

## **Pricing American derivatives and interest rates derivatives based on characteristic function of the underlying asset returns**

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DERIVATIVE >

Pricing American ~~Stock~~ and Interest & Rate Derivatives Based  
On Characteristic Function of the Underlying Asset Returns

by

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## ABSTRACT

In this thesis I introduce a new methodology for pricing American options when the underlying model of the asset price allows for stochastic volatility and/or it has a multi-factor structure. Our approach is based on a decomposition of an American option price into its European options counterpart price and the early exercise premium, paid by the option holder in order to keep the right of exercising the option at any time-point before its expiration date.

Based on closed form solutions of the joint characteristic function of the state variables driving the underlying model, the thesis provides analytic, integral solutions of the early exercise premium (and hence of the American option price) which enable us to build up fast and accurate numerical approximation procedures for calculating options prices. The analytic solutions that I derive express the optimal early exercise boundary in terms of prices of Arrow-Debreu type of securities reflecting the values of the options additional payoffs if they are exercised earlier, or not.

Numerical results reported in the thesis show that our approach can price American options on stocks, bonds and interest rates derivatives efficiently and very fast, compared with existing methods. The efficiency gains of our method stem from the fact that it involves only one step of approximation, as the European prices embodied in the American option prices can be calculated analytically. The gains of computational speed come from the fact that our method can reduce the integral dimensions of the early exercise premium considerably.

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## CHAPTER 1. INTRODUCTION

Financial derivatives constitute securities (instruments) whose payoffs depend on other financial securities, from which they derive their price. Their use in the commercial activities can be traced back several centuries ago. Since the abandonment of the Brendon Woods exchange rate mechanism in year 1973, financial derivatives have become one of the most popular categories of securities. This happened because both corporations and financial institutions have been starting to being exposed to high volatility risks stemming from the changes in interest rates, exchange rates, commodities and equities. In year 1973 the first organised exchange options market opened, the Chicago Board Options Exchange (CBOE). Since then, the growth of world's derivative markets shows no sign of slowing down. According to the recent statistics on derivative markets' volume, reported by the Bank of International Settlements (BIS), at the end of 2002 the outstanding value of derivative contracts was \$13.54-trillion, without taking into account the \$10.34-trillion value of futures contracts and the trading volume of derivatives traded in over-the-counter markets.

Parallel to the development of traded markets for derivatives, over the last three decades there was an explosive amount of research introducing and pricing correctly derivative securities. Since the seminal work of Black and Scholes (1973), derivatives have become one of the hottest areas in financial economics and econometrics. In particular, new models of option and asset pricing have been developed to allow for stochastic volatility and/or jump processes, or other phenomena which can better describe the dynamics of stock and derivative prices. However, most of the new models suggested do not provide closed form solutions for derivative prices, especially for the American type of derivatives which is the main topic of this thesis. This makes the computation and empirical evaluation problem of derivative prices a very difficult task. There are two sources of difficulties for pricing American derivatives. The first comes from the fact that this category of derivatives gives the option holder

the right to exercise the option contract at any time before the expiration date which, in turn, implies that the interim payoffs and optimal stopping (exercising) time of the derivatives, over the maturity interval, must be estimated. The lack of analytical closed form solution, is also very severe for interest rates derivatives, such as caps and swaptions, where it is assumed that the underlying interest rates are driven by a multi factor, affine model of state variables and that the interest rate is no longer deterministic.

Although the above aforementioned derivative pricing problems have been known to the practitioners and academics for a long time, there are few models providing analytic solutions for American call or put options. And these are available for simple, Gaussian asset pricing models. The aim of this thesis is to fill this gap, and to provide analytic and computationally attractive closed form solutions for American call (or put) option prices for more complicated asset pricing models, used in real life. To derive these solutions, we exploit the recent developments in asset pricing theory which indicate that, in general, asset prices and derivatives can be spanned by the same characteristic functions (CFs). Closed form solutions of these CFs are derived in the thesis. The closed form solution are used to derive analytic formulas for option prices on stocks, bonds and interest rates under stochastic volatility or jumps. The closed form solutions of the CF enable us to explicitly calculate the expected values of the payoffs implied by an early exercise date of the American option contracts. These prices are referred to in the literature as Arrow-Debreu prices. They constitute the cornerstone of the modern asset pricing theory, as they can determine the risk neutral probability density function of the underlying asset price explicitly. The analytic formulas provided by the thesis for pricing American options unbundle the option prices to a portfolio of Arrow-Debreu type of securities, which can make the evaluation of the option prices very fast and efficient.

The thesis is organised as follows. In Chapter 2, we provide a survey of the existing numerical approximation methods in the literature for pricing American op-

tions. These methods are divided into the following categories: the Binomial tree, the finite difference, optimal exercise boundary and the simulation based methods. From all of these categories, the optimal exercise boundary methods seem to be the best one in terms of pricing accuracy and computation speed. This is the main reason for employing this approach in approximating the analytic, integral formulas of the derivative prices derived in the following chapters of the thesis.

In Chapter 3, we present a new option pricing formula for pricing American calls on stocks under stochastic volatility. This is done by assuming that the optimal exercise boundary price above which the option contract is exercised is a log-linear function of the stochastic volatility. This assumption can be justified by recent empirical evidence based on parametric estimates of the optimal exercise boundary function. This relationship of the boundary enable us to decompose the American option price into its European option counterpart price and the early exercise premium that the option holder would like to pay for having the right to exercise early the option. The latter is derived in analytic form and is exploited in order to develop an optimal exercise boundary approximation approach of the exercise boundary based on Chebyshev polynomials. Numerical results indicate that our boundary approximation method can equally perform with other exercise boundary approximation methods, suggested in the literature for the simple, log-normal model of stock returns. Indeed, it can price American call options under stochastic volatility very fast and efficiently.

In Chapter 4, we present a new optimal approximation method of the exercise boundary with the aim to price American bond put options for multi-factor affine term structure models. American bond option prices are far more difficult to be calculated, compared with the American stock options. This happens because: first, the discounted factor of the option payoffs is no longer deterministic and, second, the multi-factor structure of the underlying bond price (or interest rates) tends to make the dimensionality problem of the analytic formulas more difficult to cope with.

As in Chapter 3, the suggested approximation method for pricing American

bond options is based on a decomposition of the bond option price into its European counterpart plus the early exercise premium, which can be expressed as a portfolio of Arrow-Debreu securities. To derive the analytic formulas of the Arrow-Debreu security prices, we used the extended CF suggested by Duffie, Pan and Singleton (2000). Since the space of the state variables is spanned by many variables, now our exercise boundary approximation method is based on a multi-region and multi-piece approximation of the optimal exercise boundary function. As shown, in our Monte Carlo experiments this approach can better approximate the true exercise boundary function, compared with a single region approximation of the exercise boundary state space.

In Chapter 5, we present an option model for pricing interest rates derivatives, such as caps, floors and swaptions. Since some of these derivatives critically depend on the date-to-date announcements in interests rates, we adopt Heath, Jarrow and Morton's (1992) (HJM) model of forward rates for describing the dynamics of the term structure model of interest rates. This model is extended to allow for stochastic volatility and marked point (jumps) processes. One of our motivations to adopt this model is that it can accommodate factors of the implied volatility of caps and swaptions prices which can not be spanned by affine models of the term structure, as recent evidence suggests.

As in the previous chapters, we employ the same methodology in developing a numerical approximation method for pricing the above interest rate derivatives. To evaluate whether the HJM model and the exercise boundary approximation scheme can consistently price the caps and swaptions, we fit alternative specifications of the extended HJM model into caps and swaptions data employing our approximation method. Our results indicate that allowing for stochastic volatility and jumps can significantly reduce the pricing errors of interest rates derivatives. Our results also show that the risk neutral intensity of the jump is very high, which implies the high price of jump risk, and can be compared with that estimated based on stock options

data.

In Chapter 6, we conclude the thesis.

## CHAPTER 2.

### NUMERICAL METHODS OF EVALUATING AMERICAN OPTIONS

American options give the holder the right, but not obligation, to buy (for call options) or sell (for put options) the underlying security at a fixed price, known as strike price, at any period of time before the expiration date of the option contract. As the European options, the evaluation of American options can be done by calculating the risk neutral expectation of the appropriately discounted future cash flow (payoff) of the option. This cash flow is the difference between the selling (buying) price of the underlying security for a call (put) option minus the strike price. But, in contrast to the European, the American options imply a continuum of cash flow during the maturity interval. These cash flows can come from an early exercise of the option. Since the time of exercising an American option contract is unknown, the evaluation problem of an American option is more complicated than that for a European option.

Mathematically, the evaluation problem of an American option is known as the optimal stopping (decision) time problem. Below, we briefly describe this problem for an American put option on a security with maturity date  $T$  and strike price  $K$ , denoted as  $P_A(X_t, T)$ .

Consider the probability space  $(P, \mathcal{F}, \Omega)$ , with the filtration  $\mathcal{F}_{s \in [t, T]}$  satisfying the usual conditions [see Protter (1990)]. Assume that the dynamics of interest rates and the price of the underlying security are driven by an  $N$ -dimension vector of state variables  $X_s \in \mathcal{R}^N$  and that the elements of  $X_s$  are adapted under filtration  $\mathcal{F}_s$ . Let  $G(X_s) : \mathcal{R}^N \rightarrow \mathcal{R}$ ,  $\forall s \in [t, T]$  denote the underlying asset determined by the state variables,  $g(X_s) : \mathcal{R}^N \rightarrow \mathcal{R}$ ,  $\forall s \in [t, T]$  denote the payoff function of the American option and  $\tau \in [t, T]$  denote a time-point of the maturity interval where the option can be exercised. Then, we can write the mathematical problem for calculating the American put option price  $P_A(X_t, T)$  as

$$P_A(X_t, T) = \sup_{\tau \in \Gamma = [t, T]} E_t^Q \left[ e^{-\int_t^T r(X_s, s) ds} g(X_T) | \mathcal{F}_t \right], \quad (2..1)$$

where  $Q$  stands for the risk neutral probability measure and  $\Gamma$  constitutes the set of all, possible stopping (exercise) time-points. The optimal exercise time is defined as

$$\tau^* = \inf \{s \in [t, T] : g(X_s) = P_A(X_s, T)\} \quad (2..2)$$

The above mathematical problem is referred to as the Snell envelop [see Karatzas (1988)]. Note that  $e^{-\int_t^T r(X_s, s) ds} P_A(X_t, T)$  constitutes a supermartingale, while the corresponding product for a European option price constitutes a martingale.

To obtain an optimal solution for  $P_A(X_t, T)$ , given (2..1), we need to make three important assumptions for the security markets: (i) there are no arbitrage opportunities, (ii) the markets are efficient and (iii) the markets are frictionless. The no profitable arbitrage assumption is a sufficient and necessary condition for the existence of a the martingale measure  $Q$ , under which we can price the options. Without this condition, we could not find an optimal solution. The efficiency of security and options markets assumption implies that all market participants (i.e. holders or writers of the American options) share the same information when forming expectations about the future paths of security prices and potential exercise time-points. Finally, the frictionless of the market assumption implies that there are not any transaction costs or other type of costs which may trivially complicate the evaluation problem of the option.

The above assumptions are not adequate to guarantee that a mathematical solution of problem (2..1) will always exist. To this end, we need to impose a critical boundary condition [see Duffie (1992)], stated below.

**Condition 1 (American Regularity Conditions)** *If  $g(X_s)$  is non-negative continuous process and  $E[(g^*)^q] < \infty$  for some  $q > 2$ , where  $g^* = \sup_{s \in [t, T]} g(X_s)$ , then the instantaneous interest rate, denoted as  $r(X_s, s)$ , is bounded.*

This condition guarantees that the instantaneous interest rate, which is used to discount the payoff of the option, is bounded, and thus guarantees the existence

of a solution for problem (2..1).

To complete the description of the mathematical problem for calculating the American put option price  $P_A(X_t, T)$ , next we make the following definitions. First, we define the exercise region of the space of the vector of the state variables  $X_t$ . This is given by a subset of the state space of  $X_t$  into which the option holder exercises the option contract, defined as

$$\eta_t = \{X_t \in \mathcal{R}^N : g(X_t) = P_A(X_t, T)\}.$$

The complementary of the above set, defined as

$$\bar{\eta}_t = \{X_t \in \mathcal{R}^N : g(X_t) < P_A(X_t)\},$$

constitutes the continuation region into which the holder keeps the option alive until its expiration date. The subset of  $\mathcal{R}^N$  which divides the space  $\mathcal{R}^N$  into the exercise and continuation regions is known as the optimal exercise boundary. This is a region of critical values of  $X_t$  where the option holder is indifferent of exercising or holding the option alive.

## 2.1 Mathematical representations of problem (2..1)

The optimization problem given by equation (2..1) can be represented in two ways. The first is based on a partial differential equation (PDE) of the bond price  $P_A(X_t, T)$  and the second is based on a Bellman equation. Both of these representations can be proved very useful in building up numerical methods for calculating the option price  $P_A(X_t, T)$ , given that it is difficult to find out analytic methods.

2.1.1 PDE representation Let us, for exposition, assume that the vector of the state variables  $X_t$  consists of one state variable, denoted  $X_t$ , which follows an Ito process in  $\mathcal{R}$  which satisfies the stochastic differential equation (SDE) under the risk neutral



probability measure  $Q$

$$dX_t = a(X_t)dt + b(X_t)dW_t^Q,$$

where  $W_t^Q$  is a Brownian motion process under the  $Q$  measure. According to Ito's lemma, we have

$$dP_A(X_t, T) = \mathcal{D}P_A(X_t, T)dt + \frac{\partial P_A(X_t, T)}{\partial X_t} b(X_t)dW_t^Q,$$

where  $\mathcal{D}$  is the differential operator, defined as

$$\begin{aligned} \mathcal{D} = & a(X_t) \frac{\partial P_A(X_t, T)}{\partial X_t} \\ & + \frac{1}{2} \text{tr} \left[ \frac{\partial^2 P_A(X_t, T)}{\partial X_t^2} b(X_t) b(X_t)' \right] + \frac{\partial P_A(X_t, T)}{\partial t}. \end{aligned}$$

Define  $h(X_t, T) = e^{-\int_t^T r(X_s, s)ds} P_A(X_t, T)$ , then it can be easily see that  $E_t^Q [h(X_s, T)] \leq h(X_s, T), \forall s > t$ , because  $h(X_t, t)$  is the supermartingale. This implies that  $h(X_t, t)$  should satisfy the following PDE inequality

$$\frac{\mathcal{D}h(X_t, T)}{e^{-\int_t^T r(X_s, s)ds}} = \mathcal{D}P_A(X_t, T) - r(X_t, T)P_A(X_t, T) \leq 0.$$

Since in the continuation region,  $\bar{\eta}_t$ ,  $h(X_\tau, \tau)$  is a local martingale, we have that  $P_A(X_t, T)$  should also satisfy the following PDE

$$\mathcal{D}P_A(X_t, T) - r(X_t, t)P_A(X_t, T) = 0.$$

Combining the above two relations, we can derive a PDE with a variant inequality

$$[\mathcal{D}P_A(X_t, T) - r(X_t, t)P_A(X_t, T)] [P_A(X_t, T) - g(X_t)] = 0 \quad (2..3)$$

with  $\mathcal{D}P_A(X_t, T) - r(X_t, t)P_A(X_t, T) \leq 0$  and  $P_A(X_t, T) \geq g(X_t)$ ,

subject to the boundary condition  $P_A(X_T, T) = g(X_T)$ . The above PDE can be numerically solved out to derive the option price  $P_A(X_t, T)$ .

2.1.2 Bellman Equation representation This representation of the American option pricing problem (2..1) is often used in discrete-time frameworks of the evaluation of the American option. In this framework, we assume that the set of optimal stopping time-points is given by the finite indexed set  $\gamma = \{t = s_0, s_1, \dots, s_n = T\}$ . Then, problem (2..1) implies the following discrete-time Bellman equation for the American option price

$$\begin{aligned} & P_A(X_{s_i}, T) \\ &= \max \left( g(X_{s_i}), E_{s_i}^Q \left[ e^{-\int_{s_i}^{s_{i+1}} r(X_u, u) du} P_A(X_{s_{i+1}}, T) | \mathcal{F}_{s_i} \right] \right), \end{aligned} \quad (2..4)$$

$\forall i = 0, 1, \dots, n$ . As the PDE, the above Bellman equation can be solved out to obtain a solution of the price  $P_A(X_{s_i}, T)$ .

## 2.2 Numerical methods for pricing American options

The mathematical representations of the American option pricing problem, given by equations (2..3) and (2..4), indicate that it is difficult to derive analytic, closed form solutions for the option price  $P_A(X_t, T)$ . Therefore, numerical approximation methods have been suggested in the options pricing literature. In this section we survey these methods starting, first, with the Binomial tree method, which is the oldest method.

2.2.1 Binomial tree numerical method The Binomial tree (BT) method was introduced by Cox, Ross and Rubinstein (1979). This method provides a simple and powerful approach to price American options. The BT method assumes that at the

end of each node of the maturity interval (step of the tree, as is said alternatively), the underlying asset can take one out of two possible outcomes. The method exploits the fact that an option can be replicated by a portfolio consisting of a riskless asset and the underlying risky asset (say a stock). This portfolio is known as replicating (or synthetic) portfolio. Ruling out arbitrage implies that the value of the option should be equal to the value of the synthetic portfolio which can be used to price the option contract.

To present the BT method, we first consider the case of a European put option and we assume that the underlying asset (stock) price follows a one-step Binomial tree. If the initial stock price is identical with the state variable  $X_t$  and the European put option price is  $P_t$ , then there are two possible outcomes of the stock price in the next period, when the option contract expires: either the stock price to move up to the level  $X_{t+1}^u = S_t u$  or move down to the level  $X_{t+1}^d = X_t d$ , with  $(u > 1 > d > 0)$ . The put option prices corresponding to the above two outcomes are respectively given as  $P_{t+1}^u = \max(K - X_t u, 0)$  and  $P_{t+1}^d = \max(K - X_t d, 0)$ . Under this scenario, we can show that a synthetic portfolio consisting of  $w = \frac{P_u - P_d}{S_u - S_d}$  number of stocks and one European put option is riskless and has an instantaneous rate of return equal to the riskless interest rate  $r$  [see Hull (2000), *inter alia*]. Based on this portfolio, we can derive the current price of the option as

$$\begin{aligned}
 P_t &= e^{-rT} [qP_{t+1}^u + (1 - q)P_{t+1}^d], \\
 \text{with } q &= \frac{e^{rT} - d}{u - d},
 \end{aligned}
 \tag{2..5}$$

where  $T$  denotes the maturity interval of the option (here,  $T = 1$ ). The analytic solution of the option price given by the above equation can be thought of as a weighted average of the possible one-period (step) ahead future payoffs of the option in the expansion (bull) and recession (bear) states of the stock market, respectively. The attached weights,  $q$  and  $1 - q$ , are known as risk neutral probabilities, since the

evaluation of  $P_t$  in (2.5) is made after ruling out arbitrage. These probabilities are also known as the prices of Arrow-Debreu type of securities which pay \$1 in bull state, and zero otherwise.

Multi-step Binomial tree In reality, the BT model is more complicated than the one-step tree presented above. It can involve a larger number of steps (say  $n$ ). In this situation, we need to divide the maturity interval into  $n$  equal subintervals of length  $\Delta t = \frac{T-t}{n}$  and calculate the number of all possible outcomes of the underlying asset price process at each step  $i$ ,  $i = 1, 2, \dots, n$  of the tree  $t + i\Delta t$ . We will refer to the  $j^{\text{th}}$  node (branch) of the tree at the step  $t + i\Delta t$  as the  $(i, j)$  node. In this case we can easily show that the risk neutral probability  $q$  at each step of the tree can be calculated as

$$q = \frac{e^{r\Delta t} - d}{u - d},$$

where  $u = e^{\sigma\sqrt{\Delta t}}$  and  $d = e^{-\sigma\sqrt{\Delta t}}$ .

Suppose that all future possible outcomes of the stock price are determined by a sequence of independent and identically distributed random variables  $Y_{t+1}, Y_{t+2}, \dots, Y_{t+n}$  measured on the probability space  $(q, 1 - q)$ , i.e.  $Y_i \in (u, d)$ . At the  $i^{\text{th}}$  step of the tree, the stock price will be given by the product  $X_{t+i\Delta t} = X_t Y_{t+1\Delta t} Y_{t+2\Delta t} \dots Y_{t+i\Delta t}$ , and it will take the following  $i + 1$  possible values  $X_t d^i, X_t d^{i-1}u, \dots, X_t u^i$  with probabilities  $(1 - q)^i, \frac{i!}{(i-1)!1!}q(1 - q)^{i-1}, \frac{i!}{(i-2)!2!}q^2(1 - q)^{i-2}, \dots, q^i$ , respectively.<sup>1</sup> Denote the stock price at the node  $(i, j)$  of the tree as  $X_{t+i\Delta t}^{(j)} = X_t d^{i-j}u^j$ . Then, based on the same

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<sup>1</sup> The convergence of the BT model to the BS model can be shown by assume  $i = n$ , taking logarithms of  $S_{t+n}$ , implying

$$\log S_{t+n} = \log X_t + \sum_{i=1}^n \log Y_{t+i},$$

and applying the central limit theorem to the above sum appropriately scaled.

risk neutral arguments as in the one-step case, we can show that the option price  $P_t$  can be calculated as

$$\begin{aligned} P_t &= \sum_{m=1}^n e^{-r(T-t)} \frac{n!}{(n-m)!m!} q^m (1-q)^{n-m} \max(K - X_{t+n\Delta t}^{(m)}, 0) \\ &= \sum_{m=1}^n e^{-r(T-t)} \frac{n!}{(n-m)!m!} q^m (1-q)^{n-m} \max(K - X_t u^m d^{n-m}, 0). \end{aligned}$$

Value back down tree: Pricing American options with the BT method Options' payoffs do not only depend on the stock price at the expiration date, as assumed above. For American options, they will also depend on the prices of stocks and options at any step  $i$  of the tree, before the expiration date of the option contract. These payoffs should be taken into account when calculating the option price. To price the option contract under these circumstances, we need to calculate the payoffs of the option at each step of the tree, separately, as the option can be exercised at any time-point of the maturity interval. The American option price can then be calculated by the maximum discounted present value of these payoffs.

The BT approach for pricing the American option under the above circumstances is known as the value back down BT approach. According to this, we start, first, calculating the option price and its payoffs at the end of the tree ( $n^{th}$  step), and then we move backwards to the first step. Note that, at the expiration date, the price of the American option at the  $(n, j)$  node (branch) of the tree can be calculated as

$$\begin{aligned} P_{A,t+n\Delta t}^{(j)} &= \max(K - X_{t+n\Delta t}^{(j)}, 0) \\ &= \max(K - S_t u^j d^{n-j}, 0), \end{aligned} \tag{2.6}$$

for all nodes  $j$  of the  $n^{th}$  step of the tree. Given the above value of the option, at the  $n^{th}$  step, we can move backwards to calculate the American price at the steps  $(n-1)^{th}$ ,  $(n-2)^{th}$  and so on. In doing so, notice that the risk neutral probability  $q$  of the stock price movement from the  $(i, j)$ ,  $i = 1, 2, \dots, n$ , node at time  $t+i\Delta t$  to the

$(i + 1, j + 1)$  node at time  $t + (i + 1)\Delta t$  and the risk neutral probability  $(1 - q)$  of the stock price movement from the  $(i, j)$  node at time  $t + i\Delta t$  to the  $(i + 1, j)$  node at time  $t + (i + 1)\Delta t$  are the same with those for the standard BT approach, given by equation (2..5). Thus, the American option price for the node  $(i, j)$  of the tree can be calculated as

$$P_{A,t+i\Delta t}^{(j)} = \max \left[ K - X_{t+i\Delta t}^{(j)}, e^{-r\Delta t} \left( qP_{A,t+(i+1)\Delta t}^{(j+1)} + (1 - q)P_{A,t+(i+1)\Delta t}^{(j)} \right), 0 \right] \quad (2..7)$$

where  $e^{-r\Delta t} \left( qP_{A,t+(i+1)\Delta t}^{(j+1)} + (1 - q)P_{A,t+(i+1)\Delta t}^{(j)} \right)$  denotes the time  $t + i\Delta t$  discounted payoff of the American option of the  $t + (i + 1)\Delta t$  step of the tree if it is kept alive and  $K - X_{t+i\Delta t}^{(j)}$  constitutes the payoff of the option if it is immediately exercised. Moving backwards, we can calculate the option price at the current time  $t$ .

Appraisal of the BT method Although the BT method for pricing options constitutes a very simple and popular method, it suffers from some problems listed below.

The first well known drawback of this approach is that the BT can give negative risk neutral probabilities. This happens when the steps of the tree are not sufficiently small [see Rendleman and Bartter (1979)]. To avoid this problem, Jarrow and Rudd (1983) suggest that  $q = 0.5$ , rather than  $ud = 1$ , which has been suggested by Cox, Ross and Rubinstein (1979). But, this is not an economically justified restriction of the risk neutral probability,  $q$ .

The second drawback of the BT method is that it is computationally very demanding and time-consuming. To circumvent this problem, Hull and White(1994), and Figlewski and Gao (1999) suggested modifications of the BT methods, such as the trinomial tree and the adaptive mesh tree. The trinomial tree assumes that every node of the tree potential outcomes, compared with the two of the binomial tree. This modification of the tree can increase the computation speed because it considers far more outcomes of the stock price movements at every node of the tree, and thus

increase computation speed as it can allow for less steps. The adaptive mesh tree allows for not equally distanced steps.

However, the above modifications of the BT approach can not significantly reduce the computation speed, which remains the main disadvantage of this approach. Apart from this drawback, the BT approach is difficult to be applied for cases where the underlying security price is driven by more than one state variables, since the movements of the state variables at each branch of the tree will exponentially increase with the number of state variables.

2.2.2 Finite Difference (FD) The finite difference approach [see Brennan and Schwartz (1978), *inter alia*] is entailed in pricing American options by exploiting the PDE given by equation (2..3), with its variant inequality. For expositional simplicity, we assume that the stock price conditional on its current price follows the log-normal Black and Scholes (1993) model. Then, (2..3) implies the following PDE

$$rX_t \frac{\partial P_A(X_t, t)}{\partial X_t} + \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 P_A(X_t, t)}{\partial X_t^2} - \frac{\partial P_A(X_t, t)}{\partial t} = rP_A(X_t, t), \quad (2..8)$$

subject to the boundary conditions

$$\begin{aligned} \lim_{t \rightarrow T} P_A(X_t, t) &= \max(0, K - X_t), \\ \lim_{X_t \rightarrow 0} P_A(X_t, t) &= K, \\ \lim_{X_t \rightarrow B_t} P_A(X_t, t) &= K - B_t, \\ \lim_{X_t \rightarrow B_t} \frac{\partial P_A(X_t, t)}{\partial X_t} &= -1, \end{aligned}$$

where  $X_t$  is the underlying stock price,  $r$  is the instantaneous interest rate,  $\sigma$  is the standard deviation of the stock price changes and  $B_t$  denotes the critical stock price where the option holder is indifferent between exercising and continuing to hold the option contract, known as optimal exercise boundary price.

The above PDE indicates that it is difficult to derive closed form solutions

for the option price  $P_A(X_t, t)$ . This happens because the optimal exercise boundary price,  $B_t$ , entering into this is unknown. Therefore, numerical methods have been suggested to solve out (2..8) under its boundary conditions.

The most straightforward and easy to be applied numerical approach which has been suggested is the finite difference (FD) method. To implement this method, we need to set up a grid search [see Hull (2000)]. In particular, we assume that  $t = 0$ , and that the maturity interval,  $T$ , can be divided into  $n$  subintervals of the same length  $\Delta t = \frac{T}{n}$ . Suppose that  $X_{\max}$  denotes a sufficiently high stock price with zero occurrence probability. If we divide the space of the underlying stock price  $[0, X_{\max}]$  into  $M$  equal intervals and let  $\Delta X = \frac{X_{\max}}{M}$ , we have the following sequence of  $M + 1$  stock prices

$$0, \Delta X, 2\Delta X, \dots, X_{\max}.$$

If we refer to the node that the stock price is  $j\Delta X$  at  $i\Delta t$  as the  $(i, j)$  node. Then, the above division of the stock prices and the maturity interval implies  $(M + 1)(n + 1)$  nodes of the grid search.

Armed with the above definitions, the FD approach assumes that the derivatives involved in (2..8) can be approximated as follows:

$$\begin{aligned} \frac{\partial P_A(X_t, t)}{\partial X_t} &= \frac{P_{i+1, j+1} - P_{i+1, j-1}}{2\Delta X}, \\ \frac{\partial^2 P(X_t, t)}{\partial X_t^2} &= \frac{P_{i+1, j+1} + P_{i+1, j-1} - 2P_{i+1, j}}{\Delta X^2}, \\ \text{and} \quad \frac{\partial P(X_t, t)}{\partial t} &= \frac{P_{i+1, j} - P_{i, j}}{2\Delta t}, \end{aligned}$$

where  $P_{i, j}$  denotes the American price at each node  $(i, j)$ , and that the PDE (2..8) at



each node  $(i, j)$  can be written as

$$\begin{aligned}
 P_{i,j} &= a_{1,j}P_{i+1,j-1} + a_{2,j}P_{i+1,j} + a_{3,j}P_{i+1,j+1}, \quad \text{with} \\
 a_{1,j} &= \frac{1}{1+r\Delta t} \left( -\frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2j^2\Delta t \right), \\
 a_{2,j} &= \frac{1}{1+r\Delta t} (1 - \sigma^2j^2\Delta t) \quad \text{and} \\
 a_{3,j} &= \frac{1}{1+r\Delta t} \left( \frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2j^2\Delta t \right).
 \end{aligned} \tag{2..9}$$

Given that the option price at the expiration date is equal to  $\max(K - X_T, 0)$ , we can write the option price at node  $(n, j)$ , for all  $j$ , at the expiration date of the option as

$$P_{n,j} = \max(K - j\Delta X, 0), \quad j = 0, 1, 2, \dots, M. \tag{2..10}$$

Exploiting (2..10), equation (2..9) can be solved backwards to determine the option prices at each node  $(i, j)$ , as we did for the BT approach.

The FD method is more efficient than the BT method in terms of computation time because it can limit the grid search for calculating the American option prices into a subset of the space spanned by the values of the underlying stock price. One drawback of this method, mentioned in the literature [see Hull and White (1990)], is that the coefficients  $a_{1,j}$ ,  $a_{2,j}$  and  $a_{3,j}$  in (2..9), representing the risk neutral probabilities of moving from one node of step  $i$  to another node of step  $i + 1$  in the grid, can take negative values. Note that if one of coefficients  $a_{1,j}$ ,  $a_{2,j}$  and  $a_{3,j}$  become negative, then the FD method does not converge. To overcome this problem, Hull and White (*ibid*) recommended to set up very small time-steps for the grid.

The FD approach, presented above, does not constitute the only method which has been suggested in the literature for solving PDE (2..8). Alternative methods have been also suggested by Huang and Pang (1998) and Dempster and Hutton (1999). In particular, Huang and Pang (*ibid*) suggested that we use the implicit finite difference [or Crank-Nicolson] method. According to this method, the derivatives in (2..8) can

be approximated as

$$\begin{aligned}\frac{\partial P_A(X_t, t)}{\partial X_t} &= \frac{P_{i,j+1} - P_{i,j-1}}{2\Delta X} \text{ and} \\ \frac{\partial^2 P(X_t, t)}{\partial X_t^2} &= \frac{P_{i,j+1} + P_{i,j-1} - 2P_{i,j}}{\Delta X^2},\end{aligned}$$

while the PDE (2..8) itself can be approximated as

$$P_{i+1,j} = a_{1,j}P_{i,j-1} + a_{2,j}P_{i,j} + a_{3,j}P_{i,j+1},$$

where

$$a_{1,j} = \frac{1}{2}rj\Delta t - \frac{1}{2}\sigma^2j^2\Delta t, a_{2,j} = 1 + \sigma^2j^2\Delta t + r\Delta t, a_{3,j} = -\frac{1}{2}rj\Delta t - \frac{1}{2}\sigma^2j^2\Delta t.$$

The last approximation method does not suffer from any nonconvergence problems. The only disadvantage of this method is that it requires a system of  $M - 1$  linear equations to be solved at each time-step of the grid. This will have a consequence to significantly increase the computation time needed for the calculations. Indeed, this remains to be the major disadvantage of all FD methods suggested in the literature. As for the BT method, this problem becomes more severe when multi-state, factor asset pricing models are considered.

2.2.3 Monte Carlo simulation based methods To present this method, we write the evaluation problem of an American put option (2..1) as

$$P_A(X_t, t) = E_t^Q \left[ e^{-\int_t^{\tau^*} r_s ds} g(X_{\tau^*}) \right].$$

where  $\tau^*$  is defined by the equation (2..2) as the optimal time. To calculate the American option price  $P_A(X_t, t)$ , the simulation based methods generate paths of the underlying asset price under risk neutrality, and then calculate the option price by

taking the average of the generated payoffs appropriately discounted at each time-point of the maturity interval for each simulation experiment.

The Monte Carlo simulation based method is a very simple, transparent and flexible, as it can generate future paths of stock prices and their implied payoffs for any asset pricing model, with stochastic volatility and/or jumps. One of the drawbacks of this method is that it can not identify the optimal time-point of the maturity interval where the option contract can be exercised, knowing as optimal stopping time. To this end, a few new modifications of the simulation method have been recently suggested in the literature [see the methods suggested by Barraquand and Martineau (1995), Broadie and Glasserman (1997), Garcia (2000), Ibanez and Zapatero (1999), and Longstaff and Scharwtz (2001)]. Among them, the method suggested by Longstaff and Scharwtz (2001) seems to be the most efficient one. To deal with the problem of the optimal stopping time, this method calculates the conditional expectation of the payoff from keeping option alive. This expectation is assumed to be a function of the underlying state variables (here the stock price) and is calculated by the cross-sectional regression of the simulated payoffs on the values of the state variable, at each time-point of the maturity interval. The conditional expectation of this regression is used to determine whether the option is exercisable, or not, at each point of the maturity interval, and thus can be used to identify the optimal stopping time.

Although the simulation based methods seem to be more efficient than the other numerical methods, surveyed before, they are still time consuming, and require a great amount of computation effort, especially for calculating the hedging parameters of options prices. This can be thought of as the main drawback of this category of methods.

2.2.4 Optimal exercise boundary approximation methods This method is based on a decomposition of the American option price into its European option price counterpart and a premium allowing for an early exercise of the option, known as the early exercise premium [see Barone-Adesi and Whaley (1987), Jacka (1991), Mynenni

(1992), Jamshidian (1992) and Kim (1990), *inter alia*]. Below, we present this type of decomposition for an American call option price for the simple case of the lognormal model of the stock price, derived in the literature [see Kim (1990)],

$$\begin{aligned}
 P_A(X_t, T) &= P_E(X_t, T) & (2..11) \\
 &+ \int_t^T [rKe^{-r(s-t)}N(-d_2(X_s, B_s, s-t)) - \\
 &\quad - \delta X_t e^{-\delta(s-t)}N(-d_1(X_t, B_s, s-t))] ds,
 \end{aligned}$$

where  $P_E(X_t, T)$  denotes the European option price,  $B_s$  is the optimal exercise boundary price above which the option contract will be exercised,  $\delta$  denotes the dividend rate,  $N(\cdot)$  stands for the normal distribution, with  $d_1(X_t, B_s, s-t)$  and  $d_2(X_t, B_s, s-t)$  are defined as follows

$$d_1(X_t, B_s, s-t) = \frac{\log(X_t/B_s) + (r - \delta + 0.5\sigma^2)(s-t)}{\sigma\sqrt{s-t}}$$

and  $d_2(X_t, B_s, s-t) = d_1(X_t, B_s, s-t) - \sigma\sqrt{s-t}$ .

At any point  $s \in [t, T]$ , the optimal exercise boundary price satisfies the following recursive equation

$$\begin{aligned}
 B_s - K &= P_A(B_s, T-s) & (2..12) \\
 &+ \int_s^T [rKe^{-r(h-s)}N(-d_2(B_s, B_h, h-s)) \\
 &\quad - \delta B_s e^{-\delta(h-s)}N(-d_1(B_s, B_h, h-s))] dh
 \end{aligned}$$

subject to the boundary condition  $B_T = \min(K, \frac{r}{\delta}K)$ .

The above decomposition of the option price  $P_A(X_t, T)$  implies an analytic, integral solution of the early exercise premium, given as

$$\int_t^T rKe^{-r(s-t)}N(-d_2(X_s, B_s, s-t)) - \delta X_t e^{-\delta(s-t)}N(-d_1(X_t, B_s, T-s)) ds.$$

Equations (2.11) and (2.12) indicate that the only difficulty in calculating the option price  $P_A(S_t, T)$  based on its decomposition given above is that the optimal exercise boundary price, which is defined recursively, constitutes a part of its analytical solution. However, once this price is determined, the calculation problem of  $P_A(S_t, T)$  becomes very trivial.

To calculate the optimal exercise boundary, a number of methods have been suggested in the literature [see Huang, Subrahmanyam and Yu (1996) and Ju (1998)]. In particular, Huang, Subrahmanyam and Yu (ibid) suggested to approximate the optimal exercise boundary function using a flat line at each piece of the maturity interval and Ju (ibid) recommended to employ an exponential function. These approximation schemes seem to work efficiently and to calculate American option prices very fast, compared with the other numerical methods surveyed above [see Ju (ibid)]. The success of this method stems from the fact that it involves only one step of approximation; the European price can be calculated analytically. Another source of the efficiency of the method is that, based on the above approximation schemes, we can significantly reduce the dimensionality of the integrals involved in the pricing formula.

2.2.5 Other methods The above categories of numerical methods suggested for pricing American options do not exhaust the whole set of the available method in the literature. See, for instance, the methods suggested by Barone-Adesi and Whaley (1987), Geske and Johnson (1984), Lim and Guo (2000) and Sullivan (2000). However, these methods can to some degree be thought of as mixed versions of the methods surveyed in this chapter. We do not review these methods for reasons of space.

## 2.3 Conclusions

The aim of this chapter has been to present alternative numerical approximation methods for pricing American call (or put) options. In our survey of these methods, we focus our discussion on both the computational and pricing accuracy

(efficiency) issues related to each category of methods.

CHAPTER 3.  
PRICING AMERICAN OPTIONS UNDER STOCHASTIC  
VOLATILITY: A NEW METHOD USING CHEBYSHEV  
POLYNOMIALS TO APPROXIMATE THE EARLY EXERCISE  
BOUNDARY

3.1 Introduction

Pricing American options is one of the most difficult problems in option pricing literature. The difficulty stems from the fact that, unlike a European, an American call (or put) option has no explicit closed form solution. This happens because the optimal boundary above which the American call option will be exercised is unknown and part of the option price solution. Therefore, efforts have been concentrated on developing numerical approximation schemes which can price the American options accurately and faster than the lattice or simulation based methods, which are time consuming and computationally more demanding. These schemes are based on integral representations of the American option evaluation formula or they exploit the partial differential equation satisfied by the option prices.<sup>1</sup>

The existing approximation schemes for pricing American call (or put) options in the literature are valid only under the assumptions of the Black and Scholes (1973) option pricing model, which assert that the stock price of the underlying stock is log-normally distributed conditional on the current stock price and has constant volatility. However, these assumptions are in contrast to most of the empirical evidence of the option and stock pricing empirical literature, which indicates that stocks' prices volatility is stochastic and stocks' returns distributions deviate from lognormality [see Ghysels, Harvey and Renault (1996), for a survey].

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<sup>1</sup> Examples of such type of numerical methods include the Barone-Adesi and Whaley (1987) analytical approximation method, the approximating methods of Geske-Johnson (1984) and Bunch and Johnson (1992), the Gaussian quadrature method of Sullivan (2000), *inter alia*, and the recently developed exercise boundary approximation methods of Subrahmanyam and Yu (1996), and Ju (1998).

The aim of this chapter is to develop a new numerical method for pricing American call option prices for the case that the underlying stock's price volatility is stochastic. The lack of such type of methods in the literature of the American options is primarily due to the fact that, under stochastic volatility, the optimal exercise boundary depends, in addition to time, on the paths of the volatility [see Broadie *et al* (2000)]. This considerably complicates the derivation of a suitable, analytic representation for an American call option price upon which a numerical approximation method can be build up. Our strategy of circumventing this problem is to approximate the optimal exercise boundary function with a log-linear function with respect to volatility changes over different pieces of the maturity interval.<sup>2</sup> Based on this approximation, we derive an analytic, integral representation of the early exercise premium of the American call option price. This representation unbundles the early exercise premium (and hence the American call option price) into a portfolio of Arrow-Debreu type of securities [see Bakshi and Madan (2000), for a European call option price]. The prices of these securities (and thus the American call option price) can be calculated based on the joint characteristic function of the stock price and its conditional volatility process which is derived in closed form in the chapter. To complete our numerical method for evaluating the American call option under stochastic volatility, we employ Chebyshev polynomials to estimate the optimal exercise boundary function. With these polynomials, we can efficiently approximate any non-linear pattern of the optimal exercise boundary function, over the different pieces of the maturity interval, because we can choose the point with the minimum approximation error to fit a high-degree polynomial approximating function into the true optimal exercise boundary function.

