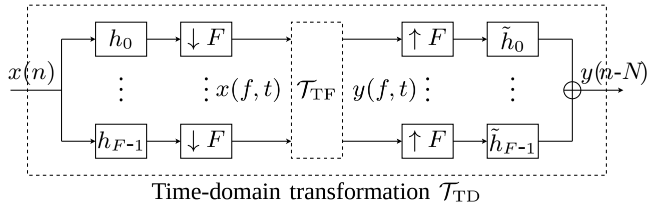
In this paper, HR-NMF is extended to multichannel signals and to convolutive mixtures. Contrary to the multichannel NMF [26] where convolution was approximated, convolution is here accurately implemented in the TF domain by following the exact approach proposed in [27]. Consequently, correlations *within* and *between* frequency bands are both taken into account. In order to estimate this multichannel HR-NMF model, we propose a fast variational EM algorithm. This paper further develops a previous work presented in [28], by providing a theoretical ground for the TF implementation of convolution.

The paper is structured as follows. The filter bank used to compute the TF representation is presented in Section II. We then show in Section III how convolutions in the original time domain can be accurately implemented in the TF domain. The multichannel HR-NMF model is introduced in section IV, and the variational EM algorithm is derived in section V. An application to a stereophonic piano signal is presented in section VI. Finally, conclusions are drawn in section VII.

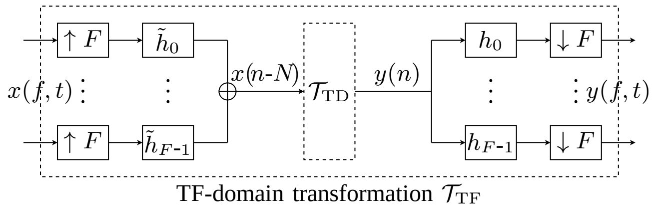
II. Time-frequency analysis

In the literature, the STFT [29] is often preferred to other existing TF transforms, because under some smoothness assumptions it allows the approximation of linear filtering by multiplying each column of the STFT by the frequency response of the filter. Instead, we propose to use the more general and flexible framework of perfect reconstruction (PR) filter banks [29]. Indeed, we will show in Section III that PR actually permits us to *accurately* implement convolutions in the TF domain.

We thus consider a filter bank [29], which transforms an input signal $x(n) \in l^\infty(\mathbb{F})$ in the original time domain $n \in \mathbb{Z}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $l^\infty(\mathbb{F})$ denotes the space of bounded sequences on $\mathbb{F}$) into a 2D-array $x(f,t) \in l^\infty(\mathbb{F})$ $\forall f \in [0 \dots F-1]$ in the TF domain $(f,t) \in [0 \dots F-1] \times \mathbb{Z}$. More precisely, $x(f,t)$ is defined as $x(f,t) = (h_f * x)(Dt)$, where D is the decimation factor, $*$ denotes standard convolution, and $h_f(n)$ is an analysis filter of support $[0 \dots N-1]$ with $N = LD$ and $L \in \mathbb{N}$. The synthesis filters $\tilde{h}_f(n)$ of same support $[0 \dots N-1]$ are designed so as to guarantee PR. This means that the output, defined as $x'(n) = \sum_{f=0}^{F-1} \sum_{t \in \mathbb{Z}} \tilde{h}_f(n-Dt)x(f,t)$, satisfies $x'(n) = x(n-N)$, which corresponds to an overall delay of N samples. Let $H_f(\nu) = \sum_{n \in \mathbb{Z}} h_f(n) e^{-2i\pi\nu n}$ (with an upper case letter) denote the discrete time Fourier transform (DTFT) of $h_f(n)$ over $\nu \in \mathbb{R}$. Considering that the time supports of $h_f(Dt_1 - n)$ and $h_f(Dt_2 - n)$ do not overlap provided that $|t_1 - t_2| \geq L$, we similarly define a whole number K, such that the overlap between the frequency supports of $H_{f_1}(\nu)$ and $H_{f_2}(\nu)$ can be neglected provided that $|f_1 - f_2| \geq K$, due to high rejection in the stopband.



(a) Applying a TF transformation to a TD signal



(b) Applying a TD transformation to TF data

Fig. 1. Time-frequency vs. time domain transformations

III. TF implementation of convolution

In this section, we consider a stable filter of impulse response $g(n) \in l^1(\mathbb{F})$ (where $l^1(\mathbb{F})$ denotes the space of sequences on $\mathbb{F}$ whose series is absolutely convergent) and two signals $x(n) \in l^\infty(\mathbb{F})$ and $y(n) \in l^\infty(\mathbb{F})$, such that $y(n) = (g * x)(n)$. Our purpose is to directly express the TF representation $y(f,t)$ of $y(n)$ as a function of $x(f,t)$, *i.e.* to find a TF transformation $\mathcal{T}_{\mathrm{TF}}$ in Figure 1(a) such that if the input of the filter bank is $x(n)$, then the output is $y(n-N)$ (y is delayed by N samples in order to take the overall delay of the filter bank into account). The following developments further investigate and generalize the study presented in [27], which focused on the particular case of critically sampled PR cosine modulated filterbanks. The general case of stable linear filters is first addressed in section III-A, then the particular case of stable recursive filters is addressed in section III-B.

A. Stable linear filters

The PR property of the filter bank implies that the relationship between $y(f,t)$ and $x(f,t)$ is given by the transformation $\mathcal{T}_{\mathrm{TF}}$ described in the larger frame in Figure 1(b), where the input is $x(f,t)$, the output is $y(f,t)$, and transformation $\mathcal{T}_{\mathrm{TD}}$ is defined as the time-domain convolution by $g(n+N)$. The resulting mathematical expression is given in Proposition 1.

