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On the intuitionistic force of classical search (extended abstract)

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Abstract

The combinatorics of proof-search in classical propositional logic lies at the heart of most efficient proof procedures because the logic admits *least-commitment* search. The key to extending such methods to quantifiers and non-classical connectives is the problem of recovering this least-commitment principle in the context of the non-classical/non-propositional logic; *i.e.*, characterizing when a least-commitment (classical) search yields sufficient evidence for provability in the (non-classical) logic.

In this paper, we present such a characterization for the $(\supset, \wedge)$-fragment of intuitionistic logic using the $\lambda\mu$-calculus: a system of realizers for *classical free deduction* (*cf. natural deduction*) due to Parigot.

We show how this characterization can be used to define a notion of *uniform proof*, and a corresponding proof procedure, which extends that of Miller *et al.* to multiple-conclusioned sequent systems. The procedure is sound and complete for the fragment of intuitionistic logic considered and enjoys the combinatorial advantages of search in classical logic.

1 Introduction

1.1 Proof procedures and search

A proof procedure for a logic is an effective method of computing evidence for or against putative consequences in the logic. The traditional decomposition of proof procedure into *inference system* and *search strategy*, as discussed for example in [11], reflects the computational reality that logical consequence in logics of interest is typically semi-decidable. The inference system serves as the major organizing principle for the proof procedure, and is the primary focus when seeking gains in efficiency.

When tableaux rules, or inverted sequent rules, are used as the basis for an inference system, standard modifications are swiftly introduced with the aim of reducing redundancy in the search space; Skolemization and unification (*e.g.*, [8, 4]) being the two most favoured.

These modifications focus on the combinatorics of quantifiers and are particularly easy to justify in the case of classical logic owing to the extensive equivalences available in that setting which support a complete separation of the propositional and quantificational structure of a formula. This property is summarized variously by the Prenex Normal Form Theorem, Herbrand's Theorem [6], Gentzen's Midsequent Theorem [5], or Smullyan's Fundamental Theorem [24]. The existence of most general unifiers then permits the least-commitment properties of classical propositional search to be recovered in the quantificational setting.

Extension of these methods to quantified non-classical logics has been considered problematic since a separation of propositional and quantificational structure is not achievable while maintaining the principle that the constructors for the states of search, *i.e.*, the inverted inference rules, remain sound. Indeed, interest in tableaux methods for non-classical logics can be said to stem from the fact that such methods permit *local* treatment of propositional structure by means of formulae on the tableau, together with a *global* treatment of quantification using the tableau structure itself. This approximates the efficiency achievable for classical quantifiers. Various authors have explored the limits of these hybrid techniques in non-classical logics while respecting this restriction [4, 3, 23].

We pointed out in [25] that this restriction to propositional fidelity in tableaux and inverted sequential methods is a curious half-way-house; the benefits of replacing local conditions on quantifiers (eigenvariable conditions and local choice of instantiating terms) with global ones (the occurs-check and unification) is a general technique applicable to the problematic propositional structure in a non-classical logic. This observation was developed in [26] and [27] to give a comprehensive treatment of first-order modal and intuitionistic logics, and subsequently applied to a system of first-order dependent types [18, 21].

Such an approach makes the least-commitment combinatorics found in classical propositional logic available for organizing the search space in a non-classical logic such as intuitionistic logic. From the point of view of efficiency, this is important since many disjunctive choices in the naïve intuitionistic search space can be represented by a single state in the classical search space. The local propositional soundness of the naïve approach is replaced by a global condition on the information associated with each (classical) state in the search space. In effect, the proof procedure calculates *classical* realizers (see below) which are then subject to a soundness check specific to intuitionistic logic.[1]

Our aim in this paper is to study this relationship in a little more detail for the $(\supset, \wedge)$-fragment of intuitionistic logic, making some of these notions explicit, and to apply the understanding gained in the design of a resolution proof procedure.

1.2 Overview of the paper

In § 2, we review the idea of proof objects and realizers, outline the $\lambda\mu$-calculus [14], and extend it by adding conjunctive (or product) types and an operation of explicit substitution. The extended term calculus we call $\lambda\mu\epsilon$.

The formulation of logic with proof-objects in place has certain advantages from the point of view of proof-search. Specifically, it is possible to determine from the structure of the realizing object whether or not a classical search has determined (the existence of) an intuitionistic derivation. This is the global soundness condition referred to above. The details are developed in § 3.

In § 4, we consider briefly an application to (hereditary Harrop) analytic resolution.

[1] An alternative view, sufficient for non-classical logics with a classical propositional basis, is to view the relationship as one of embedding the truth conditions for the non-classical logic in classical logic. See [13].

2 Proof-objects and $\lambda\mu$-calculus

2.1 Proof-objects and realizers

For the $(\supset, \wedge, \forall)$-fragment of intuitionistic logic proofs of a sequent $\Gamma \longrightarrow A$, within a single-conclusioned calculus of sequents LJ, can be interpreted as constructions of natural deductions of the succedent formula A from the antecedent formulae in Γ [28, 17, 1]. Such a natural deduction ϕ can be seen as a proof-object *realizing* (*i.e.*, providing evidence for) the consequence $\Gamma \vdash^{\phi} A$. ϕ describes how to obtain natural deduction proofs of A from natural deduction proofs of the formulae in Γ.

In such a fragment of intuitionistic logic, the relationship between the proof-object ϕ and the formulae in $\Gamma \vdash A$ is particularly intimate. Specifically, if $\Gamma = A_1, \ldots, A_m$ and if each A_i is labelled with an assumption marker, x_i, then A corresponds to a λ-term of type A, built out of variables $x_1, \ldots, x_m$.

This correspondence, between natural deduction proofs and λ-terms on the one hand and propositions and types on the other, does not hold for classical natural deduction. However, Parigot's $\lambda\mu$-calculus [14] provides an elegant language of proof-objects based on an algorithmic interpretation of classical sequent calculus provided by cut-elimination. The proof-objects are realizers for multiple-conclusioned sequents $\Gamma \longrightarrow A, \Delta$, where A is a distinguished, or active, formula. $\lambda\mu$-terms provide combinatorial evidence for the existence of classical sequent derivations.

2.2 The $\lambda\mu$-calculus.

We begin by introducing a variation on Parigot's $\lambda\mu$-calculus [14], which we shall refer to as $\lambda\mu\epsilon$. In addition to implicational types, we include conjunctive types and *explicit substitutions* $u\{t/x\}$. The latter are used in the analysis of search below to give suitable representatives for possibly incompletable sequent derivations.