To appraise the pricing performance of our method, the chapter reports numerical results of the speed and accuracy of the method in comparison with benchmark

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<sup>2</sup> Note that this approach is consistent with recent evidence suggesting that, when volatility is stochastic, the exercise boundary is smooth with respect to volatility changes [see Broadie *et al* (2000)].



methods. We also compare the pricing performance of the method for the case that volatility is constant with other exercise boundary approximation methods for the log-normal model, which are frequently used in practice. The results of the numerical evaluations are very encouraging. They show that a very parsimonious, two degree approximating function of the exercise boundary based on Chebyshev polynomials can satisfactorily price American call options for a broad class of stock and exercise prices considered in our numerical experiments. This is true both under stochastic and constant volatility. Our results show that the pricing errors of our method are very close to zero, and they are of the same order of magnitude independently on whether the volatility is constant or stochastic. In the constant volatility case, we find that the pricing errors of our method can become smaller in magnitude than the other approximation methods compared with, especially when the curvature of the true optimal exercise boundary function is high.

The chapter is organised as follows. In Section 3.2, we present the evaluation framework for the American call option price under stochastic volatility and derive an analytic, integral representation of the American call price. In Section 3.3, we show how to implement Chebyshev polynomials to approximate the optimal exercise boundary function for the lognormal and stochastic volatility models, respectively. In Section 3.4, we list and discuss numerical results of the performance of our method to price the options. Section 3.5 summarizes and concludes the chapter.

### 3.2 Analytic evaluation of American call options under stochastic volatility

In this section, in order to derive an analytic evaluation formula for an American call option we assume that the price of the underlying stock follows a geometric stochastic volatility process. This model of the stock price is known in the literature as the stochastic volatility (SV) model [see Heston (1993), *inter alia*]. The analysis of the section proceeds as follows. First, we present a general evaluation framework for pricing an American call option under stochastic volatility which is in line with

that of Broadie *et al* (2000). Based on this framework, we next derive an analytic, integral representation of the American call option price.

3.2.1 The valuation framework Assume that the dynamics of underlying stock's price, denoted  $P_t$  at time  $t$ , follow Heston's (1993) specification of the stochastic volatility (SV). For analytic convenience, assume that dividends are paid at the constant rate  $\delta$  and that the riskless interest rate,  $r$ , is constant. Then, the SV model implies that the spot stock price should satisfy the following risk-neutralised process

$$\frac{dP_t}{P_t} = (r - \delta) dt + \sqrt{V_t} dW_{1,t}, \quad (3.1)$$

where the instantaneous conditional variance (volatility),  $V_t$ , follows the mean reverting square root process

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dW_{2,t}, \quad (3.2)$$

where  $k$  is adjusted by the market price of volatility risk,  $\{W_{j,t}, t > 0\}$ ,  $j = 1, 2$ , are two correlated standard Brownian motion processes, with correlation coefficient given by  $Corr(dW_{1,t}; dW_{2,t}) = \rho dt$ ,  $\rho \in (-1, 1)$ .

Consider now an American call option contract for the above stock with maturity date  $T$  and strike price  $K$ , at the exercise time. This contract gives the holder the right of exercising the call option at any time  $h$  in the maturity interval  $[t, T]$ , i.e.  $h \in [t, T]$ . The critical stock price above which the American call will be exercised is referred to as the optimal exercise boundary. Since the price of the underlying stock depends on the paths of the volatility process  $V_t$ , we will hereafter denote the time  $t$ , which represents the current time price of the American call option contract (i.e. the American call option price) as  $C_A(P_t, V_t, T - t)$ , while the optimal exercise boundary will be denoted as  $B(V_h, h)$ ,  $\forall h \in [t, T]$ .

The American call option price  $C_A(P_t, V_t, T - t)$  can be calculated by the

maximum value of the discounted payoffs from the option where the maximum is taken over all possible stopping (exercise) times, denoted  $\tau$ , in the maturity interval,  $[t, T]$ . Define the optimal stopping time as

$$\tau^* = \inf \{ \tau \in [t, T] : C_A(P_t, V_t, T - t) = (P_t - K)_+ \}. \quad (3.3)$$

Then, the American call option pricing problem can be represented by the Snell envelop<sup>3</sup>

$$C_A(P_t, V_t, T - t) = \sup_{\tau \in \mathcal{S}_{[t, T]}} E^Q \left( e^{-\int_t^\tau r ds} (P_\tau - K)_+ \right), \quad (3.4)$$

where  $\mathcal{S}_{[t, T]}$  is the set of stopping times in the maturity interval,  $[t, T]$ ,  $E_t^Q$  denotes the time  $t$  conditional expectation under the equivalent martingale measure  $Q$ , and  $(P_\tau - K)_+$  is the payoff of the American call option at the stopping time  $\tau$ .

The following theorem characterises the optimal solution of the problem defined by equation (3.4).

**Theorem 2** *Let the stock price satisfy processes (3.1) and (3.2). Then, the American call option price  $C_A(P_t, V_t, T - t)$  can be written as*

$$C_A(P_t, V_t, T - t) = C_E(P_t, V_t, T - t) + E_t^Q \left\{ \int_t^T e^{-r(s-t)} (\delta P_s - rK) I_{\{P_s \geq B(V_s, s)\}} ds \right\}, \quad (3.5)$$

where  $C_E(P_t, V_t, T - t)$  is the value of a European call price with maturity date  $T$  and strike price  $K$ ,  $B(V_s, s)$  denotes the value of the optimal exercise boundary, at time  $s \in [t, T]$ , and  $I_A$  is the indicator function of the set  $A$ , defined as  $A = \{P_s : P_s \geq B(V_s, s) \text{ and } V_s \in \mathcal{R}^+\}$ , which contains the prices of the stock at which the

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<sup>3</sup> See Karatzas (1988), *inter alia*.

American call will be exercised. The optimal exercise boundary  $B(V_h, h)$  entered into the American call option price formula (3.5) should satisfy the following recursive equation

$$\begin{aligned}
 & B(V_h, h) - K & (3.6) \\
 = & C_E(B(V_h, h), K, V_h, T - h) \\
 & + E_h^Q \left[ \int_h^T e^{-r(s-h)} (\delta P_s - rK) I_{\{P_s \geq B(V_s, s)\}} ds \right], \quad \forall s \geq h \in [t, T],
 \end{aligned}$$

with terminal condition

$$B(V_T, T) - K = \max \{K, rK/\delta\}. \quad (3.7)$$

In Appendix A.1, we give a proof of Theorem 2 based on a decomposition of the optimal stopping problem (3.4) in terms of the optimal exercise boundary [see Myneni (1992)].

Theorem 2 shows that the American call option price  $C_A(P_t, V_t, T - t)$  can be evaluated using formula (3.5), once the values of optimal exercise boundary,  $B(V_h, h)$ , are provided. However, these values are not available due to the recursive nature of  $B(V_h, h)$ , which shows that the optimal exercise boundary is determined as part of the American call option price solution. To circumvent this difficult evaluation problem, Huang, Subrahmanyam and Yu (1996), and Ju (1998) suggested efficient numerical approximation methods of the optimal exercise boundary function for the case of the lognormal model. These methods are based on an approximation of  $B(V_h, h)$ , over different pieces of the maturity interval, with a constant and an exponential function of time,  $h$ . With these methods, we can derive analytic, integral representations of the early exercise premium embedded into the option price, given by  $E_t^Q \int_t^T e^{-r(s-t)} (\delta P_s - rK) I_{\{P_s \geq B(V_s, s)\}} ds$  [see (3.5)], and then calculate the American call option price, using equation (3.5). However, these methods can not be applied

to the case of the SV model. This happens because, under stochastic volatility, the optimal exercise boundary should also depend on the volatility changes. In the next subsection, we therefore derive an analytic, integral representation of the American call price and the optimal exercise boundary recursive equation assuming that the optimal exercise boundary function, in addition to time  $h$ , depends on the volatility changes of the underlying stock.

### 3.2.2 An integral representation of the American call option price for the SV model

Suppose that the logarithm of the optimal exercise boundary, at time  $h$ , defined as  $b(V_h, h) \equiv \ln B(V_h, h)$ , is given by the following linear in volatility,  $V_h$ , relationship

$$b(V_h, h) = b_0(h) + b_1(h)(V_h - E_t V_h). \quad (3.8)$$

The above relationship can be thought of as a first-order log-linear approximation of the optimal exercise boundary function around the time  $t$  conditional mean of volatility, denoted  $E_t V_h$ . It asserts that, for small changes of  $V_h$  around  $E_t V_h$ , the true optimal exercise boundary function is an exponentially smooth surface with respect to volatility changes. This assumption can be justified by recent evidence provided by Broadie *et al* (2000), who recovered the American call option price and the exercise boundary reduced forms from the data following a non parametric statistical approach.

Given relationship (3.8), in the next theorem we derive an integral representation of the American call option pricing formula (3.5) for the SV model upon which we can build up a numerical method for evaluating the American options, under stochastic volatility.

**Theorem 3** *For relationship (3.8), the American call option price  $C_A(P_t, V_t, T - t)$*

can be calculated as

$$\begin{aligned}
& C_A(P_t, V_t, T - t) \\
&= C_E(P_t, V_t, T - t) + \int_t^T \delta P_t e^{-\delta(s-t)} \Pi_1(b_0(s), b_1(s) | P_t, V_t) ds \\
&\quad - \int_t^T r K e^{-r(s-t)} \Pi_2(b_0(s), b_1(s) | P_t, V_t) ds, \tag{3..9}
\end{aligned}$$

where analytic, integral solutions of  $\Pi_1(\cdot)$  and  $\Pi_2(\cdot)$  are given in Appendix B. The optimal exercise boundary  $B(V_h, h)$  satisfies the following recursive equation

$$\begin{aligned}
& B(V_h, h) - K = C_E(P_h, V_h, T - h) \\
&\quad + \int_h^T \delta B(V_h, h) e^{-\delta(s-h)} \Pi'_1(b_0(s), b_1(s) | B(V_h, h), V_h) ds \\
&\quad - \int_h^T r K e^{-r(s-h)} \Pi'_2(b_0(s), b_1(s) | B(V_h, h), V_h) ds, \tag{3..10}
\end{aligned}$$

$\forall s \geq h \in [t, T]$ , with terminal condition

$$B(V_T, T) - K = \max\{K, rk/\delta\},$$

where  $\Pi'_1(\cdot)$  and  $\Pi'_2(\cdot)$  are given in Appendix A.2.

The proof of the Theorem is given in Appendix A.2.

The integral representation of the American option price  $C_A(P_t, V_t, T - t)$  and its associated exercise boundary recursive equation (3..10), given by Theorem 2, unbundles the early exercise boundary premium (and hence the American call option) into a portfolio of Arrow-Debreu type of securities. The prices of these securities, denoted by the Greek letter  $\Pi(\cdot)$ , can be derived by calculating the following risk neutral expectations:

$$\Pi_1(b_0(s), b_1(s) | P_t, V_t) = E_t^Q \left[ \frac{P_s I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}}}{E_t^Q [P_s]} | P_t, V_t \right], \tag{3..11}$$

or using the transformed measure  $Q_1$  with  $\frac{dQ_1}{dQ} = \frac{P_s}{E_h^Q[P_s]}$  as

$$\Pi_1(b_0(s), b_1(s)|P_t, V_t) = E_t^{Q_1} [I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | P_t, V_t], \quad (3..12)$$

and

$$\Pi_2(b_0(s), b_1(s)|P_t, V_t) = E_t^Q [I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | P_t, V_t]. \quad (3..13)$$

Closed form (analytic) solutions of the above prices  $\Pi_1(b_0(s), b_1(s)|P_t, V_t)$  and  $\Pi_2(b_0(s), b_1(s)|P_t, V_t)$ , given by equations (3..12) and (3..13) respectively, are given in Appendix A.2. These are derived based on the joint characteristic function of the stock price,  $P_t$ , and volatility,  $V_t$ , which is derived in closed form in Appendix A.2. The economic intuition of these prices (and thus their characterisation as Arrow-Debreu state prices) can be derived by equations (3..12) and (3..13). These show that  $\Pi_1(b_0(s), b_1(s)|P_t, V_t)$  and  $\Pi_2(b_0(s), b_1(s)|P_t, V_t)$  constitute the market prices of a security which pays \$1 in state  $\{(P_s, V_s) : P_s \geq B(V_s, s)\}$  and 0 otherwise under the risk neutral measures  $Q_1$  and  $Q$ , respectively. In the risk neutral asset pricing context, these prices are equal to the risk neutral probabilities of the state  $\{(P_s, V_s) : P_s \geq B(V_s, s)\}$ . Below, we show this for case of the lognormal model.

The integral representation of the American call option given by Theorem 2 can be reduced to that for the lognormal model, derived by Kim (2000) by setting  $k = \theta = \sigma = 0$  in equations (3..9) and (3..10) and noticing that, under the assumptions of the log-normal model, the exercise boundary equation (3..8) is given by  $B(V_h, h) = \exp[b_0(h)]$ . Then, it can be easily seen that equation (3..9) reduces to

$$C_A(P_t, T-t) = C_E(P_t, T-t) + \int_t^T \delta P_t e^{-\delta(s-t)} \Pi_1(B(s)|P_t) ds \quad (3..14) \\ - \int_t^T r K e^{-r(s-t)} \Pi_2(B(s)|P_t) ds, \text{ for } s \geq h \in [t, T],$$

while equation (3.10) reduces to

$$\begin{aligned}
 & B(h) - K & (3.15) \\
 = & C_E(P_h, T-t) + \int_t^T \delta B(h) e^{-\delta(s-h)} \Pi'_1(B(s)|B(h)) ds \\
 & - \int_t^T rK e^{-r(s-h)} \Pi'_2(B(s)|B(h)) ds,
 \end{aligned}$$

where now  $\Pi_1(B(s)|P_t)$  and  $\Pi_2(B(s)|P_t)$  are given by

$$\begin{aligned}
 & \Pi_1(B(s)|P_t) \\
 = & \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[ \frac{e^{-i\phi \log B(s)} F(i\phi_X, 0, s-h | \ln P_t, 0)}{i\phi} \right] d\phi \\
 = & N \left( \frac{\log(P_t/B(s)) + (r - \delta + \frac{1}{2}\sigma^2)(s-t)}{\sigma\sqrt{s-t}} \right) & (3.16)
 \end{aligned}$$

and

$$\begin{aligned}
 & \Pi_2(B(s)|P_t) \\
 = & \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[ \frac{e^{-i\phi \log B(s)} F(\phi_X, 0, s-h | \ln P_t, 0)}{i\phi} \right] d\phi \\
 = & N \left( \frac{\log(P_t/B(s)) + (r - \delta - \frac{1}{2}\sigma^2)(s-t)}{\sigma\sqrt{s-t}} \right), & (3.17)
 \end{aligned}$$

respectively.<sup>4</sup> Equations (3.16) and (3.17) clearly indicate that the prices of the Arrow-Debreu type of securities  $\Pi_1(B(s)|P_t)$  and  $\Pi_2(B(s)|P_t)$  reflect the prices of a security which pays \$1 in the state  $\{P_s \geq B(s)\}$  and 0 otherwise under the measures  $Q_1$  and  $Q$ , respectively. These prices can be calculated as the probabilities of the standardized normal distribution at the values of  $\log(B(s)/P_t)$  adjusted by the quantities  $(r - \delta + \frac{1}{2}\sigma^2)(s-t)$  and  $(r - \delta - \frac{1}{2}\sigma^2)(s-t)$  under the measures  $Q_1$  and

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<sup>4</sup> The prices  $\Pi'_1(B(s)|B(h))$  and  $\Pi'_2(B(s)|B(h))$  can be defined analogously.



$Q$ , respectively.<sup>5</sup>

### 3.3 Numerical evaluation of American call options using Chebyshev polynomial functions to approximate the exercise boundary

As argued before, the integral representation of the American call option price and its associated recursive optimal exercise boundary relationship given by Theorem 3 can be used to build up a numerical approximation method for pricing American options under stochastic volatility. In this section we introduce such a method based on an approximation of the optimal exercise boundary function using Chebyshev polynomials.

Our motivation to implement a numerical approach to approximate the optimal exercise boundary rather than to directly approximate the whole American value formula stems from recent evidence suggesting that this numerical group of methods can considerably increase the computation speed of calculations without losing much in accuracy [see Huang, Subrahmanyam and Yu (1996), and Ju (1998)]. This happens because the boundary approximation methods can separate the estimation problem of the optimal exercise boundary function from that of the American call option. This can increase the computation speed while, simultaneously, avoid accumulating pricing errors through the evaluation steps of the American option risk neutral pricing formula. Our motivation to employ Chebyshev polynomials to approximate the true optimal exercise boundary function stems from the fact that, with these polynomials, we can efficiently approximate any non-linear function by choosing the point with the minimum approximation error to fit a high-degree polynomial approximating function to the true function.<sup>6</sup> Note that the accuracy of this approach will increase with the

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<sup>5</sup> Note that the above two quantities differ by  $\sigma^2$  which reflects the fact that the price of risk under the measure  $Q_1$  is smaller than under measure  $Q$ . This can be attributed to the fact that under measure  $Q_1$  the payoff of the Arrow-Debreu price is scaled by the stock price [see equations (3..12) and (3..13)].

<sup>6</sup> A brief description of the Chebyshev function approximation is given in Appendix A.3.

number of the Chebyshev polynomials used in the approximating function.

To better understand how to employ the Chebyshev polynomials to approximate the optimal exercise boundary, which will be hereafter referred to as the CB method, we first start our analysis with the case of the lognormal model. We next extend the analysis to the SV model.

3.3.1 The case of the lognormal model To implement the CB method for the lognormal model, notice that the optimal exercise boundary equation (3.15) can be reduced to the one-dimension integral relationship:

$$\begin{aligned}
& B(h) - K \\
&= C_E(P_h, T - h) - B(h)e^{-\delta(T-h)}N(d_1(B(h), B(T), T - h)) + B(h)N(\xi) \\
&+ Ke^{-r(T-h)}N(d_1(B(h), B(T), T - h)) - KN(\xi) \\
&+ \int_t^T B(h)e^{-\delta(s-h)}n(d_1(B(h), B(s), s - h))\frac{\partial d_1(B(h), B(s), s - h)}{\partial s}ds \\
&- \int_h^T Ke^{-r(s-h)}n(d_2(B(h), B(T), s - h))\frac{\partial d_1(B(h), B(s), s - h)}{\partial s}ds, \quad (3.18)
\end{aligned}$$

where  $\xi = \lim_{s \rightarrow h} \frac{\ln(B_h) - \ln(B(s))}{\sigma\sqrt{s-h}} = 0$ ,  $N(\cdot)$  and  $n(\cdot)$  denote the cumulative standard normal distribution and its associated probability density function, respectively.

Let  $\tilde{b}(h) = \log \tilde{B}(h)$  denote an approximating function of the logarithm of the optimal exercise boundary which consists of  $\nu$ -Chebyshev polynomials terms. The functional form of  $\tilde{b}(h)$  is given in Appendix A.3. Substituting  $\tilde{b}(h)$  into equation (3.18) implies the following system of equations for the optimal exercise boundary recursion

$$\begin{aligned}
& \tilde{B}(h) - K \\
&= C_E(P_h, T - h) - \tilde{B}(h)e^{-\delta(T-h)}N\left(d_1\left(\tilde{B}(h), \tilde{B}(T), T - h\right)\right) + \frac{1}{2}\tilde{B}(h) \\
&+ Ke^{-r(T-h)}N\left(d_2\left(\tilde{B}(h), \tilde{B}(T), T - h\right)\right) - \frac{1}{2}K \\
&+ \int_t^T \tilde{B}(h)e^{-\delta(s-h)}n\left(d_1\left(\tilde{B}(h), \tilde{B}(s), s - h\right)\right)\left(\sum_{i=0}^{\nu-2}\alpha_i\frac{s^i}{\sqrt{s-h}}\right)ds \\
&- \int_h^T Ke^{-r(s-h)}n\left(d_2\left(\tilde{B}(h), \tilde{B}(s), s - h\right)\right)\left(\sum_{i=0}^{\nu-2}\gamma_i\frac{s^i}{\sqrt{s-h}}\right)ds, \tag{3..19}
\end{aligned}$$

where  $a_i$  and  $\gamma_i$  satisfy the following recursive equations

$$\begin{aligned}
\alpha_i &= \gamma_i = \left(2ic_i - \sum_{j=i+1}^{\nu} c_j h^{j-i}\right) / 2\sigma, \quad \text{for } i = 2, 3, \dots, \nu - 1, \\
\alpha_0 &= \frac{1}{2\sigma} \left[2c_1 - \sum_{j=2}^{\nu} c_j h^{j-1} + r - \delta + 0.5\sigma^2\right] \quad \text{and} \quad \gamma_0 = \alpha_0 - \frac{\sigma}{2},
\end{aligned}$$

where  $c_{i+1}$  (or  $c_j$ ) are the coefficients of the approximating function  $\tilde{b}(h)$  [see Appendix A.3].

The system of equations defined by (3..19) consists of  $\nu$ -nonlinear equations with  $\nu$ -unknown  $c_i$ , for  $i = 0, 1, 2, \dots, \nu - 1$ , coefficients. Based on the minmax criterion, we can solve out this system for  $c_i$ , and determine the optimal exercise boundary approximating function,  $\tilde{B}(h)$ . The above numerical approximation method guarantees that  $\tilde{B}(h)$  converges to its true value,  $B(h)$ , as the number of the polynomial terms ( $\nu$ ) of the approximating function increases. This happens because, according to the minmax criterion,  $\tilde{B}(h)$  is chosen in order to be equal to the true function  $B(h)$  at  $\nu$ -zero points, where  $\tilde{B}(h)$  cuts off  $B(h)$ . As  $\nu$  increases,  $\tilde{B}(h)$  converges to  $B(h)$  by Weierstrass theorem.

To increase the computation speed of the CB method without significantly

losing in accuracy, we can employ Richardson's extrapolation scheme [see Ju (1988), *inter alia*]. According to this scheme, we need to calculate the optimal exercise boundary approximating function  $\tilde{B}(h)$  over the whole maturity interval, which is divided into  $\lambda = 1, \dots, \Lambda$  pieces, where  $\Lambda$  denotes the maximum number of pieces. The values of the American price corresponding to the maturity interval with  $\lambda$  pieces will be hereafter denoted as  $C_{A,\lambda}(P_t, T - t)$ . Below, we introduce all necessary notation in order to show how to calculate the American call price  $C_{A,\lambda}(P_t, T - t)$ .

Let  $\tilde{B}_{\lambda l}(h)$ , where  $l = 1, 2, \dots, \lambda$ , denote the value of  $\tilde{B}(h)$  over the  $l^{\text{th}}$ -sub-interval of the  $\lambda$  pieces maturity interval. Denote by  $\tilde{B}_{\lambda l}(z_j)$ , for  $j = 1, 2, \dots, \nu$ , the  $\nu$ -zero points of  $\tilde{B}_{\lambda l}(h)$  and by  $\Delta$  the fraction of the maturity interval  $\Delta = \frac{T-t}{\lambda}$ . Then, system (3.19) evaluated at the  $\nu$ -zero points implies the following  $\nu \times \lambda$  dimension system of equations:

$$\begin{aligned}
& \tilde{B}_{\lambda l}(z_j) - K = C_E \left( \tilde{B}_{\lambda l}(z_j), T - z_j \right) \\
& - \tilde{B}_{\lambda \lambda}(T) N \left( d_1 \left( \tilde{B}_{\lambda l}(z_j), \tilde{B}_{\lambda \lambda}(T), T - z_j \right) \right) \\
& + \frac{1}{2} \tilde{B}_{\lambda \lambda}(T) + KN \left( d_2 \left( \tilde{B}_{\lambda l}(z_j), \tilde{B}_{\lambda \lambda}(T), T - z_j \right) \right) - \frac{1}{2} K \\
& + \int_{z_j}^{t+l\Delta} \tilde{B}_{\lambda l}(z_j) e^{-\delta(s-z_j)} n \left( d_1 \left( \tilde{B}_{\lambda l}(z_j), \tilde{B}_{\lambda l}(s), s - z_j \right) \sum_{i=0}^{\nu-2} \alpha_i \frac{s}{\sqrt{s-z_i}} ds \right) \\
& - \int_{z_j}^{t+l\Delta} K e^{-r(s-z_j)} n \left( d_2 \left( \tilde{B}_{\lambda l}(z_j), \tilde{B}_{\lambda l}(s), s - z_j \right) \sum_{i=0}^{\nu-2} \gamma_i \frac{s}{\sqrt{s-z_i}} ds \right) \\
& + \sum_{h=l+1}^{\lambda} \int_{t+(h-1)\Delta}^{t+h\Delta} \tilde{B}_{\lambda l}(z_j) e^{-\delta(s-z_j)} n \left( d_1 \left( \tilde{B}_{\lambda l}(z_j), \tilde{B}_{\lambda h}(s), s - z_j \right) \sum_{i=0}^{\nu-2} \alpha_i \frac{s}{\sqrt{s-z_j}} ds \right) \\
& - \sum_{h=l+1}^{\lambda} \int_{t+(h-1)\Delta}^{t+h\Delta} K e^{-r(s-z_j)} n \left( d_2 \left( \tilde{B}_{\lambda l}(z_j), \tilde{B}_{\lambda h}(s), s - z_j \right) \sum_{i=0}^{\nu-2} \gamma_i \frac{s}{\sqrt{s-z_j}} ds \right),
\end{aligned} \tag{3..20}$$

for  $j = 1, 2, \dots, \nu$  and  $l = 1, 2, \dots, \lambda$ . The above system can be solved out in the

same way as system (3.19) in order to determine the optimal exercise boundary approximating function  $\tilde{B}_{\lambda l}(h)$ , corresponding to the maturity interval with the  $\lambda$  pieces. Given the approximation of the optimal exercise boundary  $\tilde{B}_{\lambda l}(h)$  for  $l = 1, 2, \dots, \lambda$ , The American call option price  $C_{A,\lambda}(P_t, T - t)$  can be calculated based on the equation (3.14) as

$$\begin{aligned}
C_{A,\lambda}(P_t, T - t) &= C_E(P_t, T - t) \\
&+ \sum_{l=1}^{\lambda} \int_{t+(l-1)\Delta}^{t+l\Delta} \delta P_t e^{-\delta(s-t)} N\left(d_1\left(P_t, \tilde{B}_{\lambda l}(s), s-t\right)\right) ds \\
&- \sum_{l=1}^{\lambda} \int_{t+(l-1)\Delta}^{t+l\Delta} rK e^{-r(s-t)} N\left(d_2\left(P_t, \tilde{B}_{\lambda l}(s), s-t\right)\right) ds. \tag{3.21}
\end{aligned}$$

In order to accelerate the computational speed, we can employ the Richardson extrapolation scheme to  $C_{A,\lambda}(P_t, T - t)$  for  $\lambda = 1, \dots, \Lambda$ .

3.3.2 The case of the SV model The implementation of the CB method to the stochastic volatility case is slightly more complicated than the constant volatility case, described in the previous subsection. This happens because the optimal exercise boundary now is a function in two dimensions: the time and volatility. According to equation (3.8), this means that we need to approximate the functional forms of the two coefficients  $b_0(h)$  and  $b_1(h)$  in order to approximate the optimal exercise boundary function  $B(V_h, h)$ .

Let us denote the approximating functional forms of these coefficients as  $\tilde{b}_0(h)$  and  $\tilde{b}_1(h)$ , respectively. Then, equation (3.8) implies that  $\tilde{b}_0(h)$  and  $\tilde{b}_1(h)$  can be determined once two distinct values of the conditional variance (say  $V_{h,0}$  and  $V_{h,1}$ ) and their associated optimal exercise boundary are provided. Denote the approximating boundary function by the CB method at the above two values of the conditional variance as  $\tilde{B}(V_{h,i}, h)$ ,  $i = 0, 1$ , respectively. Then, the coefficients  $\tilde{b}_0(h)$  and  $\tilde{b}_1(h)$

can be calculated based on the following system of two equations:

$$\tilde{b}_1(h) = \frac{\ln[\tilde{B}(V_{h,1}, h) / \tilde{B}(V_{h,0}, h)]}{V_{h,1} - V_{h,0}} \quad (3..22)$$

and

$$\tilde{b}_0(h) = \frac{\ln[V_{h,0} \tilde{B}(V_{h,1}, h) / \tilde{B}(V_{h,0}, h) V_{h,1}]}{V_{h,0} - V_{h,1}}. \quad (3..23)$$

For an American call option with maturity date  $T$ , natural choices of  $V_{h,0}$  and  $V_{h,1}$  can be taken to be the time  $t$  variance  $V_t$  and time  $t$  expected values of the conditional variance  $E_t V_T$ , respectively. These are values around which the future values of the conditional variance over the maturity horizon  $[t, T]$  are expected to fluctuate. Equations (3..22) and (3..23) indicate that the optimal exercise boundary approximating function  $B(V_h, h)$  over the variance  $V_h$  at time  $h \in [t, T]$ , can be estimated based on the equations(3..22) and (3..23) to approximate the exercise boundary at the two values of the conditional variance  $V_{h,0}$  and  $V_{h,1}$ , i.e.  $\tilde{B}(V_{h,i}, h)$ ,  $i = 0, 1$ , respectively. The function  $\tilde{B}(V_{h,i}, h)$  over the time  $h \in [t, T]$  can further be approximated by the Chebyshev polynomial function of time  $h$ , having  $v$  unknown coefficients for the order  $v$  polynomial function. Because we need estimate  $\tilde{B}(V_{h,i}, h)$  for both  $i = 0, 1$ , we need to estimate  $2v$  coefficients for two Chebyshev polynomial function. These coefficients can be obtained by solving  $2v$  equations which are constructed by setting the approximation of  $\tilde{B}(V_{h,i}, h)$  to cut the true function  $\tilde{B}(V_h, h)$  at time zero-points  $z_j$  and variance  $V_{z_j, i}$  for  $j = 1, 2, \dots, v$  and  $i = 0, 1$ . For a maturity interval with  $\lambda$

pieces, this implies that we have the following  $2(\nu \times \lambda)$  system of equations

$$\begin{aligned}
& \tilde{B}_{\lambda l}(V_{z_j, i}, z_j) - K = C_E \left( \tilde{B}_{\lambda l}(V_{z_j, i}, z_j), K, V_{z_j, i}, T - z_j \right) \\
& + \int_{z_j}^{t+l\Delta} \delta \tilde{B}_{\lambda l}(V_{z_j, i}, z_j) e^{-\delta(s-z_j)} \Pi'_1 \left( \tilde{b}_{\lambda l, 0}(s), \tilde{b}_{\lambda l, 1}(s) \mid \tilde{B}_{\lambda l}(V_{z_j, i}, z_j), V_{z_j, i} \right) ds \\
& - \int_{z_j}^{t+l\Delta} rK e^{-r(s-z_j)} \Pi'_2 \left( \tilde{b}_{\lambda l, 0}(s), \tilde{b}_{\lambda l, 1}(s) \mid \tilde{B}_{\lambda l}(V_{z_j, i}, z_j), V_{z_j, i} \right) ds \\
& + \sum_{m=l+1}^{\lambda} \int_{t+m\Delta}^{t+m\Delta+\Delta} \delta \tilde{B}_{\lambda l}(V_{z_j, i}, z_j) e^{-\delta(s-z_j)} \Pi'_1 \left( \tilde{b}_{\lambda m, 0}(s), \tilde{b}_{\lambda m, 1}(s) \mid \tilde{B}_{\lambda l}(V_{z_j, i}, z_j), V_{z_j, i} \right) ds \\
& - \sum_{m=l+1}^{\lambda} \int_{t+m\Delta}^{t+m\Delta+\Delta} rK e^{-r(s-z_j)} \Pi'_2 \left( \tilde{b}_{\lambda m, 0}(s), \tilde{b}_{\lambda m, 1}(s) \mid \tilde{B}_{\lambda l}(V_{z_j, i}, z_j), V_{z_j, i} \right) ds,
\end{aligned} \tag{3.24}$$

for  $j = 1, 2, \dots, \nu$  and  $i = 0, 1$ , should be satisfied. Solving out this system with respect to the coefficients of the boundary approximating functions  $\tilde{B}(V_{h, i}, h)$ , for  $i = 0, 1$ , we can estimate the optimal exercise boundary approximating function  $\tilde{B}(V_h, h)$ , using equations (3.22) and (3.23). Given  $\tilde{B}(V_h, h)$ , then the American call option price corresponding to the maturity interval with  $\lambda$  pieces can be calculated as

$$\begin{aligned}
C_{A, \lambda}(P_t, V_t, T - t) &= C_E(P_t, V_t, T - t) \\
&+ \sum_{l=1}^{\lambda} \int_{t+(l-1)\Delta}^{t+l\Delta} \delta P_t e^{-\delta(s-t)} \Pi_1 \left( \tilde{b}_{\lambda l, 0}(s), \tilde{b}_{\lambda l, 1}(s) \mid P_t, V_t \right) ds \\
&- \sum_{l=1}^{\lambda} \int_{t+(l-1)\Delta}^{t+l\Delta} rK e^{-r(s-t)} \Pi_2 \left( \tilde{b}_{\lambda l, 0}(s), \tilde{b}_{\lambda l, 1}(s) \mid P_t, V_t \right) ds,
\end{aligned} \tag{3.25}$$

and the Richardson's extrapolation scheme can be employed to calculate  $C_A(P_t, V_t, T - t)$ , as the case of lognormal model.

### 3.4 Numerical results of the Chebyshev approximation method

In this section we report numerical results to evaluate the performance of the CB approximation method of the exercise boundary, developed in the previous section, to price American call options both for the stochastic volatility and lognormal models. The performance of the method is measured in terms of both the speed and accuracy with which it can price American call options in comparison with benchmark models. For the lognormal model, we compare the method with other existing numerical methods for pricing American call options based on an approximation of the optimal exercise boundary. These are the methods suggested by Huang, Subrahmanyam and Yu (1996) (hereafter HSY-3) and the exponential exercise boundary approximation method suggested by Ju (1998) (hereafter EXP-3). The aim of these comparisons is to investigate whether the CB method can improve upon the other optimal exercise boundary approximation methods, which are available for the lognormal model.<sup>7</sup> The section has the following order. We present first the numerical results for the lognormal model and, second, for the stochastic volatility model.

3.4.1 Numerical results for the lognormal model To assess the ability of the CB method to price American call options satisfactorily, compared with the other two approximation methods of the early exercise boundary function, we calculate the prices of  $J = 1250$  American call options, denoted  $C_{A,j}(P_t, V_t, T - t)$ ,  $j = 1, 2, \dots, J$ , based on the above methods and a benchmark method.<sup>8</sup> The parameters of the

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<sup>7</sup> A detail comparison of the optimal exercise boundary approximating methods with the other numerical methods for pricing American call options, based on the evaluation of the whole American call option risk neutral relationship or the finite difference methods, can be found in Ju (1998). This study clearly shows that the exercise boundary approximation methods are superior both in terms of accuracy and speed.

<sup>8</sup> Note that in order to implement the HSY-3 method, we have slightly modified the procedure suggested by Huang, Subrahmanyam and Yu (1996). We have only used the HSY method to approximate the exercise boundary. The integral terms of the American call option evaluation formula are calculated numerically, as in our method. We have found that this modification of the HSY method considerably reduces the pricing errors of the method.



lognormal stock price model that we use in calculating the options prices are randomly generated from the uniform distribution over the following intervals: [85, 115] for the current stock price ( $P_t$ ), [0.0, 0.10] for the dividend ( $\delta$ ) and interest rates ( $r$ ), [0.1, 0.6] for the volatility ( $V_t = \sigma$ ) and [0.1, 3.0] of years for the maturity interval. The strike price ( $K$ ) is set as fixed, at the level of  $K = 100$ . The above intervals of the parameters of the lognormal model cover a set of estimates that have been reported by many studies in the empirical literature of option pricing. As a benchmark model, we use the binomial-tree model of Cox, Ross and Rubinstein (1979) with  $N = 10,000$  time steps, denoted as BT. To evaluate the relative performance of the CB method as the degrees of the polynomial approximating function of the optimal exercise boundary increases, we employ the CB method with two and three degrees, denoted CB-2 and CB-3, respectively. For all the numerical methods employed, we evaluate the American call option prices over three-points of the maturity interval. Then, we use the three-point Richardson extrapolation scheme to calculate the American call options prices over the whole maturity interval.

The computational speed of each method is measured by the CPU time (in seconds) required for the calculation of the whole set of the American call options generated in all ( $J = 1250$ ) experiments. The accuracy of each method compared with the benchmark model is assessed by calculating, over the whole set of generated option prices, the following two measures: the root mean squared error ( $RMSE$ ), which is defined  $RMSE = \sqrt{\frac{\sum_{j=1}^J (C_{A,j}(\cdot) - BT_j)^2}{J}}$ , and the maximum of the absolute pricing errors ( $MAE$ ), which is defined as  $MAE = \max\{|C_{A,1}(\cdot) - BT_1|, |C_{A,2}(\cdot) - BT_2|, \dots, |C_{A,J}(\cdot) - BT_J|\}$ . We also calculate the above two measures for the option pricing errors as a percentage of the option prices of the benchmark model, i.e.  $100 \cdot \frac{C_{A,j}(\cdot) - BT_j}{BT_j}$ . These measures are denoted as  $RMSE\%$  and  $MAE\%$ , respectively. The numerical results of the above all measures and the CPU time can be found in Table 3.1.

As was expected, the results of the table clearly show that there is a trade off between accuracy and computational speed across all the approximation methods. In

|           | HSY-3   | EXP-3   | CB-2    | CB-3    |
|-----------|---------|---------|---------|---------|
| RMSE      | 0.0059  | 0.0029  | 0.0026  | 0.0012  |
| MAE       | 0.0679  | 0.0178  | 0.0163  | 0.0089  |
| RMSE%     | 0.0673% | 0.0251% | 0.0236% | 0.0171% |
| MAE%      | 0.474%  | 0.163%  | 0.142%  | 0.087%  |
| CPU(secs) | 3.17    | 9.75    | 9.65    | 67.71   |

Table 3..1: Numerical Result for the Lognormal Model

terms of accuracy, the CB-2 method can be compared with the EXP-3 method. The estimates of *RMSE* and *MAE* measures, as well as of their counterparts for the percentage pricing errors, indicate that both the CB-2 and EXP-3 methods approximate adequately the option prices and clearly outperform the HSY-3 method; with the CB-2 method performing slightly better than the EXP-3 method. The HSY-3 seems to be superior only in terms of computational speed, which is obviously due to its functional simplicity. But this is at the cost of larger pricing errors. Note that accuracy of the CB method increases considerably as the degrees of the polynomial approximation  $\nu$  increases, which is consistent with the predictions of the Weierstrass' theorem. Comparing the results of the table with those of Ju(1998), we can conclude that the CB-2 and EXP-3 methods perform much better than other numerical methods for pricing American call options based on the approximation of the whole American call option risk neutral relationship, or on the finite difference numerical methods.