Proposition 1. *Let $g(n) \in l^1(\mathbb{F})$ be the impulse response of a stable linear filter, and $x(n) \in l^\infty(\mathbb{F})$ and $y(n) \in l^\infty(\mathbb{F})$ two signals such that $y(n) = (g * x)(n)$. Let $y(f,t)$ and $x(f,t)$ be the TF representations of these signals as defined in Section II. Then*

$$y(f,t) = \sum_{\varphi \in \mathbb{Z}} \sum_{\tau \in \mathbb{Z}} c_g(f,\varphi,\tau)\; x(f-\varphi, t-\tau) \qquad (1)$$

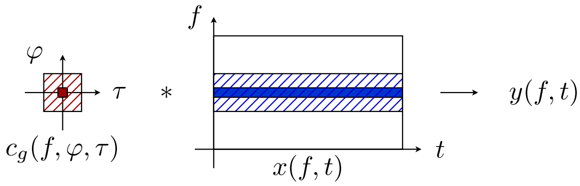


Fig. 2. TF implementation of convolution

where $\forall f \in [0 \dots F-1]$, $\forall \varphi \in \mathbb{Z}$, $\forall \tau \in \mathbb{Z}$,

$$c_g(f,\varphi,\tau) = (h_f * \tilde{h}_{f-\varphi} * g)(D(\tau+L)), \quad (2)$$

with the convention $\forall f \notin [0 \dots F-1]$, $h_f = 0$.

Proof: Equations (1) and (2) are obtained by successively substituting equations $y(n) = (g * x)(n)$ and $x(n) = \sum_{f=0}^{F-1} \sum_{t \in \mathbb{Z}} \tilde{h}_f(n\text{-}D(t\text{-}L))x(f,t)$ into $y(f,t) = (h_f * y)(Dt)$. ■

Remark 1. As mentioned in section II, if $|\varphi| \geq K$, then subbands f and $f-\varphi$ do not overlap, thus $c_g(f,\varphi,\tau)$ can be neglected.

Equation (1) shows that a convolution in the original time domain is equivalent to a 2D-convolution in the TF domain, which is stationary w.r.t. time, and non-stationary w.r.t. frequency, as illustrated in Figure 2.

B. Stable recursive filters

In this section, we introduce a parametric family of TF filters based on a state space representation, and we show a relationship between these TF filters and equation (1).

Definition 1. *Stable recursive filtering in TF domain is defined by the following state space representation:*
$\forall f \in [0 \dots F-1], t \in \mathbb{Z}$,

$$\begin{aligned} z(f,t) &= x(f,t) - \sum_{\tau=1}^{Q_a} a_g(f,\tau) z(f,t-\tau) \\ y(f,t) &= \sum_{\varphi=-P_b}^{P_b} \sum_{\tau \in \mathbb{Z}} b_g(f,\varphi,\tau)\ z(f-\varphi,t-\tau) \end{aligned} \quad (3)$$

where $Q_a \in \mathbb{N}$, $P_b \in \mathbb{N}$, *and* $\forall f \in [0 \dots F-1]$, $x(f,t) \in l^\infty(\mathbb{F})$ *is the sequence of input variables,* $z(f,t) \in l^\infty(\mathbb{F})$ *is the sequence of state variables, and* $y(f,t) \in l^\infty(\mathbb{F})$ *is the sequence of output variables. The autoregressive term* $a_g(f,\tau) \in \mathbb{F}$ *is a causal sequence of support* $[0 \dots Q_a]$ *w.r.t.* τ *(with* $a_g(f,0) = 1$*), having only simple poles lying inside the unit circle. The moving average term* $b_g(f,\varphi,\tau) \in \mathbb{F}$ *is a sequence of finite support w.r.t.* τ*, and* $\forall f \in [0 \dots F-1]$, $\forall \varphi \in [-P_b \dots P_b]$, $b_g(f,\varphi,\tau) = 0$ *provided that* $f - \varphi \notin [0 \dots F-1]$.

Proposition 2. *If* $g(n) \in l^1(\mathbb{F})$ *is the impulse response of a causal and stable recursive filter, then the TF input/output system defined in Proposition 1 admits the state space representation* (3)*, where* $P_b = K-1$ *and* $\forall f \in [0 \dots F-1]$, $\forall \varphi \in [-P_b, P_b]$, $b_g(f,\varphi,\tau)$ *is a sequence of support* $[-L+1 \dots -L+1+Q_b]$ *w.r.t.* τ*, where* $Q_b \geq 2L + Q_a - 1$.

Proposition 2 is proved in Appendix A.

Proposition 3. *In Definition 1, equation* (3) *can be rewritten in the form of equation* (1)*, where* $\forall f \in [0 \dots F-1]$, $\forall \tau \in \mathbb{Z}$, $c_g(f,\varphi,\tau) = 0$ *if* $|\varphi| > P_b$*, and* $\forall f \in [0 \dots F-1]$, $\forall \varphi \in [-P_b \dots P_b]$*, filter* $c_g(f,\varphi,\tau)$ *is defined as the only stable (bounded-input, bounded-output) solution of the following recursion:*

$$\forall \tau \in \mathbb{Z}, \ \sum_{u=0}^{Q_a} a_g(f-\varphi,u) c_g(f,\varphi,\tau-u) = b_g(f,\varphi,\tau). \quad (4)$$

Proposition 3 is proved in Appendix A.

Remark 2. In Definition 1, $a_g(f,\tau)$ and $b_g(f,\varphi,\tau)$ are overparametrised compared to $g(n)$ in Proposition 1. Consequently, if $a_g(f,\tau)$ and $b_g(f,\varphi,\tau)$ are arbitrary, then it is possible that no filter $g(n)$ exists such that equation (2) holds, which means that this state space representation does no longer correspond to a convolution in the original time domain. In this case, we will say that the TF transformation defined in equation (3) is *inconsistent*.

IV. MULTICHANNEL HR-NMF

In this section we present the multichannel HR-NMF model, initially introduced in [28]. Before defining HR-NMF in the TF domain in section IV-B, we first provide an intuitive interpretation of this model in the time domain.

A. HR-NMF in the time domain

The HR-NMF model of a multichannel signal $y_m(n) \in \mathbb{F}$ is defined for all channels $m \in [0 \dots M-1]$ and times $n \in \mathbb{Z}$, as the sum of S source images $y_{ms}(n) \in \mathbb{F}$ plus a Gaussian noise $n_m(n) \in \mathbb{F}$: $y_m(n) = n_m(n) + \sum_{s=0}^{S-1} y_{ms}(n)$. Moreover, each source image $y_{ms}(f,t)$ for any $s \in [0 \dots S-1]$ is defined as $y_{ms}(n) = (g_{ms} * x_s)(n)$, where g_{ms} is the impulse response of a causal and stable recursive filter, and $x_s(n)$ is a Gaussian process[1]. Additionally, processes x_s and n_m for all s and m are mutually independent. In order to make this model identifiable, we will further assume that the spectrum of $x_s(n)$ is flat, because the variability of source s w.r.t. frequency can be modelled within filters g_{ms} for all m. Thus filter g_{ms} represents both the transfer from source s to sensor m and the spectrum of source s. In section IV-B, recursive filters g_{ms} will be directly implemented in the TF domain via equations (7) and (8), following Definition 1[2].