The raw terms of the $\lambda\mu\epsilon$-calculus are given by the following grammar:

$$\begin{aligned} t \;::=\; & x \mid \lambda x{:}A\,.\,t \mid tt \mid \\ & \mid \langle t,t\rangle \mid \pi(t) \mid \pi'(t) \\ & \mid [\alpha]t \mid \mu\alpha.t \mid t\,\{xt/y\} \end{aligned}$$

The rules for well-formed terms are given in Figure 1. The reduction rules, which are those of $\lambda\mu$ together with those necessary to avoid interference between $\lambda\mu$-reductions and explicit substitution, are given in Figure 2. Note that the non-interference reductions, second column Figure 2, *do not* have a base case of the form $x\{yt/x\} \rightsquigarrow yt$: we do not reduce the ϵ-construct, $\{-/-\}$, itself. Moreover, there is no case of the form $t\{ys/x\}\{y't'/x'\} \rightsquigarrow \ldots$: we do not compose explicit substitutions. Informally speaking, ϵ-constructs occur in *normal* $\lambda\mu\epsilon$-terms either (i) immediately to the right of variables, or (ii) immediately to the right of another ϵ-construct, *e.g.*, (i) $(xt)\{ys/x\} \rightsquigarrow x\{ys/x\}t'$, where $t\{ys/w\} \rightsquigarrow^{*} t'$ (the normal form of $t\{ys/w\}$) or (ii) $x\{ys/w\}\{y's'/w'\}$.

The $\lambda\mu$-calculus provides an account of classical free deduction, which is natural deduction extended to multi-conclusioned sequents: *i.e.*, the terms are realizers for a calculus in which multiple-conclusioned sequents can be derived without impure constraints [2]. Consequently, the form of the typing judgment in the $\lambda\mu$-calculus is $\Gamma \vdash t : A, \Delta$, where Γ is a context familiar from the typed λ-calculus and Δ

$$\frac{}{\Gamma, x{:}\, A \vdash x{:}\, A, \Delta}\ Ax$$

$$\frac{\Gamma, x{:}\, A \vdash t{:}\, B, \Delta}{\Gamma \vdash \lambda x{:}\, A.t{:}\, A \supset B, \Delta}\ \supset I \qquad \frac{\Gamma \vdash t{:}\, A \supset B, \Delta \qquad \Gamma \vdash s{:}\, A, \Delta}{\Gamma \vdash ts{:}\, B, \Delta}\ \supset E$$

$$\frac{\Gamma \vdash t{:}\, A, \Delta}{\Gamma \vdash [\alpha]t{:}\, A^{\alpha}, \Delta}\ [\_] \qquad \frac{\Gamma \vdash t{:}\, A^{\alpha}, \Delta}{\Gamma \vdash \mu\alpha.t{:}\, A, \Delta}\ \mu$$

$$\frac{\Gamma \vdash t{:}\, A, A^{\alpha}, \Delta}{\Gamma \vdash [\alpha]t{:}\, A^{\alpha}, \Delta}\ [\_] \qquad \frac{\Gamma \vdash t{:}\, \Delta}{\Gamma \vdash \mu\alpha.t{:}\, A, \Delta}\ \mu$$

$$\frac{\Gamma, w{:}\, B \vdash t{:}\, C, \Delta \qquad \Gamma \vdash s{:}\, A, \Delta}{\Gamma, x{:}\, A \supset B \vdash t\,\{xs/w\} : C, \Delta}\ \epsilon L$$

$$\frac{\Gamma \vdash t{:}\, A, \Delta \qquad \Gamma \vdash s{:}\, B, \Delta}{\Gamma \vdash \langle t, s\rangle{:}\, A \wedge B, \Delta}\ \wedge I \qquad \frac{\Gamma \vdash t{:}\, A \wedge B, \Delta}{\Gamma \vdash \pi(t){:}\, A, \Delta}\ \wedge E \qquad \frac{\Gamma \vdash t{:}\, A \wedge B, \Delta}{\Gamma \vdash \pi'(t){:}\, B, \Delta}\ \wedge E$$

The second instances of the rules [_] and μ model contraction and weakening respectively.

Figure 1: Rules for well-typed $\lambda\mu\epsilon$-terms

β	$(\lambda x{:}\, A.t)s$	$\rightsquigarrow$	$t[s/x]$
$\mu-\nu$	$(\mu\alpha^{A\supset B}.t)s$	$\rightsquigarrow$	$\mu\beta^{B}.t[[\beta]us/[\alpha]u]$
$\mu-\eta$	$\mu\alpha.[\alpha]s$	$\rightsquigarrow$	s if α not free in s
$\mu-\beta$	$[\gamma](\mu\alpha.s)$	$\rightsquigarrow$	$s[\gamma/\alpha]$
$\mu-prod$	$\pi(\mu\alpha^{A\times B}.s)$	$\rightsquigarrow$	$\mu\beta^{A}.t[[\beta]\pi(u)/[\alpha]u]$
	$\pi'(\mu\alpha^{A\times B}.s)$	$\rightsquigarrow$	$\mu\gamma^{B}.t[[\gamma]\pi'(u)/[\alpha]u]$
$proj$	$\pi(\langle t, s\rangle)$	$\rightsquigarrow$	t
	$\pi'(\langle t, s\rangle)$	$\rightsquigarrow$	s

$(\lambda x{:}\, A.t)\{ys/z\} \rightsquigarrow \lambda x{:}\, A.t\{ys/z\}$
$(ts)\{yu/z\} \rightsquigarrow t\{yu/z\}s\{yu/z\}$
$([\alpha]t)\{ys/z\} \rightsquigarrow [\alpha]t\{ys/z\}$
$(\mu\alpha.t)\{ys/z\} \rightsquigarrow \mu\alpha.t\{ys/z\}$
Also obvious cases for conjunctive terms. Standard variable-capture conditions assumed.

The term $t[s/[\alpha]u]$ indicates the term t with all occurrences of a subterm of the form $[\alpha]u$ replaced by s.