The potential gains of CB method, compared with the two other approximation methods of the optimal exercise boundary function, for pricing American call options can be better understood with the help of Figure 3.1. This figure presents estimates of the optimal exercise boundary function by the CB-2 (line'\*\*\*'), HSY-3(line 'xxx') and EXP-3(line'+++') methods, as well as those by the benchmark method(line'...') , for the following set of parameters of the lognormal model:  $\{P_t = 100, K = 100, r = 0.03, r - \delta = -0.04, \sigma = 0.4 \text{ and } T - t = 0.5\}$ . For this set of parameters, we found that the lognormal model can generate a highly concave function of the optimal exercise

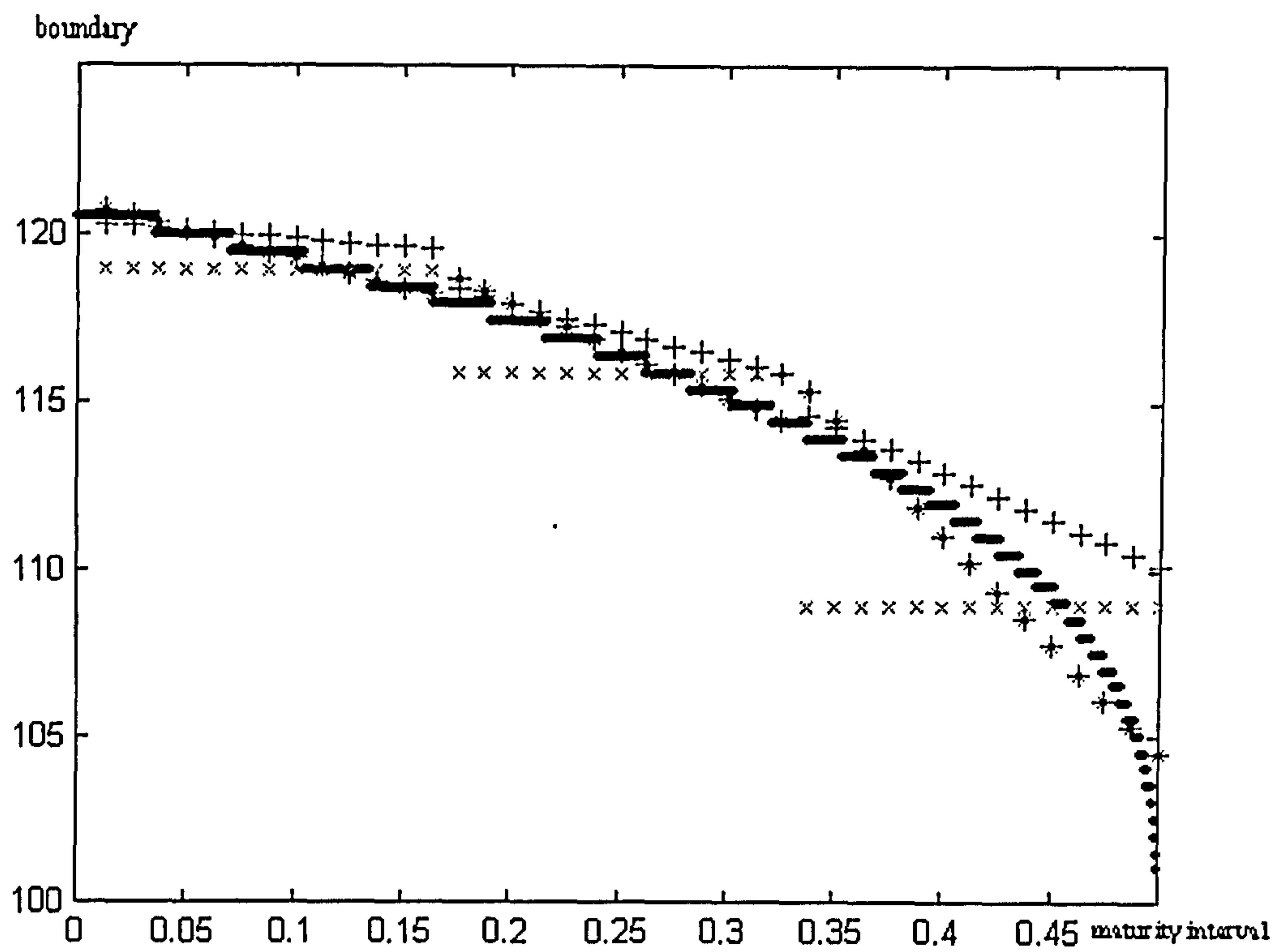


Figure 3.1: The Optimal exercise boundary for the BS model

boundary function with respect to the maturity interval. Inspection of the graphs of the figure indicate that the magnitude of the pricing errors of the CB method are clearly smaller than those of the HSY-3 and EXP-3 methods. The benefits, in terms of accuracy, of the CB method is due to the fact that it achieves a good approximation error of the true optimal exercise boundary. It does this by choosing the point with the minimum error to fit an approximating polynomial function into the true function, over the different pieces of the maturity interval, according to the minmax criterion. This will have a better pricing performance the more concave the optimal exercise boundary function is. In contrast, the HSY-3 method approximates the optimal exercise boundary function by fitting a straight line within each piece of the maturity interval, while the EXP-3 method uses a tangent line at the initial point of each piece of the interval. This will have as a consequence that the HSY method will result in higher pricing errors compared to the other two methods when the true optimal exercise boundary function is concave. The pricing errors of the EXP-3 method will depend on the degree of concavity of the optimal exercise boundary function.

Overall, the results of this section indicate that approximating the optimal exercise boundary by the CB-2 has proved to be a very fast and accurate method for pricing American call options for the lognormal model. It can be compared with other efficient approximation methods introduced in the literature, for this model.

3.4.2 Numerical results for the stochastic volatility model To assess the performance of the CB method for the SV model, we focus on the CB-2 model, with two degrees, which is found to perform very well in the case of the lognormal model. To evaluate the method, we follow steps similar to those in the previous section. We calculate the prices of  $J = 1250$  American call option prices by drawing the parameters of the SV model from the uniform distribution over the following intervals:  $[90, 110]$  for  $P_t$ ,  $[0.0005, 0.5]$  for  $V_t$ ,  $[-1.0, 1.0]$  for the correlation coefficient ( $\rho$ ),  $[0.0, 0.1]$  for  $r$  and  $\delta$ ,  $[0.1, 3.0]$  for  $k$ ,  $[0.01, 0.2]$  for  $\theta$ ,  $[0.1, 0.5]$  for  $\sigma$  and  $[0.1, 3.0]$  years for  $T - t$ . As previously, the strike price is assumed to be fixed,  $K = 100$ , in all experiments.

|      | RMSE   | MAE    | RMSE%  | MAE%   | CPU (secs)             |
|------|--------|--------|--------|--------|------------------------|
| CB-2 | 0.0035 | 0.0123 | 0.063% | 0.191% | 801.51 (or 13.35 mins) |

Table 3..2: Numerical Result For the SV Model

The accuracy and speed performance of the CB-2 method are evaluated based on the *RMSE* and *MAE* measures of the options pricing errors (as well as their *RMSE%* and *MAE%* counterparts for the pricing errors percentages), and the CPU time. To calculate the pricing errors, we use the lattice model suggested by Britten-Jones and Neuberger (2000) with  $N = 200$  steps, denoted BJ-N, as benchmark model. In Table 3.2 we report the results.

The results of the table clearly show that the CB-2 method can be successfully applied to price American call options under the SV model. The *RMSE* and *MAE* measures, as well as their *RMSE%* and *MAE%* counterparts, indicate that the magnitude of the pricing errors is very small. Note that it is almost of the same order as that for the lognormal model. In terms of computation time, the benefits of the CB-2 method are enormous. It only takes 13.35 minutes to calculate the whole set of the American call options. To make these calculations, we need about 6.0 hours by the benchmark model.

The success of the CB-2 method in pricing American call options under stochastic volatility can be attributed to fact that this method successfully approximates the optimal exercise boundary surface. This can also justify the assumption made in deriving Theorem 2 that the optimal exercise boundary surface is smooth with respect to volatility changes. To confirm this, in Figures 3.2-3.3, we present three-dimension graphs of the optimal exercise boundary surface implied by the SV model. This is done for the benchmark and CB-2 methods, respectively, based on the following set of parameters of the SV model:  $\{r = 0.03, r - \delta = 0.01, k = 1.0, \theta = 0.03, \rho = 0.00, \sigma = 0.1\}$ .<sup>9</sup> In Figure 3.4, we present a section of the estimated surfaces at the level of

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<sup>9</sup> This is a set of parameters used by Heston (1993) to calibrate the SV model.

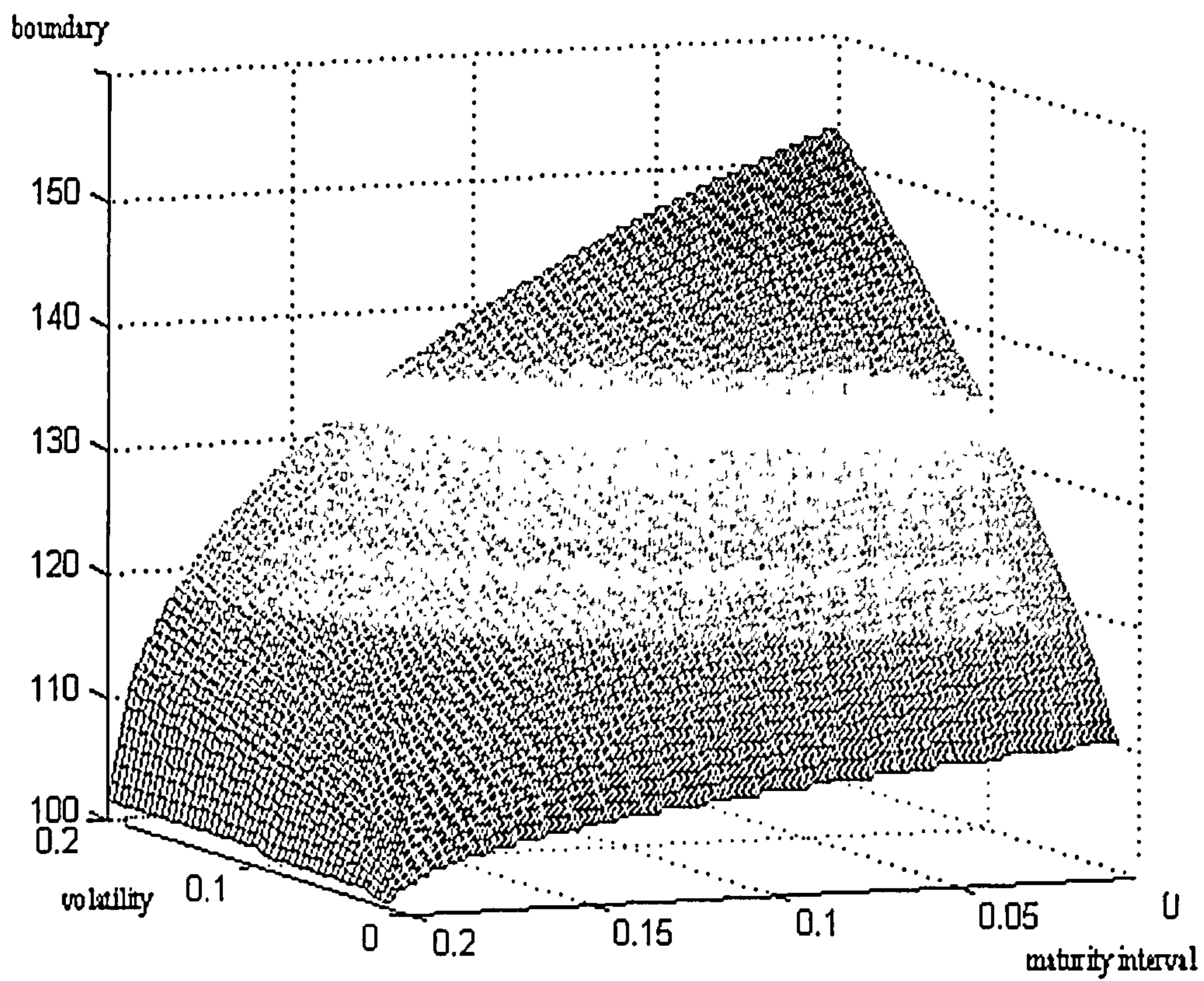


Figure 3.2: The Estimated Optimal Exercise Boudnary for the Benchmark Model

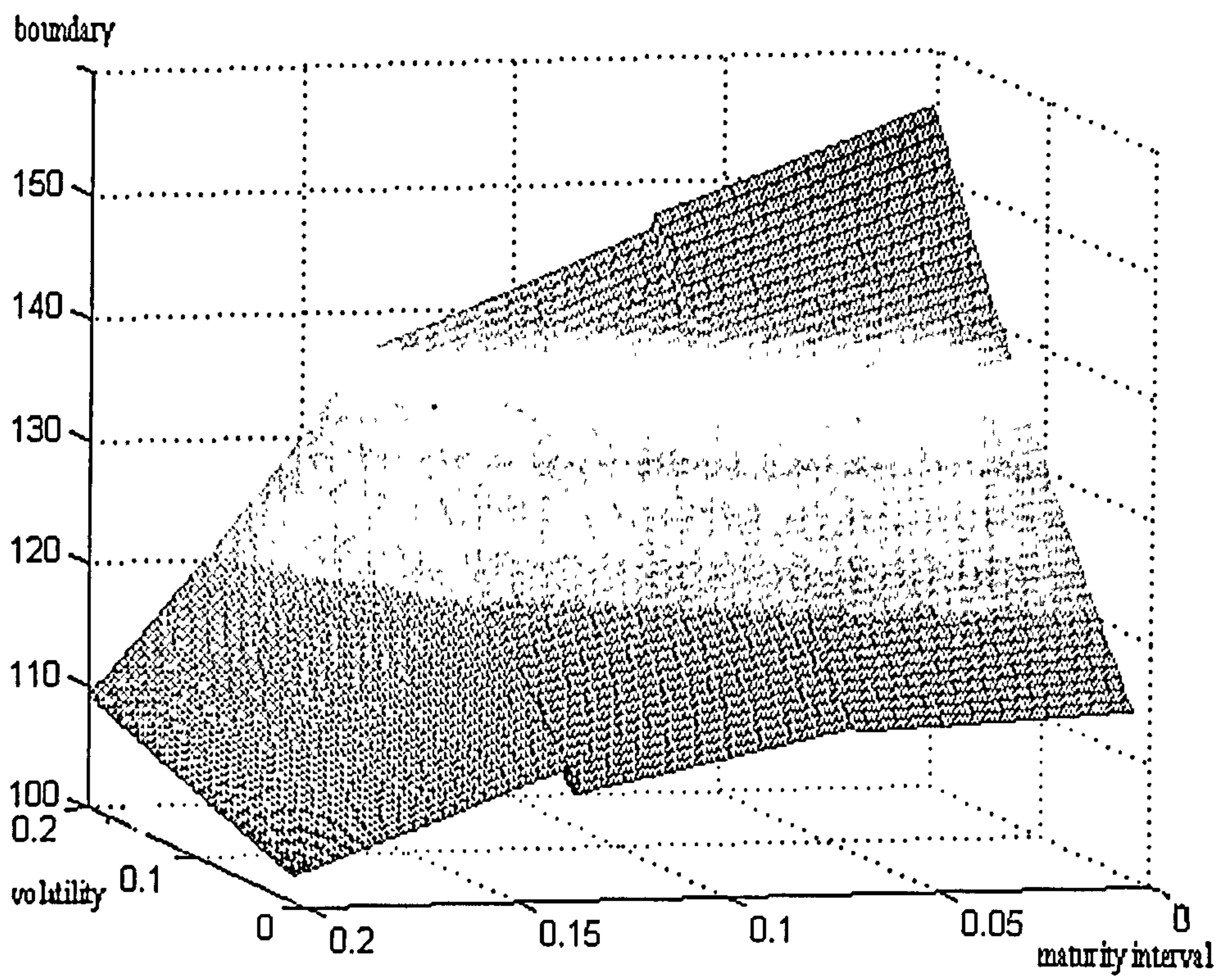


Figure 3.3: The Estimated Optimal Exercise Boundary for CB-2

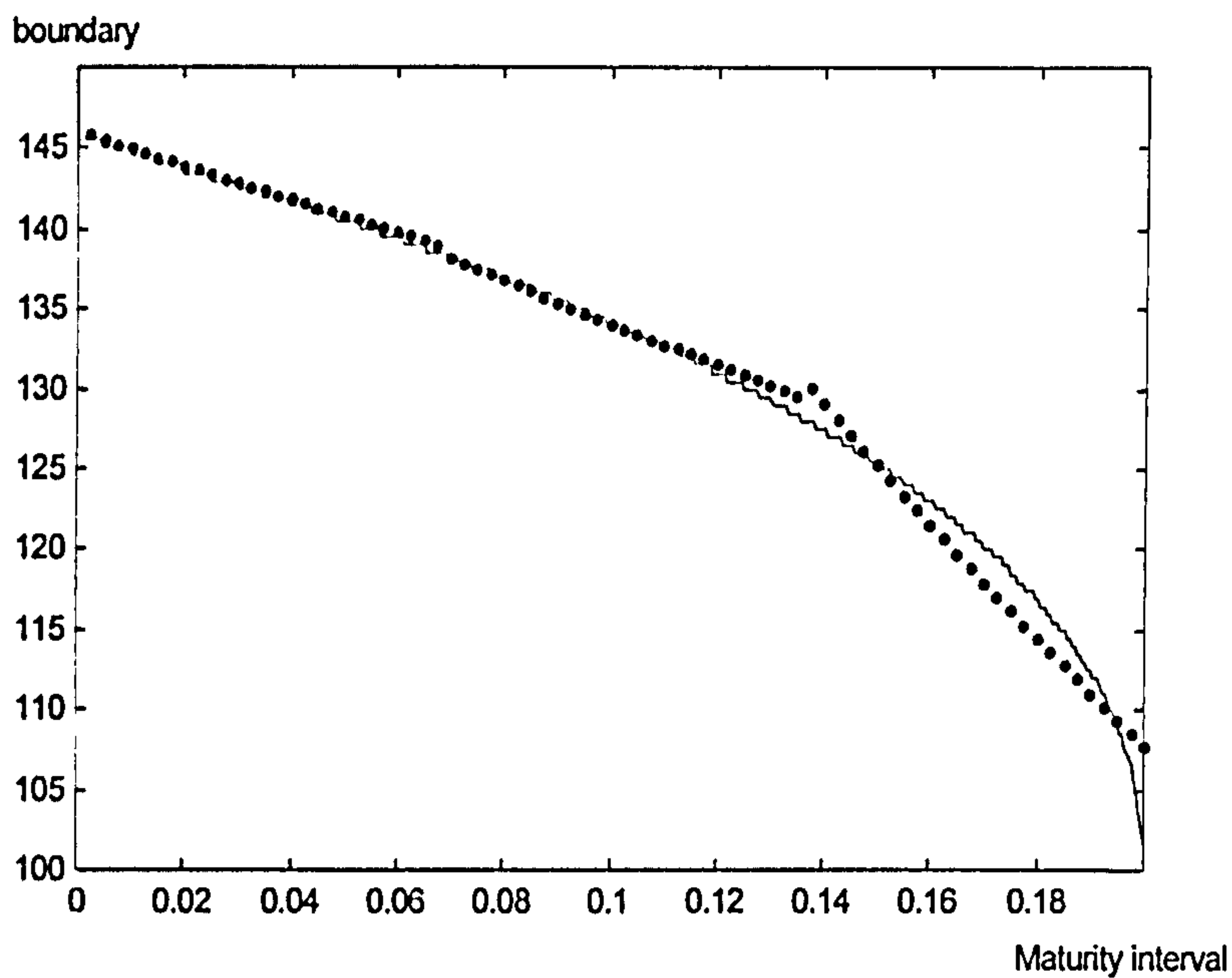


Figure 3.4: A Section of the Optimal Exercise Boundary

volatility  $V_t = 0.16$ , line '...' for the benchmark and line '...' for CB-2.<sup>10</sup> Indeed, inspection of the graphs of all the figures leads to the conclusion that a surface of the exercise boundary which is log-linear with respect to volatility changes can adequately approximate the true optimal exercise boundary. This justifies the assumption made in Theorem 3. From these graphs, it can be seen that the success of the CB-2 method in effectively pricing the options prices can be attributed to its ability to efficiently approximate the true optimal exercise boundary for the SV model. As the graphs of Figure 3.4 indicate, the approximation of the optimal exercise boundary by the CB-2 method under stochastic volatility is as closely as under constant volatility.

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<sup>10</sup> Note that these graphs are indicative. Similar graphs are taken at any other level of the volatility.



### 3.5 Conclusions

In this chapter we introduced a new numerical method of pricing an American call option under stochastic volatility. The method is based on an approximation of the optimal exercise boundary by Chebyshev polynomials. To implement the method we derived an analytic, integral representation for the American call option price under stochastic volatility employing a log-linear function of the optimal exercise boundary with respect to the volatility changes. This representation unbundles the early exercise premium (and hence the American call option price) into a portfolio of Arrow-Debreu type of securities. The prices of these securities can be calculated by the joint characteristic function of the price of the underlying stock and its conditional variance. The analytic form of this function is derived in closed form in the chapter. The chapter presented a set of numerical results which show that our method can approximate American call option prices very quickly and efficiently both under stochastic and constant volatility. The numerical results show that our method is very efficient even for cases where the curvature of the true optimal exercise boundary function is high.



## CHAPTER 4.

### PRICING AMERICAN BOND OPTIONS FOR AFFINE TERM STRUCTURE MODELS

#### 4.1 Introduction

It is well known that pricing American options is one of the most difficult tasks in academic and practitioner circles in the area of financial economics and econometrics. Among the American type of options, those on bonds (and/or interest rates) are the most sophisticated. This is because the interest rates which determine both the discount factor and the options payoffs are not assumed to be deterministic, as in most of the stock options pricing models. With the exception of a few studies [see Chesney, Elliott and Gibson (1993), Jorgensen (1994), *inter alia*], which derive closed form solutions for bond option prices for very simple cases, such as the Gaussian model of Vasicek, no closed form solutions of bond option prices are available in the literature for more complicated, multi-factor term structure of interest rates, frequently used in practice.

The other reason which makes the calculation of American bond option prices difficult is the curse of dimensionality of the term structure. This raises the computational burden of numerical methods for calculating American bond options, such as the binomial, finite difference and lattice methods, especially for multi-factor term structure models. For example, we need to calculate  $2^n$  node option prices for an  $n$ -step single factor model based on the binomial tree,  $4^n$  for a two-factor model and  $2^{mn}$  for an  $m$ -factor term rate models. Although simulation based methods [see Longstaff and Schwartz (2001), Ibanez and Zapatero (1998), Barraquand and Martineau (1995), *inter alia*] can resolve the dimensionality problem to some extent, these methods suffer from other problems. In particular, they are time-consuming and employ a perturbation scheme to calculate the hedge parameters: the delta and gamma, which can potentially increase the computational burden. For example, the perturbation scheme needed to calculate  $2n$  option prices for the estimation of the

delta parameter and  $3n(n+1)/2$  option prices for the gamma parameter. Moreover, based on a perturbation scheme may not result in complete hedging.

In this chapter, we suggest a new approach for calculating American bond put option prices for multi-factor, affine term structure of interest rates models [see Duffie, Pan and Singleton (2000), *inter alia*]. These models are widely used to price interest rates and bond prices. Our approach is based on a decomposition of the American bond put price into its European option counterpart price and the early exercise premium, for exercising the option before its expiration date. This decomposition enables to develop a very fast and accurate approximation scheme of the early exercise premium based on an approximating of the optimal exercise boundary function with a hyperplane, estimated over different regions of the space of the state variables (factors), driving the term structure of interest rates, and different pieces of the maturity interval of the option. Since our method involves only one part of approximation (the European option price can be calculated exactly), it can significantly reduce the computation effort and time, without losing in accuracy.

To implement our method, we unbundle the early exercise premium into a portfolio of Arrow-Debreu type of security prices. These prices calculate the values of early exercising the American bond option contract over all points of the maturity interval. We provide explicit solutions for these prices based on the joint characteristic function of the state variables underlying the term structure of interest rates. This function is derived in closed form in the appendix B of the chapter.

The chapter is organised as follows. Section 4.2 presents the affine term structure model, with its necessary notation, and provides the analytic formulas for pricing the American bond options. Section 4.3 suggests an algorithm for a multi-region and multi-piece approximation the optimal exercise boundary price function, and it shows how to calculate the hedging parameters. Section 4.4 presents numerical results evaluating the pricing performance of our method, compared with a benchmark method. Section 4.5 concludes the chapter. All the proofs are given in the appendix.

## 4.2 Pricing American bond options for affine term structure models

4.2.1 The affine term structure model Consider the probability space  $(\Omega, \mathcal{F}, Q)$  restricted to the time interval  $[t, T_2]$  and define  $\mathcal{F}_s = \{\mathcal{F}_s; s \in [t, T_2]\}$ , with  $\mathcal{F} = \mathcal{F}_T$ , to be the filtration generated by the relevant bond prices (or interest rates) in the economy. Then, the general affine term structure model of interest rates [see Duffie and Kan (1996) and Dai and Singleton (2000)] assumes that the risk neutral instantaneous interest rate,  $r_s$ , of the economy is given as a linear combination of  $N$  unobservable state variables, collected in the  $N$ -dimension vector  $X_s = (X_{1,s}, X_{2,s}, \dots, X_{N,s})' \in \mathcal{R}^N$  -the  $N$ -dimension Euclidean space, that is

$$r_s = \alpha_0 + \sum_{n=1}^N \alpha_n X_{n,s} = \alpha_0 + \alpha' X_s, \quad (4.1)$$

where  $\alpha_0$  is a constant scalar,  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)'$  is an  $N$ -dimension vector of constants and the dynamics of  $X_s$  are described by the following system of stochastic differential equations:

$$dX_s = \kappa (\theta - X_s) ds + \sigma \sqrt{V_s} dW_s^Q,$$

where  $\theta$  is a  $N$ -dimension vector of constants,  $\kappa$  and  $\sigma$  is non-singular  $(N \times N)$  dimension matrices,  $W_t^Q$  is a  $N$ -dimension vector of independent Brownian motion processes under the risk neutral measure, denoted  $Q$ , and  $V_t$  is an  $(N \times N)$  dimension diagonal matrix with elements

$$V_{t,ii} = \varrho_i + \beta_i' X_s,$$

where  $\varrho_i$  is a constant and  $\beta_i'$  is  $N$ -dimension vector of constants.

For the above affine model, the current time  $t$  price of the zero-coupon bond

with maturity  $T_2 - t$ , under the  $Q$  measure can be written as

$$B(X_t, T_2 - t) = E_t^Q \left( e^{-\int_t^{T_2} r_s ds} \right), \quad (4.2)$$

and can be calculated by solving the following partial differential equation (PDE):

$$\mathcal{L}B(X_t, T_2 - t) = r_t B(X_t, T_2 - t), \quad (4.3)$$

where  $\mathcal{L}$  is the differential operator, defined as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \sum_{n_1=1}^N \sum_{n_2=1}^N \left( \sum_{i=1}^N \sigma_{n_1 i} \sigma_{n_2 i} V_{t, ii} \right) \frac{\partial^2}{\partial X_{n_1, t} \partial X_{n_2, t}} \\ & + \sum_{n_1=1}^N \sum_{n_2=1}^N \kappa_{n_1 n_2} (\theta_{n_2} - X_{n_2, t}) \frac{\partial}{\partial X_{n_1, t}} + \frac{\partial}{\partial t}. \end{aligned} \quad (4.4)$$

From Duffie and Kan (1996), we know that the bond price  $B(X_t, T_2 - t)$  can be written as an exponential function of the state variables  $X_{t, n}$ ,

$$B(X_t, T_2 - t) = \exp \left( \mathcal{A}(T_2 - t) + \sum_{n=1}^N \mathcal{C}_n(T_2 - t) X_{n, t} \right), \quad (4.5)$$

where  $\mathcal{A}(T_2 - t)$  and  $\mathcal{C}_n(T_2 - t)$  are deterministic functions of the maturity interval,  $T_2 - t$ . An explicit solution for  $B(X_t, T_2 - t)$  can be derived by substituting (4.5) into (4.3) and solving the resulting following two ordinary differential equations (ODEs):

$$\begin{aligned} \frac{\partial \mathcal{A}(T_2 - t)}{\partial (T_2 - t)} = & -\alpha_0 + \sum_{n_1=1}^N \sum_{n_2=1}^N \kappa_{n_1 n_2} \theta_{n_2} \mathcal{C}_{n_1}(T_2 - t) \\ & + \frac{1}{2} \sum_{n_1=1}^N \sum_{n_2=1}^N \left( \sum_{i=1}^N \sigma_{n_1 i} \sigma_{n_2 i} \rho_i \right) \mathcal{C}_{n_1}(T_2 - t) \mathcal{C}_{n_2}(T_2 - t), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial C_n(T_2 - t)}{\partial (T_2 - t)} = & -\alpha_n - \sum_{n_1=1}^N \sum_{n_2=1}^N \kappa_{n_1 n_2} C_{n_1}(T_2 - t) \\ & + \frac{1}{2} \sum_{n_1=1}^N \sum_{n_2=1}^N \left( \sum_{i=1}^N \sigma_{n_1 i} \sigma_{n_2 i} \beta_i \right) C_{n_1}(T_2 - t) C_{n_2}(T_2 - t), \quad (4.6) \end{aligned}$$

with initial conditions  $A(0) = 0$  and  $C_n(0) = 0$ , and terminal condition  $B(X_{T_2}, 0) = 1$ .

4.2.2 Pricing American bond put options Having introduced the affine model of the term structure of interest rates with its necessary notation, we now direct our attention to the pricing of an American put option on a zero coupon bond with maturity  $T_2$ , as underlying asset. We focus our attention to this category of options because the price of an American call bond option is the same with that of a European call option.

Let  $P_A(X_t, T_1; T_2)$  denote the time  $t$  price of an American put option expiring at time  $T_1 \leq T_2$  and strike price  $K$  on a bond expiring at date  $T_2$ . At any point  $s \in [t, T_1]$ , i.e. during the life of the option, we can split the space  $\mathcal{R}^N$  spanned by the state variables into two regions: the continuation region into which the option is kept alive, defined as  $\eta_s = \{X_s \in \mathcal{R}^N : K - B(X_s, T_2 - s) < P_A(X_s, T_1; T_2)\}$ , and the early exercise region into which the option is exercised, defined as  $\bar{\eta}_s = \{X_s \in \mathcal{R}^N : K - B(X_s, T_2 - s) = P_A(X_s, T_1; T_2)\}$ . Note that  $\bar{\eta}_s$  is the complementary set of  $\eta_s$ .

Since  $X_s$  is the vector of the state variables driving bonds prices and interest rates, it is convenient to write the continuation and exercise sets in terms of the state variables. This can be done by exploiting the existence of a relationship between a state variable of the state vector  $X_s$  (say  $X_{N,s} \in \mathcal{R}^1$  -the 1 dimension Euclidean space) and the remaining state variables collected in the  $(N-1)$ -dimension vector  $Y_s = (X_{1,s}, X_{2,s}, \dots, X_{N-1,s})' \in \mathcal{R}^{N-1}$  - the  $(N-1)$ -dimension Euclidean space. Then, we can write the continuation and exercise regions as  $\eta_s = \{X_s \in \mathcal{R}^N : X_{N,s} \leq G(Y_s, s)\}$

and  $\bar{\eta}_s = \{X_s \in \mathcal{R}^N : X_{N,s} \geq G(Y_s, s)\}$ , respectively, where  $G(., .) : \mathcal{R}^{N-1} \times [t, T] \rightarrow \mathcal{R}$  is a function relating  $X_{N,s}$  to the remaining state variables, collected in  $Y_s$ .

In order to derive an analytic solution for the bond put option price, we need to introduce the concept of the optimal exercise boundary. This is a collection of the critical bond prices, denoted  $B_s^*$ ,  $\forall s \in [t, T]$ , where the holder of the option will be indifferent between exercising and continuing to hold the option contract. Let  $X_s^*$  denote the vector of the state variables corresponding to the critical bond price  $B_s^*$ , at time  $s$ . According to the affine term structure model [see equations (4.1) and (4.2)],  $B_s^*$  should be a function of the vector of state variables  $X_s^*$  and  $T_2 - s$ , i.e.  $B_s^* \equiv B(X_s^*, T_2 - s)$ . At the optimal exercise boundary price, the payoff from exercising the option, given by  $K - B_s^*$ , should be equal to the value of the American option  $P_A(X_s^*, T_1; T_2)$ , i.e.

$$K - B(X_s^*, T_2 - s) = P_A(X_s^*, T_1; T_2), \quad (4.7)$$

otherwise arbitrage opportunities will be arisen. As we did for the continuation and exercise regions, it is convenient for our later derivations to define the optimal exercise boundary set, denoted as  $\mathcal{B}_s$ , in terms of the state variables

$$\mathcal{B}_s = \{X_s^* = (X_{N,s}^*, Y_s) \in \mathcal{R}^N : X_{N,s}^* = G(Y_s, s) \quad \forall Y_s \in \mathcal{R}^{N-1}\}, \quad (4.8)$$

since both the underlying bond price and the American option price are solely determined by the vector of state variables  $X_s^*$ . Relationship (4.8) indicates that  $\mathcal{B}_s$  can be thought of as the set of the critical values of  $X_s^*$  which split the space  $\mathcal{R}^N$  into two complementary regions: the exercise region, given by set  $\bar{\eta}_s$ , and continuation region, given by set  $\eta_s$ . Relationship (4.8), which follows from the definitions of the continuation and early exercise regions as  $\eta_s = \{X_s \in \mathcal{R}^N : X_{N,s} \leq G(Y_s, s)\}$  and  $\bar{\eta}_s = \{X_s \in \mathcal{R}^N : X_{N,s} \geq G(Y_s, s)\}$ , respectively, indicates that the  $\mathcal{B}_s$  can be defined as the set of critical values of the state variable  $X_{N,s}^*$  satisfying the relationship

$$X_{N,s}^* = G(Y_s, s), \forall Y_s \in \mathcal{R}^{N-1}.$$

Since the option price  $P_A(X_t, T_1; T_2)$  constitutes a martingale in the continuation region, its solution can be derived based on Feynman-Kac's theorem by solving the following partial differential equation (PDE)

$$\mathcal{L}P_A(X_t, T_1; T_2) = r_t P_A(X_t, T_1; T_2), \quad (4.9)$$

subject to the following terminal and boundary conditions

$$\begin{aligned} \lim_{t \rightarrow T_1} P_A(X_t, T_1; T_2) &= \max(0, K - B(X_{T_1}, T_2 - T_1)), \\ \lim_{B(X_t, T_2 - t) \rightarrow \infty} P_A(X_t, T_1; T_2) &= 0, \\ \lim_{X_t \rightarrow X_t^* \in \mathcal{B}} P_A(X_t, T_1; T_2) &= K - B(X_t, T_2 - t), \\ \text{and} \quad \lim_{X_t \rightarrow X_t^* \in \mathcal{B}} \frac{\partial P_A(X_t, T_1; T_2)}{\partial B(X_t, T_2 - t)} &= -1. \end{aligned}$$

Given that it is difficult to derive an analytic solution for the above PDE, many authors [working mainly in the area of American options pricing for stocks, see Kim (1990), and Myneni (1992), *inter alia*] have recently suggested numerical approximation methods for calculating the option price  $P_A(X_t, T_1; T_2)$  based on a decomposition of  $P_A(X_t, T_1; T_2)$  into its European option price counterpart plus the early exercise premium. This method can be proved faster and more accurate than numerical methods based on a numerical solution of the PDE (4.9) [see Ju (1998), *inter alia*]. In the bond option pricing literature, such a type of decomposition of the American option price has been suggested by Jumshidian (1992) and Jorgensen (1996), *inter alia*, but this is done for the single factor Gaussian interest rate model. In the next theorem, we extend this decomposition of the American bond option price for the case of an  $N$ -dimension affine term structure model of interest rates.

**Theorem 4** *For the affine term structure model, defined by (4.1), the time  $t$  price of an American put option  $P_A(X_t, T_1; T_2)$ , with expiration date  $T_1$  and strike price*



$K$ , on a bond expiring at date  $T_2 \geq T_1$  can be decomposed as

$$P_A(X_t, T_1; T_2) = P_E(X_t, T_1; T_2) + \int_t^{T_1} E_t^Q \left[ e^{\int_t^s r_u du} r_s K \Lambda_{\{X_{N,s} \geq G(Y_s, s)\}} | X_t \right] ds, \quad (4..10)$$

where  $P_E(X_t, T_1; T_2)$  is the price of the European put option counterpart of the American option,  $\Lambda_A$  is an indicator function of a set  $A$  (here,  $A \equiv \bar{\eta}_s = \{X_s : X_{N,s} \geq G(Y_s, s)\}$ ). At any time  $h \in [t, T]$ , where  $h \leq s$ , the optimal exercise boundary,  $B(X_s^*, T_2 - s)$ , should satisfy the following recursive equation

$$\begin{aligned} & B(X_h^*, T_2 - h) - K \\ &= P_E(X_h^*, T_1; T_2) \\ &+ \int_h^{T_1} E_h^Q \left[ e^{\int_h^s r_u du} r_s K \Lambda_{\{X_{N,s} \geq G(Y_s, s)\}} | X_h^* \right] ds. \end{aligned} \quad (4..11)$$

The proof is given in Appendix B.1. An analytic formula for the European put option price  $P_E(X_t, T_1; T_2)$ , which is needed for the calculation of  $P_A(X_t, T_1; T_2)$ , is derived by Chacho and Das (1999).

Although Theorem 4 characterises the solution of the bond put option price  $P_A(X_t, T_1; T_2)$ , it does not provide an analytic formula which can be used to calculate this price. This is due to the fact that the option price  $P_A(X_t, T_1; T_2)$  and the optimal exercise boundary are related through the recursive relationship (4..11). However, the decomposition of  $P_A(X_t, T_1; T_2)$  into the European price  $P_E(X_t, T_1; T_2)$  and the early exercise premium, given by the difference  $P_A(X_t, T_1; T_2) - P_E(X_t, T_1; T_2) = \int_t^{T_1} E_t^Q \left[ e^{\int_t^s r_u du} r_s K \Lambda_{\{X_{N,s} \geq G(Y_s, s)\}} | \mathcal{F}_t \right] ds$ , enables us to develop a fast and efficient numerical approximation method for pricing the American option based on an approximation of the early optimal exercise boundary function, along the lines of the approximation methods suggested by Huang, Subrahmanyam and Yu (1996), and Ju

(1998), for American stock options. As we will see later, this method can lead to a more accurate approximation of American bond option prices because it involves only one approximation step; the calculation of the European option price can be done exactly.

To develop the exercise boundary approximation method, we first need to specify a functional form for  $G(Y_s, s)$ , determining the critical values of  $X_{N,s}^*$  and, hence, the optimal exercise boundary set  $\mathcal{B}_s$ . For the affine term structure model, a natural choice for  $G(Y_s, s)$  can be the linear relationship

$$G(Y_s, s) = \lambda_{0,s} + \sum_{n=1}^{N-1} \lambda_{n,s} X_{n,s}, \quad (4.12)$$

which is assumed that holds at the moment for the whole maturity interval, for simplicity. Given (4.12), in the next theorem we provide a two-dimension integral, analytic solution for the option price  $P_A(X_t, T_1; T_2)$ , which can be used to build up our exercise boundary approximation method.

**Theorem 5** *Assume that  $X_{N,s}^* = G(Y_s, s)$  is given by the linear function (4.12). Then, the option price  $P_A(X_t, T_1; T_2)$  can be calculated as*

$$P_A(X_t, T_1; T_2) = P_E(X_t, T_1; T_2) + K \int_t^{T_1} \Phi(X_t, s-t; V_0, V_1) \Pi(X_t, s-t) ds, \quad (4.13)$$

where the optimal exercise boundary satisfies the following recursive relationship

$$\begin{aligned} & B((Y_h, X_{N,h}^*)', T_2 - h) - K \\ &= P_E((Y_h, X_{N,h}^*)', T_1; T_2) \\ &+ K \int_h^{T_1} \Phi(X_h, s-h; V_0, V_1) \Pi((Y_h, X_{N,s}^*)', s-h) ds, \end{aligned} \quad (4.14)$$

where  $V_0$  and  $V_1$  are  $N$ -dimension vectors of zeros and unities, respectively, and  $\Phi(X_t, s - t; \phi, \varphi)$  is the extended characteristic function defined as

$$\Phi(X_t, s - t; \phi, \varphi) = E_t \left( \exp\left(\int_t^s r_u du\right) \varphi X_s e^{\phi X_s} \right)$$

The proof of the theorem is given in Appendix B.2.

Theorem 5 indicates that the early exercise premium (and hence the American put bond option price) can be unbundled into a portfolio of Arrow-Debreu type of securities. The prices of these securities, denoted by the Greek letter  $\Pi(\cdot)$ , can be calculated by evaluating the following risk neutral expectation

$$\Pi((Y_h, X_{h,s}^*)', s - h) = E_h^Q \left[ \frac{e^{\int_h^s r_u du} r_s \Lambda\{X_{N,s} \geq G(Y_s, s)\}}{E_h^Q(e^{\int_h^s r_u du} r_s)} | (Y_h, X_{N,h}^*)' \right], \quad (4.15)$$

or using the transformed measure  $Q_1$ , where  $\frac{dQ_1}{dQ} = \frac{e^{\int_h^s r_u du} r_s}{E_h^Q(e^{\int_h^s r_u du} r_s)}$ , the expectation

$$\Pi((Y_h, X_{N,h}^*)', s - h) = E_h^{Q_1} \left[ \Lambda\{X_{N,s} \geq G(Y_s, s)\} | (Y_h, X_{N,h}^*)' \right]. \quad (4.16)$$

Closed form solutions of prices  $\Pi((Y_h, X_{N,h}^*)', s - h)$  are given in Appendix B.2. These are derived based on the extended transform of the joint conditional characteristic function (CCF) of the state variables  $X_{n,s}$ , given  $X_{n,h}$ . These prices have an interesting economic interpretation. As equation (4.16) indicates, they constitute the market prices of a security under the measures  $Q_1$ , respectively, which pay \$1 in state  $\{X_s : X_{N,s} \geq G(Y_s, s)\}$  and 0 otherwise. In the asset pricing context, they are equal to the probabilities of the state  $\{X_s : X_{N,s} \geq G(Y_s, s)\}$  under measure  $Q_1$ .