B. HR-NMF in the TF domain

The multichannel HR-NMF model of TF data $y_m(f,t) \in \mathbb{F}$ is defined for all channels $m \in [0 \dots M-1]$, discrete frequencies $f \in [0 \dots F-1]$, and times $t \in [0 \dots T-1]$, as the sum of S source images $y_{ms}(f,t) \in \mathbb{F}$ plus a 2D-white noise

$$n_m(f,t) \sim \mathcal{N}_{\mathbb{F}}(0, \sigma_y^2), \quad (5)$$

[1]The probability distributions of processes $n_m(n)$ and $x_s(n)$ will be defined in the TF domain in section IV-B.

[2]More precisely, compared to the result of Proposition 2, processes $z_s(f,t)$ and $x_s(f,t)$ as defined in section IV-B are shifted $L-1$ samples backward, in order to write $b_{ms}(f,\varphi,\tau)$ in a causal form. This does not alter the definition of HR-NMF, since equation (8) is unaltered by this time shift, and $y_{ms}(f,t)$ is unchanged in equation (7).

where $\mathcal{N}_{\mathbb{F}}(0,\sigma_y^2)$ denotes a real (if $\mathbb{F}=\mathbb{R}$) or circular complex (if $\mathbb{F}=\mathbb{C}$) normal distribution of mean 0 and variance σ_y^2:

$$y_m(f,t) = n_m(f,t) + \sum_{s=0}^{S-1} y_{ms}(f,t). \tag{6}$$

Each source image $y_{ms}(f,t)$ for $s \in [0 \dots S-1]$ is defined as

$$y_{ms}(f,t) = \sum_{\varphi=-P_b}^{P_b} \sum_{\tau=0}^{Q_b} b_{ms}(f,\varphi,\tau)\; z_s(f-\varphi, t-\tau) \tag{7}$$

where $P_b, Q_b \in \mathbb{N}$, $b_{ms}(f,\varphi,\tau)=0$ if $f-\varphi \notin [0 \dots F-1]$, and the latent components $z_s(f,t) \in \mathbb{F}$ are defined as follows:

- $\forall t \in [0 \dots T-1]$, $x_s(f,t) \sim \mathcal{N}_{\mathbb{F}}(0, \sigma_{x_s}^2(t))$ and

$$z_s(f,t) = x_s(f,t) - \sum_{\tau=1}^{Q_a} a_s(f,\tau) z_s(f,t-\tau) \tag{8}$$

  where $Q_a \in \mathbb{N}$ and $a_s(f,\tau)$ defines a stable autoregressive filter,
- $\forall t \in [-Q_z \dots -1]$ where $Q_z = \max(Q_b, Q_a)$,

$$z_s(f,t) \sim \mathcal{N}(\mu_s(f,t), 1/\rho_s(f,t)). \tag{9}$$

Moreover, the random variables $n_m(f_1,t_1)$ and $x_s(f_2,t_2)$ for all s,m,f_1,f_2,t_1,t_2 are assumed mutually independent. Additionally, $\forall m \in [0 \dots M-1]$, $\forall f \in [0 \dots F-1]$, $\forall t \in [-Q_z \dots -1]$, $y_m(f,t)$ is unobserved, and $\forall s \in [0 \dots S-1]$, the prior mean $\mu_s(f,t) \in \mathbb{F}$ and the prior precision (inverse variance) $\rho_s(f,t) > 0$ of the latent variable $z_s(f,t)$ are considered to be fixed parameters.

The set θ of parameters to be estimated consists of:

- the **autoregressive parameters** $a_s(f,\tau) \in \mathbb{F}$ for $s \in [0 \dots S-1]$, $f \in [0 \dots F-1]$, $\tau \in [1 \dots Q_a]$ (we further define $a_s(f,0)=1$),
- the **moving average parameters** $b_{ms}(f,\varphi,\tau) \in \mathbb{F}$ for $m \in [0 \dots M-1]$, $s \in [0 \dots S-1]$, $f \in [0 \dots F-1]$, $\varphi \in [-P_b \dots P_b]$, and $\tau \in [0 \dots Q_b]$,
- the **variance parameters** $\sigma_y^2 > 0$ and $\sigma_{x_s}^2(t) > 0$ for $s \in [0 \dots S-1]$ and $t \in [0 \dots T-1]$.

We thus have $\theta = \{\sigma_y^2, \sigma_{x_s}^2, a_s, b_{ms}\}_{s\in[0\dots S-1], m\in[0\dots M-1]}$. This model encompasses the following special cases:

- If $M=1$, $\sigma_y^2=0$ and $P_b=Q_b=Q_a=0$, then equation (6) reduces to $y_0(f,t) = \sum_{s=0}^{S-1} b_{0s}(f,0,0) x_s(f,t)$, thus $y_0(f,t) \sim \mathcal{N}_{\mathbb{F}}(0, \widehat{V}_{ft})$, where matrix $\widehat{\boldsymbol{V}}$ of coefficients $\widehat{V}_{ft}$ is defined by the NMF $\widehat{\boldsymbol{V}} = \boldsymbol{W}\boldsymbol{H}$ with $W_{fs} = |b_{0s}(f,0,0)|^2$ and $H_{st} = \sigma_{x_s}^2(t)$. The maximum likelihood estimation of $\boldsymbol{W}$ and $\boldsymbol{H}$ is then equivalent to the minimization of the Itakura-Saito (IS) divergence between matrix $\widehat{\boldsymbol{V}}$ and spectrogram $\boldsymbol{V}$ (where $V_{ft} = |y_0(f,t)|^2$), hence this model is referred to as **IS-NMF** [14].
- If $M=1$ and $P_b=Q_b=0$, then $y_0(f,t)$ follows the monochannel **HR-NMF** model [23], [24], [30] involving variance σ_y^2, autoregressive parameters $a_s(f,\tau)$ for all $s \in [0 \dots S-1]$, $f \in [0 \dots F-1]$ and $\tau \in [1 \dots Q_a]$, and the NMF $\widehat{\boldsymbol{V}} = \boldsymbol{W}\boldsymbol{H}$.
- If $S=1$, $\sigma_y^2=0$, $P_b=0$, $\sigma_{x_0}^2(t)=1\ \forall t \in [0 \dots T-1]$, and $\mu_s(f,t)=0$ and $\rho_s(f,t)=1\ \forall t \in [-Q_z \dots -1]$, then $\forall m \in [0 \dots M-1]$, $\forall f \in [0 \dots F-1]$, $y_m(f,t)$ is an autoregressive moving average **(ARMA) process** [25, Section 3.6].
- If $S=1$, $\sigma_y^2=0$, $P_b=0$, $Q_a>0$, $Q_b=Q_a-1$, $\forall t \in [-Q_z \dots -1]$, $\mu_0(f,t)=0$, $\rho_0(f,t) \gg 1$, and $\sigma_0^2(t) = \mathbb{1}_{\{t=0\}}$ (where $\mathbb{1}_{\mathcal{S}}$ denotes the indicator function of a set $\mathcal{S}$), then $\forall m \in [0 \dots M-1]$, $\forall f \in [0 \dots F-1]$, $y_m(f,t)$ can be written in the form $y_m(f,t) = \sum_{\tau=1}^{Q_a} \alpha_{m\tau}\, z_\tau(f)^t$, where $z_1(f) \dots z_{Q_a}(f)$ are the roots of the polynomial $z^{Q_a} + \sum_{\tau=1}^{Q_a} a_0(f,\tau) z^{Q_a-\tau}$. This corresponds to the **Exponential Sinusoidal Model (ESM)** commonly used in HR spectral analysis of time series [25].