Figure 2: Reduction rules of the $\lambda\mu\epsilon$-calculus

$$\frac{}{\Gamma, A \longrightarrow A, \Delta}\ Ax$$

$$\frac{\Gamma, A, B \longrightarrow \Delta}{\Gamma, A \wedge B \longrightarrow \Delta}\ \wedge L \qquad \frac{\Gamma_1 \longrightarrow A, \Delta_1 \qquad \Gamma_2 \longrightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \longrightarrow A \wedge B, \Delta_1, \Delta_2}\ \wedge R$$

$$\frac{\Gamma_1 A \longrightarrow \Delta_1 \qquad \Gamma_2, B \longrightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \longrightarrow \Delta_1, \Delta_2}\ \vee L \qquad \frac{\Gamma \longrightarrow A, B, \Delta}{\Gamma \longrightarrow A \vee B, \Delta}\ \vee R$$

$$\frac{\Gamma_1 \longrightarrow A, \Delta_1 \qquad \Gamma_2, B \longrightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \supset B \longrightarrow \Delta_1, \Delta_2}\ \supset L \qquad \frac{\Gamma, A, \longrightarrow B}{\Gamma \longrightarrow A \supset B, \Delta}\ \supset R$$

$$\frac{\Gamma \longrightarrow A, \Delta}{\Gamma, \neg A \longrightarrow \Delta}\ \neg L \qquad \frac{\Gamma, A \longrightarrow}{\Gamma \longrightarrow \neg A, \Delta}\ \neg R$$

Figure 3: Cut-free multiple-conclusioned sequent calculus for intuitionistic logic

is a context containing types indexed by names, $\alpha, \beta, \ldots$, which are distinct from variables. The idea is that each $\lambda\mu$-sequent has exactly one principal formula, A, on the right-hand side, the leftmost one, which is the formula upon which all introduction and elimination rules operate. This formula is the type of the term t.

The term $[\alpha]t$ realizes the introduction of a name. The term $\mu\alpha.[\beta]t$ realizes the exchange operation: if A^α was part of Δ before the exchange, then A is the principal formula of the succedent after the exchange. Taken together, these terms also provide a notation for the realizers of contractions and weakenings on the right of a multiple-conclusioned calculus. It is also easy to detect whether a formula B^β in the right-hand side is, in fact, superfluous, *i.e.*, that there is a derivation of $\Gamma \vdash t{:}\, A, \Delta'$ in which Δ' does not contain B; it is superfluous if β is not a free name in t. This observation is exploited in the sequel.

Our extension of the $\lambda\mu$-calculus involves adding conjunction and a form of *explicit substitution*. The former extension is trivial; the latter deserves some discussion. The presentation of the $\lambda\mu$-calculus in [14] is as a system of linearized natural deduction for multiple conclusions, with implicational types both introduced and eliminated on the right-hand side. An alternative formulation of Parigot's system, not affecting the structure of the derivable terms, would be as a sequent calculus, with the elimination of implicational types on the right replaced by the introduction of implicational types on the left, as follows:

$$\lambda\mu{\supset}\mathrm{L} \quad \frac{\Gamma, w{:}\, B \vdash t{:}\, C, \Delta \qquad \Gamma \vdash s{:}\, A, \Delta}{\Gamma, x{:}\, A{\supset}B \vdash t[xs/w]{:}\, C, \Delta}.$$

Such a rule is admissible in Parigot's system since the cut rule,

$$\lambda\mu\mathrm{Cut} \quad \frac{\Gamma \vdash s{:}\, A, \Delta \qquad \Gamma, w{:}\, A \vdash t{:}\, B, \Delta}{\Gamma \vdash t[s/w]{:}\, B, \Delta},$$

is also admissible. In these rules the substitution $[t/x]$ is the usual implicit, meta-theoretic one. An analysis such as this for a system of first-order dependent function types is presented in [19] and exploited as a basis for a theory of proof-search in [21].

The rule (ϵL) of Figure 1, which introduces the explicit substitution $u\{xs/w\}$, corresponds exactly to the usual left rule for implication, but with explicit substitution replacing implicit substitution. The $\lambda\mu\epsilon$-calculus, presented in Figure 1, contains this left rule for explicit substitution together with the usual introduction and elimination rules for the implication.

$\lambda\mu\epsilon$ terms are thus $\lambda\mu$ terms enriched by the presence of explicit substitutions and pairing. If the substitution were implicit, and so carried out when introduced, some parts of a derivation would not be represented by the corresponding term. This happens if the variable being replaced does not occur in the term. The rule for explicit substitution (ϵL) can thus be used to model the $\supset L$ rule of the classical sequent calculus directly. In [22], a similar analysis is provided for a proof system for SLD-resolution over propositional implicational Horn clauses. Herbelin [7] also uses explicit substitutions, for a similar reason, in his version of a translation of intuitionistic sequent calculus (LJ) into a modified λ-calculus. His concern, however, is to restrict LJ so as obtain a bijective correspondence between λ-terms and LJ-derivations.

The choice of a distinguished formula on the right hand side of the sequent is enough to ensure strong normalization and confluence. Parigot's proof [16] extends to the conjunctive types and explicit substitution.

Theorem 1. *The $\lambda\mu\epsilon$-calculus is strongly normalizing.*

Proof sketch. Since there are no reductions $\rightsquigarrow$ that introduce nested explicit substitutions, Parigot's reducibility proof [16] of strong normalization can be extended to $\lambda\mu\epsilon$. Local confluence can be checked by joining all critical pairs which are generated by the rules for explicit substitution. Strong normalization then implies confluence (by Newman's Lemma [10]). □

3 Representation of sequent derivations in $\lambda\mu\epsilon$

In this section, we describe the use of the $\lambda\mu\epsilon$-calculus to represent sequent proofs. The classical nature of $\lambda\mu\epsilon$-calculus influences the way in which it can be used to represent intuitionistic sequent derivations. Hence we begin with some observations about the relationship between intuitionistic and classical sequent derivations.

In general, every intuitionistic derivation arises as a subderivation of a classical derivation. Because the $\supset R$ rule allows multiple succedents in the premiss, two different intuitionistic sequent derivations, which are not identical up to a permutation of inference rules, can be subderivations of the same classical sequent derivation up to a choice of axioms. For example, consider the two intuitionistic derivations[2]

$$\dfrac{\dfrac{}{B, A \longrightarrow B}\,Ax}{B \longrightarrow A\supset B, C\supset B}\,\supset R$$

and

$$\dfrac{\dfrac{}{B, C \longrightarrow B}\,Ax}{B \longrightarrow C\supset B, A\supset B}\,\supset R.$$

They arise as restrictions to intuitionistic logic from the following classical derivation:

$$\dfrac{\dfrac{\dfrac{}{B, A, C \longrightarrow B, B}\,Ax}{B, C \longrightarrow A\supset B, B}\,\supset R}{B \longrightarrow A\supset B, C\supset B}\,\supset R.$$

In this case, both derivations are proofs even in intuitionistic logic, and hence the order in which the $\supset R$ rules are executed does not matter. In general, however, this order matters [27]. As an easy example, take the sequent $B \longrightarrow A\supset B, D\supset E$. If the formula $A\supset B$ is reduced first working from root to leaves then the search succeeds, otherwise it fails. However, in classical logic the order does not matter. So it becomes apparent already that the search in the classical sequent calculus, when viewed as a search for intuitionistic proofs, proceeds in parallel: one classical sequent derivation may have many intuitionistic subderivations which are not permutations of each other.

