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Analytic two-loop form factors in $\mathcal{N} = 4$ SYM

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Abstract

We derive a compact expression for the three-point MHV form factors of half-BPS operators in $\mathcal{N} = 4$ super Yang-Mills at two loops. The main tools of our calculation are generalised unitarity applied at the form factor level, and the compact expressions for supersymmetric tree-level form factors and amplitudes entering the cuts. We confirm that infrared divergences exponentiate as expected, and that collinear factorisation is entirely captured by an ABDK/BDS ansatz. Next, we construct the two-loop remainder function obtained by subtracting this ansatz from the full two-loop form factor and compute it numerically. Using symbolology, combined with various physical constraints and symmetries, we find a unique solution for its symbol. With this input we construct a remarkably compact analytic expression for the remainder function, which contains only classical polylogarithms, and compare it to our numerical results. Furthermore, we make the surprising observation that our remainder is equal to the maximally transcendental piece of the two-loop Higgs plus three-gluon scattering amplitudes in QCD.

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1 Introduction

The last years have witnessed dramatic progress in our understanding of seemingly unrelated physical quantities in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM), in particular scattering amplitudes, Wilson loops and correlations functions. One important lesson is that these objects are actually related in unexpected ways, which has led to major conceptual insights, vastly improved techniques to calculate them and the discovery of new structures and symmetries.

In this context it is natural to consider simple, interesting generalisations of the quantities mentioned above, such as form factors. In a gauge theory, one considers the overlap of a state created by a gauge-invariant operator, $\mathcal{O}(x)|0\rangle$, with a multiparticle state $\langle 1 \cdots n|$. The form factor is then defined as

$$\int d^4x e^{-iqx} \langle 1 \cdots n | \mathcal{O}(x) | 0 \rangle = \delta^{(4)}(q - \sum_{i=1}^n p_i) F(1, \dots, n), \quad (1.1)$$

where $F(1, \dots, n) := \langle 1 \cdots n | \mathcal{O}(0) | 0 \rangle$.

For $n = 2$, the latter is the celebrated Sudakov form factor, which is a famous example of exponentiation – a phenomenon discovered more recently in scattering amplitudes of $\mathcal{N} = 4$ SYM [1, 2], and is intimately linked to scattering amplitudes, whose universal infrared divergences are captured by Sudakov form factors. An n -point form factor can also be viewed as describing the decay of a state created by the gauge-invariant operator \mathcal{O} into an n -particle state. For $n = 2$ the kinematic dependence is trivial, and the form factor can be expressed in terms of the cusp and collinear anomalous dimensions. In planar $\mathcal{N} = 4$ SYM, the former is known to all orders in the gauge coupling [3].

In this paper we will mainly be concerned with form factors with $n > 2$, which have highly non-trivial kinematic dependence. While form factors share several properties with scattering amplitudes, such as soft and collinear factorisation, they also display interesting and important differences. This is due to the insertion of a gauge-invariant operator, which carries momentum that can be injected at any location in Feynman diagrams, and is thus not restricted by the colour ordering of the external, on-shell particles. In practice this leads to a more complicated colour management and to the appearance of non-planar loop integral topologies, even if we restrict ourselves to the planar limit, as we do in this paper. Furthermore, this also implies that the dual conformal (super)symmetry of planar amplitudes in $\mathcal{N} = 4$ SYM is not a property of form factors.

Form factors at strong coupling in $\mathcal{N} = 4$ SYM were recently studied in [4, 5]. The particular choice of the operator considered in those papers is immaterial, as long as its conformal dimension is small compared to $\sqrt{\lambda}$. Specifically, in [5] form factors were analysed in great detail and the explicit expression of the four-point form factor was determined in the case of $(1 + 1)$ -dimensional kinematics. This is the first example of a nontrivial form factor remainder at strong coupling. In [6, 7] we have begun a systematic investigation of form factors of bilinear half-BPS operators at weak coupling, following a pioneering paper of Van Neerven [8].¹ In particular, in [6] we determined the explicit one-loop expressions for these form factors for an arbitrary number of external particles with an MHV helicity configuration.

In this paper we will focus on the two-loop calculation of the three-point form factor. In QCD, similar quantities have been calculated at one [11] and, more recently, two loops [12, 13] using Feynman diagrams. These form factors are phenomenologically important since they are related to the scattering of $e^+e^- \rightarrow 3$ jets and $H \rightarrow 3$ jets. Here we will use instead (generalised) unitarity [14–17] applied directly at the level of the form factor [6] in order to determine the two-loop, three-point form factor from its cuts.

¹A parallel investigation was carried out in [9, 10].

In Section 2 we will set the stage by presenting the details of a simpler calculation, namely that of the Sudakov form factor at two loops. Section 3 describes the calculation of our three-point form factor at two loops from four-dimensional generalised cuts. We obtain a compact result, presented in (3.11), expressed in terms of planar and non-planar two-loop integral functions. Most (but not all) of the analytic expressions for these functions have been derived in [18–20]. For this reason, we have resorted to numerical methods in order to collect data about this form factor which can then be compared to analytic expressions. The main tool here is the Mellin-Barnes representation of higher-loop integrals (see for example [21]), and their evaluation using the powerful numerical algorithm of [22].

In order to present the two-loop form factor in an efficient way, we construct in Section 4 a remainder function very much in the way as for the case of scattering amplitudes. At two loops, this remainder is

$$\mathcal{R}_n^{(2)} := \mathcal{G}_n^{(2)}(\epsilon) - \frac{1}{2}(\mathcal{G}_n^{(1)}(\epsilon))^2 - f^{(2)}(\epsilon)\mathcal{G}_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon) \quad , \quad (1.2)$$

where $\mathcal{G}_n^{(L)}$ is the colour-stripped L -loop form factor divided by its tree-level expression, $C^{(2)}$ is an n -independent constant and $f^{(2)}(\epsilon)$ contains the cusp and collinear anomalous dimensions. One important reason to introduce this remainder is that, as we show explicitly in Section 4, it has the correct collinear limits. More precisely, if $C^{(2)}$ is chosen appropriately then $\mathcal{R}_2^{(2)} = 0$ and in a simple collinear limit $\mathcal{R}_n^{(2)} \rightarrow \mathcal{R}_{n-1}^{(2)}$. In other words, the ABDK/BDK ansatz for the form factor captures correctly the collinear behaviour of the two-loop form factor. The remaining task is then to determine this remainder function which is a function of scale invariant ratios of Mandelstam variables only.

Our main weapon to attack this remainder analytically is the use of symbols [23]. This is a very powerful concept that allows to re-express complicated identities between polylogarithmic functions in terms of simple algebraic identities. It was used in [23] to rewrite the Wilson loop six-point remainder function found in [24] in a very compact form. Because of the amplitude/Wilson loop duality [25–27], this remainder is also equal to the six-point MHV amplitude remainder at two loops, as was confirmed numerically in [28, 29]. Recent interesting applications of the symbol to the construction of various remainder functions in $\mathcal{N} = 4$ SYM were presented in [30–35].

As we will discuss in Section 4, we can impose several physical constraints, such as collinear limits and various requirements on the first, second and last entry of the symbol, directly at the level of the symbol. In the present instance, we find that the various physical properties are so stringent that the symbol of the two-loop remainder function at three points is determined uniquely; we present its explicit expression in (4.28). An important point we would like to anticipate is that the symbol of the three-point form factor at two loops depends on a very restricted set of variables, as we discuss in detail in Section 4. Furthermore, it obeys a particular symmetry condition [23] which guarantees that the corresponding function can be written in terms of classical polylogarithms up to degree four only. Using this information, we will construct a surprisingly simple analytic expression for the complete remainder function, shown in (4.32). We note that beyond-the-symbol ambiguities can easily be fixed by imposing correct behaviour in collinear limits. Within errors, our result is in good agreement with our numerical results obtained for various kinematic points.

We also observe an interesting relation between the symbol of the three-point form factor remainder and a piece of the six-point MHV amplitude remainder of [23] upon appropriate identification of the kinematic variables. This is reminiscent of a result found in [5], where the four-point form factor in $(1+1)$ -dimensional kinematics was evaluated at strong coupling and expressed in terms of the eight-point MHV amplitude remainder in $(1+1)$ -dimensional kinematics.

We will conclude by establishing an intriguing relation between our remainder and the finite remainder function (defined with a prescription introduced by Catani [36]) of the two-loop amplitude $H \rightarrow ggg$ in QCD, a quantity that was recently presented in [12, 13] in the large top mass limit. Interestingly, if we translate the latter into a ABDK/BDS remainder function and extract the terms of highest transcendentality, which is four at two loops, we find a function that has the same symbol as the remainder of the three-point form factor calculated in this paper. In fact, this observation provides us with an alternative way to find the analytic $\mathcal{N} = 4$ SYM remainder, although our expression is much more compact – specifically, the analytic formula obtained pursuing this alternative strategy contains also two-dimensional harmonic polylogarithms (2dHPL’s) [19, 20] which, as we mentioned, are not present in our expression (4.32). Remaining beyond-the-symbol ambiguities can again be fixed using collinear limits and symmetry. We have checked by explicit comparison that these two representations of the remainder are in complete agreement.

To our knowledge, this is the first occurrence of the principle of maximal transcendentality [37], which relates the maximally transcendental part of a quantity in QCD to the same quantity in planar $\mathcal{N} = 4$ SYM, in an observable with non-trivial kinematic dependence. It would be interesting to find more examples of this, and understand whether this observation can be used to simplify the calculation and expressions of other scattering amplitudes in QCD.

2 Warm up: the Sudakov form factor at two loops

In this section we present a very simple, unitarity-based derivation of the two-loop Sudakov form factor $F(q^2) := \langle \phi_{12}(p_1)\phi_{12}(p_2) | \text{Tr}(\phi_{12}\phi_{12})(0) | 0 \rangle$, where $q := p_1 + p_2$. This quantity was calculated at one and two loops in $\mathcal{N} = 4$ SYM in [8] using unitarity applied to Feynman diagrams [42], see also [6, 38, 39], and recently at three loops in [40]. Here we will use an approach based on unitarity directly applied at the level of the form factor. This approach was introduced in [6] and is a straightforward generalisation of the unitarity-based method of [14, 15].

There is only one kinematic channel, the q^2 channel, hence it is sufficient to consider two-particle cuts, and lift the cut integral to a full D -dimensional integral, as in [45]. There are two distinct configurations contributing to such a cut: in the first one, a tree-level, two-point form factor and a one-loop amplitude enter the cut, whereas in the second one we have a one-loop two-point form factor, and a tree-level four-point amplitude. Let us focus on the first possibility. Furthermore, for the sake of illustration, we will perform the calculation in two different ways: firstly, using the conventional representation of the

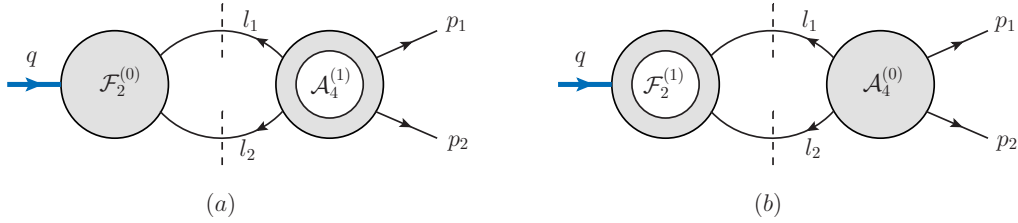


Figure 1: *The two two-particle cuts contributing to the Sudakov form factor.*

one-loop amplitude entering the cut in terms of fundamental colour generators, and then using its representation in terms of adjoint generators [48].