Because it generalizes both IS-NMF and ESM models to multichannel data, the model defined in equation (6) is called multichannel HR-NMF.

V. Variational EM algorithm

In early works that focused on monochannel HR-NMF [23], [24], in order to estimate the model parameters we proposed to resort to an expectation-maximization (EM) algorithm based on a Kalman filter/smoother. The approach proved to be appropriate for modelling audio signals in applications such as source separation and audio inpainting. However, its computational cost was high, dominated by the Kalman filter/smoother, and prohibitive when dealing with high-dimensional signals.

In order to make the estimation of HR-NMF faster, we then proposed two different strategies. The first approach aimed to improve the convergence rate, by replacing the M-step of the EM algorithm by multiplicative update rules [31]. However we observed that the resulting algorithm presented some numerical stability issues. The second approach aimed to reduce the computational cost, by using a variational EM algorithm, where we introduced two different variational approximations [30]. We observed that the mean field approximation led to both improved performance and maximal decrease of computational complexity.

In this section, we thus generalize the variational EM algorithm based on mean field approximation to the multichannel HR-NMF model introduced in section IV-B, as proposed in [28]. Compared to [30], novelties also include a reduced computational complexity and a parallel implementation.

A. Review of variational EM algorithm

Variational inference [32] is now a classical approach for estimating a probabilistic model involving both observed variables y and latent variables z, determined by a set θ of parameters. Let $\mathcal{F}$ be a set of probability density functions (PDFs) over the latent variables z. For any PDF $q \in \mathcal{F}$ and any function $f(z)$, we note $\langle f \rangle_q = \int f(z) q(z) dz$. Then for any set of parameters θ, the *variational free energy* is defined as

$$\mathcal{L}(q;\theta) = \left\langle \ln\left(\frac{p(y,z;\theta)}{q(z)}\right)\right\rangle_q. \tag{10}$$

The variational EM algorithm is a recursive algorithm for estimating θ. It consists of the two following steps at each iteration i:

- Expectation (E)-step (update q):

$$q^\star = \underset{q\in\mathcal{F}}{\text{argmax}}\ \mathcal{L}(q;\theta_{i-1}) \tag{11}$$

- Maximization (E)-step (update θ):

$$\theta_i = \underset{\theta}{\text{argmax}}\ \mathcal{L}(q^\star;\theta). \tag{12}$$

In the case of multichannel HR-NMF, θ has been specified in section IV-B. We further define $\delta_m(f,t) = 1$ if $y_m(f,t)$ is observed, otherwise $\delta_m(f,t) = 0$, in particular $\delta_m(f,t) = 0$ $\forall (f,t) \notin [0\dots F-1]\times[0\dots T-1]$. The complete set of variables consists of:

- the set y of **observed variables** $y_m(f,t)$ for $m \in [0\dots M-1]$ and for all f and t such that $\delta_m(f,t) = 1$,
- the set z of **latent variables** $z_s(f,t)$ for $s\in[0\dots S-1]$, $f\in[0\dots F-1]$, and $t\in[-Q_z\dots T-1]$.

We use a *mean field approximation* [32]: $\mathcal{F}$ is defined as the set of PDFs which can be factorized in the form

$$q(z) = \prod_{s=0}^{S-1}\prod_{f=0}^{F-1}\prod_{t=-Q_z}^{T-1} q_{sft}(z_s(f,t)). \tag{13}$$

With this particular factorization of $q(z)$, the solution of (11) is such that each PDF q_{sft} is Gaussian: $z_s(f,t) \sim \mathcal{N}_{\mathbb{F}}(\overline{z_s}(f,t), \gamma_{z_s}(f,t))$.

B. Variational free energy

Let $\alpha = 1$ if $\mathbb{F} = \mathbb{C}$, and $\alpha = 2$ if $\mathbb{F} = \mathbb{R}$. Let $D_y = \sum_{m=0}^{M-1}\sum_{f=0}^{F-1}\sum_{t=0}^{T-1}\delta_m(f,t)$ be the number of observations, and

$$\begin{aligned}
I(f,t) &= \mathbb{1}_{\{0\leq f<F,\, 0\leq t<T\}},\\
e_{y_m}(f,t) &= \delta_m(f,t)\left(y_m(f,t) - \sum_{s=0}^{S-1} y_{ms}(f,t)\right),\\
x_s(f,t) &= I(f,t)\Big(\sum_{\tau=0}^{Q_a} a_s(f,\tau)z_s(f,t-\tau)\Big).
\end{aligned}$$