#### 4.2.3 Multi-region and multi-piece approximation of the optimal exercise boundary

The analytic solution for the American put bond option price presented in the previous section is based on the assumption that the optimal exercise boundary function

can be determined through the linear relationship (4.12). However, if this is not true (as it may happen in reality), then approximating the optimal exercise boundary relationship  $X_{N,s}^* = G(Y_s, s)$  by the linear function (4.12) may lead to significant pricing errors. To reduce this type of errors, in this section we suggest a method to approximate the function of  $X_{N,s}^* = G(Y_s, s)$ , over  $M^{N-1}$  different regions (segments) of the state space of  $Y_s \in \mathcal{R}^{N-1}$  and  $L$  different pieces of the maturity interval,  $[t, T]$ , based on the linear relationship (4.12). As recently pointed out by Singleton and Umantsev (2002), who suggested a similar method for approximating swaptions prices, this approach can lead to a very accurate approximation of the exercise boundary function based on a few segments of  $\mathcal{R}^{N-1}$ , i.e.  $M = 2$ . This happens because the values of the state variables,  $X_{n,s}$ , of the affine term structure model tend to be concentrated on a small subset of  $\mathcal{R}^{N-1}$  with non-zero probability of occurrence.

To present our approach more analytically, first define the set  $\Theta \subseteq \mathcal{R}^{N-1}$ , which contains the values of  $Y_s$  with non-zero probability of occurrence.  $\Theta$  can be written as the Cartesian product  $\Theta = \chi_1 \times \chi_2 \times \dots \times \chi_{N-1}$  of  $(N-1)$ -subsets of the state variables, where  $\chi_n = [X_n^{\min}, X_n^{\max}]$   $n = 1, 2, \dots, N-1$ , where  $X_n^{\min}$  and  $X_n^{\max}$  denote the minimum and maximum values of  $X_{n,s}$  with non-zero probability of occurrence, respectively. If we divide each subset  $\chi_n$  into  $M$  equal intervals  $[\chi_n^0 = X_n^{\min}, \chi_n^1], [\chi_n^1, \chi_n^2], \dots, [\chi_n^{M-1}, \chi_n^M = X_n^{\max}]$ , then  $\Theta$  can be divided into  $M^{N-1}$  equal finer subsets (regions)  $\Xi_{j_1, j_2, \dots, j_{N-1}} = [X_1^{j_1}, X_1^{j_1+1}] \times [X_2^{j_2}, X_2^{j_2+1}] \times \dots \times [X_{N-1}^{j_{N-1}}, X_{N-1}^{j_{N-1}+1}]$ , where  $(j_1 = 0, 1, 2, \dots, M, j_2 = 0, 1, 2, \dots, M, \dots, j_{N-1} = 0, 1, 2, \dots, M)$ , and thus can be written as the sum of subsets  $\Xi_{j_1, j_2, \dots, j_{N-1}}$ , i.e.  $\Theta = \sum_{j_1=1}^M \sum_{j_2=1}^M \dots \sum_{j_{N-1}=1}^M \Xi_{j_1, j_2, \dots, j_{N-1}}$ .

Suppose that the optimal exercise boundary function  $X_{N,s}^* = G(Y_s, s)$  is determined in each region  $\Xi_{j_1, j_2, \dots, j_{N-1}}$  by the hyperplane

$$G_{j_1, j_2, \dots, j_{N-1}}(Y_s, s) = \lambda_0^{\Xi_{j_1, j_2, \dots, j_{N-1}}} + \sum_{n=1}^{N-1} \lambda_{n,s}^{\Xi_{j_1, j_2, \dots, j_{N-1}}} X_{n,s}, \quad (4.17)$$

then the functional form of the optimal boundary relationship  $X_{N,s}^* = G(Y_s, s)$  can

be approximated over the whole space  $\Theta$  by the sum of functions  $G_{j_1, j_2, \dots, j_{N-1}}(Y_s, s)$ , taken over different intervals  $M$  of the space spanned by  $Y_s$ , i.e.

$$G^M(Y_s, s) = \sum_{j_1=1}^M \sum_{j_2=1}^M \dots \sum_{j_{N-1}=1}^M G_{j_1, j_2, \dots, j_{N-1}}(Y_s, s) \Lambda_{\Xi_{j_1, j_2, \dots, j_{N-1}}}, \quad (4..18)$$

where  $\Lambda_{\Xi_{j_1, j_2, \dots, j_{N-1}}}$  denotes an indicator function for the set  $\Xi_{j_1, j_2, \dots, j_{N-1}}$ .

As  $M$  approaches to infinite, we can easily show that the approximating function  $G^M(Y_s, s)$  converges to the true function  $G(X_s, s)$ , i.e.

$$\lim_{M \rightarrow \infty} G^M(Y_s, s) = G(Y_s, s).$$

This last result guarantees that if we divide the space spanned by the vector  $Y_s$  adequately small enough, then the approximating function  $G^M(Y_s, s)$  will constitute an adequate approximation of the true function  $G(X_s, s)$ . The above result follows immediately by noticing that, for continuous and differentiable optimal exercise functions, we have that  $|G^M(Y_s, s) - G(Y_s, s)| \sim o(\Delta\chi)$ , where  $\Delta\chi = \prod_{n=1}^{N-1} \frac{X_n^{\max} - X_n^{\min}}{M}$ , thus implying  $\lim_{M \rightarrow \infty} G^M(Y_s, s) = G(X_s, s)$ .

In the next theorem, we extend the results of Theorem 5 to the case that the optimal exercise boundary function  $X_{N,s}^* = G(Y_s, s)$  is approximated by the multi-region (multi-segment) function  $G^M(Y_s, s)$ .

**Theorem 6** *Let us consider that the optimal early exercise boundary function is approximated by the multi-region function  $G^M(Y_s, s)$ , given by equation (4..18). Then, the American put option price  $P_A(X_t, T_1; T_2)$  can be approximated by*

$$P_A^M(X_t, T_1; T_2) = P_E(X_t, T_1; T_2) + K \int_t^{T_1} \Phi(X_t, s - t; V_0, V_1) \Pi^M(X_t, s - t) ds, \quad (4..19)$$

where the early exercise boundary satisfies the following recursive equation

$$\begin{aligned}
 & B^M(X_h^*, T_2 - h) - K \\
 = & P_E(X_h^*, T_1; T_2) + \\
 & K \int_h^{T_1} \Phi(X_h^*, s - h; V_0, V_1) \Pi^M((Y_h, X_{N,h}^*)', s - h) ds.
 \end{aligned} \tag{4..20}$$

As expected the results of the theorem are analogous to those of Theorem 2. The proof of the theorem and the analytic formulas of the state prices  $\Pi^M(\cdot)$ , involved in the integrals of equations (4..19) and (4..20), are given in Appendix B.3.

The above results show how to approximate the American option price  $P_A(X_t, T_1; T_2)$  based on a multi-region approximation of the optimal exercise boundary relationship  $X_{N,s}^* = G(Y_s, s)$ , over the whole maturity interval of the option  $T - t$ . To improve the efficiency performance of this approximation method, we may also consider its implementation over different pieces of the maturity interval. To this end, next present the steps of an algorithm which show how this method can be applied both over different regions of the space of the state variables and different pieces of the maturity interval.

1. First, we divide the maturity interval  $T_1 - t$  into  $L$  equal pieces  $[s_0 \equiv t, s_1]$ ,  $(s_1, s_2], \dots, (s_{L-1}, s_L \equiv T_1]$ , with  $s_l = t + l \frac{T_1 - t}{L}$ , and assume that the optimal exercise boundary function is flat for each piece  $(s_{l-1}, s_l]$ , i.e.  $G^M(Y_s, s) = G^M(Y_{s_l}, s_l)$ ,  $\forall s \in (s_{l-1}, s_l]$ .

2. Due to the recursive nature of the optimal exercise boundary and option price formulas, in order to approximate the hyperplane  $G^M(Y_s, s)$ ,  $\forall s \in (s_{l-1}, s_l]$ , we proceed backwards starting with the last piece of the interval,  $(s_{L-1}, s_L \equiv T]$ , and then moving backwards to the first piece  $[s_0 \equiv t, s_1]$ .

In order to calculate  $G^M(Y_s, s)$  for the piece  $(s_{L-1}, s_L \equiv T]$ , note that at the expiration date of the option,  $T_1$ , the optimal exercise boundary is given by  $B(X_{N,T_1}^*, T_2 - T_1) = K$ ,  $\forall Y_{T_1} \in \Xi_{j_1, j_2, \dots, j_{N-1}}$ . According to equation (4..5), this price

can be calculated as

$$B(X_{N,T_1}^*, T_2 - T_1) \\ = \exp \left( \mathcal{A}(T_2 - T_1) + \sum_{n=1}^{N-1} C_n (T_2 - T_1) X_{n,T_1} + C_N (T_2 - T_1) X_{N,T_1} \right).$$

The last relationship implies that the optimal value of the state variable  $X_{N,T_1}$  can be calculated as

$$X_{N,T_1}^* \\ = G^M(Y_{T_1}, T_1) = \frac{1}{C_N (T_2 - T_1)} \left( \log K - \mathcal{A}(T_2 - T_1) - \sum_{n=1}^{N-1} C_n (T_2 - T_1) X_{n,T_1} \right).$$

3. Having calculated  $X_{N,T_1}^*$ , in the next step we calculate  $X_{N,s_{L-1}}^* = G^M(Y_{s_{L-1}}, s_{L-1})$ , with  $s_{L-1} \notin (s_{L-1}, s_L]$ . This can be done as follows. In each region  $\Xi_{j_1, j_2, \dots, j_{N-1}} = [\chi_1^{j_1}, \chi_1^{j_1+1}] \times [\chi_2^{j_2}, \chi_2^{j_2+1}] \times \dots \times [\chi_{N-1}^{j_{N-1}}, \chi_{N-1}^{j_{N-1}+1}]$ , we collect the  $X_n^{\min}$  or  $X_n^{\max}$  values of the state variables  $X_{n,s}$ ,  $n = 1, 2, \dots, N-1$ , in a  $N-1$ -dimension vector. In total, we can yield  $2^{N-1}$  such type of vectors, denoted as  $\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{2^{N-1}}$ . Substituting the values of these vectors into the recursive boundary equation (4.20) for the date  $T_1$ , we can calculate their corresponding values of  $X_{N,s_{L-1}}^*$ , denoted as  $\tilde{X}_1^*, \tilde{X}_2^*, \dots, \tilde{X}_{2^{N-1}}^*$ , respectively, by solving out the recursive equation (4.20) for  $X_{N,s_{L-1}}^*$  using the Newton-Raphson nonlinear numerical method. The parameters of the hyperplane  $G_{j_1, j_2, \dots, j_{N-1}}(Y_{s_{L-1}}, s_{L-1}) = \lambda_0^{\Xi_{j_1, j_2, \dots, j_{N-1}}} + \sum_{n=1}^{N-1} \lambda_{n,s}^{\Xi_{j_1, j_2, \dots, j_{N-1}}} X_{n,s}$  can be derived by regressing  $\tilde{X}_1^*, \tilde{X}_2^*, \dots, \tilde{X}_{2^{N-1}}^*$  on  $\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{2^{N-1}}$  with the method of least squares.

4. We repeat step 3 backwards to calculate the hyperplanes  $G_{j_1, j_2, \dots, j_{N-1}}(Y_{s_l}, s_l)$ , over all the remaining pieces of the maturity interval, i.e.  $[s_0 \equiv t, s_1], (s_1, s_2], \dots, (s_{L-3}, s_{L-2}]$ . These are then used to calculate the optimal exercise boundary approximating function  $G^M(Y_{s_1}, s_1)$ , over the  $M^{N-1}$ -different regions of set  $\Theta$ .

5. Substituting  $G^M(Y_{s_1}, s_1)$ , obtained from step 4, into the equation (4.19),

we can calculate the American option price  $P_A(X_t, T_1; T_2)$ .

4.2.4 Calculation of the hedging parameters: the deltas and gammas The analytic solution of the American put bond option price given by Theorem 2 enables us to derive analytic formulas of the hedging parameters: the deltas ( $\delta_n$ ) and gammas ( $\gamma_{nm}$ ), for hedging against the changes of each of the state variables  $X_{n,t}$ . These formulas are given by

$$\begin{aligned}\delta_n &= \frac{\partial}{\partial X_{n,t}} P_A(X_t, T_1; T_2) \\ &= \frac{\partial}{\partial X_{n,t}} P_E(X_t, T_1; T_2) + \int_t^{T_1} \Pi_{X_{n,t}}^M(X_t, s-t) ds\end{aligned}$$

and

$$\begin{aligned}\gamma_{nm} &= \frac{\partial^2}{\partial X_{n,t} \partial X_{m,t}} P_E(X_t, T_1; T_2) \\ &= \frac{\partial}{\partial X_{n,t} \partial X_{m,t}} P_E(X_t, T_1; T_2) + \int_t^{T_1} \Pi_{X_{n,t} X_{m,t}}^M(X_t, s-t) ds\end{aligned}$$

[see Appendix B.4 for proofs]. In contrast to the simulation and/or other numerical methods which are very time consuming and computationally demanding, the above formulas of the hedging parameters will enable us to compute the parameters  $\delta_n$  and  $\gamma_{nm}$  very easily and fast.

### 4.3 Numerical results

In this section we report numerical results to evaluate the performance of the approximation method of the exercise boundary developed in the previous section, denoted as LBA, for pricing American put bond options. This is done for three frequently used, in practice, term structure models of interests: a two and three factors extensions of Vasicek's model [see De Jong and Santa-Clara (1999)], a two-factor



extension of the CIR model [see Chen and Scott (1992)] and for the three-factor affine model suggested by Dai and Singleton (2000), referred to with the acronym ATSM. For the above three models, we compare the pricing performance of our method with a benchmark model using measures of pricing errors and computation time. Due to the curse of dimensionality and the convergence problems arisen by the multi-factor term structure models, we used the least-squares Monte Carlo (LS-MC) method of Longstaff and Schwartz (2001), instead of a binomial tree method, as a benchmark method. As has shown by Longstaff and Schwartz (2001), the LS-MC method can very accurately approximate American bond option prices, especially for multifactor term structure models with bearable time-consuming.

To implement our method, we approximate the boundary function over  $L = 8$  pieces of the maturity interval. For each piece, we limit the space of the state variables  $X_{n,s}$  values in the region  $[E_t(X_{j,s}) - 5Var_t(X_{j,s}), E_t(X_{j,s}) + 5Var_t(X_{j,s})]$ , as the probability of a value of  $X_{n,s}$  to lie outside this region is very small (almost zero) for Gaussian models, and we provide two sets of numerical results: The first set assumes that  $M = 1$ , i.e the exercise boundary approximating function takes values in the whole region of  $\Theta$  (thus implying  $\Xi_{1,1} = \lambda_0^{\Xi_{1,1}} + \sum_{l=1}^{N-1} \lambda_{l,s}^{\Xi_{1,1}} X_{l,s}$ ). The second set assumes that  $M = 2$ , i.e. the exercise boundary function is linearly approximated over each of the  $M^{N-1} = 2^{N-1}$  different subsets of  $\Theta$  (thus implying  $\Xi_{j_1,j_2} = \lambda_0^{\Xi_{j_1,j_2}} + \sum_{l=1}^{N-1} \lambda_{l,s}^{\Xi_{j_1,j_2}} X_{l,s}$   $j_1 = 1, 2$  and  $j_2 = 1, 2$ ). The CFs needed to calculate the Arrow-Debreu type of prices of the state variables can be found in Appendix B.5.

To implement the LS-MC method, we assume that there are 50 exercise points per year period of the maturity interval and we carry out 100,000 (50,000 plus 50,000 antithetic) simulation paths to calculate the put bond option prices [see Longstaff and Schwartz (2001)]. For each path, we identify the optimal exercise time by comparing the immediate exercise value of the bond option with that of a non-exercisable in-the-money bond option, for all possible exercising points  $s$  of the maturity interval.

The latter is calculated by approximating the time  $s$  conditional expectation of the continuation value of the in-the-money option time  $s + 1$  cash flows by a countable set of linear basis functions, measured at point  $s$ . For the two-factor models, we use the following set of variables  $\{1, X_{1,t}, X_{1,t}^2, X_{2,t}, X_{2,t}^2, X_{1,t}X_{2,t}\}$  as the basis functions to estimate the function of the time  $s$  conditional expectation, while for the three-factor models we use  $\{1, X_{1,t}, X_{1,t}^2, X_{2,t}, X_{2,t}^2, X_{1,t}X_{2,t}, X_{3,t}, X_{3,t}^2, X_{3,t}X_{1,t}, X_{3,t}X_{2,t}\}$ .<sup>1</sup> Once the exercise time is determined for each simulation path, we value the bond option price by appropriately discounting the resulting payoff from exercising the option contract. According to the LS-MC approach, the American put bond option price is then calculated by taking the average of the put bond option prices over all simulation paths.

For each of the term structure models considered, we give 10 examples of American put bond option prices given known parameters and we also generated  $J = 1000$  American put bond option prices for randomly selecting parameters of the models from some reasonable range. The computational speed of each approximation method is measured by the CPU time (in seconds) required for the calculation of the whole set of the American put option prices generated, for all ( $J = 1000$ ) experiments. The accuracy of our method is assessed by calculating, over the whole set of generated option prices, the root mean of squared relative error  $RMSRE = \sqrt{\frac{1}{J} \sum_{j=1}^J \left( \frac{P_{A,j}^{LBA}(\cdot) - P_{A,j}^{LS-MC}(\cdot)}{P_{A,j}^{LS-MC}(\cdot)} \right)^2}$ , which is defined relatively to the LSM American option price.

Tables 4.1-4.4 report the numerical results. PA, PE, Early and Diff respectively denote the American option price, the European option price, the early exercise premium and the difference between the valuation of American option by LBA and that by LS-MC. Tables 4.1 and 4.2 report the results for the two and three factors

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<sup>1</sup> Note that these sets of variables include terms in  $X_{n,s}$ , their powers and their cross-products. The number of these terms will grow exponentially with the dimensionality of the affine model of the term structure.

extensions of Vasicek's model, Table 4.3 reports the results for the two-factor CIR's and, finally, Table 4.4 reports the results for the three-factor ATSM's models. The maturity intervals that we consider for the American bond option price and the zero-coupon bonds are respectively:  $T_1 = 4$  months and  $T_2 = 5$  years, for the Vasicek model, and  $T_1 = 4$  months and  $T_2 = 7$  years, for the other two term structure models. To generate the example American option prices, we are based on the following sets of parameters:  $\{K = 0.8, \kappa_1 = 0.0337, \kappa_2 = 0.4861, \theta_1 = 0.0415, \theta_2 = 0.0230, \sigma_1 = 0.0103, \sigma_2 = 0.0127\}$  for the two-factor Vasicek model [see De Long and Santa-Clara (1999)],  $\{K = 0.65, \kappa_1 = 0.0525, \kappa_2 = 0.0705, \kappa_3 = 0.6553, \theta_1 = 0.0415, \theta_2 = 0.0230, \theta_3 = 0.1012, \sigma_1 = 0.0163, \sigma_2 = 0.0189, \sigma_3 = 0.0214\}$  for the three-factor Vasicek model [see De Long and Santa-Clara (1999)],  $\{K = 0.7, \kappa_1 = 0.1574, \kappa_2 = 0.8103, \theta_1 = 0.0188, \theta_2 = 0.0324, \sigma_1 = 0.0775, \sigma_2 = 0.0798\}$  for the two-factor CIR model [see De Long and Santa-Clara (1999)],  $\{K = 0.68, \kappa_r = 2.19, \kappa_m = 0.0757, \kappa_v = 1.24, \theta_m = 0.0416, \theta_v = 0.000206, \sigma_m^2 = 0.00253, \text{ and } \sigma_v^2 = 0.000393\}$  for the ATSM model [see Dai and Singleton (2000)]. For 1000 random experiments, we randomly generate the parameters of the state variables underlying the term structure from the uniform distribution over the interval:  $\{X_1 \in [0.01, 0.2], X_2 \in [0.01, 0.2], \kappa_1 \in [0.01, 0.99], \kappa_2 \in [0.01, 0.99], \theta_1 \in [0.01, 0.2], \theta_2 \in [0.01, 0.2], \sigma_1 \in [0.01, 0.2], \sigma_2 \in [0.01, 0.2]\}$  for two factor Vasicek model,  $\{X_1 \in [0.01, 0.2], X_2 \in [0.01, 0.2], X_3 \in [0.01, 0.2], \kappa_1 \in [0.01, 0.99], \kappa_2 \in [0.01, 0.99], \kappa_3 \in [0.01, 0.99], \theta_1 \in [0.01, 0.2], \theta_2 \in [0.01, 0.2], \theta_3 \in [0.01, 0.2], \sigma_1 \in [0.01, 0.2], \sigma_2 \in [0.01, 0.2], \sigma_3 \in [0.01, 0.2]\}$  for three factor Vasicek model,  $\{X_1 \in [0.01, 0.2], X_2 \in [0.01, 0.2], \kappa_1 \in [0.01, 0.99], \kappa_2 \in [0.01, 0.99], \theta_1 \in [0.01, 0.2], \theta_2 \in [0.01, 0.2], \sigma_1 \in [0.01, 0.2], \sigma_2 \in [0.01, 0.2]\}$  for two factor CIR,  $\{r \in [0.01, 0.2], m \in [0.01, 0.2], v \in [0.01, 0.2], \kappa_r \in [0.01, 5], \kappa_m \in [0.01, 2], \kappa_v \in [0.01, 2], \theta_m \in [0.01, 0.2], \theta_v \in [0.01, 0.2], \sigma_m \in [0.01, 0.2], \sigma_v \in [0.01, 0.2]\}$  for the ATSM model. Note that, in addition to the *RMSRE* and CPU pricing performance measures, the table reports results on option prices for specific values of the state variables.

| $(X_1, X_2)$  | PE       | PA       | Early    | PA       | Early    | Diff      | PA       | Early    | Diff      |
|---------------|----------|----------|----------|----------|----------|-----------|----------|----------|-----------|
|               |          | LS-MC    | LS-MC    | LBA(M=1) | LBA(M=1) | $10^{-6}$ | LBA(M=2) | LBA(M=2) | $10^{-6}$ |
| (0.022,0.022) | 0.005465 | 0.007120 | 0.001655 | 0.007227 | 0.001762 | 107       | 0.007167 | 0.001702 | 47        |
| (0.022,0.024) | 0.006546 | 0.008724 | 0.002178 | 0.008805 | 0.002259 | 81        | 0.008763 | 0.002217 | 39        |
| (0.022,0.026) | 0.007778 | 0.010615 | 0.002837 | 0.010680 | 0.002902 | 65        | 0.010647 | 0.002869 | 32        |
| (0.022,0.028) | 0.009163 | 0.012921 | 0.003748 | 0.012970 | 0.003797 | 49        | 0.012948 | 0.003775 | 27        |
| (0.022,0.030) | 0.010696 | 0.015869 | 0.005173 | 0.015869 | 0.005173 | 0         | 0.015869 | 0.005173 | 0         |
| (0.018,0.032) | 0.005157 | 0.006981 | 0.001824 | 0.007078 | 0.001921 | 97        | 0.007029 | 0.001872 | 48        |
| (0.018,0.034) | 0.006190 | 0.008676 | 0.002486 | 0.008762 | 0.002572 | 86        | 0.008713 | 0.002523 | 37        |
| (0.018,0.036) | 0.007375 | 0.010736 | 0.003361 | 0.010807 | 0.003432 | 71        | 0.010759 | 0.003384 | 23        |
| (0.018,0.038) | 0.008711 | 0.013201 | 0.004490 | 0.013201 | 0.004490 | 0         | 0.013201 | 0.004490 | 0         |
| (0.018,0.040) | 0.010198 | 0.016148 | 0.005950 | 0.016148 | 0.005950 | 0         | 0.016148 | 0.005950 | 0         |
| RMSRE         |          |          |          | 1.59%    | 6.24%    |           | 1.28%    | 3.71%    |           |
| CPU(Sec)      |          | 4298.21  |          | 22.89    |          |           | 46.96    |          |           |

Table 4.1: Two Factor Vasicek Model

| $(X_1, X_2, X_3)$    | PE       | PA       | Early    | PA       | Early    | Diff      | PA        | Early    | Diff      |
|----------------------|----------|----------|----------|----------|----------|-----------|-----------|----------|-----------|
|                      |          | LS-MC    | LS-MC    | LBA(M=1) | LBA(M=1) | $10^{-6}$ | LBA(M=2)  | LBA(M=2) | $10^{-6}$ |
| (0.007,0.006,0.028)  | 0.031112 | 0.034918 | 0.003806 | 0.034918 | 0.003806 | 0         | 0.034918  | 0.003806 | 0         |
| (0.007,0.006,0.0255) | 0.029645 | 0.033006 | 0.003361 | 0.033085 | 0.003440 | 79        | 0.0033062 | 0.003417 | 56        |
| (0.007,0.006,0.023)  | 0.028206 | 0.031235 | 0.003029 | 0.031345 | 0.003139 | 110       | 0.031310  | 0.003104 | 75        |
| (0.007,0.006,0.0205) | 0.026797 | 0.029067 | 0.002270 | 0.029208 | 0.002411 | 141       | 0.029156  | 0.002359 | 89        |
| (0.007,0.006,0.018)  | 0.025418 | 0.027553 | 0.002135 | 0.027732 | 0.002314 | 179       | 0.027665  | 0.002247 | 112       |
| (0.006,0.007,0.028)  | 0.031023 | 0.034710 | 0.003687 | 0.034710 | 0.003687 | 0         | 0.034710  | 0.003687 | 0         |
| (0.006,0.007,0.0255) | 0.029558 | 0.032972 | 0.003414 | 0.033047 | 0.003489 | 75        | 0.033030  | 0.003472 | 58        |
| (0.006,0.007,0.023)  | 0.028121 | 0.030676 | 0.002555 | 0.030779 | 0.002658 | 103       | 0.030747  | 0.002626 | 71        |
| (0.006,0.007,0.0205) | 0.026713 | 0.029043 | 0.002330 | 0.029182 | 0.002469 | 139       | 0.029130  | 0.002417 | 87        |
| (0.006,0.007,0.018)  | 0.025337 | 0.027535 | 0.002198 | 0.027707 | 0.002370 | 172       | 0.027644  | 0.002307 | 109       |
| RMSRE                |          |          |          | 2.39%    | 7.37%    |           | 1.78%     | 4.51%    |           |
| CPU(sec)             |          | 8598.33  |          | 38.89    |          |           | 178.78    |          |           |

Table 4.2: Three Factor Vasicek Model

| $(X_1, X_2)$   | PE       | PA       | Early    | PA       | Early    | Diff      | PA       | Early    | Diff      |
|----------------|----------|----------|----------|----------|----------|-----------|----------|----------|-----------|
|                |          | LS-MC    | LS-MC    | LBA(M=1) | LBA(M=1) | $10^{-6}$ | LBA(M=2) | LBA(M=2) | $10^{-6}$ |
| (0.03,0.013)   | 0.010904 | 0.014153 | 0.003249 | 0.014153 | 0.003249 | 0         | 0.014153 | 0.003249 | 0         |
| (0.03,0.0115)  | 0.010249 | 0.013170 | 0.002921 | 0.013199 | 0.002950 | 29        | 0.013195 | 0.002946 | 25        |
| (0.03,0.01)    | 0.009721 | 0.012130 | 0.002409 | 0.012163 | 0.002442 | 33        | 0.012157 | 0.002436 | 27        |
| (0.03,0.0085)  | 0.009031 | 0.011067 | 0.002034 | 0.011108 | 0.002075 | 41        | 0.011098 | 0.002065 | 31        |
| (0.03,0.007)   | 0.008499 | 0.010110 | 0.001611 | 0.010149 | 0.001650 | 39        | 0.010143 | 0.001644 | 33        |
| (0.034,0.013)  | 0.018926 | 0.025378 | 0.006452 | 0.025378 | 0.006452 | 0         | 0.025378 | 0.006452 | 0         |
| (0.034,0.0115) | 0.018155 | 0.024138 | 0.005983 | 0.024138 | 0.005983 | 0         | 0.024138 | 0.005983 | 0         |
| (0.034,0.01)   | 0.017019 | 0.022896 | 0.005877 | 0.022934 | 0.005915 | 38        | 0.022923 | 0.005904 | 27        |
| (0.034,0.0085) | 0.016334 | 0.021651 | 0.005317 | 0.021704 | 0.005370 | 53        | 0.021685 | 0.005351 | 34        |
| (0.034,0.007)  | 0.015960 | 0.020404 | 0.004444 | 0.020451 | 0.004491 | 47        | 0.020441 | 0.004481 | 37        |
| RMSRE          |          |          |          | 1.21%    | 4.15%    |           | 0.98%    | 2.71%    |           |
| CPU(sec)       |          | 7298.23  |          | 18.89    |          |           | 38.96    |          |           |

Table 4.3: Two Factor CIR Model

The results of the tables lead to the following general conclusions. First, the pricing performance of the LBA method is very satisfactory, across all the term structure models and the maturity intervals. The pricing errors of the LBA approach constitute only a very small percentage of the LS-MC prices. Note that, for  $M = 1$ , the biggest differences in prices between the LBA and LS-MC approaches are found to be: 0.000107 for the two-factor Vasicek model, 0.000179 for three-factors Vasicek model, 0.000053 for the to two-factors CIR model and 0.000121 for the ATSM model, which are very small. As expected, these differences reduce tremendously as  $M$  increases.

The second conclusion which can be drawn from the table is that there important computation time benefits of the LBA approach, compared with the LS-MC approach. In contrast to the high computation time required by the LS-MC approach, it takes only a few minutes to price the options with the LBA approach. Summing up, the above results suggests that the LBA method can be proved a very fast and efficient approach of pricing American put option prices.

| $(r, \theta, m)$     | PE       | PA       | Early    | PA       | Early    | Diff      | AmPut    | Early    | Diff      |
|----------------------|----------|----------|----------|----------|----------|-----------|----------|----------|-----------|
|                      |          | LS-MC    | LS-MC    | LBA(M=1) | LBA(M=1) | $10^{-6}$ | LBA(M=2) | LBA(M=2) | $10^{-6}$ |
| (0.049,0.01,0.005)   | 0.039222 | 0.048584 | 0.009362 | 0.048584 | 0.009362 | 0         | 0.048584 | 0.009362 | 0         |
| (0.0475,0.01,0.005)  | 0.035676 | 0.044665 | 0.008989 | 0.044665 | 0.008989 | 0         | 0.044665 | 0.008989 | 0         |
| (0.046,0.01,0.005)   | 0.031971 | 0.040721 | 0.008750 | 0.040797 | 0.008826 | 76        | 0.040777 | 0.008806 | 56        |
| (0.0445,0.01,0.005)  | 0.028350 | 0.036753 | 0.008403 | 0.036846 | 0.008496 | 93        | 0.036827 | 0.008477 | 74        |
| (0.043,0.01,0.005)   | 0.024812 | 0.032760 | 0.007947 | 0.032881 | 0.008068 | 121       | 0.032851 | 0.008038 | 91        |
| (0.049,0.007,0.008)  | 0.028563 | 0.036640 | 0.008077 | 0.036640 | 0.008077 | 0         | 0.036640 | 0.008077 | 0         |
| (0.0475,0.007,0.008) | 0.024925 | 0.032646 | 0.007721 | 0.032700 | 0.007775 | 54        | 0.032682 | 0.007757 | 36        |
| (0.046,0.007,0.008)  | 0.021609 | 0.028628 | 0.007019 | 0.028699 | 0.007090 | 71        | 0.028674 | 0.007065 | 46        |
| (0.0445,0.007,0.008) | 0.018330 | 0.024584 | 0.006254 | 0.024670 | 0.006340 | 86        | 0.024649 | 0.006319 | 65        |
| (0.043,0.007,0.008)  | 0.015338 | 0.020516 | 0.005177 | 0.020625 | 0.005286 | 109       | 0.020599 | 0.005260 | 83        |
| RMSRE                |          |          |          | 2.12%    | 6.13%    |           | 1.48%    | 4.75%    |           |
| CPU(sec)             |          | 41598.12 |          | 40.71    |          |           | 182.93   |          |           |

Table 4..4: Three Factor ATSM Model

#### 4.4 Conclusions

In this chapter we introduced a new numerical method of pricing an American put option on bonds for the class of affine term structure models, frequently used in practice. The method is based on a decomposition of the American bond put option price into its European option counterpart price and the early exercise boundary premium. The latter is expressed in terms of prices of Arrow-Debreu type of securities and its approximation is based on a multi-region and multi-piece approximating function (hyperplane). To derive the analytic solutions of the prices of the Arrow-Debreu type of securities, we derive the closed form of the joint characteristic function of the state variables driving the term structure of interest rates.

To evaluate the pricing performance of our method, compared with a benchmark method the least-squares Monte Carlo method recently suggested by Longstaff and Schwartz (2001), we carry out some experiments some known term structure models. The numerical results of these experiments indicate that our method can very accurately and fast price American bond options prices, compared with the benchmark method.

## CHAPTER 5.

### PRICING CAPS AND SWAPTIONS BASED ON HJM MODELS

#### 5.1 Introduction

Pricing interest rates derivatives is one of the most important areas of financial economics. It requires an appropriate model capturing the dynamics of the term structure of interest rates. Despite the plethora of term structure of interest rates models available, recent evidence suggests that none of these models can price the term structure of interest rates and contingent claims on them consistently. As aptly noted by Collin-Dufresne and Goldstein (2002a), the affine term structure models recently developed by Duffie and Kan (1997), and Dai and Singleton (2000) with, or without stochastic volatility, can explain only a small proportion of caps' and swaptions' returns. Moreover, Heidari and Wu (2001) show that a three factor model of the term structure, which can explain almost 99% of swaps' rates variations, is not able to explain more than 60% variations of swaptions prices. This evidence is consistent with that provided by studies on the implied volatilities of caps and swaptions prices [see De Jong, Driessen and Pelsser (2002), Jagannathan, Keplin and Sun (2001), Collin-Dufresne and Goldstein (2002a), *inter alia*] supporting the view that the term structure of the implied volatilities by the above derivatives are not consistent with the movements of interest rates, across different maturity horizons. Furthermore, these studies provide evidence that there may be a systematic factor, referred to as unspanned implied volatility factor by Collin-Dufresne and Goldstein (2002a), which is necessary to jointly capture the dynamics of the caps and swaptions prices and their underlying term structure of interest rates. Actually, Collin-Dufresne and Goldstein (2002b) went further to show that, from the existing term structure models, only the Heath, Jarrow and Morton's (1992) (hereafter HJM) model of forward rates can accommodate the unspanned implied volatility factor.

In this chapter I suggest an extension of the standard HJM model which allows for stochastic volatility and/or jumps with the aim to capture the unspanned factor of

the implied volatilities by the caps and swaptions prices. My motivations to consider for such type of extensions of the standard HJM model can be justified by recent evidence of empirical studies on the term structure of interest rates [see Babbs and Webber (1993), El-Jahel, Lindberg and Perraudin (1997), Johannes (1999), and Das and Sundaram (1999), *inter alia*] suggesting that the presence of jumps in interest rates dynamics are necessary in order to capture the time series properties of interest rates and the smirks of the implied volatility of caps and swaptions, especially at the short end of the term structure.<sup>1</sup> These jumps can be attributed economically to monetary news which affect the term structure of interest rates.

The chapter is organised as follows. In section 5.2, I present the extension of HJM model allowing for stochastic volatility and marked point (jump) processes, and I derive the dynamics of the bond prices and forward rates under the physical and risk neutral measures. In section 5.3, I derive analytical solutions of the caps and swaptions prices for the extended HJM model. I provide a closed form solution for caps and an analytical approximation solution of swaptions, suggested by Singleton and Umantsev (2002) and similar to mine in pricing American option in the previous chapters, based on the characteristic function of the underlying bond prices. This function is derived in closed form in the appendix. The analytic solutions of the caps and swaptions prices are derived under the forward probability measure. In Section 5.4, I evaluate the empirical performance of the model to jointly price interest rates, caps and swaptions. The results of this section indicate that allowing for stochastic volatility and jumps in forward rates can significantly improve the performance of the model upon the standard HJM model. Section 5.5 concludes the chapter.

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<sup>1</sup> Models of interest rates derivatives which consider for smiles are suggested by Das (1997a,1997b,1999), Das and Foresi (1996) and Duffie and Kan (1996) and Shirakawa (1991).



## 5.2 The HJM model with stochastic volatility and marked point process

5.2.1 Model set up In this section, I present a version of the HJM model of the term structure of interest rates with stochastic volatility and a marked point process. This model will be used in the next section to price different types of interest rates derivatives.

Consider the probability space  $(\Omega, \mathcal{F}, \mathbf{F}, P)$  with the filtration  $\mathbf{F} = (\mathcal{F}_s)_{t \leq s \leq T}$  and let the instantaneous  $T$ -period forward rate contracted at time  $t$ , denoted  $f(t, T)$ , follow the dynamic process:

$$df(t, T) = \mu_f(t, T)dt + \sum_{m=1}^M \sigma_{m,f}(t, T) \sqrt{V_m(t)} dZ_m(t) + dJ(t, T), \quad (5.1)$$

with volatility processes  $V_m(t)$ ,  $m = 1, 2, \dots, M$ , given by the diffusions

$$dV_m(t) = \kappa_m(\theta_m - V_m(t)) + \sqrt{V_m(t)} d\zeta_m(t) \quad (5.2)$$

and a marked point process, denoted  $J(t, T)$ , defined as

$$J(t, T) = \sum_{i=1}^L J^{(i)}(t, T), \quad (5.3)$$

where  $\mu_f(t, T)$  and  $\sigma_{m,f}(t, T)$  are deterministic functions,  $Z_m(t)$  denote independent Wiener processes,  $\zeta_m(t)$  is also an independent Wiener process determining the volatility changes independently from  $Z_m(t)$ , and  $J^{(i)}(t, T)$  represents an individual marked point process,  $J^{(i)}(t, T)$ , which aims to capture random over time jumps (or shifts) at the levels of forward rates coming from  $i$  independent sources, such as central banks interventions, earthquakes, bankruptcies etc., marked with the random variables  $X^{(i)}$  taking values in the measurable space  $(G, \mathcal{G})$ , where  $G$  is the set of the states of the marked process, defined as  $G \equiv (-\infty, \infty)$ , and  $\mathcal{G}$  is a collection of all subsets of  $G$ . If we denote the time point of  $n$ -th potential jump

as  $\tau_n^{(i)}$  and its corresponding marked variable as  $X_n^{(i)}$ , for  $n = 1, 2, \dots, \infty$ , where  $\tau_n^{(i)} \in \mathcal{R}^+ \equiv (0, \infty)$ , with  $\tau_1^{(i)} < \tau_2^{(i)} < \dots < \tau_\infty^{(i)} = \infty$  [see Glasserman and Kou (2002)], then the marked point process  $J^{(i)}(t, T)$  can be characterized by the double sequence  $\{(\tau_n^{(i)}, X_n^{(i)}), n = 1, 2, \dots\}$ .

The above model extends the standard HJM into two directions. First, it allows for the forward rate volatility to be stochastic, which is consistent with recent evidence [see Collin-Dufresne and Goldstein (2001a,b)] and, second, it allows the forward rates, in addition to the Wiener processes, to be driven by a number,  $L$ , of independent marked jump processes, which is consistent with evidence that interest rates are subject to pure jumps (or switching regime), over time [see Lindberg, Orzang and Perraudin (1995), and Tzavalis and Wickens (1997), *inter alia*].

In order to complete the description of the marked point process,  $J^{(i)}(t, T)$ , define the number of jumps,  $N_t^{(i)}$ , as  $N^{(i)}(t) = \sup \{n \geq 0 : \tau_n^{(i)} < t\}$ , with an intensity function (which gives the arrival rate of the marked  $X^{(i)}$  jump within each of the subsets  $dx$  of  $G$ , at any time)  $\lambda^{(i)}(dx, t)$  and a magnitude function  $h^{(i)}(X_n^{(i)}, \tau_n^{(i)}, T)$ , which depends on  $X_n^{(i)}, \tau_n^{(i)}$  and the maturity interval of the forward rate,  $T - t$ . Note that  $\lambda^{(i)}(dx, t)$  is a measurable function  $\lambda^{(i)}(dx, t) : \mathcal{G} \times (0, \infty) \rightarrow \underline{R}$ , and magnitude . Given the above notation, we can write  $J^{(i)}(t, T)$  as  $J^{(i)}(t, T) = \sum_{n=1}^{N_t^{(i)}} h^{(i)}(X_n^{(i)}, \tau_n^{(i)}, T)$ . Our assumption that the jump magnitude depends on the maturity interval is quite general. It can allow different impacts of a jump on forward rates, across the whole maturity spectrum.