2.1 Calculation with fundamental generators

The first quantity entering the cut is the four-point one-loop amplitude, whose expression in terms of fundamental colour generators is given in (A.1) with $n = 4$. This expression has to be convoluted with a tree-level two-point form factor, simply given by $\delta^{a_1 a_2}$, see Figure 1a. We focus first on the contribution from the planar amplitude, i.e. the first line in (A.1). There are six possible permutations to consider, which give rise to the single-trace structures²

$$\begin{aligned} & \text{Tr}(1, 2, l_1, l_2), \quad \text{Tr}(1, 2, l_2, l_1), \quad \text{Tr}(1, l_1, l_2, 2), \quad \text{Tr}(1, l_2, l_1, 2), \\ & \text{Tr}(1, l_1, 2, l_2), \quad \text{Tr}(1, l_2, 2, l_1). \end{aligned} \quad (2.1)$$

When contracting with the tree form factor $\langle \phi_{12}^{a_1}(l_1) \phi_{12}^{a_2}(p_2) | \text{Tr}(\phi_{12} \phi_{12})(0) | 0 \rangle = \delta^{a_1 a_2}$, and using $\delta^{ab} T_{ij}^a T_{lm}^b = \delta_{im} \delta_{jl}$ and $\delta^{ab} T_{ij}^a T_{jm}^b = N \delta_{im}$ we see that only the first line of (2.1) is leading in colour. The four traces give an identical result, namely $N \text{Tr}(a_1 a_2) = N^2 \delta^{a_1 a_2}$. The contribution to the cut from the planar one-loop amplitude is then

$$N^2 \delta^{a_1 a_2} \left[A_{4;1}^{(1)}(1, 2, l_1, l_2) + A_{4;1}^{(1)}(1, 2, l_2, l_1) + A_{4;1}^{(1)}(2, 1, l_1, l_2) + A_{4;1}^{(1)}(2, 1, l_2, l_1) \right]. \quad (2.2)$$

We now consider the non-planar part of the one-loop amplitude (see section A for details). The corresponding contribution has the form given in (A.2), where the COP for $\{\alpha\} = \{2, 1\}$ and $\{\beta\} = \{3, 4\}$ (corresponding to $c = 3$ in (A.2)) are³

$$(2, 1, 3, 4), \quad (2, 3, 1, 4), \quad (3, 2, 1, 4), \quad (1, 2, 3, 4), \quad (1, 3, 2, 4), \quad (3, 1, 2, 4), \quad (2.3)$$

and the non-planar one-loop piece (second line of (A.1)) is

$$\begin{aligned} \mathcal{A}_{\text{NP}}^{(1)}(1, \dots, 4) &= \text{Tr}(1, 2) \text{Tr}(l_1, l_2) A_{4;3}^{(1)}(1, 2, l_1, l_2) + \text{Tr}(1, l_1) \text{Tr}(2, l_2) A_{4;3}^{(1)}(1, l_1, 2, l_2) \\ &+ \text{Tr}(1, l_2) \text{Tr}(2, l_1) A_{4;3}^{(1)}(1, l_2, 2, l_1), \end{aligned} \quad (2.4)$$

²To keep the notation simple we define $\text{Tr}(1, 2, l_1, \dots) := \text{Tr}(T^{a_1} T^{a_2} T^{a_{l_1}} \dots)$ etc.

³The meaning of the notation $\text{COP}\{\alpha\}\{\beta\}$ is explained below (A.2).

with

$$\begin{aligned}
A_{4;3}^{(1)}(1, 2, l_1, l_2) &= A_{4;1}^{(1)}(2, 1, l_1, l_2) + A_{4;1}^{(1)}(2, l_1, 1, l_2) + A_{4;1}^{(1)}(l_1, 2, 1, l_2) \\
&+ A_{4;1}^{(1)}(1, 2, l_1, l_2) + A_{4;1}^{(1)}(1, l_1, 2, l_2) + A_{4;1}^{(1)}(l_1, 1, 2, l_2). \quad (2.5)
\end{aligned}$$

Contracting with the tree-level form factor, we see that the leading structures in colour are of the form $\delta^{a_1 a_2} \text{Tr}(l_1 l_2) \text{Tr}(12) = N^2 \delta^{a_1 a_2}$. Collecting terms, we obtain a two-loop cut integrand

$$F^{(2)a_1 a_2}(q^2) \Big|_{q^2\text{-cut}} = 2 N^2 \delta^{a_1 a_2} \int d\text{LIPS}(l_1, l_2; q) \left[4 A_{4;1}^{(1)}(1, 2, l_1, l_2) + A_{4;1}^{(1)}(1, l_1, 2, l_2) \right], \quad (2.6)$$

where the Lorentz invariant phase space measure in dimensional regularisation is

$$d\text{LIPS}(l_1, l_2; q) := d^D l_1 d^D l_2 \delta^+(l_1^2) \delta^+(l_2^2) \delta^D(l_1 + l_2 + q), \quad (2.7)$$

with $D = 4 - 2\epsilon$. The one-loop component amplitudes appearing in (2.6) are given, to all orders in the dimensional regularisation parameter ϵ by

$$\begin{aligned}
A_{4;1}^{(1)}(1, 2, l_1, l_2) &= A^{(0)}(\phi_{12}(p_1), \phi_{12}(p_2), \phi_{34}(l_1), \phi_{34}(l_2)) F^{0\text{m}}(1, 2, l_1, l_2) \\
A_{4;1}^{(1)}(1, l_1, 2, l_2) &= A^{(0)}(\phi_{12}(p_1), \phi_{34}(l_1), \phi_{12}(p_2), \phi_{34}(l_2)) F^{0\text{m}}(1, l_1, 2, l_2) \quad (2.8)
\end{aligned}$$

where $F^{0\text{m}}(1, \dots, 4) = s_{12} s_{23} I^{0\text{m}}(1, \dots, 4)$ is a zero-mass box function. The tree amplitudes $A^{(0)}(\phi_{12}(p_1), \phi_{12}(p_2), \phi_{34}(l_1), \phi_{34}(l_2))$ and $A^{(0)}(\phi_{12}(p_1), \phi_{34}(l_1), \phi_{12}(p_2), \phi_{34}(l_2))$ can be extracted from Nair's superamplitude [49]

$$\mathcal{A}_{\text{MHV}} := g^{n-2} (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i \right) \delta^{(8)} \left(\sum_{i=1}^n \lambda_i \eta_i \right) \prod_{i=1}^n \frac{1}{\langle ii+1 \rangle}, \quad (2.9)$$

where $\lambda_{n+1} \equiv \lambda_1$, giving⁴

$$A^{(0)}(\phi_{12}(p_1), \phi_{12}(p_2), \phi_{34}(l_1), \phi_{34}(l_2)) = \frac{\langle l_1 l_2 \rangle \langle 12 \rangle}{\langle l_2 1 \rangle \langle 2 l_1 \rangle}, \quad (2.10)$$

$$A^{(0)}(\phi_{12}(p_1), \phi_{34}(l_1), \phi_{12}(p_2), \phi_{34}(l_2)) = \frac{\langle 12 \rangle^2 \langle l_1 l_2 \rangle^2}{\langle 1 l_1 \rangle \langle l_1 2 \rangle \langle 2 l_2 \rangle \langle l_2 1 \rangle}. \quad (2.11)$$

On the cut one easily finds

$$\frac{\langle 12 \rangle \langle l_1 l_2 \rangle}{\langle 2 l_1 \rangle \langle l_2 1 \rangle} = -\frac{q^2}{(l_2 + p_1)^2}. \quad (2.12)$$

Using this relation we easily arrive at

$$F^{(2)ab}(q^2) = N^2 \delta^{ab} F^{(2)}(q^2), \quad (2.13)$$

with

$$F^{(2)}(q^2) = 2 \left[4 \text{LT}(q^2, \epsilon) + \text{CT}(q^2, \epsilon) \right], \quad (2.14)$$

⁴Note that $s_{12} s_{2l_1} A^{(0)}(\phi_{12}(p_1), \phi_{12}(p_2), \phi_{34}(l_1), \phi_{34}(l_2)) = s_{1l_1} s_{l_1 2} A^{(0)}(\phi_{12}(p_1), \phi_{34}(l_1), \phi_{12}(p_2), \phi_{34}(l_2))$, which simplifies the cut algebra.

where the two-loop ladder and crossed triangle, $\text{LT}(q^2, \epsilon)$, $\text{CT}(q^2, \epsilon)$ respectively, are given by [41–44, 21]⁵

$$\begin{aligned} \text{LT}(q^2, \epsilon) &= (-q^2)^{-2\epsilon} e^{2\gamma\epsilon} \left\{ \frac{1}{\epsilon} \left[\frac{1}{2\epsilon} G(2, 2) G_3(2 + \epsilon, 1, 1) \right. \right. \\ &\quad \left. \left. - G(2, 1) \left[\frac{1}{\epsilon} G_3(2, 1, 1 + \epsilon) + G_3(1, 1, 1) \right] \right] \right\} \end{aligned} \quad (2.15)$$

$$\begin{aligned} &= (-q^2)^{-2\epsilon} \left[\frac{1}{4\epsilon^4} + \frac{5\pi^2}{24\epsilon^2} + \frac{29}{6\epsilon} \zeta_3 + \frac{3}{32} \pi^4 + \mathcal{O}(\epsilon) \right], \\ \text{CT}(q^2, \epsilon) &= (-q^2)^{-2\epsilon} \left[\frac{1}{\epsilon^4} - \frac{\pi^2}{\epsilon^2} - \frac{83}{3\epsilon} \zeta_3 - \frac{59}{120} \pi^4 + \mathcal{O}(\epsilon) \right], \end{aligned} \quad (2.16)$$

where

$$G(x, y) = \frac{\Gamma(x + y + \epsilon - 2) \Gamma(2 - \epsilon - x) \Gamma(2 - \epsilon - y)}{\Gamma(x) \Gamma(y) \Gamma(4 - x - y - 2\epsilon)}, \quad (2.17)$$

$$G_3(x, y, z) = \frac{\Gamma(2 - x - z - \epsilon) \Gamma(2 - y - z - \epsilon) \Gamma(-2 + x + y + z + \epsilon)}{\Gamma(x) \Gamma(y) \Gamma(4 - x - y - z - 2\epsilon)}. \quad (2.18)$$

Going through the same steps for the cut in Figure 1*b* one can easily confirm the result we have derived here.

2.2 Calculation with adjoint generators

The same calculation outlined in the previous section can be described in a very efficient way using the representation of one-loop amplitudes in terms of adjoint generators [48] described in (A.3). We consider again a two-particle cut with a tree-level form factor and a one-loop four-point amplitude

$$\mathcal{A}^{(1)}(1, 2, l_1, l_2) = \sum_{\sigma \in S_4 / (\mathbb{Z}_4 \times \mathcal{R})} \text{Tr}(F^{a_{\sigma_1}} \dots F^{a_{\sigma_4}}) A_{4;1}^{(1)}(\sigma_1, \dots, \sigma_4), \quad (2.19)$$

where $(\sigma_1, \dots, \sigma_4)$ are permutations of $(1, 2, l_1, l_2)$. The sum contains three terms, associated with the orderings $(1, 2, l_1, l_2)$, $(1, 2, l_2, l_1)$ $(1, l_1, 2, l_2)$. On the cut, we contract the colour labels a_{l_1} , a_{l_2} of the loop legs with a $\delta^{a_{l_1} a_{l_2}}$ from the two-point tree-level form factor, as already described in the previous section. Doing this, the first two, and the last colour orderings will give rise to the following two colour traces

$$\delta^{a_{l_1} a_{l_2}} \text{Tr}(F^{a_1} F^{a_2} F^{a_{l_1}} F^{a_{l_2}}), \quad \delta^{a_{l_1} a_{l_2}} \text{Tr}(F^{a_1} F^{a_{l_1}} F^{a_2} F^{a_{l_2}}), \quad (2.20)$$

respectively, where⁶

$$\text{Tr}(F^{a_1} F^{a_2} F^a F^a) = 4 N^2 \delta^{a_1 a_2}, \quad \delta^{ab} \text{Tr}(F^{a_1} F^a F^{a_2} F^a) = 4 \frac{N^2}{2} \delta^{a_1 a_2}. \quad (2.21)$$

⁵In the following formulae we actually divide these functions by a power of q^2 per loop.

⁶Because of the slightly unconventional normalisation of the adjoint generators, we have $\text{Tr}(F^a F^b) = 2N\delta^{ab}$, and $(F^a F^a)_{bc} = 2N\delta_{bc}$. This different normalisation for the adjoint generators is responsible for the appearance of factors of 4 in (2.20).

Taking into account these relations, one instantly arrives at (2.6).

Before closing this section, we also quote two useful formulae when dealing with adjoint traces, namely

$$\mathrm{Tr}(F^a A F^a B) = \mathrm{Tr}(F^a A^t F^a B) + \mathrm{Tr}(F^a A) \mathrm{Tr}(F^a B) , \quad (2.22)$$

which, in the case where $A^t = -A$, simplifies to

$$\mathrm{Tr}(F^a A F^a B) = \frac{1}{2} \mathrm{Tr}(F^a A) \mathrm{Tr}(F^a B) \quad \text{if } A^t = -A . \quad (2.23)$$

This explains the factor of $1/2$ in the second relation in (2.21).