Then using equations (5) to (9), the joint log-probability distribution $L = \log(p(y,z;\theta))$ of the complete set of variables satisfies

$$\begin{aligned}
-\alpha L = &\ -\alpha\left(\ln(p(y|z;\theta)) + \ln(p(z;\theta))\right)\\
= &\ (D_y + SF(T+Q_z))\ln(\alpha\pi)\\
&+ D_y\ln(\sigma_y^2) + \tfrac{1}{\sigma_y^2}\sum_{m=0}^{M-1}\sum_{f=0}^{F-1}\sum_{t=0}^{T-1}|e_{y_m}(f,t)|^2\\
&+ \sum_{s=0}^{S-1}\sum_{f=0}^{F-1}\sum_{t=-Q_z}^{-1}\ln\!\left(\tfrac{1}{\rho_s(f,t)}\right)\\
&+ \sum_{s=0}^{S-1}\sum_{f=0}^{F-1}\sum_{t=-Q_z}^{-1}\rho_s(f,t)|z_s(f,t)\text{-}\mu_s(f,t)|^2\\
&+ \sum_{s=0}^{S-1}\sum_{f=0}^{F-1}\sum_{t=0}^{T-1}\ln(\sigma_{x_s}^2(t)) + \tfrac{1}{\sigma_{x_s}^2(t)}|x_s(f,t)|^2.
\end{aligned}$$

Thus the variational free energy defined in (10) satisfies

$$\begin{aligned}
-\alpha\mathcal{L} = &\ D_y\ln(\alpha\pi) - SF(T+Q_z)\\
&+ D_y\ln(\sigma_y^2) + \sum_{m=0}^{M-1}\sum_{f=0}^{F-1}\sum_{t=0}^{T-1}\tfrac{\gamma_{e_{y_m}}(f,t)+|\overline{e}_{y_m}(f,t)|^2}{\sigma_y^2}\\
&+ \sum_{s=0}^{S-1}\sum_{f=0}^{F-1}\sum_{t=-Q_z}^{-1} - \ln(\rho_s(f,t)\gamma_{z_s}(f,t))\\
&+ \rho_s(f,t)\left(\gamma_{z_s}(f,t) + |\overline{z}_s(f,t)-\mu_s(f,t)|^2\right)\\
&+ \sum_{s=0}^{S-1}\sum_{f=0}^{F-1}\sum_{t=0}^{T-1}\ln\left(\tfrac{\sigma_{x_s}^2(t)}{\gamma_{z_s}(f,t)}\right) + \tfrac{\gamma_{x_s}(f,t)+|\overline{x}_s(f,t)|^2}{\sigma_{x_s}^2(t)}
\end{aligned} \tag{14}$$

where $\forall f\in[0\dots F-1]$, $\forall t\in[0\dots T-1]$,

$$\begin{aligned}
&\gamma_{e_{y_m}}(f,t) = \delta_m(f,t)\sum_{s=0}^{S-1}\gamma_{y_{ms}}(f,t),\\
&\gamma_{y_{ms}}(f,t) = \sum_{\varphi=-P_b}^{P_b}\sum_{\tau=0}^{Q_b}|b_{ms}(f,\varphi,\tau)|^2\gamma_{z_s}(f-\varphi,t-\tau),\\
&\overline{e}_{y_m}(f,t) = \delta_m(f,t)\left(y_m(f,t) - \sum_{s=0}^{S-1}\overline{y}_{ms}(f,t)\right),\\
&\overline{y}_{ms}(f,t) = \sum_{\varphi=-P_b}^{P_b}\sum_{\tau=0}^{Q_b} b_{ms}(f,\varphi,\tau)\ \overline{z}_s(f-\varphi,t-\tau),\\
&\gamma_{x_s}(f,t) = I(f,t)\Big(\sum_{\tau=0}^{Q_a}|a_s(f,\tau)|^2\gamma_{z_s}(f,t-\tau)\Big),\\
&\overline{x}_s(f,t) = I(f,t)\Big(\sum_{\tau=0}^{Q_a}a_s(f,\tau)\overline{z}_s(f,t-\tau)\Big).
\end{aligned}$$

C. Variational EM algorithm for multichannel HR-NMF

According to the mean field approximation, the maximizations in equations (11) and (12) are performed for each scalar parameter in turn [32]. The dominant complexity of each iteration of the resulting variational EM algorithm is $4MFST\Delta f\Delta t$, where $\Delta f = 1 + 2P_b$ and $\Delta t = 1 + Q_z$. However we highlight a possible parallel implementation, by making a difference between **parfor** loops which can be implemented in parallel, and **for** loops which have to be implemented sequentially.

1) E-step: For all $s\in[0\dots S-1]$, $f\in[0\dots F-1]$, $t\notin[-Q_z,-1]$, let $\rho_s(f,t) = 0$. Considering the mean field approximation (13), the E-step defined in equation (11) leads to the updates described in Table I (where $*$ denotes complex conjugation). Note that $\overline{z}_s(f,t)$ has to be updated after $\gamma_{z_s}(f,t)$.

2) M-step: The M-step defined in (12) leads to the updates described in Table II. The updates of the four parameters can be processed in parallel.

VI. Simulation results

In this section, we present a basic proof of concept of the multichannel HR-NMF model introduced in section IV-B. The following experiments deal with a single source ($S = 1$) formed of a real piano sound sampled at 11025 Hz. A 1.25ms-short stereophonic signal ($M = 2$) has been synthesized by filtering the monophonic recording of a C3 piano note with two room impulse responses simulated using the Matlab code

parfor $s \in [0 \dots S\text{-}1]$, $f \in [0 \dots F\text{-}1]$, $t \in [\text{-}Q_z \dots T\text{-}1]$ **do**

$$\gamma_{z_s}(f,t)^{-1} = \rho_s(f,t) + \sum_{\tau=0}^{Q_a} \frac{I(f,t+\tau)|a_s(f,\tau)|^2}{\sigma_{x_s}^2(t+\tau)} + \sum_{m=0}^{M-1} \sum_{\varphi=-P_b}^{P_b} \sum_{\tau=0}^{Q_b} \frac{\delta_m(f+\varphi,t+\tau)|b_{ms}(f+\varphi,\varphi,\tau)|^2}{\sigma_y^2}$$

end parfor

for $s \in [0 \dots S\text{-}1]$, $f_0 \in [0 \dots \Delta f\text{-}1]$, $t_0 \in [\text{-}Q_z \dots \text{-}Q_z + \Delta t\text{-}1]$ **do**

parfor $\frac{f-f_0}{\Delta f} \in [0 \dots \lfloor \frac{F-1-f_0}{\Delta_f} \rfloor]$, $\frac{t-t_0}{\Delta t} \in [0 \dots \lfloor \frac{T-1-t_0}{\Delta t} \rfloor]$ **do**