As in Bjork, Kabanov and Runggaldier (1997), we can rewrite the individual marked process  $J^{(i)}(t, T)$  as

$$\begin{aligned}
J^{(i)}(t, T) &= \sum_{n=1}^{N_t^{(i)}} h^{(i)}(X_n^{(i)}, \tau_n^{(i)}, T) \\
&= \int_t^T \int_G h^{(i)}(x, s, T) \mu^{(i)}(dx, ds),
\end{aligned}$$

where  $h^{(i)}(x, s, T) = h^{(i)}(X_n^{(i)}, \tau_n^{(i)}, T)$ , with  $X_n^{(i)}$  taking values  $x$  in the set  $G$  and  $\tau_n^{(i)}$  taking value  $s$ , and  $\mu^{(i)}(dx, ds) : \mathcal{G} \times (0, \infty) \rightarrow \mathcal{R}$  is a function which measures how many jumps happen in the subset  $dx \times ds$ .

For the intensity function  $\lambda^{(i)}(dx, s)$ , we can notice that it satisfies the following relationship

$$\begin{aligned}
&\int_t^T \int_E h^{(i)}(x, s, T) \left[ \mu^{(i)}(dx, ds) - \lambda^{(i)}(dx, s) ds \right] \\
&= \int_t^T \int_0^\infty h^{(i)}(x, s, T) \left[ \mu^{(i)}(dx, ds) - \lambda^{(i)}(dx, s) ds \right], \forall i,
\end{aligned}$$

according to the project theorem given by Bremaud (1981). This relationship constitutes a martingale.

5.2.2 Pricing Bond under the Risk Neutral Measure In order to derive the pricing formulas on interest rates derivatives for the HJM model, defined by equations (5.1)-(5.2), my analysis starts with describing the dynamics of a zero coupon bond price with maturity date  $T$  (i.e. maturity interval  $T - t$ ), at time  $t$ , denoted as  $P(t, T)$ , under the physical (objective) and the risk neutral, denoted  $Q$ , measures.

According to HJM model, the bond price  $P(t, T)$  can be calculated as  $P(t, T) = e^{-\int_t^T f(t, u) du}$ . Define its log-price, as  $p(t, T) \equiv \log P(t, T)$ . In the next proposition I present the diffusions describing the dynamics of the prices  $p(t, T)$  and  $P(t, T)$  under the physical measure.

**Proposition 7** *Let the forward rate satisfy equations (5.1)-(5.2), with their underlying assumptions. Then, the dynamics of the log-bond price  $p(t, T - t)$  and the bond price  $P(t, T - t)$  can be respectively described by the following diffusions*

$$\begin{aligned} dp(t, T) &= (r_t + A(t, T)) dt + \sum_{m=1}^M S_m(t, T) dZ_m(t) \\ &\quad + \sum_{i=1}^L \int_G D^{(i)}(x, t, T) \mu^{(i)}(dx, dt) \end{aligned} \quad (5.4)$$

and

$$\begin{aligned} dP(t, T) &= P(t, T) \left\{ r_t + A(t, T) + \frac{1}{2} \|S(t, T)\|^2 \right\} dt + P(t, T) \sum_{m=1}^M S_m(t, T) dZ_m(t) \\ &\quad + P(t_-, T) \sum_{i=1}^L \int_G \left( e^{D^{(i)}(x, t, T)} - 1 \right) \mu^{(i)}(dx, dt), \end{aligned} \quad (5.5)$$

where  $r_t$  is the instantaneous risk-free interest rate and

$$\begin{aligned} A(t, T) &= - \int_t^T \mu_f(t, s) ds, \\ S_m(t, T) &= - \sqrt{V_m(t)} \int_t^T \sigma_{m,f}(t, s) ds, \\ S(t, T) &= [S_1(t, T), S_2(t, T), \dots, S_M(t, T)]' \quad \text{and} \\ D^{(i)}(x, t, T) &= - \int_t^T h^{(i)}(x, t, s) ds. \end{aligned}$$

The proof of the proposition can be derived by applying Ito's Lemma to the functions of the prices  $p(t, T)$  and  $P(t, T - t)$  [see Bjork, Kabanov and Runggaldier (1997)].

Assuming that the market is complete and that there exists a unique mar-

$$r_t + A(t, T) + \frac{1}{2} \|S(t, T)\|^2 + \Gamma_t \cdot S(t, T) + \sum_{i=1}^L \left[ \int_G \left( e^{D^{(i)}(x, t, T)} - 1 \right) \lambda^{(i)Q}(dx, t) \right],$$

should be equal to the instantaneous interest rate,  $r_t$ , which in turn implies that the following relationship

$$\begin{aligned} & A(t, T) + \frac{1}{2} \|S(t, T)\|^2 + \Gamma_t \cdot S(t, T) \\ & + \sum_{i=1}^L \left[ \int_G \left( e^{D^{(i)}(x, t, T)} - 1 \right) \Phi^{(i)}(t, x) \lambda^{(i)}(dx, t) \right] \\ = & 0 \end{aligned} \tag{5..6}$$

should hold. Since we assume that the market is complete and that there exists a unique martingale, equation (5..6) implies that there will exist unique solutions for  $\Gamma_t$  and  $\Phi^{(i)}(t, x) \forall i$ .

Using (5..6), we can easily show that the dynamics of the bond price  $P(t, T)$  under the  $Q$  measure can be described by the following diffusion

$$\begin{aligned} dP(t, T) = & P(t, T)r_t dt + P(t, T) \sum_{m=1}^M S_m(t, T) dZ_m^Q(t) \\ & + P(t_-, T) dJ^Q(t, T), \end{aligned} \tag{5..7}$$

where

$$d^{(i)}(x, t, T) = e^{D^{(i)}(x, t, T)} - 1$$

and

$$dJ^Q(t, T) = \sum_{i=1}^L \left\{ \int_G d^{(i)}(x, t, s) \left[ \mu^{(i)}(dx, dt) - \Phi^{(i)}(t, x) \lambda^{(i)}(dx, t) dt \right] \right\},$$

which constitutes a  $Q$ -local martingale which is unique, since the prices of all risks are uniquely determined by the absence of arbitrage condition. This implies that a unique risk neutral (no-arbitrage) bond price can be described by equation (5.7).

In contrast with the risk price parameters which determine the drift of the bond prices  $p(t, T)$  and  $P(t, T)$  diffusions, the arbitrage condition alone is not enough in identifying the price of risks parameters associated with the stochastic volatility processes of the bond prices [see also Collin-Dufresne and Goldstein (2002(a),(b))]. To identify these parameters, we need additional information. This can be retrieved by the prices of futures contracts on the bonds' prices volatilities [see Collin-Dufresne and Shi (ibid)]. Such type of instruments can be replicated by a portfolio of futures contracts prices on the bonds' yields traded in Chicago Mercantile Exchange. The drift terms of these future prices are equal to zero under measure  $Q$ , and hence the parameters of the prices of risks associated with the stochastic volatilities of the underlying bond price can be uniquely determined [see Collin-Dufresne and Goldstein (ibid)]. In general, the volatility processes  $V_m(t)$  under measure  $Q$  can be written as

$$dV_m(t) = \kappa_m^Q (\theta_m^Q - V_m(t)) + \sqrt{V_m(t)} d\zeta_m^Q(t), \quad (5.8)$$

where  $\kappa_m^Q$  and  $\theta_m^Q$  respectively denote the mean-reversion and long-run variance parameters of the volatility processes under measure  $Q$ .

Having derived the diffusion for  $dP(t, T)$  under  $Q$ , next I present the diffusion describing the dynamics of the forward rate under the  $Q$  measure. This is done by

exploiting the arbitrage condition for  $dP(t, T)$ . This diffusion is given by

$$df(t, T) = \mu_f^Q(t, T)dt + \sum_{m=1}^M \sigma_{m,f}(t, T) \sqrt{V_m(t)} dZ_m^Q(t) + \sum_{i=1}^L \left[ \int_G h^{(i)}(x, t, T) \mu^{(i)}(dx, dt) \right], \quad (5.9)$$

where

$$\mu_f^Q(t, T) = \sum_{m=1}^M V_m(t) \sigma_{m,f}(t, T) \int_t^T \sigma_{m,f}(t, s) ds - \sum_{i=1}^L \left[ \int_G h^{(i)}(x, t, T) e^{D^{(i)}(x, t, T)} \lambda^{(i)Q}(dx, t) \right].$$

This diffusion indicates that the drift parameter of the forward rate under  $Q$  is determined by both the stochastic volatility and marked point processes.

### 5.3 Pricing interest rate contingent Claims

Having presented the diffusion processes of the forward rates and bond prices under the risk neutral measure for the HJM model, in this section I direct my analysis on pricing interest rates derivatives, such as caps and floors, and swaptions. To derive the pricing formulas for these derivatives, it is convenient at this point to transform the  $Q$  measure to the forward measure, denoted as  $U$  [see Jamshidian (1997), and Musiela and Rutkowski (1997), *inter alia*]. Under the concept of the forward measure, I can write the bond price  $P(t, T)$  in terms of another bond price with maturity date  $U$ , referred to as numeraire price, as

$$P^U(t, T) = \frac{P(t, T)}{P(t, U)},$$

while the log price  $p(t, T)$  can be written as

$$p^U(t, T) \equiv \log P^U(t, T) = \log \frac{P(t, T)}{P(t, U)}$$

.It can be shown that  $P^U(t, T)$  constitutes a martingale process under measure  $U$  [see Musiela and Rutkowski (ibid)]. Following similar steps with those before transforming the bond price  $P(t, T)$  from the physical measure to the risk neutral,  $Q$  measure, I can transform  $P(t, T)$  from the  $Q$  measure to the  $U$  measure. We can easily show that the diffusion describing the dynamics of  $P(t, T)$  under the  $U$  measure has a drift parameter which is equal to zero. This diffusion is given in the next theorem.

**Theorem 8** *Under the  $U$ -forward measure, I have*

$$\begin{aligned} & dP^U(t, T) \\ = & P^U(t, T) \sum_{m=1}^M (S_m(t, T) - S_m(t, U)) dZ_m^U \\ & + P^U(t_-, T) \sum_{i=1}^L \left[ \int_G \frac{d^{(i)}(x, t, T) - d^{(i)}(x, t, U)}{d^{(i)}(x, t, U) + 1} \right. \\ & \left. (\mu^{(i)}(dx, dt) - \lambda^{(i)U}(dx, t)dt) \right], \end{aligned} \quad (5..10)$$

where

$$\begin{aligned} \lambda^U(dx, t) &= [d^{(i)}(x, t, U) + 1] \lambda^{(i)Q}(dx, t) \\ \text{and} \quad dZ_m^U(t) &= dZ_m^Q(t) - S_m(t, U)dt. \end{aligned}$$

*The proof this theorem is given in Appendix C.1.*

The results of this theorem indicate that under the  $U$  measure both the jump intensity and the jump magnitude functions are different from those under the  $Q$  measure. The happens because, when the bond price  $P(t, T)$  is measured in terms of



the relative bond price  $P(t, U)$ , the drift parameter of the bond price  $P(t, T)$  diffusion changes.

5.3.1 Interest rates caps and floors Interest rates caps constitute the most popular interest rate derivatives traded in over-the-counter markets. These derivatives are contracts which provide the insurance against the rate of interest on a floating note (usually the Libor rate or a Treasury bill rate) going above some prespecified level known as cap rate, denoted as  $K_r$ .

Let  $r(t, T)$  denote the time  $t$  interest rate on the floating note (say a Treasury bill) with maturity  $T$ . An interest rate cap with maturity  $T_\omega$  pays the amount  $\max\{r(T_{s-1}, T_s) - K_r, 0\} \delta B$  to the holder of the cap contract at equally distanced fixed points of times  $T_s$ ,  $s = 1, 2, \dots, \omega$ , over the maturity interval, i.e.  $T_s - T_{s-1} = \delta$ , for all  $s$ , where  $B$  denotes the principal amount of the floating rate note. Given the cap's definition, we can write the cap price, denoted as  $Cap(t, T_\omega)$ , as the sum of the prices of  $\omega$  individual caps expiring at the time points  $T_s$ , referred to as caplets, i.e.

$$Cap(t, T_\omega) = \sum_{s=1}^{\omega} Caplet(t, T_s), \quad (5.11)$$

where

$$Caplet(t, T_s) = B\delta E_t^Q \left[ e^{-\int_t^{T_s} r(u)du} \max(r(T_{s-1}, T_s) - K_r, 0) \right]. \quad (5.12)$$

The time  $T_{s-1}$  floating rate with maturity date  $T_s$ , denoted  $r(T_{s-1}, T_s)$ , is given by

$$r(T_{s-1}, T_s) = \frac{1}{\delta} \left( \frac{1}{P(T_{s-1}, T_s)} - 1 \right),$$

the caplet price  $Caplet_s(t)$  can be rewritten as

$$\begin{aligned} & Caplet(t, T_s) \\ &= B(1 + \delta K_r) E_t^Q \left( e^{-\int_t^{T_{s-1}} r(u) du} \max\left(\frac{1}{(1 + \delta K_r)} - P(T_{s-1}, T_s), 0\right) \right). \end{aligned} \quad (5..13)$$

This equation indicates that the caplet price  $Caplet(t, T_s)$  can be calculated as the price of a European put option on a zero-coupon bond. Choosing the bond price  $P(T_t, T_{s-1})$  and  $P(T_t, T_s)$  respectively as a numeraire price of the forward measures  $U = T_{s-1}$  and  $U = T_s$ , we can write the price  $Caplet(t, T_s)$  at time  $t$

$$\begin{aligned} & Caplet(t, T_s) \\ &= B(1 + \delta K_r) E_t^Q \left[ e^{-\int_t^{T_{s-1}} r(u) du} \max\left(\frac{1}{(1 + \delta K_r)} - P(T_{s-1}, T_s), 0\right) \right] \\ &= BP(t, T_{s-1}) E_t^Q \left( \frac{e^{-\int_t^{T_{s-1}} r(u) du}}{P(t, T_{s-1})} \Lambda_{\{P(T_{s-1}, T_s) \leq \frac{1}{(1 + \delta K_r)}\}} \right) \\ &\quad - B(1 + \delta K_r) P(t, T_s) E_t^Q \left( \frac{e^{-\int_t^{T_s} r(u) du}}{P(t, T_s)} \Lambda_{\{P(T_{s-1}, T_s) \leq \frac{1}{(1 + \delta K_r)}\}} \right) \\ &= BP(t, T_{s-1}) E_t^{T_{s-1}} \left( \Lambda_{\{P(T_{s-1}, T_s) \leq \frac{1}{(1 + \delta K_r)}\}} \right) \\ &\quad - B(1 + \delta K_r) P(t, T_s) E_t^{T_s} \left( \Lambda_{\{P(T_{s-1}, T_s) \leq \frac{1}{(1 + \delta K_r)}\}} \right), \end{aligned} \quad (5..14)$$

where  $\Lambda_{\{ \cdot \}}$  is an indicator function, and the forward measure expectations  $E_t^{T_{s-1}}(\Lambda_{P(T_{s-1}, T_s) \leq \frac{1}{(1 + \delta K_r)}})$  and  $E_t^{T_s}(\Lambda_{P(T_{s-1}, T_s) \leq \frac{1}{(1 + \delta K_r)}})$  constitute the prices of Arrow-Debreu type of securities which pay \$1 in the state  $P(T_{s-1}, T_s) \leq \frac{1}{(1 + \delta K_r)}$ , and zero otherwise.

Based on the characteristic functions of the log-price  $p^{T_{s-1}}(T_{s-1}, \delta) = \log \frac{P(T_{s-1}, \delta)}{P(T_{s-1}, T_{s-1})}$  under the  $U \equiv T_{s-1}$  and  $U = T_s$  forward measures, which are defined as

$$\Psi^{T_{s-1}}(\phi, T_s | p^{T_{s-1}}(t, T_s), V(t)) = E^{T_{s-1}} \left[ \exp(\phi p^{T_{s-1}}(T_{s-1}, T_s)) \right],$$

and

$$\Psi^{T_s}(\phi, T_s | p^{T_{s-1}}(t, T_s), \mathbf{V}(t)) = E^{T_s} \left[ \exp(\phi p^{T_{s-1}}(T_{s-1}, T_s)) \right].$$

They are derived in closed form solution in Appendix C.2, we can analytically calculate the expectations  $E_t^{T_{s-1}} \left( \Lambda_{\{P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)}\}} \right)$  and  $E_t^{T_s} \left( \Lambda_{P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)}} \right)$ , and thus we can write the caplet price  $Caplet(t, T_s)$  in closed form solution as

$$\begin{aligned} & Caplet_s(t) \\ &= BP(t, T_{s-1}) \Pi^{T_{s-1}} \left( P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)} \right) \\ & \quad - B(1+\delta K_r) P(t, T_s) \Pi^{T_s} \left( P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)} \right), \end{aligned} \quad (5.15)$$

where

$$\begin{aligned} & \Pi^{T_{s-1}} \left( P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)} \right) \\ &= \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{e^{-i\phi \frac{1}{(1+\delta K_r)}} \Psi^{T_{s-1}}(\phi, T_{s-1}, T_s | p^{T_{s-1}}(t, T_s), \mathbf{V}(t))}{i\phi} \right) d\phi \end{aligned}$$

and

$$\begin{aligned} & \Pi^{T_s} \left( P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)} \right) \\ &= \left( \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{e^{-i\phi \frac{1}{(1+\delta K_r)}} \Psi^{T_s}(\phi, T_{s-1}, T_s | p^{T_{s-1}}(t, T_s), \mathbf{V}(t))}{i\phi} \right) d\phi \right), \end{aligned}$$

where  $\Psi^{T_{s-1}}(\phi, T_{s-1}, T_s | p^{T_{s-1}}(t, T_s), \mathbf{V}(t))$  and  $\Psi^{T_s}(\phi, T_{s-1}, T_s | p^{T_{s-1}}(t, T_s), \mathbf{V}(t))$  denote the CF of the bond prices  $p^{T_{s-1}}(T_{s-1}, T_s)$ , for  $s = 1, 2, \dots, T_\omega$ , conditional on the current log-prices  $p^{T_{s-1}}(t, T_s)$  respectively under  $U = T_{s-1}$  and  $U = T_s$  forward measure and the  $M$ -dimension vector  $\mathbf{V}(t)$  of the volatility processes, defined as  $\mathbf{V}(t) = (V_1(t), V_2(t), \dots, V_M(t))'$ .  $\Pi^{T_{s-1}} \left( P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)} \right)$  and  $\Pi^{T_s} \left( P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)} \right)$

constitute the prices of the Arrow-Debreu type of securities, defined above. Using equation (5.15), we can calculate  $cap(t, T_\omega)$ , analytically, based on equation (5.11).

Following similar steps with above, we can analytically calculate the price of floor interest rate derivative, denoted as  $floor(t, T_\omega)$ . A floor derivative is analogous to a cap derivative, with the exception that a floor places a lower limit on the floating note rate that is charged. Therefore, its price can be written as

$$Floor(t, T_\omega) = B(1 + \delta K_r) \left\{ \sum_{s=1}^{\omega} P(t, T_s) \Pi^{T_s} \left( P(T_{s-1}, T_s) \geq \frac{1}{(1 + \delta K_s)} \right) - \sum_{s=1}^{\omega} P(t, T_{s-1}) \Pi^{T_{s-1}} \left( P(T_{s-1}, T_s) \geq \frac{1}{(1 + \delta K_s)} \right) \right\}. \quad (5.16)$$

As with the cap price  $cap(t, T_\omega)$ , an analytic solution for the floor price  $Floor(t, T_\omega)$  can be derived based on the CF  $\Psi^{T_s}(\phi, T_{s-1}, T_s | p^{T_{s-1}}(t, T_s), V(t))$ , defined above.

**5.3.2 Swaptions** Swaptions constitute options on interest rate swaps which give the holder the right to enter into a prespecified swap at fixed time in the future,  $T_\omega$ , a certain interest rate swap. There are two types of a swaption. The first type involves paying floating rate payments and receiving fixed rate payments, while the second type involves paying fixed rate payments and receiving fixed rate payments, inversely. We only concentrate on the first type.

The first ( $I$ ) type of swaption can be thought of as a call option on a coupon bearing bond with strike price \$1 and maturity date  $T_0 \leq T_\omega$  that to enter the swap. The coupon rate of this bond is the fixed rate of the swap, denoted as  $c$ . Thus, the price of swaption  $I$  where the principal amount is paid off can be calculated as

$$swaption_I(t, T_0, T_\omega) = E^Q \left[ e^{-\int_t^{T_s} r(u) du} \max(P^c(T_0, T_\omega) - 1, 0) \right], \quad (5.17)$$

where  $P^c(T_0, T_\omega)$  is the time  $T_0$  price of a coupon bearing bond price with maturity

date  $T_\omega$  paying coupons at the fixed points of time  $T_1, T_2, \dots, T_\omega$ . The price of this coupon bond can be written as a portfolio of zero-coupon bond prices

$$P^c(T_0, T_\omega) = \sum_{s=1}^{\omega} cP(T_0, T_s) + P(T, T_\omega). \quad (5.18)$$

Substituting equation (5.18) into equation (5.17), we can rewrite equation (5.17) as follows

$$\begin{aligned} & \text{Swaption}_I(t, T_0, T_\omega) \\ &= E^Q \left[ e^{-\int_t^{T_0} r(u)du} \max \left( \sum_{s=1}^{\omega} cP(T_0, T_s) + P(T_0, T_\omega) - 1, 0 \right) \right] \\ &= \sum_{s=1}^{\omega} cP(t, T_s) E^Q \left[ \frac{e^{-\int_t^{T_s} r(u)du}}{P(t, T_s)} \Lambda_{\{P^c(T_0, T_\omega) > 1\}} \right] \\ & \quad + P(t, T_\omega) E^Q \left[ \frac{e^{-\int_t^{T_\omega} r(u)du}}{P(t, T_\omega)} \Lambda_{\{P^c(T_0, T_\omega) > 1\}} \right] \\ & \quad - P(t, T_0) E^Q \left[ \frac{e^{-\int_t^{T_0} r(u)du}}{P(t, T_0)} \Lambda_{\{P^c(T_0, T_\omega) > 1\}} \right] \\ &= \sum_{s=1}^{\omega} cP(t, T_s) E^{T_s} [\Lambda_{\{P^c(T_0, T_\omega) > 1\}}] + P(t, T_\omega) E^{T_\omega} [\Lambda_{\{P^c(T_0, T_\omega) > 1\}}] \\ & \quad - P(t, T_0) E^t [\Lambda_{\{P^c(T_0, T_\omega) > 1\}}], \end{aligned}$$

where  $E^{T_s} [\Lambda_{\{P^c(T_0, T_\omega) > 1.0\}}]$  constitute the prices of Arrow-Debreu type of securities which pay \$1 if the state  $P^c(T_0, T_\omega) > 1.0$  occurs, and zero otherwise. As the caps and floors prices, the prices of the Arrow-Debreu type of securities  $E^{T_s} [\Lambda_{\{P^c(T_0, T_\omega) > 1\}}]$ , implying the swaption price, can be spanned by the joint CF of the bond prices  $p^{T_0}(T_0, T_s)$ ,  $s = T_1, T_2, \dots, T_\omega$ , under the  $T_0$  forward measure, which is defined as

$$\Psi^{T_s}(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) = E^{T_s} \left[ \exp \left( \sum_{s=1}^{\omega} \phi_j p^{T_0}(T_0, T_s) \right) \right],$$

where  $\phi = (\phi_1, \phi_2, \dots, \phi_\omega)'$ ,  $\mathbf{T}_s = (T_1, T_2, \dots, T_\omega)'$ ,  $\mathbf{p}^{T_0}(t, T_\omega) = (p^{T_0}(T_0, T_1), p^{T_0}(T_0, T_2), \dots, p^{T_0}(T_0, T_\omega))'$  and  $\mathbf{V}(t) = (V_1(t), V_2(t), \dots, V_M(t))'$ .<sup>2</sup> However, in contrast to the caps and floors prices, I can not use the analytic formula of the above CF and Levy's inversion Lemma to calculate prices of the Arrow-Debreu state prices  $E^{T_s} [\Lambda_{\{P(T_0, T_\omega) > 1.0\}}]$ , implied by the swaption price  $Swaption_I(t, T_\omega)$ . This is due to the fact that the exercise region of the swaption is not explicitly defined, and thus the expectation  $E^{T_s} [\Lambda_{\{P(T_0, T_\omega) > 1.0\}}]$  needs to be calculated over an implicit subspace of the state space. For this reason, I will adopt the numerical approximation scheme suggested by Singleton and Umantsev (2002).

#### 5.4 Estimation and Evaluation of the HJM model

The goal of this section is to evaluate the empirical performance of the extension of the HJM model with stochastic volatility and marked point processes, suggested in the previous section, to jointly price interest rates derivatives. In particular, the aim of our analysis is to investigate whether the model can span data on forward rates, caps and swaptions satisfactorily. To appraise the relative performance of alternative specifications of the model, which is with, or without, stochastic volatility and with, or without, marked point processes, I will be based on in-and-out-of-sample statistical measures, which can evaluate the pricing performance of the alternative specifications considered.

The analysis of this section proceeds as follows. First, I present the alternative

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<sup>2</sup> Note that, here, we have defined the characteristic function as  $E^{T_s} \left[ \exp \left( \sum_{s=1}^{\omega} \phi_j p^{T_0}(T_0, T_s) \right) \right]$  rather than  $E^{T_s} \left[ \exp \left( \sum_{s=1}^{\omega} \phi_j p(T_0, T_s) \right) \right]$ . This happens because the drift of the diffusion for the log-price  $p(T_0, T_s)$  is a function of the instantaneous interest rate which is path dependent for the HJM model. Since we have  $E^{T_s} \left[ \exp \left( \sum_{s=1}^{\omega} \phi_j p^{T_0}(T_0, T_s) \right) \right] = E^{T_s} \left[ \exp \left( \sum_{s=1}^{\omega} \phi_j p(T_0, T_s) \right) \right]$ , to avoid specifying the path of the instantaneous interest rate, we can calculate the CF of the price  $p^{T_0}(T_0, T_s)$ , whose drift parameter does not depend on the instantaneous interest rate.

specifications of the HJM model that we consider. Second, I describe the data and report the estimation results for the alternative specifications considered.

5.4.1 Empirical specifications of the HJM model To evaluate the empirical performance of alternative specifications of the HJM model, given by equations (5.1)-(5.2), we first need to specify the functional form of the volatility parameters  $\sigma_{m,f}(t, T)$ , and the marked point process  $J(t)$ , with its intensity and magnitude parameters. For  $\sigma_{m,f}(t, T)$ , we assume the following specification

$$\sigma_{m,f}(t, T) = \frac{\sigma_m}{\beta_m} [1 - \exp(-\beta_m(T - t))]. \quad (5.19)$$

There are two reasons for adopting this specification for  $\sigma_{m,f}(t, T)$ . The first is that it can capture the type of shapes of the volatility, as evidence suggests [see Driessen, Klaassen and Melenberg (2000)]. The second is that it enables us to derive closed form solutions of the CF  $\Psi^{T_{s-1}}(\phi, T_{s-1}, T_s | p^{T_{s-1}}(t, T_s), \mathbf{V}(t))$  and  $\Psi^{T_s}(\phi, T_{s-1}, T_s | p^{T_{s-1}}(t, T_s), \mathbf{V}(t))$ , which are needed to calculate the prices  $\Pi^{T_{s-1}}\left(P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)}\right)$  and  $\Pi^{T_s}\left(P(T_{s-1}, T_s) \leq \frac{1}{(1+\delta K_r)}\right)$  of the Arrow-Debrue securities.

For the specification of the marked point process, we consider the widely used Poisson process with an intensity parameter specified as

$$\widehat{\lambda}^{(i)Q}(dx, t) = \widehat{\lambda}^{(i)} \gamma^{(i)}(x) e^{-tx} dx$$

where  $\gamma^{(i)}(x)$  is the normal density function,  $\gamma^{(i)}(x) = \frac{1}{\sigma_x^{(i)} \sqrt{2\pi}} e^{-\frac{(x-m_x^{(i)})^2}{2\sigma_x^{(i)2}}$ , and we assume that the magnitude of the jumps is specified by the function  $h^{(i)}(x, t, T) = x$ . The closed form solution of the CF  $\Psi^{T_s}(\phi, T_{s-1}, T_s | p^{T_{s-1}}(t, T_s), \mathbf{V}(t))$  for the above specifications of the volatility and intensity parameters,  $\sigma_{m,f}(t, T)$  and  $\lambda_Q^{(i)}(dx, t)$ , respectively, is given in Appendix C.2.

5.4.2 Data and estimation results Our data set consists of weekly time-series observations from January 1997 to January 2003 on US forward rates, cap prices and

swaptions with maturities 0.25 (3-months), 0.50 (6-months), 0.75 (9-months) for the 1, 2, 3, 4, 5 and 10 years. As forwards rates, we use the future contracts and swaps rates, for the above set of maturities. These notes' rates are used to construct the entire term structure interest rates curve. In particular, we used the swaps rates with maturities from 1-year to 15-year to retrieve the long-term interest rates, i.e. for 1, 2, ...,10 years. The short term interest rates, for 3,6 and 9 months, we used the futures contracts interest rates, since there are no swaps rates available for these maturity intervals.

In Table 5.1-5.3 , I present descriptive statistics, i.e. the mean and standard deviation, of our data set series. Note that the table reports the implied volatilities of the caps and swaptions. The prices of them, used in estimation, are retrieved based on the BS formula. The results of the table indicate a number of known features of the data, found in other studies [see Jong, Driessen and Pelsser (2001), *inter alia*]. First, the unconditional means of the future contracts and swaps rates increase smoothly at a decaying rate from the three-months to ten-years maturities, with their standard deviations being not bigger than 2. As in many empirical studies on the term structure of interest rates [see De Jong, Driessen and Pelsser (ibid), *inter alia*], the interest rates at the short end of the term structure are more volatile than those at the long end. The prices of the implied volatilities of the swaptions seems to decrease smoothly with the maturity interval, while those of the caps follow a humped type of shape, first increasing and then decreasing, as often observed in reality [see De Jong, Driessen and Pelsser (ibid)]. Like futures contracts' and swaps' rates, the shapes of both the swaptions and caps implied volatilities seem to be more volatile at the short end of the term structure, rather than at the long end.

To estimate the model, we follow the method suggested by Bakshi, Cao and Chen (1995). According to this, we regard the state variables,  $V_m(t)$ , determining the volatilities of forward rates, as unobservable parameters which are estimated together with the remaining parameters of the model, collected in vector  $\Theta = (\sigma_m, \beta_m, \kappa_m^Q, \theta_m^Q,$



| Maturity      | Mean  | St. Dev. |
|---------------|-------|----------|
| Futures rates |       |          |
| 3 months      | 4.694 | 1.742    |
| 6 months      | 4.746 | 1.750    |
| 9 months      | 4.819 | 1.734    |
| Swaps rates   |       |          |
| 1             | 4.910 | 1.704    |
| 2             | 5.225 | 1.462    |
| 3             | 5.466 | 1.268    |
| 4             | 5.634 | 1.135    |
| 5             | 5.762 | 1.038    |
| 6             | 5.864 | 0.963    |
| 7             | 5.949 | 0.905    |
| 8             | 6.017 | 0.859    |
| 9             | 6.078 | 0.822    |
| 10            | 6.131 | 0.793    |

Table 5..1: Summary statistics of Futures and swaps rates

| Maturity (yrs) | Mean    | St. Dev. |
|----------------|---------|----------|
| 1              | 20.4074 | 13.3802  |
| 2              | 21.8381 | 11.6603  |
| 3              | 21.3032 | 8.9177   |
| 4              | 20.7360 | 7.2306   |
| 5              | 20.1887 | 6.0587   |
| 7              | 19.2244 | 4.6180   |
| 10             | 18.1217 | 3.5103   |

Table 5..2: Summary statistics of implied volatilities for caps

| Maturity (yrs) | 1       | 2       | 3        | 4       | 5       |
|----------------|---------|---------|----------|---------|---------|
|                | swap    | swap    | swap     | swap    | swap    |
|                |         |         | mean     |         |         |
| 0.25           | 22.1276 | 22.0509 | 20.7172  | 19.9634 | 19.4226 |
| 0.5            | 22.4019 | 21.5124 | 20.3655  | 19.6757 | 19.1735 |
| 1              | 21.6289 | 20.4696 | 19.5697  | 18.9170 | 18.4224 |
| 2              | 19.8643 | 19.1274 | 18.5281  | 18.0628 | 17.6481 |
| 3              | 18.8957 | 18.2371 | 17.7495  | 17.3310 | 16.9850 |
| 4              | 18.1100 | 17.5131 | 17.0672  | 16.7151 | 16.3704 |
| 5              | 17.4898 | 16.8934 | 16.4964  | 16.1289 | 15.7874 |
|                |         |         | St. Dev. |         |         |
| 0.25           | 14.6194 | 12.6533 | 10.2171  | 8.9341  | 8.1292  |
| 0.5            | 14.0099 | 11.2449 | 9.2559   | 8.1345  | 7.4118  |
| 1              | 10.8845 | 8.6035  | 7.3405   | 6.4712  | 5.8666  |
| 2              | 6.2670  | 5.6011  | 5.0004   | 4.5725  | 4.2477  |
| 3              | 4.6973  | 4.2563  | 3.9156   | 3.6558  | 3.4558  |
| 4              | 3.6302  | 3.4033  | 3.1848   | 3.0154  | 2.8904  |
| 5              | 3.0218  | 2.8421  | 2.6919   | 2.5853  | 2.4618  |

Table 5..3: Summary statistics of implied volatilities for swaptions

$\lambda^{(i)}, m_x^{(i)}, \sigma_x^{(i)}$   $i = 1$  for one marked point process, by applying the non linear least squares method over their cross-sectional dimension of our observations, i.e. the caps and swaptions prices, over different maturities, at each point of time. In contrast to Bakshi *et al* (ibid), we obtain the parameters' estimates by minimizing the sum of the square pricing errors between the observed and estimates of the caps and swaps prices relative to their observed prices, denoted  $SSRE(t)$ , rather than their absolute errors. Mathematically, we can write our estimation problem as

$$\begin{aligned}
& SSRE(t) \\
= & \arg \min_{V_m(t), \Theta(t)} \sum_{T_w \in T_c} \left( \frac{Cap(t, T_w)^{Obs} - Cap(t, T_w)}{Cap(t, T_w)^{Obs}} \right)^2 + \sum_{T_0 \in T_{ST}} \sum_{T_w \in T_S} \\
& \left( \frac{Swaption_I(t, T_0, T_w)^{Obs} - Swaption_I(t, T_0, T_w)}{Swaption_I(t, T_0, T_w)^{Obs}} \right)^2, \forall t \quad (5..20)
\end{aligned}$$

where  $T_c$  is the set of maturity date of caps  $T_c = (1, 2, 3, 4, 5, 7, 10)$ ,  $T_{ST}$  is the set of the time when entering swaps  $T_{ST} = (0.25, 0.5, 1, 2, 3, 4, 5)$ ,  $T_S$  is the set of maturity

|                       | HJM            | HJM-SV(1)      | HJM-SVJ(1,1)                  | HJM-SVJ(2,1)                   |
|-----------------------|----------------|----------------|-------------------------------|--------------------------------|
| $\sigma_1$            | 0.0081(0.0018) | 0.7749(0.0056) | 0.7772(0.0040)                | 0.7134( 0.0126)                |
| $\beta_1$             | 0.0421(0.0201) | 0.0131(0.0023) | 0.0152(0.0043)                | 0.0113(0.0051)                 |
| $\kappa_1^Q$          |                | 0.0700(0.0019) | 0.0567(0.0051)                | 0.1107(0.0114)                 |
| $\theta_1^Q$          |                | 0.0490(0.0046) | 0.0486(0.0027)                | 0.0201( 0.0013)                |
| $\sigma_2$            |                |                |                               | 0.1007(0.0117)                 |
| $\beta_2$             |                |                |                               | 0.0912(0.0167)                 |
| $\kappa_2^Q$          |                |                |                               | 0.5481(0.0077)                 |
| $\theta_2^Q$          |                |                |                               | 0.0016(0.0003)                 |
| $\hat{\lambda}^{(1)}$ |                |                | 0.1870(0.0077)                | 0.1437(0.0075)                 |
| $m_x$                 |                |                | 0.0118(0.0024)                | 0.0083(0.0021)                 |
| $\sigma_x$            |                |                | 0.0039(3.4*10 <sup>-4</sup> ) | 0.0021(4.47*10 <sup>-4</sup> ) |
| $\bar{V}_1$           |                | 0.0140(0.0011) | 0.0121(0.0023)                | 0.0097( 0.0015)                |
| $\bar{V}_2$           |                |                |                               | 0.0061(0.0021)                 |
| $\overline{SSRE}$     | 27.31(1.00)    | 7.54(1.91)     | 5.49(0.24)                    | 4.87(0.33)                     |

Table 5..4: Estimates of the HJM specifications

intervals of swaps  $T_S = (1, 2, 3, 4, 5)$ ,  $Cap(t, T_w)^{Obs}$  and  $Swaption_I(t, T_0, T_w)^{Obs}$  denote the observations of  $Cap(t, T_w)$  and  $Swaption_I(t, T_0, T_\tau)$ , at any time-point  $t$  of our sample. The whole sample estimates of the vector of parameters of interest  $\Theta$  can be obtained by taking the average of the values of  $\Theta(t)$ , over the different points of the sample,  $t$ . Averaging  $V_m(t)$ , across  $t$ , will give us the average estimates of volatilities, over the whole sample, denoted  $\bar{V}_m$ .

To avoid any problems of overparameterization (overfitting), in estimation we consider parsimonious alternative specifications of the model (5..1)-(5..2). The most general specification considers that there is only one jump process, i.e.  $L = 1$ , and two independent volatility processes  $V_m(t)$ , i.e.  $M = 2$ . This specification implies that we need to estimate 13 parameters, thus leaving us 29 degrees of freedom.

The in-sample estimates of the vector of parameters  $\Theta$  together with their standard deviations, reported in parentheses, can be found in Table 5.4. The table reports estimation results for the following alternative four specifications of the model: (i) the standard HJM model, which assumes volatility deterministic and zero jumps,

(ii) the HJM model with one stochastic volatility (SV) process, denoted as HJM-SV(1)  
 (iii) the HJM model with one SV and one jump processes, denoted as HJM-SVJ(1,1)  
 and (iv) the most general specification, which assumes two SV and one jump processes,  
 denoted HJM-SVJ(2,1). To evaluate the pricing error performance, the table reports  
 the average, over the whole sample, estimate of the  $SSRE(t)$ , denoted as  $\overline{SSRE}$ ,  
 with its standard deviation in parentheses.

The general conclusion that can be drawn from the results of the table is  
 that the specifications of the HJM model which allow for stochastic volatility or  
 stochastic volatility and jumps constitute a very important improvement upon the  
 standard HJM specification. They result in a reduction of the pricing errors by around  
 20%. Note that the extension of the model which allow, in addition for stochastic  
 volatility, for a jump constitutes the superior specification, but the benefits of this  
 specification in terms of pricing errors do not seem to be very important (see the  
 standard deviations of  $SSRE\%$  in parentheses).

The parameters estimates indicate that there is high persistency in the under-  
 lying stochastic volatility processes, for all the alternative specifications of the HJM  
 model considered. The estimates of the risk neutral intensity of the jump parameter,  
 $\lambda_1$ , is quite high ( $\lambda_1 = 18.70\%$ ) implying a high risk aversion attitude of the investors  
 participating into the interest rates derivatives markets. This is very close to that for  
 options data on stocks [see Pan (2002)].