3 Generalised unitarity for planar form factors

In this section we compute higher-point form factors using generalised unitarity in four dimensions [16, 17]. The explicit calculation will focus on the MHV three-point two-loop form factor,

$$F_3^{\mathrm{MHV},(2)}(1, 2, 3) := \langle \phi_{12}(p_1) \phi_{12}(p_2) g^+(p_3) | \mathrm{Tr}(\phi_{12} \phi_{12})(0) | 0 \rangle , \quad (3.1)$$

for which we will now derive the complete result. Recall that the operator $\mathrm{Tr}(\phi_{12} \phi_{12})$ is a half-BPS operator and is the lowest component of the stress-tensor multiplet. Hence, we expect the result to be the same if this operator is replaced by any other operator in the stress-tensor multiplet [7]. A different choice of operator usually requires a different choice of external states as well. We also notice that in $\mathcal{N} = 4$ SYM there is nothing special about the particular choice of helicities of the external states in (3.1) – any other choice with the same total helicity and R -charge are possible. In fact, we could have worked in a more invariant way by using supersymmetric form factors as introduced in [7], and in the case of MHV form factors all helicity dependence factors out into the MHV tree-level (super) form factor,

$$F_3^{\mathrm{MHV},(2)}(1, 2, 3) = F_3^{\mathrm{MHV},(0)}(1, 2, 3) \mathcal{G}_3^{(2)}(1, 2, 3) , \quad (3.2)$$

where the helicity-blind function $\mathcal{G}_3^{(2)}(1, 2, 3)$ depends only on the momenta through $s_{12} := 2p_1 \cdot p_2$, $s_{23} := 2p_2 \cdot p_3$, $s_{31} := 2p_3 \cdot p_1$, and $q^2 := s_{12} + s_{23} + s_{31}$; finally, $F_3^{\mathrm{MHV},(0)}(1, 2, 3)$ is the tree-level (super) form factor. For the particular helicity configuration in (3.1) this tree-level form factor is $\langle 12 \rangle / (\langle 23 \rangle \langle 31 \rangle)$. It is easy to see that in all calculations this tree-level piece factors out naturally, and therefore the goal is to determine $\mathcal{G}_3^{(2)}(1, 2, 3)$. We mention that this ratio function is also shared by the NMHV three-point form factor defined by factoring out the NMHV tree-level form factor, due to the chiral symmetry of three-point case [7]. We also note in passing that three-point form factors with adjoint fields always have a colour factor proportional to the structure constants $f^{a_1 a_2 a_3}$.

We begin by reconsidering the cut we are familiar with from the two-particle two-loop form factor, namely the two-particle cut in the q^2 channel, where the cut integrand is given as the product of a tree-level two-point form factor and a one-loop amplitude. As

we have seen in the two-point case, the double-trace non-planar amplitudes also contribute to planar form factors because of the colour index contraction

$$\mathrm{Tr}(T^{a_{l_1}} T^{a_{l_2}}) [\mathrm{Tr}(T^{a_{l_1}} T^{a_{l_2}}) \mathrm{Tr}(T^{a_1} \dots T^{a_n})] = N^2 \mathrm{Tr}(T^{a_1} \dots T^{a_n}). \quad (3.3)$$

Using the general formulae for the one-loop amplitudes (A.1)-(A.2), we immediately obtain the corresponding cut integrand of the planar n -point form factor,

$$\begin{aligned} F_n^{(2)}(q^2) \Big|_{q^2\text{-cut}} &= \int d\mathrm{LIPS}(l_1, l_2; q) \sum_{\sigma \in \mathcal{S}_n} \left[4 A_{n;1}(l_1, l_2, \sigma(1), \dots, \sigma(n)) \right. \\ &\quad + 2 \sum_{i=1}^{\lfloor n-1 \rfloor / 2} A_{n;1}(l_1, \sigma(1), \dots, \sigma(i), l_2, \sigma(i+1), \dots, \sigma(n)) \\ &\quad \left. + A_{n;1}(l_1, \sigma(1), \dots, \sigma(n/2), l_2, \sigma(n/2+1), \dots, \sigma(n)) \right], \end{aligned} \quad (3.4)$$

where the last line contributes only when n is even. In particular, for three-point form factors we have

$$\begin{aligned} F_3^{(2)}(q^2) \Big|_{q^2\text{-cut}} &= \int d\mathrm{LIPS}(l_1, l_2; q) \\ &\quad \sum_{\sigma \in \mathcal{S}_3} \left[4A_{5;1}(l_1, l_2, \sigma(1), \sigma(2), \sigma(3)) + 2A_{5;1}(l_1, \sigma(1), l_2, \sigma(2), \sigma(3)) \right]. \end{aligned} \quad (3.5)$$

Note that in the above presentation we have fixed the order of the external on-shell states, and stripped off the colour factor $N^2 \mathrm{Tr}(T^{a_1} T^{a_2} \dots T^{a_n})$. We call this quantity a colour-stripped form factor, or colour-ordered form factor. The complete planar form factor can be expressed in terms of these colour-stripped form factor times the appropriate colour factor, and summing over non-cyclic permutation of external on-shell states. It is this colour-ordered form factor that we will focus on in the later calculation.

We would also like to point out that colour-ordered form factors can also be computed directly using colour-stripped Feynman rules, which are simple generalisation of the well-known colour-stripped Feynman rules for amplitudes. The relevant diagrams are planar diagrams – in the sense that we fix the ordering of the external on-shell particles, while the off-shell momentum q of the gauge-invariant operator can appear in any possible position. Hence, in momentum space, diagrams can look non-planar, while in fact in colour-space they are planar.

We will now focus on the computation of the three-point form factor. Unlike the two-point case, it is technically more difficult to lift the two-particle q^2 cut directly to the two-loop form factor, because of the presence of the one-loop five-point amplitude (3.5). However, we have used these cuts and two-particle cuts in other channels as important cross-checks of our final result.

For n -point form factors, not much is known about the set of two-loop integral functions appearing (except for $n = 3$ [19, 20]), and since dual conformal symmetry is not a feature of form factors it cannot be used to reduce the set of allowed integrals. Furthermore, planar form factors, as already seen in the case of the Sudakov form factor, contain non-planar integral topologies. Here we follow a two-step procedure that does not rely on

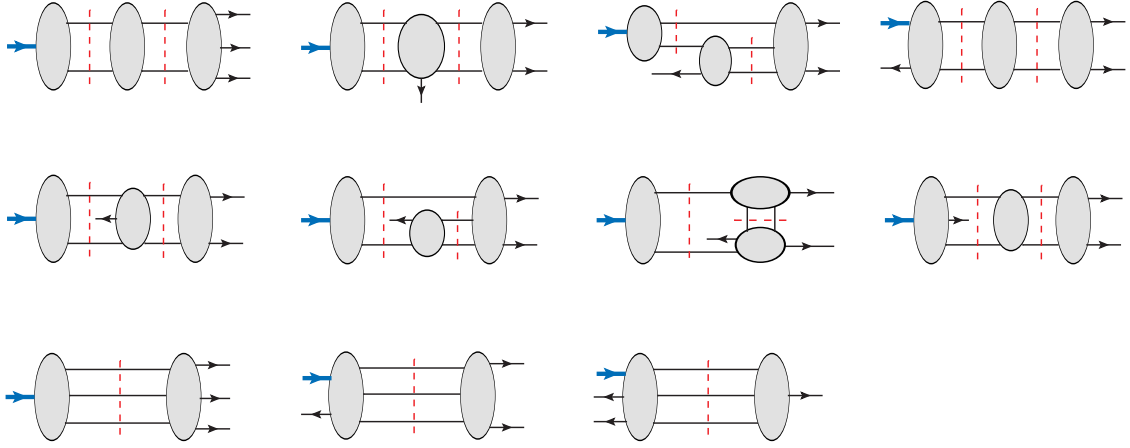


Figure 2: *Double two-particle cuts and three-particle cuts of three-point form factor.*

the knowledge of a particular basis of integral functions: in the first step we use double two-particle cuts to identify a basis. This set of cuts leads to sufficiently simple integrands so that their tensor reductions can be performed with relative ease and a simple set of integral functions appears. In this step we determine all possible integral topologies together with their coefficients, but are left with ambiguities in the numerators of the form $\sim l_i^2$, where l_i is one of the cut momenta. In the second step we perform more intricate and stringent triple cuts to fix any remaining ambiguities. One additional complication here is that sums over the helicities of the internal particles have to be performed. However, this can be dealt with elegantly using superamplitudes and super form factors [7]. This procedure leads to a unique answer that passes all consistency checks.

More concretely, following the strategy outlined above we determined the integrals by considering the set of double two-particle cuts and three-particle cuts shown in Figure 2. These cuts are sufficient to determine the three-point form factor completely. We have also considered various two-particle cuts with one-loop form factor times tree-level amplitudes, and tree-level form factor times one-loop amplitudes which provided important additional cross checks, but did not lead to new integral functions.

We also note that no parity-odd terms can be present in the final result for the three-point form factor since there are not enough independent momenta to form a non-vanishing contraction with $\epsilon_{\mu\nu\rho\lambda}$. We also expect there should be no μ^2 -terms in the expressions of the cuts for the three-point case, and hence it is sufficient to use four-dimensional unitarity. For higher points, we expect that the μ^2 -terms will appear and cuts will then have to be performed in D dimensions.

We now provide some more details on the procedure we followed. Starting from the two-particle cut expression (3.5), one can apply a further two-particle cut to the one-loop five-point amplitudes. The cut integrand is then given by the product of a two-point tree-level form factor and two tree amplitudes. We also consider the cuts which are given by a three-point tree form factor and two tree amplitudes. These types of cuts are depicted in the first two lines of Figure 2. The cut integrands are simple enough to perform the necessary tensor reduction directly. In this way, we find a set of simple integral functions with simple coefficients containing all integrals that appear in the final answer, given in

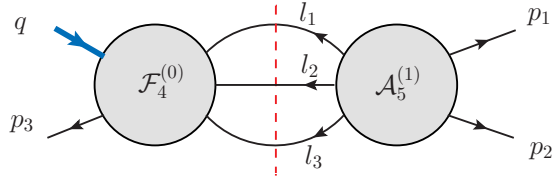


Figure 3: A particular three-particle cut of the three-point form factor.

Figure 6⁷.

However, we are left with certain ambiguities due to $l_{1,2}^2$ terms in the numerator of integrals such as *DBox* and *NBox* in Figure 6, where $l_{1,2}$ correspond to cut propagators. Such terms are not detected in the double two-particle cuts considered. Besides, there are also integrals which are not detected by these double two-particle cuts. Both problems can be fixed by considering three-particle cuts.

The three-particle cuts on their own involve several integral topologies.⁸ The cuts we have considered are shown in the third line of Figure 2, and involve up to six-point NMHV amplitudes and five-point NMHV form factors. The cut integrands are therefore much more complicated compared to double two-particle cuts, which makes it much harder to perform the tensor reduction directly in order to obtain a set of simple integral functions. That is why we chose to use first double two-particle cuts to write down an ansatz, which we then verify and refine using three-particle cuts.

For these checks we do not need the reduction of the triple-cut expressions as we can make analytic comparisons of the *integrand*s arising from the triple cut and those coming from the ansatz directly, by choosing a basis for spinors and expressing both integrands in this basis.

To be more explicit, let us consider the particular three-particle cut shown in Figure 3. The cut integrand is given as a product of a four-point tree form factor, and a five-point tree amplitude. There are two kinds of contributions to the cut integrand, depending on which one is next-to-MHV:

$$\int d^4\eta_{l_1} d^4\eta_{l_2} d^4\eta_{l_3} \left[\mathcal{F}_4^{\text{MHV},(0)}(-l_1, -l_2, -l_3, 3) \mathcal{A}_5^{\text{NMHV},(0)}(1, 2, l_3, l_2, l_1) + \mathcal{F}_4^{\text{NMHV},(0)}(-l_1, -l_2, -l_3, 3) \mathcal{A}_5^{\text{MHV},(0)}(1, 2, l_3, l_2, l_1) \right], \quad (3.6)$$

where in the last equation we have used supersymmetric amplitudes and form factors [7] in order to perform the sum over internal helicities efficiently [46, 47]. The fermionic integration can now be performed easily, and after switching back to component amplitudes

⁷There are also additional non-planar integrals arising from the reduction of the double trace term in (3.5), some of which are not allowed by the colour structure. However, these are all canceled after performing cyclic summation, and do not appear in the final result.