$$\overline{z}_s(f,t) = \overline{z}_s(f,t) - \gamma_{z_s}(f,t)\Big(\rho_s(f,t)(\overline{z}_s(f,t) - \mu_s(f,t)) + \sum_{\tau=0}^{Q_a} \frac{a_s(f,\tau)^* \, \overline{x}_s(f,t+\tau)}{\sigma_{x_s}^2(t+\tau)} - \sum_{m=0}^{M-1} \sum_{\varphi=-P_b}^{P_b} \sum_{\tau=0}^{Q_b} \frac{b_{ms}(f+\varphi,\varphi,\tau)^* \, \overline{e}_{y_m}(f+\varphi,t+\tau)}{\sigma_y^2}\Big)$$

end parfor

end for

TABLE I
E-STEP OF THE VARIATIONAL EM ALGORITHM

$$\sigma_y^2 = \tfrac{1}{D_y} \sum_{m=0}^{M-1} \sum_{f=0}^{F-1} \sum_{t=0}^{T-1} \gamma_{e_{y_m}}(f,t) + |\overline{e}_{y_m}(f,t)|^2$$

parfor $s \in [0 \dots S-1]$, $t \in [0 \dots T-1]$ **do**

$$\sigma_{x_s}^2(t) = \tfrac{1}{F} \sum_{f=0}^{F-1} \gamma_{x_s}(f,t) + |\overline{x}_s(f,t)|^2$$

end parfor

for $\tau \in [1 \dots Q_a]$ **do**

parfor $s \in [0 \dots S-1]$, $f \in [0 \dots F-1]$ **do**

$$a_s(f,\tau) = \frac{\sum_{t=0}^{T-1} \frac{1}{\sigma_{x_s}^2(t)} \big(\overline{z}_s(f,t-\tau)^* (a_s(f,\tau)\overline{z}_s(f,t-\tau) - \overline{x}_s(f,t))\big)}{\sum_{t=0}^{T-1} \frac{1}{\sigma_{x_s}^2(t)} \big(\gamma_{z_s}(f,t-\tau) + |\overline{z}_s(f,t-\tau)|^2\big)}$$

end parfor

end for

for $s \in [0 \dots S-1]$, $\varphi \in [-P_b \dots P_b]$, $\tau \in [0 \dots Q_b]$ **do**

parfor $m \in [0 \dots M\text{-}1]$, $f \in [\max(0,\varphi) \dots F\text{-}1+\min(0,\varphi)]$ **do**

$$b_{ms}(f,\varphi,\tau) = \frac{\sum_{t=0}^{T-1} \overline{z}_s(f\text{-}\varphi,t\text{-}\tau)^* \big(\delta_m(f,t) b_{ms}(f,\varphi,\tau)\overline{z}_s(f\text{-}\varphi,t\text{-}\tau) + \overline{e}_{y_m}(f,t)\big)}{\sum_{t=0}^{T-1} \delta_m(f,t)\big(\gamma_{z_s}(f\text{-}\varphi,t\text{-}\tau) + |\overline{z}_s(f\text{-}\varphi,t\text{-}\tau)|^2\big)}$$

end parfor

end for

TABLE II
M-STEP OF THE VARIATIONAL EM ALGORITHM

presented in [33][3]. The TF representation $y_m(f,t)$ of this signal has then been computed by applying a critically sampled PR cosine modulated filter bank ($\mathbb{F} = \mathbb{R}$) with $F = 201$ frequency bands, involving filters of length $8F = 1608$ samples. The resulting TF representation, of dimension $F \times T$ with $T = 77$, is displayed in Figure 3. In particular, it can be noticed that the two channels are not synchronous (the starting time in the left channel is ≈ 0.04s, whereas it is ≈ 0.02s in the right channel), which suggests that the order Q_b of filters $b_{ms}(f,\varphi,\tau)$ should be chosen greater than zero.

In the following experiments, we have set $\mu_s(f,t) = 0$ and $\rho_s(f,t) = 10^5$. These values force $\overline{z}_s(f,t)$ to be close to zero $\forall t \in [-Q_z \dots -1]$ (since the prior mean and variance of $\overline{z}_s(f,t)$ are $\mu_s(f,t) = 0$ and $1/\rho_s(f,t) = 10^{-5}$), which is relevant if the observed sound is preceded by silence. The variational EM algorithm is initialized with the neutral values $\overline{z}_s(f,t) = 0$, $\gamma_{z_s}(f,t) = \sigma_y^2 = \sigma_{x_s}^2(t) = 1$, $a_s(f,\tau) = \mathbb{1}_{\{\tau=0\}}$, and $b_{ms}(f,\varphi,\tau) = \mathbb{1}_{\{\varphi=0,\tau=0\}}$. In order to illustrate the capability of the multichannel HR-NMF model to synthesize realistic audio data, we address the case of missing observations. We suppose that all TF points within the frame in Figure 3 are unobserved: $\delta_m(f,t) = 0$ $\forall t \in [26 \dots 50]$ (which corresponds to the time range 0.47s-0.91s), and $\delta_m(f,t) = 1$ for all other t in $[0 \dots T-1]$. In each experiment, 100 iterations of the algorithm are performed, and the restored signal is returned as $\overline{y}_{ms}(f,t)$.

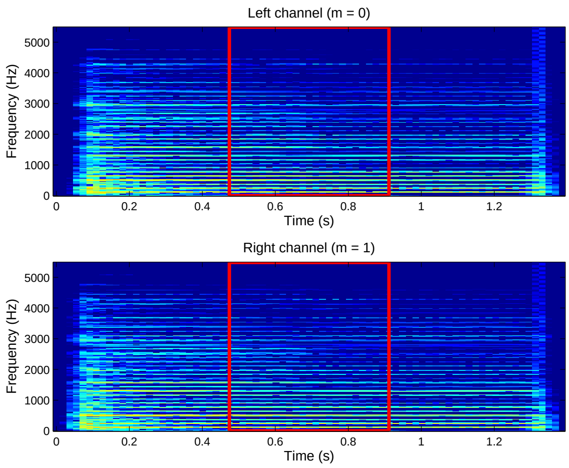


Fig. 3. Input stereo signal $y_m(f,t)$.