[2]These two inferences can either be considered to be instances of $\supset$R in our multiple-conclusioned intuitionistic sequent calculus given in Figure 3 (*cf.* [27]) or combinations of explicit weakenings and $\supset$R instances in Dummett's system [2].

Although inferences in classical logic can be freely permuted [9], the property of a classical sequent derivation having an intuitionistic subderivation is not always invariant under permutation. Examples of this phenomenon are a bit more complicated. Consider the sequent

$$x{:}\,A\supset B, y{:}\,(A\supset B)\supset B \longrightarrow B\,,$$

where we have attached variables to the antecedents to make it easier to refer to a specific formula. If first x is reduced and then y, there is no way of identifying an intuitionistic subderivation. However, if we reduce first y, and then x, then we obtain an intuitionistic derivation. Both derivations are shown in Figures 6 and 7 respectively (see page 17).

Below, we show how to formulate a condition on classical derivations to determine when they have intuitionistic subderivations. This is formulated as a condition on a $\lambda\mu\epsilon$-term that interprets the classical derivation (see Definition 4). Subsequently, we show how transformations on the $\lambda\mu\epsilon$-terms can be used to characterize the search space over a given endsequent (see Theorem 11). We prove the completeness of a particular search strategy for classical logic with respect to intuitionistic provability. Again, the formulation of this strategy uses $\lambda\mu\epsilon$-terms (see Theorem 15).

3.1 Translation into $\lambda\mu\epsilon$

We start by giving the translation from classical sequent derivations into the $\lambda\mu\epsilon$-calculus. Note that the classical sequent derivations have to be suitably annotated for the definition. Firstly, each sequent has one principal formula in the succedent together with an arbitrary number of additional formulae. We introduce a *name* for each additional formula in the succedent and a *variable* for each formula in the antecedent. Secondly, the translation has to take the explicit exchange rule in the $\lambda\mu\epsilon$-calculus into account. For example, the axiom $\Gamma, x{:}\,A \longrightarrow A, B^\beta$ can be translated to the variable x; on the other hand, the axiom $\Gamma, x{:}\,A \longrightarrow B, A^\alpha$, must be translated to the $\lambda\mu\epsilon$-term $\mu\alpha.[\beta]x$.

We shall use the following notation: if ϕ is a derivation whose last rule is R applied to the derivations $\phi_1, \ldots, \phi_n$, we write $(\phi_1, \ldots, \phi_n); R$ for ϕ.

Definition 2. *Let $\phi{:}\,\Gamma \longrightarrow A, \Delta$ be a classical sequent derivation and suppose that each occurrence of a formula in Γ and Δ has a label, i.e., we have $\Gamma = x_1{:}\,A_1, \ldots, x_n{:}\,A_n$ and $\Delta = B_1^{\beta_1}, \ldots, B_m^{\beta^m}$. (These labels turn into variables and names in the $\lambda\mu\epsilon$-calculus, hence we also use them for the derivations.) We define a $\lambda\mu\epsilon$-term $[\![\phi]\!]$ by induction over the structure of ϕ as follows (note the clause for the exchange rule):*

Axiom: *Suppose $\phi : \Gamma, x{:}\,A \longrightarrow A, \Delta$ is an axiom, then $[\![\phi]\!] \stackrel{\text{def}}{=} x$;*

Exchange: *Suppose $\phi{:}\,\Gamma \longrightarrow A, B^\beta, \Delta$, and*

$$\phi' = \phi; \mathbf{exc}{:}\,\Gamma \longrightarrow B, A^\alpha, \Delta.$$

We define $[\![\phi']\!]$ to be the contractum of the term $\mu\beta.[\alpha][\![\phi]\!]$ with respect to the rules μ-ν and μ-η;

$\wedge L$: *Suppose we have the derivation*

$$\frac{\phi{:}\,\Gamma, x{:}\,A, y{:}\,B \longrightarrow A, \Delta}{\phi; \wedge L{:}\,\Gamma, z{:}\,A \wedge B \longrightarrow A, \Delta}\wedge L,$$

then the corresponding $\lambda\mu\epsilon$-term is

$$[\![\phi; \wedge L]\!] \stackrel{\text{def}}{=} [\![\phi]\!][\pi(z)/x, \pi'(z)/y];$$

$\wedge R$: *Suppose we have the derivation*

$$\frac{\phi\colon \Gamma \longrightarrow A, \Delta \qquad \psi\colon \Gamma \longrightarrow B, \Delta}{(\phi,\psi); \wedge R\colon \Gamma \longrightarrow A \wedge B, \Delta} \wedge R,$$

then we define

$$[\![(\phi,\psi); \wedge R]\!] \stackrel{\text{def}}{=} \langle [\![\phi]\!], [\![\psi]\!] \rangle;$$

$\supset L$: *Suppose we have the derivation*

$$\frac{\phi\colon \Gamma \rightarrow A, C^{\gamma}, \Delta \qquad \psi\colon \Gamma, w\colon B \rightarrow C, \Delta}{(\phi,\psi); \supset L\colon \Gamma, x\colon A \supset B \rightarrow C, \Delta} \supset L$$

then we define $[\![(\phi,\psi); \supset L]\!]$ *to be the contractum of* $\mu\gamma.[\gamma][\![\psi]\!]\,\{x[\![\phi]\!]/w\}$ *with respect to the reduction rules* μ-ν *and* μ-η, *via appropriate reductions for* $\{x[\![\phi]\!]/w\}$;

$\supset R$: *Suppose we have the derivation*

$$\frac{\phi\colon \Gamma, x\colon A \longrightarrow B, \Delta}{\phi; \supset R\colon \Gamma \longrightarrow A \supset B, \Delta} \supset R,$$

then we define $[\![\phi; \supset R]\!]$ *to be* $\lambda x\colon A.[\![\phi]\!]$.