Apart from examining whether can reduce the in-sample pricing errors, another  
 way to assess the pricing performance of the alternative specifications of the HJM  
 model, considered here, is to investigate their out-of-sample pricing error performance.  
 Therefore, in Table 5.5 I report out-of-sample average estimates of the  $SSRE(t)$  mea-  
 sure, over the whole sample, separately for each maturity<sup>3</sup>, denoted  $\overline{SSRE}^{(T-t)}$ , and  
 for the whole set of maturities, denoted as  $\overline{SSRE}$ . These measures are calculated

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<sup>3</sup> For caps,  $T_\omega$  is defined in equation (5.11). For swaptions,  $T_0$  the date that enters the swap and  
 $T_\omega$  is the maturity date of the swap, see equation (5.17).

| Maturity (yrs)                | HJM   | HJM-SV(1)                 | HJM-SVJ(1,1) | HJM-SV(2,1) |
|-------------------------------|-------|---------------------------|--------------|-------------|
|                               |       | $\overline{SSRE}^{(T-t)}$ |              |             |
|                               |       | Caps                      |              |             |
| $T_\omega = 1$                | 17.21 | 21.58                     | 12.56        | 9.67        |
| 3                             | 9.81  | 2.83                      | 2.22         | 2.03        |
| 5                             | 7.72  | 2.64                      | 2.53         | 2.25        |
| 10                            | 14.40 | 6.79                      | 5.75         | 2.39        |
|                               |       | Swaptions                 |              |             |
| $(T_0, T_\omega) = (0.25, 1)$ | 34.04 | 3.17                      | 3.29         | 4.98        |
| (0.25, 3)                     | 29.65 | 4.70                      | 4.58         | 3.67        |
| (0.25, 5)                     | 29.78 | 5.57                      | 6.47         | 5.57        |
| (1, 1)                        | 35.20 | 2.91                      | 3.84         | 5.13        |
| (1, 3)                        | 31.69 | 4.42                      | 2.59         | 2.47        |
| (1, 5)                        | 32.91 | 6.17                      | 4.62         | 5.50        |
| (3, 1)                        | 39.47 | 6.42                      | 6.59         | 5.02        |
| (3, 3)                        | 32.22 | 9.26                      | 2.57         | 2.62        |
| (3, 5)                        | 32.99 | 9.47                      | 5.18         | 5.25        |
| (5, 1)                        | 41.56 | 6.93                      | 6.73         | 8.42        |
| (5, 3)                        | 36.34 | 10.31                     | 3.01         | 4.25        |
| (5, 5)                        | 35.87 | 10.87                     | 4.99         | 5.41        |
|                               |       | $\overline{SSRE}$         |              |             |
|                               | 33.31 | 8.47                      | 6.88         | 6.59        |

Table 5.5: Out-of-sample estimates

based on a recursive estimation procedure. According to this, the estimates of parameters for week  $t$  of the sample are used to calculate the time  $t$  predictions for the caps' and swaptions' prices for week  $t + 1$ . Notes: .

The results of Table 5.5 indicate that the overall out-of-sample performance of the alternative specifications of the HJM is analogous to the in-sample-one. There are only a small differences in the values of  $\overline{SSRE}$  which can contributed to the prediction errors. Among the alternative specifications considered, the estimates of the  $\overline{SSRE}^{(T-t)}$  indicate that the HJM-SVJ(1,1) seems to better price the caps and swaptions at both the short and long ends of the maturity structure.

Summing up, the results of this section indicate that the specification of the HJM model which allows for stochastic volatility and jump processes seems to ad-

equately prices the caps and swaptions interest rates derivatives, especially at the short and long ends of the maturity structure.

## 5.5 Conclusions

The aim of this chapter has been to introduce extensions of the standard HJM model for pricing caps and swaptions interest rates derivatives which consider for stochastic volatility and marked point, such as jumps, processes. The chapter suggests analytical formulas for calculating caps' and swaptions' prices, and evaluates the pricing performance of various alternative extensions of the HJM model based on US data on future contracts, swaps, swaptions and caps.

To derive the analytic formulas of the caps and swaptions prices, I decomposed the caps' and swaptions' prices to a portfolio of Arrow-Debreu state type of securities. For the caps, I suggested a new approach of calculating these prices based on the joint characteristic function of the underlying stochastic processes driving the forward rates in the HJM model. For the swaptions, I suggested a numerical approach along the lines of Singleton and Umantsev (2002).

My empirical evaluation of the alternative specifications of the HJM model that I suggested indicate that the extension of the model to allow for stochastic volatility and jumps consider an important improvement upon its standard specification in spanning adequately caps' and swaptions' prices across different maturity intervals. My empirical finding suggest that the inclusion, in additional to the stochastic volatility, of a jump process into the model can significantly reduce the pricing errors of the model. The estimates of the risk neutral intensity coefficients of the jump are found to be analogous to those found for stock options data.

## CHAPTER 6.

### CONCLUSION

In this thesis, we provide the analytic and computationally attractive closed form solutions for the prices of American stock and interest rate derivatives under more complicated asset pricing models, stochastic volatility or jump models. To derive these solutions, we unbundle American options and interest rate derivatives into Arrow-Debreu type of securities and exploit the recent developments in the asset pricing theory which indicate that the Arrow-Debreu type of securities can be spanned by the characteristic functions. The closed form solutions of these CFs are derived in the thesis and enable us to evaluate the option prices very quickly and efficiently.

For pricing American stock options under stochastic volatility, we introduced a new numerical method which is based on an approximation of the optimal exercise boundary by Chebyshev polynomial function with respect to the maturity interval and a log-linear function with respect to volatility. The numerical results presented in chapter 3 show that our method can approximate American call option prices very quickly and efficiently both under stochastic volatility and constant volatility.

For pricing American bond options under the class of Affine term structure models, we introduced a new numerical method which is based on an approximation of the optimal exercise boundary by multi-region and multi-piece approximating function (hyperplane). For evaluating the performance of our method, we carry out some experiments. The numerical results of these experiments presented in chapter 4 indicate that our method can price American bond options very accurately and quickly.

For pricing caps and swaptions interest rate derivatives, we introduce an extension of the standard Heath-Jarrow-Morton (HJM) model which allows for stochastic volatility and marked point jumps processes. In chapter 5, we provide the analytical formulas for calculating caps' and swaptions' prices and evaluate the pricing performance of various alternative extensions of HJM model based on US data on future

contracts, swaps, swaptions and caps. Our empirical evaluation indicates that the inclusion of unspanned stochastic volatility and jumps can significantly reduce the errors of jointly pricing caps and swaptions. The estimates of the risk neutral intensity coefficients of jumps are found to be analogous to those found for stock options data.



APPENDIX A.  
PROOFS OF CHAPTER 3

In this appendix, we proof the results given in the chapter 3.

Appendix A.1:Proof of Theorem 2 Proof. To prove the theorem we follow similar steps with Myneni (1992), who decomposed the optimal problem (3.4) for an American put option under the assumptions of the lognormal model in terms of the exercise (stopping) boundary. To this end, notice that (3.4) implies

$$\begin{aligned}
 C_A(P_t, V_t, T-t) &= \sup_{\tau \in \mathcal{S}_{[t, T]}} E_t^Q \left( e^{-\int_t^\tau r ds} (P_\tau - K)_+ \right) & (1.01) \\
 &= E_t^Q \left( e^{-\int_t^{\tau_t^*} r ds} (P_{\tau_t^*} - K)_+ \right) \\
 &= E_t^Q \left( e^{-\int_t^T r ds} (P_T - K)_+ \right) \\
 &\quad + E_t^Q \left( e^{-\int_t^{\tau_t^*} r ds} (P_{\tau_t^*} - K)_+ - e^{-\int_t^T r ds} (P_T - K)_+ \right) \\
 &= E_t^Q \left( e^{-\int_t^T r ds} (P_T - K)_+ \right) \\
 &\quad - E_t^Q \left( \int_{\tau_t^*}^T d \left( e^{-\int_t^s r du} (P_s - K)_+ \right) \right),
 \end{aligned}$$

where  $d(\cdot)$  is the differential operator. Note that first term in the last equation represents the value European option, while the second term constitutes the value of the early exercise premium. Using differentiation rules, the integral term of the early exercise premium term can be written as

$$\begin{aligned}
 &\int_{\tau_t^*}^T d \left( e^{-\int_t^s r du} (P_s - K)_+ \right) \\
 &= \int_{\tau_t^*}^T e^{-\int_t^s r du} d(P_s - K)_+ - \int_{\tau_t^*}^T r e^{-\int_t^s r du} (P_s - K)_+ ds. & (1.02)
 \end{aligned}$$

Using Tanaka's formula and *local time* for Brownian motion at the point  $K$ , the differential  $d(P_\tau - K)_+$  can be written as

$$d(P_s - K)_+ = dL_s^P(K) + I_{(P_s \geq K)} dP_s, \quad (1.03)$$

where  $L_s^P(K)$  is the *local time* for Brownian motion at the value  $K$  of the stock price  $P_s$  and  $I_A$  is the indicator function of the set  $A$ , defined in Theorem 2. Using (1.03) and applying Ito's Lemma, equation (1.02) can be decomposed as follows

$$\begin{aligned} & \int_{\tau_i^*}^T d\left(e^{-\int_t^s r du} (P_s - K)_+\right) \\ &= \int_{\tau_i^*}^T e^{-\int_t^s r du} dL_s^P(K) + \int_{\tau_i^*}^T e^{-\int_t^s r du} I_{(P_s \geq K)} dP_s - \int_{\tau_i^*}^T r e^{-\int_t^s r du} (P_s - K)_+ ds \\ &= \int_{\tau_i^*}^T e^{-\int_t^s r du} dL_s^P(K) + \int_{\tau_i^*}^T e^{-\int_t^s r du} I_{(P_s \geq K)} (r - \delta) P_s ds \\ & \quad + \int_{\tau_i^*}^T e^{-\int_t^s r du} I_{(P_s \geq K)} \sqrt{V_s} P_s dW_{1,s} - \int_{\tau_i^*}^T r e^{-\int_t^s r du} (P_s - K)_+ ds \\ &= \int_{\tau_i^*}^T e^{-\int_t^s r du} dL_s^P(K) + \int_{\tau_i^*}^T e^{-\int_t^s r du} I_{(P_s \geq K)} \sqrt{V_s} P_s dW_{1,s} \\ & \quad + \int_{\tau_i^*}^T e^{-\int_t^s r du} I_{(P_s \geq K)} (rK - \delta P_s) ds. \end{aligned} \quad (1.04)$$

As shown by Mynenni(1992), the expectation of  $\int_{\tau_i^*}^T e^{-\int_t^s r du} dL_s^P(K)$  is zero. Therefore, taking the conditional expectation of the last equation with respect the measure  $Q$

yields

$$E_t^Q \left( \int_{\tau_i^*}^T d \left( e^{-\int_t^s r du} (P_s - K)_+ \right) \right) = E_t^Q \left( \int_{\tau_i^*}^T e^{-\int_t^s r du} I_{(P_s \geq K)} (rK - \delta P_s) du \right), \quad (1.05)$$

since  $E_t^Q (dW_{1,s}) = 0$ .

Notice that, by the un-connected property of the optimal exercise boundary [see Broadie *et al* (2000), for a proof], the optimal exercise time  $\tau^*$  [see equation (3.3)] can be defined as

$$\tau^* = \inf \{ \tau \in [t, T] : P_s \geq B(V_s, s) \}, \quad \forall s \in [t, T]. \quad (1.06)$$

Using the above definition of  $\tau^*$ , equation (1.05) can be written as

$$\begin{aligned} & E_t^Q \left( \int_{\tau_i^*}^T d \left( e^{-\int_t^s r ds} (P_\tau - K)_+ \right) \right) \\ &= E_t^Q \left( \int_t^T e^{-\int_t^s r du} I_{(P_s \geq K)} I_{(P_s \geq B(V_s, s))} (rK - \delta P_s) du \right). \end{aligned} \quad (1.07)$$

By the property of the exercise boundary that  $B(V_s, s) > K, \forall s \in [t, T]$ , the last equation implies

$$\begin{aligned} & E_t^Q \left( \int_{\tau_i^*}^T d \left( e^{-\int_t^s r ds} (P_\tau - K)_+ \right) \right) \\ &= E_t^Q \left( \int_t^T e^{-\int_t^s r du} I_{(P_s \geq B(V_s, s))} (rK - \delta P_s) ds \right). \end{aligned} \quad (1.08)$$

Substituting equation (1.08) into (1.01) proves the result of equation (3.5), given by Theorem 1. The optimal exercise boundary recursive equation (3.6) can be derived

based on (3.5) and using the following arbitrage condition:  $C_A(B(V_h, h), V_h, T - h) = B(V_h, h) - K, \forall h \in [t, T]$ . ■

Appendix A.2: Proof of Theorem 3 In this appendix, we prove Theorem 3 of the thesis. To this end, we first derive the joint conditional characteristic function (CF) of the logarithm of the stock price  $\ln P_s$  adjusted by the term  $(r - \delta)(s - h), \forall s \in [t, T]$ , and the variance  $V_s$  conditional on the past values of  $\ln P_h$  and  $V_h$ , for  $s \geq h \in [t, T]$ . This is given in the following Lemma.

**Lemma 9** *Let the SV model, defined by processes (3.1) and (3.2), hold. Define  $Y_{s,h} = \ln P_s - (r - \delta)(s - h), \forall s \geq h \in [t, T]$ . Then, the joint characteristic function of  $Y_s$  and  $V_s$  conditional on the values of  $Y_{h,h} = \ln P_h$  and  $V_h$  is given by*

$$F(\phi_Y, \phi_V, s - h | Y_{h,h}, V_h) = e^{g_0(\phi_Y, \phi_V, s - h) + g_1(\phi_Y, \phi_V, s - h)Y_{h,h} + g_2(\phi_Y, \phi_V, s - h)V_h},$$

where

$$g_0(\phi_Y, \phi_V, s - h) = -\frac{k\theta}{\sigma^2} \left\{ (D + B)(s - h) + 2 \ln \left[ 1 - \frac{D + B + \sigma^2 i \phi_V}{2D} (1 - e^{-D(s-h)}) \right] \right\}$$

$$g_1(\phi_Y, \phi_V, s - h) = i\phi_Y$$

$$g_2(\phi_Y, \phi_V, s - h) = \frac{C(1 - e^{-D(s-h)}) + i\phi_V [2D - (D - B)(1 - e^{-D(s-h)})]}{2D - (D + B)(1 - e^{-D(s-h)}) - \phi_V \sigma^2 (1 - e^{-D(s-h)})}$$

and

$$A = \frac{1}{2}\sigma^2, B = \rho\sigma i\phi_Y - k, C = -\frac{1}{2}\phi_Y^2 - \frac{1}{2}i\phi_Y \text{ and } D = \sqrt{B^2 - 4AC}.$$

Proof. By Ito's Lemma, we can write

$$dY_{s,h} = -\frac{1}{2}V_s ds + \sqrt{V_s} dW_{1,s}. \quad (1.09)$$

To derive a closed form solution of the joint CF  $F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h)$ , consider the following general affine solution:

$$F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h) = e^{g_0(\phi_Y, \phi_V, s-h) + g_1(\phi_Y, \phi_V, s-h)Y_{h,h} + g_2(\phi_Y, \phi_V, s-h)V_h}. \quad (1.010)$$

The coefficients of the CF  $g_0(\phi_Y, \phi_V, s-h)$ ,  $g_1(\phi_Y, \phi_V, s-h)$  and  $g_2(\phi_Y, \phi_V, s-h)$  can be derived by noticing that  $F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h)$  should satisfy the following partial differential equation (PDE)

$$\begin{aligned} & \frac{\partial F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h)}{\partial h} + \frac{1}{2}V_h \frac{\partial^2 F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h)}{\partial Y_{h,h}^2} \\ & + \rho\sigma V_h \frac{\partial^2 F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h)}{\partial Y_{h,h} \partial V_h} + \frac{1}{2}\sigma^2 V_h \frac{\partial^2 F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h)}{\partial V_h^2} \\ & - \frac{1}{2}V_h \frac{\partial F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h)}{\partial Y_{h,h}} + k(\theta - V_h) \frac{\partial F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h)}{\partial V_h} \\ = & 0 \end{aligned} \quad (1.011)$$

Substituting (1.010) into (1.011) yields

$$\begin{aligned} & V_h \left[ \frac{\partial g_2(\phi_Y, \phi_V, s-h)}{\partial h} + \frac{1}{2}g_1^2(\phi_Y, \phi_V, s-h) \right. \\ & + \rho\sigma g_1(\phi_Y, \phi_V, s-h)g_2(\phi_Y, \phi_V, s-h) \\ & + \left. \frac{1}{2}\sigma^2 g_2^2(\phi_Y, \phi_V, s-h) - \frac{1}{2}g_1(\phi_Y, \phi_V, s-h) - kg_2(\phi_Y, \phi_V, s-h) \right] \\ & Y_{h,h} \frac{\partial g_1(\phi_Y, \phi_V, s-h)}{\partial h} + \left[ \frac{\partial g_0(\phi_Y, \phi_V, s-h)}{\partial h} + k\theta g_2(\phi_Y, \phi_V, s-h) \right] \\ = & 0 \end{aligned} \quad (1.012)$$

From the PDE (1.012), we can see that the coefficients  $g_0(\phi_Y, \phi_V, s-h)$ ,  $g_1(\phi_Y, \phi_V, s-h)$  and  $g_2(\phi_Y, \phi_V, s-h)$  can be derived by solving out the following three ordinary differential equations (ODE), implied by (1.012),

$$\frac{\partial g_1(\phi_Y, \phi_V, s-h)}{\partial h} = 0, \quad (1.013)$$

$$\begin{aligned} & \frac{\partial g_2(\phi_Y, \phi_V, s-h)}{\partial h} \\ = & -\frac{1}{2}\sigma^2 g_2^2(\phi_Y, \phi_V, s-h) - g_2(\phi_Y, \phi_V, s-h) (\rho\sigma g_1(\phi_Y, \phi_V, s-h) - k) \\ & - \left( \frac{1}{2}g_1^2(\phi_Y, \phi_V, s-h) - \frac{1}{2}g_1(\phi_Y, \phi_V, s-h) \right), \end{aligned} \quad (1.014)$$

and

$$\frac{\partial g_0(\phi_Y, \phi_V, s-h)}{\partial s} = -k\theta g_2(\phi_Y, \phi_V, s-h), \quad (1.015)$$

subject to the following boundary conditions  $g_0(\phi_Y, \phi_V, 0) = 0$ ,  $g_1(\phi_Y, \phi_V, 0) = i\phi_Y$ , and  $g_2(\phi_Y, \phi_V, 0) = i\phi_V$ .

Solving out the ODE (1.013) for  $g_1(\phi_Y, \phi_V, s-h)$  yields

$$g_1(\phi_Y, \phi_V, s-h) = i\phi_Y. \quad (1.016)$$

To derive the coefficient  $g_2(\phi_Y, \phi_V, s-h)$ , substitute (1.016) into (1.014). This gives

$$\begin{aligned} & \frac{\partial g_2(\phi_Y, \phi_V, s-h)}{\partial h} \\ = & -\frac{1}{2}\sigma^2 g_2^2(\phi_Y, \phi_V, s-h) - g_2(\phi_Y, \phi_V, s-h) (\rho\sigma i\phi_Y - k) \\ & - \left( -\frac{1}{2}\phi_Y^2 - \frac{1}{2}i\phi_Y \right) \\ = & -\frac{1}{2}\sigma^2 (g_2(\phi_Y, \phi_V, s-h) - x_1) (g_2(\phi_Y, \phi_V, s-h) - x_2), \end{aligned} \quad (1.017)$$

where  $x_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$ ,  $x_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$ ,  $A = \frac{1}{2}\sigma^2$ ,  $B = \rho\sigma i\phi_X - k$ ,  $C = -\frac{1}{2}\phi_Y^2 - \frac{1}{2}i\phi_Y$  and  $D = \sqrt{B^2 - 4AC}$ . Rearranging terms in the ODE (1.017) and integrating both sides of the resulting equation yields

$$\frac{1}{D} \int \left( \frac{1}{g_2(\phi_Y, \phi_V, s-h) - x_1} - \frac{1}{g_2(\phi_Y, \phi_V, s-h) - x_2} \right) dg_2(\phi_Y, \phi_V, s-h) = \int dh$$

.Using the boundary conditions on the coefficients of the CF, the last equation implies that the closed form solution for  $g_2(\phi_Y, \phi_V, s-h)$  is given by

$$g_2(\phi_Y, \phi_V, s-h) = \frac{C(1 - e^{-D(s-h)}) + i\phi_V [2D - (D - B)(1 - e^{-D(s-h)})]}{2D - (D + B)(1 - e^{-D(s-h)}) - \phi_V\sigma^2(1 - e^{-D(s-h)})}. \quad (1.018)$$

Substituting the closed form solutions of the coefficients  $g_1(\phi_Y, \phi_V, s-h)$  and  $g_2(\phi_Y, \phi_V, s-h)$ , given by equations (1.016) and (1.018), respectively, into the ODE (1.015) and integrating gives the closed form solution for the coefficient  $g_0(\phi_Y, \phi_V, s-h)$ :

$$g_0(\phi_Y, \phi_V, s-h) = -\frac{k\theta}{\sigma^2} \left\{ (D + B)(s-h) + 2 \ln \left[ 1 - \frac{D + B + \sigma^2 i\phi_V}{2D} (1 - e^{-D(s-h)}) \right] \right\}.$$

■

Having derived the closed form solution of the CF  $F(\phi_Y, \phi_V, s-h | Y_{h,h}, V_h)$ , we next prove Theorem 3.

**Proof.** (Proof of Theorem 3). To prove the theorem, we need to derive an integral representation of the early exercise premium

$$E_h^Q \left[ \int_h^T e^{-r(s-h)} (\delta P_s - rK) I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} ds \mid B(V_h, h), V_h \right], \quad (1.019)$$

defined in equation (3.6). This can be done as follows.

Using the law of iterated expectations, write equation (1.019) as

$$\int_h^T E_h^Q \left[ e^{-r(s-h)} (\delta P_s - rK) I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h \right] ds, \quad (1.020)$$

where  $E_h^Q \left[ e^{-r(s-h)} (\delta P_s - rK) I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h \right]$  represents the present value of the risk neutral continuous payoff of the early exercise at time  $h$ . This can be written as follows

$$\begin{aligned} E_h^Q \left[ e^{-r(s-h)} (\delta P_s - rK) I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h \right] = \\ \delta e^{-r(s-h)} E_h^Q [P_s] E_h^Q \left[ \frac{P_s I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}}}{E_h^Q [P_s]} | B(V_h, h), V_h \right] \\ - rK e^{-r(s-h)} E_h^Q \left[ I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h \right]. \end{aligned} \quad (1.021)$$

Equation (1.021) indicates that the present value of the risk neutral continuous payoff of the early exercise time  $t$  can be unbundled into a portfolio of the Arrow-Debreu type of securities [see Bakshi and Madan (2000)]. The prices of these securities are defined as

$$E_h^Q \left[ \frac{P_s I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}}}{E_h^Q [P_s]} | B(V_h, h), V_h \right] \quad (1.022)$$

and

$$E_h^Q \left[ I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h \right], \quad (1.023)$$

respectively. Below, we derive analytic, integral representations (solutions) of these prices based on the closed form solution of the CF  $F(\phi_Y, \phi_V, s - h | Y_{h,h}, V_h)$ , given by Lemma 3. Substituting these solutions into equation (1.021) will give us an analytic,



integral representation of the optimal exercise premium (1.019). To derive an analytic, integral representation of the security price defined by equation (1.022), write

$E_h^Q \left[ \frac{P_s}{E_h^Q [P_s]} I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h \right]$  as

$$E_h^Q \left[ \frac{P_s}{E_h^Q [P_s]} I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h \right] = \int_{-\infty}^{\infty} dV_s \int_{\log(B(V_s, s)) - (r-\delta)(s-h)}^{\infty} \frac{e^{(r-\delta)(s-h) + Y_{s,h}}}{E_h^Q [P_s]} \pi(Y_{s,h}, V_s | \ln(B(V_h, h)), V_h) dY_{s,h}, \quad (1.024)$$

where  $\pi(Y_{s,h}, V_s | \ln(B(V_h, h)), V_h)$  is the joint probability density function of  $Y_{s,h}$  and  $V_s$  conditional on the values of  $Y_{h,h}$  and  $V_h$ , where  $Y_{s,h}$  and  $Y_{h,h}$  are now considered at the optimal exercise boundary prices, i.e.  $Y_{s,h} = \ln(B(V_s, s)) + (r - \delta)(s - h)$  and  $Y_{h,h} = \ln(B(V_h, h))$ . Denote the marginal characteristic function of  $F(\phi_Y, \phi_V, s - h | \ln(B(V_h, h)), V_h)$  with respect to  $V_h$  as  $F_V(\phi_Y, V_s | \ln(B(V_h, h)), V_h)$ . This function is defined as

$$F_V(\phi_Y, V_s | \ln(B(V_h, h)), V_h) = \int_{-\infty}^{\infty} e^{i\phi_Y Y_{s,h}} \pi(Y_{s,h}, V_s | \ln(B(V_h, h)), V_h) dY_{s,h}.$$

Using the definition of  $F_V(\phi_Y, V_s | \ln(B(V_h, h)), V_h)$ , the exercise boundary relationship (3.8) implies that equation (1.024) can be written in terms of one-dimension

integrals as

$$\begin{aligned}
& E_h^Q \left[ \frac{P_s}{E_h^Q [P_s]} I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h \right] = \\
& \int_{-\infty}^{\infty} dV_s \left\{ \frac{1}{2} \pi_V(V_s | \ln(B(V_h, h)), V_h) + \left( \frac{e^{(r-\delta)(s-h)}}{2\pi E_h^Q [P_s]} \right) \right. \\
& \left. \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\phi_Y [b_0(s) + b_1(s)V_s - (r-\delta)(s-h)]} F_V(\phi_Y - i, V_s | \ln(B(V_h, h)), V_h)}{i\phi_Y} \right) d\phi_Y \right\}, \tag{1.025}
\end{aligned}$$

where  $\pi_V(V_s | \ln(B(V_h, h)), V_h)$  is the marginal density function of joint probability density  $\pi(Y_{s,h}, V_s | \ln(B(V_h, h)), V_h)$  with respect to  $V_h$ , defined as

$$\pi_V(V_s | \ln(B(V_h, h)), V_h) = \int_{-\infty}^{\infty} \frac{P_s}{E_h^Q [P_s]} \pi(Y_{s,h}, V_s | \ln(B(V_h, h)), V_h) dY_{s,h}.$$

Noticing that the CF  $F(\phi_Y, \phi_V, s-h | \ln(B(V_h, h)), V_h)$  and its marginal CF  $F_V(\phi_Y, V_s | \ln(B(V_h, h)), V_h)$  are linked through the following relationship

$$F(\phi_Y, \phi_V, s-h | \ln(B(V_h, h)), V_h) = \int_{-\infty}^{\infty} e^{i\phi_V V_s} F_V(\phi_Y, V_s | \ln(B(V_h, h)), V_h) dV_s,$$

equation (1.025) can be expressed in terms of one-dimension integrals as

$$\begin{aligned}
& E_h^Q \left[ \frac{P_s}{E_h^Q [P_s]} I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h \right] \\
&= \frac{1}{2} + \left( \frac{e^{(r-\delta)(s-h)}}{2\pi E_h^Q [P_s]} \right) \\
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\phi_Y [b_0(s) + b_1(s)V_s - (r-\delta)(s-h)]} F_V(\phi_Y - i, V_s | \ln(B(V_h, h)), V_h)}{i\phi_Y} \right) dV_s d\phi_Y \\
&= \frac{1}{2} + \left( \frac{e^{(r-\delta)(s-h)}}{2\pi E_h^Q [P_s]} \right) \\
& \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\phi_Y [b_0(s) - (r-\delta)(s-h)]} F(\phi_Y - i, -b_1(s)\phi_Y, s - h | \ln(B(V_h, h)), V_h)}{i\phi_Y} \right) d\phi_Y.
\end{aligned} \tag{1.026}$$

Having derived the above integral representation for the risk neutral expectation  $E_h^Q [P_s I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h]$ , the state price defined by equation (1.022) can be calculated once a closed form solution for  $E_h^Q [P_s]$  is derived. This can be done by setting  $\phi_Y = -i$  and  $\phi_V = 0$  in  $F(\phi_Y, \phi_V, s - h | Y_{h,h}, V_h)$ , which yields

$$\begin{aligned}
E_h^Q [P_s] &= e^{(r-\delta)(s-h)} F(-i, 0, s - h | \ln(B(V_h, h)), V_h) \\
&= e^{(r-\delta)(s-h)} B(V_h, h).
\end{aligned} \tag{1.027}$$

Substituting equations (1.026) and (1.027) into (1.022) gives the analytic solution for the state price given by equation (1.022)

$$\begin{aligned}
& \Pi'_1(b_0(s), b_1(s) | B(V_h, h), V_h) \\
\equiv & E_h^Q \left[ \frac{P_s I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}}}{E_h^Q [P_s]} | B(V_h, h), V_h \right] \\
= & \frac{1}{2} + \left( \frac{1}{2\pi B(V_h, h)} \right) \\
& \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\phi_Y [b_0(s) - (r-\delta)(s-h)]} F(\phi_Y - i, -b_1(s)\phi_Y, s-h | \ln(B(V_h, h)), V_h)}{i\phi_Y} \right) d\phi_Y.
\end{aligned} \tag{1.028}$$

Following similar steps to those above, we can derive the following analytic solution for the state price defined by equation (1.023)

$$\begin{aligned}
& \Pi'_2(b_0(s), b_1(s) | B(V_h, h), V_h) \\
\equiv & E_h^Q [I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | B(V_h, h), V_h] = \\
& \frac{1}{2} + \left( \frac{1}{2\pi} \right) \\
& \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\phi_Y [b_0(s) - (r-\delta)(s-h)]} F(\phi_Y, -b_1(s)\phi_Y, s-h | \ln(B(V_h, h)), V_h)}{i\phi_Y} \right) d\phi_Y.
\end{aligned} \tag{1.029}$$

Substituting (1.028) and (1.029) into (1.021) proves the boundary recursive equation (3.10), given by Theorem 2.

The closed form solutions of the Arrow-Debreu security prices which enter into the American call option price evaluation formula (3.9), i.e.  $\Pi_1(b_0(s), b_1(s) | P_t, V_t)$  and  $\Pi_2(b_0(s), b_1(s) | P_t, V_t)$ , can be derived by following similar steps to those above setting  $Y_{s,t}$  and  $Y_{t,t}$  as  $Y_{s,t} = \ln P_s + (r - \delta)(s - t)$  and  $Y_{t,t} = \ln P_t$ . This will give us

the following analytic solutions:

$$\begin{aligned}
& \Pi_1(b_0(s), b_1(s) | P_t, V_t) \\
& \equiv E_t^Q \left[ \frac{P_s I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}}}{E_t^Q [P_s]} | P_t, V_t \right] \\
& = \frac{1}{2} + \left( \frac{1}{2\pi P_h} \right) \\
& \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\phi_Y [b_0(s) - (r-\delta)(s-t)]} F(\phi_Y - i, -b_1(s)\phi_Y, s - t | \ln P_t, V_t)}{i\phi_Y} \right) d\phi_Y
\end{aligned}$$

and

$$\begin{aligned}
& \Pi_2(b_0(s), b_1(s) | P_t, V_t) \\
& \equiv E_t^Q [I_{\{(P_s, V_s): P_s \geq B(V_s, s)\}} | P_t, V_t] \\
& = \frac{1}{2} + \left( \frac{1}{2\pi} \right) \\
& \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\phi_Y [b_0(s) - (r-\delta)(s-t)]} F(\phi_Y, -b_1(s)\phi_Y, s - t | \ln P_h, V_h)}{i\phi_Y} \right) d\phi_Y,
\end{aligned}$$

which are similar in terms of functional form with those for the prices  $\Pi_1'(b_0(s), b_1(s) | B(V_h, h), V_h)$  and  $\Pi_2'(b_0(s), b_1(s) | B(V_h, h), V_h)$ . ■

Appendix A.3: Chebyshev approximation According to the CB method, any continuous function  $b(x)$ , where  $x \in [-1, 1]$ , can be approximated by a linear combination of  $\nu$ -Chebyshev polynomials, denoted  $w_j(x)$ , as follows

$$\tilde{b}(x) = \sum_{j=1}^{\nu} q_j w_j(x), \tag{1.030}$$

where  $w_j(x)$  denotes the  $i^{\text{th}}$  Chebyshev polynomial, defined as

$$w_j(x) = \cos(j \arccos(x)), \tag{1.031}$$

with  $w_j(x)$  satisfying the recurrence

$$w_{j+1}(x) = 2xw_j(x) - w_{j-1}(x), \quad (1.032)$$

with  $w_0 = 1$  and  $w_1 = x$ .

The Chebyshev polynomials satisfy the Weierstrass theorem and meet the minmax criterion. According to this criterion, the Chebyshev approximating function, denoted  $\tilde{b}(x)$ , is one that equals the true function  $b(x)$  at the set of  $\nu$  zeros values of  $w_j(x)$ , taken for  $x = \cos(\pi(j - 0.5)/\nu)$ ,  $j = 1, 2, \dots, \nu$ . The  $\nu$  zeros values of  $w_j(x)$  imply a system of the  $\nu$  equations with  $\nu$  unknown coefficients  $q_j$ . Solving out this system with respect to  $q_j$  can determine the approximating function.

Although the Chebyshev approximating function  $\tilde{b}(x)$  is defined in the finite interval  $[-1, 1]$ , we can approximate other function  $\tilde{b}(h)$ , where  $h$  is defined in the interval  $[t, T]$ , by rescaling the values of  $x$  to  $h$  as  $h = \frac{1}{2}((T - t)x + T + t)$ . This implies that

$$\tilde{b}(h) = \tilde{b}\left(x = \frac{2h - T - t}{T - t}\right). \quad (1.033)$$

Substituting  $x = \frac{2h - T - t}{T - t}$  into equation (1.030), the new function  $\tilde{b}(h)$  can be written as

$$\tilde{b}(h) = \sum_{i=0}^{\nu} c_i h^i.$$

## APPENDIX B.

### PROOFS OF CHAPTER 4

In this appendix, we prove the results given in the chapter 4.

Appendix B.1: The decomposition of the American put bond option price Proof.

Following the Carr, Jarrow and Myneni (1992), to prove the theorem, we consider the price  $P_A^*(X_s, T_1; T_2) = e^{-\int_t^s r_u du} P_A(X_s, T_1; T_2)$ , instead of  $P_A(X_s, T_1; T_2)$ . Applying Ito's Lemma,  $P_A^*(X_s, T_1; T_2)$  can be written as

$$\begin{aligned} & dP_A^*(X_s, T_1; T_2) \\ &= P_A(X_s, T_1; T_2) de^{-\int_t^s r_u du} + e^{-\int_t^s r_u du} dP_A(X_s, T_1; T_2) \\ &= -r_s e^{-\int_t^s r_u du} P_A(X_s, T_1; T_2) ds \\ &\quad + e^{-\int_t^s r_u du} \left( \mathcal{L}P_A(X_s, T_1; T_2) ds + \left[ \frac{\partial P_A(X_s, T_1; T_2)}{\partial X_s} \right]' \sigma \sqrt{R_s} dW_s^Q \right) \end{aligned}$$

Integrating the above stochastic differential equation (SDE) over the interval  $[t, T_1]$  yields

$$\begin{aligned} & P_A^*(X_{T_1}, T_1; T_2) - P_A^*(X_t, T_1; T_2) \\ &= \int_t^{T_1} e^{-\int_t^s r_u du} [\mathcal{L}P_A(X_s, T_1; T_2) - r_s P_A(X_s, T_1; T_2)] ds \\ &\quad + \int_t^{T_1} e^{-\int_t^s r_u du} \left[ \frac{\partial P_A(X_s, T_1; T_2)}{\partial X_s} \right]' \sigma \sqrt{R_s} dW_s^Q \\ &= \int_t^{T_1} e^{-\int_t^s r_u du} [\mathcal{L}P_A(X_s, T_1; T_2) - r_s P_A(X_s, T_1; T_2)] \Lambda_{\eta_s} ds \\ &\quad + \int_t^{T_1} e^{-\int_t^s r_u du} [\mathcal{L}P_A(X_s, T_1; T_2) - r_s P_A(X_s, T_1; T_2)] \Lambda_{\bar{\eta}_s} ds \\ &\quad + \int_t^{T_1} e^{-\int_t^s r_u du} \left[ \frac{\partial P_A(X_s, T_1; T_2)}{\partial X_s} \right]' \sigma \sqrt{R_s} dW_s^Q, \end{aligned} \tag{2.01}$$

where  $\Lambda_{\eta_s}$  and  $\Lambda_{\bar{\eta}_s}$  are appropriately index functions indicating that  $X_s$  lies in the continuation and exercise regions, respectively.

Since in the continuation region,  $\eta_s$ , the American option is alive, the option price satisfies the PDE (4.9)

$$\mathcal{L}P_A(X_s, T_1, T_2) = r_s P_A(X_s, T_1; T_2). \quad (2.02)$$

In the exercise (stopping) region,  $\bar{\eta}_s$ , we have

$$P_A(X_s, T_1, T_2) = K - B(X_s, T_2 - s) \quad (2.03)$$

Substituting (2.02) and (2.03) into (2.01) yields

$$\begin{aligned} & P_A^*(X_{T_1}, T_1; T_2) - P_A^*(X_t, T_1; T_2) \\ &= \int_t^{T_1} e^{-\int_t^s r_u du} \{-r_s K - [\mathcal{L}B(X_s, T_2 - s) - r_s B(X_s, T_2 - s)]\} \Lambda_{\bar{\eta}_s} ds \\ & \quad + \int_t^{T_1} e^{-\int_t^s r_u du} \left[ \frac{\partial P_A(X_s, T_1; T_2)}{\partial X_s} \right]' \sigma \sqrt{R_s} dW_s^Q \\ &= - \int_t^{T_1} e^{-\int_t^s r_u du} r_s K \Lambda_{\bar{\eta}_s} ds \\ & \quad + \int_t^{T_1} e^{-\int_t^s r_u du} \left[ \frac{\partial P_A(X_s, T_1; T_2)}{\partial X_s} \right]' \sigma \sqrt{R_s} dW_s^Q \end{aligned} \quad (2.04)$$

Taking expectations of both sides of equation (2.04) gives

$$\begin{aligned} & E_t^Q \left\{ e^{-\int_t^{T_1} r_u du} [K - B(X_{T_1}, T_2 - T_1)] \right\} - P_A(X_t, T_1; T_2) \\ &= P_E(X_t, T_1; T_2) - P_A(X_t, T_1; T_2) \\ &= -E_t^Q \left( \int_t^{T_1} e^{-\int_t^s r_u du} r_s K \Lambda_{\bar{\eta}_s} ds \right). \end{aligned}$$



Rearranging terms in the last equation yields

$$P_A(X_t, T_1; T_2) = P_E(X_t, T_1; T_2) + E_t^Q \left( \int_t^{T_1} e^{-\int_t^s r_u du} r_s K \Lambda_{\bar{\eta}_s} ds \right).$$

From the last equation, we can derive equation (4.10) of the theorem by noticing that the exercise region can be expressed in terms of the state variables as  $\bar{\eta}_s = \{X_{N,s} : X_{N,s} \geq G(Y_s, s)\}$ . The recursive equation for the optimal exercise boundary, given by equation (4.11) of the theorem, can be derived by using the arbitrage condition  $K - B(Y_s, G(Y_s)', T_2 - s) = P_A(Y_s, T_1; T_2)$ , holding for any time  $s \in [t, T]$ .

■

Appendix B.2: Integral representation of the early exercise premium In this part of the appendix, we present the proof of Theorem 5. To this end, we first present the joint CF and its extended transform of the state variables for the affine term structure model [see Duffie, Pan and Singleton (2000)].