⁸Integrals which are simple products of one-loop integrals are not detected by three-particle cut, but they are ruled out by the double two-particle cuts.

and form factors the result is

$$\mathcal{F}_3^{\text{MHV},(0)}(1, 2, 3) \left[\frac{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}{\langle l_1 l_2 \rangle \langle l_2 l_3 \rangle \langle l_3 l_1 \rangle} A_5^{\text{NMHV},(0)}(1^+, 2^+, l_3^-, l_2^-, l_1^-) + \frac{\langle 23 \rangle \langle 31 \rangle}{\langle 2l_3 \rangle \langle l_3 l_2 \rangle \langle l_2 l_1 \rangle \langle l_1 l_1 \rangle} F_{4,\text{SD}}^{\text{NMHV},(0)}(-l_1^-, -l_2^-, -l_3^-, 3^+) \right]. \quad (3.7)$$

Note that the amplitudes and form factor in the bracket are the bosonic components with fixed helicities. $F_{4,\text{SD}}^{\text{NMHV},(0)}$ is the form factor with an insertion of the operator $\text{Tr}(F_{SD}^2) + \dots$ (the dots refer to additional cubic and quartic terms) which is part of the stress-energy multiplet. The appearance of this form factor is due to the fact that both $\mathcal{F}_{4,\text{SD}}^{\text{NMHV},(0)}$ and $\mathcal{F}_4^{\text{NMHV},(0)}$ share the same NMHV factor, i.e.

$$R_4^{\text{NMHV},(0)} = \frac{\mathcal{F}_{4,\text{SD}}^{\text{NMHV},(0)}}{\mathcal{F}_{4,\text{SD}}^{\text{MHV},(0)}} = \frac{\mathcal{F}_4^{\text{NMHV},(0)}}{\mathcal{F}_4^{\text{MHV},(0)}}, \quad (3.8)$$

where the former is the super form factor with an insertion of the Lagrangian $\text{Tr}(F_{SD}^2) + \dots$ and the latter with $\text{Tr}(\phi^2)$ [7]. By performing the fermionic integrations we extract the $\eta_{l_1}^4 \eta_{l_2}^4 \eta_{l_3}^4$ component, and obtain $F_{4,\text{SD}}^{\text{NMHV},(0)}(-l_1^-, -l_2^-, -l_3^-, 3^+)$. This form factor can be calculated using BCFW recursion relations as in [7], with the result

$$F_{4,\text{SD}}^{\text{NMHV},(0)}(1^-, 2^-, 3^-, 4^+) = \frac{\langle 3|p_{12}|4 \rangle^3}{s_{412}[41][12]\langle 3|p_{41}|2 \rangle} + \frac{\langle 1|p_{23}|4 \rangle^3}{s_{234}[23][34]\langle 1|p_{34}|2 \rangle} + \frac{\langle 13 \rangle^4 s_{1234}^2}{s_{341}\langle 34 \rangle \langle 41 \rangle \langle 3|p_{41}|2 \rangle \langle 1|p_{34}|2 \rangle}. \quad (3.9)$$

We compared this cut integrand with the expression obtained from the three-particle cut of the integral presentation of the final result given in Figure 6. As shown in Figure 4, there are in total 20 contributing cut integrals. Remarkably, this cut involves all integral topologies appearing in Figure 6 and we find complete agreement with the above non-trivial cut integrand. This provides one of the strongest consistency checks that our result summarised in Figure 6 is indeed correct.

There is an important subtlety we would like to point out. As we mentioned before, we used double two-particle cuts on (3.5) in order to first detect all the integrals which can appear. We find that *TriPent* and *DTri* actually receive two identical contributions from the single-trace and the double-trace term in (3.5). We have drawn the two contributions from the planar and non-planar part in Figures 5(a) and 5(b), respectively. Similar comments apply to *DTri*. This observation is relevant when we consider unitarity cuts. For example, in the three-particle example we considered above, two different presentation of *TriPent* give a different contribution to the cut integrand – indeed, as shown in Figure 5, the positions of l_1 and l_2 are swapped, and therefore the total contribution to the cut integrand is

$$\frac{1}{2} \frac{q^2 s_{12} s_{23}}{(l_3 + p_2)^2 (l_3 - p_3)^2 (l_1 + l_2)^2 (l_1 + q)^2} + \frac{1}{2} \frac{q^2 s_{12} s_{23}}{(l_3 + p_2)^2 (l_3 - p_3)^2 (l_1 + l_2)^2 (l_2 + q)^2}. \quad (3.10)$$

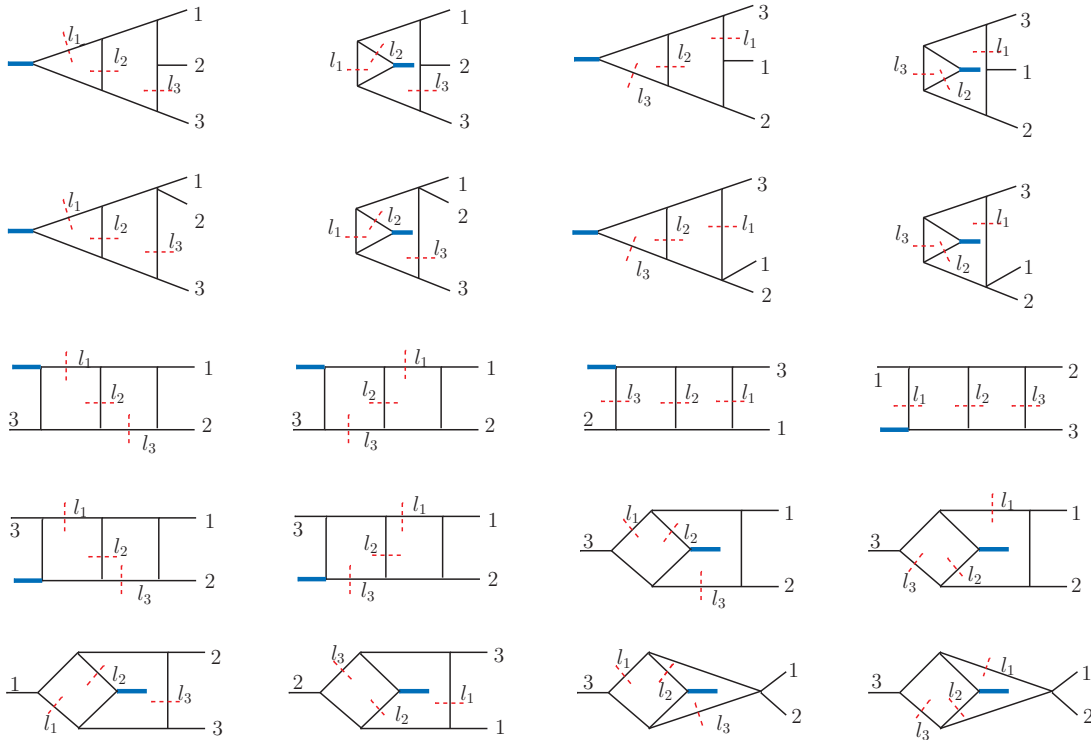


Figure 4: *There are twenty integrals contributing to the three-particle cut example discussed in the text.*

We conclude this section by presenting our complete result for the two-loop three-particle form factor in $\mathcal{N} = 4$ SYM obtained from generalised unitarity cuts:

$$\mathcal{G}_3^{(2)} = \sum_{i=1}^2 (D\text{Tri}_i + D\text{Box}_i) + \text{TriPent} + N\text{Box} + N\text{Tri} + \text{cyclic}, \quad (3.11)$$

where in Figure 6 we have presented compactly all integrals, including their precise coefficients and numerators some of which are loop dependent.

The integral functions appearing in (3.11) have six or seven propagators and notably both planar and non-planar topologies appear, some with irreducible, loop-dependent numerators. The maximum power of loop momentum appearing in the numerators is one. We will now proceed to discuss the numerical evaluation of these integrals.

3.1 Evaluation of the integral functions

Several of the integral functions appearing in our result (3.11) have been computed and explicit expressions can be found in [18–20] – in particular all the six-propagator integrals are known. For the remaining integrals *TriPent*, *DBox_i* and *NBox* explicit analytic expressions are not known, in part due to the peculiar numerators we found. In principle, there exist algorithms that allow to reduce these integrals to known master integrals

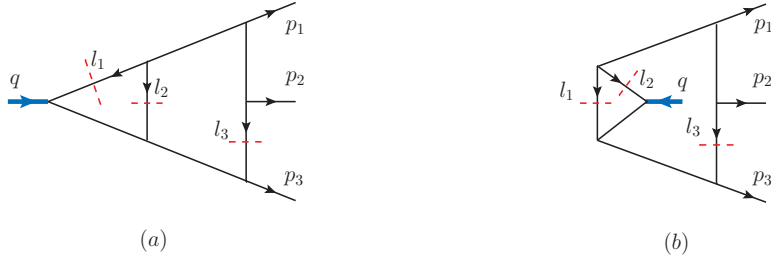


Figure 5: *The two appearances of the integral TriPent in the three-particle cut. They give different contributions to the cut integrand.*

[18–20]. However, in this work we chose to use an independent route, and expressed them in terms of Mellin-Barnes (MB) representations (see [21] for a pedagogical introduction) and used well-established codes [22] to evaluate them numerically to high precision.

The MB form of the planar integrals can readily be constructed with public programmes like AMBRE [50, 51] and we will not quote their explicit forms here. However, the MB representation of the *NBox* integral cannot be found with AMBRE due to well-known issues of this programme with non-planar topologies with several scales. Therefore, we constructed an MB representation of *NBox* directly from its Feynman parameter form. The result is an eight-fold MB representation of the form

$$\begin{aligned}
& \frac{(-q^2)^{-2\epsilon}}{2(2\pi i)^8 \Gamma(-1-3\epsilon)} \int \prod_{i=1}^8 (dz_i \Gamma(-z_i)) u^{z_5+1} v^{z_{678}+1} w^{-3-2\epsilon-z_{12345678}} \times \\
& \Gamma(-\epsilon-z_{34}) \Gamma(-\epsilon+z_4) \Gamma(1+z_{13456}) \Gamma(1+z_{157}) \Gamma(-1-\epsilon+z_3-z_8) \times \\
& \Gamma(-2-2\epsilon-z_1-z_{568}) \Gamma(-2-2\epsilon-z_{134578}) \Gamma(-2-2\epsilon-z_{1234678}) \times \\
& \frac{\Gamma(-2\epsilon-z_3+z_8) \Gamma(1+z_{168}) \Gamma(1+z_{278}) \Gamma(3+2\epsilon+z_{12345678})}{\Gamma(-2\epsilon-z_3) \Gamma(-1-2\epsilon-z_3-z_{48}) \Gamma(-1-2\epsilon+z_{34}-z_8) \Gamma(-2\epsilon-z_{34}+z_8)}, \tag{3.12}
\end{aligned}$$

where we have introduced the shorthand notation $z_{ij\dots k} = z_i + z_j + \dots + z_k$, and

$$u =: \frac{s_{12}}{q^2}, \quad v := \frac{s_{23}}{q^2}, \quad w := \frac{s_{31}}{q^2}, \tag{3.13}$$

with

$$u + v + w = 1. \tag{3.14}$$

Note that for sake of brevity we have dropped here the terms of the numerator which are linear in loop momentum ℓ ; they lead to a number of similar eight-fold MB integrals. Furthermore, due to the $\Gamma(-1-3\epsilon)$ denominator the integral effectively becomes seven-dimensional [21]. In this sense this integral is the most complicated and numerically the most challenging contribution to the form factor, since the planar topologies *DBox_i* and *TriPent* require at most three- and four-fold MB integrals, respectively.