The labelling of the assumptions has one important consequence, namely that there are several possible translations for the same classical sequent derivation. As an example, take the sequent derivation

$$\cfrac{\cfrac{\cfrac{}{B, C, A \longrightarrow B, B}\,Ax}{B, A \longrightarrow C \supset B, B}\supset R}{B \longrightarrow A \supset B, C \supset B}\supset R.$$

There are two possible $\lambda\mu\epsilon$-terms corresponding to this derivation, namely

$$\lambda x\colon A.\mu\beta.[\gamma]\lambda y\colon C.\mu\delta.[\beta]b$$

and

$$\lambda x\colon A.\mu\beta.[\gamma]\lambda y\colon C.b\,,$$

where we use the name b to denote the variable corresponding to the formula B on the left-hand side. (We will often use the lower-case version of the name of a formula as the name of the corresponding variable.) The first proof term uses the second occurrence of B at the leaf for the axiom, whereas the second uses the first occurrence of B in the succedent. In this case the difference does not matter — both derivations contain intuitionistic subderivations — but this is not generally true.

3.2 Intuitionistic provability

We consider a sequent calculus presentation of intuitionistic logic with multiple formulae on the right with weakening built into the inference rules, as in [27]. The rules are given in Figure 3. They are a restriction of the classical sequent calculus in which $\supset$R and $\neg$R are permitted for only for, respectively, singleton and empty succedents.

In deciding when a classically derivation indicates that its endsequent is intuitionistically provable, the requirement is to detect superfluous inferences. Consider again the sequent $B \longrightarrow A \supset B, D \supset E$. This sequent has an intuitionistic proof in which $A \supset B$ is reduced first. There is also the following classical proof of this sequent:

$$\cfrac{\cfrac{\cfrac{}{B, A, D \longrightarrow B, E}\,Ax}{B, A \longrightarrow B, D \supset E}\supset R}{B \longrightarrow A \supset B, D \supset E}\supset R.$$

We want to be able to detect that the use of the $\supset R$ rule to reduce the formula $D\supset E$ is superfluous by using the $\lambda\mu\epsilon$-term corresponding to this proof, we can then conclude that there is an intuitionistic proof of this sequent. The $\lambda\mu\epsilon$-term representing this derivation

$$\lambda x\colon A.\mu\beta.[\gamma]\lambda y\colon D.\mu\epsilon.[\beta]b\ ;$$

this amounts to determining when a subterm (here the λ-abstraction over D) models weakening on the right. The technical details follow below.

Definition 3. *We define* weakening terms *and* weakening occurrences of names *by induction over the structure of terms as follows:*

(i) *$\mu\alpha.t$ is a weakening term if all occurrences of α in t are weakening occurrences;*

(ii) *$\langle t, s\rangle$ is a weakening term if t and s are weakening terms;*

(iii) *$\lambda x\colon A.t$ is a weakening term if t is a weakening term and if x is not free in t;*

(iv) *The outermost occurrence of α in $[\alpha]t$ is a weakening occurrence if t is a weakening term;*

(v) *$t\,\{u/x\}$ is a weakening term if t is a weakening term.*

Now we can define our first criterion for when a classical sequent derivation determines the existence of an intuitionistic one.

Definition 4. *Call a $\lambda\mu\epsilon$-term* intuitionistic *if in any subterm $\lambda x\colon A\,.\,t'$ which is not a weakening term, all occurrences of free names are weakening occurrences.*

Let us reconsider the examples at the beginning of this section. There are two $\lambda\mu\epsilon$-terms corresponding to the two derivations of $B \longrightarrow A\supset B, D\supset E$. The first one, which corresponds to reducing $A\supset B$ first, is the term

$$\lambda x\colon A.\mu\beta.[\gamma]\lambda y\colon D.\mu\epsilon.[\beta]b\ ,$$

and the second one, which corresponds to reducing $D\supset E$ first, is the term

$$\lambda y\colon D.\mu\delta.[\alpha]\lambda x\colon A.b\ .$$

In both cases we have an intuitionistic λ-term because the λ-abstraction over D is a weakening term. This example shows the parallelism obtained by using a classical sequent calculus: both intuitionistic subderivations of either of the classical proofs are considered simultaneously without any need for backtracking.

As an example of a non-intuitionistic term, consider Peirce's formula, $((A\supset B)\supset A)\supset A$. The classical proof of this formula is

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \longrightarrow B, A}\;Ax}{\longrightarrow A\supset B, A}\;\supset R \qquad \cfrac{}{A \longrightarrow A}\;Ax}{(A\supset B)\supset A \longrightarrow A}\;\supset L}{\longrightarrow ((A\supset B)\supset A)\supset A}\;\supset R.$$

If this proof is translated into the $\lambda\mu\epsilon$-calculus, the term obtained is

$$\lambda x\colon (A\supset B)\supset A.\mu\alpha.[\alpha]a\,\{x(\lambda y\colon A.\mu\beta.[\alpha]y)/a\}\,.$$

The name α has a non-weakening occurrence in the λ-abstraction over A; hence this term is not intuitionistic.

Next we show the correctness of the criterion. The crucial point is that a weakening term corresponds to a superfluous subderivation. The following lemma makes this precise.

Lemma 5. *Let* ϕ *be a derivation* $\phi: \Gamma, A_1, \ldots, A_n \longrightarrow A, B_1, \ldots, B_m, \Delta$ *such that* $\Gamma, a_1: A_1, \ldots, a_n: A_n \vdash [\![\phi]\!]: A, B_1^{\beta_1}, \ldots, B_m^{\beta_m}, \Delta$ *holds. If the variables* a_i *do not occur in* $[\![\phi]\!]$ *and if the* β_j *have only weakening occurrences, then there is a procedure to construct a sequent derivation of* $\Gamma \longrightarrow A, \Delta$. *Moreover, if* $[\![\phi]\!]$ *is a weakening term, then there is a procedure to construct a derivation of* $\Gamma \longrightarrow \Delta$. *These procedures transform sequent derivations which have an intuitionistic subderivation into those with the same property.*

Proof. By induction over the structure of sequent derivations. We give the case of a $\supset L$ rule to illustrate the argument. Suppose we are given a proof ending with

$$\frac{\Gamma \longrightarrow C, A^{\alpha}, \Delta \qquad \Gamma, D \longrightarrow A, \Delta}{\Gamma, x: C \supset D \longrightarrow A, \Delta} \supset L$$

and suppose that its $\lambda\mu\epsilon$-term is $\mu\alpha.[\alpha]t\,\{xs/w\}$. The only interesting case arises if this term is a weakening term. In this case, the name α has only weakening occurrences in t and in s, and t is a weakening term. By the induction hypothesis, we obtain derivations of $\Gamma \longrightarrow C, \Delta$ and $\Gamma, D \longrightarrow \Delta$ and hence also a derivation of $\Gamma, C \supset D \longrightarrow \Delta$. □

Finally, we are in a position to show the correctness of the criterion.