B.2.1 The Extended Transform and CF of the affine term structure model The extended transform of the CF of the state variables of the affine term structure model [see equation (4.1)], is defined as

$$\Phi(X_t, T_2 - t; \phi, \varphi) = E_t \left( \exp\left(\int_t^{T_2} r_s ds\right) \varphi X_{T_2} e^{\phi X_{T_2}} \right),$$

while the CF is defined as

$$\Gamma(X_t, T_2 - t; \phi) = E_t \left( \exp\left(\int_t^{T_2} r_s ds\right) e^{\phi X_{T_2}} \right),$$

where  $\phi = (\phi_1, \phi_2, \dots, \phi_N)'$  and  $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_N)'$  are  $N$ -dimension vectors of constant parameters. Duffie, Pan and Singleton (2000) show that the solution of the

extended transform of the CF is given by

$$\begin{aligned}
& \Phi(X_t, T_2 - t; \phi, \varphi) \\
&= \left( \zeta(T_2 - t; \phi, \varphi) + \sum_{n=1}^N \xi_n(T_2 - t; \phi, \varphi) X_{n,t} \right) \Gamma(X_t, T_2 - t; \phi) \\
&= \left( \zeta(T_2 - t; \phi, \varphi) + \sum_{n=1}^N \xi_n(T_2 - t; \phi, \varphi) X_{n,t} \right) \exp(\mathcal{D}(T_2 - t; \phi)) \\
&+ \sum_{n=1}^N \mathcal{E}_n(T_2 - t; \phi) X_{n,t},
\end{aligned}$$

where  $\zeta(T_2 - t; \phi, \varphi)$  and  $\xi_n(T_2 - t; \phi, \varphi)$  should satisfy the following ODEs:

$$\begin{aligned}
\frac{\partial \mathcal{A}(T_2 - t)}{\partial (T_2 - t)} &= -\alpha_0 + \sum_{n_1=1}^N \sum_{n_2=1}^N \kappa_{n_1 n_2} \theta_{n_2} \mathcal{C}_{n_1}(T_2 - t) \\
&+ \frac{1}{2} \sum_{n_1=1}^N \sum_{n_2=1}^N \left( \sum_{i=1}^N \sigma_{n_1 i} \sigma_{n_2 i} \rho_i \right) \mathcal{C}_{n_1}(T_2 - t) \mathcal{C}_{n_2}(T_2 - t), \quad \text{and} \quad (2.05)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{C}_n(T_2 - t)}{\partial (T_2 - t)} &= -\alpha_n - \sum_{n_1=1}^N \sum_{n_2=1}^N \kappa_{n_1 n_2} \mathcal{C}_{n_1}(T_2 - t) \\
&+ \frac{1}{2} \sum_{n_1=1}^N \sum_{n_2=1}^N \left( \sum_{i=1}^N \sigma_{n_1 i} \sigma_{n_2 i} \beta_i \right) \mathcal{C}_{n_1}(T_2 - t) \mathcal{C}_{n_2}(T_2 - t), \quad (2.06)
\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial \zeta (T_2 - t; \phi, \varphi)}{\partial (T_2 - t)} &= \sum_{n_1=1}^N \sum_{n_2=1}^N \kappa_{n_1 n_2} \theta_{n_2} \xi_{n_1} (T_2 - t; \phi, \varphi) \\
&+ \frac{1}{2} \sum_{n_1=1}^N \sum_{n_2=1}^N \left( \sum_{i=1}^N \sigma_{n_1 i} \sigma_{n_2 i} \rho_i \right) \mathcal{E}_{n_1} (T_2 - t; \phi) \xi_{n_2} (T_2 - t; \phi, \varphi) \\
\frac{\partial \xi_n (T_2 - t; \phi, \varphi)}{\partial (T_2 - t)} &= - \sum_{n_1=1}^N \sum_{n_2=1}^N \kappa_{n_1 n_2} \xi_{n_1} (T_2 - t; \phi, \varphi) \\
&+ \frac{1}{2} \sum_{n_1=1}^N \sum_{n_2=1}^N \left( \sum_{i=1}^N \sigma_{n_1 i} \sigma_{n_2 i} \beta_i \right) \mathcal{E}_{n_1} (T_2 - t; \phi) \xi_{n_2} (T_2 - t; \phi, \varphi)
\end{aligned}$$

with the initial conditions  $\zeta (\tau; \varphi) = 0$  and  $\xi_i (\tau; \varphi) = \varphi_i, i = 1, 2, \dots, N$ . For the CF,  $\mathcal{D} (T_2 - t; \phi)$  and  $\mathcal{E}_i (\tau; \phi)$  must satisfy the following ODE

$$\begin{aligned}
\frac{\partial \mathcal{D} (T_2 - t; \phi)}{\partial (T_2 - t)} &= \alpha_0 - \sum_{n_1=1}^N \sum_{n_2=1}^N \kappa_{n_1 n_2} \theta_{n_2} \mathcal{E}_{n_1} (T_2 - t; \phi) \\
&+ \frac{1}{2} \sum_{n_1=1}^N \sum_{n_2=1}^N \left( \sum_{i=1}^N \sigma_{n_1 i} \sigma_{n_2 i} \rho_i \right) \mathcal{E}_{n_1} (T_2 - t; \phi) \mathcal{E}_{n_2} (T_2 - t; \phi), \quad (2.07)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \mathcal{E}_n (T_2 - t; \phi)}{\partial (T_2 - t)} &= -\alpha_n - \sum_{n_1=1}^N \sum_{n_2=1}^N \kappa_{n_1 n_2} \mathcal{E}_{n_1} (T_2 - t; \phi) \\
&+ \frac{1}{2} \sum_{n_1=1}^N \sum_{n_2=1}^N \left( \sum_{i=1}^N \sigma_{n_1 i} \sigma_{n_2 i} \beta_i \right) \mathcal{E}_{n_1} (T_2 - t; \phi) \mathcal{E}_{n_2} (T_2 - t; \phi), \quad (2.08)
\end{aligned}$$

subject to the initial conditions  $\mathcal{E}_n (0; \phi) = \phi_n$  and  $\mathcal{D} (0; \phi) = 0$ . Note that the functional forms of the PDEs (4.6) are the same, but they differ only in the initial conditions.

B.2.2 Proof for Theorem 5 Proof. First notice that the solution of the European price in equation (4.10) is given by

$$P_E(X_t, T_1, T_2) = KB(X_t, T_1 - t) \Upsilon_1(X_t, T_1 - t) - B(X_t, T_2 - t) \Upsilon_2(X_t, T_2 - t), \quad (2.09)$$

where

$$\Upsilon_j(X_t, T_j) = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{1}{i\nu} e^{-i\nu \log K} \tilde{\Upsilon}_j(X_t, T_j - t; \nu_j) \right) d\nu, \text{ for } j = 1, 2$$

with

$$\begin{aligned} \tilde{\Upsilon}_1(X_t, T_1 - t; \nu_1) &= \frac{1}{B(X_t, T_1 - t)} \Gamma(X_t, T_1 - t; \nu_1) \\ \tilde{\Upsilon}_2(X_t, T_2 - t; \nu_2) &= \frac{1}{B(X_t, T_2 - t)} \Gamma(X_t, T_2 - t; \nu_2) \end{aligned}$$

$$\nu_1 = [i\nu C_1(T_2 - t), i\nu C_2(T_2 - t), \dots, i\nu C_N(T_2 - t)]'$$

$$\nu_2 = [(1 + i\nu) C_1(T_2 - t), (1 + i\nu) C_2(T_2 - t), \dots, (1 + i\nu) C_N(T_2 - t)]',$$

[see Chacko and Das (1999)].

Notice that the early exercise premium can be written as

$$E_t^Q \left( \int_t^{T_1} e^{-\int_t^s r_u du} r_s K \Lambda_{\bar{\eta}_s} ds \right) = K \int_t^{T_1} E_t^Q \left( e^{-\int_t^s r_u du} r_s \Lambda_{\bar{\eta}_s} \right) ds. \quad (2.010)$$

Let price  $\Pi(X_t, s - t)$  be given by

$$\Pi(X_t, s - t) = E_t^Q \left[ \frac{e^{\int_t^s r_u du} r_s \Lambda\{X_{N,s} \geq G(Y_s, s)\}}{E_t^Q(e^{\int_t^s r_u du} r_s)} | X_t, t \right],$$

then (2.010) can be written as

$$\begin{aligned} & E_t^Q \left( \int_t^{T_1} e^{-\int_t^s r_u du} r_s K \Lambda_{\bar{\eta}_s} ds \right) \\ &= K \int_t^{T_1} E_t^Q(e^{\int_t^s r_u du} r_s) \Pi(X_t, s - t) ds \\ &= K \int_t^{T_1} \Phi(X_t, s - t; V_0, V_1) \Pi(X_t, s - t) ds, \end{aligned} \quad (2.011)$$

where  $V_0 = (0, 0, \dots, 0)'$ ,  $V_1 = (\alpha_1, \alpha_2, \dots, \alpha_N)'$ .

Given the closed form of the CF [see section (B.1)], the closed form solution of  $\Pi(X_t, s - t)$  can be calculated as follows:

$$\begin{aligned} & \Pi(X_t, s - t) \\ &= E_t^Q \left[ \frac{e^{\int_t^s r_u du} r_s \Lambda\{X_{N,s} \geq G(Y_s, s)\}}{E_t^Q(e^{\int_t^s r_u du} r_s)} | X_t, t \right] \\ &= \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} dX_{1,s} \int_{-\infty}^{\infty} dX_{2,s} \dots \int_{-\infty}^{\infty} dX_{N-1,s} \int_{-\infty}^{\infty} e^{\int_t^s r_u^{(N)} du} \\ & \quad r_s^{(N)} \operatorname{Re} \left( \frac{e^{-i\nu G(Y_s, s)}}{i\nu \Phi(X_t, s - t; V(0), V(1))} \Gamma_N(X_t, s - t; i\nu, Y_s) \right) d\nu \\ & \quad + \frac{1}{2} + \frac{\alpha_N}{2\pi} \int_{-\infty}^{\infty} dX_{1,s} \int_{-\infty}^{\infty} dX_{2,s} \dots \int_{-\infty}^{\infty} dX_{N-1,s} \int_{-\infty}^{\infty} e^{\int_t^s r_u^{(N)} du} \\ & \quad \operatorname{Re} \left( \frac{e^{-i\nu G(Y_s, s)}}{i\nu \Phi(X_t, s - t; V(0), V(1))} \frac{\partial}{\partial \nu} \Gamma_N(X_t, s - t; i\nu, Y_s) \right) d\nu \end{aligned} \quad (2.012)$$

where

$$r_u^{(N)} = \alpha_0 + \sum_{n=1}^{N-1} \alpha_n X_{n,t},$$

$$\Gamma_N(X_t, s-t; i\nu, Y_s) = \int_{-\infty}^{\infty} e^{\alpha_N \int_t^s X_{N,u} du} e^{i\nu X_{N,s}} f((Y_s, X_{N,s})', s; (Y_t, X_{N,t})', t) dX_{N,s},$$

and

$f((Y_s, X_{N,s})', s; (Y_t, X_{N,t})', t)$  is the transition probability density function.

Under the assumption that the early exercise boundary is a linear function of the state variables, i.e.  $G(Y_s, s) = \lambda_{0,s} + \sum_{n=1}^{N-1} \lambda_{n,s} X_{n,t}$ , equation (2.012) implies

$$\begin{aligned} & \Pi(X_t, s-t) \\ &= \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\int_t^s r_u^{(N)} du} r_s^{(N)} \\ & \quad \operatorname{Re} \left( \frac{\exp \left( -i\nu \left( \lambda_{0,s} + \sum_{n=1}^{N-1} \lambda_{n,s} X_{n,t} \right) \right)}{i\nu \Phi(X_t, s-t; V(0), V(1))} \Gamma_N(X_t, s-t; i\nu, Y_s) \right) \\ & \quad dX_{1,s} dX_{2,s} \dots dX_{N-1,s} d\nu + \frac{1}{2} + \frac{\alpha_N}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\int_t^s r_u^{(N)} du} \\ & \quad \operatorname{Re} \left( \frac{\exp \left( -i\nu \left( \lambda_{0,s} + \sum_{n=1}^{N-1} \lambda_{n,s} X_{n,t} \right) \right)}{i\nu \Phi(X_t, s-t; V(0), V(1))} \frac{\partial}{\partial i\nu} \Gamma_N(X_t, s-t; i\nu, Y_s) \right) \\ & \quad dX_{1,s} dX_{2,s} \dots dX_{N-1,s} d\nu. \end{aligned} \tag{2.013}$$

The closed form solution of the extended transform of CF [see B.1] implies that

$$\begin{aligned} & \Phi(X_t, s-t; \phi_0, \varphi_0) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{\int_t^s r_u^{(N)} du} r_s^{(N)} \exp \left( -i\nu \sum_{n=1}^{N-1} \lambda_{n,s} X_{n,s} \right) \\ & \quad \Gamma_N(X_t, s-t; i\nu, Y_s) dX_{1,s} dX_{2,s} \dots dX_{N-1,s}, \end{aligned} \tag{2.014}$$

where  $\phi_0 = (-i\nu\lambda_{1,s}, -i\nu\lambda_{2,s}, \dots, -i\nu\lambda_{N-1,s}, i\nu)$  and  $\varphi_0 = (\alpha_1, \alpha_2, \dots, \alpha_{N-1}, 0)$

$$\begin{aligned} & \Gamma(X_t, s-t; \phi_0) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{\int_t^s r_u^{(N)} du} \exp\left(-i\nu \sum_{n=1}^{N-1} \lambda_{n,s} X_{n,s}\right) \\ & \quad \Gamma_N(X_t, s-t; i\nu, Y_s) dX_{1,s} dX_{2,s} \dots dX_{N-1,s}. \end{aligned} \quad (2.015)$$

Substituting equations (2.014) and (2.015) into (2.013) yields

$$\begin{aligned} & \Pi(X_t, s-t) \\ &= 1 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\nu\lambda_{0,s}} \Phi(X_t, s-t; \phi_0, \varphi_0)}{i\nu \Phi(X_t, s-t; V(0), V(1))} \right) d\nu \\ & \quad + \frac{a_N}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\nu\lambda_{0,s}} Z(\phi_0, s-t) \Gamma(X_t, s-t; \phi_0)}{\nu \Phi(X_t, s-t; V(0), V(1))} \right) d\nu, \end{aligned}$$

with

$$\begin{aligned} & Z(s-t, \phi_0) \\ &= \left( \frac{\partial \mathcal{D}(\phi, s-t)}{\partial \phi_N} + \sum_{n=1}^N \frac{\partial \mathcal{E}_n(\phi, s-t)}{\partial \phi_N} \right) \Big|_{\phi=\phi_0}. \end{aligned}$$

Using the boundary arbitrage conditions, we derive the boundary recursive equation given by Theorem 5. ■

Appendix B.3: Proof for Theorem 6 In this part of the appendix, we present the proof of Theorem 6. To this end, we first need to prove the following lemma.

**Lemma 10** *Let*

$$\Gamma_n(X_t, s-t; i\nu, X_s^{(n)}) = \int_{-\infty}^{\infty} \exp\left(\int_t^s \alpha_n X_{n,u} du + i\nu X_{n,s}\right) f(X_s, s; X_t, t) dX_{n,s} \quad (2.016)$$

and define sample characteristic function as

$$\begin{aligned} & \widehat{\Gamma}_{n,m}(X_t, s-t; i\nu, X_s^{(n)}) \\ &= \int_{-\infty}^{\infty} \exp\left(\int_t^s \alpha_n X_{n,u} du + i\nu X_{n,s}\right) \sum_{l=-\infty}^{\infty} \Lambda_{[l\delta X_n + \chi_n^m, l\delta X_n + \chi_n^{m+1}]} f(X_s, s; X_t, t) dX_{n,s}, \end{aligned} \quad (2.017)$$

where  $X_s^{(n)} = (X_{1,s}, X_{2,s}, \dots, X_{n-1,s}, X_{n+1,s}, \dots, X_{N,s})$ ,  $\delta X_n = X_n^{\max} - X_n^{\min}$  and  $\delta \chi_n = \chi_n^{m+1} - \chi_n^m$ , then

$$\widehat{\Gamma}_{n,m}(X_t, s-t; i\nu, X_s^{(n)}) = \sum_{j=-\infty}^{\infty} \widehat{\vartheta}_{n,m}(j) \Gamma_n(X_t, s-t; i(\nu - jv_{n,0}), X_s^{(n)})$$

where  $\nu_{n,0} = \frac{2\pi}{\delta X_n}$ ,

$$\widehat{\vartheta}_{n,m}(j) = e^{ij\nu_{n,0}(\chi_n^m + \chi_n^{m+1})/2} \frac{\delta \chi_n}{\delta X_n} Sa\left(\frac{1}{2}j\nu_{n,0}\delta \chi\right)$$

and

$$Sa\left(\frac{1}{2}j\nu_{n,0}\delta \chi\right) = \frac{\sin\left(\frac{1}{2}j\nu_{n,0}\delta \chi\right)}{\frac{1}{2}j\nu_{n,0}\delta \chi}$$

**Proof.** First, define the Fourier series  $\vartheta_{n,m} = \sum_{l=-\infty}^{\infty} \Lambda_{[l\delta X_n + \chi_n^m, l\delta X_n + \chi_n^{m+1}]}$ , where  $\vartheta_{n,m}$  is the periodic function with period  $\delta X_n$ . The Fourier transform of  $\vartheta_{n,m}$  is defined as

$$\begin{aligned} \widehat{\vartheta}_{n,m}(j) &= \frac{1}{\delta X_n} \int_{X_n^{\min}}^{X_n^{\max}} e^{-ij\nu_{n,0}X_{n,s}} \Lambda_{[\chi_n^m, \chi_n^{m+1}]} dX_s^n \\ &= e^{ij\nu_{n,0}(\chi_n^m + \chi_n^{m+1})/2} \frac{\delta \chi_n}{\delta X_n} Sa\left(\frac{1}{2}j\nu_{n,0}(\chi_{m+1}^n - \chi_m^n)\right), \end{aligned}$$



where  $\nu_{n,0} = \frac{2\pi}{\delta X_n}$  and  $Sa(x) = \frac{\sin(x)}{x}$ . Given  $\widehat{\vartheta}_{n,m}(j)$ , we write the Fourier series as

$$\vartheta_{n,m} = \sum_{l=-\infty}^{\infty} \Lambda[l\delta X_n + \chi_n^m, l\delta X_n + \chi_n^{m+1}] = \sum_{j=-\infty}^{\infty} \widehat{\vartheta}_{n,m}(j) e^{in\nu_{j,0}t}. \quad (2.018)$$

Substituting equation (2.018) into (2.017) yields

$$\widehat{\Gamma}_{n,m}(X_t, s-t; i\nu, X_s^{(n)}) = \sum_{j=-\infty}^{\infty} \widehat{\vartheta}_{n,m}(j) \Gamma_n(X_t, s-t; i(\nu - j\nu_{n,0}), X_s^{(n)})$$

Since  $Sa(x) = \frac{\sin(x)}{x}$  exponentially collapse to zero in both tails, we do not need many terms  $\widehat{\vartheta}_{n,m}(j) \Gamma_n(X_t, s-t; X_s^{(n)}, i(\nu - j\nu_{n,0}))$  in the above expansion. ■

Next we prove Theorem 6.

**Proof.** To prove the theorem, we will follow similar steps with those that we follow to prove Theorem 5. Here, we will only concentrate on deriving the analytic forms of the prices of the Arrow-Debreu securities. Given that now  $G(Y_s, s)$  is approximated by

$$\begin{aligned} G(Y_s, s) &\equiv G^M(Y_s, s) \\ &= \sum_{l_1=1}^M \sum_{l_2=1}^M \dots \sum_{l_{N-1}=1}^M G_{l_1, l_2, \dots, l_{N-1}}(Y_s, s) \Lambda_{\Xi_{l_1, l_2, \dots, l_{N-1}}} \end{aligned}$$

the prices of the Arrow-Debreu securities can be calculated as follows

$$\begin{aligned}
& \Pi^M (X_t, s - t) \\
&= E_t^Q \left[ \frac{e^{\int_t^s r_u du} r_s \Lambda \{X_{N,s} \geq G(Y_s, s)\}}{E_t^Q (e^{\int_t^s r_u du} r_s)} \middle| X_t, t \right] \\
&= \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\int_t^s r_u^{(N)} du} r_s^{(N)} \\
& \quad \text{Re} \left[ \frac{\exp \left( -i\nu \left( \sum_{l_1=1}^M \sum_{l_2=1}^M \dots \sum_{l_{N-1}=1}^M G_{l_1, l_2, \dots, l_{N-1}} (Y_s, s) \Lambda_{\Xi_{l_1, l_2, \dots, l_{N-1}}} \right) \right)}{i\nu \Phi (X_t, s - t; V_0, V_1)} \right] \\
& \quad \Gamma_N (X_t, s - t; i\nu, Y_s) dX_{1,s} dX_{2,s} \dots dX_{N-1,s} d\nu \\
& + \frac{1}{2} + \frac{\alpha_N}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\int_t^s r_u^{(N)} du} \\
& \quad \text{Re} \left[ \frac{\exp \left( -i\nu \left( \sum_{l_1=1}^M \sum_{l_2=1}^M \dots \sum_{l_{N-1}=1}^M G_{l_1, l_2, \dots, l_{N-1}} (Y_s, s) \Lambda_{\delta_{l_1, l_2, \dots, l_{N-1}}} \right) \right)}{i\nu \Phi (X_t, s - t; V_0, V_1)} \right] \\
& \quad \frac{\partial}{i\partial\nu} \Gamma_N (X_t, s - t; i\nu, Y_s) \Big] dX_{1,s} dX_{2,s} \dots dX_{N-1,s} d\nu
\end{aligned}$$

Since the two tails of probability density of  $X_{n,s}$  exponentially approach to zero [see Lemma 10], if we appropriately truncate  $X_{n,s}$  as  $[X_n^{\min}, X_n^{\max}]$ , we have

$$\Lambda_{[X_n^m, X_n^{m+1}]} \Gamma_N (X_t, s - t; i\nu, Y_s) \cong \sum_{l=-\infty}^{\infty} \Lambda_{[l\delta X_j + \chi_j^m, l\delta X_j + \chi_j^{m+1}]} \Gamma_N (X_t, s - t; i\nu, Y_s) \tag{2.019}$$

Then,  $\Pi^M(X_t, s - t)$  can be written as

$$\begin{aligned} & \Pi^M(X_t, s - t) \\ &= 1 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\nu\lambda_{0,s}} \widehat{\Phi}(X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}, \varphi_0)}{i\nu\Phi(X_t, s - t; V_0, V_1)} \right) d\nu \\ & \quad + \frac{a_N}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\nu\lambda_{0,s}} \widehat{\Psi}(X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}, \varphi_0)}{\nu\Phi(X_t, s - t; V_0, V_1)} \right) d\nu, \end{aligned}$$

where

$$\begin{aligned} & \widehat{\Phi}(X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}, \varphi_0) \\ &= \sum_{l_1=1}^M \sum_{l_2=1}^M \dots \sum_{l_{N-1}=1}^M \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \dots \sum_{n_{N-1}=-\infty}^{\infty} \widehat{\vartheta}_{1, l_1}(n_1) \\ & \quad \widehat{\vartheta}_{2, l_2}(n_2) \dots \widehat{\vartheta}_{N-1, l_{N-1}}(n_{N-1}) \Phi(X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}^*, \varphi_0) \end{aligned}$$

and

$$\begin{aligned} & \widehat{\Psi}(X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}, \varphi_0) \\ &= \sum_{l_1=1}^M \sum_{l_2=1}^M \dots \sum_{l_{N-1}=1}^M \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \dots \sum_{n_{N-1}=-\infty}^{\infty} \\ & \quad \widehat{\vartheta}_{1, l_1}(n_1) \widehat{\vartheta}_{2, l_2}(n_2) \dots \widehat{\vartheta}_{N-1, l_{N-1}}(n_{N-1}) Z(s - t, \phi_{l_1, l_2, \dots, l_{N-1}}^*) \Gamma(X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}^*) \end{aligned}$$

$$\phi_{l_1, l_2, \dots, l_{N-1}} = \left( -i\nu\lambda_{1,s}^{l_1}, -i\nu\lambda_{2,s}^{l_2}, \dots, -i\nu\lambda_{N-1,s}^{l_{N-1}}, i\nu \right)$$

$$\phi_{l_1, l_2, \dots, l_{N-1}}^* = \left( -i\nu\lambda_{1,s}^{l_1} - in_1\nu_{1,0}, -i\nu\lambda_{2,s}^{l_2} - in_2\nu_{2,0}, \dots, -i\nu\lambda_{N-1,s}^{l_{N-1}} - in_{N-1}\nu_{N-1,0}, i\nu \right).$$

■

Appendix B.4: Calculation of the Hedging Parameters: the deltas and gammas In this appendix, we derive the analytic formulas of the hedging parameters: the delta and gamma. To this end, we need to derive the first derivative of the American bond put option with respect to the state variable  $X_{n,t}$ . This constitutes the sums of the derivatives of the European call price  $P_E(X_t, T_1; T_2)$  and the early exercise premium with respect to  $X_{n,t}$ .

The first of the European bond put option price with respect to  $X_{n,t}$  can be calculated as

$$\begin{aligned}
& \frac{\partial}{\partial X_{n,t}} P_E(X_t, T_1; T_2) \\
&= \frac{K}{2} \frac{\partial}{\partial X_{n,t}} B(X_t, T_1 - t) + \frac{K}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{1}{i\nu} e^{-i\nu \log K} \frac{\partial}{\partial X_{n,t}} \Gamma(X_t, T_1 - t; \nu_1) \right) d\nu \\
&\quad - \frac{1}{2} \frac{\partial}{\partial X_{n,t}} B(X_t, T_2 - t) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{1}{i\nu} e^{-i\nu \log K} \frac{\partial}{\partial X_{n,t}} \Gamma(X_t, T_2 - t; \nu_2) \right) \\
&= \frac{K}{2} B(X_t, \tau_1) C_n(T_1 - t) + \frac{K}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{1}{i\nu} e^{-i\nu \log K} \mathcal{E}_n(T_1 - t; \nu_1) \Gamma(X_t, T_1 - t; \nu_1) \right) \\
&\quad - \frac{1}{2} B(X_t, \tau_2) C_n(T_2 - t) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{1}{i\nu} e^{-i\nu \log K} \mathcal{E}_n(T_2 - t; \nu_2) \Gamma(X_t, T_2 - t; \nu_2) \right),
\end{aligned}$$

while that of the early exercise premium can be calculated as follows

$$\begin{aligned}
& \frac{\partial}{\partial X_{n,t}} \Pi^M(X_t, t, s) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\nu \lambda_{0,s}}}{i\nu} \frac{\partial}{\partial X_{n,t}} \widehat{\Phi} \left( X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}, \varphi_0 \right) \right) d\nu \\
&\quad - \frac{a_N}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\nu \lambda_{0,s}}}{\nu} \frac{\partial}{\partial X_{n,t}} \widehat{\Psi} \left( X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}} \right) \right) d\nu,
\end{aligned}$$

where

$$\begin{aligned}
& \frac{\partial}{\partial X_{n,t}} \widehat{\Phi} \left( X_t, s-t; \phi_{l_1, l_2, \dots, l_{N-1}}, \varphi_0 \right) \\
= & \sum_{l_1=1}^M \sum_{l_2=1}^M \dots \sum_{l_{N-1}=1}^M \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \dots \sum_{n_{N-1}=-\infty}^{\infty} \widehat{\vartheta}_{1, l_1}(n_1) \widehat{\vartheta}_{2, l_2}(n_2) \dots \widehat{\vartheta}_{N-1, l_{N-1}}(n_{N-1}) \\
& \times \left[ \xi_n(s-t; \varphi_0) + \left( \zeta(s-t; \varphi_0) + \sum_{n_1=1}^N \xi_{n_1}(s-t; \varphi_0) X_{n_1, t} \right) \mathcal{E}_n \left( s-t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) \right] \\
& \times \Gamma \left( X_t, s-t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial}{\partial X_{n,t}} \widehat{\Psi} \left( X_t, s-t; \phi_{l_1, l_2, \dots, l_{N-1}} \right) \\
= & \sum_{l_1=1}^M \sum_{l_2=1}^M \dots \sum_{l_{N-1}=1}^M \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \dots \sum_{n_{N-1}=-\infty}^{\infty} \widehat{\vartheta}_{1, l_1}(n_1) \widehat{\vartheta}_{2, l_2}(n_2) \dots \widehat{\vartheta}_{N-1, l_{N-1}}(n_{N-1}) \\
& Z \left( s-t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) \mathcal{E}_n \left( s-t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) \Gamma \left( X_t, s-t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right).
\end{aligned}$$

Adding the above two derivations will give us the delta hedging parameter. To derive the gamma parameter, we need to derive the second, cross derivative of European option and early exercise premium with respect to  $X_{j,t}$  and  $X_{l,t}$ . The former of these

derivatives can be calculated as follows:

$$\begin{aligned}
& \frac{\partial^2}{\partial X_{n,t} \partial X_{m,t}} P_E(X_t, T_1; T_2) \\
= & \frac{K}{2} B(X_t, T_1 - t) C_n(T_1 - t) C_m(T_1 - t) + \\
& \frac{K}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{1}{i\nu} e^{-i\nu \log K} \mathcal{E}_n(T_1 - t; \nu_1) \mathcal{E}_m(T_1 - t; \nu_1) \Gamma(X_t, T_1 - t; \nu_1) \right) \\
& - \frac{1}{2} B(X_t, T_2 - t) C_n(T_2 - t) C_m(T_2 - t) \\
& - \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{1}{i\nu} e^{-i\nu \log K} \mathcal{E}_n(T_1 - t; \nu_2) \mathcal{E}_m(T_1 - t; \nu_2) \Gamma(X_t, T_1 - t; \nu_2) \right),
\end{aligned}$$

while the latter can be calculated as follows

$$\begin{aligned}
& \frac{\partial^2}{\partial X_{n,t} \partial X_{m,t}} \Pi^M(X_t, s - t) \\
= & \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\nu \lambda_{0,s}}}{i\nu} \frac{\partial^2}{\partial X_{n,t} \partial X_{m,t}} \widehat{\Phi} \left( X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}, \varphi_0 \right) \right) d\nu \\
& - \frac{a_N}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{e^{-i\nu \lambda_{0,s}}}{\nu} \frac{\partial^2}{\partial X_{n,t} \partial X_{m,t}} \widehat{\Psi} \left( X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}, \varphi_0 \right) \right) d\nu,
\end{aligned}$$

where

$$\begin{aligned}
& \frac{\partial^2}{\partial X_{n,t} \partial X_{m,t}} \widehat{\Phi} \left( X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}, \varphi_0 \right) \\
= & \sum_{l_1=1}^M \sum_{l_2=1}^M \dots \sum_{l_{N-1}=1}^M \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \dots \sum_{n_{N-1}=-\infty}^{\infty} \widehat{\vartheta}_{1, l_1}(n_1) \widehat{\vartheta}_{2, l_2}(n_2) \dots \widehat{\vartheta}_{N-1, l_{N-1}}(n_{N-1}) \\
& \left\{ \xi_m(s - t; \varphi_0) \mathcal{E}_n \left( s - t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) + \xi_n(s - t; \varphi_0) \mathcal{E}_m \left( s - t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) \right. \\
& + \left( \zeta(s - t; \varphi_0) + \sum_{n=1}^N \xi_n(s - t; \varphi_0) X_{n,t} \right) \mathcal{E}_m \left( s - t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) \\
& \left. \mathcal{E}_n \left( s - t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) \right\} \Gamma \left( X_t, s - t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial^2}{\partial X_{n,t} \partial X_{m,t}} \widehat{\Psi} \left( X_t, s-t; \phi_{l_1, l_2, \dots, l_{N-1}} \right) \\
= & \sum_{l_1=1}^M \sum_{l_2=1}^M \dots \sum_{l_{N-1}=1}^M \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \dots \sum_{n_{N-1}=-\infty}^{\infty} \\
& \widehat{\vartheta}_{1, l_1}(n_1) \widehat{\vartheta}_{2, l_2}(n_2) \dots \widehat{\vartheta}_{N-1, l_{N-1}}(n_{N-1}) Z \left( s-t, \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) \\
& \mathcal{E}_n \left( s-t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) \mathcal{E}_m \left( s-t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right) \Gamma \left( X_t, s-t; \phi_{l_1, l_2, \dots, l_{N-1}}^* \right)
\end{aligned}$$

Adding the above two derivatives gives us the closed form solution for the gamma hedging parameter.

## Appendix B.5: CFs for Vasicek's, CIR's and ATSM models

B.5.1 Vasicek Model In an  $n$ -factor extension of Vasicek's model, the instantaneous interest rate is determined by  $n$  state variables as

$$r_t = \sum_{j=1}^N \alpha_j X_{j,t} = \alpha_0 + \alpha' X_t$$

where  $\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_N)$  and  $X_{j,t}$ ,  $j = 1 \dots N$ , under risk neutrality follows the O-U process

$$dX_{j,t} = \kappa_j (\theta_j - X_{j,t}) dt + \sigma_j dW_{j,t}^Q.$$

For simplicity, we assume that  $\text{Corr}(dW_{j,t}^Q dW_{i,t}^Q) = 0$   $i \neq j$  and let  $\alpha_0 = 0$  and  $\alpha_j = 1$ , for all  $j$ . As we know, O-U process implies that  $X_{j,t}$  is normally distributed with mean

$$E_t(X_{j,s}) = X_{j,t} e^{-\kappa_j(s-t)} + \theta_j (1 - e^{-\kappa_j(s-t)})$$

and variance

$$\text{Var}(X_{j,s}) = \frac{\sigma^2}{\kappa_j} (1 - e^{-\kappa_j(s-t)}).$$

Denote  $\tau = T - t$ , then the CF for the  $N$ -factor Vasicek model is given

$$\begin{aligned} \Gamma(X_t, \tau; \phi) &= E_t \left( \exp\left(\int_t^T r_s ds\right) e^{\phi X_T} \right) \\ &= \exp\left(\sum_{j=1}^N [\mathcal{E}_j(\tau; \phi) X_{j,t} + \mathcal{D}_j(\tau; \phi)]\right), \end{aligned}$$

where

$$\mathcal{E}_j(\tau; \phi) = \phi_j b_j(t, T) - B_j(t, T),$$

$$\begin{aligned} \mathcal{D}_j(\tau; \phi) &= \frac{1}{\kappa_j^2} (B_j(t, T) - \tau) \left( \kappa_j^2 \theta_j - \frac{\sigma_j^2}{2} \right) - \frac{\sigma_j B_j(t, T)}{4\kappa_j} + \kappa_j \theta_j \phi_j B_j(t, T) \\ &\quad + \frac{\sigma_j^2 \phi_j}{2} B_j^2(t, T) + \frac{\sigma_j^2 \phi_j^2}{4} (2B_j^2(t, T) - \kappa_j B_j(t, T)), \end{aligned}$$

$$B_j(t, T) = \frac{1}{\kappa_j} (1 - e^{-\kappa_j \tau}),$$

and

$$b_j(t, T) = e^{-\kappa_j \tau}.$$



The extended CF for this model is given by

$$\begin{aligned}\Phi(X_t, \tau; \phi, \varphi) &= E_t \left( \exp\left(\int_t^T r_s ds\right) \varphi X_T e^{\phi X_T} \right) \\ &= \left( \sum_{n=1}^N (\xi_n(\tau; \phi, \varphi) X_{n,t} + \zeta_n(\tau; \phi, \varphi)) \right) \\ &\quad \exp\left(\sum_{j=1}^N \mathcal{E}_j(\tau; \phi) X_{j,t} + \mathcal{D}_j(\tau; \phi)\right),\end{aligned}$$

where

$$\xi_n(T-t; \phi, \varphi) = \varphi_n b_n(t, T)$$

and

$$\zeta_n(\tau; \phi, \varphi) = \varphi_n \left( \kappa_j \theta_j B_j(t, T) + \frac{\sigma_j^2}{2} B_j^2(t, T) + \frac{\sigma_j^2 \phi_j}{2} (2B_j^2(t, T) - \kappa_j B_j(t, T)) \right).$$

**B.5.2 CIR model** For the CIR model with  $N$ -state variables, the instantaneous interest rate is given by

$$r_t = \sum_{j=1}^N \alpha_j X_{j,t} = \alpha_0 + \alpha' X_t,$$

where  $X_{j,t}$  follows the risk neutral process of CIR model

$$dX_{j,t} = \kappa_j (\theta_j - X_{j,t}) dt + \sigma_j \sqrt{X_{j,t}} dW_{j,t}^Q.$$

The CF of the instantaneous interest rate for this model is given by

$$\begin{aligned}\Gamma(X_t, \tau; \phi) &= E_t \left( \exp\left(\int_t^T r_s ds\right) e^{\phi X_T} \right) \\ &= \exp\left(\sum_{j=1}^N [\mathcal{E}_j(\tau; \phi) X_{j,t} + \mathcal{D}_j(\tau; \phi)]\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}_j(\tau; \phi) &= \frac{2}{\sigma_j^2} \left( \frac{(2d_j + \phi_j \sigma_j^2) c_j e^{c_j \tau} - (2c_j + \phi_j \sigma_j^2) d_j e^{d_j \tau}}{(2c_j + \phi_j \sigma_j^2) e^{d_j \tau} - (2d_j + \phi_j \sigma_j^2) e^{c_j \tau}} \right), \\ \mathcal{D}_j(\tau; \phi) &= \frac{2\kappa_j \theta_j}{\sigma_j^2} \ln \left( \frac{2(c_j - d_j)}{(2c_j + \phi_j \sigma_j^2) e^{d_j \tau} - (2d_j + \phi_j \sigma_j^2) e^{c_j \tau}} \right), \\ c_j &= \frac{-\kappa_j + \sqrt{\kappa_j^2 + 2\sigma_j^2}}{2} \quad \text{and} \quad c_j = \frac{-\kappa_j - \sqrt{\kappa_j^2 + 2\sigma_j^2}}{2}\end{aligned}$$

The extended CF is given by

$$\begin{aligned}\Phi(X_t, \tau; \phi, \varphi) &= E_t \left( \exp\left(\int_t^T r_s ds\right) \varphi X_T e^{\phi X_T} \right) \\ &= \left( \sum_{n=1}^N (\xi_n(\tau; \phi, \varphi) X_{n,t} + \zeta_n(T-t; \phi, \varphi)) \right) \\ &\quad \exp\left(\sum_{j=1}^N \mathcal{E}_j(\tau; \phi) X_{j,t} + \mathcal{D}_j(\tau; \phi)\right),\end{aligned}$$

where

$$\begin{aligned}\xi_n(\tau; \phi, \varphi) &= \varphi_n \frac{4(\kappa_n^2 + 2\sigma_n^2) e^{-\kappa_n \tau}}{((2c_n + \phi_n \sigma_n^2) e^{d_n \tau} - (2d_n + \phi_n \sigma_n^2) e^{c_n \tau})^2}, \quad n = 1, 2, \dots, N \\ \text{and} \\ \zeta_n(\tau; \phi, \varphi) &= \frac{-2\kappa_n \theta_n \varphi_n}{(2c_n + \phi_n \sigma_n^2) e^{d_n \tau} - (2d_n + \phi_n \sigma_n^2) e^{c_n \tau}}.\end{aligned}$$