We have evaluated (3.11) by expressing all six-propagator integrals by their known analytic formulae and using `MB.m` by [22] for numerical evaluations of the MB representations of all seven-propagator topologies. The result is a power series starting as ϵ^{-4} which we have computed up to and including finite $\mathcal{O}(\epsilon^0)$ terms. We present here a few results

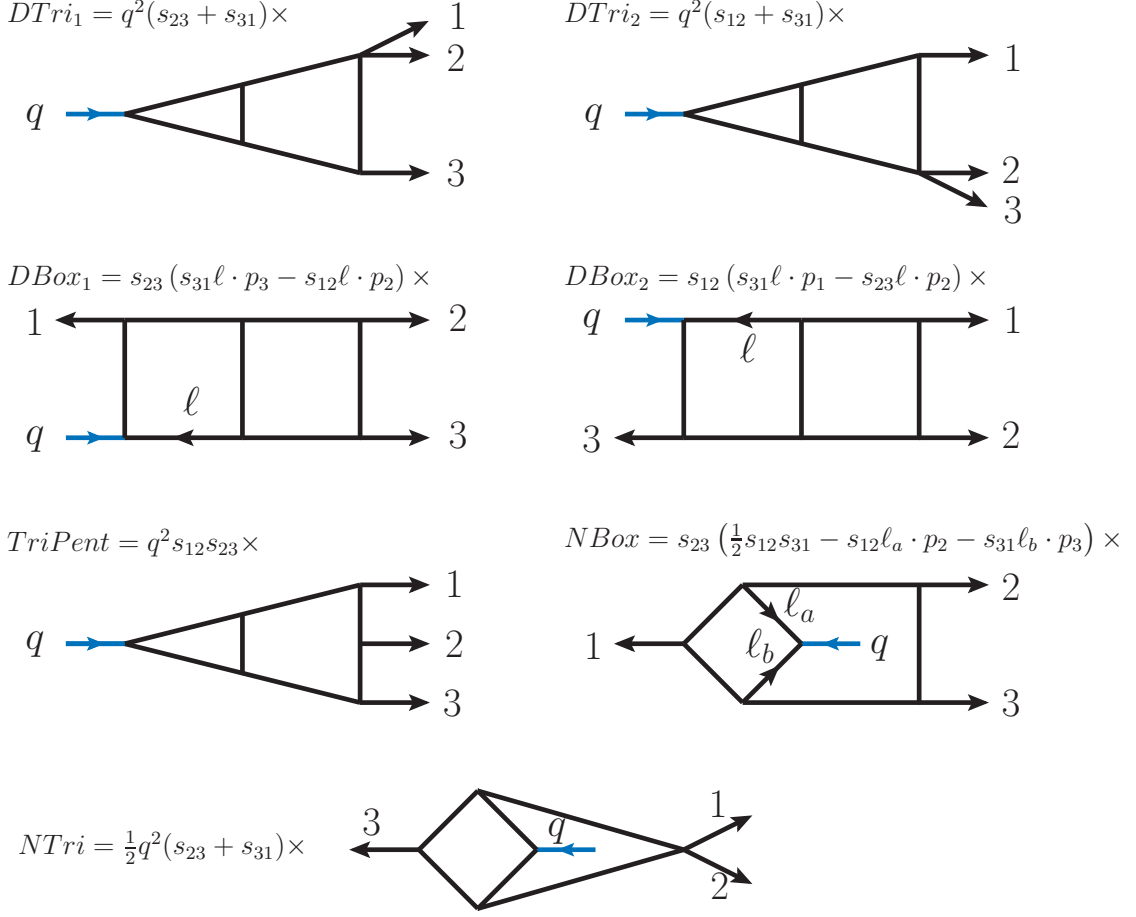


Figure 6: The integral expansion of our final result for the three-point form factor $\mathcal{G}_3^{(2)}$.

of our numerical evaluation at four kinematic points $(-s_{12}, -s_{23}, -s_{31})$:

$$\begin{aligned}
(1, 1, 1) &: \frac{4.5}{\epsilon^4} + \frac{0.}{\epsilon^3} + \frac{6.12223}{\epsilon^2} - \frac{16.7052}{\epsilon} - 18.2484 \pm 0.02 + \mathcal{O}(\epsilon), & (3.15) \\
(1, 1, 2) &: \frac{4.5}{\epsilon^4} - \frac{2.07944}{\epsilon^3} + \frac{7.98765}{\epsilon^2} - \frac{18.9491}{\epsilon} - 7.3182 \pm 0.02 + \mathcal{O}(\epsilon), \\
(1, 2, 2) &: \frac{4.5}{\epsilon^4} - \frac{4.15888}{\epsilon^3} + \frac{9.2099}{\epsilon^2} - \frac{23.0025}{\epsilon} + 1.8686 \pm 0.02 + \mathcal{O}(\epsilon), \\
(1, 2, 3) &: \frac{4.5}{\epsilon^4} - \frac{5.37528}{\epsilon^3} + \frac{11.6703}{\epsilon^2} - \frac{25.9714}{\epsilon} + 10.6624 \pm 0.03 + \mathcal{O}(\epsilon).
\end{aligned}$$

Here we have only quoted the errors of the finite terms. The numerical error of the $1/\epsilon^4$ and $1/\epsilon^3$ are negligible, while for all kinematic points we have investigated, a conservative estimate of the errors of the $1/\epsilon^2$ and $1/\epsilon$ terms is 10^{-13} and 10^{-7} , respectively.

In the following we will study this two-loop result in detail and, motivated by the fact that form factors have universal collinear limits, we will define a finite remainder very much in the spirit of [1, 2]. We will then determine the symbol [23] of the remainder function, and from that derive its analytic expression, which we will compare to our numerical results.

4 Exponentiation of the form factor and the remainder function

In this section we want to consider the possibility that higher-loop form factors in $\mathcal{N} = 4$ SYM obey a similar exponentiation relation as MHV loop amplitudes [1, 2] for small numbers of on-shell particles, and understand from which number of particles a remainder function has to be added. We expect this remainder to appear for a smaller number of particles compared to amplitudes because only Lorentz symmetry and dilatations are unbroken (up to infrared divergences). This leaves us with $3n - 7$ parameters on which a remainder might depend. Indeed we will find that the three-point remainder function depends on two variables. Similar observations were made at strong coupling for $(1 + 1)$ -dimensional kinematics in [5], where a non-trivial remainder appears at four points.

For the Sudakov form factor, exponentiation is evidently true even in QCD, and in $\mathcal{N} = 4$ this was explicitly proved in [8]. Specifically, one finds

$$F^{(2)}(q^2, \epsilon) - \frac{1}{2}(F^{(1)}(q^2, \epsilon))^2 = (-q^2)^{-2\epsilon} \left[\frac{\zeta_2}{\epsilon^2} + \frac{\zeta_3}{\epsilon} + \mathcal{O}(\epsilon) \right], \quad (4.1)$$

where $F^{(L)}$ is the L -loop Sudakov form factor. The result in (4.1) was re-derived in appendix B of [6], where it was recast in a slightly more modern language as

$$F^{(2)}(q^2, \epsilon) - \frac{1}{2}(F^{(1)}(q^2, \epsilon))^2 = f^{(2)}(\epsilon)F^{(1)}(q^2, 2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon), \quad (4.2)$$

where $f^{(2)}(\epsilon) = f_0^{(2)} + f_1^{(2)}\epsilon + f_2^{(2)}\epsilon^2$, with

$$f_0^{(2)} = -2\zeta_2, \quad f_1^{(2)} = -2\zeta_3, \quad (4.3)$$

and with one relation between $f_2^{(2)}$ and $C^{(2)}$, namely⁹

$$C^{(2)} = \frac{f_2^{(2)}}{2} + \frac{\pi^4}{18}. \quad (4.4)$$

For comparison with amplitudes, we recall that the four-point MHV amplitude (divided by the tree-level amplitude) satisfies [1, 2]

$$\mathcal{M}_4^{(2)}(\epsilon) - \frac{1}{2}(\mathcal{M}_4^{(1)}(\epsilon))^2 = f_{\text{amp}}^{(2)}(\epsilon)\mathcal{M}_4^{(1)}(2\epsilon) + C_{\text{amp}}^{(2)} + \mathcal{O}(\epsilon), \quad (4.5)$$

with $f_{\text{amp}}^{(2)}(\epsilon) = \tilde{f}_0^{(2)} + \epsilon\tilde{f}_1^{(2)} + \epsilon^2\tilde{f}_2^{(2)}$, and¹⁰

$$\tilde{f}_0^{(2)} = -\zeta_2 = \frac{f_0^{(2)}}{2}, \quad \tilde{f}_1^{(2)} = -\zeta_3 = \frac{f_1^{(2)}}{2}. \quad (4.6)$$

⁹We note that $f_2^{(2)}$ will be fixed later in (4.20) from collinear factorisation.

¹⁰As observed in [6], the factor of 1/2 in the result (4.6) is a matter of convention – it can be understood once one recalls that $f_0^{(2)}$ and $f_1^{(2)}$ are written in a convention where the 't Hooft coupling a_{BDS} is twice as that used in [8, 6]. It is the combination $af^{(2)}$ which must be independent of any conventions used to define the coupling.

4.1 Collinear factorisation

In this section we would like to discuss collinear factorisation and the exponentiation of infrared divergences of form factors. This will lead us to the definition of a finite, scaling-invariant and regulator-independent remainder function with trivial collinear limits.

We begin by recalling that an important hint that MHV amplitudes in $\mathcal{N} = 4$ may exponentiate came from the study of collinear limits [1] where two adjacent momenta a and b become parallel. It is well known that scattering amplitudes have a universal collinear factorisation behaviour, which is governed by splitting amplitudes [14, 15, 52]. These quantities only depend on the helicities and the momentum fractions carried by the two legs becoming collinear, but are completely blind to other details of the process. If we consider $\mathcal{N} = 4$ SYM and focus on the case of MHV amplitudes, the helicity-blind ratio $A_n^{(L)}/A_n^{(0)} = \mathcal{M}_n^{(L)}(p_1, \dots, p_n)$ therefore obeys the following factorisation:

$$\mathcal{M}_n^{(L)}(p_1, \dots, p_a, p_b, \dots, p_n) \xrightarrow{a\parallel b} \sum_{l=0}^L \mathcal{M}_{n-1}^{(l)}(p_1, \dots, p_a + p_b, \dots, p_n) r^{(L-l)}(\epsilon; z, s_{ab}) , \quad (4.7)$$

where $r^{(0)} = 1$, $\mathcal{M}_n^{(0)} = 1$, and

$$r^{(1)}(\epsilon; z, s_{ab}) := \frac{c_\Gamma}{\epsilon^2} \left(\frac{-s_{ab}}{\mu^2} \right)^{-\epsilon} \left[1 - {}_2F_1 \left(1, -\epsilon, 1 - \epsilon, \frac{z-1}{z} \right) - {}_2F_1 \left(1, -\epsilon, 1 - \epsilon, \frac{z}{z-1} \right) \right] , \quad (4.8)$$

to all orders in the dimensional regularisation parameter ϵ [53, 54].¹¹

For general large- N gauge theories, a unitarity-based proof of (4.7) at any loop order has been given in [45]. The important point we would like to make is that the main steps of that proof apply directly to form factors as well, which therefore share the universal factorisation properties of amplitudes, including (4.7), which we write as

$$\mathcal{G}_n^{(L)}(p_1, \dots, p_a, p_b, \dots, p_n) \xrightarrow{a\parallel b} \sum_{l=0}^L \mathcal{G}_{n-1}^{(l)}(p_1, \dots, p_a + p_b, \dots, p_n) r^{(L-l)}(\epsilon; z, s_{ab}) , \quad (4.9)$$

where we have defined

$$\mathcal{G}_n^{(L)} := F_n^{(L)} / F_n^{(0)} . \quad (4.10)$$

We stress that the splitting amplitudes appearing in (4.9) are the same as for amplitudes.¹²

Remarkably, loop splitting amplitudes in $\mathcal{N} = 4$ SYM obey a cross-order relation which at two loops is [1]

$$r^{(2)}(\epsilon; z, s) = \frac{1}{2} \left(r^{(1)}(\epsilon; z, s) \right)^2 + f^{(2)}(\epsilon) r^{(1)}(2\epsilon; z, s) + \mathcal{O}(\epsilon) , \quad (4.11)$$

¹¹The all-orders in ϵ expression for $r^{(1)}(\epsilon; z, s_{ab})$ in (4.8) was also rederived in [55] using one-loop MHV diagrams.