Theorem 6. *Let* $\phi: \Gamma \longrightarrow A, \Delta$ *be a classical sequent derivation. If* $[\![\phi]\!]$ *is an intuitionistic* $\lambda\mu\epsilon$*-term, then there exists an intuitionistic derivation of* $\Gamma \longrightarrow A, \Delta$.

Proof. We proceed by induction over the structure of derivations of sequents. Suppose the last rule is the rule $\supset R$ to obtain a sequent $\Gamma \longrightarrow A \supset B, \Delta$. By the induction hypothesis, we have an intuitionistic sequent derivation of $\Gamma, A \longrightarrow B, \Delta$. Let $[\![\phi]\!] = \lambda a: A.t$. Either $[\![\phi]\!]$ is a weakening term, in which case Lemma 5 implies that there is also an intuitionistic derivation of $\Gamma \longrightarrow \Delta$, and hence also of $\Gamma \longrightarrow A \supset B, \Delta$. If $[\![\phi]\!]$ is not a weakening term, then there are no free names in $[\![\phi]\!]$ that have a non-weakening occurrence. Hence by Lemma 5 again, there is an intuitionistic derivation $\Gamma, A \longrightarrow B$. Now the intuitionistic $\supset R$ rule yields the result. □

3.3 Representation of uniform proofs

In this subsection and the next, we show that a certain classical proof procedure is sound and complete for intuitionistic provability in the fragment $(\supset, \wedge)$. The proof procedure is a extension of Miller's notion of *uniform proof* to multiple-conclusioned systems.

A uniform proof [12] is a sequent derivation in which, when read from root to leaves, all right rules are applied whenever it is possible so to do, except for axioms with non-atomic principal formulae.[3] We call a proof *fully uniform* if right rules are preferred even over axioms. The notion of a uniform proof leads to a simple, highly determinsitic search algorithm: first apply all possible right-rules; then select an appropriate left-rule. Note that Miller *et al.* define uniform proofs for the full, single-conclusioned calculus LJ [5]. In

[3] An axiom is said to be *atomic* just in case its principal formula is atomic.

this case, not every LJ-provable propositional sequent has a uniform proof. The reason is that it may be necessary to apply the $\vee L$ rule before the $\vee R$ rule to obtain a proof. In the $(\supset, \wedge)$-fragment this argument fails and we will be able to show that every provable sequent in this fragment has a uniform proof. If we were to add $\vee$ a restriction to hereditary Harrop formulae, as used by Miller *et al.*, would seem to be necessary.

As a first step we examine how the $\lambda\mu\epsilon$-calculus models uniform proofs. We start with important properties of $\lambda\mu\epsilon$-terms which are the translations of uniform classical proofs.

Definition 7. *Let t be any $\lambda\mu\epsilon$-term such that $\Gamma \vdash t\colon A, \Delta$. A* uniform term *$C^A(t_1, \ldots, t_n)$* for *$t$ is a $\lambda\mu\epsilon$-term with parameters (holes) $t_1, \ldots, t_n$ defined by induction over the structure of A as follows:*

(i) *If A is a base type, then $C^A(t) = s$, where s is the normal form of t;*

(ii) *For a function type $A \supset B$, define $C^{A \supset B}(t_1, \ldots, t_n)$ to be $\lambda x\colon A \,.\, C^B(t_1, \ldots, t_n)$, where $C^B(t_1, \ldots, t_n)$ is the uniform term for tx;*

(iii) *For a product type $A \wedge B$, define $C^{A \wedge B}(t_1, \ldots, t_n, s_1, \ldots, s_m)$ to be*

$$\langle C^A(t_1, \ldots, t_n), C^B(s_1, \ldots, s_m) \rangle \,,$$

where $C^A(t_1, \ldots, t_n)$ and $C^B(s_1, \ldots, s_m)$ are the uniform terms for $\pi_1(t)$ and $\pi_2(t)$, respectively.

Lemma 8. *Suppose $t\,\{xs/w\}$ is a subterm of $[\![\phi]\!]$, where ϕ is a fully uniform classical proof. Then s is equal to its uniform term $C^A(s_1, \ldots, s_m)$.*

Proof. By induction over the structure of derivations. □

The notion of a uniform classical proof generalizes the corresponding notion for intuitionistic logic [12]. This is made precise in the following:

Proposition 9. *For the $(\supset, \wedge)$-fragment: every LJ sequent derivation ϕ translates under $[\![-]\!]$ into a $\lambda\mu\epsilon$-term with no names. Moreover, if ϕ is fully uniform, then $[\![\phi]\!]$ is a λ-term in long $\beta\eta$-normal form (after replacing all occurrences of $t\,\{xs/w\}$ by $t[xs/w]$).*

Proof. The absence of names in $[\![\phi]\!]$ is a direct consequence of the absence of any structural right-rules in ϕ. The uniform term of a λ-term is its long $\beta\eta$-normal form, which is well-typed in the usual simply typed λ-calculus [1]. □

3.4 Permutations

We shall now analyse the effect of permutations on classical uniform proofs. This is important because there are (well-known) non-permutabilities in intuitionistic logic. We have seen examples of this already, namely with the sequents $B \longrightarrow A \supset B, D \supset E$ and $(A \supset B) \supset B, A \supset B \longrightarrow B$. The first case covers the exchange of two right-rules. There, the order in which the two right-rules were executed did not matter. The second case concerns the exchange of $\supset L$ rules. Whereas in the first case, where there is a general strategy which renders an exhaustive search of all permutation variants superfluous, in the second case we do have to take into account all possible permutations of $\supset L$ rules for completeness. The invariance under right-rules is covered by the following lemma.

Lemma 10. *Let ϕ be a classical sequent derivation such that $[\![\phi]\!]$ is an intuitionistic $\lambda\mu\epsilon$-term.*

(i) *If ψ is the derivation resulting from interchanging two $\supset R$ rules in ϕ, then $[\![\psi]\!]$ is an intuitionistic term.*

(ii) *If ϕ is the derivation*

$$\dfrac{\dfrac{\Gamma, A \longrightarrow B, C, \Delta \qquad \Gamma, A \longrightarrow B, D, \Delta}{\Gamma, A \longrightarrow B, C \wedge D, \Delta}\ \wedge R}{\Gamma \longrightarrow A \supset B, C \wedge D, \Delta}\ \supset R,$$

then the derivation ψ obtained by permuting the $\supset R$ rule over the $\wedge R$ rule, towards the leaves, has an intuitionistic $\lambda\mu\epsilon$-term $[\![\psi]\!]$. Conversely, if we start with a ψ such that $[\![\psi]\!]$ is an intuitionistic $\lambda\mu\epsilon$-term, and permute the rules other way around, then at least one of the $\lambda\mu\epsilon$-terms that results from a different choice of axioms in the permuted derivation is intuitionistic.