B.5.3 ATSM models For the Affine term structure models (ATSM) of Dai and Singleton (2000), for simplification reasons we assume that the instantaneous interest rate under risk-neutrality determined by  $N=3$  state variables as

$$\begin{aligned} dr_t &= \kappa_r (m_t - r_t) dt + \sqrt{V_t} dW_{r,t}^Q, \\ dV_t &= \kappa_v (\theta_v - V_t) dt + \sigma_v \sqrt{V_t} dW_{v,t}^Q, \\ &\text{and} \\ d\theta_t &= \kappa_m (\theta_m - m_t) dt + \sigma_m \sqrt{m_t} dW_{m,t}^Q, \end{aligned}$$

where  $Corr(dW_{r,t}^Q dW_{v,t}^Q) = \rho$  and  $Corr(dW_{r,t}^Q dW_{m,t}^Q) = Corr(dW_{v,t}^Q dW_{m,t}^Q) = 0$ . The CF of the instantaneous interest rate for this model is given by

$$\begin{aligned} \Gamma(X_t, \tau; \phi) &= E_t \left( \exp\left(\int_t^T r_s ds\right) e^{\phi X_T} \right) \\ &= \exp(\mathcal{E}_r(\tau; \phi) r_t + \mathcal{E}_v(\tau; \phi) V_t + \mathcal{E}_m(\tau; \phi) m_t + \mathcal{D}(\tau; \phi)), \end{aligned}$$

where

$$\begin{aligned} \frac{\partial \mathcal{E}_r(\tau; \phi)}{\partial \tau} &= -1 - \kappa_r \mathcal{E}_r(\tau; \phi), \\ \frac{\partial \mathcal{E}_v(\tau; \phi)}{\partial \tau} &= -\kappa_v \mathcal{E}_v(\tau; \phi) + \frac{1}{2} \mathcal{E}_r^2(\tau; \phi) + \rho V_t \mathcal{E}_r(\tau; \phi) \mathcal{E}_v(\tau; \phi) + \frac{1}{2} \mathcal{E}_v^2(\tau; \phi), \\ \frac{\partial \mathcal{E}_m(\tau; \phi)}{\partial \tau} &= \kappa_r \mathcal{E}_r(\tau; \phi) - \kappa_m \mathcal{E}_m(\tau; \phi) + \frac{1}{2} \mathcal{E}_m^2(\tau; \phi) \quad \text{and} \\ \frac{\partial \mathcal{D}(\tau; \phi)}{\partial \tau} &= \kappa_v \theta_v \mathcal{E}_v(\tau; \phi) + \kappa_m \theta_m \mathcal{E}_m(\tau; \phi), \end{aligned}$$

subject to the following restrictions  $\mathcal{E}_r(\tau; \phi) = \phi_r$ ,  $\mathcal{E}_v(\tau; \phi) = \phi_v$  and  $\mathcal{E}_m(\tau; \phi) = \phi_m$  and  $\mathcal{D}(0; \phi) = 0$ . Solving the above ODEs, we can derive the following closed form

solutions  $\mathcal{E}_r(\tau; \phi)$ ,  $\mathcal{E}_v(\tau; \phi)$ ,  $\mathcal{E}_m(\tau; \phi)$  and  $\mathcal{D}(\tau; \phi)$ :

$$\begin{aligned}\mathcal{E}_r(\tau; \phi) &= \left( \frac{1}{\kappa_r} + \phi_r \right) e^{-\kappa_r \tau} - \frac{1}{\kappa_r}, \\ \mathcal{E}_v(\tau; \phi) &= \frac{\delta_1 [h e^{-\kappa_r \tau} \mathcal{J}_1(\tau) + p e^{-\kappa_r \tau} \mathcal{J}'_1(\tau)]}{\left( \frac{2\kappa_r}{\sigma_m^2} \right) \left[ \delta_1 \mathcal{J}_1(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau) \right]} + \\ &\quad \frac{e^{-\kappa_r(1-b)\tau} [(h e^{-\kappa_r \tau} + 1 - b) \mathcal{J}_2(\tau) + p e^{-\kappa_r \tau} \mathcal{J}'_2(\tau)]}{\left( \frac{2\kappa_r}{\sigma_m^2} \right) \left[ \delta_1 \mathcal{J}_1(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau) \right]}, \\ \mathcal{E}_m(\tau; \phi) &= \frac{\kappa_m \mathcal{Z}_q(w_\tau) - u_\tau [\mathcal{Z}_{q-1}(w_\tau) - \mathcal{Z}_{q+1}(w_\tau)]}{\sigma_m^2 \mathcal{Z}_q(w_\tau)} \text{ and} \\ \mathcal{D}(\tau; \phi) &= \frac{2\kappa_v \theta_v}{\sigma_v^2} \ln \left[ \frac{\delta_1 e^{h(1-e^{-\kappa_r \tau})} \mathcal{J}_1(0) + e^{h(1-e^{-\kappa_r \tau})} \mathcal{J}_2(0)}{\delta_1 \mathcal{J}_1(\tau) + e^{-\kappa_r(1-b)\tau} \mathcal{J}_2(\tau)} \right] \\ &\quad + \frac{2\kappa_m \theta_m}{\sigma_m^2} \ln \left[ \frac{\mathcal{Z}_q(0)}{e^{-\frac{1}{2}\kappa_m \tau} \mathcal{Z}_q(w_\tau)} \right],\end{aligned}$$

where  $\mathcal{J}_1(\tau) = H_{a,b}(p e^{-\kappa_r \tau})$ ,  $\mathcal{J}'_1(T-t) = \frac{a}{b} H_{a+1,b+1}(p e^{-\kappa_r \tau})$ ,  $\mathcal{J}_2(\tau) = H_{a-b+1,2-b}(p e^{-\kappa_r \tau})$ ,  $\mathcal{J}'_2(\tau) = \frac{a-b+1}{2-b} H_{a-b+2,3-b}(p e^{-\kappa_r \tau})$  and  $\mathcal{Z}_q(w_\tau) = \delta_2 J_q(w_t) + \mathcal{N}_q(w_t)$ , where  $H_q(\cdot)$ ,  $J_q(\cdot)$ ,  $Y_q(\cdot)$ , and  $\mathcal{N}_q(\cdot)$  respectively denote the confluent hypergeometric function, first kind Bessel function, second kind Bessel function and Neumann function. The remaining parameters  $h$ ,  $p$ ,  $a$ ,  $b$ ,  $q$ ,  $w_\tau$ ,  $u_\tau$ ,  $\delta_1$ , and  $\delta_2$  are defined as follows:

$$\begin{aligned}h &= \frac{-\sigma_v \phi_r}{2\kappa_r} \left( \rho + i\sqrt{1-\rho^2} \right), p = -2h - \frac{\rho \sigma_v \phi_r}{\kappa_r}, a = \frac{2\kappa_r^2 b h}{4\kappa_r^2 h + 2\rho \sigma_v \kappa_r \phi_r} \\ b &= 1 - \frac{\kappa_v}{\kappa_r}, q = -\frac{\kappa_m}{\kappa_r}, w_\tau = -\frac{\sqrt{2\phi_v}}{\kappa_r} \sigma_m e^{-\frac{1}{2}\kappa_r \tau}, u_\tau = \frac{\sqrt{2\phi_m}}{2} \sigma_m e^{-\frac{1}{2}\kappa_r(T-t)} \\ \delta_1 &= \frac{[-\kappa_r(h+1-b) + \frac{1}{2}\sigma_v^2 \phi_r] \mathcal{J}_2(0) - \kappa_r p \mathcal{J}'_2(0)}{[-\kappa_r h + \frac{1}{2}\sigma_v^2 \phi_r] \mathcal{J}_1(0) - \kappa_r p \mathcal{J}'_1(0)} \\ \delta_2 &= \frac{(\kappa_m - \sigma_m^2 \phi_r) Y_q(w_0) + u_0 Y_{q-1}(w_0) - u_0 Y_{q+1}(w_0)}{(\kappa_m - \sigma_m^2 \phi_r) J_q(w_0) + u_0 J_{q-1}(w_0) - u_0 J_{q+1}(w_0)}\end{aligned}$$

Taking the derivative of CF yields the extended CF, which is given by

$$\begin{aligned}\Phi(, X_t, \tau; \phi, \varphi) &= E_t \left( \exp\left(\int_t^T r_s ds\right) \varphi X_T e^{\phi X_T} \right) \\ &= (\xi_r(\tau; \phi, \varphi) r_t + \xi_v(\tau; \phi, \varphi) V_t + \xi_m(\tau; \phi, \varphi) m_t + \zeta(\tau; \phi, \varphi)) \\ &\quad \exp\left(\sum_{j=1}^N \mathcal{E}_j(\tau; \phi) X_{j,t} + \mathcal{D}_j(\tau; \phi)\right),\end{aligned}$$

where

$$\xi_r(\tau; \phi, \varphi) = \varphi_r e^{-\kappa_r \tau},$$

$$\begin{aligned}\xi_v(\tau; \phi, \varphi) &= \varphi_r \left\{ \frac{\delta_{1r} [h e^{-\kappa_r \tau} \mathcal{J}_1(\tau) + p e^{-\kappa_r \tau} \mathcal{J}'_1(\tau)]}{\left(\frac{2\kappa_r}{\sigma_m^2}\right) [\delta_1 \mathcal{J}_1(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau)]} + \right. \\ &\quad \frac{\delta_1 [h_r e^{-\kappa_r \tau} \mathcal{J}_1(\tau) + p_r e^{-\kappa_r \tau} \mathcal{J}'_1(\tau)]}{\left(\frac{2\kappa_r}{\sigma_m^2}\right) [\delta_1 \mathcal{J}_1(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau)]} + \\ &\quad \frac{\delta_1 [h e^{-\kappa_r \tau} \mathcal{J}_{1r}(\tau) + p e^{-\kappa_r \tau} \mathcal{J}'_{1r}(\tau)]}{\left(\frac{2\kappa_r}{\sigma_m^2}\right) [\delta_1 \mathcal{J}_1(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau)]} + \\ &\quad \frac{e^{-\kappa_r(1-b)\tau} [h_r e^{-\kappa_r \tau} \mathcal{J}_2(\tau) + p_r e^{-\kappa_r \tau} \mathcal{J}(\tau)]}{\left(\frac{2\kappa_r}{\sigma_m^2}\right) [\delta_1 \mathcal{J}_1(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau)]} + \\ &\quad \frac{e^{-\kappa_r(1-b)\tau} [(h e^{-\kappa_r \tau} + 1 - b) \mathcal{J}_{2r}(\tau) + p e^{-\kappa_r \tau} \mathcal{J}'_{2r}(\tau)]}{\left(\frac{2\kappa_r}{\sigma_m^2}\right) [\delta_1 \mathcal{J}_1(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau)]} \\ &\quad \left. \frac{\mathcal{E}_v(\tau; \phi) [\delta_{1r} \mathcal{J}_1(\tau) + (p_r e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau)]}{\delta_1 \mathcal{J}_1(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau)} \right\}, \\ &\quad \left. \frac{\mathcal{E}_v(\tau; \phi) [\delta_1 \mathcal{J}_{1r}(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_{2r}(\tau)]}{\delta_1 \mathcal{J}_1(\tau) + (p e^{-\kappa_r \tau})^{1-b} \mathcal{J}_2(\tau)} \right\},\end{aligned}$$

$$\begin{aligned} \xi_m(\tau; \phi, \varphi) = & \frac{-\varphi_m u_{\tau m} [\mathcal{Z}_{q-1}(w_\tau) - \mathcal{Z}_{q+1}(w_\tau)]}{\sigma_m^2 \mathcal{Z}_q(w_\tau)} + \\ & \varphi_v \frac{\kappa_m \mathcal{Z}_{q,v}(w_\tau) - u_\tau [\mathcal{Z}_{q-1,v}(w_\tau) - \mathcal{Z}_{q+1,v}(w_\tau)]}{\sigma_m^2 \mathcal{Z}_q(w_\tau)} - \varphi_v \frac{\mathcal{E}_m(\tau; \phi)}{\mathcal{Z}_q(w_\tau)} \mathcal{Z}_{qv}(w_\tau) \\ & + \varphi_r \delta_{2r} \frac{\kappa_m J_q(w_t) - u_\tau [J_{q-1}(w_t) - J_{q+1}(w_t)]}{\sigma_m^2 \mathcal{Z}_q(w_\tau)} - \varphi_r \delta_{2r} \frac{\mathcal{E}_m(\tau; \phi)}{\mathcal{Z}_q(w_\tau)} J_q(w_t), \end{aligned}$$

$$\begin{aligned} \zeta(\tau; \phi, \varphi) = & -\varphi_v \frac{2\kappa_m \theta_m \mathcal{Z}_{q,v}(w_\tau)}{\sigma_m^2 \mathcal{Z}_q(w_\tau)} + \varphi_r \frac{2\kappa_v \theta_v}{\sigma_v^2} e^{-h(1-e^{-\kappa_r \tau})} \left\{ \frac{h_r \mathcal{J}_2(0) + \mathcal{J}_{2r}(0)}{\delta_1 \mathcal{J}_1(\tau) + e^{-\kappa_r(1-b)\tau} \mathcal{J}_2(\tau)} \right. \\ & + \frac{\delta_{1r} \mathcal{J}_1(0) + \delta_1 h_r \mathcal{J}_1(0) + \delta_1 \mathcal{J}_{1r}(0)}{\delta_1 \mathcal{J}_1(\tau) + e^{-\kappa_r(1-b)\tau} \mathcal{J}_2(\tau)} \\ & - \frac{\delta_1 e^{h(1-e^{-\kappa_r \tau})} \mathcal{J}_1(0) + e^{h(1-e^{-\kappa_r \tau})} \mathcal{J}_2(0)}{[\delta_1 \mathcal{J}_1(\tau) + e^{-\kappa_r(1-b)\tau} \mathcal{J}_2(\tau)]^2} e^{h(1-e^{-\kappa_r \tau})} [\delta_{1r} \mathcal{J}_1(\tau) \\ & \left. + \delta_1 \mathcal{J}_{1r}(\tau) + e^{-\kappa_r(1-b)\tau} \mathcal{J}_{2r}(\tau)] \right\} \end{aligned}$$

and

$$h_r = \frac{-\sigma_v}{2\kappa_r} \left( \rho + i\sqrt{1-\rho^2} \right), p_r = -2h_r - \frac{\rho\sigma_v}{\kappa_r}, a = \frac{4\kappa_r^3 b \rho \sigma_v (h_r \phi_r - h)}{(4\kappa_r^2 h + 2\rho\sigma_v \kappa_r \phi_r)^2},$$

$$w_\tau = -\frac{1}{\kappa_r \sqrt{2\phi_v}} \sigma_m e^{-\frac{1}{2}\kappa_r \tau}, u_\tau = \frac{1}{2\sqrt{2\phi_m}} \sigma_m e^{-\frac{1}{2}\kappa_r \tau},$$

$$\begin{aligned} \delta_{1r} = & -\frac{[-\kappa_r h_r + \frac{1}{2}\sigma_v^2] \mathcal{J}_2(0) - \kappa_r p_r \mathcal{J}_2'(0) - \kappa_r p \mathcal{J}_{2r}'(0)}{[-\kappa_r h + \frac{1}{2}\sigma_v^2 \phi_r] \mathcal{J}_1(0) - \kappa_r p \mathcal{J}_1'(0)} \\ & - \frac{[-\kappa_r(h+1-b) + \frac{1}{2}\sigma_v^2 \phi_r] \mathcal{J}_{2r}(0)}{[-\kappa_r h + \frac{1}{2}\sigma_v^2 \phi_r] \mathcal{J}_1(0) - \kappa_r p \mathcal{J}_1'(0)} + \frac{\delta_1 [-\kappa_r h_r + \frac{1}{2}\sigma_v^2] \mathcal{J}_1(0) - \kappa_r p_r \mathcal{J}_1'(0)}{[-\kappa_r h + \frac{1}{2}\sigma_v^2 \phi_r] \mathcal{J}_1(0) - \kappa_r p \mathcal{J}_1'(0)} \\ & + \frac{\delta_1 [-\kappa_r h + \frac{1}{2}\sigma_v^2 \phi_r] \mathcal{J}_{1r}(0) - \kappa_r p \mathcal{J}_{1r}'(0)}{[-\kappa_r h + \frac{1}{2}\sigma_v^2 \phi_r] \mathcal{J}_1(0) - \kappa_r p \mathcal{J}_1'(0)}, \end{aligned}$$

$$\delta_{2r} = \frac{(\kappa_m - \sigma_m^2) Y_q(w_0)}{(\kappa_m - \sigma_m^2 \phi_r) J_q(w_0) + u_0 J_{q-1}(w_0) - u_0 J_{q+1}(w_0)} \cdot \frac{\delta_2 (\kappa_m - \sigma_m^2) J_q(w_0)}{[(\kappa_m - \sigma_m^2 \phi_r) J_q(w_0) + u_0 J_{q-1}(w_0) - u_0 J_{q+1}(w_0)]},$$

$$\delta_{2v} = \frac{(\kappa_m - \sigma_m^2 \phi_r) Y_{q,v}(w_0) + u_0 Y_{q-1,v}(w_0) - u_0 Y_{q+1,v}(w_0)}{(\kappa_m - \sigma_m^2 \phi_r) J_q(w_0) + u_0 J_{q-1}(w_0) - u_0 J_{q+1}(w_0)} \cdot \frac{\delta_2 (\kappa_m - \sigma_m^2 \phi_r) J_{q,v}(w_0) + u_0 J_{q-1,v}(w_0) - u_0 J_{q+1,v}(w_0)}{\delta_2 (\kappa_m - \sigma_m^2 \phi_r) J_q(w_0) + u_0 J_{q-1}(w_0) - u_0 J_{q+1}(w_0)},$$

$$\delta_{2m} = \frac{u_{0m} Y_{q-1}(w_0) - u_{0m} Y_{q+1}(w_0)}{(\kappa_m - \sigma_m^2 \phi_r) J_q(w_0) + u_0 J_{q-1}(w_0) - u_0 J_{q+1}(w_0)} \cdot \frac{\delta_2 (u_{0m} J_{q-1}(w_0) - u_{0m} J_{q+1}(w_0))}{\delta_2 (\kappa_m - \sigma_m^2 \phi_r) J_q(w_0) + u_0 J_{q-1}(w_0) - u_0 J_{q+1}(w_0)}.$$

where  $\mathcal{J}_{1r}(\tau) = H_{a,b}(pe^{-\kappa_r \tau}) + \frac{a}{b} H_{a+1,b+1}(pe^{-\kappa_r \tau}) p_r$ ,  $\mathcal{J}'_1(\tau) = \frac{a}{b} H_{a+1,b+1}(pe^{-\kappa_r \tau})$ ,  $\mathcal{J}_2(\tau) = H_{a-b+1,2-b}(pe^{-\kappa_r \tau})$ ,  $\mathcal{J}'_2(\tau) = \frac{a-b+1}{2-b} H_{a-b+2,3-b}(pe^{-\kappa_r \tau})$  and  $\mathcal{Z}_q(w_\tau) = \delta_2 J_q(w_t) + \mathcal{N}_q(w_t) \cdot H_q(\cdot)$ .

APPENDIX C.  
PROOFS OF CHAPTER 5

In this appendix, we proof the results given in the chapter 5.

Appendix C.1: Proof of Theorem 8 Proof. Let  $p^U(t, T) = \log P^U(t, T) = - \int_U^T f(t, u) du$ , where  $P^U(t, T) = \frac{P(t, T)}{P(t, U)} = \exp(p^U(t, T))$ . To prove the theorem, first apply Ito's lemma for a jump process to the forward log-bond price  $p^U(t, T)$ . This yields

$$\begin{aligned}
 & dp^U(t, T) \\
 = & - \int_U^T \mu_f^Q(t, u) du dt - \sum_{m=1}^M \sqrt{V_m(t)} \int_U^T \sigma_{m,f}(t, u) du dZ_m^Q(t) \\
 & - \sum_{i=1}^L \int_G \int_U^T h^{(i)}(x, t, u) du \mu^{(i)}(dx, dt) \\
 = & - \sum_{m=1}^M \frac{1}{2} (S_m^2(t, T) - S_m^2(t, U)) dt \\
 & - \sum_{i=1}^L \int_G (d^{(i)}(x, t, T) - d^{(i)}(x, t, U)) \lambda^{(i)Q}(dx, t) dt \\
 & + \sum_{m=1}^M (S_m(t, T) - S_m(t, U)) dZ_m^Q(t) \\
 & + \sum_{i=1}^L \int_G (D^{(i)}(x, t, T) - D^{(i)}(x, t, U)) \mu^{(i)}(dx, dt).
 \end{aligned}$$



Applying Ito's Lemma to  $P^U(t, T)$  implies

$$\begin{aligned}
& dP^U(t, T) \\
&= -\exp(-p^U(t, T))dp^U(t, T) + \frac{1}{2}\exp(-p^U(t, T))d^2p^U(t, T) \\
&\quad + \exp(-p^U(t_-, T)) \sum_{i=1}^L \int_G \{\exp(D^{(i)}(x, t, T) - D^{(i)}(x, t, U)) - 1\} \mu^{(i)}(dx, dt) \\
&= -P^U(t, T) \sum_{m=1}^M \frac{1}{2} (S_m^2(t, T) - S_m^2(t, U)) dt + P^U(t, T) \sum_{m=1}^M (S_m(t, T) - S_m(t, U)) dZ_m^Q(t) \\
&\quad - P^U(t, T) \sum_{i=1}^L \int_G (d^{(i)}(x, t, T) - d^{(i)}(x, t, U)) \lambda^{(i)Q}(dx, t) dt \\
&\quad + \frac{1}{2} P^U(t, T) \sum_{m=1}^M (S_m(t, T) - S_m(t, U))^2 dt \\
&\quad + P^U(t_-, T) \sum_{i=1}^L \int_G \{\exp(D^{(i)}(x, t, T) - D^{(i)}(x, t, U)) - 1\} \mu^{(i)}(dx, dt) \\
&= P^U(t, T) \sum_{m=1}^M (S_m(t, T) - S_m(t, U)) (dZ_m^Q(t) - S_m(t, U) dt) \\
&\quad + P^U(t_-, T) \sum_{i=1}^L \int_G \frac{d^{(i)}(x, t, T) - d^{(i)}(x, t, U)}{d^{(i)}(x, t, U) + 1} [\mu^{(i)}(dx, dt) \\
&\quad - (d^{(i)}(x, t, U) + 1) \lambda^{(i)Q}(dx, t) dt].
\end{aligned}$$

Notice that under the  $U$ -forward measure,  $P^U(t, T)$  is martingale. Thus, transformation from the risk-neutral,  $Q$  measure to the  $U$ -forward measure implies

$$\begin{aligned}
dZ_m^U(t) &= dZ_m^Q(t) - S_m(t, U) dt, \quad \text{and} \\
\lambda^{(i)U}(dx, t) &= (d^{(i)}(x, t, U) + 1) \lambda^{(i)Q}(dx, t),
\end{aligned}$$

which completes the proof of the theorem. ■

Appendix C.2: Closed form solution of the CF  $\Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t))$  In this subsection of the appendix I derive the closed form solution of the CF  $\Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t))$ .

To this end, I first need to prove the following Lemma. This Lemma gives the closed form solution of  $\Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t))$  under the general assumptions of the HJM model, defined by equations (5.1)-(5.2).

**Lemma 11 (B.1)** *Given the definition of characteristic function,*

$$\Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) = E^U \left[ \exp \left( \sum_{s=1}^{\omega} \phi_j p^{T_0}(T_0, T_s) \right) \right],$$

*the closed form solution of  $\Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t))$  is given by*

$$\Psi_t^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) = \exp \left( \sum_{s=1}^{\omega} \phi_j p^{T_0}(t, T_s) + A(t) + \sum_{m=1}^M B_m(t) V_m(t) \right), \quad (3.01)$$

*where  $A(t)$  and  $B_m(t)$ , for  $m=1, 2, \dots, M$ , satisfy the following ODEs:*

$$\begin{aligned} -\frac{\partial B_m(t)}{\partial t} &= \sum_{s=1}^{\omega} \phi_s \left[ \frac{1}{2} \mathbf{S}_m^2(t, T_s) - \frac{1}{2} \mathbf{S}_m^2(t, T_0) - \mathbf{S}_m(t, U) (\mathbf{S}_m(t, T_s) - \mathbf{S}_m(t, T_0)) \right] \\ &\quad + \frac{1}{2} \sum_{s_1=1}^{\omega} \sum_{s_2=1}^{\omega} \phi_{s_1} \phi_{s_2} (\mathbf{S}_m(t, T_{s_1}) - \mathbf{S}_m(t, T_0)) (\mathbf{S}_m(t, T_{s_2}) - \mathbf{S}_m(t, T_0)) \\ &\quad - \mathbf{S}_m(t, T_0) - \kappa_m^* B_m(t) + \frac{1}{2} B_m^2(t) \end{aligned} \quad (3.02)$$

$$\begin{aligned} -\frac{\partial A(t)}{\partial t} &= \sum_{m=1}^M B_m(t) \kappa_m^Q \theta_m^Q \\ &\quad + \sum_{s=1}^{\omega} \sum_{i=1}^L \phi_s \int_G (d^{(i)}(x, t, T_s) - d^{(i)}(x, t, T_0)) \lambda^{(i)Q}(dx, t) \\ &\quad + \sum_{i=1}^L \int_G \left\{ \exp \left( \sum_{s=1}^{\omega} \phi_s (D^{(i)}(x, t, T_n) - D^{(i)}(x, t, T_0)) \right) - 1 \right\} \lambda^{(i)U}(dx, t) \end{aligned} \quad (3.03)$$

*subject to  $A(T_0) = 0$  and  $B_j(T_0) = 0$  for  $j = 1, 2, \dots, M$*

**Proof of Lemma.** Under the U-forward measure, the dynamics of the log-price  $p^{T_0}(t, T_s)$ , given by  $p^{T_0}(t, T_s) = - \int_{T_0}^{T_s} f(t, u) du$ , can be calculated as follows:

$$\begin{aligned}
& dp^{T_0}(t, T_s) \\
= & - \sum_{m=1}^M \sqrt{V_m(t)} \left[ \int_{T_0}^{T_s} \sigma_{m,f}(t, u) du \right] dZ_m^U(t) - \sum_{i=1}^L \int_G \int_{T_0}^{T_s} h^{(i)}(x, t, u) du \mu^{(i)}(dx, dt) - \\
& \sum_{m=1}^M V_m(t) \left( \int_{T_0}^{T_s} \sigma_{m,f}(t, y) \int_t^y \sigma_{m,f}(t, u) du dy - \int_{T_0}^{T_s} \sigma_{m,f}(t, y) \int_t^U \sigma_{m,f}(t, u) du dy \right) dt \\
& + \sum_{i=1}^L \int_G \int_{T_0}^{T_s} h^{(i)}(x, t, u) e^{D^{(i)}(x, t, u)} du \lambda^{(i)Q}(dx, t) dt \\
= & - \sum_{m=1}^M \frac{1}{2} (S_m(t, T_s) - S_m(t, T_0)) (S_m(t, T_s) + S_m(t, T_0) - 2S_m(t, U)) dt \\
& - \sum_{i=1}^L \int_G (d^{(i)}(x, t, T_s) - d^{(i)}(x, t, T_0)) \lambda^{(i)Q}(dx, t) dt \\
& + \sum_{m=1}^M (S_m(t, T_s) - S_m(t, T_0)) dZ_m^U(t) \\
& + \sum_{i=1}^L \int_G (D^{(i)}(x, t, T_s) - D^{(i)}(x, t, T_0)) \mu^{(i)}(dx, dt).
\end{aligned}$$

Let  $\mathbf{S}_m(t, T) = \int_t^T \sigma_m(t, u) du$ , then the above equation can be written as

$$\begin{aligned}
dp^{T_0}(t, T_s) &= -\frac{1}{2} \sum_{m=1}^M V_m(t) (\mathbf{S}_m(t, T_s) - \mathbf{S}_m(t, T_0)) \\
&\quad (\mathbf{S}_m(t, T_s) + \mathbf{S}_m(t, T_0) - 2\mathbf{S}_m(t, T_0)) dt \\
&\quad - \sum_{i=1}^L \int_G (d^{(i)}(x, t, T_s) - d^{(i)}(x, t, T_0)) \lambda^{(i)Q}(dx, t) dt \\
&\quad - \sum_{m=1}^M \sqrt{V_m(t)} (\mathbf{S}_m(t, T_s) - \mathbf{S}_m(t, T_0)) dZ_m^U(t) \\
&\quad + \sum_{i=1}^L \int_G (D^{(i)}(x, t, T_s) - D^{(i)}(x, t, T_0)) \mu^{(i)}(dx, dt).
\end{aligned}$$

Since the characteristic function should constitute a martingale under the U-forward measure, the following PDE should hold, derived after applying Ito's lemma to the

characteristic function,

$$\begin{aligned}
0 = & \Psi_t^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) + \sum_{m=1}^M \Psi_{V_m}^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) (\kappa_m^* \theta_m^* - \kappa_m^* V_m) + \\
& - \sum_{s=1}^{\omega} \Psi_{p^{T_0}(t, T_s)}^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) \sum_{m=1}^M V_m(t) \\
& \left[ \frac{1}{2} \mathbf{S}_m^2(t, T_s) - \frac{1}{2} \mathbf{S}_m^2(t, T_0) - \mathbf{S}_m(t, U) (\mathbf{S}_m(t, T_s) - \mathbf{S}_m(t, T_0)) \right] \\
& - \sum_{s=1}^{\omega} \Psi_{p^{T_0}(t, T_s)}^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) \\
& \sum_{i=1}^L \int_G (d^{(i)}(x, t, T_s) - d^{(i)}(x, t, T_0)) \lambda^{(i)Q}(dx, t) \\
& + \frac{1}{2} \sum_{m=1}^M \Psi_{V_m V_m}^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) V_m \\
& + \frac{1}{2} \sum_{s_1=1}^{\omega} \sum_{s_2=1}^{\omega} \Psi_{p^{T_0}(t, T_{s_1}) p^{T_0}(t, T_{s_2})}^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) \\
& \sum_{m=1}^M V_m (\mathbf{S}_m(t, T_{s_1}) - \mathbf{S}_m(t, T_0)) (\mathbf{S}_m(t, T_{s_2}) - \mathbf{S}_m(t, T_0)) \\
& + \Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) \\
& \sum_{i=1}^L \int_G \left\{ \exp \left( \phi \sum_{s=1}^{\omega} (D^{(i)}(x, t, T_s) - D^{(i)}(x, t, T_0)) \right) - 1 \right\} \lambda^{(i)U}(dx, t),
\end{aligned}$$

$$\text{subject to } \Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(T_0, T_\omega), \mathbf{V}(T_0)) = \exp \left( \sum_{s=1}^{\omega} \phi_s p^{T_0}(T_0, T_s) \right).$$

Following Duffie et al (2000), the closed form formula of  $\Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t))$  is given by  $\Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), \mathbf{V}(t)) = \exp \left( \sum_{s=1}^{\omega} \phi_s p^{T_0}(t, T_s) + A(t) + \sum_{m=1}^M B_m(t) V_m(t) \right)$ , subject to  $A(T_0) = 0$  and  $B_m(T_0) = 0$ , for  $m = 1, 2, \dots, M$ . Substituting this formula

into the above PDE, I can transform the resulting PDE to an ODE as follows:

$$\begin{aligned}
-\frac{\partial B_m(t)}{\partial t} &= \sum_{s=1}^{\omega} \phi_s \left[ \frac{1}{2} \mathbf{S}_m^2(t, T_s) - \frac{1}{2} \mathbf{S}_m^2(t, T_0) - \mathbf{S}_m(t, U) (\mathbf{S}_m(t, T_s) - \mathbf{S}_m(t, T_0)) \right] \\
&\quad + \sum_{s_1=1}^{\omega} \sum_{s_2=1}^{\omega} \frac{\phi_{s_1} \phi_{s_2}}{2} (\mathbf{S}_m(t, T_{s_1}) - \mathbf{S}_m(t, T_0)) \\
&\quad (\mathbf{S}_m(t, T_{s_2}) - \mathbf{S}_m(t, T_0)) - \kappa_m^* B_m(t) + \frac{1}{2} B_m^2(t)
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial A(t)}{\partial t} &= \sum_{m=1}^M B_m(t) \kappa_m^Q \theta_m^Q + \sum_{s=1}^{\omega} \sum_{i=1}^L \phi_s \int_G (d^{(i)}(x, t, T_s) - d^{(i)}(x, t, T_0)) \lambda^{(i)Q}(dx, t) \\
&\quad + \sum_{i=1}^L \int_G \left\{ \exp \left( \sum_{s=1}^{\omega} \phi_s (D^{(i)}(x, t, T_n) - D^{(i)}(x, t, T_0)) \right) - 1 \right\} \lambda^{(i)U}(dx, t),
\end{aligned}$$

which proves the result of the Lemma. ■

Given the above Lemma, I now can derive the formula of the CF  $\Psi^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}$

$(T_0, T_\omega), V(T_0))$  for the case that the volatility and the intensity parameters,  $\sigma_{m,f}(t, T)$  and  $\widehat{\lambda}^{(i)Q}(dx, t)$ , are respectively given by  $\sigma_{m,f}(t, T) = \frac{\sigma_m}{\beta_m} [1 - \exp(-\beta_m(T - t))]$  and  $\widehat{\lambda}^{(i)Q}(dx, t) = \widehat{\lambda}^{(i)} \gamma^{(i)}(x) e^{-tx} dx$ , and the jump magnitude function is given by  $h^{(i)}(x, t, T) = x$ . This CF is given in the next Lemma.

**Lemma 12** *Provided the above specifications of  $\sigma_{m,f}(t, T)$ ,  $\widehat{\lambda}^{(i)Q}(dx, t)$  and  $h^{(i)}(x, t, T) = x$ , the CF  $\Psi_t^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), V(t))$ , given by above Lemma can be written as*

$$\Psi_t^U(\phi, \mathbf{T}_s | \mathbf{p}^{T_0}(t, T_\omega), V(t)) = \exp \left( \sum_{s=1}^{\omega} \phi_j p^{T_0}(t, T_s) + A(t) + \sum_{m=1}^M B_m(t) V_m(t) \right),$$

where

$$\tau = T_0 - t,$$

$$B_m(\tau) = k + \sqrt{2\alpha} e^{-\beta_m \tau} \frac{\Omega J'_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} e^{-\beta_m \tau} \right) + Y'_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} e^{-\beta_m \tau} \right)}{\Omega J_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} e^{-\beta_m \tau} \right) + Y_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} e^{-\beta_m \tau} \right)}$$

$$A(\tau) = \sum_{m=1}^M -2\kappa_m^Q \theta_m^Q \ln \left( \frac{e^{-k\tau/2} \left[ \Omega J_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} e^{-\beta_m \tau} \right) + Y_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} e^{-\beta_m \tau} \right) \right]}{\Omega J_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} \right) + Y_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} \right)} \right) + \psi \tau,$$

$$\alpha = \sum_{s=1}^{\omega} \frac{\phi_s}{2} (\exp(-\beta_m(T_s - t)) + 1 - \exp(-\beta_m(U - t))) (\exp(-\beta_m(T_s - t)) - 1) \\ + \sum_{s_1=1}^{\omega} \sum_{s_2=1}^{\omega} \frac{\phi_{s_1} \phi_{s_2}}{2} (\exp(-\beta_m(T_{s_1} - t)) - 1) (\exp(-\beta_m(T_{s_2} - t)) - 1),$$

$$\Omega = \frac{a_1}{a_2} = - \frac{kY_{\frac{k}{2\beta_m}}(\sqrt{\frac{\alpha}{2\beta_m^2}}) + \frac{\sqrt{2\alpha}}{2} \left( Y_{\frac{k}{2\beta_m}-1}(\sqrt{\frac{\alpha}{2\beta_m^2}}) - Y_{\frac{k}{2\beta_m}+1}(\sqrt{\frac{\alpha}{2\beta_m^2}}) \right)}{kJ_{\frac{k}{2\beta_m}}(\sqrt{\frac{\alpha}{2\beta_m^2}}) + \frac{\sqrt{2\alpha}}{2} \left( J_{\frac{k}{2\beta_m}-1}(\sqrt{\frac{\alpha}{2\beta_m^2}}) - J_{\frac{k}{2\beta_m}+1}(\sqrt{\frac{\alpha}{2\beta_m^2}}) \right)}$$

and

$$\psi = - \sum_{i=1}^L \sum_{s=1}^{\omega} \hat{\lambda}^{(i)} \phi_s \left[ e^{-T_s m_x^{(i)} + \frac{T_s^2 \sigma_x^{(i)2}}{2}} - e^{-T_s m_x^{(i)} + \frac{T_s^2 \sigma_x^{(i)2}}{2}} \right] \\ + \sum_{i=1}^L \hat{\lambda}^{(i)} \left\{ \exp \left( - \left( \sum_{s=1}^{\omega} \phi_s (T_s - T_0) + U \right) m_x^{(i)} + \frac{(\sum_{s=1}^{\omega} \phi_s (T_s - T_0) + U)^2 \sigma_x^{(i)2}}{2} \right) \right. \\ \left. - \exp \left( -U m_x^{(i)} + \frac{U^2 \sigma_x^{(i)2}}{2} \right) \right\}.$$

**Proof.** Given that  $s_m(t, T) = \frac{\sigma_m}{\beta_m} [1 - \exp(-\beta_m(T - s))]$ ,  $\tau = T_0 - t$ , then ODE in Lemma (B1) can be written as

$$\frac{\partial B_m(\tau)}{\partial \tau} = \frac{1}{2} B_m^2(\tau) - k B_m(\tau) + \alpha e^{-2\beta_m \tau}, \quad (3.04)$$

where

$$\alpha = \sum_{s=1}^{\omega} \frac{\phi_s \sigma_m^2}{2\beta_m^2} (\exp(-\beta_m(T_s - T_0)) + 1 - 2 \exp(-\beta_m(U - T_0))) (\exp(-\beta_m(T_s - T_0)) - 1) \\ + \sum_{s_1=1}^{\omega} \sum_{s_2=1}^{\omega} \frac{\phi_{s_1} \phi_{s_2} \sigma_m^2}{2\beta_m^2} (\exp(-\beta_m(T_{s_1} - T_0)) - 1) (\exp(-\beta_m(T_{s_2} - T_0)) - 1).$$



Equation (3.04) is of Ricatti's form, and thus its solution is given

$$B_m(\tau) = -\frac{2u'_m(\tau)}{u_m(\tau)}, \quad (3.05)$$

where  $u_m(\tau)$  satisfies the ODE

$$u''_m(\tau) + ku'_m(\tau) + \frac{\alpha}{2}e^{-2\beta_m\tau} = 0. \quad (3.06)$$

Let us define  $X = e^{-\beta_m\tau}$  and  $u_m(\tau) = W(X)$ , then equation (3.06) can be written as

$$X^2 \frac{\partial^2 W(X)}{\partial X^2} + \left(1 - \frac{k}{\beta_m}\right) X \frac{\partial W(X)}{\partial X} + \frac{\alpha}{2\beta_m^2} X^2 W(X) = 0. \quad (3.07)$$

If I denote  $W(X) = X^{\frac{k}{2\beta_m}} \Psi\left(\sqrt{\frac{\alpha}{2\beta_m^2}} X\right)$ , where  $Z = \sqrt{\frac{\alpha}{2\beta_m^2}} X$ , the last equation can be written as

$$Z^2 \frac{\partial^2 \Psi(Z)}{\partial Z^2} + Z \frac{\partial \Psi(Z)}{\partial Z} + \left(Z^2 - \frac{k^2}{4\beta_m^2}\right) \Psi(Z) = 0. \quad (3.08)$$

This is Bessel's equation of order  $\frac{k}{2\beta_m}$ , and its solution is given by

$$\Psi(Z) = a_1 J_{\frac{k}{2\beta_m}}(Z) + a_2 Y_{\frac{k}{2\beta_m}}(Z). \quad (3.09)$$

Given (3.09), the solution of  $u_m(\tau)$  [see equation (3.06)] is

$$u_m(\tau) = X^{\frac{k}{2\beta_m}} \left( a_1 J_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}} X\right) + a_2 Y_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}} X\right) \right).$$

Notice that  $B_m(0) = 0$  is equivalent to  $u'_m(0) = 0$ . According to this initial condition, I can write

$$\Omega = \frac{a_1}{a_2} = -\frac{kY_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}\right) + \frac{\sqrt{2\alpha}}{2}\left(Y_{\frac{k}{2\beta_m}-1}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}\right) - Y_{\frac{k}{2\beta_m}+1}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}\right)\right)}{kJ_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}\right) + \frac{\sqrt{2\alpha}}{2}\left(J_{\frac{k}{2\beta_m}-1}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}\right) - J_{\frac{k}{2\beta_m}+1}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}\right)\right)}$$

and

$$B_m(\tau) = k + \sqrt{2\alpha}e^{-\beta_m\tau} \frac{\Omega J'_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right) + Y'_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right)}{\Omega J_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right) + Y_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right)},$$

where

$$\begin{aligned} J'_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right) &= \frac{1}{2}\left(J_{\frac{k}{2\beta_m}-1}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right) - J_{\frac{k}{2\beta_m}+1}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right)\right) \\ Y'_{\frac{k}{2\beta_m}}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right) &= \frac{1}{2}\left(Y_{\frac{k}{2\beta_m}-1}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right) - Y_{\frac{k}{2\beta_m}+1}\left(\sqrt{\frac{\alpha}{2\beta_m^2}}e^{-\beta_m\tau}\right)\right). \end{aligned}$$

Given that  $h^{(i)}(x, t, T) = x$  and  $\lambda^{(i)Q}(dx, t) = \widehat{\lambda}^{(i)}\gamma^{(i)}(x)e^{-xt}dx$ , where  $\gamma^{(i)}(x) = \frac{1}{\sigma_x^{(i)}\sqrt{2\pi}}e^{-\frac{(x-u_x^{(i)})^2}{2\sigma_x^{(i)2}}$ , ODE (3.03) in Lemma (B1) becomes

$$\frac{\partial A(\tau)}{\partial \tau} = \sum_{m=1}^M -\frac{2\kappa_m^Q\theta_m^Q u'_m(\tau)}{u_m(\tau)} + \psi,$$

where

$$\begin{aligned}
\psi = & - \sum_{i=1}^L \sum_{s=1}^{\omega} \hat{\lambda}^{(i)} \phi_s \left[ e^{-T_s m_x^{(i)} + \frac{T_s^2 \sigma_x^{(i)2}}{2}} - e^{-T_s m_x^{(i)} + \frac{T_s^2 \sigma_x^{(i)2}}{2}} \right] \\
& + \sum_{i=1}^L \hat{\lambda}^{(i)} \left\{ \exp \left( - \left( \sum_{s=1}^{\omega} \phi_s (T_s - T_0) + U \right) m_x^{(i)} + \frac{\left( \sum_{s=1}^{\omega} \phi_s (T_s - T_0) + U \right)^2 \sigma_x^{(i)2}}{2} \right) \right. \\
& \left. - \exp \left( -U m_x^{(i)} + \frac{U^2 \sigma_x^{(i)2}}{2} \right) \right\}.
\end{aligned}$$

The solution of the above ODE yields

$$A(\tau) = \sum_{m=1}^M -2\kappa_m^Q \theta_m^Q \ln \left( \frac{e^{-k\tau/2} \left[ \Omega J_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} e^{-\beta_m \tau} \right) + Y_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} e^{-\beta_m \tau} \right) \right]}{\Omega J_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} \right) + Y_{\frac{k}{2\beta_m}} \left( \sqrt{\frac{\alpha}{2\beta_m^2}} \right)} \right) + \psi \tau.$$

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