¹²We have checked this for bilinear operators following the proof of [45], however we caution the reader that the proof presented in that paper does not extend automatically to operators such as $\text{Tr}\phi^n$ for $n > 2$, and collinear factorisation should be re-examined carefully in these cases.

which has generalisations to higher loops [2]. Importantly, this implies that the correct collinear factorisation of the two-loop amplitude $\mathcal{M}_n^{[2]}$ is captured by the combination

$$\widetilde{\mathcal{M}}_n^{(2)} := \frac{1}{2}(\mathcal{M}_n^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)\mathcal{M}_n^{(1)}(2\epsilon), \quad (4.12)$$

which is given purely in terms of the one-loop ratio functions and $f^{(2)}(\epsilon)$. The only allowed discrepancy between $\widetilde{\mathcal{M}}_n^{(2)}$ and the full two-loop ratio $\mathcal{M}_n^{(2)}$ would be a function that is finite in all collinear limits. As is well known, in $\mathcal{N} = 4$ SYM this difference turns out to be a constant for $n = 4, 5$ and a non-trivial function $n > 5$ – the remainder function. This behaviour is linked to the fact that loop amplitudes in $\mathcal{N} = 4$ SYM obey an anomalous dual conformal Ward identity [56, 57], of which $\widetilde{\mathcal{M}}_n^{(2)}$ is a particular solution. For $n = 4, 5$ no dual conformal cross-ratios exists, and the only homogeneous solution is a constant, while for $n > 5$ there are $3n - 15$ cross ratios and explicit calculations have confirmed the necessity to add a remainder function $\mathcal{R}_n(u_1, \dots, u_{3n-15})$ [28, 29].

Given the similarities between amplitudes and form factors, specifically concerning their collinear factorisation properties and the universality and exponentiation of their infrared divergences [36, 58–61], it is natural to expect that loop form factors should have a similar exponentiated form. Let us investigate this point in more detail.

To begin with, we have checked explicitly in a specific example our claim based on a simple generalisation of [45] that form factors enjoy collinear factorisation properties identical to those of amplitudes. Specifically, we have confirmed that the one-loop MHV form factors derived in [6] indeed obey the anticipated collinear factorisation (4.9). For concreteness we now consider the one-loop three-point form factor ratio function (see (3.17) in [6]),

$$\begin{aligned} \mathcal{G}_3^{(1)} = & -\frac{c_\Gamma}{\epsilon^2} \left[(-s_{12})^{-\epsilon} + (-s_{23})^{-\epsilon} + (-s_{31})^{-\epsilon} \right. \\ & + (-s_{12})^{-\epsilon} h\left(-\frac{s_{31}}{s_{23}}\right) + (-s_{23})^{-\epsilon} h\left(-\frac{s_{31}}{s_{12}}\right) - (-q^2)^{-\epsilon} h\left(-\frac{s_{31}q^2}{s_{12}s_{23}}\right) \\ & + (-s_{23})^{-\epsilon} h\left(-\frac{s_{12}}{s_{31}}\right) + (-s_{31})^{-\epsilon} h\left(-\frac{s_{12}}{s_{23}}\right) - (-q^2)^{-\epsilon} h\left(-\frac{s_{12}q^2}{s_{23}s_{31}}\right) \\ & \left. + (-s_{31})^{-\epsilon} h\left(-\frac{s_{23}}{s_{12}}\right) + (-s_{12})^{-\epsilon} h\left(-\frac{s_{23}}{s_{31}}\right) - (-q^2)^{-\epsilon} h\left(-\frac{s_{23}q^2}{s_{31}s_{12}}\right) \right], \end{aligned} \quad (4.13)$$

with $h(x) = {}_2F_1(1, -\epsilon, 1 - \epsilon, x) - 1$ and $q^2 = (p_1 + p_2 + p_3)^2 = s_{12} + s_{23} + s_{31}$.

In the collinear limit $1||2$ we have $s_{12} \rightarrow 0$, $q^2 \rightarrow s_{23} + s_{31}$. Setting, in the limit, $s_{31}/q^2 \rightarrow z$ and $s_{23}/q^2 \rightarrow 1 - z$, we find

$$\mathcal{G}_3^{(1)} \xrightarrow{1||2} r^{(1)}(\epsilon; s_{12}, z) - \frac{2c_\Gamma}{\epsilon^2} (-(P + p_3)^2)^{-\epsilon}, \quad (4.14)$$

with $P = p_1 + p_2$. Note that the splitting amplitude comes from terms 1, 4 and 11 in (4.13), and the second term is nothing but the one-loop two-point (Sudakov) form factor. Reproducing the expected factorisation requires a delicate conspiracy among various terms in (4.13).

4.2 Iterative structure at higher loops

Having established the correct collinear factorisation at one loop, we can now write down a general two-loop MHV form factor ansatz as follows,

$$\mathcal{G}_n^{(2)} = \frac{1}{2}(\mathcal{G}_n^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)\mathcal{G}_n^{(1)}(2\epsilon) + \mathcal{R}_n^{(2)}, \quad (4.15)$$

where the ratio $\mathcal{G}_n^{(L)}$ is defined in (4.10). Note that we allow for a potential form factor remainder function $\mathcal{R}_n^{(2)}$ on the right-hand side of (4.15). A few key properties of the form factor remainder are:

1. It must be free of infrared divergences, and
2. It must be finite in all collinear limits. Upon properly normalising the remainder function by adding to it an n -independent, transcendentality-four constant, we expect that the normalised remainder behaves as

$$\mathcal{R}_n \rightarrow \mathcal{R}_{n-1}, \quad (4.16)$$

in a simple collinear limit. Furthermore, we also expect that

3. It must be expressed in terms of transcendentality-four functions, of which we will shortly discuss the symbol.
4. It is a rescaling invariant function and hence it depends on Mandelstam variables only through their ratios. But due to the lack of dual conformal (super) symmetry of form factors it does not enjoy the more restricted dependence on conformal cross ratios of amplitude remainders in planar $\mathcal{N} = 4$ SYM. For this reason, a non-trivial form factor remainder can already appear at $n = 3$.

The first property follows from the exponentiation of infrared divergences. Pleasingly, we have checked it explicitly in the case of the three-point, two-loop form factor.

The second property follows from the analysis of collinear limits at higher loops [28]. This analysis was performed in the case of amplitudes in Section 7 of [62], following [28], and it is instantly extended to form factors because their collinear factorisation, as expressed in (4.9), works in the same way as in the amplitude case (4.7). In particular, using (4.9), one finds that under a simple collinear limit, the scalar functions $\mathcal{G}_n^{(1)}$ and $\mathcal{G}_n^{(2)}$ behave as

$$\begin{aligned} \mathcal{G}_n^{(1)} &\rightarrow \mathcal{G}_{n-1}^{(1)} + r^{(1)}(\epsilon; z, s_{ab}), \\ \mathcal{G}_n^{(2)} &\rightarrow \mathcal{G}_{n-1}^{(2)} + r^{(1)}(\epsilon; z, s_{ab})\mathcal{G}_{n-1}^{(1)} + r^{(2)}(\epsilon; z, s_{ab}). \end{aligned} \quad (4.17)$$

Using the fact that splitting amplitudes obey an iterative formula identical to the homogeneous form of the BDS conjecture for the amplitude [1], expressed by (4.11), we conclude that under a simple collinear limit,

$$\begin{aligned} \mathcal{G}_n^{(2)}(\epsilon) - \frac{1}{2}(\mathcal{G}_n^{(1)}(\epsilon))^2 - f^{(2)}(\epsilon)\mathcal{G}_n^{(1)}(2\epsilon) \\ \rightarrow \mathcal{G}_{n-1}^{(2)}(\epsilon) - \frac{1}{2}(\mathcal{G}_{n-1}^{(1)}(\epsilon))^2 - f^{(2)}(\epsilon)\mathcal{G}_{n-1}^{(1)}(2\epsilon). \end{aligned} \quad (4.18)$$

Equation (4.15) defines the form factor remainder $\mathcal{R}_n^{(2)}$ as

$$\mathcal{R}_n^{(2)} := \mathcal{G}_n^{(2)}(\epsilon) - \frac{1}{2}(\mathcal{G}_n^{(1)}(\epsilon))^2 - f^{(2)}(\epsilon)\mathcal{G}_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon) \quad , \quad (4.19)$$

and it follows from (4.18) that in the simple collinear limit $\mathcal{R}_n \rightarrow \mathcal{R}_{n-1}$, as anticipated in (4.16). The outcome of this brief analysis is that writing the L -loop form factor as an L -loop BDS-like ansatz plus an L -loop remainder gives a very efficient way to organise the presentation of the result – the BDS part captures infrared divergences and simple collinear factorisation, while the remainder, is finite in four dimensions and in the collinear limit flows smoothly from the n -point to the $n-1$ -point remainder. An important comment is in order here. The iterative structure in (4.18) is driven by the iteration of the splitting amplitude (4.11). For this reason, the function $f^{(2)}(\epsilon)$ is completely determined to be the same as in the amplitude case. In our normalisations for the coupling constant, we have

$$f^{(2)}(\epsilon) = -2\zeta_2 - 2\zeta_3\epsilon - 2\zeta_4\epsilon^2 \quad . \quad (4.20)$$

In turn, this implies that we can determine the constant $C^{(2)}$ introduced in the iteration of the Sudakov form factor, see (4.2). Plugging $f_2^{(2)} = -2\zeta_4$ into (4.4), we find

$$C^{(2)} = 4\zeta_4 \quad . \quad (4.21)$$

Importantly, the constant $C^{(2)}$ is independent of the number of legs. The consequence of this observation is that the three-point form factor remainder function as defined in (4.19) (for $n = 3$) with the value for $C^{(2)}$ just found, must have a very precise collinear limit:

$$\mathcal{R}_3^{(2)} \rightarrow 0 \quad , \quad (4.22)$$

in any of the three possible simple collinear limits. The third property is simply a consequence of maximal transcendentality.

4.3 Numerical results for the three-point remainder at two loops

As explained in the previous section, it is natural to define a remainder function, which at three points is given by

$$\mathcal{R}_3^{(2)} := \mathcal{G}_3^{(2)}(\epsilon) - \frac{1}{2}(\mathcal{G}_3^{(1)}(\epsilon))^2 - f^{(2)}(\epsilon)\mathcal{G}_3^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon) \quad . \quad (4.23)$$

This quantity is finite, regulator-independent and vanishes in all soft and collinear limits.

Our numerical evaluations confirmed that all infrared-divergent ϵ^{-p} terms in $\mathcal{R}_3^{(2)}$ do indeed cancel: for $p = 3, 4$ numerical errors are negligible, while for $p = 2$ and $p = 1$ we have confirmed this up to about 10^{-13} and 10^{-7} , respectively. This is a stringent check that our result (3.11) obtained using generalised unitarity is indeed correct. We have collected in Table 1 a few values of the remainder function obtained with our numerical programmes. The remainder is defined in (4.23) where we have set $C^{(2)} = 4\zeta_4$ and $f_2^{(2)} = -2\zeta_4$. We will later compare this numerical remainder to our analytical expression, see Table 2.

(u, v, w)	numerical $\mathcal{R}_3^{(2)}$	est. error
$(1/3, 1/3, 1/3)$	-0.1519	0.02
$(1/4, 1/4, 1/2)$	-0.1203	0.02
$(1/5, 2/5, 2/5)$	-0.1301	0.02
$(1/2, 1/3, 1/6)$	-0.1080	0.03

Table 1: *Some numerical values of the remainder $\mathcal{R}_3^{(2)}$ defined in (4.23), with $C^{(2)} = 4\zeta_4$ and $f^{(2)} = -2\zeta_2 - 2\zeta_3\epsilon - 2\zeta_4\epsilon^2$. Infrared poles cancel with negligible numerical errors.*

4.4 The symbol of the three-point remainder at two loops

The question we would like to discuss now is how we can constrain the form factor remainder function $\mathcal{R}_n^{(2)}$ defined in (4.19) using its symbol [23].¹³ We will focus on the case of the three-point form factor calculated explicitly in this paper. The conclusion of the analysis presented in this section is that, in the three-point case, there is a unique non-trivial solution for the symbol of the remainder function (up to an overall constant) to be presented shortly.