Proof. By induction over the structure of derivations. The additional statement in (*ii*) arises from the fact that if the term $\lambda x{:}\, A.\mu\beta.[\gamma]t$ is not a weakening term, then in $[\gamma]t$ the name γ has only weakening occurrences. Now we use Lemma 5 to show that in this case $\Gamma, A \longrightarrow B$ has a intuitionistic sequent proof. The derivation is now obvious. □

There are cases in which moving an $\supset R$ rule below a $\wedge R$ rule can lead to a derivation which has no intuitionistic $\lambda\mu\epsilon$-term assigned to it. As an example, consider the (permuted) derivation

$$\dfrac{\dfrac{\dfrac{}{B, D, A \longrightarrow B, C}\ Ax \qquad \dfrac{}{B, D, A \longrightarrow B, D}\ Ax}{B, D, A \longrightarrow B, C \wedge D}\ \wedge R}{B, D \longrightarrow A \supset B, C \wedge D}\ \supset R$$

If we choose the axiom with principal formula D to close the second leaf sequent, the resulting $\lambda\mu\epsilon$-term is not intuitionistic. However, with the other choice, namely the axiom with principal formula B, we do obtain an intuitionistic proof.

We have completeness:

Theorem 11. *If the sequent $\Gamma \longrightarrow A, \Delta$ is intuitionistically provable, then, for any possible order of right-rules applied to the succedent, there exists a fully uniform (classical) proof ψ of the sequent with this order of right rules such that $[\![\psi]\!]$ is intuitionistic.*

Proof. Since the sequent $\Gamma \longrightarrow A, \Delta$ is intuitionistically provable, there exists a formula B in A, Δ such that ϕ is a fully uniform LJ-proof of $\Gamma \longrightarrow B$, and where each leaf of ϕ is atomic. Note that Proposition 9 implies that $[\![\phi]\!]$ has no names. Now show by an induction over the structure of formulae that for any such derivation ϕ and any antecedent Γ' and succedent Δ', any order of right rules applied to B, Δ', there is a fully uniform proof $\psi : \Gamma, \Gamma' \longrightarrow B, \Delta'$, with the order of the right rules such that the following three conditions are met:

(i) $[\![\psi]\!]$ is intuitionistic;

(ii) ψ has only weakening occurrences of free names except possibly a name for the formula B, and all subterms corresponding to right rules reducing formulas in Δ' are weakening terms;

(iii) the variables occurring in Γ' do not occur in $[\![\psi]\!]$.

The proof is concluded by setting $\Delta' = \Delta''$, where Δ'' is obtained from Δ by possible exchange of A and B. □

This proof does not extend to the fragment containing $\vee$ rules. The reason is that Proposition 9 no longer holds as the uniform proof of the sequent $A \vee B \longrightarrow A \vee B$ introduces a non-weakening name for B.

4 Application to (hereditary Harrop) analytic resolution

In this section, we apply the above results to an analytic resolution procedure for intuitionistically provable hereditary Harrop formulae based on the $\supset$L rule. The restriction to the hereditary Harrop fragment facilitates the search procedure: in an application of a $\supset L$ rule to the formula $B \supset A$, the formula A is always atomic, and hence can be matched with a formula in the succedent. There is no loss of generality in this restriction because every intuitionistically valid formula over $\supset$ and $\wedge$ is equivalent to a hereditary Harrop formula.

The definition of propositional hereditary Harrop formulae in the absence of disjunction (*cf.* [12, 20]) is as follows:

Definition 12. *Define* goal formulae G *and* definite formulae D *by*

$$\begin{array}{lcl} G & ::= & A \mid G \wedge G \mid D \supset G \\ D & ::= & A \mid G \supset A \mid D \wedge D, \end{array}$$

where A is atomic. Call a sequent $\Gamma \longrightarrow \Delta$ hereditary Harrop *if Γ consists of just D-formulae and Δ consists of just G-formulae.*

Definition 13. *A sequent derivation is called a* resolution derivation *if it satisfies the following constraints for rule applications:*

(i) *An $\supset R$ rule is applied only if no formula on the right-hand side is a conjunction;*

(ii) *An $\supset L$ rule, with principal formula $G \supset A$, is applied only if all formulae on the right-hand side are atomic and A occurs on the right-hand side;*

(iii) *A $\wedge L$ rule is applied only if all formulae on the right-hand side are atomic;*

(iv) *An $\supset L$ rule is applied only if no formula on the left-hand side is a conjunction.*

We include condition (iv) only for consistency with the usual definition [12, 20]. It is inessential for the analysis presented here.

The primary difference between a fully uniform proof and a resolution proof is the requirement in the latter that the atomic matrix of the principal formula of each $\supset$L rule match with an atom on the succedent of the conclusion of the rule. Note also that the application of both the left and right rules has to be in a specified order —

conjunction first — in the case of the latter.

Lemma 10 implies that if the restricted order in which the right rules are applied does not succeed in obtaining an intuitionistic proof, then no other ordering will. Moreover, resolution proofs are complete for intuitionistic provability of propositional hereditary Harrop formulae without disjunctions.

Corollary 14. *If $\Gamma \longrightarrow \Delta$ is an intuitionistically provable hereditary Harrop sequent, then there exists a resolution proof ψ of this sequent such that $[\![\psi]\!]$ is intuitionistic.*

Proof. From Theorem 11, since any resolution proof is uniform. □

So, in order to search for an intuitionistic proof of the sequent $\Gamma \longrightarrow \Delta$ it is enough to construct a resolution proof and then check, for all possible axiom instances and all possible exchanges of $\supset L$ rules, whether the corresponding $\lambda\mu\epsilon$-terms are intuitionistic. Working on the $\lambda\mu\epsilon$-terms, the first step consists in replacing a variable x by $\mu\alpha.[\beta]y$ or *vice versa*. The second step is a lot more complicated to capture. The reason is that the $\supset L$ rules introduce arbitrarily complex formulae in the succedent: these formulae must be decomposed.