In the first experiment, a multichannel HR-NMF with $Q_a = Q_b = P_b = 0$ is estimated. Similarly to the example provided in section IV-B, this is equivalent to modelling the two channels by two rank-1 IS-NMF models [14] having distinct spectral atoms $\boldsymbol{W}$ and sharing the same temporal activation $\boldsymbol{H}$, or by a rank 1 multichannel NMF [26]. The resulting TF representation $\overline{y}_{ms}(f,t)$ is displayed in Figure 4. It can be noticed that wherever $y_m(f,t)$ is observed ($\delta_m(f,t) = 1$), $\overline{y}_{ms}(f,t)$ does not accurately fit $y_m(f,t)$ (this is particularly visible in high frequencies), because the length Q_b of filters $b_{ms}(f,\varphi,\tau)$ has been underestimated: the source to distortion ratio (SDR) in the observed area is 11.7dB. In other respects, the missing observations ($\delta_m(f,t) = 0$) could not be restored ($\overline{y}_{ms}(f,t)$ is zero inside the frame, resulting in an SDR of 0dB in this area), because the correlations between contiguous TF coefficients in $y_m(f,t)$ have not been taken into account.

In the second experiment, a multichannel HR-NMF model with $Q_a = 2$, $Q_b = 3$, and $P_b = 1$ is estimated. The resulting TF representation $\overline{y}_{ms}(f,t)$ is displayed in Figure 5. It can be noticed that wherever $y_m(f,t)$ is observed, $\overline{y}_{ms}(f,t)$ better fits $y_m(f,t)$: the SDR is 36.8dB in the observed area. Besides, the missing observations have been better estimated: the SDR is 4.8dB inside the frame. Actually, choosing $P_b > 0$ was necessary to obtain this result, which means that the spectral overlap between frequency bands cannot be neglected.

[3]Those impulse responses were simulated using 15625 virtual sources. The dimensions of the room were [20m, 19m, 21m], the coordinates of the two microphones were [19m, 18m 1.6m] and [15m, 11m, 10m], and those of the source were [5m, 2m, 1m]. The reflection coefficient of the walls was 0.3.

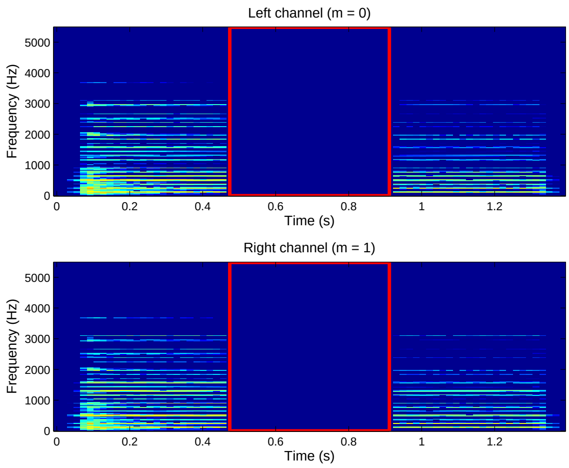


Fig. 4. Stereo signal $\overline{y}_{ms}(f,t)$ estimated with filters of length 1.

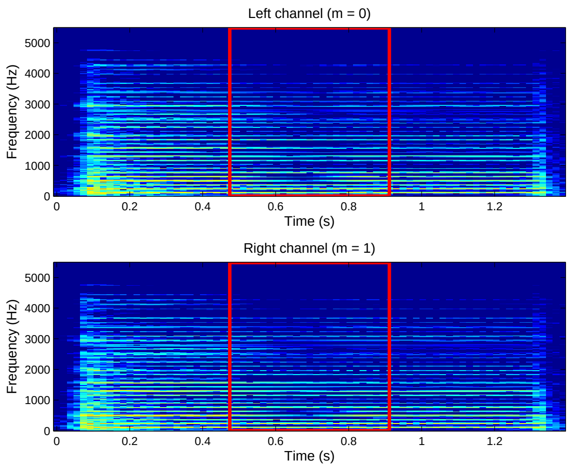


Fig. 5. Stereo signal $\overline{y}_{ms}(f,t)$ estimated with longer filters.

VII. Conclusions

In this paper, we have shown that convolution can be accurately implemented in the TF domain, by applying 2D-filters to a TF representation obtained as the output of a PR filter bank. In the particular case of recursive filters, we have also shown that filtering can be implemented by means of a state space representation in the TF domain. These results have then been used to extend the monochannel HR-NMF model initially proposed in [23], [24] to multichannel signals and convolutive mixtures. The resulting multichannel HR-NMF model can accurately represent the transfer from each source to each sensor, as well as the spectrum of each source. It also takes the correlations over frequencies into account. In order to estimate this model from real audio data, a variational EM algorithm has been proposed, which has a reduced computational complexity and a parallel implementation compared to [30]. This algorithm has been successfully applied to a stereophonic piano signal, and has been capable of modelling reverberation due to room impulse response, and restoring missing observations.

Because audio signals are sparse in the time-frequency domain, we observed that the multichannel HR-NMF model involves a small number of non-zero parameters in practice. In future work, we will investigate enforcing this property, for instance by introducing an a priori distribution of the parameters inducing sparsity [34]. In order to deal with more realistic music signals, the estimation of HR-NMF should be performed in a more informed way, for instance by means of semi-supervised learning, or by using any kind of prior information about the sources. For instance, harmonicity and temporal or spectral smoothness could be enforced by introducing some prior distributions of the parameters, or by re-parametrising the model. The model could also be extended in several ways, for instance by taking the correlations over latent components into account, or by using other types of TF transforms, *e.g.* wavelet transforms. Other Bayesian estimation techniques such as Markov chain Monte Carlo (MCMC) methods and message passing algorithms [32] might prove more effective than the variational EM algorithm. Lastly, the proposed approach could be used in a variety of applications, such as source separation, source coding, audio inpainting, and automatic music transcription.

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Appendix A
TF implementation of stable recursive filtering

Proof of Proposition 2: We consider the TF implementation of convolution given in Proposition 1, and we define $g(n)$ as the impulse response of a causal and stable recursive filter, having only simple poles. Then the partial fraction expansion of its transfer function [35] shows that it can be written in the form $g(n) = g_0(n) + \sum_{k=1}^{Q} g_k(n)$, where $Q \in \mathbb{N}$, $g_0(n)$ is a causal sequence of support $[0 \dots N_0 - 1]$ (with $N_0 \in \mathbb{N}$), and $\forall k \in [1 \dots Q]$,

$$g_k(n) = A_k e^{\delta_k n} \cos(2\pi\nu_k n + \psi_k) \mathbb{1}_{n\geq 0}$$

where $A_k > 0$, $\delta_k < 0$, $\nu_k \in [0, \frac{1}{2}]$, $\psi_k \in \mathbb{R}$.