In order to write down an ansatz for the symbol of the remainder function, we make the following physical considerations which lead to important constraints on its structure:

1. We should understand which variables the remainder function can depend upon.
2. The symbol must satisfy a certain first-entry condition [30], arising from the requirement that the two-loop form factor remainder must have discontinuities only across the physical branch cuts starting at $P_j^2 = 0$, where P_j^2 's are appropriate kinematic invariants made of sums of external momenta (in the three-point case considered here, these could be s_{12} , s_{23} and s_{13}). This implies that the symbol $\mathcal{S}^{(2)}$ of the two-loop remainder must be of the form

$$\mathcal{S}^{(2)} = \sum_{\text{disc}_J} P_J^2 \otimes \mathcal{S}[\mathcal{D}_J^{(2)}], \quad (4.24)$$

where $\mathcal{D}_J^{(2)}$ is the discontinuity of the two-loop remainder in the P_J^2 -channel, and the sum is extended to all channels where there is a physical discontinuity.

3. The second and the last entries of the symbol must also satisfy certain additional constraints, to be discussed below.
4. The symbol must have trivial collinear limits.
5. The symbol must satisfy all relevant symmetries.
6. The ansatz must be the symbol of a local function, which in turn means that the symbol must satisfy certain integrability conditions.

¹³Similar discussions have appeared for the symbol of amplitude remainder functions in [30–35].

We now focus on the three-point case and discuss the implementation of the properties listed above.

1. To begin with, we wish to claim that the symbol of the three-point form factor remainder function takes its entries only from the following set of six kinematic variables

$$\{u, v, w; 1 - u, 1 - v, 1 - w\} , \quad (4.25)$$

where u , v and w are the scale invariant ratios defined in (3.13). This statement is highly non-trivial and is an important constraint, as it does not allow for dependence on any other variables such as $1 + u$, or $1 + uv$ etc. Our claim is substantiated by explicit inspection of the two-loop master integrals with one off-shell and three on-shell legs [19, 20] in terms of which our two-loop, three-point form factors is expanded. We have calculated the symbol of the relevant integral functions, with the result that they always depend on the variables listed in (4.25) only and that the first entry of the symbol is always taken from the subset $\{u, v, w\}$.

2. The first-entry condition requires that the first entry of the symbol must be either u , v , or w , but the remaining three variables $1 - u$, $1 - v$, $1 - w$ cannot appear as possible first entries. This is also confirmed by inspection of the integral functions appearing in [19, 20] as mentioned under **1**.

3. Furthermore, the second entry of the symbol is restricted in the following way [30]. If the first entry is u , then the second entry can be $1 - u$ or v or w , but not u or $1 - v$ or $1 - w$; similar conditions in the case when first entry is v or w are obtained by cyclic permutations of u , v and w . Finally, we impose a last entry condition on the symbol [34, 31], by requiring it to be always of the form $u/(1-u)$, or $v/(1-v)$, or $w/(1-w)$. If we insist in using the variables in (4.25) as entries for our symbol, the above conditions relate pairs of coefficients, namely the coefficient of a term in the symbol whose last entry is u is the opposite of the coefficient of the term whose first three entries are the same, with the last one equal to $1 - u$. And similarly for the pairs v and $1 - v$, and w and $1 - w$.

At this point, we observe that imposing the first three conditions, one obtains an ansatz for the symbol in terms of 324 independent parameters.

4. Next, we recall that the three-point remainder must vanish in all collinear limits.¹⁴ Imposing this constraint term by term on the symbol is too restrictive, as there can be cancellations among different terms. Therefore, we keep the most general ansatz and impose that it vanishes in any of the three simple collinear limits. We also recall that, in order for a symbol to vanish, it is enough that one of its entries smoothly goes to one in the limit.

For definiteness, let us focus on the $s_{12} \rightarrow 0$, or $u \rightarrow 0$ limit, which we take as follows: we leave untouched every entry equal to u in the symbol; however we can replace $1 - u \rightarrow 1$, and furthermore, since $v + w = 1$ in the collinear limit, we can replace $w \rightarrow 1 - v$. Requiring the symbol to vanish in this limit, we obtain three sets of equations, for the three independent simple collinear limits $s_{12} \rightarrow 0$ (or $u \rightarrow 0$), $s_{23} \rightarrow 0$ (or $v \rightarrow 0$) and $s_{31} \rightarrow 0$ ($w \rightarrow 0$). Altogether, the collinear equations impose 99 independent constraints on the symbol.

¹⁴We would like to thank Paul Heslop for a very useful conversation on collinear limits of symbols.

5. Next, we impose that the remainder $\mathcal{R}_3^{(2)}$ must be totally symmetric in u, v, w . This leads to 52 linear relations between the coefficients.

6. Finally, we impose the integrability condition on the ansatz, namely the constraint that the symbol is derived from a local function. This is implemented as follows [63, 30]: one takes the symbol

$$\mathcal{S} = \sum w_1 \otimes \cdots \otimes w_n, \quad (4.26)$$

and replaces two consecutive entries w_i, w_{i+1} by $dw_i \wedge dw_{i+1}$. The resulting expression must vanish:

$$\sum dw_i \wedge dw_{i+1} \left(w_1 \otimes \cdots \otimes w_{i-1} \otimes w_{i+2} \otimes \cdots \otimes w_n \right) = 0. \quad (4.27)$$

There are three different integrability conditions we have to impose in our case.¹⁵ Remarkably, by requiring these to be obeyed, we find that there is a unique solution for the symbol of the two-loop remainder (up to an overall normalisation). It is given by the following compact expression:

$$\begin{aligned} \mathcal{S}^{(2)} = & -2u \otimes (1-u) \otimes (1-u) \otimes \frac{1-u}{u} + u \otimes (1-u) \otimes u \otimes \frac{1-u}{u} \\ & -u \otimes (1-u) \otimes v \otimes \frac{1-v}{v} - u \otimes (1-u) \otimes w \otimes \frac{1-w}{w} \\ & -u \otimes v \otimes (1-u) \otimes \frac{1-v}{v} - u \otimes v \otimes (1-v) \otimes \frac{1-u}{u} \\ & +u \otimes v \otimes w \otimes \frac{1-u}{u} + u \otimes v \otimes w \otimes \frac{1-v}{v} \\ & +u \otimes v \otimes w \otimes \frac{1-w}{w} - u \otimes w \otimes (1-u) \otimes \frac{1-w}{w} \\ & +u \otimes w \otimes v \otimes \frac{1-u}{u} + u \otimes w \otimes v \otimes \frac{1-v}{v} \\ & +u \otimes w \otimes v \otimes \frac{1-w}{w} - u \otimes w \otimes (1-w) \otimes \frac{1-u}{u} \\ & + \text{cyclic permutations}. \end{aligned} \quad (4.28)$$

The next challenge is twofold: firstly, we wish to determine the function whose symbol is given by (4.28); and secondly, we wish to determine terms missed by the symbol, e.g. terms of the form $\pi^2 \times F_2$ where F_2 is a sum of transcendentality-two functions with rational coefficients.

In this respect, there is an additional piece of information about (4.28) that we would like to mention. Our symbol $\mathcal{S}^{(2)}$ defined in (4.28) satisfies an important symmetry constraint [63] discussed in [23], namely

$$\mathcal{S}_{abcd}^{(2)} - \mathcal{S}_{bacd}^{(2)} - \mathcal{S}_{abdc}^{(2)} + \mathcal{S}_{badc}^{(2)} - (a \leftrightarrow c, b \leftrightarrow d) = 0. \quad (4.29)$$

According to a conjecture of Goncharov, symbols with this peculiar property can always be obtained from a function involving logarithms and classical polylogarithms Li_k 's with $k \leq 4$ only [63, 23]. The explicit solution we will present in the next section will confirm

¹⁵In general, there are $p-1$ integrability conditions for a transcendentality p symbol.

this expectation beautifully. As we will show in the final part of this paper, there is an alternative way to obtain an analytic result of the form factor remainder in terms of two-dimensional harmonic polylogarithms [64]. This is due to a remarkable relation between the $\mathcal{N} = 4$ form factor and the planar, maximally transcendental part of the two-loop QCD amplitude for $H \rightarrow ggg$ recently obtained in [12, 13].

4.5 The analytic remainder function

The remaining task now is to find a transcendentality-four function whose symbol is given by (4.28). Recall that the symbol only takes entries from the list $\{u, v, w, 1-u, 1-v, 1-w\}$ and has the symmetry (4.29), which implies the result should be expressed purely in terms of classical polylogarithms of degree up to four and logarithms [63, 23]. This however does not fix a priori the allowed arguments of these functions, but the arguments of individual functions must be such that the symbol of that function has only entries from that list. Taking these considerations into account, the most general ansatz will be built from the following set of functions:

$$\log x_1 \log x_2 \log x_3 \log x_4, \text{Li}_2(x_1) \log x_2 \log x_3, \text{Li}_2(x_1)\text{Li}_2(x_2), \text{Li}_3(x_1) \log x_2 \text{ and } \text{Li}_4(x_i), \quad (4.30)$$

where we found it sufficient to take the possible arguments x_i from the list

$$\left\{ u, v, w, 1-u, 1-v, 1-w, 1-\frac{1}{u}, 1-\frac{1}{v}, 1-\frac{1}{w}, -\frac{uv}{w}, -\frac{vw}{u}, -\frac{wu}{v} \right\}. \quad (4.31)$$

Imposing the constraint that the ansatz has the same symbol as (4.28) one can easily find a solution. We have then applied various polylogarithm identities to simplify the *raw* solution obtained in this way. The final result takes the remarkably simple and compact form

$$\begin{aligned} \mathcal{R}_3^{(2)} = & -2 \left[\text{J}_4\left(-\frac{uv}{w}\right) + \text{J}_4\left(-\frac{vw}{u}\right) + \text{J}_4\left(-\frac{wu}{v}\right) \right] - 8 \sum_{i=1}^3 \left[\text{Li}_4(1-u_i^{-1}) + \frac{\log^4 u_i}{4!} \right] \\ & - 2 \left[\sum_{i=1}^3 \text{Li}_2(1-u_i^{-1}) \right]^2 + \frac{1}{2} \left[\sum_{i=1}^3 \log^2 u_i \right]^2 - \frac{\log^4(uvw)}{4!} - \frac{23}{2} \zeta_4, \end{aligned} \quad (4.32)$$

where $u_1 = u$, $u_2 = v$ and $u_3 = w$ and we have introduced the function

$$\text{J}_4(z) := \text{Li}_4(z) - \log(-z)\text{Li}_3(z) + \frac{\log^2(-z)}{2!}\text{Li}_2(z) - \frac{\log^3(-z)}{3!}\text{Li}_1(z) - \frac{\log^4(-z)}{48}. \quad (4.33)$$

It is curious to note here that $\text{J}_4(z)$ is almost identical to the function $\text{D}_4(z)$ introduced by Ramakrishnan. The functions $\text{D}_m(z)$, $m > 2$, are generalisations of the Bloch-Wigner functions (see [67] for an inspirational exposition of these topics and references).

In the representation obtained above we have already taken into account beyond-the-symbol ambiguities which arise due to the fact that the symbol is blind to transcendentality-four terms of the form π^4 or $\pi^2 \times \{\log x_i \log x_j, \text{Li}_2(x_i)\}$. It is a simple task to fix these

ambiguities using constraints from permutation symmetry and collinear limits. In our case it was sufficient to add the ζ_4 term to get a symmetric function, that is smooth throughout the Euclidean region defined as $0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1$ and $u + v + w = 1$, and vanishes in all collinear and soft limits.

Finally, we have collected in Table 2 results from our numerical evaluations in Section 3.1 and compared them with the exact result (4.32). This also serves as confirmation of the overall normalisation of the remainder, which is not fixed by the symbol alone.

(u, v, w)	numerical $\mathcal{R}_3^{(2)}$	est. error	analytic $\mathcal{R}_3^{(2)}$
(1/3, 1/3, 1/3)	-0.1519	0.02	-0.148966
(1/4, 1/4, 1/2)	-0.1203	0.02	-0.134873
(1/5, 2/5, 2/5)	-0.1301	0.02	-0.136454
(1/2, 1/3, 1/6)	-0.1080	0.03	-0.125366

Table 2: Comparison of numerical and analytic values of the remainder $\mathcal{R}_3^{(2)}$.