To see the necessity of exchanging $\supset L$ rules, consider the sequent

$$x : A \supset B,\ y : (A \supset B) \supset B \longrightarrow B\ .$$

One possible derivation is given by Figure 6, in which x is reduced first. The derivation in Figure 7 is obtained from the first one by exchanging the two occurrences of the $\supset L$ rule, *i.e.*, exchanging the order of reduction of x and y, and then pushing the right-rules to the root of the derivation, thereby obtaining a uniform derivation. The corresponding $\lambda\mu\epsilon$-terms are $\mu\beta.[\beta]b\,\{x(\mu\alpha.[\beta]b\,\{y(\lambda a{:}\,A.\mu\theta.[\alpha]a)/b\})/b\}$ and $b\,\{y(\lambda a{:}\,A.b\,\{xa/b\})/b\}$. The first is not an intuitionistic $\lambda\mu\epsilon$-term because the λ-abstraction over A is not a weakening term, and yet the occurrence of $[\alpha]$ is not a weakening occurrence. The second one is an intuitionistic $\lambda\mu\epsilon$-term because there are no names (in fact, it is the uniform derivation in the single-conclusioned calculus LJ).

Note that both derivations are not only uniform but are also resolution derivations. This implies that the second premiss in the $\supset L$ rule is always an axiom. However both premisses of the $\supset L$ rule are important for determining when a resolution derivation is intuitionistic. The reason is that the choice of the axiom at the right premiss matters. This is not the case for single-conclusioned intuitionistic resolutions.

Now we describe the general situation. Consider Figures 4 and 5. The former is intended to be a classically valid uniform derivation. The latter is intended to be an intuitionistically valid uniform derivation obtained from the former by permuting ⊃L rules with respect to one another and by inserting any right-rules so induced.

Theorem 15. *Let ϕ be the uniform derivation given in Figure 4 and let*

$$t'\,\{yC^C(t_j)/v\}\\ \{x(C^A(s_i\,\{yC^C(u_{i,j})/v\}))/w\}$$

be the corresponding $\lambda\mu\epsilon$-term. Then the $\lambda\mu\epsilon$-term corresponding to the ex-

changed derivation, given in Figure 5, is the term

$$t' \{xC^A(s_i)/w\} \\ \{y(C^C(t_j \{xC^A(u'_{i,j})/w\}))/v\} ,$$

where $u'_{i,j}$ *is the uniform term corresponding to* $\mu\alpha_i.[\gamma]u_{i,j}$. *If the first derivation is a resolution derivation, so is the second one.*

Proof. By induction over the structure of the formulae A and C. □

5 Conclusions

We have presented a characterization of provability in the $(\supset, \wedge)$-fragment of intuitionistic logic in terms of the $\lambda\mu\epsilon$-calculus — a variant of Parigot's $\lambda\mu$-calculus. This calculus provides a system of realizers for the (cut-free) classical sequent calculus. Moreover, we have formulated a condition on the realizers for when a classical derivation yields sufficient evidence to judge the provability of the endsequent in intuitionistic logic.

The characterization allows us to obtain search procedures for intuitionistic logic from search procedures for classical logic. We have exploited this by showing how an analytic resolution procedure for intuitionistically provable hereditary Harrop formulae can be obtained by extending the notion of uniform proof [12] to a multiple-conclusioned setting. The conbinatorics of the classical calculus can then be used to compute realizers on which the test for intuitionistic provability can be performed.

There are at least two directions for further work. A first is to extend the treatment to $\vee$ and first-order quantifiers. The restriction to hereditary Harrop formulae would then become essential for formulating a sound and complete analytic resolution procedure. The addition of $\vee$ requires additional work in the $\lambda\mu\epsilon$-calculus because the introduction of formulae of the form $A \vee B$ to a sequent effects the properties of the names occurring in the sequent. When we add quantifiers we encounter a variety familiar issues (*cf.* [27, 21]) connected with the calculation of witnesses via unification. For example, we must identify suitable global correctness criteria that do not require significant backtracking [21, 23, 27].

A second direction concerns applications. One promising line is to analyse the intuitionistic force of standard classical proof procedures such as various resolution methods, model elimination and tableaux methods, by representing these procedures as methods for constructing classical proof-objects (*i.e.*, $\lambda\mu\epsilon$-terms).

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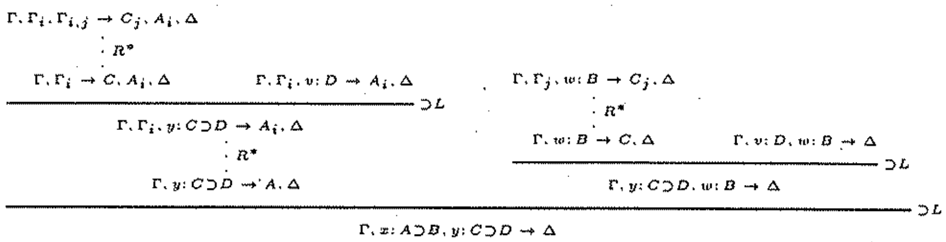

Figure 4: Derivation before Permutation

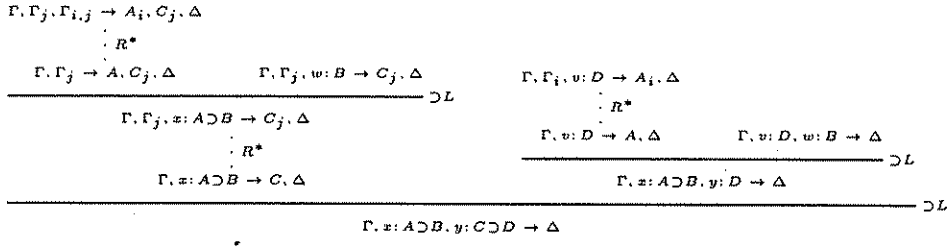

Figure 5: Derivation after Permutation

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \longrightarrow A, B, B}\,Ax}{\longrightarrow A, A\supset B, B}\,\supset R \qquad \dfrac{}{B \longrightarrow A, B}\,Ax}{y{:}\,(A\supset B)\supset B \longrightarrow A, B}\,\supset L \qquad \dfrac{}{y{:}\,(A\supset B)\supset B, B \longrightarrow B}\,Ax}{x{:}\,A\supset B, y{:}\,(A\supset B)\supset B \longrightarrow B}\,\supset L$$

Figure 6: Example derivation before permutation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \longrightarrow A, B, B}\,Ax \qquad \dfrac{}{A, B \longrightarrow B, B}\,Ax}{x{:}\,A\supset B, A \longrightarrow B, B}\,\supset L}{x{:}\,A\supset B \longrightarrow A\supset B, B}\,\supset R \qquad \dfrac{}{A\supset B, B \longrightarrow B}\,Ax}{x{:}\,A\supset B, y{:}\,(A\supset B)\supset B \longrightarrow B}\,\supset L$$

Figure 7: Example derivation after permutation

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