Then $\forall f \in [0 \dots F-1]$, equation (2) yields $c_g(f,\varphi,\tau) = \sum_{k=0}^{Q} c_{g_k}(f,\varphi,\tau)$ with

$$c_{g_0}(f,\varphi,\tau) = (h_f * \tilde{h}_{f-\varphi} * g_0)(D(\tau + L))$$

and $\forall k \in [1 \dots Q]$,

$$\begin{aligned} c_{g_k}(f,\varphi,\tau) &= e^{\delta_k D\tau}(A_k(f,\varphi,\tau)\cos(2\pi\nu_k D\tau) \\ &\quad + B_k(f,\varphi,\tau)\sin(2\pi\nu_k D\tau)) \end{aligned}$$

where we have defined

$$\begin{aligned} A_k(f,\varphi,\tau) &= A_k \sum_{n=-N+1}^{N-1} (h_f * \tilde{h}_{f-\varphi})(n+N) \\ &\quad \times e^{-\delta_k n} \cos(2\pi\nu_k n - \psi_k) \mathbb{1}_{n \leq D\tau}, \\ B_k(f,\varphi,\tau) &= A_k \sum_{n=-N+1}^{N-1} (h_f * \tilde{h}_{f-\varphi})(n+N) \\ &\quad \times e^{-\delta_k n} \sin(2\pi\nu_k n - \psi_k) \mathbb{1}_{n \leq D\tau}. \end{aligned}$$

It can be easily proved that $\forall f \in [0 \dots F-1]$, $\forall \varphi \in \mathbb{Z}$,

- the support of $c_{g_0}(f,\varphi,\tau)$ is $[-L+1\dots L+\lceil\frac{N_0-2}{D}\rceil]$,
- if $\tau\le -L$, then $c_{g_0}(f,\varphi,\tau)$, $A_k(f,\varphi,\tau)$ and $B_k(f,\varphi,\tau)$ are zero, thus $c_g(f,\varphi,\tau)=0$,
- if $\tau\ge L$, then $A_k(f,\varphi,\tau)$ and $B_k(f,\varphi,\tau)$ don't depend on τ.

Therefore $\forall f\in[0\dots F-1]$, $\forall\varphi\in\mathbb{Z}$, $c_g(f,\varphi,\tau-L+1)$ is the impulse response of a causal and stable recursive filter, whose transfer function has a denominator of order $2Q$ and a numerator of order $2L+2Q-1+\lceil\frac{N_0-2}{D}\rceil]$.

As a particular case, suppose that $\forall k\in[1\dots Q]$, $|\delta_k|\ll 1$. If $\tau\ge L$, then $A_k(f,\varphi,\tau)$ and $B_k(f,\varphi,\tau)$ can be neglected as soon as ν_k does not lie in the supports of both $H_f(\nu)$ and $H_{f-\varphi}(\nu)$. Thus for each f and φ, there is a limited number $Q(f,\varphi)\le Q$ (possibly 0) of $c_{g_k}(f,\varphi,\tau)$ which contribute to $c_g(f,\varphi,\tau)$. In the general case, we can still consider without loss of generality that $\forall f\in[0\dots F-1]$, $\forall\varphi\in\mathbb{Z}$, there is a limited number $Q(f,\varphi)\le Q$ of $c_{g_k}(f,\varphi,\tau)$ which contribute to $c_g(f,\varphi,\tau)$. We then define $Q_a=2\max\limits_{f,\varphi}Q(f,\varphi)$ and $Q_b=2L+Q_a-1+\lceil\frac{N_0-2}{D}\rceil$. Then $\forall f\in[0\dots F-1]$, $\forall\varphi\in\mathbb{Z}$, $c_g(f,\varphi,\tau-L+1)$ is the impulse response of a causal and stable recursive filter, whose transfer function has a denominator of order Q_a and a numerator of order Q_b. Considering Remark 1, we conclude that the input/output system described in equation (1) is equivalent to the state space representation (3), where $P_b=K-1$. ■

Proof of Proposition 3: We consider the state space representation in Definition 1, and we first assume that $\forall f\in[0\dots F-1]$, sequences $x(f,t)$, $y(f,t)$, and $z(f,t)$ belong to $l^1(\mathbb{Z})$. Then the following DTFTs are well-defined:

$$\begin{array}{rcl} Y(f,\nu) &=& \sum_{t\in\mathbb{Z}} y(f,t)e^{-2i\pi\nu t},\\ X(f,\nu) &=& \sum_{t\in\mathbb{Z}} x(f,t)e^{-2i\pi\nu t},\\ B_g(f,\varphi,\nu) &=& \sum_{\tau\in\mathbb{Z}} b_g(f,\varphi,\tau)e^{-2i\pi\nu\tau},\\ A_g(f,\nu) &=& \sum_{\tau=0}^{Q_a} a_g(f,\tau)e^{-2i\pi\nu\tau}. \end{array}$$

Then applying the DTFT to equation (3) yields $Z(f,\nu)=\frac{1}{A_g(f,\nu)}X(f,\nu)$ and $Y(f,\nu)=\sum\limits_{\varphi=-P_b}^{P_b}B_g(f,\varphi,\nu)Z(f-\varphi,\nu)$. Therefore

$$Y(f,\nu)=\sum_{\varphi=-P_b}^{P_b} C_g(f,\varphi,\nu)X(f-\varphi,\nu), \quad (15)$$

where

$$C_g(f,\varphi,\nu)=\frac{B_g(f,\varphi,\nu)}{A_g(f-\varphi,\nu)} \quad (16)$$

is the frequency response of a recursive filter. Since this frequency response is twice continuously differentiable, then this filter is stable, which means that its impulse response $c_g(f,\varphi,\tau)=\int_0^1 C_g(f,\varphi,\nu)e^{+2i\pi\nu\tau}d\nu$ belongs to $l^1(\mathbb{F})$. Equations (1) and (4) are then obtained by applying an inverse DTFT to (15) and (16). Finally, even if $x(f,t)$, $y(f,t)$, and $z(f,t)$ belong to $l^\infty(\mathbb{Z})$ but not to $l^1(\mathbb{Z})$, equations (1) and (3) are still well-defined, and the same filter $c_g(f,\varphi,\tau)\in l^1(\mathbb{F})$ is still the only stable solution of equation (4). ■

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