4.6 Connections between amplitude and form factor remainders

We have seen earlier that the three-point form factor remainder depends on three variables u, v and w defined in (3.13), where $u + v + w = 1$. In (4.28) we have presented the symbol of this remainder. An a priori entirely different calculation is that of the six-point MHV amplitude remainder, whose symbol was explicitly evaluated in [23] using the results of [24]. This symbol can be expressed in terms of the three independent cross-ratios u, v and w one can write down with six lightlike momenta satisfying $\sum_{i=1}^6 p_i = 0$, as well as three additional variables y_u, y_v, y_w . An explicit compact expression for this six-point amplitude remainder symbol is given in Eq. (21) of [32].¹⁶ It can be seen from that formula that the symbol is naturally decomposed into two terms,

$$\mathcal{S}_{6, \text{ampl}}^{(2)} = \hat{\mathcal{S}}_{6, \text{ampl}}^{(2)}(u, v, w) + \tilde{\mathcal{S}}_{6, \text{ampl}}^{(2)}(u, v, w; y_u, y_v, y_w). \quad (4.34)$$

The second term in (4.34) is given by [32]

$$\begin{aligned} \tilde{\mathcal{S}}_{6, \text{ampl}}^{(2)}(u, v, w; y_u, y_v, y_w) &= -\frac{1}{8} [u \otimes (1-u) \otimes y_u y_v y_w - 2u \otimes v \otimes y_w] \otimes y_u y_v y_w \\ &+ \text{permutations.} \end{aligned} \quad (4.35)$$

On the other hand, it is straightforward to check that $\hat{\mathcal{S}}_{6, \text{ampl}}^{(2)}(u, v, w)$ is in fact identical (up to an overall multiplicative constant) to the symbol of our two-loop three-point form factor remainder (4.28) upon identifying the three (unconstrained) cross-ratios of the amplitude calculation with our three (constrained) variables u, v and w defined in (3.13).

Note that $\hat{\mathcal{S}}_{6, \text{ampl}}^{(2)}(u, v, w)$ has trivial collinear limits both when it is part of a form factor symbol, where $u + v + w = 1$, and when it is part of the six-point remainder

¹⁶The definitions of the y variables can also be found in Eq. (15) of [32]. We will not need their explicit expressions here.

symbol. In the latter case the three cross-ratios u , v and w are in general unconstrained, but in the collinear limit they become dependent – for instance, in the collinear limit $w \rightarrow 0$ one also has that $u + v \rightarrow 1$.

This is reminiscent of a similar interesting coincidence observed at strong coupling in [5]. There, the four-point form factor remainder was evaluated in the particular case of $(1+1)$ -dimensional kinematics, and found to be expressible in terms of the eight-point MHV amplitude remainder in $(1+1)$ -kinematics, which was determined in [65, 66]. We note that the relations between form factor and amplitude remainders at strong and weak coupling seen in [5] and in the present paper can by no means be anticipated from the explicit calculation. For instance, in our weak coupling calculation we find that the three-point form factor remainder contains several integral functions – in particular non-planar ones – which do not appear in the corresponding six-point amplitude calculation.

Incidentally, we note that while the four-point form factor remainder and the eight-point MHV amplitude remainder in of $(1+1)$ -dimensional kinematics depend both on two independent variables,¹⁷ in our case the number of independent variables is different – the form factor remainder depends on two independent simple ratios, and the amplitude’s on three cross-ratios.

Perhaps this connection between amplitude and form factor remainders is another indication that the paradigm “loop observable = rational coefficient \times integral function” may obscure the simplicity of the final result of a loop calculation. Furthermore it is very likely that the symbol technology will allow to simplify the expressions of the (higher-loop) integral functions themselves, such as those appearing in (4.28) and in QCD. For a recent example in QCD see [68].

4.7 A surprising relation with QCD

In this final section we wish to discuss an intriguing connection of our result with the recent work of [12]. In that paper, the two-loop helicity amplitudes for $H \rightarrow ggg$ and $H \rightarrow q\bar{q}g$ were computed in the large top mass limit. In this approximation the top quark can be integrated out at one loop and produces a new effective vertex of the form Hgg . If we consider the process $H \rightarrow ggg$, which is the case of interest for our discussion here, the calculation is equivalent to that of a three-gluon form-factor with a $\text{Tr}F^2$ operator insertion, where $F_{\mu\nu}$ is the gluon field strength. As we noted earlier, the operator $\text{Tr}\phi_{12}^2$ – whose form factor we consider in this paper – is the lowest component of the stress-tensor multiplet, which among many other operators contains the on-shell Lagrangian $\mathcal{L}_{\text{o.s.}} = \text{Tr}F_{\text{SD}}^2 + \dots$, where F_{SD} is the self-dual part of the field strength, and the dots stand for cubic and quartic terms in the Lagrangian. Up to a universal helicity-dependent prefactor the form factor of the operator with three-gluons $\langle g^-(p_1)g^-(p_2)g^+(p_3)|\mathcal{L}_{\text{o.s.}}(0)|0\rangle$ is equal to the form factor we have been discussing in this paper, and on the other hand is related to the object considered in [12] in QCD.

The full QCD result of [12] is of course very complicated and has no resemblance to

¹⁷Recall that for this restricted kinematics, the $2n$ -point form factor remainder depends on $2n - 2$ variables, whereas the amplitude remainder depends on $2n - 6$ cross-ratios.

(4.32). The story becomes more interesting if we focus on the planar part of the finite two-loop remainder $A_{\Omega}^{(2)}$ defined in (5.8) and (5.20) of [12], and with explicit formulae in terms of 2dHPL's given in Eqns. (B.1) and (B.7) of the same reference for the two different gluon helicity configurations.¹⁸ It is important to note that these finite remainders are defined using the formalism of [36] widely used in QCD to remove the universal two-loop infrared divergences, and are different from the BDS finite remainders used in this paper [1, 2]; in particular the former does not vanish in collinear limits. However, it is easy to relate the two definitions in $\mathcal{N} = 4$ (see [1]), because one simply has to subtract $\frac{1}{2}(F_{\text{fin}}^{(1)})^2 - 2\zeta_2 F_{\text{fin}}^{(1)}$ from Catani's remainder [36] in order to arrive at the BDS remainder.

The interesting observation arises as follows: First, take the finite two-loop remainders $A_{\alpha}^{(2)}$ or $A_{\beta}^{(2)}$ from Eqns. (B.1) and (B.7) of [12], corresponding to the helicity configurations $(g^+g^+g^+)$ and $(g^+g^+g^-)$ for the external gluons, respectively, and extract the terms containing functions of maximal transcendentality degree (four in this case). Then map them to BDS remainders using the prescription outlined above with $F_{\text{fin}}^{(1)}$ replaced by the degree-two parts of the one-loop amplitudes $A_{\alpha}^{(1)}$ and $A_{\beta}^{(1)}$ (Eqns. (A.1) and (A.4) in [12]), which coincides up to a factor of two with the ϵ^0 term of our one-loop form factor in (4.13). Finally, calculate the symbol of the resulting expression. Surprisingly, one finds exactly (4.28), for both helicity configurations.¹⁹ This implies that our form factor remainder (4.32) is contained in the full QCD result of [12], and allows to present the latter in a more compact form. It would be interesting to see if symbols can be used effectively to simplify the terms of lower degree (transcendentality) in the expressions of [12], or in other QCD results. We also point out that in [40] it was observed that the maximal transcendentality principle applies also to the three-loop Sudakov form factor.

Alternatively, the connection we have described between QCD and $\mathcal{N} = 4$ SYM three-point form factors provides us with an additional strategy to reconstruct a function from the symbol (4.32) by simply projecting out the functions of maximal degree and performing the subtraction described above to get the BDS remainder. The result obtained in this way is given in terms of 2dHPL's and the remaining beyond-the-symbol ambiguities can be fixed by considering symmetries and collinear limits. In this case the necessary correction terms can also be read off directly from e.g. $A_{\beta}^{(2)}$ in (B.6) of [12]: one has to keep all terms of the form $\pi^2 \times F_2$ where F_2 is a sum of degree-two functions, while terms of the form $\zeta_3 \times F_1$ where F_1 is a sum of logarithms have to be dropped because they spoil the correct collinear behaviour. We also note that these correction terms are essential to restore the symmetry of the remainder $\mathcal{R}_3^{(2)}$ under permutations of (u, v, w) . We have checked that the two formulae for the remainder match exactly up to a factor of four.²⁰

¹⁸Note that $A_{\Omega}^{(2)}$ are the coefficients of the N^2 terms in the finite remainder defined in (5.20) of [12]. The QCD result with external gluons does not have subleading in N terms, but does have contributions from quarks which are proportional to powers of the number of quark flavours N_F . In order to compare to the $\mathcal{N} = 4$ form factor calculation we have suppressed such N_F -dependent terms. Diagrams in the QCD calculation where gluons run in the loops are of course in common with the $\mathcal{N} = 4$ form factor calculation, and are accounted for by $A_{\Omega}^{(2)}$.

¹⁹In this respect we note that in $\mathcal{N} = 4$ SYM the two form factors with MHV $(g^+g^+g^-)$ and maximally non-MHV $(g^+g^+g^+)$ helicity configurations are actually related, a fact that was explained in Section 3.2.3 of [7].

²⁰The relative factors of four and two for two-loop and one-loop quantities respectively are due to different normalisation conventions of the gauge coupling constant.

We believe that this is the first example where the principle of maximal transcendentality, first observed in [37] for anomalous dimensions of operators, relates physical quantities with non-trivial kinematic dependence in planar QCD and $\mathcal{N} = 4$ SYM. It would be exciting to identify more such examples and we are confident that the technology of symbols will turn into a powerful tool also in QCD.

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A Complete one-loop amplitudes with fundamental and adjoint generators

At one loop, complete (planar plus non-planar) amplitudes can be written as [14]

$$\begin{aligned} \mathcal{A}^{(1)}(1, \dots, n) &= \mathcal{A}_{\text{P}}^{(1)}(1, \dots, n) + \mathcal{A}_{\text{NP}}^{(1)}(1, \dots, n) \\ &= N \sum_{\sigma \in S_n / \mathbb{Z}_n} \text{Tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) A_{n;1}^{(1)}(\sigma_1, \dots, \sigma_n) \\ &\quad + \sum_{\sigma \in S_n / S_{n;c}} \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \text{Tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_{c-1}}}) \text{Tr}(T^{a_{\sigma_c}} \dots T^{a_{\sigma_n}}) A_{n;c}^{(1)}(\sigma_1, \dots, \sigma_n). \end{aligned} \quad (\text{A.1})$$

Here S_n is the set of permutations of n objects, and $S_{n;c}$ is the subset of permutations which leaves the double-trace structure in (A.1) invariant. $A_{n;1}^{(1)}(\sigma_1, \dots, \sigma_n)$ are colour-ordered one-loop amplitudes, whereas $A_{n;c}^{(1)}(\sigma_1, \dots, \sigma_n)$ are certain linear combinations thereof, constructed as [14]

$$A_{n;c}^{(1)}(1, 2, \dots, c-1; c, c+1, \dots, n) = (-1)^{c-1} \sum_{\sigma \in \text{COP}\{\alpha\}\{\beta\}} A_{n;1}^{(1)}(\sigma), \quad (\text{A.2})$$

where $\alpha_i \in \{\alpha\} \equiv \{c-1, c-2, \dots, 2, 1\}$, $\beta_i \in \{\beta\} \equiv \{c, c+1, \dots, n-1, n\}$, and $\text{COP}\{\alpha\}\{\beta\}$ denotes the set of all permutations of $\{1, 2, \dots, n\}$ where n is held fixed, and such that the cyclic ordering of the α_i within $\{\alpha\}$ and of the β_i within $\{\beta\}$ is maintained, while allowing for all possible relative orderings of the α_i with respect to the β_i .

One can also work with the alternative, remarkably compact representation of the complete one-loop amplitudes presented in [48],

$$\mathcal{A}^{(1)}(1, \dots, n) = \sum_{\sigma \in S_n / (\mathbb{Z}_n \times \mathcal{R})} \text{Tr}(F^{a_{\sigma_1}} \dots F^{a_{\sigma_n}}) A_{n;1}^{(1)}(\sigma_1, \dots, \sigma_n), \quad (\text{A.3})$$

where F_{bc}^a are adjoint generators.²¹ The sum in (A.3) contains $(n-1)!/2$ terms. Note that planar and non-planar contributions are both contained in (A.3), where only colour-ordered amplitudes are summed.

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²¹We define the adjoint generators are defined as $F_{bc}^a := -if^{abc}$, where the structure constants f^{abc} have an additional factor of $\sqrt{2}$ compared to the usual ones. This normalisation is chosen in order to accommodate for the normalisation $\text{Tr}(T^a T^b) = \delta^{ab}$ of the fundamental generators.

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